SIAM REVIEW
Vol. 16, No. 4, October 1974



# ON THE ASSIGNMENT POLYTOPE*

M. L. BALINSKI† AND ANDREW RUSSAKOFF‡

**Abstract.** An expository, completely elementary and self-contained account is given describing several properties of the constraint polytope of the assignment problem. In particular, it is shown that the "Hirsch conjecture" holds, and that to go from any one extreme point to any other, at most 2 extreme edges need to be traversed.

**Introduction.** The $n \times n$ *assignment polytope* $P_n$ is

$$P_n = \left\{ x = (x_{ij}) : \sum_{i=1}^{n} x_{ij} = 1,\ \sum_{j=1}^{n} x_{ij} = 1,\ x_{ij} \geqq 0 \right\}.$$

The interest in this polytope comes from the fact that the well-known and much studied *assignment problem* $AP_n$ is to $\max_x\{cx = \sum_{i,j} c_{ij}x_{ij} : x \in P_n\}$, where $c = (c_{ij})$ are arbitrary real numbers. In fact, the assignment problem—and here we find its baptismal antecedent—stems itself from the following purely combinatorial problem: given $n$ men, $i = 1, \cdots, n$, and $n$ jobs, $j = 1, \cdots, n$, and the "efficiency rating" $c_{ij}$ of man $i$ performing job $j$, find an assignment of the $n$ men to the $n$ jobs which maximizes the total efficiency rating. In this raw form the problem is to find an assignment from among the $n!$ possibilities which is best.

The formation of the combinatorial problem as the (special) linear program $AP_n$ requires a demonstration that solutions $x$ of $AP_n$ exist in which each $x_{ij}$ has values 0 or 1. This is easily done by showing that the extreme points of $P_n$ have precisely this property, that is, that the extreme points $x \in P_n$, thought of as $n \times n$ matrices, are precisely the permutation matrices of "assignments". (The matrices $x \in P_n$ have columns and rows summing to 1. If some $x_{ij}$ is not 0 or 1, hence fractional, that row $i$ has some other fractional entry and so does column $j$. Continue to make this observation and thereby construct a pair of matrices $x^1, x^{11} \in P_n$, each perturbations of $x$, with $x = \frac{1}{2}x^1 + \frac{1}{2}x^{11}$).

Masses of methods for solving the combinatorial assignment problem have been published. Most, if not all, explicitly or implicitly stem from considering the problem as the (special) linear program $AP_n$ and using duality notions. What does this achieve? At best it replaces the search for some optimal permutation from among $n!$ by a relatively simple sequential search requiring at most $\binom{n}{2}$ iterations (see, for example, [1], [9]). But, as in using the celebrated simplex method for solving general linear programs, all ("primal") methods for $AP_n$ proceed, in this sequential search, from one extreme point (or 0-dimensional face) along an extreme edge (or 1-dimensional face) to another "neighboring" extreme point of $P_n$. A considerable study of general polytopes (convex polyhedral sets which are bounded) is now, and has recently been, in process (see [4], [5]). In particular, Klee and Witzgall [7] discuss various aspects of the transportation polytope, of which

* Received by the editors June 5, 1972, and in revised form January 17, 1974. This work was supported by the Army Research Office under Contract DA-31-124-ARD(D)-366.

† Graduate Center, City University of New York, New York, New York 10036.

‡ John Jay College, City University of New York, New York, New York, 10019.

$P_n$ is a specialization. Nevertheless, several rather striking results obtain for $P_n$ which are either untrue for the more general transportation polytopes or not known to be true.

The intent, then, of this paper is to give a very elementary self-contained account of these new results and, hopefully in the process, induce readers to subsequently immerse themselves in true polytopal depths.

**1. Preliminaries.** We will need some definitions and ideas which naturally arise, and are standard fodder, for linear programming, but which are introduced only in the present context. (All definitions are illustrated in Example 1.)

To do this, let $A = (a_{k,ij})$ be the $2n \times n^2$ incidence matrix of "men-jobs", $k = 1, \cdots, n$ (or men $i = 1, \cdots, n$), $n + 1, \cdots, 2n$ (or jobs $j = 1, \cdots, n$) versus "individual assignments" $(i, j) = (1, 1), \cdots, (1, n), (2, 1), \cdots, (2, n), \cdots, (n, 1), \cdots, (n, n)$, with $a_{k,ij} = 1$ if $k = i$ or $k = n + j$, and $= 0$ otherwise.

Every column $A_{ij}$ of $A$ has exactly two 1's, and the remaining entries are 0's. Using $A$, we can write $P_n = \{x; Ax = e, x \geqq 0\}$, where $e$ is a $2n$-vector of 1's.

A *basis* of $P_n$ is a maximal linearly independent set of columns of $A$. Given a basis, the variables $x$ corresponding to its columns are called *basic variables*, the remaining are *nonbasic variables*. Given any basis, a unique solution $x$ to $Ax = e$ with $x_{ij} = 0$ for $x_{ij}$ nonbasic clearly exists. A basis is *feasible*, and its corresponding solution is *feasible*, if $x \in P_n$, i.e., if $x$ solves $Ax = e$, $x \geqq 0$ with $x_{ij} = 0$ for $x_{ij}$ nonbasic. Such an $x$ is a *basic feasible solution*.

Every basic feasible solution $x$ of $P_n$ is an extreme point of $P_n$. (Argument: suppose not, then there exist $x^1$, $x^{11} \in P_n$ with $x = \frac{1}{2}x^1 + \frac{1}{2}x^{11}$. But since $x$, $x^1$, $x^{11} \geqq 0$, $x_{ij} = 0$ implies $x^1_{ij} = x^{11}_{ij} = 0$, whence $x = x^1 = x^{11}$). Thus we can assert that if $x_{ij} > 0$ for $x$ a basic feasible solution of $P_n$, then $x_{ij} = 1$.

In fact, as will be seen, every extreme point $x$ of $P_n$ has slews of "corresponding" feasible bases. For this reason, among others, it is convenient to work with the basic notion of basis and feasible basis. Any basis of $P_n$ has cardinality precisely equal to $2n - 1$. (The $2n$ rows of $A$ are linearly dependent since the sum of the first $n$ is the same as the sum of the last $n$, while any $2n - 1$ rows are linearly independent).

Two bases are *neighbors* if the cardinality of the intersection of their respective column sets is $2n - 2$, or, if their column sets are identical in all but one column. This definition comes from the following "geometry". A feasible basis corresponds to an extreme point. In the "agreeable case" where feasible bases are in one-to-one correspondence to extreme points (and some slight perturbations in the definition of $P_n$ where $e$ is taken close to the vector of 1's but "different" can achieve this), we also have a one-to-one correspondence between nonbasic sets of feasible solutions and extreme points. What is $x_{ij} = 0$? It is a hyperplane. Note that $P_n = M \bigcap_{ij} H_{ij}$, where $M = \{x; Ax = e\}$ is a linear manifold of dimension $n^2 - (2n - 1) = (n - 1)^2$, and $H_{ij} = \{x; x_{ij} \geqq 0\}$, a half-space in $R^{n^2}$. Thus if setting $(n - 1)^2$ $x_{ij}$ to zero admits $x \in P_n$, this $x$ is an extreme point. If in the agreeable case, only $(n - 1)^2 - 1$ of the $x_{ij}$ are set to zero and this admits $x \in P_n$, then a line of solutions $x \in P_n$ is admitted which must connect two extreme points of $P_n$ and is thus an extreme line (or 1-dimensional face) of $P_n$. In our (somewhat disagreeable) case, two different neighboring feasible bases can correspond to one and the same extreme point.

Two extreme points $x \neq y$ of $P_n$ are *neighbors* if $x$ and $y$ are basic feasible solutions having bases $S(x)$ and $S(y)$ which are neighbors.

The structure of bases, feasible bases, neighboring bases, etc., is particularly simple for $P_n$. The natural model through which to display this structure is a finite graph. To this purpose, let $C(n, n)$ be the complete bipartite graph having disjoint node sets $I$ and $J$, with $|I| = |J| = n$, and all edges $(i, j)$, $i \in I, j \in J$. With any $x \in P_n$ associate the bipartite subgraph $B(x)$ of $C(n, n)$ having edges $(i, j)$ if $x_{ij} > 0$. With any basis $S$ of $P_n$ associate the bipartite subgraph $B(S)$ of $C(n, n)$ having edges $(i, j)$ if $A_{ij} \in S$. An *assignment subgraph* of $C(n, n)$ is a subgraph having $n$ edges and all nodes of valency 1. A *path* is a connected graph which has two nodes of valency 1 and whose other nodes all have valency 2. A *circuit* is a connected graph all of whose nodes have valency 2.

THEOREM 1. (a) *The extreme points of $P_n$ are in 1–1 correspondence with the assignment subgraphs of $C(n, n)$.*

(b) *The bases (the feasible bases) $S$ of $P_n$ are in 1–1 correspondence with the spanning trees of $C(n, n)$ (the spanning trees which contain an assignment subgraph).*

(c) *Two feasible bases $S$, $T$ of $P_n$ are neighbors if and only if $B(S) \cup B(T)$ contains exactly one circuit.*

*Proof.* Part (a) is obvious. To show the "feasible bases" part of (b), suppose $x$ is the basic feasible solution corresponding to $S$. Then $x$ is extreme and hence contains an assignment subgraph. Suppose $B(S)$ contains a circuit. Then the circuit may be decomposed into two distinct subassignment subgraphs, say $B_1$ and $B_2$, on the nodes of the circuit in question, and $\sum_{B_1} A_{ij} = \sum_{B_2} A_{ij}$, contradicting linear independence. On the other hand, suppose $B$ is a spanning tree and $S$ the set of columns corresponding to edges $(i, j) \in B$. Consider $\sum A_{ij}\lambda_{ij} = 0$. Since $B$ is a tree it has a node, say $i$, with valency 1, say with incident edge $(i, j)$. Then $\lambda_{ij} = 0$. The argument repeats. Thus, $S$ is a basis. If $B$ contains an assignment subgraph $H$, let $x_{ij} = 1$ for $(i, j) \in H$ and $x_{ij} = 0$ otherwise. This shows $S$ must be feasible. Part (c) is now obvious.

THEOREM 2. *Two extreme points $x \neq y$ of $P_n$ are neighbors if and only if $B(x) \cup B(y)$ contains exactly one circuit.*

*Proof.* $B(x)$, $B(y)$ are assignment subgraphs. So, every node $i \in I$, $j \in J$, has valency 1 or 2 on $B(x) \cup B(y)$. Consider the maximal connected subgraphs of $B(x) \cup B(y)$: each must be either a single edge or an (even) circuit.

Suppose there are $p$ single edges $E$ and one circuit $C$, $|E| = p$, $|C| = 2(n - p)$. Let $F$ be any subset of edges of $C(n, n)$, $F \cap (B(x) \cup B(y)) = \emptyset$, $|F| = p$, which connect the $p + 1$ maximal connected subgraphs of $B(x) \cup B(y)$; and let $(r, s)$, $(r, t)$ be two edges of $C$ with $x_{rs} = y_{rt} = 1$ (hence $s \neq t \in J$). Define

$$S(x) = E \cup F \cup \{(i, j) \in C : (i, j) \neq (r, t)\}$$

and

$$S(y) = E \cup F \cup \{(i, j) \in C : (i, j) \neq (r, s)\}.$$

$S(x)$ and $S(y)$ are feasible bases corresponding to $x$ and $y$, respectively, and $B(S)x)) \cup B(S(y))$ contains exactly one circuit, so $x$ and $y$ are neighbors.

If $B(x) \cup B(y)$ contains more than one maximal connected subgroup which is a cycle, then clearly for any feasible bases $S(x), S(y)$ containing, respectively, the assignment subgraphs $B(x)$, $B(y)$, $B(S(x)) \cup B(S(y))$ must contain more than one circuit.

*Example* 1. $n = 3$.

$$A = \begin{array}{c} \begin{array}{ccccccccc} A_{11} & A_{12} & A_{13} & A_{21} & A_{22} & A_{23} & A_{31} & A_{32} & A_{33} \end{array} \\ \left[\begin{array}{ccccccccc} 1 & 1 & 1 & & & & & & \\ & & & 1 & 1 & 1 & & & \\ & & & & & & 1 & 1 & 1 \\ 1 & & & 1 & & & 1 & & \\ & 1 & & & 1 & & & 1 & \\ & & 1 & & & 1 & & & 1 \end{array}\right] \\ n^2 \end{array} \begin{array}{l} \text{(man 1)} \\ \text{(man 2)} \\ \text{(man 3)} \\ \text{(job 1)} \\ \text{(job 2)} \\ \text{(job 3)} \end{array} \; 2n$$

Feasible basis $S$: $(A_{11}, A_{13}, A_{21}, A_{22}, A_{31})$.

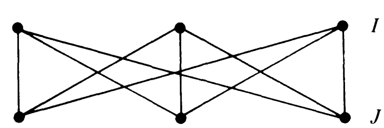
Basic feasible solution $x$ corresponding to $S$: $x_{13} = 1, x_{22} = 1, x_{31} = 1$, remaining $x_{ij} = 0$.

Feasible basis $S'$: $(A_{11}, A_{13}, A_{22}, A_{23}, A_{31})$. Again $x$ is the corresponding basic feasible solution.

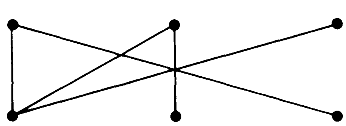
Feasible basis $T$: $(A_{11}, A_{21}, A_{22}, A_{31}, A_{33})$. Its basic feasible solution is $y$: $y_{11} = y_{22} = y_{33} = 1$, remaining $y_{ij} = 0$.

$S$ and $T$ are neighbors. Therefore $x$ and $y$ are neighbors. Let $z$ be: $z_{11} = z_{12} = z_{22} = z_{23} = z_{33} = z_{31} = \frac{1}{2}$, remaining $z_{ij} = 0$. $z \in P_3$.

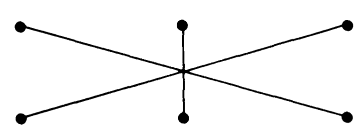
$C(3, 3)$:



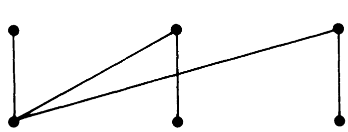

$B(S)$:



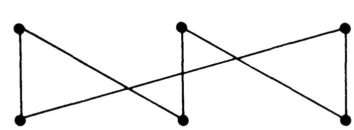
$B(x)$:



$B(T)$:



$B(z)$:



**2. The graph $G(P_n)$.** The *graph of* $P_n$, $G(P_n)$, is the (finite undirected) graph which has as its nodes the extreme points or 0-dimensional faces of $P_n$ and as its edges the extreme edges or 1-dimensional faces of $P_n$. Two nodes $x$, $y$ of a graph are *similar* if for some automorphism $\phi$ of the graph ($\phi$ is a permutation of the node set which preserves adjacency), $\phi(x) = y$. A graph is *node-symmetric* if every pair of points are similar.

THEOREM 3. *$G(P_n)$ is node-symmetric.*

*Proof.* Let $x$, $y$ be nodes of $G(P_n)$, i.e., $x, y$ are extreme points of $P_n$ or permutation matrices. Let $\bar{\varphi}$ be the permutation on the columns of $x$ (considered as a matrix) which takes $x$ into $y$. Equivalently, $\bar{\varphi}$ is the permutation of nodes $J$ taking

$B(x)$ into $B(y)$. Define $\varphi$ for any $z = (z_{ij}) \in G(P_n)$ by $\varphi(z_{ij}) = z_{i\bar{\varphi}(j)}$. $\varphi$ is clearly an adjacency preserving automorphism with $\varphi(x) = y$.

THEOREM 4. (a) *$G(P_n)$ has $n!$ nodes.*

(b) *The valency of each node of $G(P_n)$ is $N(n) = \sum_0^{n-2} \binom{n}{k} (n-k-1)!$*

(c) $\lim_{n\to\infty} N(n)/(n-1)! = e$.

*Proof.* Part (a) is obvious. By Theorem 3, all nodes of $G(P_n)$ have the same valency, and it is sufficient to count the number of neighbors of $x = (x_{ij})$, $x_{ii} = 1$ and $i$, $x_{ij} = 0$ for $i \neq j$.

The number of neighbors $y$ with $B(y)$ containing no edges in common with $B(x) = \{(i, i)\}$ is $(n - 1)!$; with exactly 1 edge in common with $B(x)$, the number is $\binom{n}{1}(n-2)!$; $\ldots$ ; with exactly $k$ edges in common with $B(x)$, the number is $\binom{n}{k}(n-k-1)!$ for $k = 0, 1, \cdots, n-2$. This establishes (b).

To see (c), consider

$$\frac{N(n)}{(n-1)!} = \sum_0^{n-2} \binom{n}{k} \frac{(n-k-1)!}{(n-1)!} = \sum_0^{n-2} \frac{n}{(n-k)k!} = \sum_0^{n-2} \frac{1}{k!} + \sum_1^{n-2} \frac{1}{(n-k)(k-1)!}.$$

Since

$$\sum_1^{n-2} \frac{1}{(n-k)(k-1)} = \frac{1}{n}\left\{\sum_1^{n-2}\left(1 + \frac{k-1}{n-k} + \frac{1}{n-k}\right)\frac{1}{(k-1)!}\right\} \leqq \frac{3}{n}\sum_1^{n-2}\frac{1}{(k-1)!},$$

and the right side goes to zero as $n$ becomes large, the result obtains.

The *distance* between a pair of nodes in a connected graph is the number of edges in a shortest path connecting the nodes. The *diameter* of a connected graph is the greatest distance between any pair of nodes of the graph.

THEOREM 5. *$G(P_n)$ has diameter* 2.

*Proof.* Let $x, y$ be nodes of $G(P_n)$ and $B(x)$, $B(y)$ the corresponding assignment subgraphs. Consider the subgraph $B(x) \cup B(y)$ of $C(n, n)$. It contains between $n + 2$ and $2n$ edges.

If $B(x) \cup B(y)$ contains exactly one circuit, $x$ and $y$ are neighbors in $G(P_n)$, i.e., their distance is 1.

Otherwise $B(x) \cup B(y)$ contains at least two circuits. If $B(x)$ and $B(y)$ have an edge in common, then by applying induction on $n$, the result obtains. Thus, assume $B(x) \cup B(y)$ has $2n$ edges and exactly $p$ disjoint circuits $C_i$, $2 \leqq p \leqq [n/2]$, each circuit's edges alternating between an edge of $B(x)$ and an edge of $B(y)$ (see Fig. 1 for example of the constructive proof which follows).

From each circuit $C_i$ choose an edge $e_i$ of $B(y)$ and drop it from $C_i$ to obtain $p$ disjoint components $\bar{C}_i$ with $B(x) \subset \bigcup \bar{C}_i$, each $\bar{C}_i$ containing one 1-valent $I$-node, one 1-valent $J$-node, and all remaining nodes 2-valent. Let $\bar{E}$ be any $p$ edges of $C(n, n)$ which connect the $p$ components $\bar{C}_i$ into a single circuit. Note the $p$ edges $E$ and the $p$ edges $e_i$ also form a single circuit. Let $\bar{e} \in \bar{E}$. Then $S(x) = \bigcup \bar{C}_i \cup (\bar{E} \sim \bar{e})$ is a basis for $x$.

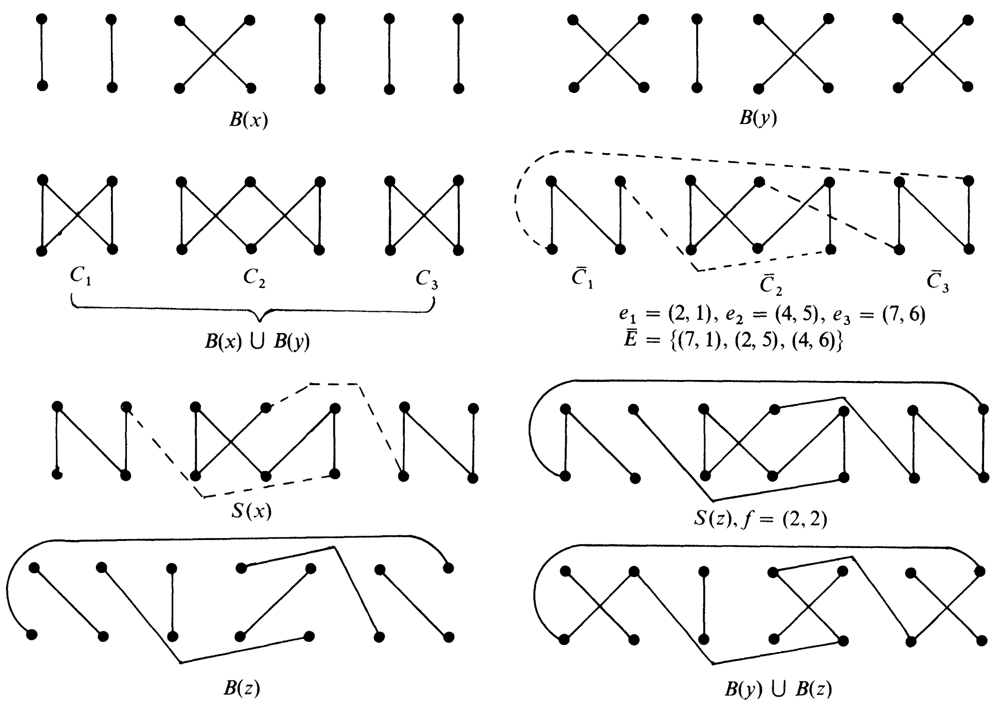



FIG. 1. *I-nodes top, J-nodes bottom*

Let $f$ be any edge of $B(x)$, i.e., $f = (i, j)$, with $x_{ij} = 1$. Consider $S(z) = (\bigcup \bar{C}_i \cup \bar{E}^2) \sim f$. It is a spanning tree of $C(n, n)$ with all nodes of valency 2 save exactly two which are of valency 1. Thus it is the feasible basis for an extreme point $z$ or node $z$ of $G(P_n)$ which is a neighbor of $x$. $B(z)$, the assignment subgraph of $S(z)$, has $n - p$ edges in common with $B(y)$ (on edges of $\bar{C}_i$) and contains the $p$ edges $\bar{E}$. $B(z)$ has no edges in common with $B(x)$.

But $z$ and $y$ are neighbors as well, since the edges of $B(z) \cup B(y) \sim (B(z) \cap B(y))$ form a single circuit consisting of the $p$ edges $e_i$ and the $p$ edges $\bar{E}$. This completes the proof.

In fact, Theorem 5 is equivalent to the following very simple statement concerning permutations, as two Davids—Gale and Walkup—were kind enough to point out.

(S1) *Every permutation is the product of two cycles. Every permutation $p$ on $n$ letters can be uniquely expressed as a product of disjoint cycles, i.e., in the form*

$$p = (j(1), \cdots, j(n_1))(j(n_1 + 1), \cdots, j(n_2)) \cdots ((jn_k + 1), \cdots, j(n)).$$

Extended search through available textbooks has failed to reveal (S1)! Its proof: $p = (j(1), \cdots, j(n))(j(n), j(n_k), \cdots, j(n_1))$.

David Walkup suggests a different manner of expressing and developing our results. Let the permutation $\varphi(x) = (j_1, \cdots, j_n)$ represent $x \in G(P_n)$, where $x_{ij} = 1$, i.e., $B(x) = \{(1, j_1), \cdots, (n, j_n)\}$. Clearly there is a one-to-one correspondence between $x$'s and $p(x)$'s. Let $P(n)$ be the set of all permutations $p$ on $n$ letters, and let

$P'(n)$ be the subset of those permutations which consist of exactly one cycle (of length as least 2). Then we have the following.

(S2) *$x, y$ are neighbors of $P_n$ if and only if* $p(x) = p(y)p(z)$, $p(z) \in P'(n)$.

In our terms, (S2) is immediate. Suppose $p(y)$ is the identity, $p(y) = (1, \cdots, n)$, or $y = (y_{ij})$, $y_{ii} = 1$ for all $i$, $y_{ij} = 0$ otherwise. Then (S2) is the same as Theorem 2. Otherwise, use Theorem 3, and the same reasoning applies to $(p(y))^{-1}p(x)$, $(p(y))^{-1}p(y)$. Thus the valency of each node of $G(P_n)$ is $|P'(n)|$. Finally, we have (S3).

(S3) *The diameter of* $G(P_n)$ *is at most* 2.

Consider two arbitrary permutations $p(x)$, $p(y)$. By (S1), $(p(x))^{-1}p(y) = p(u)p(v)$, with $p(u)$, $p(v) \in P'(n)$, thus $y$ and the vertex, say $w$, corresponding to $p(x)p(u)$ are neighbors. Also, $p(x) = p(x) \cdot p(u) \cdot (p(u))^{-1}$ and $p(u))^{-1} \in P'(n)$, so $x$ and $w$ are neighbors (unless $p(u)$ is the identity).

This construction—if poured from the bottom up—may be more elegant, but it would still leave us with the necessity of introducing the bipartite graph model for the purposes of §3.

A graph is said to be *Hamiltonian* if it contains a circuit which includes all nodes of the grapn.

THEOREM 6. *$G(P_n)$ is Hamiltonian.*

*Proof.* Note that Dirac's theorem shows $G(P_n)$ to be Hamiltonian only for $n \leqq 7$. The proof is constructive.

Each successive pair $x, y$ of the Hamiltonian circuit on $G(P_n)$ to be displayed has the property that $p(y)$ is obtained from $p(x)$ by one interchange of adjacent indices. Thus, $B(x) \cup B(y)$ contains exactly one circuit of 4 edges which, in the "natural" ordering on nodes $I$ and $J$, are always on nodes $k$, $k + 1 \in I$, and $k$, $k + 1 \in J$.

For $n = 2$, a circuit is $(1, 2), (2, 1), (1, 2)$.

For $n = 3$, a circuit is $(1, 2, 3)$, $(1, 3, 2)$, $(3, 1, 2)$, $(3, 2, 1)$, $(2, 3, 1)$, $(2, 1, 3)$, $(1, 2, 3)$.

Assume, inductively, that we have a Hamiltonian circuit for $G(P_{n-1})$, say $p_1^{(n-1)}, p_2^{(n-1)}, \cdots, p_{(n-1)!}^{(n-1)}, p_1^{(n-1)}$ on indices $1, 2, \cdots, n - 1$, and each $p_{j+1}^{(n-1)}$ is obtained from $p_j^{(n-1)}$, as well as $p_1^{(n-1)}$ being obtained from $p_{(n-1)!}^{(n-1)}$, by making one interchange of adjacent indices. We obtain a Hamiltonian circuit for $G(P_n)$ as follows.

*Step* 1. Begin with $p_1^{(n)} = (p_1^{(n-1)}, n)$, and make successive interchanges between $n$ and the index to the left of $n$, obtaining $p_2^{(n)}, \cdots, p_n^{(n)} = (n, p_1^{(n-1)})$.

*Step* 2. Go to $p_{n+1}^{(n)} = (n, p_2^{(n-1)})$ and make successive interchanges between $n$ and the index to the right of $n$, obtaining $p_{n+2}^{(n)}, \cdots, p_{2n}^{(n)} = (p_2^{(n-1)}, n)$.

*Step* 3. Go to $p_{2n+1}^{(n)} = (p_3^{(n-1)}, n)$, etc.

$\vdots$

*Step* $(n - 1)!$. Go to $p_{n!-n+1}^{(n)} = (n, p_{(n-1)!}^{(n-1)})$ and make successive interchanges between $n$ and the index to the right of $n$, obtaining $p_{n!-n+2}^{(n)}, \cdots, p_{n!}^{(n)} = (p_{(n-1)!}^{(n-1)}, n)$; then go to $p_1^{(n)} = (p_1^{(n-1)}, n)$.

The adjacent interchange property holds for $\{p_j^{(n)}\}$, so each successive pair are neighboring nodes of $G(P_n)$. All nodes of $G(P_n)$ are visited since the circuit includes $n!$ nodes and no two are the same. (Note that the circuit for $n = 3$ above is derived from the circuit for $n = 2$ by the construction given.) There are many methods for generating all permutations $P(n)$ by successive interchanges. This particular construction , which preserve adjacency on $G(P_n)$, is due, it turns out, to Selmer Johnson (see [8] for a discussion of several methods).

The *girth* of a graph is the number of edges in a shortest circuit of the graph. The *circumference* of a graph is the number of edges in a longest circuit of the graph. From the foregoing, it is obvious that the girth of $G(P_n)$ is 3, its circumference $n!$. A graph is *n-connected* provided there exist $n$ mutually disjoint paths between every pair of nodes of the graph. We conjecture that $G(P_n)$ is $N(n)$-connected. The conjecture is easily verified for $n \leqq 4$.

## 3. Feasible bases of $P_n$.

THEOREM 7. *To each extreme point of $P_n$ there correspond $2^{n-1}n^{n-2}$ feasible bases. Thus $P_n$ has $2^{n-1}n^{n-2}n!$ feasible bases, and $n^{2(n-1)}$ bases in all.*

*Proof*. Given $x$ an extreme for $P_n$, consider $B(x)$. The number of feasible bases corresponding to $x$ is the number of spanning trees of $C(n, n)$ containing $B(x)$.

Let $C(n)$ be the complete graph on $n$ nodes. Think of the $n$ edges $B(x)$ as $n$ nodes of $C(n)$ which must be spanned by a tree. There are ("Cayley's Theorem") $n^{n-2}$ such trees, and each such gives rise to $2^{n-1}$ spanning trees of $C(n, n)$ since every pair of edges of $B(x)$ may be connected by two different edges. $n^{2(n-1)}$ is the number of spanning trees of $C(n, n)$, hence the number of bases, feasible or not.

The Hirsch conjecture (see [2, pp. 160 and 168], [6]) is a long-standing conjecture originally prompted by consideration of the simplex method for linear programming. In linear programming or simplex method terminology, it asks: given $r$ independent equations in nonnegative variables—meaning a basis has $r$ basic variables—is it possible to go from any one feasible basis to any other feasible basis in $r$ pivot steps with each intermediate basis being feasible? A pivot step is the process of introducing one new column into the current basis at the price of eliminating one column from the basis: this is the work of going from one basis to any one of its neighbors. Note that it is obvious that at most $r$ pivots are necessary if intermediate bases are not required to be feasible. For $P_n$, $r = 2n - 1$. Geometrically, it asks: given a simple convex polyhedron $P$ in $p$-space defined by $q$ half-spaces, is $q - p$ an upper bound on the minimum number of extreme edges which must be traversed in going along extreme edges from any one to any other extreme point of $P$? For $P_n$, $p = (n-1)^2$, $q = n^2$ and $q - p = 2n - 1$. (Note that $P_n$ is not simple since $N(n) > n$.) The following theorem establishes the truth of the conjecture for $P_n$.

THEOREM 8. *A path of neighboring feasible bases of length at most $2n - 1$ connects any pair of feasible bases of $P_n$.*

*Proof*. Note that if $S(x)$, $S'(x)$ are two feasible bases for an extreme point $x$ of $P_n$, then a path of neighboring feasible bases of $x$ of length at most $n - 1$ link $S(x)$ and $S'(x)$. For suppose $S(x) \neq S'(x)$, and consider the spanning trees $B(S(x)) \neq B(S'(x))$ which both contain $B(x)$. Let $e \in B(S'(x))$, $e \notin B(S(x))$. Then $B(S(x)) \cup \{e\}$ contains one circuit $C$ which must contain an edge $f \in B(S(x))$, $f \notin B(S'(x))$, since otherwise $C$ would be contained in $B(S'(x))$. Therefore $B(S(x)) \cup \{f\} \sim \{e\}$ is a spanning tree and, as such, corresponds to a basis of $x$ having one more column in common with $S'(x)$ than does $S(x)$. Since $S(x)$ and $S'(x)$ have at least $n$ columns in common, at most $n - 1$ repetitions of this operation traces out a path linking $S(x)$ and $S'(x)$.

Let $x$, $y$ be extreme points of $P_n$ with bases $S(x)$, $S(y)$ respectively. $B(x) \cup B(y) = \bigcup_1^p C_i$, where each $C_i$ is a connected component which is either a circuit or an edge. Let $2k_i$ be the number of nodes of $C^i$, $\sum_1^p k_i = n$. By the preceding reasoning, a path of neighboring feasible bases of length at most $\sum_1^p (k_i - 1) = n - p$ connects $S(x)$ to a feasible basis $S'(x)$ with $S'(x)$ containing $k_i - 1$ edges from $C_i \cap B(y)$ for each $i$. Thus $S'(x)$ contains all edges of $C_i \cap B(y)$ save one, say $e_i$, for each $i$ with $k_i \geqq 2$.

Adjoin, successively, the one edge $e_i$ for each $i$ having $k_i \geqq 2$ while dropping one edge from $C_i \cap B(x)$. This traces a path of neighboring feasible bases of length at most $p$ (actually of length the number of $C_i$ with $k_i \geqq 2$) linking $S'(x)$ and $S'(y)$, where $S'(y)$ is some feasible basis for $y$. Since the length of any desired path linking $S'(y)$ and $S(y)$ is at most $n - 1$, we obtain the result since a path of length at most $n - p + p + n - 1 = 2n - 1$ links $S(x)$ and $S(y)$.

The above argument actually establishes a corollary.

COROLLARY. *If $S(x)$, $S(y)$ are feasible bases for extreme points $x$, $y$ of $P_n$, and $B(x) \cup B(y)$ forms $p$ connected components of which $q$ ($\leqq p$) are circuits, then a path of neighboring feasible bases of length at most $2n - 2(p - q) - 1$ connects $S(x)$ and $S(y)$.*

The Hirsch conjecture has been proved for an allied special case, namely, for the polytope associated with the shortest route problem [10]. It has also been established for polyhedra arising from Leontief substitution systems [3], a class of polyhedra which includes the shortest route polytope.

Klee and Walkup [6] elucidate the situation for general polyhedra of dimension less than 6. In particular, they show the Hirsch conjecture for unbounded polyhedra is false for dimensions greater than or equal to 4.

*Some statistics for $P_n$*

| $n$ | dimension | no facets | no. extreme pts. | no. neighbor extreme pts. | no. bases per extreme pt. |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 | 1 |
| 2 | 1 | 4 | 2 | 1 | 2 |
| 3 | 4 | 9 | 6 | 5 | 12 |
| 4 | 9 | 16 | 24 | 20 | 144 |
| 5 | 16 | 25 | 120 | 84 | 2000 |
| 6 | 25 | 36 | 720 | 409 | 41,472 |
| 7 | 36 | 49 | 5040 | 2540 | 1,075,648 |
| 8 | 49 | 64 | 40,320 | 16,064 | 33,554,432 |
| $n$ | $(n-1)^2$ | $n^2$ | $n!$ | $N(n)$ | $2^{n-1}n^{n-2}$ |

## REFERENCES

[1] M. L. BALINSKI AND R. E. GOMORY, *A primal method for the assignment and transportation problems*, Management Sci. 10 (1964), pp. 578–593.

[2] GEORGE B. DANTZIG, *Linear Programming and Extensions*, Princeton University Press, Princeton, N.J., 1963.

[3] RICHARD C. GRINOLD, *The Hirsch conjecture in Leontief substitution systems*, Working paper, Center for Research in Management Science, University of California at Berkeley, March 1970.

[4] BRANKO GRUNBAUM, *Convex Polytopes*, Interscience, New York, 1967.

[5] ———, *Polytopes, graphs, and complexes*, Bull. Amer. Math. Soc., 76 (1970), pp. 1131–1201.

[6] VICTOR KLEE AND DAVID W. WALKUP, *The d-step conjecture for polyhedra of dimension $d < 6$*, Acta Math., 117 (1967), pp. 53–78.

[7] VICTOR KLEE AND CHRISTOPH WITZGALL, *Facets and vertices of transportation polytopes*, Mathematics of the Decision Sciences, vol. II, George B. Dantzig and Arthur F. Veinott, Jr., eds., American Mathematical Society, Providence, R.I., 1968, pp. 257–282.

[8] D. H. LEHMER, *The machine tools of combinatorics*, Applied Combinatorial Mathematics, E. F. Beckenbach, ed., John Wiley, New York, 1964.

[9] JAMES MUNKRES, *Algorithms for the assignment and transportation problems*, J. Soc. Indust. Appl. Math., 5 (1957), pp. 32–38.

[10] ROMESH SAIGAL, *A proof of the Hirsch conjecture on the polyhedron of the shortest route problem*, SIAM J. Appl. Math., 17 (1969), pp. 1232–1238.