

A Bicycle with Flower-Shaped Wheels

Differential Geometry (MATH 474) final project

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Motivation:

Our project was inspired by the creation of a square-wheeled bicycle. We had heard of this possibility in class, and our original idea was to create our own square-wheeled bicycle. The square-wheeled bicycle rolls smoothly on a track comprised of a series of upside-down catenary-shaped humps. We found many simulations of this bicycle online, and learned that a bicycle with wheels shaped like any regular polygon can be created with a catenary-shaped track. As the number of sides of the polygon increases, the height of the upside-down catenaries decreases, and the track approaches a flat surface.

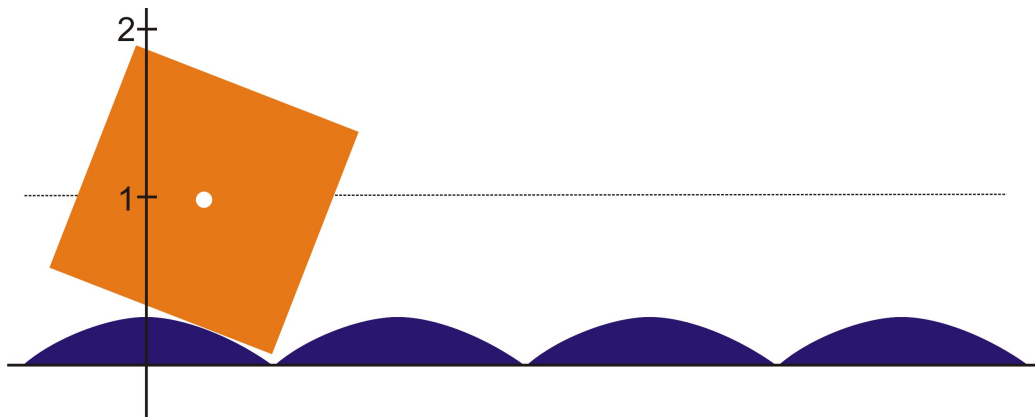


Figure 1: Square wheel and track

In our research for the square-wheeled bicycle, we came across a mathematics web page for children that mentioned briefly the possibility of creating a bicycle wheel of *any* rotationally symmetric shape and forming the appropriate track based upon this shape. This web page included no mathematical description for how this was to be done, but did include a picture of a flower-shaped wheel running on a zig-zag shaped track.

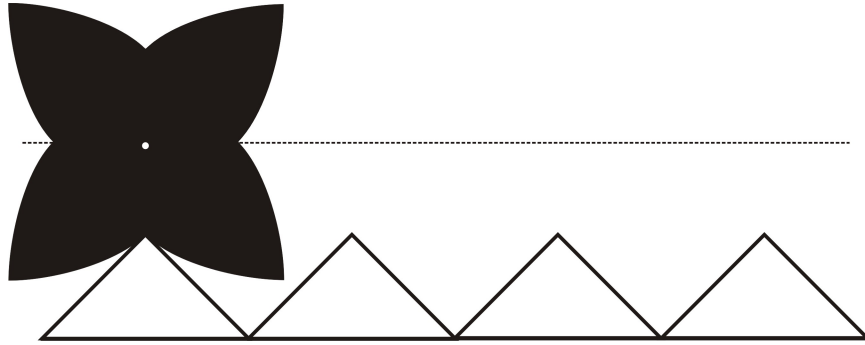


Figure 2: Flower-shaped wheel and track

This idea of a flower-shaped wheel intrigued us for several reasons. First, we thought that it would be easier to construct a right-angle zig-zag track than it would be to construct a catenary-shaped track. Second, it was something very original - as far as we could tell, no one had ever actually created a flower-wheeled bicycle before. Because of these reasons, we decided to change our project from creating a square-wheeled bicycle to creating a flower-wheeled bicycle.

Mathematics:

Before beginning our work on the flower-wheeled bicycle, we spent some time making sure that we understood the mathematics behind the square-wheeled bicycle. Our original thought was that the flower-wheeled bicycle would be a sort of inverse problem to the square-wheeled bicycle since our track was a series of squares. We assumed that our flower would be comprised of eight catenaries, two for each petal. We first approached the problem from this angle, and tried to use the geometry of the situation and the general catenary equation to extract a model for our flower. However, this method turned out to be ineffective and difficult due to the complexity of the catenary equation (the inverse of an arccosine function involves complex numbers), so we decided to approach the problem differently. Instead of assuming that the equation for each side of each petal would be the equation of a catenary on a rotated axis, we decided to make no assumptions on the nature of the curve and just use the necessary properties to extract our own differential equation. There were several things that needed to be fulfilled by the situation, and these were the restricting factors that we used to develop our differential equation:

- The arclength of one side of a petal needed to be equal to the length of one side of our track.

- The center of the flower needed to remain at the same height during the entire rotation to ensure that our bike wouldn't wobble.
- The tangent vectors of the flower at the petal tips and inner petal crease needed to form right angles

With these factors in mind, we drew up a model for the situation and created a differential equation. Our initial model looked like this:

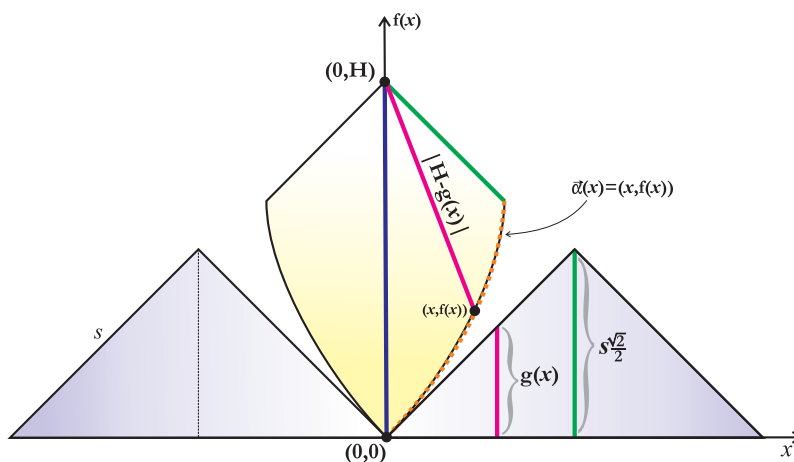


Figure 3: Model of one petal

In this model, $\vec{\alpha}(x) = (x, f(x))$ is curve that describes one half of one petal of our flower. Since the arclength of this curve must be equal to the length of the side, we took a line integral and set it equal to the side length of part of our track:

$$\vec{\alpha}'(x) = (1, f'(x))$$

$$\begin{aligned} \text{arclength of purple section of curve} &= \int_0^x \sqrt{1 + f'(x\zeta)^2} d\zeta = \text{length of purple section of track} \\ &= \sqrt{2}g(x) \end{aligned}$$

This gives us an equation in terms of the functions $f(x)$ and $g(x)$. To simplify, we need to find $g(x)$ in terms of $f(x)$ by considering the equation of the pink line. The pink line passes through the points $(x, f(x))$ and $(0, H)$, so its equation is:

$$\text{pink line: } l(\chi) = \frac{f(x)-H}{x}\chi + H$$

At $\chi = g(x)$, $l(\chi) = g(x)$ because the triangle created by the section of the track, the pink vertical line, and the x-axis is a 45-45-90 triangle. So,

$$g(x) = \frac{f(x)-H}{x}g(x) + H$$

$$g(x)[1 - \frac{f(x)-H}{x}] = H$$

$$g(x) = \frac{Hx}{x-f(x)+H}$$

This gives us an equation for $g(x)$ in terms of $f(x)$, which we can plug into our equation for arclength above.

$$\int_0^x \sqrt{1 + f'(\zeta)^2} d\zeta = \frac{\sqrt{2}Hx}{x-f(x)+H}$$

Differentiating both sides, we get

$$\sqrt{1 + f'(x)^2} - \sqrt{1 + f'(0)^2} = \frac{d}{dx} \left(\frac{\sqrt{2}Hx}{x-f(x)+H} \right)$$

But we know that the track must be tangent to the curve, so $f'(0)$ will just be the slope of the ramp, which is 1. This leaves us with

$$\sqrt{1 + f'(x)^2} - \sqrt{2} = \frac{(x-f(x)+H)\sqrt{2}H - \sqrt{2}Hx(1-f'(x))}{(x-f(x)+H)^2}$$

$$1 + f'(x)^2 = \left(\frac{-\sqrt{2}H(f(x)-xf'(x)-H)}{(x-f(x)+H)^2} + \sqrt{2} \right)^2$$

Simplifying this, our resulting differential was

$$f'(x) = \sqrt{\left(\frac{-\sqrt{2}H(f(x)-xf'(x)-H)}{(x-f(x)+H)^2} + \sqrt{2} \right)^2 - 1}$$

The solution to this differential equation would give us the equation for the curve representing one side of one petal of our flower. We need to consider this curve on the interval $x \in [0, c]$ where c is the x-value corresponding to the endpoint of the curve. This x-value will depend on the side length of our track, and also the equation of our curve. To solve for this x-value, we can set $g(x) = \frac{s\sqrt{2}}{2}$, where s is the side length of our track.

$$\frac{s\sqrt{2}}{2} = \frac{Hx}{x-f(x)+H}$$

$$x = \frac{s\sqrt{2}(f(x)-H)}{s\sqrt{2}-2H}$$

Therefore, Our differential equation gives us the equation for a graph on the interval $x \in [0, \frac{s\sqrt{2}(f(x)-H)}{s\sqrt{2}-2H}]$. To find the rest of our flower, we can first reflect this section of a curve in the y-axis, which will give us one petal of our flower. Then we can rotate this petal about the point $(0, H)$, which is the center of our petal, by $\pi/2$ three times to give us the other three petals and create the entire flower.

The problem with this method for approaching our problem is that the differential equation that we found is not an easy differential equation to solve. We attempted to use numerical methods in maple and mathematica to get a graph of our situation, but we were unsuccessful. But this did not stop us from actually designing and creating our bicycle. Instead of using a specific equation to model the situation, we used the three necessary properties that I mentioned earlier and some vector modeling software to create a template.

Vector modeling:

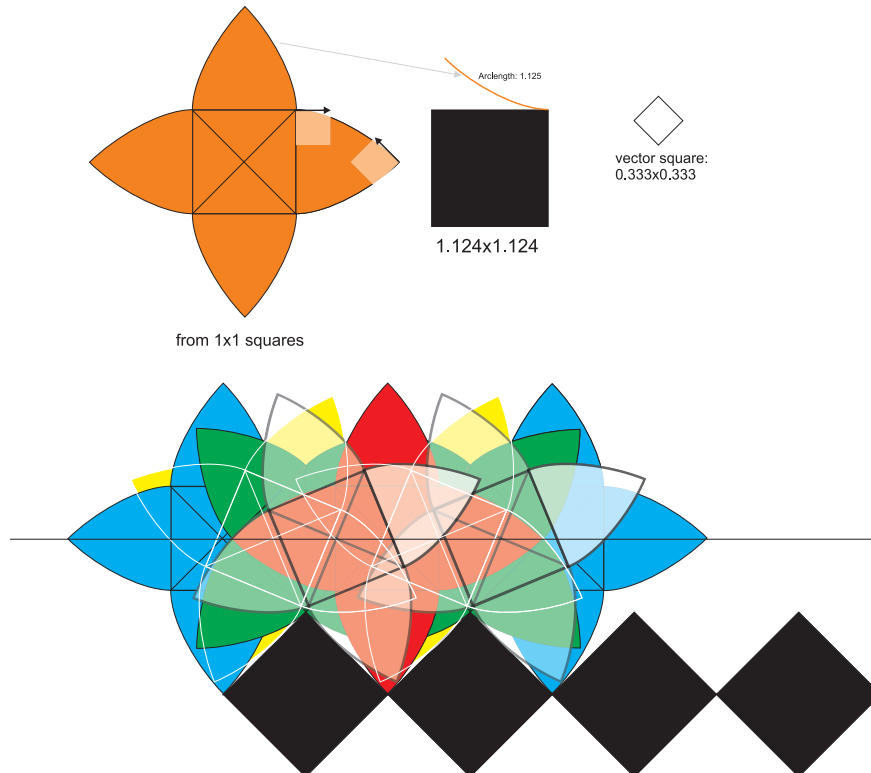


Figure 4: Vector models of flower wheel and track

In our attempt at creating a viable model for our wheel track, we pursued the idea of using vector graphics software to model our curve. In vector graphic software one may define images as nodes, lines, and tangent vectors at the nodes, which define the curvature of the lines connecting them. Our idea, then, was to begin with an isosceles triangle, representing one petal of our wheel, and define the vectors from each node to ensure the angles we required at each point. Unfortunately, our software was intended for graphic design artists, not mathematicians. Therefore, one cannot define vectors by their equations. (Notice the “vector square” defining the petals tangent vectors in figure 4.)

Now, with a somewhat arbitrary triangle height (we chose 1” so all calculations that followed might be made more simple), we could determine the length of each square from which would form our track. Unfortunately, this gave rise to the question of, in defining the petal curvature, what length vectors we should use to create an arc length equal to our newly defined track. The solution to this was founded more in intuition than in our not-yet-solved differential equation. As seen in figure 4, the pedal was embedded in five 1”x1” squares, yielding a track square of 1.124”. We then arbitrarily defined vectors for the petal which, to no surprise, did not produce a sufficient arc length. Then, based on two parts visual intuition, one part luck, and three parts mathemagic, we decided to define the curve by vectors of $1/3$ ”. Amazingly, this gave us the perfect arc for our now defined track.

Utilizing the same software, we tested the design for the wheel by placing it at different rotations on a track of proper length and checked the following:

1. That the center of the wheel remained at the same height no matter the rotation angle.
2. That the tip of the petal never passes through the track (which, of course, would have made a physical model impossible).
3. That it would look neat if done in different colors.

Fortunately, all three criteria were met, and we proceeded to model the wheel in 3D in order to animate it (the animation can be seen at <http://www.youtube.com/watch?v=twHyywnJrr4>), thereby further validating the design. After a successful 3D modeling, we decided to move the wheel and track into production.

Construction:

The main part of our project was the actual construction of this bicycle. Being mathematicians, this was a task that was somewhat foreign to us. Our first step was to decide what type of material

that we wanted to use. We decided that 3/4" plywood would be strong enough to hold up the bicycle frame, and that we could make our track out of thicker boards. A trip to Home Depot supplied us with one sheet of 3/4" plywood, two 12" boards, and six cans of spray paint. A friend donated an old bicycle frame to our cause, giving us all of our necessary materials.

Our next step was to make an accurate, to-scale stencil of our flower. We measured the size of the bicycle wheels in the bike, and made this the tip-to-tip distance of our flower. The vector-constructed flower was made for a 1-by-1 square track, making the conversions simple matters of multiplication. To create this stencil, we projected our vector image onto a blank white wall, and adjusted the size until it was accurate. Then we held up a piece of poster board and traced the projected image, creating our wheel template.

Armed with our to-scale template, we spent a series of Saturdays constructing. Using the template and a jig saw, we cut out our flower-shaped wheels from the plywood. We also started constructing the track by cutting the boards into pieces and screwing them together at right angles. Once we had made a few pieces of the track, we tried out our wheels and they rolled very well. From here, we continued and made as long of a track as we could with our two boards. Once the track was constructed, we decided to use the extra plywood to make siding for the track to hold it in place and ensure that the right angles stayed as right angles and didn't give under the weight of the bicycle.

The next obvious step in our construction was decoration. We painted our track and wheels, and decided on designs of cats and canaries, since we think the curve is a rotated catenary. (Cat + Canary = Catenary!) We also needed to attach some grip to the track and the wheels because wood rolling on wood is too slippery, and the wheels didn't have enough traction to turn properly. We used some sticky cabinet grip that Anna's roommate happened to have lying around the kitchen as the grip on the track, and attached it with spray adhesive.

Our next task was to attach the wheels to the bicycle frame, which proved to be the most difficult part of construction. Luckily, a friend of Dave's with some knowledge in bicycle repair agreed to help us. We decided that the best way to attach our wheels, while still letting them spin, was to fix dowels to the bike frame, and have the wheels rotate around the fixed dowels. First, we drilled holes barely larger than the dowels in the center of each wheel. We then attached extra plywood to the wheels, at either side of the holes, for added stability. Initially, after pushing the dowels through the wheels, the fit was too snug for proper rotation. After some sanding, the wheels began to rotate as needed. All we then had to do was detach the brakes which were now rendered useless. Another thing that was

rendered useless by this construction were the pedals. We had spent some time debating whether or not to actually make the bike rideable, and in the end decided against it. To begin with, we were a little nervous about the plywood wheels having the strength to hold a person's weight. And making the brakes and gears function with these new wheels would have been a whole new task in itself. So we decided that it would be better to make a bike that would roll over the tracks, but not that someone would actually ride.

Our final task was to attach some sort of grip onto the wheels, and hope that our whole contraption worked. We chose bike tubes instead of bike tires, because we wanted to keep the shape of the wheels as well as possible. We cut open an old bike tube and used a staple gun to attach it to the edges of the wheels. Then, we held our breath and tried out the finished product.

Unfortunately, it didn't work too well. The grip on the track was tearing when the wheels hit it, and the bike didn't roll very smoothly. We decided to take the grip off the track and try the contraption on just wood. This did make the bike roll more smoothly, but it still wasn't the perfect fit that we were expecting. We were somewhat upset about this result, but we were still left with a cool looking bike, so we accepted our shortcomings and decided to analyze our errors.

Error Analysis:

As "pure" mathematicians, dealing with rounded figures was somewhat foreign to us. While the curve we came up with in the vector software required an irrational track length, that same software required we round our figures to three decimal places. Then, to make the wheel, accuracy was lost in tracing the initial template. Further error came in mapping that template to the plywood, and, finally, in cutting the plywood with a rather old jigsaw. Used to dealing with exact figures, we were somewhat worried that all this error would amount to a botched product. Much to our surprise, however, we found that real life allows for quite a bit of error tolerance. When we began to make the track, introducing more error terms, the wheel seemed to work on it quite nicely.

While we cannot say exactly what our error tolerance was, we do know we started by rounding to three decimal places. This, along with the error from tracing and cutting, did not seem to adversely affect the project. Unfortunately, however, the tread we added to the wheel did adversely affect the fit of the wheel on the track. While its added thickness has made for a somewhat "clunkier" ride, it was needed to prevent too much slipping. The steep angle of the track made it very difficult to create enough traction for the wheel to roll smoothly without slipping down the side of the track.

Conclusion:

In the end, our project turned out as well as we could have expected it to. We realized once we had chosen our topic and put some considerable effort into it that we had taken on an immense task. When we were researching the square-wheeled bicycle, we found that many college classes will take the entire semester to design and construct a square-wheeled bicycle. So the two of us attempted a task on our own in two months that it took an entire class an entire semester to complete. So with that in mind, we were very happy with our not-so-perfect results. As mathematics students who lean towards the “pure” side of the math spectrum, it was a great experience for us to use our math to design something real, and then try to create it. This “real-world” application was something that we did not have much experience with, and we learned a lot about the inevitable addition of error tolerance in construction. And finally, this project also enabled us to get out of the darkness of the TILT building, and spend a few Saturdays sitting outside in the sun getting covered in saw dust, which was probably very beneficial to our general well-being.