Series Solution of Weakly-Singular Kernel Volterra Integro-Differential Equations by the Combined Laplace-Adomian Method

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Abstract: *To solve the weakly-singular Volterra integro-differential equations, the combined method of the Laplace Transform Method and the Adomian Decomposition Method is used. As a result, series solutions of the equations are constructed. In order to explore the rapid decay of the equations, the pade approximation is used. The results present validity and great potential of the method as a powerful algorithm in order to present series solutions for singular kind of differential equations.*

Keywords: Volterra integro-differential equations, Laplace Transform Method, Adomian Decomposition Method, Weakly-singular equations

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1. Introduction

To introduce behavior and effects of many phenomena in engineering as well as physics, different types of differential equations are presented. Solving and obtaining solutions of the equations are important area of research in the field of mathematical sciences. As a result, the obtained solutions can be used to solve complex mathematical models from engineering as well as physics problems.

The weakly-singular kernel Volterra Integro differential equation is written as:

$$\sum_{i=0}^{n} a_i(x) y^{(i)}(x) = f(x) + \int_a^x k(t,x) y^m(t) dz, \quad a \le x \le b, \qquad m \ge 1 \tag{1}$$

Where $k(t,x)$ is the kernel of the integral equation. It is usually assumed that the functions $y(x)$ and $f(x)$ are continuous or square integral which can be obtained on $[a,b]$.

The first order of the equation is when $n = 1$ *and* $m \ge 1$. When $n = 2$ *and* $m = 1$, the equation is second order. Also, in case of $t \to \infty$, the Kernel $k(t,x) = \frac{1}{(x-t)^\alpha}$ is singular.

Eq. (1) is important equation in physics sciences as well as engineering in order to describe behaviors of many phenomena such as neutron diffusion and biological species.

Wazwaz studied nonlinear Volterra integro–differential equations by combining the Laplace transform–Adomian decomposition method [1]. Brunner has obtained numerical solution of nonlinear Volterra integro-differential equations [2]. Contea and Preteb used fast collocation methods for Volterra integral equations of

convolution type [3]. The application of spectral Jacobi-collocation methods to a certain class of weakly singular Volterra integral equations is presented by Ma et al. [4]. A numerical solution of weakly singular Volterra integral equations including the Abels equations by the second Chebyshev wavelet method is presented by Zhu and wang [5]. Yi and Huang [6] presented CAS wavelet method for solving the fractional integro-differential equation with a weakly singular kernel. Bernstein series solution of a class of linear integro-differential equations with weakly singular kernel is presented by Işik et al. [7]. Application of the homotopy perturbation method for nonlinear differential equations is presented by Nourazar et al. [8-10]. Soori et al. [11] , [12] presented application of the Variational Iteration Method and the Homotopy Perturbation Method to the fisher type equation.
This paper presents an application for the combined Laplace Transform Method and Adomian Decomposition Method in order to establish series solutions for the weakly-singular kernel Volterra integro-differential equations.
Section 2 presents the idea of combined Laplace-Adomian method. Section 3 presents application of the combined Laplace-Adomian method in order to establish series solutions for the weakly-singular kernel Volterra integro-differential equations.

2. The combined Laplace-Adomian method

To explain idea of the combined Laplace-Adomian method, consider the nonlinear Volterra integral equation:

$$u(x) = f(x) + \lambda \int_0^x k(x-t) f(u(t)) dz, \tag{2}$$

Then the Laplace transform is applied to the both side of Eq. (2),

$$U(s) = L\{f(x)\} + \lambda L\{K(x-t)\} L\{f\big(u(t)\big)\}, \tag{3}$$

For handling and addressing the nonlinear term $f\big(u(x)\big)$, the adomian decomposition and the adomian polynomials can be utilized. Then, the linear term $U(s)$ can be presented by an infinite series of components as Eq. (4),

$$U(s) = \sum_{n=0}^{\infty} U_n(s), \tag{4}$$

And similarly,

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{5}$$

Where the components $U_n(s), n \geq 0$ will be determined recursively. The nonolinear term $F\big(u(x)\big)$ in Eq. (3) are presented by an infinite series of Adomian polynomials A_n as,

$$F\big(u(x)\big) = \sum_{n=0}^{\infty} A_n(x), \tag{6}$$

Where $A_n, n \geq 0$ are defined by,

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F\left(\sum_{i=0}^{n} \lambda^i u_i \right) \right]_{\lambda=0}, n = 0,1,2,3, \ldots \tag{7}$$

We can evaluate the adomian polynomials A_n for all terms of nonlinearity. It is assumed that the nonlinear function is $F\big(u(x)\big)$. So, the adomian polynomials can be achieved by,

$$\begin{aligned}
A_0 &= F(u_0), \\
A_1 &= u_1 F'(u_0), \\
A_2 &= u_2 F'(u_0) + \frac{1}{2!} u_1^2 \; F''(u_0), \\
A_3 &= u_3 F'(u_0) + u_1 u_2 \, F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0), \\
A_4 &= u_4 F'(u_0) + \left(\frac{1}{2!} u_2^2 + u_1 u_3 \right) F''(u_0) + \frac{1}{2!} u_1^2 u_2 F'''(u_0) + \frac{1}{4!} u_1^4 F^{(iv)}(u_0),
\end{aligned} \tag{8}$$

Substituting Eq. (5) and Eq. (6) in to Eq. (3) we have,

$$L\{\sum_{n=0}^{\infty} U_n(s)\} = L\{f(x) + \lambda L\{K(x-t)\}L\{\sum_{n=0}^{\infty} A_n(x)\}, \quad (9)$$

By using the adomian decomposition method the recursive relation can be shown as,

$$U_0(s) = L\{f(x)\}, \quad (10)$$

$$U_{k+1}(s) = \lambda L\{K(x-t)\}L\{A_k(x)\}, K \geq 0,$$

By applying inverse Laplace transform to the first part of Eq. (10), we can obtain $u_0(x)$. Then, $A_0(x)$ can be defined by using $u_0(x)$. Also, $u_1(x)$ can be found by using $A_0(x)$. The determination of $u_0(x)$ *and* $u_1(x)$ can be used to construct $A_1(x)$ which will be used to calculate $u_2(x)$ and so on. The calculations can determine the component of $u_k(x), k \geq 0$.

Obtaining the series solution can lead us to find exact solution of problem if the solution exists for the equation. Else, this series solution can be tested by pade approximations.

3. The Volterra integro-differential equation

The second-order of the Eq. (1) can be shown as Eq. (11).

$$u(x) = f(x) + \int_a^x \frac{y(t)}{(x-t)^{\alpha}} dz, \quad 0 \leq \alpha \leq 1, \quad (11)$$

For $\alpha = \frac{1}{2}$, $a = 0, f(x) = x + 1, y(t) = u^2(t)$, the Eq. (11) can be presented as:

$$u(x) = x + 1 + \int_0^x \frac{u^2(t)}{(x-t)^{\frac{1}{2}}} dt, \quad (12)$$

Where

$$u(0) = 1, \lim_{t \to \infty} u(t) = 0, \quad (13)$$

The Laplace transform method is applied to the both sides of the Eq. (12). Then we have,

$$L\{u(x)\} = L\{x+1\} + L\left\{x^{-\frac{1}{2}}\right\} L\{u^2(x)\}, \quad (14)$$

Therefore we can write Eq. (14) as,

$$U(s) = \frac{1}{s^2} + \frac{1}{s} + L\left\{x^{-\frac{1}{2}}\right\} L\{u^2(x)\}, \quad (15)$$

Or,

$$U(s) = \frac{1}{s^2} + \frac{1}{s} + \frac{\sqrt{\pi}}{\sqrt{s}} L\{u^2(x)\}, \quad (16)$$

By using the adomian decomposition method, it is assumed that,

$$U_0(s) = \frac{1}{s^2} + \frac{1}{s},$$

$$L(u_{K+1}(x)) = \frac{\sqrt{\pi}}{\sqrt{s}} L\{A_k(x)\}, \qquad k \geq 0, \quad (17)$$

Where $A_k(x)$ are the adomian polynomials for the nonlinear term $u^2(x)$.

The linear term $u(x)$ can be determined as the series of Eq. (18) by using the adomian method.

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad (18)$$

The nonlinear term $u^2(x)$ can be showed by the series,

$$u^2(x) = \sum_{n=0}^{\infty} A_n(x), \tag{19}$$

Some of the adomian polynomials for $u^2(x)$ are as,

$$\begin{aligned} A_0 &= u_0^2, \\ A_1 &= 2u_1u_0, \\ A_2 &= 2u_2u_0 + u_1^2 \ , \\ A_3 &= 2u_3u_0 + 2u_1u_2 \ , \\ A_4 &= 2u_4u_0 + \frac{1}{2}u_2^2 + u_1u_3 \ , \end{aligned} \tag{20}$$

By using the recurrence relation in the Eq. (16) and the inverse Laplace transform of U_0,

$$\begin{aligned} u_0(x) &= x + 1, \\ u_1(x) &= 2x^{\frac{1}{2}} + \frac{8}{3}x^{\frac{3}{2}} + \frac{16}{15}x^{\frac{5}{2}}, \\ u_2(x) &= \frac{118336}{675}x^{\frac{3}{2}} + \frac{253184}{675}x^{\frac{5}{2}} + \frac{520192}{1575}x^{\frac{7}{2}} + \frac{2097152}{14175}x^{\frac{9}{2}} + \frac{1048576}{31185}\ x^{\frac{11}{2}} \\ &\qquad + \frac{2097152}{675675}\ x^{\frac{13}{2}} \end{aligned} \tag{21}$$

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Therefore, the series solution of the Eq. (12) can be presented as,

$$u(x) = 1 + 2x^{\frac{1}{2}} + x + 117.9792593x^{\frac{3}{2}} + 376.1540741x^{\frac{5}{2}} + 330.2806349x^{\frac{7}{2}} + 147.947231x^{\frac{9}{2}} + 33.62437069x^{\frac{11}{2}} + 3.103788064x^{\frac{13}{2}} + \cdots \tag{22}$$

It can be concluded that this series solution admits the first condition $u(0) = 1$.

To evaluate validity of the obtained u(x) in the Eq. (22), pade approximations can be used. Also, it is shown that pade approximations can give results with a smaller error bounds in comparison to the approximation by polynomials.

For obtaining pade approximations we first set $t = x^{\frac{1}{2}}$, therefore we have,

$$\begin{aligned} u(x) = 1 + 2t + t^2 &+ 117.9792593t^3 + 376.1540741t^5 + 330.2806349t^7 + 147.947231t^9 \\ &+ 33.62437069t^{11} + 3.103788064t^{13} \\ &+ \cdots \end{aligned} \tag{23}$$

By using maple package the [3,3] and [4,4] pade approximations are given as,

$$[3,3] = \frac{1 + 12.047354t + 48.62018t^2 + 21.3521884t^3}{1 + 15.427492t + 59.23065t^2 + 62.4501442t^3} \tag{24}$$

$$[4,4] = \frac{1 + 19.632847t + 83.562245t^2 + 105.76623595t^3 + 79.3427124t^4}{1 + 24.9053263t + 93.845013t^2 + 174.9457034t^3 + 142.4720455342t^4} \tag{25}$$

The graphs of the Pade approximants for [3,3] and [4,4] is shown in Fig. 1, where the upper graph is for [3,3]. The graph presents the rapid decay of the weakly-singular Volterra Integro equations. Also, the second condition $\lim_{t\to\infty} u(t) = 0,$ is justified.

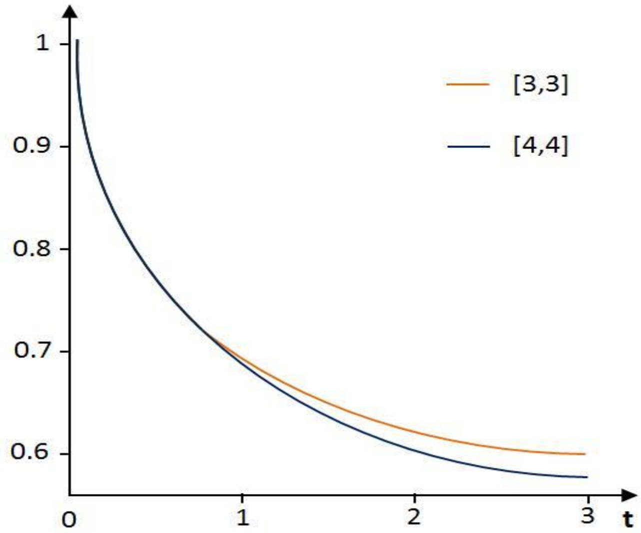

Fig. 1. The pade approximation for [3,3] and [4,4] of $u(t)$, $t = x^{\frac{1}{2}}$

4. Conclusion

In this paper, developed method of combined form of the Laplace transform method with the Adomian decomposition method is used to construct series solutions of the weakly-singular kernel Volterra integro-differential equations. The rapid convergence of the obtained serious toward the exact solutions of the equation is also numerically shown by using the pade approximations. The results present validity and great potential of the method as a powerful algorithm in order to obtain the series solution of singular kernel differential equations.

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