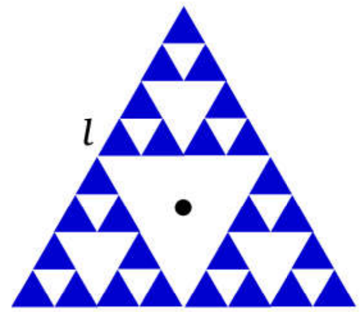
Moment of a Fractal

Take an equilateral triangle of side ℓ, and remove the middle triangle (1/4 of the area). Then remove the middle triangle from each of the remaining three triangles (as shown), and so on, forever. Let the final object have mass M. Find the moment of inertia of this object, around an axis through its center and perpendicular to its plane.



Answer of problem **Moment of a Fractal**

$$\boxed{I = \frac{1}{9}M\ell^2}$$

We'll use scaling arguments and parallel axis theorem, but instead of taking out parts of the triangle, we'll reverse the procedure and we'll be building our shape starting from some tiny-tiny (presumably) trangle of mass m_0 with side length ℓ_0. The moment of inertia of this 'elementary' building block is $I_0 = Cm_0\ell_0^2$ - and it turns out that the exact coefficient here $C \sim 1$ does not matter in the end! (It kind of makes sense because after infinite number of triangle 'dilutions' it is not quite clear what kind of elementary block we get!)

Now we step by step will be building our shape, and in $n \to \infty$ limit we'll get the needed result. Assume at some building level n we know the mass, size and moment of inertia of our shape, so their values at the next level will be

$$m_{n+1} = 3m_n \qquad \ell_{n+1} = 2\ell_n \qquad I_{n+1} = 3(I_n + m_n a_n^2) = 3I_n + m_n\ell_n^2$$

where $a_n = \ell_n/\sqrt{3}$ is the distance between the center of the figure at level n and the center of the new figure.

For mass and size at level n we have

$$m_n = 3^n m_0 \underset{n\to\infty}{\to} M \qquad \ell_n = 2^n\ell_0 \underset{n\to\infty}{\to} \ell \qquad \Rightarrow \qquad m_n\ell_n^2 = 12^n m_0\ell_0^2 \underset{n\to\infty}{\to} M\ell^2$$

The moment of inertia at this level is found by iterations,

$$\begin{aligned}
I_1 &= 3Cm_0\ell_0^2 + m_0^2\ell_0^2 = (3C+1)m_0\ell_0^2 \\
I_2 &= 3I_1 + m_1\ell_1^2 = (3^2C + 3 + 12)m_0\ell_0^2 \\
I_3 &= 3I_2 + m_2\ell_2^2 = (3^3C + 3^2 + 3*12 + 12^2)m_0\ell_0^2 \\
&\vdots \\
I_n &= (12^{n-1} + 12^{n-2}3 + \dots + 3^{n-1} + 3^nC)m_0\ell_0^2
\end{aligned} \qquad (1)$$

Take limit $n \to \infty$ of

$$I = \lim_{n\to\infty} I_n = \lim_{n\to\infty} 12^{n-1}m_0\ell_0^2\left(1 + \frac{1}{4} + \frac{1}{4^2} + \dots + \frac{1}{4^{n-1}} + \frac{1}{4^n}C\right)$$

the last term, dependent on C, plays no role in the infinite series which results in

$$= \frac{1}{12}M\ell^2\left(1 + \frac{1}{4} + \frac{1}{4^2} + \dots + \frac{1}{4^n} + \dots\right) = \frac{1}{12}M\ell^2\frac{1}{1-1/4} = \frac{1}{12}\frac{4}{3}M\ell^2 = \boxed{\frac{1}{9}M\ell^2}$$

As an exercise do a similar scaling analysis to determine numerical value of coefficient C in moment of inertias $I = CmL^2$ of solid triangle and solid square of mass m and length of the side L. Check your answer by direct integration.