# Advanced Electromagnetic Waves 

M.S. Abrishamian<br>K.N. Toosi University of Technology

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## Chapter 1

## Basic Electromagnetic Concepts

```
"Everything should be made as simple as possible, but not simpler."
Albert Einstein
```


### 1.1 Introduction

Computation is the basic tools for getting results and analysis of our ordinary problems. Computation techniques allow engineers to model complicated objects which cannot be done analytically. In electrical engineering we will face with many problems:

- Voltage breakdown of insulators,
- Radar cross section of missiles and aircraft vehicle,
- Resonance frequency of cavities and modes of waveguide,
- Radiation pattern and input impedance of wire antenna,
- Mutual interaction of human body and electromagnetic radiators,
- Eddy current loss in electrical machinery,
- Electromagnetic interference in modern telecommunication,
- Target recognition,
- Design and evaluation of VLSI circuits in which electromagnetic fields can corrupt signal transmission, etc.
Without using computer and mathematical tools we can not get reasonable results for these problems. In this book we will attempt to introduce some
basic techniques in electromagnetic; such as scattering of wave from metallic and dielectric objects and confirm the results by classical method for known shapes such as cylinder, sphere, disk, strip, and so on.


### 1.2 Review of Electromagnetic Fields

Many useful technologies such as radio, television, microwave oven are a consequence of the pioneer work performed on electromagnetic fields during the 19th century. In 1867 Maxwell postulation unified, the comprehensive theory of electromagnetics(EM). There are eight equations that completely describe all electromagnetic phenomena. Four Maxwell's equations in differential form are :

1) Faraday's Law and Maxwell's first equation:

$$
\begin{equation*}
\nabla \times \mathbf{E}(\mathbf{r}, t)=-\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \tag{1.1}
\end{equation*}
$$

where $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ are the electric field intensity and magnetic flux density are as a function of time and space in volts per meter and Webers per square meter, respectively.
2) Ampere-Maxwell's Law and Maxwell's second equation:

$$
\begin{equation*}
\nabla \times \mathbf{H}(\mathbf{r}, t)=\mathbf{J}_{g}(\mathbf{r}, t)+\mathbf{J}_{c}(\mathbf{r}, t)+\frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} \tag{1.2}
\end{equation*}
$$

where $\mathbf{H}(\mathbf{r}, t)$ and $\mathbf{D}(\mathbf{r}, t)$ are the magnetic field intensity and the electric flux density as a function of time in Ampere per meter and Coulombs per square meter respectively. $\mathbf{J}_{g}(\mathbf{r}, t)$ and $\mathbf{J}_{c}(\mathbf{r}, t)$ are the electric current density due to source generator and electric current density due to conductivity of medium are as a function of time and space in Ampere per square meter for both.
3) Gauss's Law (electric):

$$
\begin{equation*}
\nabla \cdot \mathbf{D}(\mathbf{r}, t)=\rho(\mathbf{r}, t) \tag{1.3}
\end{equation*}
$$

where $\rho(\mathbf{r}, t)$ is the electric charge density as a function of time and space in Coulombs per cubic meter.
4) Gauss's Law (magnetic):

$$
\begin{equation*}
\nabla \cdot \mathbf{B}(\mathbf{r}, t)=0 \tag{1.4}
\end{equation*}
$$

and four Maxwell's equations in integral form are:
1)Faraday's law:

$$
\begin{equation*}
\oint_{c} \mathbf{E}(\mathbf{r}, t) \cdot d \mathbf{L}=-\frac{\partial}{\partial t} \int_{s} \mathbf{B}(\mathbf{r}, t) \cdot d \mathbf{S} \tag{1.5}
\end{equation*}
$$

2)Ampere-Maxwell's Law:

$$
\begin{equation*}
\oint_{c} \mathbf{H}(\mathbf{r}, t) \cdot d \mathbf{L}=\int_{s} \mathbf{J}_{v}(\mathbf{r}, t) \cdot d \mathbf{S}+\frac{\partial}{\partial t} \int_{s} \mathbf{D}(\mathbf{r}, t) \cdot d \mathbf{S} \tag{1.6}
\end{equation*}
$$

where $\mathbf{J}_{v}(\mathbf{r}, t)$ is the total electric current density.
3) Gauss's Law (electric):

$$
\begin{equation*}
\oint_{s} \mathbf{D}(\mathbf{r}, t) \cdot d \mathbf{S}=\int_{v} \rho(\mathbf{r}, t) d V \tag{1.7}
\end{equation*}
$$

4) Gauss's Law (magnetic):

$$
\begin{equation*}
\oint_{s} \mathbf{B}(\mathbf{r}, t) \cdot d \mathbf{S}=0 \tag{1.8}
\end{equation*}
$$

The $\mathbf{J}_{v}(\mathbf{r}, t)$ and $\rho_{v}(\mathbf{r}, t)$ are related to each other by continuity equation:

$$
\begin{equation*}
\nabla \cdot \mathbf{J}_{v}(\mathbf{r}, t)=-\frac{\partial \rho_{v}(\mathbf{r}, t)}{\partial t} \tag{1.9}
\end{equation*}
$$

where $\rho_{v}(\mathbf{r}, t)$ is the electric charge density in Coulombs per cubic meter. This means that the net outflow of current from any volume is a measure of the time rate of decrease of electric charge inside the volume. Eq.(1.9) is also called Conservation of electric charge. Eq.(1.9) has an equivalent integral form:

$$
\begin{equation*}
\oint_{s} \mathbf{J}_{v}(\mathbf{r}, t) \cdot d \mathbf{S}=-\int_{v} \frac{\partial \rho_{v}(\mathbf{r}, t)}{\partial t} d V \tag{1.10}
\end{equation*}
$$

The last one is Lorentz force equation:

$$
\begin{equation*}
\mathbf{F}(\mathbf{r}, t)=q_{v}(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t)+q_{v}(\mathbf{r}, t) \mathbf{U} \times \mathbf{B}(\mathbf{r}, t) \tag{1.11}
\end{equation*}
$$

$q_{v}(\mathbf{r}, t)$ : total electric charge in Coulombs.
$\mathbf{F}(\mathbf{r}, t)$ : Force on charge $q_{v}(\mathbf{r}, t)$ due to electric and magnetic field in Newtons.
$\mathbf{U}$ : velocity of charge $q_{v}(\mathbf{r}, t)$ in meter per second.

### 1.2.1 Magnetic charge

A magnetic monopole is the smallest element of magnetic charge. Although, this has not been discovered yet. The concept of magnetic charge is very useful in electromagnetic theory.
Defining the macroscopic magnetic charge density $\rho_{m}(\mathbf{r}, t)$ and the magnetic current density $\mathbf{J}_{m}(\mathbf{r}, t)$ exactly as we defined with electric charge, we define the conservation of magnetic charge to provide a continuity equation.

$$
\begin{equation*}
\nabla \cdot \mathbf{J}_{m}(\mathbf{r}, t)=-\frac{\partial \rho_{m}(\mathbf{r}, t)}{\partial t} \tag{1.12}
\end{equation*}
$$

### 1.2.2 Charge Conservation

The law of conservation of charge is that
The net charge in any closed system remains constant with time.

### 1.3 Time Harmonic Electromagnetic Fields

In the previous sections Maxwell's equations in differential or integral form were presented in time domain. Now let us assume that the sources $\mathbf{J}$ and $\rho$ and the resultant electromagnetic field vary sinusoidally with time. If the angular frequency being $\omega$ then any sinusoidal time varying quantities can be represented by:

$$
\begin{equation*}
A(t)=\operatorname{Re}\left[\mathbf{A}(\omega) e^{j \omega t}\right] \tag{1.13}
\end{equation*}
$$

where $\mathbf{A}(\omega)$ is a complex quantity independent of time, and $R e[.$.$] denotes$ the real part of quantity inside the bracket. In more mathematical language, one can say that $\mathbf{A}(\omega)$ is the Fourier Transform of $A(t)$. The time derivative of a time domain function can be replaced by $j \omega$ in frequency domain. In electromagnetics one can write all the time domain fields in frequency domain.

$$
\begin{align*}
& \mathbf{E}(\mathbf{r}, t)=\operatorname{Re}\left[\mathbf{E}(\mathbf{r}, \omega) e^{j \omega t}\right]  \tag{1.14}\\
& \mathbf{H}(\mathbf{r}, t)=\operatorname{Re}\left[\mathbf{H}(\mathbf{r}, \omega) e^{j \omega t}\right]  \tag{1.15}\\
& \mathbf{D}(\mathbf{r}, t)=\operatorname{Re}\left[\mathbf{D}(\mathbf{r}, \omega) e^{j \omega t}\right] \tag{1.16}
\end{align*}
$$

$$
\begin{align*}
\mathbf{B}(\mathbf{r}, t) & =\operatorname{Re}\left[\mathbf{B}(\mathbf{r}, \omega) e^{j \omega t}\right]  \tag{1.17}\\
\mathbf{J}(\mathbf{r}, t) & =\operatorname{Re}\left[\mathbf{J}(\mathbf{r}, \omega) e^{j \omega t}\right]  \tag{1.18}\\
\rho(\mathbf{r}, t) & =\operatorname{Re}\left[\rho(\mathbf{r}, \omega) e^{j \omega t}\right] \tag{1.19}
\end{align*}
$$

If we substitute the Eq.(1.14)-Eq.(1.19) in the Maxwell's time domain equations, we will have Maxwell's equations in frequency domain. Thus

$$
\begin{equation*}
\nabla \times \mathbf{H}=\mathbf{J}_{g}+\mathbf{J}_{c}+j \omega \mathbf{D} \tag{1.20}
\end{equation*}
$$

$$
\begin{equation*}
\nabla \times \mathbf{E}=-j \omega \mathbf{B} \tag{1.21}
\end{equation*}
$$

### 1.4 Constitutive Relations

Those relations that describe the properties of medium, is called Constitutive relation. The constitutive relations in Electromagnetics can be written as:

$$
\begin{align*}
\mathbf{D} & =F_{1}(\mathbf{E}, \mathbf{H})  \tag{1.22}\\
\mathbf{B} & =F_{2}(\mathbf{E}, \mathbf{H})
\end{align*}
$$

where $F_{1}$ and $F_{2}$ are two different functions that depend on the medium and can be classified as:

- Linearity: if we apply $\mathbf{E}_{\mathbf{1}}, \mathbf{H}_{\mathbf{1}}$ in a medium and get $\mathbf{D}_{\mathbf{1}}, \mathbf{B}_{\mathbf{1}}$ and apply $\mathbf{E}_{2}, \mathbf{H}_{\mathbf{2}}$ and get $\mathbf{D}_{2}, \mathbf{B}_{2}$. Now if we apply $\mathbf{E}=\alpha_{1} \mathbf{E}_{1}+\alpha_{2} \mathbf{E}_{\mathbf{2}}, \mathbf{H}=$ $\beta_{1} \mathbf{H}_{1}+\beta_{2} \mathbf{H}_{\mathbf{2}}$ and get $\mathbf{D}=\alpha_{1} \mathbf{D}_{1}+\alpha_{2} \mathbf{D}_{\mathbf{2}}, \mathbf{B}=\beta_{1} \mathbf{B}_{1}+\beta_{2} \mathbf{B}_{\mathbf{2}}$, then the medium is called linear. Otherwise it is called nonlinear medium.
- Homogeneity: if the functional $F_{1}$ and $F_{2}$ depend on the space coordinates then the medium is called inhomogeneous, otherwise it is called homogeneous.
- Metamaterials: Metamaterials are artificial materials engineered to provide properties which "may not be readily available in nature". These materials usually gain their properties from structure rather than
composition, using the inclusion of small inhomogeneities to enact effective macroscopic behavior.[68], [69]
The primary research in metamaterials investigates materials with negative refractive index. Negative refractive index materials appear to permit the creation of 'superlenses' which can have a spatial resolution below that of the wavelength, and a form of 'invisibility' has been demonstrated at least over a narrow wave band.
- Stationary: if the functional $F_{1}$ and $F_{2}$ depend on time then the medium is called time dependent otherwise is called time invariant or stationary. Time Dispersion is common for most time dependent media. As an example, the permittivity of water drops from $80 \epsilon_{0}$ to approximately $1.8 \epsilon_{0}$ as frequency increase from static to optical ranges.


## - Dispersion

From discovery of the phenomenon of electricity, man recognized two types of material: Conductors and Insulator. The metals were identified as good conductors and many like wood, insulators. Soon they found that by inserting insulator into a capacitor, the capacity will increase. They put different insulators and got different results. As we know today, they defined Polarization, the electric moment of a system of charges $\mathbf{P}$, and it may or may not be proportional to applied $\mathbf{E}$. In linear dielectric it is obvious that it is proportional, $\chi \epsilon_{0}$, and $\chi$ depends on electronic, ionic and orientational dipole moment of dielectric material. Then they defined $\epsilon_{r}=1+\chi$ as dielectric constant or relative permittivity. Does this new born $\epsilon_{r}$ depend on the frequency of applied field? the answer is yes. Different persons found different formulas for different range of frequencies. We will discuss about this matter in the following chapter.

- Isotropic Media: suppose we have a medium with simple constitutive relations

$$
\begin{equation*}
\mathbf{D}(\mathbf{r}, t)=\epsilon \mathbf{E}(\mathbf{r}, t) \tag{1.23}
\end{equation*}
$$

where $\epsilon$ is the permittivity of the medium in Farads per meter.

$$
\begin{equation*}
\mathbf{B}(\mathbf{r}, t)=\mu \mathbf{H}(\mathbf{r}, t) \tag{1.24}
\end{equation*}
$$

where $\mu$ is the permeability of the medium in Henrys per meter. In this case $\mathbf{E}$ is parallel to $\mathbf{D}$ and $\mathbf{H}$ is parallel to $\mathbf{B}$ then the medium is called isotropic. In some media the constitutive relation are in tensor form:

$$
\begin{aligned}
& \hline \mathbf{D}=\bar{\epsilon} \cdot \mathbf{E} \\
& \mathbf{B}=\bar{\mu} \cdot \mathbf{H}
\end{aligned}
$$

where $\bar{\epsilon}$ and $\bar{\mu}$ is permittivity and permeability tensors, respectively. This medium is called electrically or magnetically anisotropic. A medium can be both electrically and magnetically anisotropic. In particular case, such as crystals, which may be described by choosing suitable coordinate system, permittivity tensor looks like:

$$
\bar{\epsilon}=\left[\begin{array}{ccc}
\epsilon_{x} & 0 & 0  \tag{1.25}\\
0 & \epsilon_{y} & 0 \\
0 & 0 & \epsilon_{z}
\end{array}\right]
$$

If the medium is dispersive, the relation between $\mathbf{D}(\mathbf{r}, t), \mathbf{E}(\mathbf{r}, t)$ and $\bar{\epsilon}(t)$ will be:

$$
\begin{equation*}
\mathbf{D}(\mathbf{r}, t)=\int_{-\infty}^{t} \bar{\epsilon}(t-\tau) \cdot \mathbf{E}(\mathbf{r}, \tau) d \tau \tag{1.26}
\end{equation*}
$$

similarly:

$$
\begin{equation*}
\mathbf{B}(\mathbf{r}, t)=\int_{-\infty}^{t} \bar{\mu}(t-\tau) \cdot \mathbf{H}(\mathbf{r}, \tau) d \tau \tag{1.27}
\end{equation*}
$$

for cubic crystals, $\epsilon_{x}=\epsilon_{y}=\epsilon_{z}$, which are isotropic. In tetragonal, hexagonal, and rhombohedral crystals, two of the parameters are equal. Such crystals are uniaxial. In orthorhombic, monoclinic, and triclinic crystals $\epsilon_{x} \neq \epsilon_{y} \neq \epsilon_{z}$ and the medium is biaxial [24].

- Bi-anisotropic Media: For isotropic or anisotropic media, the constitutive relations relate the two electric and two magnetic field vectors by either a scalar or a tensor. Such media become polarized when placed in an electric field and become magnetized when placed in magnetic field. A bi-anisotropic medium provides cross coupling between electric and magnetic fields and becomes both polarized and magnetized.

In general bi-anisotropic media can be described as:

$$
\begin{aligned}
& \mathbf{D}=\bar{\epsilon} \cdot \mathbf{E}+\bar{\xi} \cdot \mathbf{H} \\
& \mathbf{B}=\bar{\zeta} \cdot \mathbf{E}+\bar{\mu} \cdot \mathbf{H}
\end{aligned}
$$

If the four tensors $\bar{\epsilon}, \bar{\xi}, \bar{\zeta}$ and $\bar{\mu}$ become scalars, the medium is called bi-isotropic or chiral.

- Chiral Media: The word chiral comes from the Greek word chiro meaning hand and chirality refers to handedness, in other words, the asymmetry property of an object. It is a purely geometric notion, which refers to the lack of bilateral symmetry between an object and its mirror image. A chiral object is, by definition, a body that cannot be mapped on its mirror image by translation or rotation. One of the aspects characterizing chiral media is the phenomenon of optical activity. A material, which rotates the plane of polarization of incident linearly polarized light is said to be optically active. Some substances rotate the polarization of electric field clockwise, some rotate it counterclockwise. An object that is not chiral is said to be achiral. Chiral medium is a subclass of materials known as bi-isotropic materials.

We assume that the medium is: linear, isotropic, time invariant and homogeneous. Otherwise we mention the condition of the medium. When the above conditions hold, the Eq.(1.1) through Eq.(1.4) become

$$
\begin{equation*}
\nabla \times \mathbf{E}(\mathbf{r}, t)=-\mu \frac{\partial \mathbf{H}(\mathbf{r}, t)}{\partial t} \tag{1.28}
\end{equation*}
$$

$$
\begin{equation*}
\nabla \times \mathbf{H}(\mathbf{r}, t)=\mathbf{J}_{g}(\mathbf{r}, t)+\sigma \mathbf{E}(\mathbf{r}, t)+\epsilon \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} \tag{1.29}
\end{equation*}
$$

$$
\begin{equation*}
\nabla \cdot \mathbf{H}(\mathbf{r}, t)=0 \tag{1.31}
\end{equation*}
$$

$$
\begin{equation*}
\nabla \cdot \mathbf{E}(\mathbf{r}, t)=\frac{\rho(\mathbf{r}, t)}{\epsilon} \tag{1.30}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{J}_{c}(\mathbf{r}, t)=\sigma \mathbf{E}(\mathbf{r}, t) \tag{1.32}
\end{equation*}
$$

where $\sigma$ is the conductivity of the medium in Siemens per meter.

### 1.5 Scalar and Vector Potentials

For boundary value problem in EM, as aids in obtaining electric (E) and magnetic (H) fields, we usually define vector electric potential $\mathbf{F}$, magnetic vector potential A, electric scalar potential $\phi$ or $V$ and finally magnetic scalar potential $\phi_{m}$. Since $\nabla \cdot \nabla \times \mathbf{C}=0$ and $\nabla \cdot \mathbf{B}=0$ we can define:

$$
\mathbf{B}=\nabla \times \mathbf{A}
$$

but there are many vectors like $\mathbf{A}$ that has curls equal to vector $\mathbf{B}$. Which one should we select? We will find special value for it !

The first Maxwell's equation become:

$$
\begin{equation*}
\nabla \times \mathbf{E}(\mathbf{r}, t)+\frac{\partial}{\partial t} \nabla \times \mathbf{A}(\mathbf{r}, t)=0 \tag{1.33}
\end{equation*}
$$

since $\nabla \times \nabla \phi(\mathbf{r}, t)=0$ therefore:

$$
\begin{gather*}
\mathbf{E}(\mathbf{r}, t)=-\nabla \phi(\mathbf{r}, t)-\frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}  \tag{1.34}\\
\mathbf{H}(\mathbf{r}, t)=\frac{1}{\mu} \nabla \times \mathbf{A}(\mathbf{r}, t) \tag{1.35}
\end{gather*}
$$

we know that

$$
\nabla \times \nabla \times \mathbf{A}=\nabla(\nabla \cdot \mathbf{A})-\nabla^{2} \mathbf{A}=\nabla \times \mathbf{B}=\mu \nabla \times \mathbf{H}
$$

and

$$
\nabla(\nabla \cdot \mathbf{A})-\nabla^{2} \mathbf{A}=\mu \mathbf{J}+\mu \epsilon \frac{\partial \mathbf{E}}{\partial t}
$$

by using Eq.(1.34) in above equation

$$
\begin{equation*}
\nabla \times \nabla \times \mathbf{A}=\nabla(\nabla \cdot \mathbf{A})-\nabla^{2} \mathbf{A}=\mu \mathbf{J}-\mu \epsilon \nabla \frac{\partial \phi}{\partial t}-\mu \epsilon \frac{\partial^{2} \mathbf{A}}{\partial t^{2}} \tag{1.36}
\end{equation*}
$$

If we choose

$$
\begin{equation*}
\nabla \cdot \mathbf{A}(\mathbf{r}, t)+\mu \epsilon \frac{\partial \phi(\mathbf{r}, t)}{\partial t}=0 \tag{1.37}
\end{equation*}
$$

so that it satisfies the above condition and it is called Lorentz Condition, we will have

$$
\begin{equation*}
\nabla^{2} \mathbf{A}(\mathbf{r}, t)-\mu \epsilon \frac{\partial^{2} \mathbf{A}(\mathbf{r}, t)}{\partial t^{2}}=-\mu \mathbf{J}(\mathbf{r}, t) \tag{1.38}
\end{equation*}
$$

and also we can reach

$$
\begin{equation*}
\nabla^{2} \phi(\mathbf{r}, t)-\mu \epsilon \frac{\partial^{2} \phi(\mathbf{r}, t)}{\partial t^{2}}=-\frac{\rho(\mathbf{r}, t)}{\epsilon} \tag{1.39}
\end{equation*}
$$

It is possible to combine the above scalar and vector potentials and the Lorentz condition and form a single vector called Hertz Vector, from which electric and magnetic field can be calculated:

$$
\begin{equation*}
\mathbf{A}(\mathbf{r}, t)=\mu \epsilon \frac{\partial \boldsymbol{\Pi}(\mathbf{r}, t)}{\partial t}, \quad \text { and } \quad \phi(\mathbf{r}, t)=-\nabla \cdot \boldsymbol{\Pi}(\mathbf{r}, t) \tag{1.40}
\end{equation*}
$$

We combine $\mathbf{J}$ and $\rho$ consistent with the continuity equation by using electrical polarization vector $\mathbf{P}(\mathbf{r}, t)$, and it is equal to the dipole moment per unit volume of exciting source.

$$
\begin{equation*}
\mathbf{J}(\mathbf{r}, t)=\frac{\partial \mathbf{P}(\mathbf{r}, t)}{\partial t}, \quad \text { and } \quad \rho(\mathbf{r}, t)=-\nabla \cdot \mathbf{P}(\mathbf{r}, t) \tag{1.41}
\end{equation*}
$$

therefore

$$
\begin{equation*}
\nabla^{2} \boldsymbol{\Pi}(\mathbf{r}, t)-\mu \epsilon \frac{\partial^{2} \boldsymbol{\Pi}(\mathbf{r}, t)}{\partial t^{2}}=-\frac{\mathbf{P}(\mathbf{r}, t)}{\epsilon} \tag{1.42}
\end{equation*}
$$

from which we will have

$$
\begin{equation*}
\mathbf{E}=\nabla(\nabla \cdot \boldsymbol{\Pi}(\mathbf{r}, t))-\mu \epsilon \frac{\partial^{2} \boldsymbol{\Pi}(\mathbf{r}, t)}{\partial t^{2}}=\nabla \times \nabla \times \boldsymbol{\Pi}(\mathbf{r}, t)-\frac{\mathbf{P}(\mathbf{r}, t)}{\epsilon} \tag{1.43}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{H}=\epsilon \nabla \times \frac{\partial \boldsymbol{\Pi}(\mathbf{r}, t)}{\partial t} \tag{1.44}
\end{equation*}
$$

by the same procedure we can find formula for $\mathbf{F}$ and $\phi_{m}$.

### 1.5.1 Time Harmonic Scalar and Vector Potentials

In the theory of electromagnetic radiation, it is not convenient to work with the electric and magnetic fields directly, except for simple plane waves. It is more convenient to use the "scalar
potentials" and "vector potentials." In homogeneous region the two curl equation

$$
\begin{gather*}
\nabla \times \mathbf{E}=-\mathbf{M}-j \omega \mu \mathbf{H}  \tag{1.45}\\
\nabla \times \mathbf{H}=\mathbf{J}+j \omega \epsilon \mathbf{E} \tag{1.46}
\end{gather*}
$$

provide six scalar equations to be solved in order to find electric and magnetic fields. The divergence equations for the fields $\nabla \cdot \mathbf{E}=\frac{\rho}{\epsilon}$ and $\nabla \cdot \mathbf{H}=\frac{\rho_{m}}{\mu}$ must be simultaneously satisfied while also the constitutive relations $\mathbf{D}=\epsilon \mathbf{E}$ and $\mathbf{B}=\mu \mathbf{H}$ hold. For the purpose of avoiding the solution of such a large set of equations, it is generally useful to introduce some potential functions in terms of which the electromagnetic fields can be expressed. In EM theory, we have two vector potential $\mathbf{A}$ and $\mathbf{F}$ which are related to scalar potential $\phi_{e}$ and $\phi_{m}$ by Lorentz condition or "gauge":

$$
\begin{align*}
& \nabla \cdot \mathbf{A}=-j \omega \mu \epsilon \phi_{e}  \tag{1.47}\\
& \nabla \cdot \mathbf{F}=-j \omega \mu \epsilon \phi_{m} \tag{1.48}
\end{align*}
$$

where
$\mathbf{A}=$ magnetic vector potential in Webers per meter
$\mathbf{F}=$ electric vector potential in Coloumbs per meter
$\phi_{m}=$ magnetic scalar potential in Amperes
$\phi_{e}=$ electric scalar potential in Volts
$\mathbf{M}=$ ficticious magnetic current density in $\left[\mathrm{V} / \mathrm{m}^{2}\right]$
$\rho_{m}=$ ficticious magnetic charge density in [weber $/ \mathrm{m}^{3}$ ]
and we define electric and magnetic vector as

$$
\begin{align*}
\mathbf{H}_{A} & =\frac{1}{\mu} \nabla \times \mathbf{A}  \tag{1.49}\\
\mathbf{E}_{F} & =-\frac{1}{\epsilon} \nabla \times \mathbf{F} \tag{1.50}
\end{align*}
$$

and correspond electric and magnetic fields

$$
\begin{align*}
& \mathbf{E}_{A}=-\nabla \phi_{e}-j \omega \mathbf{A}=-j \omega \mathbf{A}-j \frac{1}{\omega \mu \epsilon} \nabla(\nabla \cdot \mathbf{A})  \tag{1.51}\\
& \mathbf{H}_{F}=-\nabla \phi_{m}-j \omega \mathbf{F}=-j \omega \mathbf{F}-j \frac{1}{\omega \mu \epsilon} \nabla(\nabla \cdot \mathbf{F}) \tag{1.52}
\end{align*}
$$

### 1.6 Scalar and Vector Wave Equations

For some classes of electromagnetic problems related to the scattering, Maxwell's equations assume a symmetric form. Consider a linear, time invariant, homogenous and isotropic medium without any external sources, Maxwell's
equations in frequency domain are:

$$
\begin{align*}
& \nabla \times \mathbf{E}=-j \omega \mu \mathbf{H} \\
& \nabla \times \mathbf{H}=j \omega \epsilon \mathbf{E} \\
& \nabla \cdot \mathbf{E}=0  \tag{1.53}\\
& \nabla \cdot \mathbf{H}=0
\end{align*}
$$

where $\mu$ or / and $\epsilon$ may be complex. From these equations, we can reach two vector wave equations:

$$
\begin{align*}
& \nabla \times \nabla \times \mathbf{E}-\omega^{2} \mu \epsilon \mathbf{E}=0 \\
& \nabla \times \nabla \times \mathbf{H}-\omega^{2} \mu \epsilon \mathbf{H}=0 \tag{1.54}
\end{align*}
$$

These equations are totally symmetrical and decoupled in the field variables. It is the solution to vector wave equations for specified boundary and radiation conditions that describes the scattering of electromagnetic waves. However, these equations are sufficiently difficult to solve in general.
In most classical problem, we have the solution of scalar the Helmholtz equation:

$$
\begin{equation*}
\nabla^{2} \psi+k^{2} \psi=0 \tag{1.55}
\end{equation*}
$$

where $k^{2}=\omega^{2} \mu \epsilon$. In certain curvilinear coordinate systems, such as Cartesian, spherical, cylindrical coordinate system, $\psi$ can be obtained by the technique of separation of variables. The separation procedure reduces the partial differential equation to several ordinary differential equations. The separated equations can often be cast in the form of the well known Sturm-Liouville equations, so that the solution space is guaranteed to be complete for scalar Helmholtz equation:
Let us introduce general form of vector wave equation

$$
\begin{equation*}
\nabla(\nabla \cdot \mathbf{C})-\nabla \times \nabla \times \mathbf{C}+k^{2} \mathbf{C}=0 \tag{1.56}
\end{equation*}
$$

The differential equation is satisfied by the electric and magnetic fields that satisfies the vector wave equations since the term $\nabla \cdot \mathbf{C}$ will be zero for fields that have zero divergence. We define a vector function that is obtained by taking the gradient of the scalar function $\psi,[67]$,

$$
\begin{equation*}
\mathbf{L}=\nabla \psi \tag{1.57}
\end{equation*}
$$

$\mathbf{L}$ satisfies the vector wave equation Eq.(1.56) if $\psi$ is a solution to scalar wave equation Eq.(1.55):

$$
\begin{align*}
I & =\nabla(\nabla \cdot \mathbf{L})-\nabla \times \nabla \times \mathbf{L}+k^{2} \mathbf{L}  \tag{1.58}\\
& =\nabla(\nabla \cdot \nabla \psi)-\nabla \times \nabla \times \nabla \psi+k^{2} \nabla \psi \\
& =\nabla\left(\nabla^{2} \psi+k^{2} \psi\right) \quad(\text { since } \nabla \times \nabla \psi=0) \\
& =0 \quad\left(\text { since } \nabla^{2} \psi+k^{2} \psi=0\right) .
\end{align*}
$$

Now we define a vector $\mathbf{M}=\nabla \times \mathbf{a} \psi$, where $\psi$ is the given scalar solution and $\mathbf{a}$ is an arbitrary constant vector. The divergence condition is satisfied, since divergence of curl is identically zero: $\nabla \cdot \mathbf{M}=\nabla \cdot(\nabla \times \mathbf{a} \psi)=0$. There are six orthogonal curvilinear coordinate systems that have been verified this identity $\nabla(\nabla \cdot \mathbf{M})-\nabla \times \nabla \times \mathbf{M}=\nabla^{2} \mathbf{M}$, [11]. The reader can prove it for Cartesian, cylindrical and spherical coordinate systems.
We can write:

$$
\begin{align*}
I & =\nabla(\nabla \cdot \mathbf{M})-\nabla \times \nabla \times \mathbf{M}+k^{2} \mathbf{M}  \tag{1.59}\\
& =\nabla^{2} \mathbf{M}+k^{2} \mathbf{M} \\
& =\nabla^{2}(\nabla \times \mathbf{a} \psi)+k^{2}(\nabla \times \mathbf{a} \psi) \\
& =\nabla^{2}[\nabla \psi \times \mathbf{a}+\psi(\nabla \times \mathbf{a})]+k^{2}[\nabla \psi \times \mathbf{a}+\psi(\nabla \times \mathbf{a})] \\
& =\nabla^{2}(\nabla \psi \times \mathbf{a})+k 2(\nabla \psi \times \mathbf{a}) \quad(\text { since } \nabla \times \mathbf{a}=0) \\
& =\left(\nabla^{2}(\nabla \psi)+k^{2} \nabla \psi\right) \times \mathbf{a} \quad\left(\text { since } \nabla^{2} \text { does not act on } \mathbf{a}\right) \\
& \left.=\left[\nabla\left(\nabla^{2} \psi\right)+k^{2} \psi\right] \times \mathbf{a}=0 \quad(\text { since } \psi \text { satisfies Eq. } 1.55)\right) .
\end{align*}
$$

Also, $\mathbf{M}=\nabla \times \mathbf{a} \psi=\nabla \psi \times \mathbf{a}=\mathbf{L} \times \mathbf{a}$. So, $\mathbf{M} \cdot \mathbf{L}=0$, i.e. $\mathbf{L}$ and $\mathbf{M}$ are orthogonal.
Now let us define another vector as $\mathbf{N}=\frac{1}{k} \nabla \times \mathbf{M}$. which has zero divergence, $\nabla \cdot \mathbf{M}=0$, that is a solution to the vector wave equation,Eq.(1.56). Assuming $k$ is a constant (homogenous medium), we obtain:

$$
\begin{align*}
I & =\nabla(\nabla \cdot \mathbf{N})-\nabla \times \nabla \times \mathbf{N}+k^{2} \mathbf{N}  \tag{1.60}\\
& =\frac{1}{k}\left[\nabla(\nabla \cdot \nabla \times \mathbf{M})-\nabla \times \nabla \times \nabla \times \mathbf{M}+k^{2} \nabla \times \mathbf{M}\right] \\
& =\nabla \times\left[-\nabla \times(\nabla \times \mathbf{M})+k^{2} \mathbf{M}\right] \quad[\text { since } \nabla \cdot(\nabla \times \mathbf{M})=0] \\
& =\nabla \times\left[\nabla(\nabla \cdot \mathbf{M})-\nabla \times(\nabla \times \mathbf{M})+k^{2} \mathbf{M}\right] \quad(\text { since } \nabla \cdot \mathbf{M}=0) \\
& =0 \quad \text { (since M satisfies Eq.(1.56)) } .
\end{align*}
$$

Thus, we show that $\mathbf{N}$ also satisfies the vector wave equation. Also $\nabla \cdot \mathbf{N}=0$, since divergence of curl is zero.
Since $\mathbf{N}=\frac{1}{k} \nabla \times \mathbf{M}$,

$$
\nabla \times \mathbf{N}=\frac{1}{k} \nabla \times \nabla \times \mathbf{M}
$$

Therefore

$$
\nabla \times \mathbf{N}=\frac{1}{k} k^{2} \mathbf{M}
$$

Thus $\mathbf{M}=\frac{1}{k} \nabla \times \mathbf{N}$. Clearly, $\mathbf{M}$ and $\mathbf{N}$ are distinct from $\mathbf{L}$, since the latter has nonzero divergence in general. So $\mathbf{L}$ must be linearly independent of $\{\mathrm{M}, \mathrm{N}\}$.
We have proved that $\mathbf{L}$ and $\mathbf{M}$ are orthogonal. Thus given a countably infinite set of particular solutions to Eq.(1.55), $\left\{\psi_{n}\right\}$, that are finite, continuous, single valued, and with continuous partial derivatives; associated with each $\psi_{n}$ one can obtain a triple of mutually noncoplanar vector solution $\left\{\mathbf{L}_{n}, \mathbf{M}_{n}, \mathbf{N}_{n}\right\}$, satisfying Eq.(1.56). Presumably, any arbitrary solution of vector wave equation can be express as a linear combination of these vector functions, [67].
Consider a solution $\mathbf{F}$ whose divergence is zero. let us find an expansion of $\mathbf{F}$ in terms of the basis $\left\{\mathbf{L}_{n}, \mathbf{M}_{n}, \mathbf{N}_{n}\right\}$, so that

$$
\begin{equation*}
\mathbf{F}=\sum_{n}\left\{a_{n} \mathbf{M}_{n}+b_{n} \mathbf{N}_{n}+c_{n} \mathbf{L}_{n}\right\} \tag{1.61}
\end{equation*}
$$

Taking the divergence of both sides of the above equation, we find

$$
\begin{equation*}
\nabla \cdot \mathbf{F}=\sum_{n}\left\{a_{n} \nabla \cdot \mathbf{M}_{n}+b_{n} \nabla \cdot \mathbf{N}_{n}+c_{n} \nabla \cdot \mathbf{L}_{n}\right\} \tag{1.62}
\end{equation*}
$$

or

$$
\begin{equation*}
0=\sum_{n}\left\{c_{n} \nabla \cdot \mathbf{L}_{n}\right\} \tag{1.63}
\end{equation*}
$$

Since this must hold true at all points, we conclude that all the $c_{n}$ 's must be zero. In other words, a zero divergence solution, $\{$ like $\mathbf{E}$ and $\mathbf{H}\}$, can be expressed only in terms of $\mathbf{M}$ and $\mathbf{N}$ functions.
Now let us find the vector wave equation in inhomogeneous media, $\mu=\mu_{0}$ and $\epsilon(x, y, z)$. Therefore

$$
\begin{equation*}
\nabla \times \nabla \times \mathbf{E}=-\mu_{0} \frac{\partial}{\partial t}(\nabla \times \mathbf{H})=\nabla(\nabla \cdot \mathbf{E})-\nabla^{2} \mathbf{E} \tag{1.64}
\end{equation*}
$$

or

$$
\begin{equation*}
\nabla \times \nabla \times \mathbf{E}-\mu_{0} \epsilon \frac{\partial^{2} \mathbf{E}}{\partial t^{2}}=0 \tag{1.65}
\end{equation*}
$$

or in another form

$$
\begin{equation*}
\nabla^{2} \mathbf{E}-\mu_{0} \epsilon \frac{\partial^{2} \mathbf{E}}{\partial t^{2}}=\nabla(\nabla \cdot \mathbf{E}) \tag{1.66}
\end{equation*}
$$

Let us now investigate the right hand side of Eq.(1.66) and find the term $\nabla \cdot$ E. From Gauss's law

$$
\begin{align*}
& \nabla \cdot \mathbf{D}=\nabla \cdot(\epsilon \mathbf{E})=0 \\
& \nabla \cdot \mathbf{D}=\epsilon \nabla \cdot \mathbf{E}+\mathbf{E} \cdot \nabla \epsilon=0 \tag{1.67}
\end{align*}
$$

From Eq.(1.67) we can find the $\nabla \cdot \mathbf{E}=-\mathbf{E} \cdot \frac{\nabla \epsilon}{\epsilon}$, and substituting in Eq.(1.66), we have

$$
\begin{equation*}
\nabla^{2} \mathbf{E}-\mu_{0} \epsilon \frac{\partial^{2} \mathbf{E}}{\partial t^{2}}=-\mathbf{E} \cdot \frac{\nabla \epsilon}{\epsilon} \tag{1.68}
\end{equation*}
$$

and the vector wave equation for $\mathbf{H}$ will be

$$
\begin{equation*}
\nabla \times \nabla \times \mathbf{H}+\mu_{0} \epsilon \frac{\partial^{2} \mathbf{H}}{\partial t^{2}}=\frac{\nabla \epsilon}{\epsilon} \times \nabla \times \mathbf{H} \tag{1.69}
\end{equation*}
$$

since $\nabla \cdot \mathbf{H}=0$,

$$
\begin{equation*}
\nabla^{2} \mathbf{H}-\mu_{0} \epsilon \frac{\partial^{2} \mathbf{H}}{\partial t^{2}}=-\frac{\nabla \epsilon}{\epsilon} \times \nabla \times \mathbf{H} \tag{1.70}
\end{equation*}
$$

For step-index waveguide we assume that the refractive index of the core is slightly higher than refractive index of clad, so we use $\nabla \epsilon=0$

### 1.7 Electromagnetic Boundary Conditions

In an electromagnetic problem, we usually faced with interaction of fields and matter. For example, in order to find fields inside a dielectric objects like human head or scattering of field by an metallic sphere, we should know the condition governing the electric or magnetic fields at the boundary of two different media.

### 1.7.1 Finite Conductivity Media

In the media with finite conductivities, we assume that there is no electric charges or sources along the boundary and $\sigma_{1} \neq 0, \sigma_{2} \neq 0$. Figure [1.1] shows


Figure 1.1: Tangential boundary conditions
the interface of two different media with electrical parameters $\mu_{1}, \epsilon_{1}, \sigma_{1}$ and $\mu_{2}, \epsilon_{2}, \sigma_{2}$.

$$
\begin{equation*}
\mathbf{n}_{12} \times\left(\mathbf{E}_{2}-\mathbf{E}_{1}\right)=0 \quad \Rightarrow E_{2 t}=E_{1 t} \tag{1.71}
\end{equation*}
$$

$\mathbf{n}_{12}$ : unit vector normal to boundary directed from medium 1 to medium 2.

$$
\begin{equation*}
\mathbf{n}_{12} \times\left(\mathbf{H}_{2}-\mathbf{H}_{1}\right)=0 \quad \Rightarrow H_{2 t}=H_{1 t} \tag{1.72}
\end{equation*}
$$

Both Eq.(1.71) and Eq.(1.72) state that: the tangential components of electric and magnetic field intensity across an interface of two media are continuous.

$$
\begin{array}{|l}
\hline \mathbf{n}_{12} \cdot\left(\mathbf{D}_{2}-\mathbf{D}_{1}\right)=0 \quad \Rightarrow D_{2 n}=D_{1 n} \\
 \tag{1.74}\\
\mathbf{n}_{12} \cdot\left(\mathbf{B}_{2}-\mathbf{B}_{1}\right)=0 \quad \Rightarrow B_{2 n}=B_{1 n}
\end{array}
$$

Both Eq.(1.73) and Eq.(1.74) state that: the normal components of electric and magnetic flux density across an interface of two media are continuous, Fig.(1.2).

### 1.7.2 Infinite Conductivity Media

If one of the media has infinite conductivity, $\sigma_{1}=\infty$ or $\sigma_{2}=\infty$, there must be electric surface charge density $\rho_{s}$ (source or induced) or electric surface


Figure 1.2: Normal boundary conditions
current density $\mathbf{J}_{s}$ (source or induced) across the boundary. Let us assume that $\sigma_{1}=\infty$ therefore, $\mathbf{E}_{1}=0$ and

$$
\begin{equation*}
\mathbf{n}_{12} \times \mathbf{E}_{2}=0 \quad \Rightarrow E_{2 t}=0 \tag{1.75}
\end{equation*}
$$

Eq.(1.75) states that: the tangential component of the electric field on perfect conductors vanishes. According to above assumption and Maxwell's first equation, $\mathbf{H}_{1}=0$, [why ...?], therefore the surface electric current density $\mathbf{J}_{s}$ and $\rho_{s}$ will be induced on the surface of the conductor.
where $\mathbf{J}_{s}$ is electric surface current density in Ampers per meter and $\rho_{s}$ is electric surface charge density in Coulombs per square meter.
The boundary condition for tangential component on the magnetic field is

$$
\begin{equation*}
\mathbf{n}_{12} \times \mathbf{H}_{2}=\mathbf{J}_{s} \quad \Rightarrow H_{2 t}=J_{s} \tag{1.76}
\end{equation*}
$$

This states that the tangential component of magnetic field intensity across the boundary of a perfect conductor is equal to the surface electric current density on that conductor, The boundary condition for normal component of the electric flux density is:

$$
\begin{equation*}
\mathbf{n}_{12} \cdot \mathbf{D}_{2}=\rho_{s} \quad \Rightarrow D_{2 n}=\rho_{s} \tag{1.77}
\end{equation*}
$$

This states that the normal component of electric flux density across the boundary of a perfect conductor is equal to the surface charge density. At
last, the boundary condition for normal component of magnetic flux density is:

$$
\begin{equation*}
\mathbf{n}_{12} \cdot \mathbf{B}_{2}=0 \quad \Rightarrow B_{2 n}=0 \tag{1.78}
\end{equation*}
$$

This states that the normal component of magnetic flux density across a perfect electric conductor vanishes.
The above boundary conditions are appropriate for stationary boundary. When the boundary surface are moving, the partial derivative with respect to time can be written as:

$$
\begin{equation*}
\frac{d}{d t} \int d V \mathbf{A}=\int d V \frac{\partial \mathbf{A}}{\partial t}+\oint(d \mathbf{S} \cdot \mathbf{U}) \mathbf{A} \tag{1.79}
\end{equation*}
$$

where $\mathbf{U}$ is the velocity of the boundary [24]. Therefore the new boundary will be:

$$
\begin{gather*}
\mathbf{n}_{12} \times\left(\mathbf{E}_{2}-\mathbf{E}_{1}\right)-\left(\mathbf{n}_{12} \cdot \mathbf{U}\right)\left(\mathbf{B}_{2}-\mathbf{B}_{1}\right)=0  \tag{1.80}\\
\mathbf{n}_{12} \times\left(\mathbf{H}_{2}-\mathbf{H}_{1}\right)+\left(\mathbf{n}_{12} \cdot \mathbf{U}\right)\left(\mathbf{D}_{2}-\mathbf{D}_{1}\right)=\mathbf{J}_{s}  \tag{1.81}\\
\mathbf{n}_{12} \cdot\left(\mathbf{B}_{2}-\mathbf{B}_{1}\right)=0  \tag{1.82}\\
\mathbf{n}_{12} \cdot\left(\mathbf{D}_{2}-\mathbf{D}_{1}\right)=\rho_{s} \tag{1.83}
\end{gather*}
$$

We can also write the boundary conditions in a new form:

$$
\begin{align*}
\mathbf{n}_{1} \times \mathbf{H}_{1}+\mathbf{n}_{2} \times \mathbf{H}_{2} & =\mathbf{J}_{s}  \tag{1.84}\\
\mathbf{n}_{1} \times \mathbf{E}_{1}+\mathbf{n}_{2} \times \mathbf{E}_{2} & =\mathbf{M}_{s} \\
\mathbf{n}_{1} \cdot \mathbf{D}_{1}+\mathbf{n}_{2} \cdot \mathbf{D}_{2} & =\rho_{s} \\
\mathbf{n}_{1} \cdot \mathbf{B}_{1}+\mathbf{n}_{2} \cdot \mathbf{B}_{2} & =\rho_{m s}
\end{align*}
$$

### 1.7.3 Boundary conditions for scalar and vector potentials

In case of scalar electric potential we can write

$$
\begin{equation*}
\epsilon_{2} \frac{\partial V_{2}}{\partial n}-\epsilon_{1} \frac{\partial V_{1}}{\partial n}=\rho_{s} \quad \mathbf{n}=\mathbf{n}_{12} \tag{1.85}
\end{equation*}
$$



Figure 1.3: Boundary conditions

The continuity of the scalar potential across the material interface as:

$$
\begin{equation*}
V_{2}=V_{1} \tag{1.86}
\end{equation*}
$$

where $\frac{\partial V}{\partial n}=\nabla V \cdot \mathbf{n}$. If at some boundaries $V=0$ it is called Dirichlet's Condition and if $\frac{\partial V}{\partial n}=0$ it is called Neumann's Condition.
For vector magnetic potential we will have:

$$
\begin{align*}
& \mu_{2} \frac{\partial \mathbf{A}_{2}}{\partial n}-\mu_{1} \frac{\partial \mathbf{A}_{1}}{\partial n}=\mathbf{J}_{s}  \tag{1.87}\\
& \mathbf{n}_{12} \times\left(\mathbf{A}_{2}-\mathbf{A}_{1}\right)=0 \tag{1.88}
\end{align*}
$$

### 1.7.4 Leontovich Impedance Boundary Conditions

It worth to give some remarks about boundary conditions in time harmonic case. For good conductors $\frac{\sigma}{\omega \epsilon} \gg 1$, therefore the wave can penetrate into the conductor at maximum depth of $5 \delta$, where $\delta$ is the skin depth of good conductors. If the radius of curvature of the surface is much greater than the skin depth $\delta$, the approximate boundary condition holds and is called Leontovich Impedance boundary condition. If medium 1 is a very good conductor with surface impedance $Z_{s}$ (Ohms) given approximately by:

$$
\begin{equation*}
Z_{s}=R_{s}+j X_{s}=(1+j) \sqrt{\frac{\omega \mu}{2 \sigma}} \tag{1.89}
\end{equation*}
$$

then we will have

$$
\begin{align*}
\mathbf{J}_{s} & =\mathbf{n}_{12} \times \mathbf{H}_{2} \\
\mathbf{E}_{t 2} & =Z_{s} \mathbf{J}_{s}=Z_{s} \mathbf{n}_{12} \times \mathbf{H}_{2} \tag{1.90}
\end{align*}
$$

### 1.7.5 PEMC Boundary Conditions

Perfect electric conductor (PEC) and perfect magnetic conductor (PMC) are basic concepts in electromagnetics. Lindell has recently introduced perfect electromagnetic conductor (PEMC) as generalization of the perfect electric conductor (PEC) and perfect magnetic conductor (PMC) [56]-[59]. It is well known that PEC boundary may be defined by the conditions:

$$
\begin{align*}
\mathbf{n} \times \mathbf{E} & =0 \\
\mathbf{n} \cdot \mathbf{B} & =0 \tag{1.91}
\end{align*}
$$

While PMC boundary may be defined by the boundary conditions

$$
\begin{align*}
\mathbf{n} \times \mathbf{H} & =0 \\
\mathbf{n} \cdot \mathbf{D} & =0 \tag{1.92}
\end{align*}
$$

The PEMC boundary conditions are of the more general form

$$
\begin{array}{r}
\mathbf{n} \times(\mathbf{H}+M \mathbf{E})=0 \\
\mathbf{n} \cdot(\mathbf{D}-M \mathbf{B})=0 \tag{1.93}
\end{array}
$$

where $M$ denotes the admittance of the PEMC boundary. It is obvious that PMC corresponds to $\mathrm{M}=0$, while PEC corresponds to $M=\infty$. It may be noted that PEMC boundary is non-reciprocal. Non-reciprocity of the PEMC boundary can be demonstrated by showing that the polarization of the plane wave reflected from PEMC surface is rotated. Possibilities for the realization of a PEMC boundary have also been studied [56].

### 1.8 Power and Energy

The concept of electromagnetic energy has important physical interpretations and applications. Many problems can often be simplified if an energy
viewpoint is adopted. The energy conservation of the Electromagnetic fields can be formulated as:

$$
\begin{equation*}
\oint_{s} \tilde{\mathbf{E}} \times \tilde{\mathbf{H}} \cdot d \mathbf{S}+\int_{v}\left[\tilde{\mathbf{H}} \cdot \frac{\partial \tilde{\mathbf{B}}}{\partial t}+\tilde{\mathbf{E}} \cdot \frac{\partial \tilde{\mathbf{D}}}{\partial t}+\sigma \tilde{\mathbf{E}} \cdot \tilde{\mathbf{E}}+\tilde{\mathbf{E}} \cdot \tilde{\mathbf{J}}_{g}\right] d V=0 \tag{1.94}
\end{equation*}
$$

or simply

$$
\begin{equation*}
\oint_{s} \tilde{\mathbf{E}} \times \tilde{\mathbf{H}} \cdot d \mathbf{S}+\int_{v}\left[\mu \tilde{\mathbf{H}} \cdot \frac{\partial \tilde{\mathbf{H}}}{\partial t}+\epsilon \tilde{\mathbf{E}} \cdot \frac{\partial \tilde{\mathbf{E}}}{\partial t}+\sigma \tilde{\mathbf{E}} \cdot \tilde{\mathbf{E}}+\tilde{\mathbf{E}} \cdot \tilde{\mathbf{J}}_{g}\right] d V=0 \tag{1.95}
\end{equation*}
$$

where the sign $\sim$ denotes the functionality of $\mathbf{r}$ and $t$, i.e. $\tilde{\mathbf{C}}=\mathbf{C}(\mathbf{r}, t)$. The average complex power exiting from a surface;

$$
\begin{equation*}
P_{e}=\frac{1}{2} \int\left(\mathbf{E} \times \mathbf{H}^{*}\right) \cdot d \mathbf{s} \tag{1.96}
\end{equation*}
$$

where ( $\mathbf{E}$ and $\mathbf{H}$ represent peak values), The dissipated power inside the closed surface is given by

$$
\begin{equation*}
P_{d}=\frac{1}{2} \int \sigma|\mathbf{E}|^{2} d v \tag{1.97}
\end{equation*}
$$

The time average magnetic field stored energy inside the closed surface is given by

$$
\begin{equation*}
W_{m}=\frac{1}{4} \int \mu|\mathbf{H}|^{2} d v \tag{1.98}
\end{equation*}
$$

The time average electric field stored energy inside the closed surface is given by

$$
\begin{equation*}
W_{e}=\frac{1}{4} \int \epsilon|\mathbf{E}|^{2} d v \tag{1.99}
\end{equation*}
$$

### 1.9 Classification of EM Problems

To determine the best solution method for a given EM problem it is useful to classify a it according to certain criteria. Typically we classify problems according to

1) the solution region of interest
2) the nature of the equation of the problem, and
3) the associated boundary conditions

### 1.9.1 Solution Region

The solution region for EM problems is either an interior problem, an exterior problem or a combination of them. Interior Problems consist of regions that are confined by walls with distinct boundary conditions, i.e., a perfect conductor which imposes a boundary condition that the tangential electric field must be zero. Mathematically these problems have discrete set of eigenvalues and a discrete spectrum of wavelengths. Examples of interior problems are waveguides, transmission line problems, and the microwave oven. Open problems consist of the regions where the fields can extend to infinity. Mathematically these problems have continuous set of eigenvalues and a continuous spectrum of wavelengths. Examples of open problems are radiation and scattering problems.
Hybrid problems are those that have both an interior region and an open region. Mathematically these problems have both a discrete set of eigenvalues and a continuous spectrum. Examples of hybrid problems are dielectric waveguide problems and aperture radiation problems.

### 1.9.2 Classification of Differential Equations

Electromagnetic problems are classified in terms of the equations describing them. The equations can be differential, integral or integro-differential. Here we will look at the differential form.
All deterministic partial differential equations (PDE) can be described by the operator equation

$$
\begin{equation*}
\mathcal{L}(\mathbf{f})=\mathrm{g} \tag{1.100}
\end{equation*}
$$

where, $\mathcal{L}$ is a differential operator, $\mathbf{g}$ is a known excitation or source and $\mathbf{f}$ is the unknown function to be determined. The type of problem is determined by the differential operator $\mathcal{L}$. In general $\mathcal{L}$ can have the following form:

$$
\begin{equation*}
\mathcal{L}=a \frac{\partial^{2}}{\partial x^{2}}+b \frac{\partial^{2}}{\partial x \partial y}+c \frac{\partial^{2}}{\partial y^{2}}+d \frac{\partial}{\partial x}+e \frac{\partial}{\partial y}+f \tag{1.101}
\end{equation*}
$$

The linear second-order PDE can be classified as elliptic, hyperbolic, or parabolic depending on the coefficients $a, b$, and $c$ :

$$
\begin{array}{rc}
\text { elliptic if: } & b^{2}-4 a c<0 \\
\text { hyperbolic if: } & b^{2}-4 a c>0 \\
\text { parabolic if: } & b^{2}-4 a c=0 \tag{1.102}
\end{array}
$$

These terms elliptic, hyperbolic, and parabolic are derived from the fact that the quadratic equation:

$$
\begin{equation*}
a x^{2}+b x y+c y^{2}+d x+e y+f=0 \tag{1.103}
\end{equation*}
$$

represents an ellipse, hyperbola, or parabola if the coefficients take on the form of Eq.(1.103). In each of these categories the PDE can be used to model certain physical phenomena, which are not necessarily limited to electromagnetics. In fact they can include phenomena such as heat transfer, boundary-layer flow, fluid dynamics, vibrations, elasticity, acoustics, and financial analysis. Examples of the different types of PDEs are:
Elliptic PDE's

$$
\begin{array}{cc}
\text { Laplace's Equation } & \nabla^{2} V=0 \\
\text { Poisson's Equation } & \nabla^{2} V=-\frac{\rho}{\epsilon}
\end{array}
$$

$$
\text { Magnetic Laplace's Equation } \quad \nabla^{2} \mathbf{A}=0
$$

$$
\text { Magnetic Poisson's Equation } \quad \nabla^{2} \mathbf{A}=-\mu \mathbf{J}
$$

## Parabolic PDE's

Diffusion equation $\nabla^{2} \mathbf{J}=\mu \sigma \frac{\partial \mathbf{J}}{\partial t}$ like skin effect on a metallic cylinder; If $\frac{\sigma}{\omega \epsilon} \gg 1$
Hyperbolic PDE's

$$
\text { Wave Equation } \quad \nabla^{2} \mathbf{A}-\mu \epsilon \frac{\partial^{2} \mathbf{A}}{\partial t^{2}}=-\mu \mathbf{J}
$$

### 1.9.3 Eigenvalue Problems

Problems that take the form of

$$
\begin{equation*}
\mathcal{L}(\mathbf{f})=\lambda \mathbf{f} \tag{1.104}
\end{equation*}
$$

are referred to eigenvalue problems. For these types of problems only particular values of $\lambda$, called eigenvalues, are permissible. For each permissible eigenvalue there is an associated solution called an eigenfunction f. Eigenvalue problems are usually encountered in interior problems such as waveguide and cavity problems.

### 1.9.4 Classification of Boundary Conditions

In order to obtain a unique solution to an EM problem we must apply certain conditions to the boundary of the problem. The boundary condition on surface $s$ can be classified to the following forms:

1) Dirichlet boundary conditions: $\psi(r)=p(r)$ for $r$ on $s$
2) Neumann boundary conditions: $\frac{\partial \psi}{\partial n}=q(r)$ for $r$ on $s$
3) Mixed Boundary Conditions: $\psi(r)+\frac{\partial \psi}{\partial n}=w(r)$ for $r$ on $s$

If $p(r), q(r)$, or $w(r)$ are equal to zero, the boundary conditions are said to be homogeneous otherwise it is called nonhomogeneous boundary conditions.

### 1.10 Electromagnetic Theorem

In electromagnetics there are some useful and important theorems. Sometimes a complicated electromagnetic problem can be recast into a more simple problem, or a sequence of more simple problems by using various principles or theorems. Some of the more common theorems are: superposition, uniqueness, and equivalence.

### 1.10.1 Superposition Principle

The principle of superposition is derived from the linearity of Maxwell's equations. That is to say if given an operator equations as shown in Eq.(1.100) is linear, then there is an alternate solution for it.

$$
\begin{gathered}
\mathcal{L}\left(a_{1} \mathbf{f}_{\mathbf{1}}+a_{2} \mathbf{f}_{\mathbf{2}}+\cdots\right)=\mathbf{g} \\
a_{1} \mathcal{L}\left(\mathbf{f}_{\mathbf{1}}\right)+a_{2} \mathcal{L}\left(\mathbf{f}_{\mathbf{2}}\right)+\cdots=\mathbf{g} \\
\mathbf{f}=\sum_{n=1}^{N} a_{n} \mathbf{f}_{\mathbf{n}}
\end{gathered}
$$

### 1.10.2 Uniqueness Theorem

A solution is unique when it is the only one possible solution among a given class of solutions. Importance of the uniqueness theorem:

- Tells us what information is needed to obtain the solution.
- Establish conditions for one to one correspondence of a field to its sources which allows us to calculate field from sources or sources from fields.
- Comforting to know that a solution is the only solution.

Suppose sources $\mathbf{J}^{i}, \mathbf{M}^{i}$ radiating in a lossy material medium with complex electrical parameters:

$$
\begin{aligned}
\epsilon & =\epsilon^{\prime}-j \epsilon^{\prime \prime} \\
\mu & =\mu^{\prime}-j \mu^{\prime \prime}
\end{aligned}
$$

The electric and magnetic fields radiated by $\mathbf{J}^{i}, \mathbf{M}^{i}$ satisfy:

$$
\begin{equation*}
-\nabla \times \mathbf{E}=j \omega \mu \mathbf{H}+\mathbf{M}^{i} ; \quad \nabla \times \mathbf{H}=j \omega \epsilon \mathbf{E}+\mathbf{J}^{i} \tag{1.105}
\end{equation*}
$$

Assume two solutions $\left\{\mathbf{E}_{a}, \mathbf{H}_{a}\right\} ;\left\{\mathbf{E}_{b}, \mathbf{H}_{b}\right\}$ exists that both satisfies Maxwell's equations Eq.(1.105) and boundary conditions.

$$
\begin{array}{rlrl}
-\nabla \times \mathbf{E}_{a} & =j \omega \mu \mathbf{H}_{a}+\mathbf{M}^{i} ; & & \nabla \times \mathbf{H}_{a}=j \omega \epsilon \mathbf{E}_{a}+\mathbf{J}^{i} \\
-\nabla \times \mathbf{E}_{b}=j \omega \mu \mathbf{H}_{b}+\mathbf{M}^{i} ; & & \nabla \times \mathbf{H}_{b}=j \omega \epsilon \mathbf{E}_{b}+\mathbf{J}^{i} \tag{1.107}
\end{array}
$$

Subtracting Eq.(1.107) from Eq.(1.106):

$$
\begin{equation*}
-\nabla \times \delta \mathbf{E}=j \omega \mu \delta \mathbf{H} ; \quad \nabla \times \delta \mathbf{H}=j \omega \epsilon \delta \mathbf{E} \tag{1.108}
\end{equation*}
$$

where $\delta \mathbf{E}=\mathbf{E}_{a}-\mathbf{E}_{b}, \delta \mathbf{H}=\mathbf{H}_{a}-\mathbf{H}_{b}$. If the solution is unique, then $\delta \mathbf{E}=0$ and $\delta \mathbf{H}=0$. From Eq.(1.108) we can derive the Poynting Theorem based on the difference fields:

$$
\begin{equation*}
\oint_{S}\left(\delta \mathbf{E} \times \delta \mathbf{H}^{*}+\delta \mathbf{E}^{*} \times \delta \mathbf{H}\right) \cdot d \mathbf{S}=-2 \omega \int_{V}\left[\epsilon^{\prime \prime}|\delta \mathbf{E}|^{2}+\mu^{\prime \prime}|\delta \mathbf{H}|^{2}\right] d v \tag{1.109}
\end{equation*}
$$

If the tangential field of $\mathbf{E}$ or $\mathbf{H}$ at the boundary be known, Using the properties of the triple scalar product, $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}=\mathbf{B} \cdot \mathbf{C} \times \mathbf{A}=\mathbf{C} \cdot \mathbf{A} \times \mathbf{B}$ and $d \mathbf{S}=\mathbf{n} d S$ we can write:

$$
\begin{equation*}
\delta \mathbf{H}^{*} \cdot[\mathbf{n} \times \delta \mathbf{E}]=0, \quad \delta \mathbf{E}^{*} \cdot[\mathbf{n} \times \delta \mathbf{H}]=0 \tag{1.110}
\end{equation*}
$$

therefore, the left side of Eq.(1.109) should be equal zero. Now, since $\epsilon^{\prime \prime}>0$ and $\mu^{\prime \prime}>0$, then we must have that $|\delta \mathbf{E}|^{2}=|\delta \mathbf{H}|^{2}=0 \Rightarrow \delta \mathbf{E}=\delta \mathbf{H}=0$.

It is noted that Eq.(1.110) is satisfied when:

- a) $\mathbf{n} \times \mathbf{E}$ is uniquely specified over the entire surface $S$. Then, $\mathbf{n} \times \delta \mathbf{E}=0$ over $S$, and $\mathbf{E}$ and $\mathbf{H}$ are unique.
- b) $\mathbf{n} \times \mathbf{H}$ is uniquely specified over the entire surface $S$. then $\mathbf{n} \times \delta \mathbf{H}=0$ over $S$, and $\mathbf{E}$ and $\mathbf{H}$ are unique.
- c) When $\mathbf{n} \times \mathbf{E}$ is known over a part of $S$, and $\mathbf{n} \times \mathbf{H}$ is known over the remaining part of $S$, then $\mathbf{E}$ and $\mathbf{H}$ are unique.
- d) Both $\mathbf{n} \times \mathbf{E}$ and $\mathbf{n} \times \mathbf{H}$ are uniquely specified over the entire surface $S$, then $\mathbf{E}$ and $\mathbf{H}$ are unique.

Suppose sources of finite extent $\mathbf{J}^{i}, \mathbf{M}^{i}$ are in $V$. We can uniquely specify $\mathbf{E}$ and $\mathbf{H}$ in $V$ knowing $\mathbf{J}^{i}, \mathbf{M}^{i}$ and $\mathbf{n} \times \mathbf{E}$ over $S$, and $\mathbf{n} \times \mathbf{H}$ over $S$.


Figure 1.4: Implications of Uniqueness

### 1.10.3 Equivalence Theorem

For certain problems, i.e., combinations of interior or exterior problems, it is often convenient to replace the one or the other with an equivalent source
condition via the equivalence theorem. To see this, consider the homogeneous penetrable object with its permittivity and permeability $\epsilon_{2}$ and $\mu_{2}$ embedded in unbounded homogeneous space with its permittivity and permeability $\epsilon_{1}$ and $\mu_{1}$ as shown in Figure[??]. In this case, we have used the superposition, or linearity, principle to decompose the total electric field, $\mathbf{E}$; into an incident component, $\mathbf{E}^{\text {inc }}$ and $\mathbf{H}^{\text {inc }}$, and a scattered component, $\mathbf{E}^{\text {sca }}$ and $\mathbf{H}^{s c a}$, which are both exterior to the penetrable object. In this figure, $\mathbf{E}$ and $\mathbf{H}$ represent the total fields interior to the penetrable object. By using the surface equivalence principle the original problem may be replaced by a linear superposition of the exterior problem and the interior problem as shown in Fig[??] and Fig[??] respectively, with $\mathbf{J}_{1}$ and $\mathbf{M}_{1}, \mathbf{J}_{2}$ and $\mathbf{M}_{2}$ as the equivalent surface currents for each case.
In the case of exterior problem, the permittivity and permeability $\epsilon_{1}$ and $\mu_{1}$ are assumed both interior and exterior to the object. Similarly, in the case of interior problem, the permittivity and permeability $\epsilon_{1}$ and $\mu_{1}$ are assumed throughout the whole space. Thus one may apply the respect EM equations, solve the problem and obtain the total solution from superposition .

### 1.10.4 Duality Theorem

Let us look once more to our frequently used formula given in Table(1.1). We can obtain right hand side column formulas from the other side and vise versa, by interchanging. The duality theorem help us to solve one typical problem from solution of another one. This reduces formulation and computational efforts.

### 1.10.5 Lorentz Reciprocity Theorem

Suppose $\mathbf{E}_{\mathbf{1}}$ and $\mathbf{H}_{\mathbf{1}}$ be the fields that is generated by distributed sources $\mathbf{J}_{\mathbf{1}}$ and $\mathbf{M}_{\mathbf{1}}$ inside a volume $V$ surrounded by closed surface $S$. Also suppose that $\mathbf{E}_{\mathbf{2}}$ and $\mathbf{H}_{\mathbf{2}}$ be the fields that is generated by distributed sources $\mathbf{J}_{\mathbf{2}}$ and $\mathbf{M}_{\mathbf{2}}$ at that volume. It is better to assume that the media is isotropic with $\mu$ and $\epsilon$, (can be lossy, lossless, inhomogeneous). Therefore;

$$
\begin{gather*}
-\nabla \times \mathbf{E}_{1}=j \omega \mu \mathbf{H}_{1}+\mathbf{M}_{1}  \tag{1.111}\\
\nabla \times \mathbf{H}_{1}=j \omega \epsilon \mathbf{E}_{1}+\mathbf{J}_{1} \tag{1.112}
\end{gather*}
$$

Table 1.1: Duality Equations for $\mathbf{J}$ and $\mathbf{M}$ Sources

| Electric Sources $(\mathbf{J} \neq 0, \mathbf{M}=0)$ | Magnetic Sources $(\mathbf{J}=0, \mathbf{M} \neq 0)$ |
| :---: | :---: |
| $\nabla \times \mathbf{E}_{A}=-j \omega \mu \mathbf{H}_{A}$ | $\nabla \times \mathbf{H}_{F}=j \omega \epsilon \mathbf{E}_{F}$ |
| $\nabla \times \mathbf{H}_{A}=\mathbf{J}+j \omega \epsilon \mathbf{E}_{A}$ | $-\nabla \times \mathbf{E}_{F}=\mathbf{M}+j \omega \mu \mathbf{H}_{F}$ |
| $\nabla^{2} \mathbf{A}+k^{2} \mathbf{A}=-\mu \mathbf{J}$ | $\nabla^{2} \mathbf{F}+k^{2} \mathbf{F}=-\epsilon \mathbf{M}$ |
| $\mathbf{A}=\frac{\mu}{4 \pi} \int_{V} \mathbf{J} \frac{e^{-j k R}}{R} d v^{\prime}$ | $\mathbf{F}=\frac{\epsilon}{4 \pi} \int_{V} \mathbf{M}^{\frac{e^{-j k R}}{R}} d v^{\prime}$ |
| $\mathbf{H}_{A}=\frac{1}{\mu} \nabla \times \mathbf{A}$ | $\mathbf{E}_{F}=-\frac{1}{\epsilon} \nabla \times \mathbf{F}$ |
| $\mathbf{E}_{A}=-j \omega \mathbf{A}-j \frac{1}{\omega \mu \epsilon} \nabla(\nabla \cdot \mathbf{A})$ | $\mathbf{H}_{F}=-j \omega \mathbf{F}-j \frac{1}{\omega \mu \epsilon} \nabla(\nabla \cdot \mathbf{F})$ |

and

$$
\begin{align*}
-\nabla \times \mathbf{E}_{2} & =j \omega \mu \mathbf{H}_{2}+\mathbf{M}_{2}  \tag{1.113}\\
\nabla \times \mathbf{H}_{2} & =j \omega \epsilon \mathbf{E}_{2}+\mathbf{J}_{2} \tag{1.114}
\end{align*}
$$

From Eq.(1.112) and Eq.(1.113):

$$
\begin{gather*}
\mathbf{E}_{2} \cdot \nabla \times \mathbf{H}_{1}=\mathbf{E}_{2} \cdot j \omega \epsilon \mathbf{E}_{1}+\mathbf{E}_{2} \cdot \mathbf{J}_{1}  \tag{1.115}\\
-\mathbf{H}_{1} \cdot \nabla \times \mathbf{E}_{2}=\mathbf{H}_{1} \cdot j \omega \mu \mathbf{H}_{2}+\mathbf{H}_{1} \cdot \mathbf{M}_{2} \tag{1.116}
\end{gather*}
$$

Next, add Eq.(1.115) and Eq.(1.116) and apply the vector identity:

$$
\begin{equation*}
\nabla \cdot(\mathbf{A} \times \mathbf{B})=\mathbf{B} \cdot \nabla \times \mathbf{A}-\mathbf{A} \cdot \nabla \times \mathbf{B} \tag{1.117}
\end{equation*}
$$

leading to:

$$
\begin{equation*}
\nabla \cdot\left(\mathbf{E}_{1} \times \mathbf{H}_{2}-\mathbf{E}_{2} \times \mathbf{H}_{1}\right)=\mathbf{E}_{2} \cdot \mathbf{J}_{1}-\mathbf{E}_{1} \cdot \mathbf{J}_{2}+\mathbf{H}_{1} \cdot \mathbf{M}_{2}-\mathbf{H}_{2} \cdot \mathbf{M}_{1} \tag{1.118}
\end{equation*}
$$

which is Lorentz Reciprocity Theorem in differential form. Introduce a volume V bound by $S$ that encloses the sources. We can integrate Eq.(1.118) over

Table 1.2: Duality of the $\mathbf{J}$ and $\mathbf{M}$ Sources

| $(\mathbf{J} \neq 0, \mathbf{M}=0)$ |  | $(\mathbf{J}=0, \mathbf{M} \neq 0)$ |
| :---: | :---: | :---: |
| $\mathbf{E}_{A}$ | $\Leftrightarrow$ | $\mathbf{H}_{F}$ |
| $\mathbf{H}_{A}$ | $\Leftrightarrow$ | $-\mathbf{E}_{F}$ |
| $\mathbf{J}$ | $\Leftrightarrow$ | $\mathbf{M}$ |
| $\mathbf{A}$ | $\Leftrightarrow$ | $\mathbf{F}$ |
| $\epsilon$ | $\Leftrightarrow$ | $\mu$ |
| $\mu$ | $\Leftrightarrow$ | $\epsilon$ |

V. Applying the divergence theorem, leads to:

$$
\begin{equation*}
\oint_{S}\left(\mathbf{E}_{1} \times \mathbf{H}_{2}-\mathbf{E}_{2} \times \mathbf{H}_{\mathbf{1}}\right) \cdot d \mathbf{S}=\int\left(\mathbf{E}_{\mathbf{2}} \cdot \mathbf{J}_{1}-\mathbf{E}_{\mathbf{1}} \cdot \mathbf{J}_{\mathbf{2}}+\mathbf{H}_{\mathbf{1}} \cdot \mathbf{M}_{\mathbf{2}}-\mathbf{H}_{\mathbf{2}} \cdot \mathbf{M}_{1}\right) d V \tag{1.119}
\end{equation*}
$$

At this point we may deduce that

- at any point without source

$$
\nabla \cdot\left(\mathbf{E}_{1} \times \mathbf{H}_{\mathbf{2}}-\mathbf{E}_{\mathbf{2}} \times \mathbf{H}_{1}\right)=0
$$

or source free volume $V$

$$
\begin{equation*}
\oint_{S}\left(\mathbf{E}_{1} \times \mathbf{H}_{2}-\mathbf{E}_{\mathbf{2}} \times \mathbf{H}_{\mathbf{1}}\right) \cdot d \mathbf{S}=0 \tag{1.120}
\end{equation*}
$$

This form of the reciprocity theorem states that given any closed surface $S$ that bounds no sources, than any fields radiated by independent sources must satisfy Eq.(1.120).
Now let us look once more to Eq.(1.119). If the sources are of finite extent, then as $S \rightarrow \infty$, the waves radiated by the sources can be assumed to be plane waves. Namely:

$$
\lim _{r \rightarrow \infty} \mathbf{E} \approx E_{\theta} \mathbf{a}_{\theta}+E_{\phi} \mathbf{a}_{\phi}
$$



Figure 1.5: Lorentz Reciprocity Theorem

$$
\lim _{r \rightarrow \infty} \mathbf{H} \approx \frac{\mathbf{k} \times \mathbf{E}}{\eta}=-\frac{E_{\phi}}{\eta} \mathbf{a}_{\theta}+\frac{E_{\theta}}{\eta} \mathbf{a}_{\phi}
$$

therefore,

$$
\mathbf{E}_{\mathbf{1}} \times \mathbf{H}_{\mathbf{2}}-\mathbf{E}_{\mathbf{2}} \times \mathbf{H}_{\mathbf{1}}=\frac{1}{\eta}\left(E_{\theta}^{1} E_{\theta}^{2}+E_{\phi}^{1} E_{\phi}^{2}-E_{\theta}^{2} E_{\theta}^{1}-E_{\phi}^{2} E_{\phi}^{1}\right)=0
$$

As a consequence of Eq.(1.120), Eq.(1.119) simply becomes:

$$
\begin{equation*}
\iiint\left[\mathbf{E}_{\mathbf{2}} \cdot \mathbf{J}_{1}-\mathbf{E}_{1} \cdot \mathbf{J}_{\mathbf{2}}+\mathbf{H}_{\mathbf{1}} \cdot \mathbf{M}_{\mathbf{2}}-\mathbf{H}_{\mathbf{2}} \cdot \mathbf{M}_{\mathbf{1}}\right] d v=0 \tag{1.121}
\end{equation*}
$$

where $V$ now can represent all space. This is a useful form of the reciprocity theorem that will be related to the reaction principal.

### 1.10.6 Reaction Theorem

It is emphasized that the reciprocity theorem does not represent power. Rather, it represents the reaction between sources. We can define the reaction of source 1 with source 2 as:

$$
\begin{equation*}
\langle 1,2\rangle=\int_{V}\left[\mathbf{E}_{1} \cdot \mathbf{J}_{2}-\mathbf{H}_{1} \cdot \mathbf{M}_{2}\right] d v \tag{1.122}
\end{equation*}
$$

The reaction of source 2 with source 1 is:

$$
\begin{equation*}
\langle 2,1\rangle=\int_{V}\left[\mathbf{E}_{2} \cdot \mathbf{J}_{1}-\mathbf{H}_{2} \cdot \mathbf{M}_{1}\right] d v \tag{1.123}
\end{equation*}
$$

If the surrounding medium is reciprocal, then

$$
\begin{equation*}
\langle 1,2\rangle=\langle 2,1\rangle \tag{1.124}
\end{equation*}
$$

### 1.11 Babinet's Principle

Babinet's principle states that the fields scattered by complementary surfaces are also complementary. In particular, if the electric and magnetic fields scattered by the conducting plate of figure (1.6-a), illuminated by $\mathbf{E}_{\text {inc }}$ and $\mathbf{H}_{\text {inc }}$, are $\mathbf{E}_{d i f}$ and $\mathbf{H}_{d i f}$, respectively, then the fields diffracted by the complementary aperture of figure (1.6-b), illuminated by $-\sqrt{\frac{\mu}{\epsilon}} \mathbf{H}_{\text {inc }}^{c}$ and $\sqrt{\frac{\epsilon}{\mu}} \mathbf{E}_{\text {inc }}^{c}$, will relate to their complementary as $-\sqrt{\frac{\mu}{\epsilon}} \mathbf{H}_{d i f}^{c}=-\mathbf{E}_{d i f}$ and $\sqrt{\frac{\epsilon}{\mu}} \mathbf{E}_{d i f}^{c}=-\mathbf{H}_{d i f}$.


Figure 1.6: Babinet's Principle

### 1.12 Radiation and Scattering

Radiation and scattering belong to exterior problems. In this section we will review some important points in radiation and scattering problems.

### 1.12.1 Radiation Condition

For an unbounded region it is necessary to specify the fields on a surface at infinity. By assuming that all sources are contain in a finite region, only out going waves must be present at large distance from the sources. In other words, the field behavior at large distance from the sources must meet the physical requirement that energy travel away from the sources region. This requirement is the Sommerfeld Radiation Condition and constitutes a boundary condition on the surface at infinity.
For 3D region the condition is:

$$
\begin{equation*}
\lim _{r \rightarrow \infty} r\left(\frac{\partial A}{\partial r}+j k A\right)=0 \tag{1.125}
\end{equation*}
$$

and for 2 D region

$$
\begin{equation*}
\lim _{\rho \rightarrow \infty} \sqrt{\rho}\left(\frac{\partial A}{\partial \rho}+j k A\right)=0 \tag{1.126}
\end{equation*}
$$

The above equations apply to non-dissipative media. When the media are slightly lossy, one may use the simpler requirement that all fields exited by sources in a finite region should vanish at infinity.

### 1.12.2 Edge Condition

In many cases, boundary and radiation conditions alone are not sufficient to determine the unique solution. What we need is some additional information concerning the behavior of the fields in the vicinity of edge. The Edge Condition requires that the energy density must be integrable over any finite domain even if this domain contains singularities of the electromagnetic field or in another word, the electromagnetic energy in any finite domain must be finite. [4],[5].

### 1.12.3 Radiation of Distributed Sources

a) Near Field

Let us assume that a distributed source $\mathbf{J}\left(x^{\prime}, y^{\prime}, z^{\prime}\right)=J_{x}\left(x^{\prime}, y^{\prime}, z^{\prime}\right) \mathbf{a}_{x}+J_{y}\left(x^{\prime}, y^{\prime}, z^{\prime}\right) \mathbf{a}_{y}+$ $J_{z}\left(x^{\prime}, y^{\prime}, z^{\prime}\right) \mathbf{a}_{z}$ exists. The magnetic filed at point $x, y, z$ will be found:

$$
\begin{align*}
& H_{x}=\frac{1}{4 \pi} \iiint_{V}\left[\left(z-z^{\prime}\right) J_{y}-\left(y-y^{\prime}\right) J_{z}\right] \frac{1+j \beta R}{R^{3}} e^{-j \beta R} d x^{\prime} d y^{\prime} d z^{\prime}(1  \tag{1.127}\\
& H_{y}=\frac{1}{4 \pi} \iiint_{V}\left[\left(x-x^{\prime}\right) J_{z}-\left(z-z^{\prime}\right) J_{x}\right] \frac{1+j \beta R}{R^{3}} e^{-j \beta R} d x^{\prime} d y^{\prime} d z^{\prime}(1  \tag{1.128}\\
& H_{z}=\frac{1}{4 \pi} \iiint_{V}\left[\left(y-y^{\prime}\right) J_{x}-\left(x-x^{\prime}\right) J_{y}\right] \frac{1+j \beta R}{R^{3}} e^{-j \beta R} d x^{\prime} d y^{\prime} d z^{\prime}(1
\end{align*}
$$

and the electric field at point $x, y, x$ will be:

$$
\begin{align*}
E_{x}= & -\frac{j \eta}{4 \pi \beta} \iiint\left\{G_{1} J_{x}+\left(x-x^{\prime}\right) G_{2} \times\right. \\
& {\left.\left[\left(x-x^{\prime}\right) J_{x}+\left(y-y^{\prime}\right) J_{y}+\left(z-z^{\prime}\right) J_{z}\right]\right\} e^{-j \beta R} d x^{\prime} d y^{\prime} d z^{\prime} }  \tag{1.130}\\
E_{y}= & -\frac{j \eta}{4 \pi \beta} \iiint\left\{G_{1} J_{y}+\left(y-y^{\prime}\right) G_{2} \times\right. \\
& {\left.\left[\left(x-x^{\prime}\right) J_{x}+\left(y-y^{\prime}\right) J_{y}+\left(z-z^{\prime}\right) J_{z}\right]\right\} e^{-j \beta R} d x^{\prime} d y^{\prime} d z^{\prime} }  \tag{1.131}\\
E_{z}= & -\frac{j \eta}{4 \pi \beta} \iiint\left\{G_{1} J_{z}+\left(z-z^{\prime}\right) G_{2} \times\right. \\
& {\left.\left[\left(x-x^{\prime}\right) J_{x}+\left(y-y^{\prime}\right) J_{y}+\left(z-z^{\prime}\right) J_{z}\right]\right\} e^{-j \beta R} d x^{\prime} d y^{\prime} d z^{\prime} } \tag{1.132}
\end{align*}
$$

where $R=\sqrt{\left(x-x^{\prime}\right)^{2}+\left(y-y^{\prime}\right)^{2}+\left(z-z^{\prime}\right)^{2}}$ and

$$
\begin{align*}
G_{1} & =\frac{-1-j \beta R+\beta^{2} R^{2}}{R^{3}} \\
G_{2} & =\frac{3+j 3 \beta R-\beta^{2} R^{2}}{R^{5}} \tag{1.133}
\end{align*}
$$

In case of magnetic current $\mathbf{M}$ we can use duality theorem in order to find $\mathbf{E}$ and $\mathbf{H}$.
In 2 D case, which we have $\mathbf{J}(x, y)$, the $\mathbf{E}$ and $\mathbf{H}$ will be:

$$
\begin{equation*}
\mathbf{H}=\frac{j k}{4} \iint(\hat{\rho} \times \mathbf{J}) H_{1}^{(2)}(k \rho) d x^{\prime} d y^{\prime} \tag{1.134}
\end{equation*}
$$

where $H_{1}^{(2)}(k \rho)$ is the first order Hankel function of second kind and the $\hat{\rho}$ is a unit vector directed along the line joining any point of the source and the observation point. If we have $J_{z}(x, y)$, the fields will be:

$$
\begin{align*}
& H_{x}=\frac{j k}{4} \iint\left(y-y^{\prime}\right) J_{z}\left(x^{\prime}, y^{\prime}\right) \frac{H_{1}^{(2)}(k \rho)}{\rho} d x^{\prime} d y^{\prime}  \tag{1.135}\\
& H_{y}=\frac{j k}{4} \iint\left(x-x^{\prime}\right) J_{z}\left(x^{\prime}, y^{\prime}\right) \frac{H_{1}^{(2)}(k \rho)}{\rho} d x^{\prime} d y^{\prime} \tag{1.136}
\end{align*}
$$

and the $\rho=\sqrt{\left(x-x^{\prime}\right)^{2}+\left(y-y^{\prime}\right)^{2}}$. We know that $\mathbf{E}=\frac{1}{j \omega \epsilon} \nabla \times \mathbf{H}$ therefore:

$$
\begin{equation*}
E_{z}=\frac{k}{4 \omega \epsilon} \iint\left[2 \frac{H_{1}^{(2)}(k \rho)}{\rho}+k \rho H_{1}^{\prime(2)}(k \rho)-H_{1}^{(2)}(k \rho)\right] J_{z}\left(x^{\prime}, y^{\prime}\right) d x^{\prime} d y^{\prime} \tag{1.137}
\end{equation*}
$$

We can write the same for magnetic current.

$$
\begin{equation*}
\mathbf{E}=\frac{-j k}{4} \iint(\hat{\rho} \times \mathbf{M}) H_{1}^{(2)}(k \rho) d x^{\prime} d y^{\prime} \tag{1.138}
\end{equation*}
$$

Let assume that we have only $M_{x}(x, y), M_{y}(x, y)$ component, therefore

$$
\begin{equation*}
\left.E_{z}=\frac{-j k}{4} \iint\left[x-x^{\prime}\right) M_{y}-\left(y-y^{\prime}\right) M_{x}\right] H_{1}^{(2)}(k \rho) d x^{\prime} d y^{\prime} \tag{1.139}
\end{equation*}
$$

and the corresponding magnetic field will be

$$
\begin{align*}
H_{x}= & \frac{k}{4 \omega \mu} \iint\left[\left(\frac{k\left(x-x^{\prime}\right)\left(y-y^{\prime}\right) H_{1}^{\prime(2)}(k \rho)}{\rho^{2}}-\frac{\left(x-x^{\prime}\right)\left(y-y^{\prime}\right) H_{1}^{(2)}(k \rho)}{\rho^{3}}\right) M_{y}\right. \\
& \left.+\left(\frac{-k\left(y-y^{\prime}\right)^{2} H_{1}^{\prime(2)}(k \rho)}{\rho^{2}}+\frac{\left(y-y^{\prime}\right)^{2} H_{1}^{(2)}(k \rho)}{\rho^{3}}-\frac{H_{1}^{(2)}(k \rho)}{\rho}\right) M_{x}\right] d x^{\prime} d y^{\prime} \tag{1.140}
\end{align*}
$$

$$
\begin{align*}
H_{y}= & \frac{k}{4 \omega \mu} \iint\left[\left(\frac{k\left(x-x^{\prime}\right)\left(y-y^{\prime}\right) H_{1}^{\prime(2)}(k \rho)}{\rho^{2}}-\frac{\left(x-x^{\prime}\right)\left(y-y^{\prime}\right) H_{1}^{(2)}(k \rho)}{\rho^{3}}\right) M_{x}\right. \\
& \left.+\left(\frac{-k\left(x-x^{\prime}\right)^{2} H_{1}^{\prime(2)}(k \rho)}{\rho^{2}}+\frac{\left(x-x^{\prime}\right)^{2} H_{1}^{(2)}(k \rho)}{\rho^{3}}-\frac{H_{1}^{(2)}(k \rho)}{\rho}\right) M_{y}\right] d x^{\prime} d y^{\prime} \tag{1.141}
\end{align*}
$$

## b) Far Field

Let us assume surface currents $\mathbf{J}_{s}$ and $\mathbf{M}_{s}$ on a given surface. If the observation point is far from source $\beta R \gg 1$ and $r \geq \frac{2 D^{2}}{\lambda}$ where $D$ is the largest dimension of the radiator or scatterer, then:

$$
\begin{align*}
\mathbf{A} & =\frac{\mu}{4 \pi} \iint_{S} \mathbf{J}_{s} \frac{e^{-j \beta R}}{R} d s^{\prime} \approx \frac{\mu e^{-j \beta r}}{4 \pi r} \mathbf{N} \\
\mathbf{F} & =\frac{\mu}{4 \pi} \iint_{S} \mathbf{M}_{s} \frac{e^{-j \beta R}}{R} d s^{\prime} \approx \frac{\epsilon e^{-j \beta r}}{4 \pi r} \mathbf{L} \tag{1.142}
\end{align*}
$$

where $R=\left[r^{2}+r^{\prime 2}-2 r r^{\prime} \cos \psi\right]^{1 / 2}$ and can be approximated as

$$
R \approx \begin{cases}r-r^{\prime} \cos \psi & \text { for phase variations }  \tag{1.143}\\ r & \text { for amplitude variations }\end{cases}
$$

and $\psi$ is the angle between $\mathbf{r}$ and $\mathbf{r}^{\prime}$.

$$
\begin{align*}
\mathbf{N} & =\iint_{S} \mathbf{J}_{s} e^{j \beta r^{\prime} \cos \psi} d s^{\prime} \\
\mathbf{L} & =\iint_{S} \mathbf{M}_{s} e^{j \beta r^{\prime} \cos \psi} d s^{\prime} \tag{1.144}
\end{align*}
$$

In spherical coordinate we have:

$$
\begin{align*}
& E_{r} \approx 0 \\
& E_{\theta} \approx-\frac{j \beta e^{-j \beta r}}{4 \pi r}\left(L_{\phi}+\eta N_{\theta}\right) \\
& E_{\phi} \approx+\frac{j \beta e^{-j \beta r}}{4 \pi r}\left(L_{\theta}-\eta N_{\phi}\right) \\
& H_{r} \approx 0 \\
& H_{\theta} \approx+\frac{j \beta e^{-j \beta r}}{4 \pi r}\left(N_{\phi}-\frac{L_{\theta}}{\eta}\right) \\
& H_{\phi} \approx-\frac{j \beta e^{-j \beta r}}{4 \pi r}\left(N_{\theta}+\frac{L_{\phi}}{\eta}\right) \tag{1.145}
\end{align*}
$$

In rectangular coordinate system $N_{\theta}, N_{\phi}, L_{\theta}$ and $L_{\phi}$ can be found from the following formula

$$
\begin{align*}
& N_{\theta}=\iint_{S}\left(J_{x} \cos \theta \cos \phi+J_{y} \cos \theta \sin \phi-J_{z} \sin \theta\right) e^{+j \beta r^{\prime} \cos \psi} d s^{\prime} \\
& N_{\phi}=\iint_{S}\left(-J_{x} \sin \phi+J_{y} \cos \phi\right) e^{+j \beta r^{\prime} \cos \psi} d s^{\prime} \\
& L_{\theta}=\iint_{S}\left(M_{x} \cos \theta \cos \phi+M_{y} \cos \theta \sin \phi-M_{z} \sin \theta\right) e^{+j \beta r^{\prime} \cos \psi} d s^{\prime} \\
& L_{\phi}=\iint_{S}\left(-M_{x} \sin \phi+M_{y} \cos \phi\right) e^{+j \beta r^{\prime} \cos \psi} d s^{\prime} \tag{1.146}
\end{align*}
$$

### 1.12.4 Radar Cross Section (RCS)

- Two-dimensional problems:

The geometry is independent of one coordinate. An infinitely long cylinder with constant cross section is an example. The echo width $\sigma_{2 d}$ is used for two dimensional scattering:

$$
\begin{equation*}
\sigma_{2 d}=\lim _{\rho \rightarrow \infty} 2 \pi \rho \frac{\left|\mathbf{E}_{s}\right|^{2}}{\left|\mathbf{E}_{i}\right|^{2}} \tag{1.147}
\end{equation*}
$$

- Three-dimensional problems:

Conventional $\mathrm{RCS} \sigma_{3 d}$ is used for three dimensional scattering:

$$
\begin{equation*}
\sigma_{3 d}=\lim _{r \rightarrow \infty} 4 \pi r^{2} \frac{\left|\mathbf{E}_{s}\right|^{2}}{\left|\mathbf{E}_{i}\right|^{2}} \tag{1.148}
\end{equation*}
$$

The approximate conversion from echo width to radar cross section will be:

$$
\begin{equation*}
\sigma_{3 d}=\sigma_{2 d} \frac{2 D^{2}}{\lambda} \tag{1.149}
\end{equation*}
$$

## Chapter 2

## Reflection, Refraction and Transmission

## "Anyone who has never made a mistake has never tried anything new." <br> Albert Einstein

### 2.1 Classification of Reflection and Refraction



### 2.1.1 Normal Incident: Two Media

As shown in the Fig.(2.1), there are two media with electrical parameters $\mu_{1}, \epsilon_{1}, \sigma_{1}$ and $\mu_{2}, \epsilon_{2}, \sigma_{2}$ with impedance and propagation constant of $\eta_{i}=$ $\sqrt{\frac{j \omega \mu_{i}}{\sigma_{i}+j \omega \epsilon_{i}}}$ and $\gamma_{i}=\sqrt{j \omega \mu_{i}\left(\sigma_{i}+j \omega \epsilon_{i}\right)}, i=1,2$, respectively. The incident,

50 CHAPTER 2. REFLECTION, REFRACTION AND TRANSMISSION reflected and transmitted waves propagated in the z -direction, are given by


Figure 2.1: Normal incident, transmission and reflection from the boundary of two media

$$
\begin{align*}
\mathbf{E}^{i} & =A_{1} e^{-\gamma_{1} z} \mathbf{a}_{x} \\
\mathbf{H}^{i} & =\frac{A_{1}}{\eta_{1}} e^{-\gamma_{1} z} \mathbf{a}_{y}  \tag{2.1}\\
\mathbf{E}^{r} & =B_{1} e^{\gamma_{1} z} \mathbf{a}_{x} \\
\mathbf{H}^{r} & =\frac{-B_{1}}{\eta_{1}} e^{\gamma_{1} z} \mathbf{a}_{y}  \tag{2.2}\\
\mathbf{E}^{t} & =A_{2} e^{-\gamma_{2} z} \mathbf{a}_{x} \\
\mathbf{H}^{t} & =\frac{A_{2}}{\eta_{2}} e^{-\gamma_{2} z} \mathbf{a}_{y} \tag{2.3}
\end{align*}
$$

By enforcing continuity of tangential electric and magnetic fields at $z=0$; and introducing reflection coefficient $\Gamma=\frac{B_{1}}{A_{1}}$ and transmission coefficient $\tau=\frac{A_{2}}{A_{1}}$, we have:

$$
\begin{align*}
\Gamma & =\frac{\eta_{2}-\eta_{1}}{\eta_{2}+\eta_{1}}  \tag{2.4}\\
\tau & =\frac{2 \eta_{2}}{\eta_{2}+\eta_{1}} \tag{2.5}
\end{align*}
$$

If $\sigma_{2} \rightarrow \infty$ therefore, $\Gamma=-1$ and $\tau=0$.

### 2.1.2 Normal Incident: Multilayered Media

Suppose we have $N$ media with electrical parameters $\mu_{k} \epsilon_{k}$ and $\sigma_{k}$ and $N-1$ interfaces which located at $z_{1}, z_{2}, \cdots z_{N-1}$. Let us take $N=3$. According to Fig.(2.2), we can find electric and magnetic fields of each media:


Figure 2.2: Normal incident, transmission and reflection from multilayered media

$$
\begin{align*}
\mathbf{E}^{1} & =\left(A_{1} e^{-\gamma_{1} z}+B_{1} e^{+\gamma_{1} z}\right) \mathbf{a}_{x}  \tag{2.6}\\
\mathbf{H}^{1} & =\left(\frac{A_{1}}{\eta_{1}} e^{-\gamma_{1} z}-\frac{B_{1}}{\eta_{1}} e^{+\gamma_{1} z}\right) \mathbf{a}_{y}  \tag{2.7}\\
\mathbf{E}^{2} & =\left(A_{2} e^{-\gamma_{2} z}+B_{2} e^{+\gamma_{2} z}\right) \mathbf{a}_{x}  \tag{2.8}\\
\mathbf{H}^{2} & =\left(\frac{A_{2}}{\eta_{2}} e^{-\gamma_{2} z}-\frac{B_{2}}{\eta_{2}} e^{+\gamma_{2} z}\right) \mathbf{a}_{y}  \tag{2.9}\\
\mathbf{E}^{3} & =A_{3} e^{-\gamma_{3} z} \mathbf{a}_{x}  \tag{2.10}\\
\mathbf{H}^{3} & =\frac{A_{3}}{\eta_{3}} e^{-\gamma_{3} z} \mathbf{a}_{y} \tag{2.11}
\end{align*}
$$

By applying boundary conditions and writing them in matrix form, we have:

$$
\left[\begin{array}{cccc}
-e^{\gamma_{1} z_{1}} & e^{-\gamma_{2} z_{1}} & e^{\gamma_{2} z_{1}} & 0  \tag{2.12}\\
\frac{1}{\eta_{1}} e^{\gamma_{1} z_{1}} & \frac{1}{\eta_{1}} e^{\gamma_{2} z_{1}} & \frac{1}{\eta_{2}} e^{\gamma_{2} z_{1}} & 0 \\
0 & -e^{-\gamma_{2} z_{2}} & -e^{\gamma_{2} z_{2}} & e^{\gamma_{2} z_{2}} \\
0 & \frac{1}{\eta_{2}} e^{-\gamma_{2} z_{2}} & \frac{1}{\eta_{2}} e^{\gamma_{2} z_{2}} & \frac{1}{\eta_{3}} e^{\gamma_{3} z_{2}}
\end{array}\right]\left[\begin{array}{c}
B_{1} \\
A_{2} \\
B_{2} \\
A_{3}
\end{array}\right]=\left[\begin{array}{c}
A_{1} e^{-\gamma_{1} z_{1}} \\
\frac{1}{\eta_{1}} A_{1} e^{-\gamma_{1} z_{1}} \\
0 \\
0
\end{array}\right]
$$

### 2.2 Inhomogeneous Media

Exact one dimensional solution of wave equation for linear index of refraction $n(z)=n_{0}+\frac{n_{s}-n_{0}}{D} z$ will be consider now. The wave equation for this case will be:

$$
\begin{equation*}
\frac{d^{2} E_{x}}{d z^{2}}+\left[k_{0} n(z)\right]^{2} E_{x}=0 \Rightarrow \frac{d^{2} E_{x}}{d \xi^{2}}+\left[\frac{\xi}{\alpha}\right]^{2} E_{x}=0 \tag{2.13}
\end{equation*}
$$

where $\alpha=k_{0} \frac{n_{s}-n_{0}}{D}$ and $\xi=k_{0} n(z)$ therefore the solution of wave equation will be

$$
\begin{equation*}
E_{x}=\sqrt{\xi}\left[A J_{1 / 4}\left(\frac{\xi^{2}}{2 \alpha}\right)+B Y_{1 / 4}\left(\frac{\xi^{2}}{2 \alpha}\right)\right] \tag{2.14}
\end{equation*}
$$

where $J_{\nu}(x)$ and $Y_{\nu}(x)$ are Bessel and Neumann functions. If $n(z)^{2}=n_{0}^{2}+$ $\frac{n_{s}^{2}-n_{0}^{2}}{D} z, \alpha=k_{0}^{2} \frac{n_{s}^{2}-n_{0}^{2}}{D}$ and $\xi=\left[k_{0} n(z)\right]^{2}$ the solution will be determined as

$$
\begin{equation*}
E_{x}=\sqrt{\xi}\left[A J_{1 / 3}\left(\frac{2}{3 \alpha} \xi^{3 / 2}\right)+B Y_{1 / 3}\left(\frac{2}{3 \alpha} \xi^{3 / 2}\right)\right] \tag{2.15}
\end{equation*}
$$

If $n(z)=n_{0} \exp \left[\frac{z}{D} \ln \left(\frac{n_{s}}{n_{0}}\right)\right]$ The solution will be

$$
\begin{equation*}
E_{x}=A J_{0}\left(\frac{\xi}{\alpha}\right)+B Y_{0}\left(\frac{\xi}{\alpha}\right) \tag{2.16}
\end{equation*}
$$

where $\xi=k_{0} n_{0} e^{-\alpha z}$ and $\alpha=\frac{1}{D} \ln \frac{n_{s}}{n_{0}}$

### 2.2.1 Inhomogeneous Layers

In previous section, we have considered the multilayer media which also can be used for computation of reflection and transmission in inhomogeneous
media. In this section we shall develop a new method for exact solution of 1D inhomogeneous layer by matrix similarity transformation.
We first consider a dielectric layer whose $\epsilon_{r}(z)$ profile is a function of $z$. By 1D Maxwell's equations we have:

$$
\begin{align*}
\frac{d E_{x}}{d z} & =-j \omega \mu_{0} \mu_{r}(z) H_{y} \\
\frac{d H_{y}}{d z} & =-j \omega \epsilon_{0} \epsilon_{r}(z) E_{x} \tag{2.17}
\end{align*}
$$

or in matrix form:

$$
\frac{d}{d z}\left[\begin{array}{l}
E_{x}  \tag{2.18}\\
H_{y}
\end{array}\right]=\left[\begin{array}{cc}
0 & -j \omega \mu_{0} \mu_{r}(z) \\
-j \omega \epsilon_{0} \epsilon_{r}(z) & 0
\end{array}\right]\left[\begin{array}{l}
E_{x} \\
H_{y}
\end{array}\right]
$$

In general form it can be shown as

$$
\begin{equation*}
\frac{d \mathbf{q}}{d z}=\mathbf{A}(z) \mathbf{q}(z) \tag{2.19}
\end{equation*}
$$

and its solution will be

$$
\begin{equation*}
\mathbf{q}\left(z_{2}\right)=e^{\int_{z_{1}}^{z_{2}} \mathbf{A}(z) d z} \mathbf{q}\left(z_{1}\right) \tag{2.20}
\end{equation*}
$$

The $e^{\int_{z_{1}}^{z_{2}} \mathbf{A}(z) d z}$ can be calculated numerically. The result is a matrix. If we show this result by matrix $\mathbf{B}, e^{\int_{z_{1}}^{z_{2}} \mathbf{A}(z) d z}=\mathbf{B}$, then we will have:

$$
\left[\begin{array}{ll}
B_{11} & B_{12}  \tag{2.21}\\
B_{21} & B_{22}
\end{array}\right]\left[\begin{array}{l}
E_{x}^{i}\left(z_{1}\right)+E_{x}^{r}\left(z_{1}\right) \\
H_{y}^{i}\left(z_{1}\right)+H_{y}^{r}\left(z_{1}\right)
\end{array}\right]=\left[\begin{array}{l}
E_{x}^{t}\left(z_{2}\right) \\
H_{y}^{t}\left(z_{2}\right)
\end{array}\right]
$$

By introducing $\frac{E_{x}^{r}\left(z_{1}\right)}{E_{x}^{i}\left(z_{1}\right)}=\Gamma$ and $\frac{E_{x}^{t}\left(z_{2}\right)}{E_{x}^{i}\left(z_{1}\right)}=\tau$ :

$$
\left[\begin{array}{ll}
B_{11} & B_{12}  \tag{2.22}\\
B_{21} & B_{22}
\end{array}\right]\left[\begin{array}{c}
1+\Gamma \\
(1-\Gamma) / \eta_{1}
\end{array}\right]=\left[\begin{array}{c}
\tau \\
\tau / \eta_{3}
\end{array}\right]
$$

Now we can find the transmission and reflection coefficients from homogeneous and inhomogeneous multilayer dielectric slab. Therefore:

$$
\left[\begin{array}{cc}
B_{12} / \eta_{1}-B_{11} & 1  \tag{2.23}\\
B_{22} / \eta_{1}-B_{21} & 1 / \eta_{3}
\end{array}\right]\left[\begin{array}{l}
\Gamma \\
\tau
\end{array}\right]=\left[\begin{array}{c}
B_{11}+B_{12} / \eta_{1} \\
B_{21}+B_{22} / \eta_{1}
\end{array}\right]
$$

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The matrix $\mathbf{B}$ can be calculated by numerical method. We divide the domain of integral by N segments. Let integral be from $Z_{0}$ to $Z_{N}$ so $Z_{k}=k \frac{Z_{N}-Z_{0}}{N}$, and $\Delta z=\frac{Z_{N}-Z_{0}}{N}$, therefore the matrix $\mathbf{B}=e^{\int_{z_{0}}^{z_{N}} \mathbf{A}(z) d z}$ will be:

$$
\begin{equation*}
\mathbf{B}=\prod_{k=1}^{N} e^{\mathbf{A}\left(\frac{z_{k}+Z_{k-1}}{2}\right) \Delta z} \tag{2.24}
\end{equation*}
$$

For more numerical calculation we need:

$$
e^{\mathbf{A}(z) \Delta z}=\left[\begin{array}{lr}
\cos (\bar{\beta} \Delta z) & -j \bar{\eta} \sin (\bar{\beta} \Delta z)  \tag{2.25}\\
-j \sin (\bar{\beta} \Delta z) / \bar{\eta} & \cos (\bar{\beta} \Delta z)
\end{array}\right]
$$

where $\bar{\beta}=\omega \sqrt{\bar{\mu}}, \bar{\eta}=\sqrt{\frac{\bar{\mu}}{\epsilon}}$ and $\bar{\epsilon}=\epsilon_{0} \epsilon_{r}(z), \bar{\mu}=\mu_{0} \mu_{r}(z)$

### 2.2.2 Normal reflection of a plane wave from PEMC

Suppose a plane wave incident normally to a PEMC. We want to find the reflected wave. Let us take:

$$
\begin{align*}
\mathbf{E}^{\mathbf{i}} & =\mathbf{E}_{\mathbf{0}}^{\mathbf{i}} e^{-j k_{0} z}=\left(E_{x}^{i} \mathbf{a}_{\mathbf{x}}+E_{y}^{i} \mathbf{a}_{\mathbf{y}}\right) e^{-j k_{0} z}  \tag{2.26}\\
\mathbf{H}^{\mathbf{i}} & =\frac{\mathbf{a}_{\mathbf{z}} \times \mathbf{E}_{\mathbf{0}}^{\mathbf{i}}}{\eta_{0}} e^{-j k_{0} z}
\end{align*}
$$

and the reflected wave:

$$
\begin{align*}
\mathbf{E}^{\mathbf{r}} & =\mathbf{E}_{\mathbf{0}}^{\mathbf{r}} e^{+j k_{0} z}=\left(E_{x}^{r} \mathbf{a}_{\mathbf{x}}+E_{y}^{r} \mathbf{a}_{\mathbf{y}}\right) e^{+j k_{0} z}  \tag{2.27}\\
\mathbf{H}^{\mathbf{r}} & =\frac{-\mathbf{a}_{\mathbf{z}} \times \mathbf{E}_{0}^{\mathbf{r}}}{\eta_{0}} e^{+j k_{0} z}
\end{align*}
$$

at $z=0$, we apply PEMC boundary conditions, $\mathbf{n} \times(\mathbf{H}+M \mathbf{E})=0$, therefore it will give us two linear equations

$$
\begin{align*}
& E_{y}^{i}-E_{y}^{r}-\eta_{0} M\left(E_{x}^{i}+E_{x}^{r}\right)=0  \tag{2.28}\\
& E_{x}^{i}-E_{x}^{r}+\eta_{0} M\left(E_{y}^{i}+E_{y}^{r}\right)=0
\end{align*}
$$

By solving the above equations we will find.

$$
\begin{align*}
E_{x}^{r} & =\frac{\left(1-\eta_{0}^{2} M^{2}\right) E_{x}^{i}+2 \eta_{0} M E_{y}^{i}}{1+\eta_{0}^{2} M^{2}}  \tag{2.29}\\
E_{y}^{r} & =\frac{\left(1-\eta_{0}^{2} M^{2}\right) E_{y}^{i}-2 \eta_{0} M E_{x}^{i}}{1+\eta_{0}^{2} M^{2}}
\end{align*}
$$

As these relations show, for linearly polarized incident wave, the wave reflected from PEMC boundary has both co-polarized multiple of $\mathbf{E}^{\mathbf{i}}$ ) and cross-polarized components (multiple of $\mathbf{a}_{\mathbf{z}} \times \mathbf{E}^{\mathbf{i}}$ ) in general case. For the PMC and PEC special cases ( $M=0$ and $M=\infty$, respectively), the crosspolarized component vanishes. For the special PEMC case $M=1 / \eta_{0}$, we have $\mathbf{E}^{\mathbf{r}}=-\mathbf{a}_{\mathbf{z}} \times \mathbf{E}^{\mathbf{i}}$ which means that the reflected wave appears totally cross-polarized. Thus the boundary acts as a twist polarizer which is a nonreciprocal device [lindell, sihvola].

### 2.2.3 Transmission and Reflection from Chiral Media



Figure 2.3: Chiral slab
A homogeneous isotropic chiral medium is characterized by three (complex) parameters. These are the electric permittivity $\epsilon$, the magnetic permeability $\mu$, and the chirality measure $\beta$. The constitutive relations for a chiral media can be written as

$$
\begin{align*}
\mathbf{D} & =\epsilon(\mathbf{E}+\beta \nabla \times \mathbf{E})  \tag{2.30}\\
\mathbf{B} & =\mu(\mathbf{H}+\beta \nabla \times \mathbf{H}) \tag{2.31}
\end{align*}
$$

where $\beta$ is the chirality parameter. In optical frequency $\beta$ is very small quantity around $10^{-10} \mathrm{~m}$, but at microwave frequency is big as $10^{-3} \mathrm{~m}$. There is an alternate but equivalent notation as

$$
\begin{equation*}
\mathbf{D}=\epsilon^{\prime} \mathbf{E}-j \kappa \sqrt{\mu_{0} \epsilon_{0}} \mathbf{H} \tag{2.32}
\end{equation*}
$$

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$$
\begin{equation*}
\mathbf{B}=\mu^{\prime} \mathbf{H}+j \kappa \sqrt{\mu_{0} \epsilon_{0}} \mathbf{E} \tag{2.33}
\end{equation*}
$$

where $\kappa$ is the chirality parameter and is dimensionless. The relations between notations (2.30), (2.31) and (2.32), (2.33) are:

$$
\begin{align*}
\epsilon^{\prime} & =\frac{\epsilon}{1-\omega^{2} \mu \epsilon \beta^{2}} \\
\mu^{\prime} & =\frac{\mu}{1-\omega^{2} \mu \epsilon \beta^{2}} \\
\kappa \sqrt{\mu_{0} \epsilon_{0}} & =\frac{\omega \mu \epsilon \beta}{1-\omega^{2} \mu \epsilon \beta^{2}} \tag{2.34}
\end{align*}
$$

and

$$
\begin{align*}
\epsilon & =\epsilon^{\prime}\left(1-\frac{\kappa^{2}}{n^{2}}\right) \\
\mu & =\mu^{\prime}\left(1-\frac{\kappa^{2}}{n^{2}}\right) \\
k_{0} \beta & =\frac{\kappa}{n^{2}-\kappa^{2}} \tag{2.35}
\end{align*}
$$

where $n=\sqrt{\mu^{\prime} \epsilon^{\prime}} / \sqrt{\mu_{0} \epsilon_{0}}$ and $k_{0}=\omega \sqrt{\mu_{0} \epsilon_{0}}$.
In a source free region we have

$$
\begin{align*}
\nabla \times \mathbf{E} & =-j \omega \mathbf{B} \\
\nabla \times \mathbf{H} & =j \omega \mathbf{D} \\
\nabla \cdot \mathbf{D}=\nabla \cdot \mathbf{E} & =0 \\
\nabla \cdot \mathbf{B}=\nabla \cdot \mathbf{H} & =0 \tag{2.36}
\end{align*}
$$

Therefore we can rewrite the above equations

$$
\begin{align*}
& \nabla \times\left[\begin{array}{l}
\mathbf{E} \\
\mathbf{H}
\end{array}\right]=[\mathbf{K}]\left[\begin{array}{l}
\mathbf{E} \\
\mathbf{H}
\end{array}\right]  \tag{2.37}\\
& \nabla^{2}\left[\begin{array}{l}
\mathbf{E} \\
\mathbf{H}
\end{array}\right]=[\mathbf{K}]^{2}\left[\begin{array}{l}
\mathbf{E} \\
\mathbf{H}
\end{array}\right] \tag{2.38}
\end{align*}
$$

where the matrix $\mathbf{K}$ is given by:

$$
[\mathbf{K}]=\frac{1}{1-k^{2} \beta^{2}}\left[\begin{array}{cc}
k^{2} \beta & -j \omega \mu  \tag{2.39}\\
j \omega \epsilon & k^{2} \beta
\end{array}\right]
$$

where $k=\omega \sqrt{\mu \epsilon}$ in [1/meter]. The matrix in Eq.(2.39)can be diagonalized. A linear transformation of the electromagnetic field:

$$
\left[\begin{array}{c}
\mathbf{E}  \tag{2.40}\\
\mathbf{H}
\end{array}\right]=[\mathbf{A}]\left[\begin{array}{l}
\mathbf{Q}_{R} \\
\mathbf{Q}_{L}
\end{array}\right]
$$

diagonalizes $\mathbf{K}$

$$
\begin{equation*}
[\mathbf{\Lambda}]=[\mathbf{A}]^{-1}[\boldsymbol{\Lambda}][\mathbf{A}] \tag{2.41}
\end{equation*}
$$

where

$$
\begin{align*}
& {[\boldsymbol{\Lambda}]=\left[\begin{array}{cc}
-k_{R} & 0 \\
0 & k_{L}
\end{array}\right]}  \tag{2.42}\\
& {[\mathbf{A}]=\left[\begin{array}{cc}
1 & j \eta \\
j / \eta & 1
\end{array}\right]} \tag{2.43}
\end{align*}
$$

therefore this guide us to decompose the $\mathbf{E}$ and $\mathbf{H}$ fields into two $\mathbf{Q}_{R}$ and $\mathrm{Q}_{L}$ fields:

$$
\begin{align*}
\mathbf{E} & =\mathbf{Q}_{R}+j \eta \mathbf{Q}_{L} \\
\mathbf{H} & =\mathbf{Q}_{L}+\frac{j}{\eta} \mathbf{Q}_{R} \tag{2.44}
\end{align*}
$$

where $\eta=\sqrt{\mu / \epsilon}$. We can also show that

$$
\begin{align*}
\nabla \times \mathbf{Q}_{R} & =k_{R} \mathbf{Q}_{R} \\
\nabla \times \mathbf{Q}_{L} & =-k_{L} \mathbf{Q}_{L} \\
\nabla \cdot \mathbf{Q}_{R} & =0 \\
\nabla \cdot \mathbf{Q}_{L} & =0 \\
\nabla^{2} \mathbf{Q}_{R}+k_{R}^{2} \mathbf{Q}_{R} & =0 \\
\nabla^{2} \mathbf{Q}_{L}+k_{R}^{2} \mathbf{Q}_{L} & =0 \tag{2.45}
\end{align*}
$$

where

$$
\begin{align*}
k_{R} & =\frac{k}{1-k \beta} \\
k_{L} & =\frac{k}{1+k \beta} \tag{2.46}
\end{align*}
$$

are the wave numbers of $\mathbf{Q}_{R}$ and $\mathbf{Q}_{L}$, respectively. In the above relations, $\mathrm{Q}_{L}$ represents a left hand circularly polarized (LCP) and $\mathbf{Q}_{R}$ a right hand

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circularly polarized (RCP) wave. Hence the left and right handed CP waves travel with different phase velocities and this gives rise to the rotation in the plane of polarization when linearly polarized light passes through the medium. The LCP waves travel faster than RCP waves inside the left handed medium and vice versa. If, in addition, the medium is lossy, that is $k$ is complex, the two eigenwaves will experience different attenuation resulting in an elliptically polarized wave with a rotation of the major axis of the ellipse (dichroism). Bohren [60] introduced the decomposition of $\mathbf{E}$ and $\mathbf{H}$ fields into left circular polarization $\mathbf{Q}_{L}$ and right circular polarization $\mathbf{Q}_{R}$ as defined above.


Figure 2.4: Electromagnetic Properties of Chiral Medium
Suppose we have two back to back media. The first one is free space and the second one is a chiral media. A uniform plane wave incident normally on the chiral media from the free space. We want to find the reflection and transmission of wave. We assume uniform plane wave as an incident wave:

$$
\begin{equation*}
\mathbf{E}^{i}=e^{-j k_{0} z} \mathbf{a}_{x} \quad \mathbf{H}^{i}=\frac{1}{\eta_{0}} e^{-j k_{0} z} \mathbf{a}_{y} \tag{2.47}
\end{equation*}
$$

The reflected wave could be:

$$
\begin{equation*}
\mathbf{E}^{r}=\left(\Gamma_{x} \mathbf{a}_{x}+\Gamma_{y} \mathbf{a}_{y}\right) e^{j k_{0} z} \quad \mathbf{H}^{r}=\frac{\left(\Gamma_{y} \mathbf{a}_{x}-\Gamma_{x} \mathbf{a}_{y}\right)}{\eta_{0}} e^{j k_{0} z} \tag{2.48}
\end{equation*}
$$

and the transmitted wave will be:

$$
\begin{equation*}
\mathbf{E}^{t}=\mathbf{Q}_{R}+j \eta \mathbf{Q}_{L} \quad \mathbf{H}^{t}=\mathbf{Q}_{L}+\frac{j}{\eta} \mathbf{Q}_{R} \tag{2.49}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{Q}_{R}=\tau_{R}\left(\mathbf{a}_{x}-j \mathbf{a}_{y}\right) e^{-j k_{R} z} \quad \mathbf{Q}_{L}=\tau_{L}\left(\mathbf{a}_{x}+j \mathbf{a}_{y}\right) e^{-j k_{L} z} \tag{2.50}
\end{equation*}
$$

Determining the four unknowns, we use boundary conditions at $z=0$,

$$
\begin{align*}
\mathbf{a}_{x} \cdot\left(\mathbf{E}^{i}+\mathbf{E}^{r}-\mathbf{E}^{t}\right)=0 & \mathbf{a}_{y} \cdot\left(\mathbf{E}^{i}+\mathbf{E}^{r}-\mathbf{E}^{c}\right)=0 \\
\mathbf{a}_{x} \cdot\left(\mathbf{H}^{i}+\mathbf{H}^{r}-\mathbf{H}^{t}\right)=0 & \mathbf{a}_{y} \cdot\left(\mathbf{H}^{i}+\mathbf{H}^{r}-\mathbf{H}^{t}\right)=0 \tag{2.51}
\end{align*}
$$

and the unknowns can be found as:

$$
\begin{gather*}
\Gamma_{x}=\frac{\eta-\eta_{0}}{\eta+\eta_{0}} \\
\Gamma_{y}=0 \\
\tau_{R}=\frac{\eta}{\eta+\eta_{0}}  \tag{2.52}\\
\tau_{L}=\frac{-j}{\eta+\eta_{0}}
\end{gather*}
$$

### 2.2.4 The Reflection and Transmission of EM Waves by a Chiral Slab

Following the previous analysis, in the case of normal incidence, we can calculate the reflection and transmission coefficients for a chiral slab occupying the region $0<z<d$ in free space. For a normal incident plane wave the reflected wave will be linearly polarized in the same direction, while the transmitted is elliptically polarized. Thus it has a co-polarized and a cross-polarized components with respect to the incident wave. With the co-polarized components along the $x$-direction, we have:

$$
\begin{align*}
\mathbf{E}^{i} & =e^{-j k_{0} z} \mathbf{a}_{x} \\
\mathbf{H}^{i} & =\frac{1}{\eta_{0}} e^{-j k_{0} z} \mathbf{a}_{y} \\
\mathbf{E}^{r} & =\left(\Gamma_{x} \mathbf{a}_{x}+\Gamma_{y} \mathbf{a}_{y}\right) e^{j k_{0} z} \\
\mathbf{H}^{r} & =\frac{1}{\eta_{0}}\left(\Gamma_{y} \mathbf{a}_{x}-\Gamma_{x} \mathbf{a}_{y}\right) e^{j k_{0} z} \tag{2.53}
\end{align*}
$$

and the transmitted wave will be

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Figure 2.5: Chiral slab in free space

$$
\begin{align*}
\mathbf{E}^{t} & =\left(\tau_{x} \mathbf{a}_{x}+\tau_{y} \mathbf{a}_{y}\right) e^{-j k_{0} z} \\
\mathbf{H}^{t} & =\frac{1}{\eta_{0}}\left(-\tau_{y} \mathbf{a}_{x}+\tau_{x} \mathbf{a}_{y}\right) e^{-j k_{0} z} \tag{2.54}
\end{align*}
$$

and the fields inside the chiral slab can be written as:

$$
\begin{align*}
\mathbf{E}^{c} & =\mathbf{Q}_{R}+j \eta \mathbf{Q}_{L} \\
\mathbf{H}^{c} & =\mathbf{Q}_{L}+\frac{j}{\eta} \mathbf{Q}_{R} \\
\mathbf{Q}_{R} & =A_{R}\left(\mathbf{a}_{x}-j \mathbf{a}_{y}\right) e^{-j k_{R} z}+B_{R}\left(\mathbf{a}_{x}+j \mathbf{a}_{y}\right) e^{j k_{R} z} \\
\mathbf{Q}_{L} & =A_{L}\left(\mathbf{a}_{x}+j \mathbf{a}_{y}\right) e^{-j k_{L} z}+B_{L}\left(\mathbf{a}_{x}-j \mathbf{a}_{y}\right) e^{j k_{L} z} \tag{2.55}
\end{align*}
$$

and boundary conditions at $z=0$ and $z=d$ will be:

$$
\begin{aligned}
\mathbf{a}_{x} \cdot\left(\mathbf{E}^{i}+\mathbf{E}^{r}-\mathbf{E}^{c}\right)=0 & \mathbf{a}_{y} \cdot\left(\mathbf{E}^{i}+\mathbf{E}^{r}-\mathbf{E}^{c}\right)=0 \\
\mathbf{a}_{x} \cdot\left(\mathbf{H}^{i}+\mathbf{H}^{r}-\mathbf{H}^{c}\right)=0 & \mathbf{a}_{y} \cdot\left(\mathbf{H}^{i}+\mathbf{H}^{r}-\mathbf{H}^{c}\right)=0
\end{aligned}
$$

and for $z=d$ :

$$
\begin{aligned}
\mathbf{a}_{x} \cdot\left(\mathbf{E}^{c}-\mathbf{E}^{t}\right)=0 & \mathbf{a}_{y} \cdot\left(\mathbf{E}^{c}-\mathbf{E}^{t}\right)=0 \\
\mathbf{a}_{x} \cdot\left(\mathbf{H}^{c}-\mathbf{H}^{t}\right)=0 & \mathbf{a}_{y} \cdot\left(\mathbf{H}^{c}-\mathbf{H}^{t}\right)=0
\end{aligned}
$$

then we can find eight unknowns $\Gamma_{x}, \Gamma_{y}, \tau_{x}, \tau_{y}, A_{R}, B_{R}, A_{L}$ and $B_{L}$.

### 2.2.5 Metal Backed Chiral Slab

A uniform linearly polarized plane wave incident normally on a metal backed chiral slab as shown on Fig.(2.5). we suppose that the incident and reflected wave will be:

$$
\begin{align*}
\mathbf{E}^{i} & =e^{-j k_{0} z} \mathbf{a}_{x} \\
\mathbf{H}^{i} & =\frac{1}{\eta_{0}} e^{-j k_{0} z} \mathbf{a}_{y} \\
\mathbf{E}^{r} & =\left(\Gamma_{x} \mathbf{a}_{x}+\Gamma_{y} \mathbf{a}_{y}\right) e^{j k_{0} z} \\
\mathbf{H}^{r} & =\frac{1}{\eta_{0}}\left(\Gamma_{y} \mathbf{a}_{x}-\Gamma_{x} \mathbf{a}_{y}\right) e^{j k_{0} z} \tag{2.56}
\end{align*}
$$

where $\Gamma_{x}$ is the co-polarization reflection coefficient and $\Gamma_{y}$ is the cross polarization reflection coefficient. The fields inside the chiral slab will be:


Figure 2.6: Metal Backed Chiral Slab

$$
\begin{align*}
\mathbf{E}^{c} & =\mathbf{Q}_{R}+j \eta \mathbf{Q}_{L} \\
\mathbf{H}^{c} & =\mathbf{Q}_{L}+\frac{j}{\eta} \mathbf{Q}_{R} \\
\mathbf{Q}_{R} & =A_{R}\left(\mathbf{a}_{x}-j \mathbf{a}_{y}\right) e^{-j k_{R} z}+B_{R}\left(\mathbf{a}_{x}+j \mathbf{a}_{y}\right) e^{j k_{R} z} \\
\mathbf{Q}_{L} & =A_{L}\left(\mathbf{a}_{x}+j \mathbf{a}_{y}\right) e^{-j k_{L} z}+B_{L}\left(\mathbf{a}_{x}-j \mathbf{a}_{y}\right) e^{j k_{L} z} \tag{2.57}
\end{align*}
$$

by applying boundary conditions at $z=0$ :

$$
\begin{array}{ll}
\mathbf{a}_{x} \cdot\left(\mathbf{E}^{i}+\mathbf{E}^{r}-\mathbf{E}^{c}\right)=0 & \mathbf{a}_{y} \cdot\left(\mathbf{E}^{i}+\mathbf{E}^{r}-\mathbf{E}^{c}\right)=0 \\
\mathbf{a}_{x} \cdot\left(\mathbf{H}^{i}+\mathbf{H}^{r}-\mathbf{H}^{c}\right)=0 & \mathbf{a}_{y} \cdot\left(\mathbf{H}^{i}+\mathbf{H}^{r}-\mathbf{H}^{c}\right)=0
\end{array}
$$

and $z=d$

$$
\mathbf{a}_{x} \cdot \mathbf{E}^{c}=0 \quad \mathbf{a}_{y} \cdot \mathbf{E}^{c}=0
$$

we can find six unknowns $\Gamma_{x}, \Gamma_{y}, A_{R}, B_{R}, A_{L}$ and $B_{L}$.

### 2.3 Transmission and Reflection of Obliquely Incident Plane Wave from Lossless Media

A uniform plane wave obliquely incident on a large flat surface of a media gives rise to both a reflected and a transmitted waves. The direction of transmitted wave is different from incident wave and this phenomenon is called refraction. Any plane wave can be divided into two polarization types , $T M$ and $T E$ as shown in Fig.(2.7) and Fig.(2.8). The transmission and reflection coefficients for each case can be derived as follows.
\& (TM case)
We can derive the incident wave as:


Figure 2.7: TM polarization

$$
\begin{equation*}
\mathbf{H}^{i}=\frac{\mathbf{a}_{y}}{\eta_{1}} e^{-j k_{1}\left(x \sin \theta^{i}+z \cos \theta^{i}\right)} \tag{2.58}
\end{equation*}
$$

where $\eta_{1}=\sqrt{\mu_{1} / \epsilon_{1}}$. And corresponding electric field:

$$
\begin{equation*}
\mathbf{E}^{i}=\left(\mathbf{a}_{x} \cos \theta^{i}-\mathbf{a}_{z} \sin \theta^{i}\right) e^{-j k_{1}\left(x \sin \theta^{i}+z \cos \theta^{i}\right)} \tag{2.59}
\end{equation*}
$$

According to Fig.(2.7), we can write the reflected and refracted waves as:

$$
\begin{align*}
& \mathbf{H}^{r}=\frac{\mathbf{a}_{y} \Gamma_{T M}}{T_{M}-j k_{1}\left(x \sin \theta^{r}-z \cos \theta^{r}\right)} \\
& \mathbf{H}^{t}=\frac{\mathbf{a}_{y} T M}{T_{2}} e^{-j k_{2}\left(x \sin \theta^{t}+z \cos \theta^{\theta}\right)} \tag{2.60}
\end{align*}
$$

and the corresponding electric fields

$$
\begin{align*}
& \left.\mathbf{E}^{r}=\Gamma_{T M}\left(\mathbf{a}_{x} \cos \theta^{r}+\mathbf{a}_{z} \sin \theta^{r}\right)\right) e^{-j k_{1}\left(x \sin \theta^{r}-z \cos \theta^{r}\right)} \\
& \mathbf{E}^{t}=\tau_{T M}\left(\mathbf{a}_{x} \cos \theta^{t}-\mathbf{a}_{z} \sin \theta^{t}\right) e^{-j k_{2}\left(x \sin \theta^{t}+z \cos \theta^{t}\right)} \tag{2.61}
\end{align*}
$$

By applying boundary condition, we can determine the relation between incident, reflected and transmitted angles and also both $\Gamma_{T M}$ and $\tau_{T M}$ as:

$$
\begin{gather*}
\theta^{i}=\theta^{r} \\
k_{1} \sin \theta^{i}=k_{2} \sin \theta^{t}  \tag{2.62}\\
\Gamma_{T M}=\frac{\eta_{2} \cos \theta^{t}-\eta_{1} \cos \theta^{i}}{\eta_{2} \cos \theta^{t}+\eta_{1} \cos \theta^{i}}  \tag{2.63}\\
\tau_{T M}=\frac{2 \eta_{2} \cos \theta^{i}}{\eta_{2} \cos \theta^{t}+\eta_{1} \cos \theta^{i}} \tag{2.64}
\end{gather*}
$$

## \% (TE case)

Like previous case, we assume that the incident wave is given as

$$
\begin{equation*}
\mathbf{E}^{i}=\mathbf{a}_{y} e^{-j k_{1}\left(x \sin \theta^{i}+z \cos \theta^{i}\right)} \tag{2.65}
\end{equation*}
$$

And corresponding magnetic field:

$$
\begin{equation*}
\mathbf{H}^{i}=\frac{1}{\eta_{1}}\left(-\mathbf{a}_{x} \cos \theta^{i}+\mathbf{a}_{z} \sin \theta^{i}\right) e^{-j k_{1}\left(x \sin \theta^{i}+z \cos \theta^{i}\right)} \tag{2.66}
\end{equation*}
$$

According to Fig.(2.8), we can write the reflected and refracted waves as:

$$
\begin{align*}
& \mathbf{E}^{r}=\mathbf{a}_{y} \Gamma_{T E} e^{-j k_{1}\left(x \sin \theta^{r}-z \cos \theta^{r}\right)} \\
& \mathbf{E}^{t}=\mathbf{a}_{y} \tau_{T E} e^{-j k_{2}\left(x \sin \theta^{t}+z \cos \theta^{t}\right)} \tag{2.67}
\end{align*}
$$



Figure 2.8: TE polarization
and the corresponding magnetic fields:

$$
\begin{align*}
& \left.\mathbf{H}^{r}=\frac{\Gamma_{T E}}{\eta_{1}}\left(\mathbf{a}_{x} \cos \theta^{r}+\mathbf{a}_{z} \sin \theta^{r}\right)\right) e^{-j k_{1}\left(x \sin \theta^{r}-z \cos \theta^{r}\right)} \\
& \mathbf{H}^{t}=\frac{\tau_{T E}}{\eta_{2}}\left(\mathbf{a}_{x} \cos \theta^{t}-\mathbf{a}_{z} \sin \theta^{t}\right) e^{-j k_{2}\left(x \sin \theta^{t}+z \cos \theta^{t}\right)} \tag{2.68}
\end{align*}
$$

by applying boundary conditions, we can determine the relation between incident, reflected and transmitted angles and also both $\Gamma_{T E}$ and $\tau_{T E}$ as:

$$
\begin{gather*}
\theta^{i}=\theta^{r} \\
k_{1} \sin \theta^{i}=k_{2} \sin \theta^{t}  \tag{2.69}\\
\Gamma_{T E}=\frac{\eta_{2} \cos \theta^{i}-\eta_{1} \cos \theta^{t}}{\eta_{2} \cos \theta^{i}+\eta_{1} \cos \theta^{t}}  \tag{2.70}\\
\tau_{T E}=\frac{2 \eta_{2} \cos \theta^{t}}{\eta_{2} \cos \theta^{i}+\eta_{1} \cos \theta^{t}} \tag{2.71}
\end{gather*}
$$

If the second medium be a perfect electric conductor (PEC), $\eta_{2}=0, \Gamma_{T E}=$ $\Gamma_{T M}=-1$ and $\tau_{T E}=\tau_{T M}=0$ then the surface current on PEC will be:

$$
\begin{array}{ll}
\mathbf{J}_{s}=\frac{2}{\eta_{1}} e^{-j k_{1}\left(x \sin \theta^{i}\right)} \mathbf{a}_{x} & \mathrm{TM} \\
\mathbf{J}_{s}=\frac{2 \cos \theta^{i}}{\eta_{1}} e^{-j k_{1}\left(x \sin \theta^{i}\right)} \mathbf{a}_{y} & \mathrm{TE} \tag{2.72}
\end{array}
$$

### 2.3.1 Reflection of Obliquely Incident Plane Wave from PEMC

Let us assume that the incident wave in oblique case in general form will be:

$$
\begin{align*}
& \mathbf{E}^{i}=\left(A_{H}^{i} \cos \theta^{i} \mathbf{a}_{x}+A_{E}^{i} \mathbf{a}_{y}-A_{H}^{i} \sin \theta^{i} \mathbf{a}_{z}\right) e^{-j k_{0}\left(x \sin \theta^{i}+z \cos \theta^{i}\right)} \\
& \mathbf{H}^{i}=\frac{1}{\eta_{0}}\left(-A_{E}^{i} \cos \theta^{i} \mathbf{a}_{x}+A_{H}^{i} \mathbf{a}_{y}+A_{E}^{i} \sin \theta^{i} \mathbf{a}_{z}\right) e^{-j k_{0}\left(x \sin \theta^{i}+z \cos \theta^{i}\right)} \tag{2.73}
\end{align*}
$$

And the reflected wave

$$
\begin{align*}
& \mathbf{E}^{r}=\left(A_{H}^{r} \cos \theta^{r} \mathbf{a}_{x}+A_{E}^{r} \mathbf{a}_{y}+A_{H}^{r} \sin \theta^{r} \mathbf{a}_{z}\right) e^{-j k_{0}\left(x \sin \theta^{r}-z \cos \theta^{r}\right)} \\
& \mathbf{H}^{r}=\frac{1}{\eta_{0}}\left(A_{E}^{r} \cos \theta^{r} \mathbf{a}_{x}-A_{H}^{r} \mathbf{a}_{y}+A_{E}^{r} \sin \theta^{r} \mathbf{a}_{z}\right) e^{-j k_{0}\left(x \sin \theta^{r}-z \cos \theta^{r}\right)} \tag{2.74}
\end{align*}
$$

at $z=0$, we apply PEMC boundary conditions, $\mathbf{n} \times(\mathbf{H}+M \mathbf{E})=0$, therefor it will give us: $\theta^{i}=\theta^{r}$ and

$$
\begin{align*}
& A_{E}^{r}=\frac{\left(1-\eta_{0}^{2} M^{2}\right) A_{E}-2 \eta_{0} M A_{H}}{1+\eta_{0}^{2} M^{2}}  \tag{2.75}\\
& A_{H}^{r}=\frac{\left(1-\eta_{0}^{2} M^{2}\right) A_{H}+2 \eta_{0} M A_{E}}{1+\eta_{0}^{2} M^{2}}
\end{align*}
$$

Electric and magnetic surface current on the PEMC at $z=0$ can be calculated, $\mathbf{J}_{s}=\mathbf{n} \times \mathbf{H}$ and $\mathbf{M}_{s}=-\mathbf{n} \times \mathbf{E}$ :
$\mathbf{J}_{s}=\frac{1}{\eta_{0}}\left[\mathbf{a}_{x} \frac{2 M^{2} \eta_{0}^{2} A_{H}-2 M \eta_{0} A_{E}}{1+M^{2} \eta_{0}^{2}}+\mathbf{a}_{y} \frac{2 M^{2} \eta_{0}^{2} A_{E}+2 M \eta_{0} A_{H}}{1+M^{2} \eta_{0}^{2}} \cos \theta^{i}\right] e^{-j k_{0} x \sin \theta^{i}}$
and

$$
\begin{equation*}
\mathbf{M}_{s}=\left[\mathbf{a}_{x} \frac{2 M \eta_{0} A_{H}-2 A_{E}}{1+M^{2} \eta_{0}^{2}}+\mathbf{a}_{y} \frac{2 M \eta_{0} A_{E}+2 A_{H}}{1+M^{2} \eta_{0}^{2}} \cos \theta^{i}\right] e^{-j k_{0} x \sin \theta^{i}} \tag{2.77}
\end{equation*}
$$

and for special case when $M \rightarrow \infty$ or $M \rightarrow 0$ we have

$$
\begin{align*}
\mathbf{J}_{s} & =\frac{1}{\eta_{0}}\left[2 A_{H} \mathbf{a}_{x}+2 A_{E} \cos \theta^{i} \mathbf{a}_{y}\right] e^{-j k_{0} x \sin \theta^{i}}  \tag{2.78}\\
\mathbf{M}_{s} & =\frac{1}{\eta_{0}}\left[-2 A_{E} \mathbf{a}_{x}+2 A_{H} \cos \theta^{i} \mathbf{a}_{y}\right] e^{-j k_{0} x \sin \theta^{i}} \tag{2.79}
\end{align*}
$$

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### 2.3.2 Transmission and Reflection of Obliquely Incident Plane Wave by Chiral Media

We consider the general case oblique incident on flat chiral media:

$$
\begin{align*}
& \mathbf{E}^{i}=\left(A_{H} \mathbf{a}_{x} \cos \theta^{i}+A_{E} \mathbf{a}_{y}-A_{H} \mathbf{a}_{z} \sin \theta^{i}\right) e^{-j k_{0}\left(x \sin \theta^{i}+z \cos \theta^{i}\right)} \\
& \mathbf{H}^{i}=-\frac{1}{j \omega \epsilon_{0}} \nabla \times \mathbf{E}^{i} \tag{2.80}
\end{align*}
$$

where $A_{E} \neq 0$ and $A_{H}=0$ is refereed to $T E$ polarization while $A_{H} \neq 0$ and $A_{E}=0$ is refereed to $T M$ polarization.
The reflected wave obey's Snell's law of reflection but it is circular polarization wave.

$$
\begin{align*}
& \mathbf{E}^{r}=\left(-\Gamma_{T M} A_{H} \cos \theta^{r} \mathbf{a}_{x}+\Gamma_{T E} A_{E} \mathbf{a}_{y}+\Gamma_{T M} A_{H} \sin \theta^{r} \mathbf{a}_{z}\right) e^{-j k_{0}\left(x \sin \theta^{r}-z \cos \theta^{r}\right)} \\
& \mathbf{H}^{r}=-\frac{1}{j \omega \epsilon_{0}} \nabla \times \mathbf{E}^{r} \tag{2.81}
\end{align*}
$$

The wave that propagates in chiral media has two different angle of inclination $\theta_{R}$ and $\theta_{L}$ :

$$
\begin{align*}
& \mathbf{E}^{c}=\mathbf{Q}_{R}+j \eta \mathbf{Q}_{L} \\
& \mathbf{H}^{c}=\mathbf{Q}_{L}+\frac{j}{\eta} \mathbf{Q}_{R} \tag{2.82}
\end{align*}
$$

where $\eta=\sqrt{\mu / \epsilon}$ and

$$
\begin{align*}
& \mathbf{Q}_{R}=\tau_{R}\left(\mathbf{a}_{x} \cos \theta_{R}-j \mathbf{a}_{y}-\mathbf{a}_{z} \sin \theta_{R}\right) e^{-j k_{R}\left(x \sin \theta_{R}+z \cos \theta_{R}\right)} \\
& \mathbf{Q}_{L}=\tau_{L}\left(\mathbf{a}_{x} \cos \theta_{L}+j \mathbf{a}_{y}-\mathbf{a}_{z} \sin \theta_{L}\right) e^{-j k_{L}\left(x \sin \theta_{L}+z \cos \theta_{L}\right)} \tag{2.83}
\end{align*}
$$

By applying boundary conditions at $z=0$ :

$$
\begin{gather*}
\mathbf{a}_{x} \cdot\left[\mathbf{E}^{\mathbf{i}}+\mathbf{E}^{\mathbf{r}}-\mathbf{E}^{\mathbf{c}}\right]=0 \\
\mathbf{a}_{y} \cdot\left[\mathbf{E}^{\mathbf{i}}+\mathbf{E}^{\mathbf{r}}-\mathbf{E}^{\mathbf{c}}\right]=0 \\
\mathbf{a}_{x} \cdot\left[\mathbf{H}^{\mathbf{i}}+\mathbf{H}^{\mathbf{r}}-\mathbf{H}^{\mathbf{c}}\right]=0 \\
\mathbf{a}_{y} \cdot\left[\mathbf{H}^{\mathbf{i}}+\mathbf{H}^{\mathbf{r}}-\mathbf{H}^{\mathbf{c}}\right]=0 \\
\theta^{r}=\theta^{i} \\
k_{0} \sin \theta^{i}=k_{R} \sin \theta_{R}  \tag{2.84}\\
k_{0} \sin \theta^{i}=k_{L} \sin \theta_{L}
\end{gather*}
$$

where $k_{R}=\frac{k}{1-\beta k}, k_{L}=\frac{k}{1+\beta k}, k=\omega \sqrt{\mu \epsilon}$.

$$
\left[\begin{array}{cccc}
0 & -A_{H} \cos \theta^{i} & \cos \theta_{R} & j \eta \cos \theta_{L}  \tag{2.85}\\
A_{E} & 0 & j & \eta \\
\frac{A_{E}}{\eta_{0}} \cos \theta^{i} & 0 & \frac{j}{\eta} \cos \theta_{R} & \cos \theta_{L} \\
0 & \frac{A_{H}}{\eta_{0}} & \frac{1}{\eta} & j
\end{array}\right]\left[\begin{array}{c}
\Gamma_{T E} \\
\Gamma_{T M} \\
\tau_{R} \\
\tau_{L}
\end{array}\right]=\left[\begin{array}{c}
A_{H} \cos \theta^{i} \\
-A_{E} \\
\frac{A_{E}}{\eta_{0}} \cos \theta^{i} \\
\frac{A_{H}}{\eta_{0}}
\end{array}\right]
$$

Now, we can find seven unknowns $\Gamma_{T E}, \Gamma_{T M}, \tau_{R}, \tau_{L}, \theta_{R}, \theta_{L}, \theta^{r}$.

### 2.4 Rectangular Waveguide

In this section we will study electromagnetic field behavior inside a rectangular waveguide Fig.(2.9) The solution of scaler wave equation in rectangular


Figure 2.9: Rectangular Waveguide
coordinate which wave is propagating in $z$ direction will be:

$$
\begin{equation*}
\psi=\left(A_{1} \cos k_{x} x+B_{1} \sin k_{x} x\right)\left(A_{2} \cos k_{y} y+B_{2} \sin k_{y} y\right) e^{-k_{z} z} \tag{2.86}
\end{equation*}
$$

where $k_{x}^{2}+k_{y}^{2}+k_{z}^{2}=k^{2}=\omega^{2} \mu \epsilon$

## \& $\mathrm{TE}^{\mathrm{z}}$ and $\mathrm{TM}^{\mathrm{z}}$ Modes

Let us take $\mathbf{M}=\nabla \times\left(\psi \mathbf{a}_{z}\right)$ and $\mathbf{N}=\frac{1}{k} \nabla \times \mathbf{M}$, therefore $\mathbf{E}=\mathbf{M}$ represents the $T E^{z}$ modes and $\mathbf{E}=\mathbf{N}$ give us $T M^{z}$ modes. For $T E^{z}$ i.e. $E_{z}=0$ modes

$$
\begin{equation*}
\mathbf{E}=\frac{\partial \psi}{\partial y} \mathbf{a}_{x}-\frac{\partial \psi}{\partial x} \mathbf{a}_{y} \quad \mathbf{H}=-\frac{1}{j \omega \mu} \nabla \times \mathbf{E} \tag{2.87}
\end{equation*}
$$

By imposing boundary conditions we find that $k_{x}=\frac{m \pi}{a}, k_{y}=\frac{n \pi}{b}$ and $k_{z}=$ $\sqrt{\omega^{2} \mu \epsilon-\left(\frac{m \pi}{a}\right)^{2}-\left(\frac{n \pi}{b}\right)^{2}}$ where $m=0,1,2, \cdots, n=0,1,2, \cdots$ and $m \neq n=$

$$
\begin{align*}
& \psi=A \cos \left(\frac{m \pi}{a} x\right) \cos \left(\frac{n \pi}{b} y\right) e^{-k_{z} z} \\
& \mathbf{E}=\frac{\partial \psi}{\partial y} \mathbf{a}_{x}-\frac{\partial \psi}{\partial x} \mathbf{a}_{y}  \tag{2.88}\\
& \mathbf{H}=-\frac{1}{j \omega \mu} \frac{\partial^{2} \psi}{\partial z \partial x} \mathbf{a}_{x}-\frac{1}{j \omega \mu} \frac{\partial^{2} \psi}{\partial z \partial y} \mathbf{a}_{y}-\frac{k^{2}-k_{z}^{2}}{j \omega \mu} \psi \mathbf{a}_{z}
\end{align*}
$$

For $T M^{z}$ i.e. $H_{z}=0$ modes

$$
\begin{align*}
& \psi=B \sin \left(\frac{m \pi}{a} x\right) \sin \left(\frac{n \pi}{b} y\right) e^{-k_{z} z} \quad m=1,2,3, \cdots \quad n=1,2, \cdots \\
& \mathbf{H}=\frac{\partial \psi}{\partial y} \mathbf{a}_{x}-\frac{\partial \psi}{\partial x} \mathbf{a}_{y} \\
& \mathbf{E}=\frac{1}{j \omega \epsilon} \frac{\partial^{2} \psi}{\partial z \partial x} \mathbf{a}_{x}+\frac{1}{j \omega \epsilon} \frac{\partial^{2} \psi}{\partial z \partial y} \mathbf{a}_{y}+\frac{k^{2}-k_{z}^{2}}{j \omega \epsilon} \psi \mathbf{a}_{z}  \tag{2.89}\\
& k_{z}=\sqrt{\omega^{2} \mu \epsilon-\left(\frac{m \pi}{a}\right)^{2}-\left(\frac{n \pi}{b}\right)^{2}}
\end{align*}
$$

### 2.5 Dielectric Slab Waveguide

Suppose we have three-layer slab waveguide with a one dimensional (1D) structure as shown in Fig.(2.10). In this analysis we consider the configuration and coordinate system depicted in Fig.(2.10). The scalar wave equation will be solved first. This is an eigenvalue problem. We assume some limitation to our problem. The waves propagate along $z$ axis will be assumed, i.e. $e^{-j k_{z} z}$. We have no variation along $y$ axis and dielectric slab waveguide has no limit along $y$ axis, therefore $\frac{\partial}{\partial y}=0$.

$$
\begin{equation*}
\frac{\partial^{2} \psi}{\partial x^{2}}+\frac{\partial^{2} \psi}{\partial z^{2}}=-k^{2} \psi \tag{2.90}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{\partial^{2} \psi}{\partial x^{2}}=\left(k_{z}^{2}-k^{2}\right) \psi \tag{2.91}
\end{equation*}
$$

The solution of this wave inside and outside slab will be

$$
\begin{equation*}
\psi_{1}=\left[A_{1} \cos \left(h_{1} x\right)+B_{1} \sin \left(h_{1} x\right)\right] e^{-j k_{z} z} \tag{2.92}
\end{equation*}
$$



Figure 2.10: Dielectric Slab Waveguide
where $h_{1}^{2}=k_{0}^{2} n_{1}^{2}-k_{z}^{2}, k_{0}^{2}-\omega^{2} \mu_{0} \epsilon_{0}$. And

$$
\begin{equation*}
\psi_{2}=A_{2} e^{(\alpha|x|-d)} e^{-j k_{z} z} \tag{2.93}
\end{equation*}
$$

where $h_{2}^{2}=k_{0}^{2} n_{2}^{2}-k_{z}^{2}, \alpha^{2}=-h_{2}^{2}$, why?
Now we are going to find the solution of vector wave equation inside and outside of slabe. Like previous method, we use $\mathbf{M}$ and $\mathbf{N}$ vectors. If $\mathbf{E}=$ $\mathbf{M}=\nabla \times\left(\psi \mathbf{a}_{\mathbf{z}}\right)$ we will have $T E^{z}$ modes, otherwise if $\mathbf{H}=\mathbf{M}=\nabla \times\left(\psi \mathbf{a}_{\mathbf{z}}\right)$ we will have $T M^{z}$ modes

### 2.5.1 TE Mode

According to our assumption, for the $T E^{z}$ mode we have

$$
\begin{array}{lll}
E_{z}=0, & E x=0, & H_{y}=0 \\
H_{z} \neq 0, & H_{x} \neq 0, & E_{y} \neq 0 \tag{2.94}
\end{array}
$$

For the $T E^{z}$ mode, we can write the $\mathbf{E}$ and $\mathbf{H}$ in terms of $\psi$ component.

$$
\begin{align*}
& \mathbf{E}=-\frac{\partial \psi}{\partial x} \mathbf{a}_{y} \\
& \mathbf{H}=-\frac{1}{j \omega \mu_{0}} \frac{\partial^{2} \psi}{\partial z \partial x} \mathbf{a}_{x}-\frac{k^{2}-k_{z}^{2}}{j \omega \mu_{0}} \psi \mathbf{a}_{z} \tag{2.95}
\end{align*}
$$

$E_{y}$ component inside dielectric slab:

$$
\begin{equation*}
E_{y 1}=\left[A_{1} h_{1} \sin \left(h_{1} x\right)+B_{1} h_{1} \cos \left(h_{1} x\right)\right] e^{-j k_{z} z} \tag{2.96}
\end{equation*}
$$

where $|x| \leq d, h_{1}^{2}=n_{1}^{2} k_{0}-k_{z}^{2}$ and $k_{0}^{2}=\omega \mu_{0} \epsilon_{0}$. The $\cos \left(h_{1} x\right)$ represents the even $T E^{z}$ modes and $\sin \left(h_{1} x\right)$ represents the odd $T E^{z}$ modes. Since the fields outside the slab (cladding) will be vanish as $x$ approaches infinity, we assume that the field will be evanescent as

$$
\begin{equation*}
E_{y 2}=A_{2} \alpha e^{-\alpha(|x|-d)} e^{-j k_{z} z} \quad \text { for } \quad|x| \geq d \tag{2.97}
\end{equation*}
$$

where $\alpha$ is the attenuation constant of evanescent field. If we substitute Eq.(2.93) in wave equation for cladding, i.e.

$$
\begin{equation*}
\frac{\partial^{2} \psi}{\partial x^{2}}+h_{2}^{2} \psi=0 \quad h_{2}^{2}=n_{2} k_{0}^{2}-k_{z}^{2} \tag{2.98}
\end{equation*}
$$

Then we will have a relation for $\alpha$

$$
\begin{equation*}
\alpha^{2}=k_{z}^{2}-n_{2} k_{0}^{2} \tag{2.99}
\end{equation*}
$$

Therefore the electromagnetic field components inside the slab for $T E^{z}$ even modes will be:

$$
\begin{align*}
& E_{y 1}=A \cos \left(h_{1} x\right) e^{-j k_{z} z} \\
& H_{x 1}=-A \frac{k_{z}}{\omega \mu_{0}} \cos \left(h_{1} x\right) e^{-j k_{z} z}  \tag{2.100}\\
& H_{z 1}=-A \frac{h_{1}}{\omega \mu_{0}} \sin \left(h_{1} x\right) e^{-j k_{z} z}
\end{align*}
$$

and the electromagnetic field components in cladding will be:

$$
\begin{align*}
& E_{y 2}=C e^{-\alpha(|x|-d)} e^{-j k_{z} z} \\
& H_{x 2}=C \frac{-k_{z}}{\omega \mu_{0}} e^{-\alpha(|x|-d)} e^{-j k_{z} z}  \tag{2.101}\\
& H_{z 2}=C \frac{j \alpha}{\omega \mu_{0}}\left(\frac{-x}{|x|}\right) e^{-\alpha(|x|-d)} e^{-j k_{z} z}
\end{align*}
$$

by using boundary conditions for electric and magnetic components we will reach the following characteristic equation for even $T E^{z}$ modes

$$
\begin{equation*}
\tan \left(h_{1} d\right)=\frac{\alpha}{h_{1}} \quad \text { for even } T E^{z} \text { modes } \tag{2.102}
\end{equation*}
$$

we do the same method for $T E^{z}$ odd modes, we will have

$$
\begin{equation*}
\tan \left(h_{1} d\right)=-\frac{h_{1}}{\alpha} \quad \text { for odd } T E^{z} \text { modes } \tag{2.103}
\end{equation*}
$$

where

$$
\begin{equation*}
h_{1}^{2}=n_{1}^{2} k_{0}-k_{z}^{2} \tag{2.104}
\end{equation*}
$$

by multiplying Eq.(2.99) and Eq.(2.104)by $d^{2}$ and adding

$$
\begin{equation*}
(\alpha d)^{2}+\left(h_{1} d\right)^{2}=\left(n_{1}^{2}-n_{2}^{2}\right)\left(k_{0} d\right)^{2} \tag{2.105}
\end{equation*}
$$

If we take $x=h_{1} d, y=\alpha d$ and $R^{2}=\left(n_{1}^{2}-n_{2}^{2}\right)\left(k_{0} d\right)^{2}$ then we will have to solve two nonlinear equations

$$
\begin{array}{ll}
x^{2}+y^{2}=R^{2} & \\
y=x \tan x & \text { for even } T E^{z} \text { modes }  \tag{2.106}\\
y=-x \cot x & \text { for odd } T E^{z} \text { modes }
\end{array}
$$

In this problem, we have two unknowns; $\alpha$ or $y=\alpha d$ and $k_{z}$ or $x=h_{1} d$ where $h_{1}=\sqrt{n_{1}^{2} k_{0}^{2}-k_{z}^{2}}$. If $k_{z}=0$ we can find the cutoff frequency of dielectric slab waveguide for even or odd modes.

### 2.5.2 TM Mode

For $T M^{z}$ type of modes the magnetic field component of $H_{z}=0$. According to previous assumption, we get

$$
\begin{array}{ll}
H_{z}=0, & H x=0, \tag{2.107}
\end{array} E_{y}=0, ~\left(E_{x} \neq 0, \quad H_{y} \neq 0\right.
$$

For the $T M^{z}$ mode, we can write the $\mathbf{E}$ and $\mathbf{H}$ in terms of $\psi$ component.

$$
\begin{align*}
& \mathbf{H}=-\frac{\partial \psi}{\partial x} \mathbf{a}_{y} \\
& \mathbf{E}=\frac{1}{j \omega \epsilon} \frac{\partial^{2} \psi}{\partial z \partial x} \mathbf{a}_{x}+\frac{k^{2}-k_{z}^{2}}{j \omega \epsilon} \psi \mathbf{a}_{z} \tag{2.108}
\end{align*}
$$

$H_{y}$ component inside dielectric slab:

$$
\begin{equation*}
H_{y 1}=\left[A_{1} h_{1} \sin \left(h_{1} x\right)+B_{1} h_{1} \cos \left(h_{1} x\right)\right] e^{-j k_{z} z} \tag{2.109}
\end{equation*}
$$

For the $T M^{z}$ mode, we can write the $E_{y}$ and $E_{x}$ in terms of $H_{y}$ component.

$$
\begin{align*}
& E_{x}=\frac{k_{z}}{\omega \epsilon} H_{y}  \tag{2.110}\\
& E_{z}=\frac{-j}{\omega \epsilon} \frac{\partial H_{y}}{\partial x}
\end{align*}
$$

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Therefore for inside the dielectric slab, $|x| \leq d$, electromagnetic field for even modes will be

$$
\begin{align*}
& H_{y 1}=A \cos \left(h_{1} x\right) e^{-j k_{z} z} \\
& E_{x 1}=A \frac{k_{z}}{\omega \epsilon_{1}} \cos \left(h_{1} x\right) e^{-j k_{z} z}  \tag{2.111}\\
& E_{z 1}=A \frac{j h_{1}}{\omega \epsilon_{1}} \sin \left(h_{1} x\right) e^{-j k_{z} z} \\
& h_{1}^{2}=n_{1}^{2} k_{0}^{2}-k_{z}^{2}, \quad \epsilon_{1}=n_{1}^{2} \epsilon_{0}
\end{align*}
$$

and for odd modes will be

$$
\begin{align*}
& H_{y 1}=B \sin \left(h_{1} x\right) e^{-j k_{z} z} \\
& E_{x 1}=B \frac{k_{z}}{\omega \epsilon_{1}} \sin \left(h_{1} x\right) e^{-j k_{z} z} \\
& E_{z 1}=B \frac{-j h_{1}}{\omega \epsilon_{1}} \cos \left(h_{1} x\right) e^{-j k_{z} z}  \tag{2.112}\\
& h_{1}^{2}=n_{1}^{2} k_{0}^{2}-k_{z}^{2}, \quad \epsilon_{1}=n_{1}^{2} \epsilon_{0}
\end{align*}
$$

and the electromagnetic field components in cladding, $|x| \geq d$, can be found:

$$
\begin{align*}
& H_{y 2}=C e^{-\alpha(|x|-d)} e^{-j k_{z} z} \\
& E_{x 2}=C \frac{k_{z}}{\omega \epsilon_{2}} e^{-\alpha(|x|-d)} e^{-j k_{z} z} \\
& E_{z 2}=C \frac{j \alpha}{\omega \epsilon_{2}}\left(\frac{x}{|x|}\right) e^{-\alpha(|x|-d)} e^{-j k_{z} z}  \tag{2.113}\\
& \alpha^{2}=k_{z}^{2}-n_{2}^{2} k_{0}^{2}
\end{align*}
$$

by using boundary conditions for electric and magnetic components we will reach the following characteristic equation for $T M^{z}$ modes

$$
\begin{equation*}
\tan \left(h_{1} d\right)=-\left(\frac{n_{2}}{n_{1}}\right)^{2}\left(\frac{h_{1}}{\alpha}\right) \tag{2.114}
\end{equation*}
$$

If we take $x=h_{1} d, y=\alpha d$ and $R^{2}=\left(n_{1}^{2}-n_{2}^{2}\right)\left(k_{0} d\right)^{2}$ then we will have to solve two nonlinear equations

$$
\begin{align*}
& x^{2}+y^{2}=R^{2} \\
& y=-\left(\frac{n_{2}}{n_{1}}\right)^{2} x \cot x \tag{2.115}
\end{align*}
$$

In this problem, we have two unknowns; $\alpha$ or $y=\alpha d$ and $k_{z}$ or $x=h_{1} d$ where $h_{1}=\sqrt{n_{1}^{2} k_{0}^{2}-k_{z}^{2}}$. If $k_{z}=0$ we can find the cutoff frequency of dielectric slab waveguide for even or odd modes.

### 2.6 Problems

- 1 Write a program that computes the reflection and transmission coefficient of a dielectric slab with $\epsilon_{r}=4$. Both sides of dielectric slab are air.
- 2 Use inhomogeneous formula in previous problem and certify your previous results (Problem 1).
- 3 One layer of inhomogeneous dielectric slab with linear index of refraction $n(z)=n_{0}+\frac{n_{s}-n_{0}}{L} z$ is given. Find transmission and reflection coefficient. First media is free space $n_{0}=1$ and substrate has $n_{s}=2$ and the thickness of inhomogeneous slab is $L=2 \mathrm{~cm}$. Plot normalized transmitted and reflected power with respect to frequency $(0.01 \mathrm{GHz}$ to $15 \mathrm{GHz})$.
- 4 Two layers of chiral slab with electrical parameters $\mu_{c}, \epsilon_{c}, \beta_{1}$ with thickness of $d_{1}$ and $\mu_{c}, \epsilon_{c}, \beta_{2}$ with thickness of $d_{2}$ in air are given. Find a relation between two electrical parameters and thickness of both chiral slabs that there will be no reflection if an normal incident plane wave on this two layer slab.
- 5 Show that Eq.(2.52) can be reduced to regular formulas of reflection and transmission case.
- 6 Find $T E^{x}, T E^{y}, T M^{x}$ and $T M^{y}$ modes of a rectangular waveguide. These modes are called Hybrid Modes.
- 7 We have a dielectric slab waveguide with $d=5$ millimeter and $n_{1}=2$ and $n_{2}=1$. Find the cutoff frequency for dominant mode.
- 8 Find the $T E^{z}$ and $T M^{z}$ modes of 2D dielectric slab waveguide of Fig.(2.11).


Figure 2.11: Dielectric Slab Waveguide

## Chapter 3

## Circular Cylinder

"The whole of science is nothing more than a refinement of everyday thinking."
Albert Einstein

### 3.1 Introduction

Circular waveguide, sectorial waveguide, scattering by circular conducting or dielectric cylinders, radiation of line source near or inside a cylinder are the problems considered in this chapter.

### 3.2 Solution of Helmholtz Equation in Cylindrical Coordinates

The scalar wave equation in general form in cylindrical coordinates, is

$$
\begin{equation*}
\nabla^{2} \Psi+k^{2} \Psi=0 \tag{3.1}
\end{equation*}
$$

where $k=\omega \sqrt{\mu \epsilon}$ and

$$
\begin{equation*}
\nabla^{2} \Psi=\frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Psi}{\partial \rho}\right)+\frac{1}{\rho^{2}} \frac{\partial^{2} \Psi}{\partial \phi^{2}}+\frac{\partial^{2} \Psi}{\partial z^{2}} \tag{3.2}
\end{equation*}
$$

To solve the above equation, we use the separation of variables method:

$$
\begin{equation*}
\Psi=R(\rho) F(\phi) Z(z) \tag{3.3}
\end{equation*}
$$

therefore we get three separate equations

$$
\begin{equation*}
\frac{\partial^{2} F}{\partial \phi^{2}}+\nu^{2} F=0 \tag{3.4}
\end{equation*}
$$

which has the solution:

$$
\begin{equation*}
F=c_{1} e^{-j \nu \phi}+c_{2} e^{+j \nu \phi} \tag{3.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial^{2} Z}{\partial z^{2}}+k_{z}^{2} Z=0 \tag{3.6}
\end{equation*}
$$

which has the solution:

$$
\begin{equation*}
Z(z)=c_{3} \cos \left(k_{z} z\right)+c_{4} \sin \left(k_{z} z\right) \tag{3.7}
\end{equation*}
$$

or

$$
\begin{equation*}
Z(z)=c_{3} e^{-j k_{z} z}+c_{4} e^{+j k_{z} z} \tag{3.8}
\end{equation*}
$$

and by substitution $k_{\rho}^{2}=k^{2}-k_{z}^{2}$ we have

$$
\begin{equation*}
\rho^{2} \frac{\partial^{2} R}{\partial \rho^{2}}+\rho \frac{\partial R}{\partial \rho}+\left(k_{\rho}^{2} \rho^{2}-\nu^{2}\right) R=0 \tag{3.9}
\end{equation*}
$$

which has the solutions:

$$
\begin{equation*}
R=c_{5} J_{\nu}\left(k_{\rho} \rho\right)+c_{6} Y_{\nu}\left(k_{\rho} \rho\right) \tag{3.10}
\end{equation*}
$$

or linear combination of general form of:

$$
\begin{equation*}
R=C_{7} Z_{\nu}^{(g)}\left(k_{\rho} \rho\right) \tag{3.11}
\end{equation*}
$$

where $C_{7}$ is a constant and $g=1,2,3,4$ represents types of the cylindrical Bessel's function:

$$
\begin{array}{lll}
Z_{\nu}^{(1)}\left(k_{\rho} \rho\right) & =J_{\nu}\left(k_{\rho} \rho\right) & \\
\text { Bessel function of first kind } \\
Z_{\nu}^{(2)}\left(k_{\rho} \rho\right) & =Y_{\nu}\left(k_{\rho} \rho\right) & \\
\text { Neumann function }  \tag{3.15}\\
Z_{\nu}^{(3)}\left(k_{\rho} \rho\right) & =H_{\nu}^{(1)}\left(k_{\rho} \rho\right) & \text { Hankel function of first kind } \\
Z_{\nu}^{(4)}\left(k_{\rho} \rho\right) & =H_{\nu}^{(2)}\left(k_{\rho} \rho\right) & \text { Hankel function of second kind }
\end{array}
$$

Each of these functions has special properties: $g=1$ and $g=2$ indicate standing wave while $g=3$ represents an inward traveling wave and $g=4$ an outward traveling wave. We should notice that

$$
\begin{align*}
& Z_{\nu}^{(1)}\left(k_{\rho} \rho\right)=J_{\nu}\left(k_{\rho} \rho\right)=\frac{H_{\nu}^{(1)}\left(k_{\rho} \rho\right)+H_{\nu}^{(2)}\left(k_{\rho} \rho\right)}{2}  \tag{3.16}\\
& Z_{\nu}^{(2)}\left(k_{\rho} \rho\right)=Y_{\nu}\left(k_{\rho} \rho\right)=\frac{H_{\nu}^{(1)}\left(k_{\rho} \rho\right)-H_{\nu}^{(2)}\left(k_{\rho} \rho\right)}{2 j} \tag{3.17}
\end{align*}
$$

### 3.2.1 Vector Wave Equation in Cylindrical Coordinate System

The solution of scaler wave function in cylindrical coordinate $\nabla^{2} \psi+k^{2} \psi=$ 0 , with assumption that $\partial / \partial z=0$ will be $\psi=Z_{n}^{g}(k \rho) e^{j n \phi}$ By selecting $\mathbf{M}=\nabla \times\left(\psi \mathbf{a}_{\mathbf{z}}\right)$ and therefore $\mathbf{N}=\frac{1}{k} \nabla \times \mathbf{M}, \mathbf{M}=\frac{1}{k} \nabla \times \mathbf{N}$, we will have

$$
\begin{align*}
\mathbf{M}_{n}^{g}(k \rho) & =M_{n \rho}^{g}(k \rho) \mathbf{a}_{\rho}+M_{n \phi}^{g}(k \rho) \mathbf{a}_{\phi}  \tag{3.18}\\
\mathbf{N}_{n}^{g}(k \rho) & =N_{n z}^{g}(k \rho) \mathbf{a}_{z}
\end{align*}
$$

$$
\begin{align*}
M_{\rho} & =\frac{1}{\rho} \frac{\partial \psi}{\partial \phi}  \tag{3.19}\\
M_{\phi} & =-\frac{\partial \psi}{\partial \rho} \\
N_{z} & =k \psi
\end{align*}
$$

The vector wave solution in cylindrical coordinate system will be

$$
\begin{align*}
\mathbf{E} & =E_{0} \sum_{n=-\infty}^{\infty}\left[a_{n} \mathbf{M}_{n}^{g}(k \rho)+b_{n} \mathbf{N}_{n}^{g}(k \rho)\right]  \tag{3.20}\\
\mathbf{H} & =\frac{j E_{0}}{\eta} \sum_{n=-\infty}^{\infty}\left[a_{n} \mathbf{N}_{n}^{g}(k \rho)+b_{n} \mathbf{M}_{n}^{g}(k \rho)\right] \tag{3.21}
\end{align*}
$$

which have $T E^{z}$ and $T M^{z}$ modes.


Figure 3.1: Circular waveguide

### 3.3 Circular Waveguides

The solution of scaler wave function in cylindrical coordinate $\nabla^{2} \psi+k^{2} \psi=0$ for this type of waveguide will be

$$
\begin{equation*}
\psi=J_{m}\left(k_{\rho} \rho\right)(A \cos m \phi+B \sin m \phi) e^{-j k_{z} z} \tag{3.22}
\end{equation*}
$$

where eigenvalues of them are related $k_{z}^{2}+k_{\rho}^{2}=k^{2}=\omega^{2} \mu \epsilon$. Let us first find $T M^{z}$ modes, $\mathbf{H}=\mathbf{M}=\nabla \times\left(\psi \mathbf{a}_{z}\right)$, therefore

$$
\begin{equation*}
\mathbf{H}=\frac{1}{\rho} \frac{\partial \psi}{\partial \phi} \mathbf{a}_{\rho}-\frac{\partial \psi}{\partial \rho} \mathbf{a}_{\phi} \tag{3.23}
\end{equation*}
$$

and we can find $\mathbf{E}=\frac{1}{j \omega \epsilon} \nabla \times \mathbf{H}$

$$
\begin{align*}
& \mathbf{N}=\frac{1}{k} \nabla \times \mathbf{M}=N_{\rho} \mathbf{a}_{\rho}+N_{\phi} \mathbf{a}_{\phi}+N_{z} \mathbf{a}_{z} \\
& N_{\rho}=\frac{1}{k} \frac{\partial^{2} \psi}{\partial z \partial \rho}, \quad N_{\phi}=\frac{1}{k \rho} \frac{\partial^{2} \psi}{\partial z \partial \phi}, \quad N_{z}=\frac{k^{2}-k_{z}^{2}}{k} \psi \tag{3.24}
\end{align*}
$$

Now we can apply boundary conditions, $E_{z}=0$ at $\rho=a$, then $J_{m}\left(k_{\rho} a\right)=0$ and the roots of Bessel function can be found in appendix ( $\mathbf{E}$ ), and here we call them $\alpha_{m n}$, which the propagation constant of waveguide can be calculated from: $k_{z}=\sqrt{\omega^{2} \mu \epsilon-\left(\frac{\alpha_{m n}}{a}\right)^{2}}$.

For $T E^{z}$ modes we can take $\mathbf{E}=\mathbf{M}=\nabla \times\left(\psi \mathbf{a}_{z}\right)$, and then find $\mathbf{H}=$ $\frac{-1}{j \omega \mu} \nabla \times \mathbf{E}$. By applying boundary condition; i.e. $E_{\phi}=0$ at $\rho=a$, we have $J_{m}^{\prime}\left(k_{\rho} a\right)=0$. If we name the roots of derivative of Bessel function as $\alpha_{m n}^{\prime}$, then the propagation constant can be determined by $k_{z}=\sqrt{\omega^{2} \mu \epsilon-\left(\frac{\alpha_{m n}^{\prime}}{a}\right)^{2}}$.

### 3.3.1 Circular Sectoral Waveguides

We have two types of circular sectoral waveguide. First let us find $T E^{z}$ and $T M^{z}$ modes for Fig.(3.2). The solution of scalar wave equation for this


Figure 3.2: Circular sectoral waveguide
waveguide can be:

$$
\begin{equation*}
\psi=J_{\nu}\left(k_{\rho} \rho\right)(A \cos \nu \phi+B \sin \nu \phi) e^{-j k_{z} z} \tag{3.25}
\end{equation*}
$$

The $k_{z}$ and $\nu$ can be determined from boundary conditions. Now we want to find the electric and magnetic components for $T M^{z}$ type of modes. Like previous case, $\mathbf{H}=\mathbf{M}=\nabla \times\left(\psi \mathbf{a}_{z}\right)$, therefore:

$$
\begin{equation*}
\mathbf{H}=\frac{1}{\rho} \frac{\partial \psi}{\partial \phi} \mathbf{a}_{\rho}-\frac{\partial \psi}{\partial \rho} \mathbf{a}_{\phi} \tag{3.26}
\end{equation*}
$$

and electric field can be found as $\mathbf{E}=\frac{1}{j \omega \epsilon} \nabla \times \mathbf{H}$

$$
\begin{align*}
& \mathbf{N}=\frac{1}{k} \nabla \times \mathbf{M}=N_{\rho} \mathbf{a}_{\rho}+N_{\phi} \mathbf{a}_{\phi}+N_{z} \mathbf{a}_{z} \\
& N_{\rho}=\frac{1}{k} \frac{\partial^{2} \psi}{\partial z \partial \rho}, \quad N_{\phi}=\frac{1}{k \rho} \frac{\partial^{2} \psi}{\partial z \partial \phi}, \quad N_{z}=\frac{k^{2}-k_{\ddot{z}}^{2}}{k} \psi \tag{3.27}
\end{align*}
$$

By applying boundary conditions, $E_{z}=0$ at $\rho=a, \phi=0$ and $\phi=\phi_{0}$, it will give us $\nu=\frac{m \pi}{\phi_{0}}$, where $m=1,2,3, \ldots$ and also $J_{\nu}\left(k_{\rho} a\right)=0$. If we name the roots of this type of Bessel function $\alpha_{m n}$, then $k_{z}=\sqrt{\omega^{2} \mu \epsilon-\left(\frac{\alpha_{m n}}{a}\right)^{2}}$.
For $T E^{z}$ type modes, let $\mathbf{E}=\mathbf{M}=\nabla \times\left(\psi \mathbf{a}_{z}\right)$, and then we can find $\mathbf{H}=\frac{-1}{j \omega \mu} \nabla \times \mathbf{E}$. By imposing boundary conditions; i.e. $E_{\phi}=0$ at $\rho=a$, $E_{\rho}=0$ at $\phi=0$ and $\phi=\phi_{0}$, it will give us $\nu=\frac{m \pi}{\phi_{0}}$ where $m=0,1,2, \ldots$ and also $J_{\nu}^{\prime}\left(k_{\rho} a\right)=0$. If we name the roots of derivative of Bessel function as $\alpha_{m n}^{\prime}$, then the propagation constant can be determined by $k_{z}=\sqrt{\omega^{2} \mu \epsilon-\left(\frac{\alpha_{m n}^{\prime}}{a}\right)^{2}}$. Another circular sectoral waveguide can be considered as depicted in Fig.(3.3). For this type we can write the solution of scalar wave equation as:


Figure 3.3: Circular sectoral waveguide

$$
\begin{equation*}
\psi=\left[A H_{\nu}^{(1)}\left(k_{\rho} \rho\right)+B H_{\nu}^{(2)}\left(k_{\rho} \rho\right)\right](C \cos \nu \phi+D \sin \nu \phi) e^{-j k_{z} z} \tag{3.28}
\end{equation*}
$$

With similar treatment as that of Fig.(3.2), for $T M^{z}$ modes, $\mathbf{H}=\mathbf{M}=$ $\nabla \times\left(\psi \mathbf{a}_{z}\right)$, therefore:

$$
\begin{equation*}
\mathbf{H}=\frac{1}{\rho} \frac{\partial \psi}{\partial \phi} \mathbf{a}_{\rho}-\frac{\partial \psi}{\partial \rho} \mathbf{a}_{\phi} \tag{3.29}
\end{equation*}
$$

and electric field can be found as $\mathbf{E}=\frac{1}{j \omega \epsilon} \nabla \times \mathbf{H}$

$$
\begin{align*}
& \mathbf{N}=\frac{1}{k} \nabla \times \mathbf{M}=N_{\rho} \mathbf{a}_{\rho}+N_{\phi} \mathbf{a}_{\phi}+N_{z} \mathbf{a}_{z} \\
& N_{\rho}=\frac{1}{k} \frac{\partial^{2} \psi}{z z \partial \rho}, \quad N_{\phi}=\frac{1}{k \rho} \frac{\partial^{2} \psi}{\partial z \partial \phi}, \quad N_{z}=\frac{k^{2}-k_{z}^{2}}{k} \psi \tag{3.30}
\end{align*}
$$

By applying boundary conditions, $E_{z}=0$ at $\rho=a, \rho=b \phi=0$ and $\phi=\phi_{0}$, we have $\nu=\frac{m \pi}{\phi_{0}}$, where $m=1,2,3, \ldots$ and

$$
\begin{equation*}
J_{\nu}\left(k_{\rho} a\right) Y_{\nu}\left(k_{\rho} b\right)-J_{\nu}\left(k_{\rho} b\right) Y_{\nu}\left(k_{\rho} a\right)=0 \tag{3.31}
\end{equation*}
$$

which is characteristic equation of this type of waveguide for $T M^{z}$ modes. By finding $k_{\rho}$ from equation Eq.(3.31) the propagation constant of waveguide can be calculated, $k_{z}=\sqrt{\omega^{2} \mu \epsilon-k_{\rho}^{2}}$.
For $T E^{z}$ modes we have $\nu=\frac{m \pi}{\phi_{0}}$, where $m=0,1,2, \ldots$ and characteristic equation will be

$$
\begin{equation*}
J_{\nu}^{\prime}\left(k_{\rho} a\right) Y_{\nu}^{\prime}\left(k_{\rho} b\right)-J_{\nu}^{\prime}\left(k_{\rho} b\right) Y_{\nu}^{\prime}\left(k_{\rho} a\right)=0 \tag{3.32}
\end{equation*}
$$

which give us $k_{\rho}$ and then $k_{z}=\sqrt{\omega^{2} \mu \epsilon-k_{\rho}^{2}}$.

### 3.4 Optical Fiber

An optical fiber consists of a core and a cladding Fig.(3.4). The refractive index of the core is taken slightly higher than that of the cladding in order that most of the energy of electromagnetic wave propagate inside the fiber optic. For analyzing the fields of fiber optic we start by Maxwell's equations and therefore vector wave equations. We first analyze the electromagnetic fields present in optical fibers to derive several important propagation characteristics. We assume that the fiber is lossless and the propagation is along the $z$ axis. Therefore $\mathbf{E}=\mathbf{E}_{0}(\rho, \phi) e^{-j k_{z} z}, \mathbf{H}=\mathbf{H}_{0}(\rho, \phi) e^{-j k_{z} z}$. We also assume the following fields for this type of dielectric circular cylindrical waveguide:

$$
\begin{align*}
& \mathbf{E}_{0}=E_{\rho} \mathbf{a}_{\rho}+E_{\phi} \mathbf{a}_{\phi}+E_{z} \mathbf{a}_{z} \\
& \mathbf{H}_{0}=H_{\rho} \mathbf{a}_{\rho}+H_{\phi} \mathbf{a}_{\phi}+H_{z} \mathbf{a}_{z} \tag{3.33}
\end{align*}
$$



Figure 3.4: Step-index optical fiber

Using Maxwell's equation in cylindrical coordinate, we get

$$
\begin{align*}
\frac{1}{\rho} \frac{\partial E_{z}}{\partial \phi}+j k_{z} E_{\phi} & =-j \omega \mu_{0} H_{\rho}  \tag{3.34}\\
-j k_{z} E_{\rho}-\frac{\partial E_{z}}{\partial \rho} & =-j \omega \mu_{0} H_{\phi} \\
\frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho E_{\phi}\right)-\frac{1}{\rho} \frac{\partial E_{\rho}}{\partial \phi} & =-j \omega \mu_{0} H_{z} \\
\frac{1}{\rho} \frac{\partial H_{z}}{\partial \phi}+j k_{z} H_{\phi} & =j \omega \epsilon E_{\rho} \\
-j k_{z} H_{\rho}-\frac{\partial H_{z}}{\partial \rho} & =-j \omega \epsilon E_{\phi} \\
\frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho H_{\phi}\right)-\frac{1}{\rho} \frac{\partial H_{\rho}}{\partial \phi} & =-j \omega \epsilon E_{z}
\end{align*}
$$

from these six Eq.(3.34) we can find $E_{\rho}, E_{\phi}, H_{\rho}$ and $H_{\phi}$ in terms of $E_{z}, H_{z}$. Therefore:

$$
\begin{align*}
E_{\rho} & =\frac{-j}{h^{2}}\left[k_{z} \frac{\partial E_{z}}{\partial \rho}+\omega \mu_{0} \frac{1}{\rho} \frac{\partial H_{z}}{\partial \phi}\right]  \tag{3.35}\\
E_{\phi} & =\frac{-j}{h^{2}}\left[k_{z} \frac{1}{\rho} \frac{\partial E_{z}}{\partial \phi}-\omega \mu_{0} \frac{\partial H_{z}}{\partial \rho}\right]  \tag{3.36}\\
H_{\rho} & =\frac{-j}{h^{2}}\left[k_{z} \frac{\partial H_{z}}{\partial \rho}-\omega \epsilon \frac{1}{\rho} \frac{\partial E_{z}}{\partial \phi}\right]  \tag{3.37}\\
H_{\phi} & =\frac{-j}{h^{2}}\left[k_{z} \frac{1}{\rho} \frac{\partial H_{z}}{\partial \phi}+\omega \epsilon \frac{\partial E_{z}}{\partial \rho}\right] \tag{3.38}
\end{align*}
$$

where $h^{2}=\omega^{2} \mu \epsilon-k_{z}^{2}$. Wave equation in terms of longitudinal components of $E_{z}$ or $H_{z}$ field will be:

$$
\begin{align*}
\frac{\partial^{2} E_{z}}{\partial \rho^{2}}+\frac{1}{\rho} \frac{\partial E_{z}}{\partial \rho}+\frac{1}{\rho^{2}} \frac{\partial^{2} E_{z}}{\partial \phi^{2}}+h^{2} E_{z} & =0  \tag{3.39}\\
\frac{\partial^{2} H_{z}}{\partial \rho^{2}}+\frac{1}{\rho} \frac{\partial H_{z}}{\partial \rho}+\frac{1}{\rho^{2}} \frac{\partial^{2} H_{z}}{\partial \phi^{2}}+h^{2} H_{z} & =0
\end{align*}
$$

The method of modal analysis seeks to represent the fields as the superposition of several special types of fields. There are several types of modes. First, we must reject $T E M^{z}$ (transverse electric and magnetic) modes, in which both $E_{z}=0$ and $H_{z}=0$, since no (nonzero) $T E M^{z}$ modes can lead to real power flow in this situation. $T E^{z}$ (transverse electric) modes have $E_{z}=0$, while $T M^{z}$ (transverse magnetic) modes have $H_{z}=0$. Although $T E^{z}$ or $T M^{z}$ modes lead to a somewhat simplified analysis, there are several important fields which cannot be expressed as the superposition of such modes. Therefore, we must also consider more general hybrid modes; $H E$ modes have $\left|H_{z}\right| \gg\left|E_{z}\right|>0$ and $E H$ modes have $\left|E_{z}\right| \gg\left|H_{z}\right|>0$.
The geometry of the situation leads us to seek functions periodic in $\phi$. Therefore, we assume the angle-dependence has the form $e^{j n \phi}$ where $n$ is an integer (possibly positive, negative or 0 ). Thus:

$$
\begin{equation*}
E_{z}=A F_{n}(\rho) e^{j n \phi} e^{j k_{z} z} \tag{3.40}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\frac{\partial^{2} F_{n}}{\partial \rho^{2}}+\frac{1}{\rho} \frac{\partial F_{n}}{\partial \rho}+\left(h^{2}-\frac{n^{2}}{\rho^{2}}\right) F_{n}=0 \tag{3.41}
\end{equation*}
$$

Now let us distinguish between the core $\rho \leq a$, characterized by $\epsilon_{1}, \mu_{0}$, and the cladding $\rho \geq a$ characterized by $\epsilon_{2}, \mu_{0}$. Define:

$$
\begin{align*}
k_{1} & =\omega \sqrt{\mu_{0} \epsilon_{1}}=\frac{2 \pi}{\lambda} n_{1}  \tag{3.42}\\
k_{2} & =\omega \sqrt{\mu_{0} \epsilon_{2}}=\frac{2 \pi}{\lambda} n_{2}  \tag{3.43}\\
u & =\sqrt{k_{1}^{2}-k_{z}^{2}}  \tag{3.44}\\
w & =\sqrt{k_{z}^{2}-k_{2}^{2}} \tag{3.45}
\end{align*}
$$

where $n_{1}$ and $n_{2}$ are index of refractions. Imposing that the core function stays finite at $\rho=0$, the cladding function decay to zero at $\rho \rightarrow \infty$, and that $k_{z}>0$ for real power flow, we have that $u>0$ and $w>0$. Thus, in particular, $k_{1}>k_{z}>k_{2}$. In paractice $\epsilon_{1}>\epsilon_{2}$, which is the case of real fibers. The longitudinal components of the electric and magnetic fields in the core and cladding are given by:

$$
\begin{align*}
& E_{z}=\left\{\begin{array}{lll}
A J_{n}(u \rho) e^{j n \phi} e^{-j k_{z} z} & \text { for } & \rho \leq a \\
B K_{n}(w \rho) e^{j n \phi} e^{-j k_{z} z} & \text { for } & \rho \geq a
\end{array}\right.  \tag{3.46}\\
& H_{z}=\left\{\begin{array}{lll}
C J_{n}(u \rho) e^{j n \phi} e^{-j k_{z} z} & \text { for } & \rho \leq a \\
D K_{n}(w \rho) e^{j n \phi} e^{-j k_{z} z} & \text { for } & \rho \geq a
\end{array}\right. \tag{3.47}
\end{align*}
$$

where $J_{n}(u \rho)$ and $K_{n}(w \rho)$ are Bessel's functions and modified Bessel's function. In particular, we impose the same value $n$ characterize the fields in the core and cladding. This is necessary to achieve phase match conditions at $\rho=a$; for example, $E_{z}$ must be continuous at $\rho=a$. Also, $K_{n}$ is the $n^{\text {th }}$ order Bessel function of the second kind. It can be shown that $K_{n}(w \rho) \sim e^{-w \rho}$ (asymptotically) as $\rho \rightarrow \infty$. In fact, $\left|J_{n}(j x)\right|=K_{n}(x)$ for real $x$.
Now we can summarize the fields in core and cladding.

1) In the core $(\rho \leq a)$ :

$$
\begin{align*}
& E_{z}=A J_{n}(u \rho) e^{j n \phi} e^{-j k_{z} z} \\
& E_{\rho}=\left[-A \frac{j k_{z}}{u} J_{n}^{\prime}(u \rho)+C \frac{j \omega \mu_{0}}{u^{2}} \frac{n}{\rho} J_{n}(u \rho)\right] e^{j n \phi} e^{-j k_{z} z}  \tag{3.48}\\
& E_{\phi}=\left[-A \frac{j k_{z}}{u^{2}} \frac{n}{\rho} J_{n}(u \rho)+C \frac{j \omega \mu_{0}}{u} J_{n}^{\prime}(u \rho)\right] e^{j n \phi} e^{-j k_{z} z}
\end{align*}
$$

$$
\begin{align*}
& H_{z}=C J_{n}(u \rho) e^{j n \phi} e^{-j k_{z} z} \\
& H_{\rho}=\left[A \frac{j \omega \epsilon_{1}}{u^{2}} \frac{n}{\rho} J_{n}(u \rho)-C \frac{j k_{z}}{u} J_{n}^{\prime}(u \rho)\right] e^{j n \phi} e^{-j k_{z} z}  \tag{3.49}\\
& H_{\phi}=\left[-A \frac{j \omega \epsilon_{1}}{u} J_{n}^{\prime}(u \rho)+C \frac{j k_{z}}{u^{2}} \frac{n}{\rho} J_{n}(u \rho)\right] e^{j n \phi} e^{-j k_{z} z}
\end{align*}
$$

$2)$ In the cladding $(\rho \geq a)$ :

$$
\begin{align*}
& E_{z}=B K_{n}(w \rho) e^{j n \phi} e^{-j k_{z} z} \\
& E_{\rho}=\left[B \frac{j k_{z}}{w} K_{n}^{\prime}(w \rho)-D \frac{j \omega \mu_{0}}{w^{2}} \frac{n}{\rho} K_{n}(w \rho)\right] e^{j n \phi} e^{-j k_{z} z}  \tag{3.50}\\
& E_{\phi}=\left[B \frac{j k_{z}}{w^{2}} \frac{n}{\rho} K_{n}(w \rho)-D \frac{j \omega \mu_{0}}{w} K_{n}^{\prime}(w \rho)\right] e^{j n \phi} e^{-j k_{z} z} \\
& H_{z}=D K_{n}(u \rho) e^{j n \phi} e^{-j k_{z} z} \\
& H_{\rho}=\left[-B \frac{j \omega \epsilon_{2}}{w^{2}} \frac{n}{\rho} K_{n}(w \rho)+D \frac{j k_{z}}{w} K_{n}^{\prime}(w \rho)\right] e^{j n \phi} e^{-j k_{z} z}  \tag{3.51}\\
& H_{\phi}=\left[B \frac{j \omega \epsilon_{2}}{w} K_{n}^{\prime}(w \rho)-D \frac{j k_{z}}{w^{2}} \frac{n}{\rho} K_{n}(w \rho)\right] e^{j n \phi} e^{-j k_{z} z}
\end{align*}
$$

We must impose the appropriate boundary conditions at $\rho=a$.

$$
\begin{equation*}
E_{z 1}=E_{z 2}, \quad E_{\phi 1}=E_{\phi 2} \tag{3.52}
\end{equation*}
$$

The first boundary condition leads to:

$$
\begin{equation*}
A J_{n}(u a)-B K_{n}(w a)=0 \tag{3.53}
\end{equation*}
$$

and the second to:

$$
\begin{equation*}
-A \frac{j k_{z}}{u^{2}} \frac{n}{a} J_{n}(u a)+C \frac{j \omega \mu_{0}}{u} J_{n}^{\prime}(u a)=B \frac{j k_{z}}{w^{2}} \frac{n}{a} K_{n}(w a)-D \frac{j \omega \mu_{0}}{w} K_{n}^{\prime}(w a) \tag{3.54}
\end{equation*}
$$

We must also impose:

$$
\begin{equation*}
H_{z 1}=H_{z 2}, \quad H_{\phi 1}=H_{\phi 2} \tag{3.55}
\end{equation*}
$$

We have a total of four homogeneous equations in $A, B, C$ and $D$.

$$
[\mathbf{M}]\left[\begin{array}{c}
A  \tag{3.56}\\
B \\
C \\
D
\end{array}\right]=\left[\begin{array}{l}
0 \\
0 \\
0 \\
0
\end{array}\right]
$$

Since we seek nonzero fields, we cannot have all four constants $A, B, C$ and $D$ equal to zero; hence:

$$
\begin{equation*}
\operatorname{det}[\mathbf{M}]=0 \tag{3.57}
\end{equation*}
$$

This condition must be met for the field to be exist. Define:

$$
\begin{equation*}
\left[\frac{J_{n}^{\prime}(u a)}{u J_{n}(u a)}+\frac{K_{n}^{\prime}(w a)}{w K_{n}(w a)}\right]\left[k_{1}^{2} \frac{J_{n}^{\prime}(u a)}{u J_{n}(u a)}+k_{2}^{2} \frac{K_{n}^{\prime}(w a)}{w K_{n}(w a)}\right]=\left(\frac{k_{z} n}{a}\right)^{2}\left(\frac{1}{u^{2}}+\frac{1}{w^{2}}\right)^{2} \tag{3.58}
\end{equation*}
$$

Given $k$, that is $\omega$ or $\lambda$, the parameter $u, w$ are known functions of $K_{z}$. Hence, Eq.(3.58) is specified as the function of frequency $\omega$ and parameter $n$. This equation has only discrete solutions, and in general for each $n$ there will be several roots, denoted as: $k_{z 1}^{(n)}, k_{z 2}^{(n)}, k_{z 3}^{(n)}, \cdots, k_{z M}^{(n)}$. The corresponding modes are denoted as $T E_{n m}, T M_{n m}, E H_{n m}$ or $H E_{n m}$ as appropriate.
Let us examine the $T E$ and $T M$ cases in particular. We obtain $T E$ by setting $A=C=0$, and we seek nonzero $B$ and $D$; and similarly $T M$ is obtained by setting $B=D=0$ and we seek nonzero $A$ and $B$. In each case, this requires a $2 \times 2$ sub-matrix of $M$ to have nonzero determinant, and in particular can be shown that require $n=0$. Thus, there can be no $\phi$ variation (there is radial symmetry) for $T E$ and $T M$ modes. The equation determining $k_{z}$ for $T E_{0 m}$ is:

$$
\begin{equation*}
\frac{J_{1}(u a)}{u J_{0}(u a)}+\frac{K_{1}(w a)}{w K_{0}(w a)}=0 \tag{3.59}
\end{equation*}
$$

Similarly, the equation determining $k_{z}$ for $T M_{0 m}$ is:

$$
\begin{equation*}
k_{1}^{2} \frac{J_{1}(u a)}{u J_{0}(u a)}+k_{2}^{2} \frac{K_{1}(w a)}{w K_{0}(w a)}=0 \tag{3.60}
\end{equation*}
$$

If $n \neq 0$, we do not have $T E$ or $T M$ modes, and the analysis becomes very complex. However, if $n_{1} \approx n_{2}$ or $\left(n_{1}-n_{2} \ll 1\right)$, we can apply an important class of approximations which lead to weakly guided waves.
The cutoff conditions $\left(w^{2} \rightarrow 0\right)$ for lower order modes are summarized in

Table 3.1: Cutoff Conditions for Lower Order Modes of Optical Fiber

| $n$ | Mode | Cutoff Condition |
| :---: | :---: | :---: |
| 0 | $T E_{0 m}, T M_{0 m}$ | $J_{0}(u a)=0$ |
| 1 | $H E_{0 m}, E H_{0 m}$ | $J_{1}(u a)=0$ |
| $\geq 2$ | $E H_{n m}$ | $J_{n}(u a)=0$ |
| $\geq 2$ | $H E_{n m}$ | $\left(\frac{\epsilon_{1}}{\epsilon_{2}}+1\right) J_{n-1}(u a)=\frac{u a}{n-1} J_{n}(u a)$ |

Table(3.1). We now discuss the $V$-number, also called the V -parameter or the normalized frequency. The value $V$ is defined as:

$$
\begin{equation*}
V^{2}=\left(u^{2}+w^{2}\right) a^{2}=\left(\frac{2 \pi a}{\lambda_{0}}\right)^{2}\left(n_{1}^{2}-n_{2}^{2}\right) \tag{3.61}
\end{equation*}
$$

$V$ is dimensionless. Note that the value $\left(\frac{2 \pi a}{\lambda_{0}}\right)$ is proportional to frequency (up to a factor equal to the speed of light), and hence $V$ is called the normalized frequency.
The value $V$ is related to the number of modes that a fiber can support. Also the normalized propagation constant, b can be defined as:

$$
\begin{equation*}
b=\frac{a^{2} w^{2}}{V^{2}}=\frac{\left(k_{z} / k\right)^{2}-n_{2}^{2}}{n_{1}^{2}-n_{2}^{2}}=\frac{n_{e f f}^{2}-n_{2}^{2}}{n_{1}^{2}-n_{2}^{2}} \tag{3.62}
\end{equation*}
$$

Note that $0<b<1$ corresponds to wave propagation.
A graph of $b$ or $k_{z} / k$ versus V shows that for fixed $V$, only several modes are possible. In particular, the $H E_{11}$ mode exists (corresponds to a value $b$ in the range 0 to 1 ) for all $V$, down to $V=0$. No other mode exists until $V=2.405$ (this is the smallest root of $J_{0}(x)$ ). Hence, below this value of $V$, all modes other than $H E_{11}$ are in cutoff. For this reason, $H E_{11}$ is called the dominant mode.
Two modes with the same value for $k_{z}$ are said to be degenerate. We associate degenerate modes together since they have identical propagation characteristics, although different field distributions. In other words, we consider all linear combinations of a class of degenerate modes to be a mode uncoupled to itself. We list the primary lower order modes of optical fiber according to
their degeneracies:

$$
\begin{aligned}
& H E_{11} \\
& T E_{01}, T M_{01}, H E_{21} \\
& H E_{31}, E H_{11} \\
& H E_{12} \\
& H E_{41}, E H_{21} \\
& T E_{02}, T M_{02}, H E_{22}
\end{aligned}
$$

These pairs of degenerate modes are called $L P$ (linearly polarized) modes of optical fiber, since they can be combined to yield fixed orientation. That is, in a complete set of modes, only one $E$ and one $H$ component are significant, say the $\mathbf{E}$ polarized along one axis and $\mathbf{H}$ perpendicular to it.

### 3.5 Infinite Electric Line Source

Let the electric line source Fig.(3.5) with $I_{0}$ be located at ( $\rho_{0}, \phi_{0}$ ), therefore the electric field at point $(\rho, \phi)$ will be:


Figure 3.5: Line source

$$
\begin{equation*}
E_{z}=-I_{0} \frac{\omega \mu}{4} H_{0}^{(2)}\left(k\left|\rho-\rho_{0}\right|\right) \tag{3.63}
\end{equation*}
$$

where $R=\left|\rho-\rho_{0}\right|=\sqrt{\rho^{2}+\rho_{0}^{2}-2 \rho \rho_{0} \cos \left(\phi-\phi_{0}\right)}$. The Hankel function of Eq.(3.63) can be written as:

$$
\begin{align*}
H_{0}^{(2)}\left(k\left|\rho-\rho_{\mathbf{0}}\right|\right) & =\sum_{n=-\infty}^{+\infty} J_{n}(k \rho) H_{n}^{(2)}\left(k \rho_{0}\right) e^{j n\left(\phi-\phi_{0}\right)} \tag{3.64}
\end{align*} \quad \rho \leq \rho_{0}, ~=\sum_{n=-\infty}^{+\infty} J_{n}\left(k \rho_{0}\right) H_{n}^{(2)}(k \rho) e^{j n\left(\phi-\phi_{0}\right)} \quad \rho \geq \rho_{0} .
$$

We can write $\mathbf{E}$ and $\mathbf{H}$ fields of line source in terms of $\mathbf{M}$ and $\mathbf{N}$ in cylindrical coordinates, $\frac{\partial}{\partial z}=0$.

$$
\begin{gather*}
\psi=Z_{n}^{(g)}(k \rho) e^{j n \phi} \\
\mathbf{M}=\nabla \times\left(\psi \mathbf{a}_{z}\right)=\frac{1}{\rho} \frac{\partial \psi}{\partial \phi} \mathbf{a}_{\rho}-\frac{\partial \psi}{\partial \rho} \mathbf{a}_{\phi}  \tag{3.66}\\
\mathbf{N}=\frac{1}{k} \nabla \times \mathbf{M}=k \psi \mathbf{a}_{z} \\
\mathbf{E}= \begin{cases}\sum_{n=-\infty}^{\infty} a_{n}^{(4)} \mathbf{N}_{n}^{(1)}(k \rho) & \rho \leq \rho_{0} \\
\sum_{n=-\infty}^{\infty} a_{n}^{(1)} \mathbf{N}_{n}^{(4)}(k \rho) & \rho \geq \rho_{0}\end{cases}  \tag{3.67}\\
\mathbf{H}=\frac{j}{\eta} \begin{cases}\sum_{n=-\infty}^{\infty} a_{n}^{(4)} \mathbf{M}_{n}^{(1)}(k \rho) & \rho \leq \rho_{0} \\
\sum_{n=-\infty}^{\infty} a_{n}^{(1)} \mathbf{M}_{n}^{(4)}(k \rho) & \rho \geq \rho_{0}\end{cases} \tag{3.68}
\end{gather*}
$$

where $a_{n}^{(4)}=-I_{0} \frac{\omega \mu}{4 k} H_{n}^{(2)}\left(k \rho_{0}\right) e^{-j n \phi_{0}}$ for $\rho \leq \rho_{0}$ and $a_{n}^{(1)}=-I_{0} \frac{\omega \mu}{4 k} J_{n}\left(k \rho_{0}\right) e^{-j n \phi_{0}}$ for $\rho \geq \rho_{0}$.

### 3.5.1 Line Source Near a Dielectric Circular Cylinder

Consider a circular dielectric cylinder, parallel to a line source (an incident plane wave is a particular case of such a source). The problem is two-dimensional. Let the line source be at $\rho_{0}, \phi_{0}$ and the dielectric circular cylinder have radius $a$. Now we want to find electromagnetic field inside and outside the cylinder. Outside the cylinder is medium 1 and inside the cylinder is assumed media 2. According to Fig.(3.6), it can be written:


Figure 3.6: Line source near a dielectric cylinder

$$
\begin{align*}
& \mathbf{E}^{i}=\sum_{n=-\infty}^{\infty} a_{n}^{(4)} \mathbf{N}_{n}^{(1)}\left(k_{1} \rho\right) \quad \rho \leq \rho_{0} \\
& \mathbf{H}^{i}=\frac{j}{\eta_{1}} \sum_{n=-\infty}^{\infty} a_{n}^{(4)} \mathbf{M}_{n}^{(1)}\left(k_{1} \rho\right) \tag{3.69}
\end{align*}
$$

where $a_{n}^{(4)}=-I_{0} \frac{\omega \mu_{1}}{4 k_{1}} H_{n}^{(2)}\left(k_{1} \rho_{0}\right) e^{-j n \phi_{0}}$ for $\rho \leq \rho_{0}$. The scattered fields will be

$$
\begin{align*}
& \mathbf{E}^{s}=\sum_{n=-\infty}^{\infty} a_{n}^{s} \mathbf{N}_{n}^{(4)}\left(k_{1} \rho\right) \\
& \mathbf{H}^{s}=\frac{j}{\eta_{1}} \sum_{n=-\infty}^{\infty} a_{n}^{s} \mathbf{M}_{n}^{(4)}\left(k_{1} \rho\right) \tag{3.70}
\end{align*}
$$

and the fields inside the dielectric will be:

$$
\begin{align*}
\mathbf{E}^{t} & =\sum_{n=-\infty}^{\infty} a_{n}^{t} \mathbf{N}_{n}^{(1)}\left(k_{2} \rho\right) \\
\mathbf{H}^{t} & =\frac{j}{\eta_{2}} \sum_{n=-\infty}^{\infty} a_{n}^{t} \mathbf{M}_{n}^{(1)}\left(k_{2} \rho\right) \tag{3.71}
\end{align*}
$$

Now we can apply boundary conditions for tangential electric and magnetic fields at $\rho=a$.

$$
\mathbf{a}_{z} \cdot\left(\mathbf{E}^{i}+\mathbf{E}^{s}-\mathbf{E}^{t}\right)=0
$$

$$
\begin{gather*}
\mathbf{a}_{\phi} \cdot\left(\mathbf{H}^{i}+\mathbf{H}^{s}-\mathbf{H}^{t}\right)=0 \\
\sum_{n=-\infty}^{\infty} a_{n}^{(4)} k_{1} J_{n}\left(k_{1} a\right) e^{j n \phi}+\sum_{n=-\infty}^{\infty} a_{n}^{s} k_{1} H_{n}^{(2)}\left(k_{1} a\right) e^{j n \phi}=\sum_{n=-\infty}^{\infty} a_{n}^{t} k_{2} J_{n}\left(k_{2} a\right) e^{j n \phi} \tag{3.72}
\end{gather*}
$$

we multiply both side of Eq.(3.72) by $e^{-j m \phi}$ and integrating from 0 to $2 \pi$, it yields:

$$
\begin{equation*}
a_{n}^{(4)} k_{1} J_{n}\left(k_{1} a\right)+a_{n}^{s} k_{1} H_{n}^{(2)}\left(k_{1} a\right)=a_{n}^{t} k_{2} J_{n}\left(k_{2} a\right) \tag{3.73}
\end{equation*}
$$

by the same procedure for tangential magnetic field:

$$
\begin{equation*}
\frac{k_{1}}{\eta_{1}} a_{n}^{(4)} J_{n}^{\prime}\left(k_{1} a\right)+\frac{k_{1}}{\eta_{1}} a_{n}^{s} H_{n}^{\prime(2)}\left(k_{1} a\right)=\frac{k_{2}}{\eta_{2}} a_{n}^{t} J_{n}^{\prime}\left(k_{2} a\right) \tag{3.74}
\end{equation*}
$$

From Eq.(3.73) and Eq.(3.74) we can determine two unknowns $a_{n}^{s}$ and $a_{n}^{t}$ :

$$
\left[\begin{array}{cc}
-k_{1} H_{n}^{(2)}\left(k_{1} a\right) & k_{2} J_{n}\left(k_{2} a\right)  \tag{3.75}\\
-\epsilon_{1} H_{n}^{\prime(2)}\left(k_{1} a\right) & \epsilon_{2} J_{n}^{\prime}\left(k_{2} a\right)
\end{array}\right]\left[\begin{array}{c}
a_{n}^{s} \\
a_{n}^{t}
\end{array}\right]=\left[\begin{array}{c}
a_{n}^{(4)} k_{1} J_{n}\left(k_{1} a\right) \\
\epsilon_{1} a_{n}^{(4)} J_{n}^{\prime}\left(k_{1} a\right)
\end{array}\right]
$$

### 3.5.2 Line Source Inside a Dielectric Circular Cylinder

Fig.(3.7) shows the location of a electric line source inside a dielectric cylinder. Incident or excitation field will be:

$$
\begin{align*}
\mathbf{E}^{i} & =\sum_{n=-\infty}^{\infty} a_{n}^{(1)} \mathbf{N}_{n}^{(4)}\left(k_{2} \rho\right) \quad \rho \geq \rho_{0} \\
\mathbf{H}^{i} & =\frac{j}{\eta_{2}} \sum_{n=-\infty}^{\infty} a_{n}^{(1)} \mathbf{M}_{n}^{(4)}\left(k_{2} \rho\right) \tag{3.76}
\end{align*}
$$

where $a_{n}^{(1)}=-I_{0} \frac{\omega \mu_{2}}{4 k_{2}} J_{n}\left(k_{2} \rho_{0}\right) e^{-j n \phi_{0}}$ for $\rho \geq \rho_{0}$. The reflected or scattered field from the boundary:

$$
\begin{align*}
\mathbf{E}^{s} & =\sum_{n=-\infty}^{\infty} a_{n}^{s} \mathbf{N}_{n}^{(1)}\left(k_{2} \rho\right) \\
\mathbf{H}^{s} & =\frac{j}{\eta_{2}} \sum_{n=-\infty}^{\infty} a_{n}^{s} \mathbf{M}_{n}^{(1)}\left(k_{2} \rho\right) \tag{3.77}
\end{align*}
$$



Figure 3.7: Line source inside a dielectric cylinder
and the transmitted fields outward the cylinder will be

$$
\begin{align*}
\mathbf{E}^{t} & =\sum_{n=-\infty}^{\infty} a_{n}^{t} \mathbf{N}_{n}^{(4)}\left(k_{1} \rho\right)  \tag{3.78}\\
\mathbf{H}^{t} & =\frac{j}{\eta_{1}} \sum_{n=-\infty}^{\infty} a_{n}^{t} \mathbf{M}_{n}^{(4)}\left(k_{1} \rho\right)
\end{align*}
$$

By applying boundary conditions at $\rho=a$,

$$
\begin{gathered}
\mathbf{a}_{z} \cdot\left(\mathbf{E}^{i}+\mathbf{E}^{s}-\mathbf{E}^{t}\right)=0 \\
\mathbf{a}_{\phi} \cdot\left(\mathbf{H}^{i}+\mathbf{H}^{s}-\mathbf{H}^{t}\right)=0
\end{gathered}
$$

therefore:

$$
\begin{align*}
& a_{n}^{(1)} k_{2} H_{n}^{(2)}\left(k_{2} a\right)+a_{n}^{s} k_{2} J_{n}\left(k_{2} a\right)=a_{n}^{t} k_{1} H_{n}^{(2)}\left(k_{1} a\right) \\
& \frac{k_{2}}{\eta_{2}} a_{n}^{(1)} H_{n}^{\prime(2)}\left(k_{2} a\right)+\frac{k_{2}}{\eta_{2}} a_{n}^{s} J_{n}^{\prime}\left(k_{2} a\right)=\frac{k_{1}}{\eta_{1}} a_{n}^{t} H_{n}^{\prime(2)}\left(k_{1} a\right) \tag{3.79}
\end{align*}
$$

The unknowns can be determined from the above two equations:

$$
\left[\begin{array}{cc}
-k_{2} J_{n}^{(2)}\left(k_{2} a\right) & k_{1} H_{n}^{(2)}\left(k_{1} a\right)  \tag{3.80}\\
-\epsilon_{2} J_{n}^{\prime}\left(k_{2} a\right) & \epsilon_{1} H_{n}^{\prime(2)}\left(k_{1} a\right)
\end{array}\right]\left[\begin{array}{c}
a_{n}^{s} \\
a_{n}^{t}
\end{array}\right]=\left[\begin{array}{c}
a_{n}^{(1)} k_{2} H_{n}^{(2)}\left(k_{2} a\right) \\
\epsilon_{2} a_{n}^{(1)} H_{n}^{\prime(2)}\left(k_{2} a\right)
\end{array}\right]
$$

### 3.5.3 Line Source at the Center of a Dielectric Circular Cylinder

This problem is a special case of the configuration Fig.(3.7). We do not need summation [why ?]. The excitation field will be:

$$
\begin{array}{ll}
E_{z}^{i}=a^{i} H_{0}^{(2)}\left(k_{2} \rho\right), & a^{i}=-I_{0} \frac{\omega \mu_{2}}{4}, \quad \rho \leq a  \tag{3.81}\\
H_{\phi}^{i}=\frac{1}{j \eta_{2}} a^{i} H_{0}^{(2)}\left(k_{2} \rho\right) &
\end{array}
$$

The transmitted field cab be written:

$$
\begin{align*}
& E_{z}^{t}=a^{t} H_{0}^{(2)}\left(k_{1} \rho\right), \quad \rho \geq a \\
& H_{\phi}^{t}=\frac{1}{j \eta_{1}} a^{t} H_{0}^{\prime(2)}\left(k_{1} \rho\right) \tag{3.82}
\end{align*}
$$

and finally the scattered field:

$$
\begin{align*}
& E_{z}^{s}=a^{s} J_{0}\left(k_{2} \rho\right), \\
& H_{\phi}^{s}=\frac{1}{j \eta_{2}} a^{s} J_{0}^{\prime}\left(k_{2} \rho\right) \tag{3.83}
\end{align*} \quad, \rho \leq a=a, ~
$$

Now we can apply boundary conditions, at $\rho=a$ :

$$
\begin{align*}
& a^{i} H_{0}^{(2)}\left(k_{2} a\right)+a^{t} J_{n}\left(k_{2} a\right)=a^{s} H_{0}^{(2)}\left(k_{1} a\right) \\
& \frac{1}{\eta_{2}} a^{i} H_{0}^{\prime(2)}\left(k_{2} a\right)+\frac{1}{\eta_{2}} a^{t} J_{0}^{\prime}\left(k_{2} a\right)=\frac{1}{\eta_{1}} a^{s} H_{0}^{\prime(2)}\left(k_{1} a\right) \tag{3.84}
\end{align*}
$$

or in matrix form:

$$
\left[\begin{array}{cc}
-J_{0}^{(2)}\left(k_{2} a\right) & H_{0}^{(2)}\left(k_{1} a\right)  \tag{3.85}\\
\frac{-1}{\eta_{2}} J_{0}^{\prime}\left(k_{2} a\right) & \frac{1}{\eta_{1}} H_{0}^{\prime(2)}\left(k_{1} a\right)
\end{array}\right]\left[\begin{array}{c}
a^{t} \\
a_{n}^{s}
\end{array}\right]=\left[\begin{array}{c}
a^{i} H_{0}^{(2)}\left(k_{2} a\right) \\
\frac{1}{\eta_{2}} a^{i} H_{0}^{\prime(2)}\left(k_{2} a\right)
\end{array}\right]
$$

and the unknown will be found.

### 3.6 Scattering by a Circular Cylinder

In this section we will consider some simple objects like metallic or dielectric cylinder, and in each case we will find the Radar Cross Section of them. In cylinder, the wave may hit the cylinder normally or in oblique. In this book we only consider normal case. The electric or magnetic field may be parallel to the axis of cylinder. The TE and TM polarizations will be treated in the
following section.
Scattering by cylinder $\begin{cases}\text { normal incident }\left\{\begin{array}{l}\text { TE Polarization }\left\{\begin{array}{l}\text { Dielectric } \\ \text { Conductor } \\ \text { PEMC }\end{array}\right. \\ \text { TM Polarization }\left\{\begin{array}{l}\text { Dielectric } \\ \text { Conductor } \\ \text { PEMC }\end{array}\right.\end{array}\right. \\ \text { oblique incident }\left\{\begin{array}{l}\text { TE Polarization }\left\{\begin{array}{l}\text { Dielectric } \\ \text { Conductor } \\ \text { PEMC }\end{array}\right. \\ \text { TM Polarization }\left\{\begin{array}{l}\text { Dielectric } \\ \text { Conductor } \\ \text { PEMC }\end{array}\right.\end{array}\right.\end{cases}$

### 3.6.1 Plane Wave Expansion in Cylindrical Coordinate

Suppose we have a line source that is located at $\rho_{0}, \phi_{0}$. The wave that comes toward the origin will be

$$
\begin{equation*}
E_{z}^{i}=\sum_{n=-\infty}^{\infty} a_{n}^{i} J_{n}(k \rho) e^{j n \phi} \quad \rho \leq \rho_{0} \tag{3.86}
\end{equation*}
$$

where $a_{n}^{i}=-I_{0} \frac{\omega \mu}{4} H_{n}^{(2)}\left(k \rho_{0}\right) e^{-j n \phi_{0}}$ for $\rho \leq \rho_{0}$. If line source goes farther and farther from the origin, the wave looks like the $T M^{z}$ plane wave. Now let us have a plane wave which comes towards the origin with incident angle $\phi_{0}$. It does not matter whether is $T E^{z}$ or $T M^{z}$

$$
\begin{equation*}
e^{j k\left(x \cos \phi_{0}+y \sin \phi_{0}\right)} \tag{3.87}
\end{equation*}
$$

This equation can be transformed into cylindrical coordinate

$$
\begin{equation*}
e^{j k\left(\rho \cos \phi \cos \phi_{0}+\rho \sin \phi \sin \phi_{0}\right)}=e^{j k \rho \cos \left(\phi-\phi_{0}\right)} \tag{3.88}
\end{equation*}
$$

The incident field is expanded in a Fourier series of $\phi$, whose $\rho$-dependent coefficients are found by insertion into Helmholtz equation, to satisfy Bessel's
equation. The result will be

$$
\begin{equation*}
e^{j k \rho \cos \left(\phi-\phi_{0}\right)}=\sum_{n=-\infty}^{\infty} a_{n}^{i} J_{n}(k \rho) e^{j n \phi} \tag{3.89}
\end{equation*}
$$

where $a_{n}=j^{n} e^{-j n \phi_{0}}$.

### 3.7 Scattering by a Circular Dielectric Cylinder

One of the geometries widely used in two dimensional electromagnetic wave scattering is circular cylinder, because it can be formulated with well known functions. Consider an infinitely long dielectric circular cylinder with radius $a$ and electrical parameters $\mu_{2}, \epsilon_{2}, \sigma_{2}$ located in medium $\mu_{1}, \epsilon_{1}, \sigma_{1}$ as shown in Fig.(3.8). In such condition, propagation constant is complex.


Figure 3.8: Scattering by a circular dielectric cylinder, $T E^{z}$ polarization

$$
\epsilon_{r c}=\epsilon_{r}-j \frac{\sigma}{\omega \epsilon_{0}}, \quad \epsilon=\epsilon_{r c} \epsilon_{0}, \quad k=\omega \sqrt{\mu \epsilon}
$$

The plane wave may incident at normal or oblique to cylinder. In the case of normal incident two new case can be considered: TE and TM polarizations.

### 3.7.1 Normal Incidence, TM Polarization

A plane wave incident on a dielectric circular cylinder with radius $a$, as shown in Fig.(3.9). We suppose plane wave as:


Figure 3.9: Scattering by a circular dielectric cylinder, TM polarization

$$
\begin{equation*}
\mathbf{E}^{i}=E_{0} e^{j k\left(x \cos \phi_{0}+y \sin \phi_{0}\right)} \mathbf{a}_{z}=\sum_{n=-\infty}^{\infty} E_{0} j^{n} e^{-j n \phi_{0}} J_{n}(k \rho) e^{j n \phi} \mathbf{a}_{z} \tag{3.90}
\end{equation*}
$$

This can be rewritten as:

$$
\begin{align*}
\mathbf{E}^{i} & =\sum_{n=-\infty}^{\infty} a_{n}^{i} \mathbf{N}_{n}^{(1)}\left(k_{1} \rho\right) \\
\mathbf{H}^{i} & =\frac{j}{\eta_{1}} \sum_{n=-\infty}^{\infty} a_{n}^{i} \mathbf{M}_{n}^{(1)}\left(k_{1} \rho\right) \tag{3.91}
\end{align*}
$$

where $a_{n}^{i}=\frac{E_{0} j^{n}}{k_{1}} e^{-j n \phi_{0}}$. The scattered and transmitted fields can be expressed by:

$$
\begin{align*}
& \mathbf{E}^{s}=\sum_{n=-\infty}^{\infty} b_{n}^{s} \mathbf{N}_{n}^{(4)}\left(k_{1} \rho\right) \\
& \mathbf{H}^{s}=\frac{j}{\eta_{1}} \sum_{n=-\infty}^{\infty} b_{n}^{s} \mathbf{M}_{n}^{(4)}\left(k_{1} \rho\right) \tag{3.92}
\end{align*}
$$

$$
\begin{align*}
\mathbf{E}^{t} & =\sum_{n=-\infty}^{\infty} a_{n}^{t} \mathbf{N}_{n}^{(1)}\left(k_{2} \rho\right) \\
\mathbf{H}^{t} & =\frac{j}{\eta_{2}} \sum_{n=-\infty}^{\infty} a_{n}^{t} \mathbf{M}_{n}^{(1)}\left(k_{2} \rho\right) \tag{3.93}
\end{align*}
$$

where $b_{n}^{s}$ and $a_{n}^{t}$ can be found by applying boundary conditions. At $\rho=a$ the tangential electric and magnetic fields are continuous.

$$
\begin{gathered}
\mathbf{a}_{z} \cdot\left(\mathbf{E}^{i}+\mathbf{E}^{s}-\mathbf{E}^{t}\right)=0 \\
\mathbf{a}_{\phi} \cdot\left(\mathbf{H}^{i}+\mathbf{H}^{s}-\mathbf{H}^{t}\right)=0
\end{gathered}
$$

therefore

$$
\begin{align*}
a_{n}^{i} k_{1} J_{n}\left(k_{1} a\right)+b_{n}^{s} k_{1} H_{n}^{(2)}\left(k_{1} a\right) & =a_{n}^{t} k_{2} J_{n}\left(k_{2} a\right)  \tag{3.94}\\
\frac{k_{1}}{\eta_{1}} a_{n}^{i} J_{n}^{\prime}\left(k_{1} a\right)+\frac{k_{1}}{\eta_{1}} b_{n}^{s} H_{n}^{\prime(2)}\left(k_{1} a\right) & =\frac{k_{2}}{\eta_{2}} a_{n}^{t} J_{n}^{\prime}\left(k_{2} a\right) \tag{3.95}
\end{align*}
$$

from which $b_{n}^{s}$ and $a_{n}^{t}$ will be determined.

$$
\left[\begin{array}{cc}
-k_{1} H_{n}^{(2)}\left(k_{1} a\right) & k_{2} J_{n}\left(k_{2} a\right)  \tag{3.96}\\
-\epsilon_{1} H_{n}^{\prime(2)}\left(k_{1} a\right) & \epsilon_{2} J_{n}^{\prime}\left(k_{2} a\right)
\end{array}\right]\left[\begin{array}{c}
b_{n}^{s} \\
a_{n}^{t}
\end{array}\right]=\left[\begin{array}{c}
a_{n}^{i} k_{1} J_{n}\left(k_{1} a\right) \\
\epsilon_{1} a_{n}^{i} J_{n}^{\prime}\left(k_{1} a\right)
\end{array}\right]
$$

where $\epsilon_{1}=\epsilon_{0} \epsilon_{r c 1}, \epsilon_{r c 1}=\epsilon_{r 1}-j \frac{\sigma_{1}}{\omega \epsilon_{0}}$ and $k_{1}=\omega \sqrt{\mu_{1} \epsilon_{1}}$. For dielectric cylinder: $\epsilon_{2}=\epsilon_{0} \epsilon_{r c 2}$ and $\epsilon_{r c 2}=\epsilon_{r 2}-j \frac{\sigma_{2}}{\omega \epsilon_{0}}$, therefore $k_{2}=\omega \sqrt{\mu_{2} \epsilon_{2}}$ and in general $\eta=\sqrt{\frac{\mu}{\epsilon}}$.

### 3.7.2 Scattering by Conducting Circular Cylinder, TM Polarization

If $\sigma_{2}=\infty$, we will have perfect circular cylinder. In this case the $E_{z}^{i}+E_{z}^{s}=0$ at the surface of cylinder, therefore $a_{n}^{t}=0$ and $b_{n}^{s}$ will be:

$$
\begin{equation*}
b_{n}^{s}=-\frac{J_{n}\left(k_{1} a\right)}{H_{n}^{(2)}\left(k_{1} a\right)} a_{n}^{i} \tag{3.97}
\end{equation*}
$$

### 3.7.3 Scattering by Multilayer Dielectric Cylinder TM Polarization

Like previous section, we consider a $T M^{z}$ plane wave that incident on a multilayer dielectric circular cylinder under incident angle $\phi_{0}$ as shown in Fig.(3.10). The electric field in region one with parameters $\mu_{1}, \epsilon_{1}$ and $\sigma_{1}$ is:


Figure 3.10: Scattering by Multilayer Dielectric Cylinder TM Polarization

$$
\begin{equation*}
\mathbf{E}^{i}=E_{0} e^{j k_{1}\left(x \cos \phi_{0}+y \sin \phi_{0}\right)} \mathbf{a}_{z} \tag{3.98}
\end{equation*}
$$

The corresponding magnetic field will be found from $\mathbf{H}^{i}=\frac{-1}{j \omega \mu_{1}} \nabla \times \mathbf{E}^{i}$. Now we can expand the plane waves as:

$$
\begin{align*}
& \mathbf{E}^{i}=\sum_{n=-\infty}^{\infty} a_{n}^{(1)} \mathbf{N}_{n}^{(1)}\left(k_{1} \rho\right)  \tag{3.99}\\
& \mathbf{H}^{i}=\frac{j}{\eta_{1}} \sum_{n=-\infty}^{\infty} a_{n}^{(1)} \mathbf{M}_{n}^{(1)}\left(k_{1} \rho\right)
\end{align*}
$$

where $a_{n}^{(1)}=\frac{E_{0} j^{n}}{k_{1}} e^{-j n \phi_{0}}$ and the scattered field in region 1 will be:

$$
\begin{align*}
& \mathbf{E}^{s}=\sum_{n=-\infty}^{\infty} b_{n}^{(1)} \mathbf{N}_{n}^{(4)}\left(k_{1} \rho\right) \\
& \mathbf{H}^{s}=\frac{j}{\eta_{1}} \sum_{n=-\infty}^{\infty} b_{n}^{(1)} \mathbf{M}_{n}^{(4)}\left(k_{1} \rho\right) \tag{3.100}
\end{align*}
$$

The transmitted and scattered fields in region 2 with electrical parameters $\mu_{2}, \epsilon_{2}$ and $\sigma_{2}$, are:

$$
\begin{align*}
& \mathbf{E}^{(2)}=\sum_{n=-\infty}^{\infty} a_{n}^{(2)} \mathbf{N}_{n}^{(1)}\left(k_{2} \rho\right)+b_{n}^{(2)} \mathbf{N}_{n}^{(4)}\left(k_{2} \rho\right) \\
& \mathbf{H}^{(2)}=\frac{j}{\eta_{2}} \sum_{n=-\infty}^{\infty} a_{n}^{(2)} \mathbf{M}_{n}^{(1)}\left(k_{2} \rho\right)+b_{n}^{(2)} \mathbf{M}_{n}^{(4)}\left(k_{2} \rho\right) \tag{3.101}
\end{align*}
$$

Finally, the transmitted fields in region 3 with parameters $\mu_{3}, \epsilon_{3}$ and $\sigma_{3}$ are:

$$
\begin{align*}
& \mathbf{E}^{(3)}=\sum_{n=-\infty}^{\infty} a_{n}^{(3)} \mathbf{N}_{n}^{(1)}\left(k_{3} \rho\right) \\
& \mathbf{H}^{(3)}=\frac{j}{\eta_{3}} \sum_{n=-\infty}^{\infty} a_{n}^{(3)} \mathbf{M}_{n}^{(1)}\left(k_{3} \rho\right) \tag{3.102}
\end{align*}
$$

By applying boundary conditions at $\rho=R_{1}$ and $\rho=R_{2}$ we have:

$$
\begin{align*}
a_{n}^{(1)} k_{1} J_{n}\left(k_{1} R_{1}\right)+b_{n}^{(1)} k_{1} H_{n}^{(2)}\left(k_{1} R_{1}\right) & =a_{n}^{(2)} k_{2} J_{n}\left(k_{2} R_{1}\right)+b_{n}^{(2)} k_{2} H_{n}^{(2)}\left(k_{2} R_{1}\right) \\
\frac{k_{1}}{\eta_{1}} a_{n}^{(1)} J_{n}^{\prime}\left(k_{1} R_{1}\right)+\frac{k_{1}}{\eta_{1}} b_{n}^{(1)} H_{n}^{\prime(2)}\left(k_{1} R_{1}\right) & =\frac{k_{2}}{\eta_{2}} a_{n}^{(2)} J_{n}^{\prime}\left(k_{2} R_{1}\right)+\frac{k_{2}}{\eta_{2}} b_{n}^{(2)} H_{n}^{\prime(2)}\left(k_{2} R_{1}\right) \\
a_{n}^{(2)} k_{2} J_{n}\left(k_{2} R_{2}\right)+b_{n}^{(2)} k_{2} H_{n}^{(2)}\left(k_{2} R_{2}\right) & =a_{n}^{(3)} k_{3} J_{n}\left(k_{3} R_{2}\right) \\
\frac{k_{2}}{\eta_{2}} a_{n}^{(2)} J_{n}^{\prime}\left(k_{2} R_{2}\right)+\frac{k_{2}}{\eta_{2}} b_{n}^{(2)} H_{n}^{\prime(2)}\left(k_{2} R_{2}\right) & =\frac{k_{3}}{\eta_{3}} a_{n}^{(3)} J_{n}^{\prime}\left(k_{3} R_{2}\right) \tag{3.103}
\end{align*}
$$

From these equations we can form a $4 \times 4$ matrix to determine the unknowns, $b_{n}^{(1)}, a_{n}^{(2)}, b_{n}^{(2)}$ and $a_{n}^{(3)}$.

### 3.7.4 Normal Incidence, TE Polarization

By considering Fig.(3.11), we have:

$$
\begin{equation*}
\mathbf{H}^{i}=H_{0} e^{j k\left(x \cos \phi_{0}+y \sin \phi_{0}\right)} \mathbf{a}_{z}=\sum_{n=-\infty}^{\infty} H_{0} j^{n} e^{-j n \phi_{0}} J_{n}(k \rho) e^{j n \phi} \mathbf{a}_{z} \tag{3.104}
\end{equation*}
$$

The corresponding electric field will be find from $\mathbf{E}^{i}=\frac{1}{j \omega \epsilon_{1}} \nabla \times \mathbf{H}^{i}$. Now we


Figure 3.11: Scattering by a circular dielectric cylinder for TE polarization can expand the plane waves as

$$
\begin{align*}
\mathbf{H}^{i} & =\sum_{n=-\infty}^{\infty} a_{n}^{i} \mathbf{N}_{n}^{(1)}\left(k_{1} \rho\right) \\
\mathbf{E}^{i} & =-j \eta_{1} \sum_{n=-\infty}^{\infty} a_{n}^{i} \mathbf{M}_{n}^{(1)}\left(k_{1} \rho\right) \tag{3.105}
\end{align*}
$$

where $a_{n}^{i}=\frac{H_{0} j^{n}}{k_{1}} e^{-j n \phi_{0}}$. The scattered and transmitted fields can be expressed as:

$$
\begin{align*}
& \mathbf{H}^{s}=\sum_{n=-\infty}^{\infty} b_{n}^{s} \mathbf{N}_{n}^{(4)}\left(k_{1} \rho\right) \\
& \mathbf{E}^{s}=-j \eta_{1} \sum_{n=-\infty}^{\infty} b_{n}^{s} \mathbf{M}_{n}^{(4)}\left(k_{1} \rho\right)  \tag{3.106}\\
& \mathbf{H}^{t}=\sum_{n=-\infty}^{\infty} a_{n}^{t} \mathbf{N}_{n}^{(1)}\left(k_{2} \rho\right)  \tag{3.107}\\
& \mathbf{E}^{t}=-j \eta_{2} \sum_{n=-\infty}^{\infty} a_{n}^{t} \mathbf{M}_{n}^{(1)}\left(k_{2} \rho\right)
\end{align*}
$$

where $b_{n}^{s}$ and $a_{n}^{t}$ can be found by applying boundary conditions at $\rho=a$ :

$$
\begin{align*}
& H_{z}^{t}=H_{z}^{i}+H_{z}^{s} \quad 0 \leq \phi \leq 2 \pi \\
& E_{\phi}^{t}=E_{\phi}^{i}+E_{\phi}^{s} \quad \rho=a  \tag{3.108}\\
& a_{n}^{i} k_{1} J_{n}\left(k_{1} a\right)+b_{n}^{s} k_{1} H_{n}^{(2)}\left(k_{1} a\right)=a_{n}^{t} k_{2} J_{n}\left(k_{2} a\right)  \tag{3.109}\\
& \eta_{1} k_{1} a_{n}^{i} J_{n}^{\prime}\left(k_{1} a\right)+\eta_{1} k_{1} b_{n}^{s} H_{n}^{\prime(2)}\left(k_{1} a\right)=\eta_{2} k_{2} a_{n}^{t} J_{n}^{\prime}\left(k_{2} a\right) \tag{3.110}
\end{align*}
$$

The prime sign indicates the derivative with respect to the argument of the Bessel's functions.

$$
\left[\begin{array}{cc}
-k_{1} H_{n}^{(2)}\left(k_{1} a\right) & k_{2} J_{n}\left(k_{2} a\right)  \tag{3.111}\\
-\mu_{1} H_{n}^{\prime(2)}\left(k_{1} a\right) & \mu_{2} J_{n}^{\prime}\left(k_{2} a\right)
\end{array}\right]\left[\begin{array}{c}
b_{n}^{s} \\
a_{n}^{t}
\end{array}\right]=\left[\begin{array}{c}
a_{n}^{i} k_{1} J_{n}\left(k_{1} a\right) \\
\mu_{1} a_{n}^{i} J_{n}^{\prime}\left(k_{1} a\right)
\end{array}\right]
$$

### 3.7.5 Scattering by Conducting Circular Cylinder for TE Polarization

If in Fig.(3.11) $\sigma_{2} \rightarrow \infty$, we will have perfect circular cylinder. In this case $E_{\phi}^{i}+E_{\phi}^{s}=0$, therefore $a_{n}^{t}=0$ and $b_{n}^{s}$ will be [2]:

$$
\begin{equation*}
b_{n}^{s}=-\frac{J_{n}^{\prime}\left(k_{1} a\right)}{H_{n}^{\prime(2)}\left(k_{1} a\right)} a_{n}^{i} \tag{3.112}
\end{equation*}
$$

### 3.7.6 Scattering by Multilayer Dielectric Cylinder for TE Polarization

Referring with Fig.(3.13), the magnetic field in region one with parameters $\mu_{1}, \epsilon_{1}$ and $\sigma_{1}$ is:

$$
\begin{equation*}
\mathbf{H}^{i}=H_{0} e^{j k_{1}\left(x \cos \phi_{0}+y \sin \phi_{0}\right)} \mathbf{a}_{z} \tag{3.113}
\end{equation*}
$$

The corresponding electric field will be found from $\mathbf{E}^{i}=\frac{1}{j \omega \epsilon_{1}} \nabla \times \mathbf{H}^{i}$.
Now we can expand the plane waves as:

$$
\begin{align*}
\mathbf{H}^{i} & =\sum_{n=-\infty}^{\infty} a_{n}^{(1)} \mathbf{N}_{n}^{(1)}\left(k_{1} \rho\right)  \tag{3.114}\\
\mathbf{E}^{i} & =j \eta_{1} \sum_{n=-\infty}^{\infty} a_{n}^{(1)} \mathbf{M}_{n}^{(1)}\left(k_{1} \rho\right)
\end{align*}
$$




Figure 3.12: Scattering of a TE and TM plane wave by a circular conducting cylinder
where $a_{n}^{(1)}=\frac{H_{0} j^{n}}{k_{1}} e^{-j n \phi_{0}}$ and the scattered field in region 1 will be:

$$
\begin{align*}
& \mathbf{H}^{s}=\sum_{n=-\infty}^{\infty} b_{n}^{(1)} \mathbf{N}_{n}^{(4)}\left(k_{1} \rho\right) \\
& \mathbf{E}^{s}=j \eta_{1} \sum_{n=-\infty}^{\infty} b_{n}^{(1)} \mathbf{M}_{n}^{(4)}\left(k_{1} \rho\right) \tag{3.115}
\end{align*}
$$

The transmitted and scattered fields in region 2 with electrical parameters $\mu_{2}, \epsilon_{2}$ and $\sigma_{2}$, are:

$$
\begin{align*}
\mathbf{H}^{(2)} & =\sum_{n=-\infty}^{\infty} a_{n}^{(2)} \mathbf{N}_{n}^{(1)}\left(k_{2} \rho\right)+b_{n}^{(2)} \mathbf{N}_{n}^{(4)}\left(k_{2} \rho\right) \\
\mathbf{E}^{(2)} & =j \eta_{2} \sum_{n=-\infty}^{\infty} a_{n}^{(2)} \mathbf{M}_{n}^{(1)}\left(k_{2} \rho\right)+b_{n}^{(2)} \mathbf{M}_{n}^{(4)}\left(k_{2} \rho\right) \tag{3.116}
\end{align*}
$$

Finally the transmitted fields in region 3 with parameters $\mu_{3}, \epsilon_{3}$ and $\sigma_{3}$ are:

$$
\begin{align*}
& \mathbf{H}^{(3)}=\sum_{n=-\infty}^{\infty} a_{n}^{(3)} \mathbf{N}_{n}^{(1)}\left(k_{3} \rho\right)  \tag{3.117}\\
& \mathbf{E}^{(3)}=j \eta_{3} \sum_{n=-\infty}^{\infty} a_{n}^{(3)} \mathbf{M}_{n}^{(1)}\left(k_{3} \rho\right)
\end{align*}
$$



Figure 3.13: Scattering by multilayer dielectric cylinder for TE polarization

By applying boundary conditions at $\rho=R_{1}$ and $\rho=R_{2}$ we have:

$$
\begin{align*}
a_{n}^{(1)} k_{1} J_{n}\left(k_{1} R_{1}\right)+b_{n}^{(1)} k_{1} H_{n}^{(2)}\left(k_{1} R_{1}\right) & =a_{n}^{(2)} k_{2} J_{n}\left(k_{2} R_{1}\right)+b_{n}^{(2)} k_{2} H_{n}^{(2)}\left(k_{2} R_{1}\right) \\
k_{1} \eta_{1} a_{n}^{(1)} J_{n}^{\prime}\left(k_{1} R_{1}\right)+k_{1} \eta_{1} b_{n}^{(1)} H_{n}^{\prime(2)}\left(k_{1} R_{1}\right) & =k_{2} \eta_{2} a_{n}^{(2)} J_{n}^{\prime}\left(k_{2} R_{1}\right)+k_{2} \eta_{2} b_{n}^{(2)} H_{n}^{\prime(2)}\left(k_{2} R_{1}\right) \\
a_{n}^{(2)} k_{2} J_{n}\left(k_{2} R_{2}\right)+b_{n}^{(2)} k_{2} H_{n}^{(2)}\left(k_{2} R_{2}\right) & =a_{n}^{(3)} k_{3} J_{n}\left(k_{3} R_{2}\right) \\
k_{2} \eta_{2} a_{n}^{(2)} J_{n}^{\prime}\left(k_{2} R_{2}\right)+k_{2} \eta_{2} b_{n}^{(2)} H_{n}^{\prime(2)}\left(k_{2} R_{2}\right) & =k_{3} \eta_{3} a_{n}^{(3)} J_{n}^{\prime}\left(k_{3} R_{2}\right) \tag{3.118}
\end{align*}
$$

From these equations we can form $4 \times 4$ matrix to determine the unknowns $a_{n}^{(1)}, b_{n}^{(1)}, a_{n}^{(2)}, b_{n}^{(2)}$ and $a_{n}^{(3)}$ :
$\left[\begin{array}{cccc}-k_{1} H_{n}^{(2)}\left(k_{1} R_{1}\right) & k_{2} J_{n}\left(k_{2} b\right) & k_{2} H_{n}^{(2)}\left(k_{2} R_{1}\right) & 0 \\ -\mu_{1} H_{n}^{\prime(2)}\left(k_{1} R_{1}\right) & \mu_{2} J_{n}^{\prime}\left(k_{2} R_{1}\right) & \mu_{2} H_{n}^{\prime(2)}\left(k_{2} R_{1}\right) & 0 \\ 0 & -k_{2} J_{n}\left(k_{2} R_{2}\right) & -k_{2} H_{n}^{(2)}\left(k_{2} R_{2}\right) & k_{3} J_{n}\left(k_{3} R_{2}\right) \\ 0 & \mu_{2} J_{n}^{\prime}\left(k_{2} R_{2}\right) & \mu_{2} H_{n}^{\prime(2)}\left(k_{2} a\right) & \epsilon_{3} J_{n}^{\prime}\left(k_{3} R_{2}\right)\end{array}\right]\left[\begin{array}{c}b_{n}^{(1)} \\ a_{n}^{(2)} \\ b_{n}^{(2)} \\ a_{n}^{(3)}\end{array}\right]=\left[\begin{array}{c}a_{n}^{1} k_{1} J_{n}\left(k_{1} R_{1}\right) \\ \mu_{1} a_{n}^{1} J_{n}^{\prime}\left(k_{1} R_{1}\right) \\ 0 \\ 0\end{array}\right]$


Figure 3.14: Scattering of a $T M^{z}$ plane wave by a circular dielectric cylinder

### 3.8 Scattering by a PEMC Cylinder

As we mentioned in previous section, at the surface of a Perfect Electromagnetic Conductor(PEMC), the boundary conditions are:

$$
\begin{array}{r}
\mathbf{n \times ( \mathbf { H } + M \mathbf { E } )}=0 \\
\mathbf{n} \cdot(\mathbf{D}-M \mathbf{B})=0 \tag{3.120}
\end{array}
$$

We know that in cylindrical coordinate system, the solution of two dimensional scalar wave equation will be:

$$
\begin{equation*}
\nabla^{2} \psi+k^{2} \psi=0 \quad==>\quad \psi=e^{j n \phi} \tag{3.121}
\end{equation*}
$$

where $Z_{n}^{g}(k \rho)$ is cylindrical Bessel's functions. Let us introduce two auxiliary vectors $\mathbf{M}$ and $\mathbf{N}$ :

$$
\begin{align*}
\mathbf{M} & =\nabla \times\left(\psi \mathbf{a}_{\mathbf{z}}\right) \\
\mathbf{N} & =\frac{1}{k} \nabla \times \mathbf{M} \tag{3.122}
\end{align*}
$$



Figure 3.15: Distribution of $E_{z}$ field in circular dielectric cylinder

After some manipulations

$$
\begin{align*}
\mathbf{M} & =\frac{1}{\rho} \frac{\partial \psi}{\partial \phi} \mathbf{a}_{\rho}-\frac{\partial \psi}{\partial \rho} \mathbf{a}_{\phi} \\
\mathbf{N} & =k \psi \mathbf{a}_{\mathbf{z}} \tag{3.123}
\end{align*}
$$

We assume a normal incident plane wave, and expand it in terms of $\mathbf{M}$ and $\mathbf{N}$, therefore for $T M^{z}$ polarization, the incident fields are:
$\mathbf{E}^{i}=E_{0} e^{j k\left(x \cos \phi_{0}+y \sin \phi_{0}\right)} \mathbf{a}_{\mathbf{z}}=E_{0} e^{j k \rho \cos \left(\phi-\phi_{0}\right)} \mathbf{a}_{\mathbf{z}}=E_{0} \sum_{n=-\infty}^{\infty} a_{n} J_{n}(k \rho) e^{j n \phi} \mathbf{a}_{\mathbf{z}}$
where $a_{n}=j^{n} e^{-j n \phi_{0}}$ therefore:

$$
\begin{align*}
\mathbf{E}^{i} & =\frac{E_{0}}{k} \sum_{n=-\infty}^{\infty} a_{n} \mathbf{N}_{n}^{(1)}(k \rho) \\
\mathbf{H}^{i} & =\frac{j E_{0}}{\omega \mu} \sum_{n=-\infty}^{\infty} a_{n} \mathbf{M}_{n}^{(1)}(k \rho) \tag{3.125}
\end{align*}
$$



Figure 3.16: Distribution of $E_{z}$ field in circular dielectric cylinder

The scattered fields are expanded in the form of:

$$
\begin{align*}
\mathbf{E}^{s} & =\frac{E_{0}}{k} \sum_{n=-\infty}^{\infty}\left[b_{n} \mathbf{N}_{n}^{(4)}(k \rho)+c_{n} \mathbf{M}_{n}^{(4)}(k \rho)\right] \\
\mathbf{H}^{s} & =\frac{j E_{0}}{\omega \mu} \sum_{n=-\infty}^{\infty}\left[b_{n} \mathbf{M}_{n}^{(4)}(k \rho)+c_{n} \mathbf{N}_{n}^{(4)}(k \rho)\right] \tag{3.126}
\end{align*}
$$

Since we have PEMC, there should be one more term for cross-polarized TE components. Now we can apply boundary conditions for PEMC. The tangential and normal field components have to satisfy the boundary conditions at the cylinder surface:

$$
\begin{align*}
H_{t}^{i}+H_{t}^{s}+M\left(E_{t}^{i}+E_{t}^{s}\right) & =0  \tag{3.127}\\
\epsilon\left(E_{\rho}^{i}+E_{\rho}^{s}\right)-M \mu\left(H_{\rho}^{i}+H_{\rho}^{s}\right) & =0 \tag{3.128}
\end{align*}
$$

By applying these boundary conditions, we obtain the following system of linear equations are obtained:

$$
\begin{align*}
H_{n}^{(2)}(k a) b_{n}+\frac{j}{M \eta} H_{n}^{(2)}(k a) c_{n} & =-a n J_{n}(k a)  \tag{3.129}\\
H_{n}^{\prime(2)}(k a) b_{n}-j M \eta H_{n}^{\prime(2)}(k a) c_{n} & =-a n J_{n}^{\prime}(k a) \tag{3.130}
\end{align*}
$$

where $a$ is the radius of the PEMC cylinder. By solving these equations we will have:

$$
\begin{align*}
b_{n} & =-\frac{H_{n}^{(2)}(k a) J_{n}^{\prime}(k a)+M^{2} \eta^{2} H_{n}^{\prime(2)}(k a) J_{n}(k a)}{\left(1+M^{2} \eta^{2}\right) H_{n}^{(2)}(k a) H_{n}^{\prime(2)}(k a)} a_{n}  \tag{3.131}\\
c_{n} & =\frac{2 M \eta}{\pi k a\left(1+M^{2} \eta^{2}\right) H_{n}^{(2)}(k a) H_{n}^{\prime(2)}(k a)} a_{n} \tag{3.132}
\end{align*}
$$

For $T E^{z}$ case we have:

$$
\begin{align*}
\mathbf{E}^{i} & =E_{0} e^{j k\left(x \cos \phi_{0}+y \sin \phi_{0}\right)}=\frac{j E_{0}}{k} \sum_{n=-\infty}^{\infty} a_{n} \mathbf{M}_{n}^{(1)}(k \rho) \\
\mathbf{H}^{i} & =\frac{E_{0}}{\omega \mu} \sum_{n=-\infty}^{\infty} a_{n} \mathbf{N}_{n}^{(1)}(k \rho) \tag{3.133}
\end{align*}
$$

where $a_{n}=j^{n} e^{-j n \phi_{0}}$. The scattered fields are expanded in the form of:

$$
\begin{align*}
\mathbf{E}^{s} & =\frac{j E_{0}}{k} \sum_{n=-\infty}^{\infty}\left[b_{n} \mathbf{M}_{n}^{(4)}(k \rho)+c_{n} \mathbf{N}_{n}^{(4)}(k \rho)\right] \\
\mathbf{H}^{s} & =\frac{E_{0}}{\omega \mu} \sum_{n=-\infty}^{\infty}\left[b_{n} \mathbf{N}_{n}^{(4)}(k \rho)+c_{n} \mathbf{M}_{n}^{(4)}(k \rho)\right] \tag{3.134}
\end{align*}
$$

Similar to the previous case, we have the cross-polarized term.
By applying boundary conditions for PEMC cylinder at $\rho=a$, we will have:

$$
\begin{align*}
b_{n} & =-\frac{H_{n}^{\prime(2)}(k a) J_{n}(k a)+M^{2} \eta^{2} H_{n}^{(2)}(k a) J_{n}^{\prime}(k a)}{\left(1+M^{2} \eta^{2}\right) H_{n}^{(2)}(k a) H_{n}^{\prime(2)}(k a)} a_{n}  \tag{3.135}\\
c_{n} & =\frac{2 M \eta}{\pi k a\left(1+M^{2} \eta^{2}\right) H_{n}^{(2)}(k a) H_{n}^{\prime(2)}(k a)} a_{n} \tag{3.136}
\end{align*}
$$

### 3.9 Scattering by Circular Anisotropic Cylinder

In a single crystal, the physical and mechanical properties often differ with orientation. It can be seen from looking at our models of crystalline structure
that atoms should be able to slip over one another or distort in relation to one another easier in some directions than others. When the properties of a material vary with different crystallographic orientations, the material is said to be anisotropic. Alternately, when the properties of a material are the same in all directions, the material is called to be isotropic. For modeling human anatomy, composition of fluids, composite materials, the substrates of integrated circuits and so on are anisotropic materials.
In this section we study the scattering of a plane wave by anisotropic circular cylinder with the following permittivity and permeability tensors:

$$
\epsilon=\left[\begin{array}{ccc}
\epsilon_{x x} & \epsilon_{x y} & 0  \tag{3.137}\\
\epsilon_{y x} & \epsilon_{y y} & 0 \\
0 & 0 & \epsilon_{z z}
\end{array}\right] \quad \mu=\left[\begin{array}{ccc}
\mu_{x x} & \mu_{x y} & 0 \\
\mu_{y x} & \mu_{y y} & 0 \\
0 & 0 & \mu_{z z}
\end{array}\right]
$$

where all the elements of the matrices of $\epsilon$ and $\mu$ are may be real or complex constants.
In this section we will study only the $T E^{z}$ polarization (i.e. $\mathbf{H}=H_{z} \mathbf{a}_{z}$ ); The reader may do the same derivation for $T M^{z}$ polarization as an exercise. The partial differential equation for $T M^{z}$ polarization is found to be [66]

$$
\begin{equation*}
\epsilon_{x x} \frac{\partial^{2} H_{z}}{\partial x^{2}}+\epsilon_{y y} \frac{\partial^{2} H_{z}}{\partial y^{2}}+\left(\epsilon_{y x}+\epsilon_{x y}\right) \frac{\partial^{2} H_{z}}{\partial x \partial y}+\omega^{2} \mu_{z z} \gamma H_{z}=0 \tag{3.138}
\end{equation*}
$$

where $\gamma=\epsilon_{x x} \epsilon_{y y}-\epsilon_{x y} \epsilon_{y x}$. It can be shown by the angular spectrum representation of plane waves given in [66] that the field inside a cylinder can be expressed by a finite summation of eigne plane waves as:

$$
\begin{equation*}
H_{z}(\rho, \phi)=\sum_{s=-N}^{N} a_{s} e^{j k_{s} \rho \cos \left(\phi-\phi_{s}\right)} \tag{3.139}
\end{equation*}
$$

where

$$
\begin{align*}
k_{s} & =\left(\frac{m_{z}^{2}}{\epsilon_{+}+\epsilon_{-} \cos 2 \phi_{s}+\sigma_{+} \sin 2 \phi_{s}}\right)^{1 / 2}  \tag{3.140}\\
m_{z} & =\omega \sqrt{\mu_{z z} \gamma} \\
\epsilon_{ \pm} & =\frac{1}{2}\left(\epsilon_{x x} \pm \epsilon_{y y}\right) \\
\sigma_{ \pm} & =\frac{1}{2}\left(\epsilon_{x y} \pm \epsilon_{y x}\right) \\
\phi_{s} & =\frac{2 \pi s}{2 N+1} \quad s=-N, \ldots, N
\end{align*}
$$

Each eigen plane wave in Eq.(3.139)is a wave function of the first kind in an anisotropic medium. Expanding the plane wave factor in Eq.(3.139), we obtain the series form of wavefunctions of the first kind for anisotropic medium as follows:

$$
\begin{align*}
H_{z}(\rho, \phi) & =\sum_{s=-N}^{N} a_{s} H_{z s}^{(1)}(\rho, \phi)  \tag{3.141}\\
H_{z s}^{(1)}(\rho, \phi) & =\sum_{m=-\infty}^{\infty} j^{-m} J_{m}\left(k_{s} \rho\right) e^{j m \phi_{s}} e^{-j m \phi} \tag{3.142}
\end{align*}
$$

Note that because Bessel function of other kinds satisfy the same differential equations and the same recursive relations as that of the first kind, we can give the definition of cylindrical wave functions of $g$ th kind $H_{z}^{(g)}(\rho, \phi)$ for the anisotropic medium as:

$$
\begin{align*}
& H_{z}^{(g)}(\rho, \phi)=\sum_{s=-N}^{N} a_{s}^{(g)} H_{z s}^{(g)}(\rho, \phi)  \tag{3.143}\\
& H_{z s}^{(g)}(\rho, \phi)=\sum_{m=-\infty}^{\infty} j^{-m} e^{j m \phi_{s}} Z_{m}^{(g)}\left(k_{s} \rho\right) e^{-j m \phi} \quad g=1,2,3,4 \tag{3.144}
\end{align*}
$$

Using the four wave functions given in Eq.(3.143) we can obtain solution to the problems in cylindrically layered structures. Now let us look at simple case, scattering by an anisotropic cylinder, Fig.(3.17). In free space the incident and scattered fields will be:

$$
\begin{equation*}
H_{z}^{i}(\rho, \phi)=\sum_{n=-\infty}^{\infty} j^{-n} e^{j n \phi_{0}} J_{n}\left(k_{0} \rho\right) e^{-j n \phi} \tag{3.145}
\end{equation*}
$$

and the corresponding tangential electric field will be:

$$
\begin{gather*}
E_{\phi}^{i}(\rho, \phi)=-j \eta_{0} \sum_{n=-\infty}^{\infty} j^{-n} e^{j n \phi_{0}} J_{n}^{\prime}\left(k_{0} \rho\right) e^{-j n \phi}  \tag{3.146}\\
H_{z}^{s}(\rho, \phi)=\sum_{n=-\infty}^{\infty} j^{-n} a_{n}^{s} H_{n}^{(2)}\left(k_{0} \rho\right) e^{j n \phi} \tag{3.147}
\end{gather*}
$$



Figure 3.17: anisotropic cylinder
and the corresponding tangential electric field will be:

$$
\begin{equation*}
E_{\phi}^{s}(\rho, \phi)=-j \eta_{0} \sum_{n=-\infty}^{\infty} j^{-n} a_{n}^{s} H_{n}^{\prime(2)}\left(k_{0} \rho\right) e^{j n \phi} \tag{3.148}
\end{equation*}
$$

The fields inside the anisotropic cylinder will be considered as:

$$
\begin{equation*}
H_{z}^{t}(\rho, \phi)=\sum_{n=-\infty}^{\infty} j^{-n} e^{j n \phi} \sum_{s=-N}^{N} a_{s}^{t} H_{n s}^{(1)} \tag{3.149}
\end{equation*}
$$

where

$$
\begin{equation*}
H_{n s}^{(1)}=J\left(k_{s} \rho\right) e^{-j n \phi_{s}} \tag{3.150}
\end{equation*}
$$

and the corresponding tangential electric field will be

$$
\begin{equation*}
E_{\phi}^{t}(\rho, \phi)=\sum_{n=-\infty}^{\infty} j^{-n} e^{j n \phi} \sum_{s=-N}^{N} a_{s}^{t} E_{n s}^{(1)} \tag{3.151}
\end{equation*}
$$

where

$$
\begin{align*}
& E_{n s}^{(1)}=-\frac{k_{s}\left(\phi_{s}\right)}{\omega \gamma}\left\{-j \epsilon_{\rho \rho} J_{n}^{\prime}\left[k_{s}\left(\phi_{s}\right) \rho\right]\right. \\
& +\frac{n}{k_{s}\left(\phi_{s}\right) \rho} J_{n}\left[k_{s}\left(\phi_{s}\right) \rho\right] \epsilon_{\phi \rho}\left(\phi_{s}+\frac{\pi}{2}\right\} e^{-j n \phi_{s}} \tag{3.152}
\end{align*}
$$

and

$$
\begin{align*}
\epsilon_{\rho \rho}\left(\phi_{s}\right) & =\epsilon_{+}+\epsilon_{-} \cos 2 \phi_{s}+\epsilon_{+} \sin 2 \phi_{s}  \tag{3.153}\\
\epsilon_{\phi \rho}\left(\phi_{s}\right) & =-\sigma_{-}+\sigma_{+} \cos 2 \phi_{s}-\epsilon_{-} \sin 2 \phi_{s}
\end{align*}
$$

The boundary conditions at the surface of the cylinder $\rho=a$ leads to the following equations:

$$
\begin{align*}
& \sum_{n=-\infty}^{\infty} j^{-n} e^{j n \phi} \sum_{s=-N}^{N} a_{s}^{t} J\left(k_{s} a\right) e^{-j n \phi_{s}}=  \tag{3.154}\\
& \sum_{n=-\infty}^{\infty} j^{-n} e^{j n \phi} J_{n}\left(k_{0} a\right) e^{-j n \phi_{0}}+\sum_{n=-\infty}^{\infty} j^{-n} e^{j n \phi} a_{n}^{s} H_{n}^{(2)}\left(k_{0} a\right) \\
& \sum_{n=-\infty}^{\infty} j^{-n} e^{j n \phi} \sum_{s=-N}^{N} a_{s}^{t} E_{n s}^{(1)}=  \tag{3.155}\\
& -j \eta_{0} \sum_{n=-\infty}^{\infty} j^{-n} e^{j n \phi} J^{\prime}{ }_{n}\left(k_{0} a\right) e^{-j n \phi_{0}}-j \eta_{0} \sum_{n=-\infty}^{\infty} j^{-n} e^{j n \phi} a_{n}^{s} H_{n}^{\prime(2)}\left(k_{0} a\right)
\end{align*}
$$

By using the orthogonality of sinusoidal functions in both Eq.(3.154) and Eq.(3.155), we will have:

$$
\begin{align*}
& \sum_{s=-N}^{N} a_{s}^{t} J\left(k_{s} a\right) e^{-j n \phi_{s}}+a_{n}^{s} H_{n}^{(2)}\left(k_{0} a\right)=J_{n}\left(k_{0} a\right) e^{-j n \phi_{0}}  \tag{3.156}\\
& \sum_{s=-N}^{N} a_{s}^{t} E_{n s}^{(1)}-j \eta_{0} a_{n}^{s} H_{n}^{\prime(2)}\left(k_{0} a\right)=-j \eta_{0} J_{n}^{\prime}\left(k_{0} a\right) e^{-j n \phi_{0}} \tag{3.157}
\end{align*}
$$

Two equation Eq.(3.156) and Eq.(3.157) are the matrix form and we can find the unknowns parameters $a_{s}^{t}$ and $a_{n}^{s}$.

### 3.10 Scattering by Circular Chiral Cylinder

We consider an infinitely long circular chiral cylinder. As is customary in twodimensional problems, two mutually exclusive situations will be analyzed: the $T M^{z}$-case in which the incident electric field is parallel to the cylinder axis (i.e., the z axis); and the $T E^{z}$ - case in which the incident magnetic field
is parallel to the z axis. The two situations are illustrated in Fig.(3.18). The cylinder is considered to be embedded in a space with $\mu_{1}, \epsilon_{1}$, and $k_{1}=$ $\omega \sqrt{\mu_{1} \epsilon_{1}}$, the media wave number. Bohren [60] was the first person to examine the scattering of plane waves by a chiral cylinder.


Figure 3.18: Scattering by a circular chiral cylinder

### 3.10.1 $T M^{z}$ polarization

We first consider the $T M^{z}$ case. The incident electric field is given by:

$$
\begin{equation*}
\mathbf{E}^{i}=E_{0} e^{j k_{1}\left(x \cos \phi_{0}+y \sin \phi_{0}\right)} \mathbf{a}_{z}=\sum_{n=-\infty}^{\infty} E_{0} j^{n} e^{-j n \phi_{0}} J_{n}\left(k_{1} \rho\right) e^{j n \phi} \mathbf{a}_{z} \tag{3.158}
\end{equation*}
$$

The incident wave can be written as

$$
\begin{align*}
\mathbf{E}^{i} & =\sum_{n=-\infty}^{\infty} a_{n}^{i} \mathbf{N}_{n}^{(1)}\left(k_{1} \rho\right)  \tag{3.159}\\
\mathbf{H}^{i} & =\frac{j}{\eta_{1}} \sum_{n=-\infty}^{\infty} a_{n}^{i} \mathbf{M}_{n}^{(1)}\left(k_{1} \rho\right) \tag{3.160}
\end{align*}
$$

where $a_{n}^{i}=\frac{j^{n} E_{0}}{k_{1}} e^{-j n \phi_{0}}$. The scattered field will be

$$
\begin{equation*}
\mathbf{E}^{s}=\sum_{n=-\infty}^{\infty} a_{n}^{s} \mathbf{M}_{n}^{(4)}\left(k_{1} \rho\right)+b_{n}^{s} \mathbf{N}_{n}^{(4)}\left(k_{1} \rho\right) \tag{3.161}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{H}^{s}=\frac{j}{\eta_{1}} \sum_{n=-\infty}^{\infty} a_{n}^{s} \mathbf{N}_{n}^{(4)}\left(k_{1} \rho\right)+b_{n}^{s} \mathbf{M}_{n}^{(4)}\left(k_{1} \rho\right) \tag{3.162}
\end{equation*}
$$

and the fields inside the chiral cylinder will be $\mathbf{E}^{t}=\mathbf{Q}_{R}+j \eta_{2} \mathbf{Q}_{L}, \mathbf{H}^{t}=$ $\mathbf{Q}_{L}+\frac{j}{\eta_{2}} \mathbf{Q}_{R}$

$$
\begin{align*}
\mathbf{Q}_{R} & =\sum_{n=-\infty}^{\infty} a_{n}^{t}\left[\mathbf{M}_{n}^{(1)}\left(k_{R} \rho\right)-\mathbf{N}_{n}^{(1)}\left(k_{R} \rho\right)\right]  \tag{3.163}\\
\mathbf{Q}_{L} & =\sum_{n=-\infty}^{\infty} b_{n}^{t}\left[\mathbf{M}_{n}^{(1)}\left(k_{L} \rho\right)+\mathbf{N}_{n}^{(1)}\left(k_{L} \rho\right)\right] \tag{3.164}
\end{align*}
$$

where

$$
k_{R}=\frac{\omega \sqrt{\mu_{2} \epsilon_{2}}}{1+\beta \omega \sqrt{\mu_{2} \epsilon_{2}}}
$$

and

$$
k_{L}=\frac{\omega \sqrt{\mu_{2} \epsilon_{2}}}{1-\beta \omega \sqrt{\mu_{2} \epsilon_{2}}}
$$

Now we can apply boundary conditions in order to find four unknown $a_{n}^{s}, b_{n}^{s}, a_{n}^{t}$ and $b_{n}^{t}$. At $\rho=a$;

$$
\begin{align*}
\mathbf{a}_{z} \cdot\left[\mathbf{E}^{i}+\mathbf{E}^{s}-\mathbf{E}^{t}\right] & =0 \\
\mathbf{a}_{\phi} \cdot\left[\mathbf{E}^{i}+\mathbf{E}^{s}-\mathbf{E}^{t}\right] & =0 \\
\mathbf{a}_{z} \cdot\left[\mathbf{H}^{i}+\mathbf{H}^{s}-\mathbf{H}^{t}\right] & =0 \\
\mathbf{a}_{\phi} \cdot\left[\mathbf{H}^{i}+\mathbf{H}^{s}-\mathbf{H}^{t}\right] & =0 \tag{3.165}
\end{align*}
$$

$$
\begin{align*}
a_{n}^{i} k_{1} J_{n}\left(k_{1} a\right)+b_{n}^{s} k_{1} H_{n}^{(2)}\left(k_{1} a\right)-a_{n}^{t} k_{R} J_{n}\left(k_{R} a\right)+j \eta_{2} k_{L} b_{n}^{t} J_{n}\left(k_{L} a\right) & =0 \\
+a_{n}^{s} H_{n}^{\prime(2)}\left(k_{1} a\right)-a_{n}^{t} k_{R} J_{n}^{\prime}\left(k_{R} a\right)-j \eta_{2} k_{L} b_{n}^{t} J_{n}^{\prime}\left(k_{L} a\right) & =0 \\
\frac{j}{\eta_{1}} a_{n}^{i} J_{n}^{\prime}\left(k_{1} a\right)+\frac{j}{\eta_{1}} b_{n}^{s} H_{n}^{\prime(2)}\left(k_{1} a\right)-\frac{j}{\eta_{2}} k_{R} a_{n}^{t} J_{n}^{\prime}\left(k_{R} a\right)-b_{n}^{t} k_{L} J_{n}^{\prime}\left(k_{L} a\right) & =0 \\
\frac{j}{\eta_{1}} k_{1} a_{n}^{s} H_{n}^{(2)}\left(k_{1} a\right)-\frac{1}{\eta_{2}} k_{R} a_{n}^{t} J_{n}\left(k_{R} a\right)+b_{n}^{t} k_{L} J_{n}\left(k_{L} a\right) & =0 \tag{3.166}
\end{align*}
$$

$$
\left[\begin{array}{cccc}
-k_{1} H_{n}^{(2)}\left(k_{0} a\right) & k_{R} J_{n}\left(k_{R} a\right) & -j \eta_{2} k_{L} J_{n}\left(k_{L} a\right) & 0 \\
\frac{-j k_{1}}{\eta_{1}} H_{n}^{\prime(2)}\left(k_{1} a\right) & \frac{j k_{R}}{\eta_{2}} J_{n}^{\prime}\left(k_{R} a\right) & k_{L} J_{n}^{\prime}\left(k_{L} a\right) & 0 \\
0 & \frac{j k_{R}}{\eta_{2}} J_{n}\left(k_{R} a\right) & -k_{L} J_{n}\left(k_{L} a\right) & \frac{-j}{\eta_{1}} H_{n}^{(2)}\left(k_{1} a\right) \\
0 & k_{R} J_{n}^{\prime}\left(k_{R} a\right) & j \eta_{2} k_{L} J_{n}^{\prime}\left(k_{L} a\right) & -k_{1} H_{n}^{\prime(2)}\left(k_{1} a\right)
\end{array}\right]\left[\begin{array}{c}
b_{n}^{s} \\
a_{n}^{t} \\
b_{n}^{t} \\
a_{n}^{s}
\end{array}\right]=\left[\begin{array}{c}
a_{n}^{i} k_{1} J_{n}\left(k_{1} a\right) \\
\frac{j k_{1}}{\eta_{1}} a_{n}^{i} J_{n}^{\prime}\left(k_{1} a\right) \\
0 \\
0
\end{array}\right]
$$

and from these four equations we can determine the unknowns.

### 3.10.2 $T E^{z}$ polarization

The incident magnetic field is given by:

$$
\begin{equation*}
\mathbf{H}^{i}=H_{0} e^{j k_{1}\left(x \cos \phi_{0}+y \sin \phi_{0}\right)} \mathbf{a}_{z}=\sum_{n=-\infty}^{\infty} H_{0} j^{n} e^{-j n \phi_{0}} J_{n}\left(k_{1} \rho\right) e^{j n \phi} \mathbf{a}_{z} \tag{3.167}
\end{equation*}
$$

The incident wave can be written as:

$$
\begin{gather*}
\mathbf{H}^{i}=\sum_{n=-\infty}^{\infty} a_{n}^{i} \mathbf{N}_{n}^{(1)}\left(k_{1} \rho\right)  \tag{3.168}\\
\mathbf{E}^{i}=-j \eta_{1} \sum_{n=-\infty}^{\infty} a_{n}^{i} \mathbf{M}_{n}^{(1)}\left(k_{1} \rho\right) \tag{3.169}
\end{gather*}
$$

where $a_{n}^{i}=\frac{j^{n} H_{0}}{k_{1}} e^{-j n \phi_{0}}$. The scattered field will be

$$
\begin{gather*}
\mathbf{H}^{s}=\sum_{n=-\infty}^{\infty} a_{n}^{s} \mathbf{M}_{n}^{(4)}\left(k_{1} \rho\right)+b_{n}^{s} \mathbf{N}_{n}^{(4)}\left(k_{1} \rho\right)  \tag{3.170}\\
\mathbf{E}^{s}=-j \eta_{1} \sum_{n=-\infty}^{\infty} a_{n}^{s} \mathbf{N}_{n}^{(4)}\left(k_{1} \rho\right)+b_{n}^{s} \mathbf{M}_{n}^{(4)}\left(k_{1} \rho\right) \tag{3.171}
\end{gather*}
$$

and the fields inside the cylinder will be $\mathbf{E}^{t}=\mathbf{Q}_{R}+j \eta_{2} \mathbf{Q}_{R}, \mathbf{H}^{t}=\mathbf{Q}_{L}+\frac{j}{\eta_{2}} \mathbf{Q}_{R}$

$$
\begin{equation*}
\mathbf{Q}_{R}=\sum_{n=-\infty}^{\infty} a_{n}^{t}\left[\mathbf{M}_{n}^{(1)}\left(k_{R} \rho\right)-\mathbf{N}_{n}^{(1)}\left(k_{R} \rho\right)\right] \tag{3.172}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{Q}_{L}=\sum_{n=-\infty}^{\infty} b_{n}^{t}\left[\mathbf{M}_{n}^{(1)}\left(k_{L} \rho\right)+\mathbf{N}_{n}^{(1)}\left(k_{L} \rho\right)\right] \tag{3.173}
\end{equation*}
$$

where

$$
k_{R}=\frac{\omega \sqrt{\mu_{2} \epsilon_{2}}}{1+\beta \omega \sqrt{\mu_{2} \epsilon_{2}}}
$$

and

$$
k_{L}=\frac{\omega \sqrt{\mu_{2} \epsilon_{2}}}{1-\beta \omega \sqrt{\mu_{2} \epsilon_{2}}}
$$

Now we can apply the boundary conditions at $\rho=a$ to find the four unknown parameters $a_{n}^{s}, b_{n}^{s}, a_{n}^{t}$ and $b_{n}^{t}$ :

$$
\begin{align*}
& \mathbf{a}_{z} \cdot\left[\mathbf{E}^{i}+\mathbf{E}^{s}-\mathbf{E}^{t}\right]=0 \\
& \mathbf{a}_{\phi} \cdot\left[\mathbf{E}^{i}+\mathbf{E}^{s}-\mathbf{E}^{t}\right]=0 \\
& \mathbf{a}_{z} \cdot\left[\mathbf{H}^{i}+\mathbf{H}^{s}-\mathbf{H}^{t}\right]=0 \\
& \mathbf{a}_{\phi} \cdot\left[\mathbf{H}^{i}+\mathbf{H}^{s}-\mathbf{H}^{t}\right]=0  \tag{3.174}\\
& a_{n}^{i} k_{1} J_{n}\left(k_{1} a\right)+b_{n}^{s} k_{1} H_{n}^{(2)}\left(k_{1} a\right)-\frac{j}{\eta_{2}} a_{n}^{t} k_{R} J_{n}\left(k_{R} a\right)+k_{L} b_{n}^{t} J_{n}\left(k_{L} a\right)=0 \\
&+a_{n}^{s} k_{1} H_{n}^{\prime(2)}\left(k_{1} a\right)-\frac{j}{\eta_{2}} a_{n}^{t} k_{R} J_{n}^{\prime}\left(k_{R} a\right)-k_{L} b_{n}^{t} J_{n}^{\prime}\left(k_{L} a\right)=0 \\
&-j \eta_{1} k_{1} a_{n}^{i} J_{n}^{\prime}\left(k_{1} a\right)-j \eta_{1} k_{1} b_{n}^{s} H_{n}^{\prime(2)}\left(k_{1} a\right)-k_{R} a_{n}^{t} J_{n}^{\prime}\left(k_{R} a\right)-j \eta_{2} k_{L} b_{n}^{t} J_{n}^{\prime}\left(k_{L} a\right)=0 \\
&-j \eta_{1} k_{1} a_{n}^{s} H_{n}^{(2)}\left(k_{1} a\right)-k_{R} a_{n}^{t} J_{n}\left(k_{R} a\right)+j \eta_{2} b_{n}^{t} k_{L} J_{n}\left(k_{L} a\right)=0 \tag{3.175}
\end{align*}
$$

In matrix form;

$$
\left[\begin{array}{lrrr}
-k_{1} H_{n}^{(2)}\left(k_{1} a\right) & \frac{j k_{R}}{\eta_{2}} J_{n}\left(k_{R} a\right) & -k_{L} J_{n}\left(k_{L} a\right) & 0  \tag{3.176}\\
-j k_{1} \eta_{1} H_{n}^{\prime(2)}\left(k_{1} a\right) & -k_{R} J_{n}^{\prime}\left(k_{R} a\right) & -j \eta_{2} k_{L} J_{n}^{\prime}\left(k_{L} a\right) & 0 \\
0 & k_{R} J_{n}\left(k_{R} a\right) & -j \eta_{2} k_{L} J_{n}\left(k_{L} a\right) & j \eta_{1} k_{1} H_{n}^{(2)}\left(k_{1} a\right) \\
0 & \frac{j k_{R}}{\eta_{2}} J_{n}^{\prime}\left(k_{R} a\right) & j k_{L} J_{n}^{\prime}\left(k_{L} a\right) & k_{1} H_{n}^{\prime(2)}\left(k_{1} a\right)
\end{array}\right]
$$

$$
\left[\begin{array}{c}
b_{n}^{s} \\
a_{n}^{t} \\
b_{n}^{t} \\
a_{n}^{s}
\end{array}\right]=\left[\begin{array}{c}
a_{n}^{i} k_{1} J_{n}\left(k_{1} a\right) \\
j a_{n}^{i} \eta_{1} k_{1} J_{n}^{\prime}\left(k_{1} a\right) \\
0 \\
0
\end{array}\right]
$$

### 3.10.3 Scattering of EM waves from chirally coated circular cylinders

In this section we will consider a long circular chirally coated cylinder as shown in Fig.(3.19) We first consider the $T M^{z}$ case. The incident electric


Figure 3.19: Chirally coated infinite circular cylinder
field is given by:

$$
\begin{equation*}
\mathbf{E}^{i}=E_{0} e^{j k_{3}\left(x \cos \phi_{0}+y \sin \phi_{0}\right)} \mathbf{a}_{z}=\sum_{n=-\infty}^{\infty} E_{0} j^{n} e^{-j n \phi_{0}} J_{n}\left(k_{3} \rho\right) e^{j n \phi} \mathbf{a}_{z} \tag{3.177}
\end{equation*}
$$

The incident wave can be written as:

$$
\begin{align*}
\mathbf{E}^{i} & =\sum_{n=-\infty}^{\infty} a_{n}^{i} \mathbf{N}_{n}^{(1)}\left(k_{3} \rho\right)  \tag{3.178}\\
\mathbf{H}^{i} & =\frac{j}{\eta_{3}} \sum_{n=-\infty}^{\infty} a_{n}^{i} \mathbf{M}_{n}^{(1)}\left(k_{3} \rho\right) \tag{3.179}
\end{align*}
$$

where $a_{n}^{i}=\frac{j^{n} E_{0}}{k_{3}} e^{-j n \phi_{0}}$. The scattered field will be:

$$
\begin{gather*}
\mathbf{E}^{s}=\sum_{n=-\infty}^{\infty} a_{n}^{(3)} \mathbf{M}_{n}^{(4)}\left(k_{3} \rho\right)+b_{n}^{(3)} \mathbf{N}_{n}^{(4)}\left(k_{3} \rho\right)  \tag{3.180}\\
\mathbf{H}^{s}=\frac{j}{\eta_{3}} \sum_{n=-\infty}^{\infty} a_{n}^{(3)} \mathbf{N}_{n}^{(4)}\left(k_{3} \rho\right)+b_{n}^{(3)} \mathbf{M}_{n}^{(4)}\left(k_{3} \rho\right) \tag{3.181}
\end{gather*}
$$

and the fields inside the chiral layer will be $\mathbf{E}^{2}=\mathbf{Q}_{R}+j \eta_{2} \mathbf{Q}_{L}, \mathbf{H}^{2}=\mathbf{Q}_{L}+$ $\frac{j}{\eta_{2}} \mathbf{Q}_{R}$

$$
\begin{align*}
\mathbf{Q}_{R} & =\sum_{n=-\infty}^{\infty}\left\{a_{n}^{(2)}\left[\mathbf{M}_{n}^{(1)}\left(k_{R} \rho\right)-\mathbf{N}_{n}^{(1)}\left(k_{R} \rho\right)\right]+b_{n}^{(2)}\left[\mathbf{M}_{n}^{(4)}\left(k_{R} \rho\right)-\mathbf{N}_{n}^{(4)}\left(k_{R} \rho\right)\right]\right\}  \tag{3.182}\\
\mathbf{Q}_{L} & =\sum_{n=-\infty}^{\infty}\left\{c_{n}^{(2)}\left[\mathbf{M}_{n}^{(1)}\left(k_{L} \rho\right)+\mathbf{N}_{n}^{(1)}\left(k_{L} \rho\right)\right]+d_{n}^{(2)}\left[\mathbf{M}_{n}^{(4)}\left(k_{L} \rho\right)+\mathbf{N}_{n}^{(4)}\left(k_{L} \rho\right)\right]\right\} \tag{3.183}
\end{align*}
$$

where

$$
k_{R}=\frac{\omega \sqrt{\mu_{2} \epsilon_{2}}}{1+\beta \omega \sqrt{\mu_{2} \epsilon_{2}}}
$$

and

$$
k_{L}=\frac{\omega \sqrt{\mu_{2} \epsilon_{2}}}{1-\beta \omega \sqrt{\mu_{2} \epsilon_{2}}}
$$

The fields in the innermost layer are given as follows

$$
\begin{equation*}
\mathbf{E}^{(1)}=\sum_{n=-\infty}^{\infty} a_{n}^{(1)} \mathbf{M}_{n}^{(1)}\left(k_{1} \rho\right)+b_{n}^{(1)} \mathbf{N}_{n}^{(1)}\left(k_{1} \rho\right) \tag{3.184}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{H}^{(1)}=\frac{j}{\eta_{1}} \sum_{n=-\infty}^{\infty} a_{n}^{(1)} \mathbf{N}_{n}^{(1)}\left(k_{1} \rho\right)+b_{n}^{(1)} \mathbf{M}_{n}^{(1)}\left(k_{1} \rho\right) \tag{3.185}
\end{equation*}
$$

Now the boundary conditions are applied to find the eight unknown coefficients of $a_{n}^{(3)}, b_{n}^{(3)}, a_{n}^{(2)}, b_{n}^{(2)}, c_{n}^{(2)}, d_{n}^{(2)}, a_{n}^{(1)}$ and $b_{n}^{(1)}$.

$$
\begin{aligned}
& \mathbf{a}_{z} \cdot\left[\mathbf{E}^{i}+\mathbf{E}^{s}-\mathbf{E}^{(2)}\right]=0 \mathbf{a}_{\phi} \cdot\left[\mathbf{H}^{i}+\mathbf{H}^{s}-\mathbf{H}^{(2)}\right]=0 \\
& \mathbf{a}_{z} \cdot\left[\mathbf{E}^{(2)}-\mathbf{E}^{(1)}\right]=0 \mathbf{a}_{\phi} \cdot\left[\mathbf{H}^{(2)}-\mathbf{H}^{(1)}\right]=0 \quad \rho=b \\
& \hline=a
\end{aligned}
$$

For the $T E^{z}$ case. The incident magnetic field is given by:

$$
\begin{equation*}
\mathbf{H}^{i}=H_{0} e^{j k_{3}\left(x \cos \phi_{0}+y \sin \phi_{0}\right)} \mathbf{a}_{z}=\sum_{n=-\infty}^{\infty} H_{0} j^{n} e^{-j n \phi_{0}} J_{n}\left(k_{3} \rho\right) e^{j n \phi} \mathbf{a}_{z} \tag{3.186}
\end{equation*}
$$

The incident wave can be written as:

$$
\begin{gather*}
\mathbf{H}^{i}=\sum_{n=-\infty}^{\infty} a_{n}^{i} \mathbf{N}_{n}^{(1)}\left(k_{3} \rho\right)  \tag{3.187}\\
\mathbf{E}^{i}=-j \eta_{3} \sum_{n=-\infty}^{\infty} a_{n}^{i} \mathbf{M}_{n}^{(1)}\left(k_{3} \rho\right) \tag{3.188}
\end{gather*}
$$

where $a_{n}^{i}=\frac{j^{n} H_{0}}{k_{3}} e^{-j n \phi_{0}}$. The scattered field will be:

$$
\begin{gather*}
\mathbf{H}^{s}=\sum_{n=-\infty}^{\infty} a_{n}^{(3)} \mathbf{M}_{n}^{(4)}\left(k_{3} \rho\right)+b_{n}^{(3)} \mathbf{N}_{n}^{(4)}\left(k_{3} \rho\right)  \tag{3.189}\\
\mathbf{E}^{s}=-j \eta_{3} \sum_{n=-\infty}^{\infty} a_{n}^{(3)} \mathbf{N}_{n}^{(4)}\left(k_{3} \rho\right)+b_{n}^{(3)} \mathbf{M}_{n}^{(4)}\left(k_{3} \rho\right) \tag{3.190}
\end{gather*}
$$

and the fields inside the chiral layer is the same as those for $T M^{z} ; \mathbf{E}^{(2)}=$ $\mathbf{Q}_{R}+j \eta_{2} \mathbf{Q}_{L}, \mathbf{H}^{(2)}=\mathbf{Q}_{L}+\frac{j}{\eta_{2}} \mathbf{Q}_{R}$. The fields in the innermost layer are given as follows

$$
\begin{equation*}
\mathbf{H}^{(1)}=\sum_{n=-\infty}^{\infty} a_{n}^{(1)} \mathbf{M}_{n}^{(1)}\left(k_{1} \rho\right)+b_{n}^{(1)} \mathbf{N}_{n}^{(1)}\left(k_{1} \rho\right) \tag{3.191}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{E}^{(1)}=-j \eta_{1} \sum_{n=-\infty}^{\infty} a_{n}^{(1)} \mathbf{N}_{n}^{(1)}\left(k_{1} \rho\right)+b_{n}^{(1)} \mathbf{M}_{n}^{(1)}\left(k_{1} \rho\right) \tag{3.192}
\end{equation*}
$$

Now the boundary conditions are applied to find the eight unknown coefficients of $a_{n}^{(3)}, b_{n}^{(3)}, a_{n}^{(2)}, b_{n}^{(2)}, c_{n}^{(2)}, d_{n}^{(2)}, a_{n}^{(1)}$ and $b_{n}^{(1)}$ :

$$
\begin{aligned}
& \mathbf{a}_{z} \cdot\left[\mathbf{H}^{i}+\mathbf{H}^{s}-\mathbf{H}^{(2)}\right]=0 \mathbf{a}_{\phi} \cdot\left[\mathbf{E}^{i}+\mathbf{E}^{s}-\mathbf{E}^{(2)}\right]=0 \\
& \mathbf{a}_{z} \cdot\left[\mathbf{H}^{(2)}-\mathbf{H}^{(1)}\right]=0 \mathbf{a}_{\phi} \cdot\left[\mathbf{E}^{(2)}-\mathbf{E}^{(1)}\right]=0 \quad \rho=b \\
& \hline=a
\end{aligned}
$$

### 3.11 The addition theorem for circularly cylindrical waves

Before attempting to solve any scattering of waves by two or more eccentric circular cylinder, we need to review one theorem. In Fig.(3.20) $O_{1}$ and $O_{2}$ denote the origins of two rectangular coordinate systems. The figure is on the $x y$-plane of both systems and the axes $x_{2}, y_{2}$ and $z$ through $O_{2}$ are respectively parallel to $x_{1}, y_{1}$ and $z$. We wish to express the cylindrical wave in system $O_{1}$ in terms of a sum of cylindrical wave functions to parallel $z$-axis through $O_{2}$ :


Figure 3.20: Scattering from two eccentric cylinder

$$
\begin{equation*}
J_{n}\left(\beta \rho_{2}\right) e^{j n \phi_{2}}=\sum_{m=-\infty}^{\infty} J_{m}(\beta d) e^{-j m \phi_{d}} J_{n+m}\left(\beta \rho_{1}\right) e^{j(n+m) \phi_{1}} \tag{3.193}
\end{equation*}
$$

or

$$
\begin{equation*}
J_{n}\left(\beta \rho_{1}\right) e^{j n \phi_{1}}=\sum_{m=-\infty}^{\infty} J_{m}(\beta d) e^{-j m\left(\phi_{d}+\pi\right)} J_{n+m}\left(\beta \rho_{2}\right) e^{j(n+m) \phi_{2}} \tag{3.194}
\end{equation*}
$$

If $d=0$, the two centers coincide; $J_{m}(0)=0$ for all values of $m$ except zero and $J_{0}(0)=1$. For Hankel functions we have If $\rho_{1} \geq d$ then

$$
\begin{equation*}
H_{n}^{(2)}\left(\beta \rho_{2}\right) e^{j n \phi_{2}}=\sum_{m=-\infty}^{\infty} J_{m}(\beta d) e^{-j m \phi_{d}} H_{n+m}^{(2)}\left(\beta \rho_{1}\right) e^{j(n+m) \phi_{1}} \tag{3.195}
\end{equation*}
$$

and if $\rho_{1} \leq d$ the above equation fails to converge and replaced by:

$$
\begin{equation*}
H_{n}^{(2)}\left(\beta \rho_{2}\right) e^{j n \phi_{2}}=\sum_{m=-\infty}^{\infty} H_{m}^{(2)}(\beta d) e^{-j m \phi_{d}} J_{n+m}\left(\beta \rho_{1}\right) e^{j(n+m) \phi_{1}} \tag{3.196}
\end{equation*}
$$

which is finite at $\rho_{1}=0$. For $H_{n}^{(2)}\left(\beta \rho_{1}\right) e^{j n \phi_{1}}$ we have If $\rho_{2} \geq d$ then

$$
\begin{equation*}
H_{n}^{(2)}\left(\beta \rho_{1}\right) e^{j n \phi_{1}}=\sum_{m=-\infty}^{\infty} J_{m}(\beta d) e^{-j m\left(\phi_{d}+\pi\right)} H_{n+m}^{(2)}\left(\beta \rho_{2}\right) e^{j(n+m) \phi_{2}} \tag{3.197}
\end{equation*}
$$

and if $\rho_{2} \leq d$ the above equation fails to converge and replaced by

$$
\begin{equation*}
H_{n}^{(2)}\left(\beta \rho_{1}\right) e^{j n \phi_{1}}=\sum_{m=-\infty}^{\infty} H_{m}^{(2)}(\beta d) e^{-j m\left(\phi_{d}+\pi\right)} J_{n+m}\left(\beta \rho_{2}\right) e^{j(n+m) \phi_{2}} \tag{3.198}
\end{equation*}
$$

### 3.12 Scattering of a Plane Wave by two Eccentric Circular Cylinder

Now let us use this theorem in order to find scattering of a plane wave by two eccentric circular cylinder as shown in Fig.(3.21).
We consider $T M^{z}$ polarization and the incident plane wave can be expanded


Figure 3.21: Scattering from two eccentric cylinder
as:

$$
\begin{equation*}
E_{z}^{i}=e^{k_{0}\left(x_{1} \cos \phi_{0}+y_{1} \sin \phi_{0}\right)}=e^{j k_{0} \rho_{1} \cos \left(\phi_{1}-\phi_{0}\right)}=\sum_{n=-\infty}^{\infty} a_{n}^{0} J_{n}\left(k_{0} \rho_{1}\right) e^{j n \phi_{1}} \tag{3.199}
\end{equation*}
$$

where $a_{n}^{0}=j^{n} e^{-j n \phi_{0}}$ and the scattered field in medium 0 will be:

$$
\begin{equation*}
E_{z}^{s}=\sum_{n=-\infty}^{\infty} b_{n}^{0} H_{n}^{(2)}\left(k_{0} \rho_{1}\right) e^{j n \phi_{1}} \tag{3.200}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{\phi_{1}}^{s}=\frac{1}{j \eta_{0}} \sum_{n=-\infty}^{\infty} b_{n}^{0} H_{n}^{\prime(2)}\left(k_{0} \rho_{1}\right) e^{j n \phi_{1}} \tag{3.201}
\end{equation*}
$$

Therefore in free space we have

$$
\begin{array}{r}
E_{z}^{0}=\sum_{n=-\infty}^{\infty}\left[a_{n}^{0} J_{n}\left(k_{0} \rho_{1}\right)+b_{n}^{0} H_{n}^{(2)}\left(k_{0} \rho_{1}\right)\right] e^{j n \phi_{1}} \\
H_{\phi_{1}}^{0}=\frac{1}{j \eta_{0}} \sum_{n=-\infty}^{\infty}\left[a_{n}^{0} J_{n}^{\prime}\left(k_{0} \rho_{1}\right)+b_{n}^{0} H_{n}^{\prime(2)}\left(k_{0} \rho_{1}\right)\right] e^{j n \phi_{1}} \tag{3.203}
\end{array}
$$

where $b_{n}^{0}$ is unknown. In medium one we may face with two cases. Case I
if $d<R_{2}$ :

$$
\begin{equation*}
E_{z}^{1}=\sum_{n=-\infty}^{\infty}\left[a_{n}^{1} J_{n}\left(k_{1} \rho_{1}\right)+b_{n}^{1} H_{n}^{(2)}\left(k_{1} \rho_{1}\right)\right] e^{j n \phi_{1}} \tag{3.204}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{\phi_{1}}^{1}=\frac{1}{j \eta_{1}} \sum_{n=-\infty}^{\infty}\left[a_{n}^{1} J_{n}^{\prime}\left(k_{1} \rho_{1}\right)+b_{n}^{1} H_{n}^{\prime(2)}\left(k_{1} \rho_{1}\right)\right] e^{j n \phi_{1}} \tag{3.205}
\end{equation*}
$$

Case II
if $d \geq R_{2}$ :

$$
\begin{equation*}
E_{z}^{1}=\sum_{n=-\infty}^{\infty} a_{n}^{1} J_{n}\left(k_{1} \rho_{1}\right) e^{j n \phi_{1}} \tag{3.206}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{\phi_{1}}^{1}=\frac{1}{j \eta_{1}} \sum_{n=-\infty}^{\infty} a_{n}^{1} J_{n}^{\prime}\left(k_{1} \rho_{1}\right) e^{j n \phi_{1}} \tag{3.207}
\end{equation*}
$$

where $a_{n}^{1}$ and $b_{n}^{1}$ are unknown.
In medium two, only one unknown will be found, therefore

$$
\begin{equation*}
E_{z}^{2}=\sum_{n=-\infty}^{\infty} a_{n}^{2} J_{n}\left(k_{2} \rho_{2}\right) e^{j n \phi_{2}} \tag{3.208}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{\phi_{2}}^{2}=\frac{1}{j \eta_{2}} \sum_{n=-\infty}^{\infty} a_{n}^{2} J_{n}^{\prime}\left(k_{2} \rho_{2}\right) e^{j n \phi_{2}} \tag{3.209}
\end{equation*}
$$

In order to impose boundary conditions, it is better to expand $J_{n}\left(k_{1} \rho_{1}\right) e^{j n \phi_{1}}$ and $H_{n}^{(2)}\left(k_{1} \rho_{1}\right) e^{j n \phi_{1}}$ as:

$$
\begin{equation*}
J_{n}\left(k_{1} \rho_{1}\right) e^{j n \phi_{1}}=\sum_{m=-\infty}^{\infty} J_{m}\left(k_{1} d\right) e^{-j m\left(\phi_{d}+\pi\right)} J_{n+m}\left(k_{1} \rho_{2}\right) e^{j(n+m) \phi_{2}} \tag{3.210}
\end{equation*}
$$

and if $\rho_{2} \geq d$ :

$$
\begin{equation*}
H_{n}^{(2)}\left(k_{1} \rho_{1}\right) e^{j n \phi_{1}}=\sum_{m=-\infty}^{\infty} J_{m}\left(k_{1} d\right) e^{-j m\left(\phi_{d}+\pi\right)} H_{n+m}^{(2)}\left(k_{1} \rho_{2}\right) e^{j(n+m) \phi_{2}} \tag{3.211}
\end{equation*}
$$

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 where $\phi_{d}$ is the angle that vector $\overrightarrow{o_{2} O_{1}}$ makes with vector $\overrightarrow{o_{2} x_{2}}$. If $\rho_{2} \leq d$ :$$
\begin{equation*}
H_{n}^{(2)}\left(k_{1} \rho_{1}\right) e^{j n \phi_{1}}=\sum_{m=-\infty}^{\infty} H_{m}^{(2)}\left(k_{1} d\right) e^{-j m\left(\phi_{d}+\pi\right)} J_{n+m}\left(k_{1} \rho_{2}\right) e^{j(n+m) \phi_{2}} \tag{3.212}
\end{equation*}
$$

Therefore for case I

$$
\begin{align*}
E_{z}^{1} & =\sum_{n=-\infty}^{\infty} a_{n}^{1} \sum_{m=-\infty}^{\infty} J_{m}\left(k_{1} d\right) e^{-j m\left(\phi_{d}+\pi\right)} J_{n+m}\left(k_{1} \rho_{2}\right) e^{j(n+m) \phi_{2}}  \tag{3.213}\\
& +\sum_{n=-\infty}^{\infty} b_{n}^{1} \sum_{m=-\infty}^{\infty} H_{m}^{(2)}\left(k_{1} d\right) e^{-j m\left(\phi_{d}+\pi\right)} J_{n+m}\left(k_{1} \rho_{2}\right) e^{j(n+m) \phi_{2}}
\end{align*}
$$

then

$$
\begin{align*}
H_{\phi_{2}}^{1}= & \frac{1}{j \eta_{1}}\left\{\sum_{n=-\infty}^{\infty} a_{n}^{1} \sum_{m=-\infty}^{\infty} J_{m}\left(k_{1} d\right) e^{-j m\left(\phi_{d}+\pi\right)} J_{n+m}^{\prime}\left(k_{1} \rho_{2}\right) e^{j(n+m) \phi_{2}}\right.  \tag{3.214}\\
& \left.+\sum_{n=-\infty}^{\infty} b_{n}^{1} \sum_{m=-\infty}^{\infty} H_{m}^{(2)}\left(k_{1} d\right) e^{-j m\left(\phi_{d}+\pi\right)} J_{n+m}^{\prime}\left(k_{1} \rho_{2}\right) e^{j(n+m) \phi_{2}}\right\}
\end{align*}
$$

If $\rho_{2} \geq d$ and

$$
\begin{align*}
E_{z}^{1} & =\sum_{n=-\infty}^{\infty} a_{n}^{1} \sum_{m=-\infty}^{\infty} J_{m}\left(k_{1} d\right) e^{-j m\left(\phi_{d}+\pi\right)} J_{n+m}\left(k_{1} \rho_{2}\right) e^{j(n+m) \phi_{2}}  \tag{3.215}\\
& +\sum_{n=-\infty}^{\infty} b_{n}^{1} \sum_{m=-\infty}^{\infty} J_{m}\left(k_{1} d\right) e^{-j m\left(\phi_{d}+\pi\right)} H_{n+m}^{(2)}\left(k_{1} \rho_{2}\right) e^{j(n+m) \phi_{2}}
\end{align*}
$$

then

$$
\begin{align*}
& H_{\phi_{2}}^{1}=\frac{1}{j \eta_{1}}\left[\sum_{n=-\infty}^{\infty} a_{n}^{1} \sum_{m=-\infty}^{\infty} J_{m}\left(k_{1} d\right) e^{-j m\left(\phi_{d}+\pi\right)} J_{n+m}^{\prime}\left(k_{1} \rho_{2}\right) e^{j(n+m) \phi_{2}}\right.  \tag{3.216}\\
&\left.+\sum_{n=-\infty}^{\infty} b_{n}^{1} \sum_{m=-\infty}^{\infty} J_{m}\left(k_{1} d\right) e^{-j m\left(\phi_{d}+\pi\right)} H_{n+m}^{\prime(2)}\left(k_{1} \rho_{2}\right) e^{j(n+m) \phi_{2}}\right]
\end{align*}
$$

it is clear that for case II, the $b_{n}^{1}$ will be vanished, $b_{n}^{1}=0$.
By applying boundary conditions to the continuity of the $E_{z}$ and $H_{\phi}$ fields
at the surface of outer cylinder $\rho_{1}=r_{1}$ for any $\phi_{1}$. For case I we have:

$$
\begin{align*}
& \sum_{n=-\infty}^{\infty} a_{n}^{0} J_{n}\left(k_{0} r_{1}\right) e^{j n \phi_{1}}+\sum_{n=-\infty}^{\infty} b_{n}^{0} H_{n}^{(2)}\left(k_{0} r_{1}\right) e^{j n \phi_{1}}  \tag{3.217}\\
= & \sum_{n=-\infty}^{\infty} a_{n}^{1} J_{n}\left(k_{1} r_{1}\right) e^{j n \phi_{1}}+\sum_{n=-\infty}^{\infty} b_{n}^{1} H_{n}^{(2)}\left(k_{1} r_{1}\right) e^{j n \phi_{1}}
\end{align*}
$$

By using orthogonality

$$
\begin{align*}
a_{n}^{0} J_{n}\left(k_{0} r_{1}\right)+b_{n}^{0} H_{n}^{(2)}\left(k_{0} r_{1}\right) & =a_{n}^{1} J_{n}\left(k_{1} r_{1}\right)+b_{n}^{1} H_{n}^{(2)}\left(k_{1} r_{1}\right)  \tag{3.218}\\
\frac{1}{\eta_{0}}\left[a_{n}^{0} J_{n}^{\prime}\left(k_{0} r_{1}\right)+b_{n}^{0} H_{n}^{\prime(2)}\left(k_{0} r_{1}\right)\right] & =\frac{1}{\eta_{1}}\left[a_{n}^{1} J_{n}^{\prime}\left(k_{1} r_{1}\right)+b_{n}^{1} H_{n}^{\prime(2)}\left(k_{1} r_{1}\right)\right] \tag{3.219}
\end{align*}
$$

We denote $U_{m}=J_{m}\left(k_{1} d\right) e^{j m\left(\phi_{2}-\phi_{d}-\pi\right)}$ and $V_{m}=H_{m}^{(2)}\left(k_{1} d\right) e^{j m\left(\phi_{2}-\phi_{d}-\pi\right)}$ therefore at $\rho_{2}=r_{2}$ for any $\phi_{2}$, If $r_{2} \geq d$, we have:

$$
\begin{array}{r}
a_{n}^{1} \sum_{m=-\infty}^{\infty} U_{m} J_{n+m}\left(k_{1} r_{2}\right)+b_{n}^{1} \sum_{m=-\infty}^{\infty} U_{m} H_{n+m}^{(2)}\left(k_{1} r_{2}\right)=a_{n}^{2} J_{n}\left(k_{2} r_{2}\right) \\
\frac{1}{\eta_{1}}\left[a_{n}^{1} \sum_{m=-\infty}^{\infty} U_{m} J_{n+m}^{\prime}\left(k_{1} r_{2}\right)+b_{n}^{1} \sum_{m=-\infty}^{\infty} U_{m} H_{n+m}^{\prime(2)}\left(k_{1} r_{2}\right)\right]  \tag{3.221}\\
=\frac{1}{\eta_{2}} a_{n}^{2} J_{n}^{\prime}\left(k_{2} r_{2}\right)
\end{array}
$$

and if $r_{2} \leq d$, we have

$$
\begin{array}{r}
a_{n}^{1} \sum_{m=-\infty}^{\infty} U_{m} J_{n+m}\left(k_{1} r_{2}\right)+b_{n}^{1} \sum_{m=-\infty}^{\infty} V_{m} J_{n+m}\left(k_{1} r_{2}\right)=a_{n}^{2} J_{n}\left(k_{2} r_{2}\right) \\
\frac{1}{\eta_{1}}\left[a_{n}^{1} \sum_{m=-\infty}^{\infty} U_{m} J_{n+m}^{\prime}\left(k_{1} r_{2}\right)+b_{n}^{1} \sum_{m=-\infty}^{\infty} V_{m} J_{n+m}^{\prime}\left(k_{1} r_{2}\right)\right]  \tag{3.223}\\
=\frac{1}{\eta_{2}} a_{n}^{2} J_{n}^{\prime}\left(k_{2} r_{2}\right)
\end{array}
$$

Therefore if $r_{2} \geq d$ we can find unknowns coefficients:

$$
\left[\begin{array}{cccc}
A_{n}^{11} & A_{n}^{12} & A_{n}^{13} & 0  \tag{3.224}\\
A_{n}^{21} & A_{n}^{22} & A_{n}^{23} & 0 \\
0 & A_{n}^{32} & A_{n}^{33} & A_{n}^{34} \\
0 & A_{n}^{42} & A_{n}^{43} & A_{n}^{44}
\end{array}\right]\left[\begin{array}{c}
b_{n}^{0} \\
a_{n}^{1} \\
b_{n}^{1} \\
a_{n}^{2}
\end{array}\right]=\left[\begin{array}{c}
a_{n}^{0} J_{n}\left(k_{0} r_{1}\right) \\
\frac{1}{\eta_{0}} a_{n}^{0} J_{n}^{\prime}\left(k_{0} r_{1}\right) \\
0 \\
0
\end{array}\right]
$$

where:

$$
\begin{aligned}
& A_{n}^{11}=-H_{n}^{(2)}\left(k_{0} r_{1}\right), A_{n}^{12}=J_{n}\left(k_{1} r_{1}\right), A_{n}^{13}=H_{n}^{(2)}\left(k_{1} r_{1}\right), \\
& A_{n}^{21}=\frac{-1}{\eta_{0}} H_{n}^{\prime(2)}\left(k_{0} r_{1}\right), A_{n}^{22}=\frac{1}{\eta_{1}} J_{n}^{\prime}\left(k_{1} r_{1}\right), A_{n}^{23}=\frac{1}{\eta_{1}} H_{n}^{\prime(2)}\left(k_{1} r_{1}\right) \\
& A_{n}^{32}=-\widetilde{\sum} U_{m} J_{n+m}\left(k_{1} r_{2}\right), A_{n}^{33}=-\widetilde{\sum} U_{m} H_{n+m}^{(2)}\left(k_{1} r_{2}\right), A_{n}^{34}=J_{n}\left(k_{2} r_{2}\right), \\
& A_{n}^{42}=\frac{-1}{\eta_{1}} \sum U_{m} J_{n+m}^{\prime}\left(k_{1} r_{2}\right), A_{n}^{43}=\frac{-1}{\eta_{1}} \sum U_{m} H_{n+m}^{\prime(2)}\left(k_{1} r_{2}\right), A_{n}^{44}=\frac{1}{\eta_{2}} J_{n}^{\prime}\left(k_{2} r_{2}\right)
\end{aligned}
$$

If $r_{2} \leq d$
$A_{n}^{11}=-H_{n}^{(2)}\left(k_{0} r_{1}\right), A_{n}^{12}=J_{n}\left(k_{1} r_{1}\right), A_{n}^{13}=H_{n}^{(2)}\left(k_{1} r_{1}\right)$,
$A_{n}^{21}=\frac{-1}{\eta_{0}} H_{n}^{\prime(2)}\left(k_{0} r_{1}\right), A_{n}^{22}=\frac{1}{\eta_{1}} J_{n}^{\prime}\left(k_{1} r_{1}\right), A_{n}^{23}=\frac{1}{\eta_{1}} H_{n}^{\prime(2)}\left(k_{1} r_{1}\right)$
$A_{n}^{32}=-\widetilde{\sum} U_{m} J_{n+m}\left(k_{1} r_{2}\right), A_{n}^{33}=-\widetilde{\sum} V_{m} J_{n+m}\left(k_{1} r_{2}\right), A_{n}^{34}=J_{n}\left(k_{2} r_{2}\right)$,
$A_{n}^{42}=\frac{-1}{\eta_{1}} \sum U_{m} J_{n+m}^{\prime}\left(k_{1} r_{2}\right), A_{n}^{43}=\frac{-1}{\eta_{1}} \sum V_{m} J_{n+m}^{\prime}\left(k_{1} r_{2}\right), A_{n}^{44}=\frac{1}{\eta_{2}} J_{n}^{\prime}\left(k_{2} r_{2}\right)$
where in the above matrix we denoted $\widetilde{\sum}=\sum_{m=-\infty}^{\infty}$.
For case II, since $b_{n}^{1}=0$, the matrix dimension will be reduced.

### 3.13 Scattering from a Semicircular Channel in a Ground Plane

Scattering from geometries with channels, grooves and cracks have received considerable attention due to fact that these local guiding structures may excite internal resonances which may have dramatic effects on the nearby electromagnetic structures. The investigations show that when these structures are loaded with dielectric materials, the overall scattering patterns change
significantly, and thus, it is important to obtain an analytical solution to predict exactly the new scattering behavior. Before attempting to solve this problem, let us solve a general problem. As shown in Fig.(3.22), a normally incident plane wave makes an angle $\phi^{i n c}$ with the positive $x$ axis, the circular lossy dielectric cylinder along the $z$ axis is of radius $\rho=a$ with electrical parameters $\mu_{2}, \epsilon_{2}, \sigma_{2}$, and the direction of scattering is given by the angle $\phi$ with the $x$ axis. The surrounding area have $\mu_{1}, \epsilon_{1}$ and $\sigma_{1}$. For the incident $T M$ plane wave, the $z$-component of the electric field can be expanded in cylindrical waves as:


Figure 3.22: Scattering from the semicircular channel

$$
\begin{equation*}
E_{z}^{i n c}=e^{j k_{1} \rho \cos \left(\phi-\phi^{i n c}\right)}=\sum_{n=-\infty}^{\infty} j^{n} J_{n}\left(k_{1} \rho\right) e^{j n\left(\phi-\phi^{i n c}\right)} \tag{3.225}
\end{equation*}
$$

The scattered field for $\rho \geq a$ may be expressed as the sum of two parts, the reflected $T M$ plane wave given by

$$
\begin{equation*}
E_{z}^{r e f}=-e^{j k_{1} \rho \cos \left(\phi+\phi^{i n c}\right)}=-\sum_{n=-\infty}^{\infty} j^{n} J_{n}\left(k_{1} \rho\right) e^{j n\left(\phi+\phi^{i n c}\right)} \tag{3.226}
\end{equation*}
$$

where $k_{1}=\omega \sqrt{\mu_{1} \epsilon_{0} \epsilon_{r c 1}}, \epsilon_{r c 1}=\epsilon_{r 1}-j \frac{\sigma_{1}}{\omega \epsilon_{0}}$ and the diffracted field expanded with Hankel functions as:

$$
\begin{equation*}
E_{z}^{d i f}=\sum_{n=1}^{\infty} A_{n} H_{n}^{(2)}\left(k_{1} \rho\right) \sin (n \phi) \quad(\rho \geq a) \tag{3.227}
\end{equation*}
$$

where the $A_{n}$ are the unknown modal coefficients. The diffracted field vanishes on the ground plane so that total field for $\rho \geq a$ also vanishes there. In the interior region $(\rho \leq a)$ the electric field can be expanded with Bessel function as:

$$
\begin{equation*}
E_{z}^{i n c}=\sum_{n=0}^{\infty} J_{n}\left(k_{2} \rho\right)\left[B_{n} \cos (n \phi)-C_{n} \sin (n \phi)\right] \quad\left(C_{0}=0\right) \quad(\rho \leq a) \tag{3.228}
\end{equation*}
$$

where $k_{2}=\omega \sqrt{\mu_{2} \epsilon_{0} \epsilon_{r c 2}}, \epsilon_{r c 2}=\epsilon_{r 2}-j \frac{\sigma_{2}}{\omega \epsilon_{0}}$ and the $B_{n}$ and $C_{n}$ are two more sets of modal coefficients which along with $A_{n}$, will be determined from the boundary conditions at $\rho=a$. From Maxwell's equations we write the $\phi$-component of the magnetic field as $H_{\phi}=\frac{1}{j \omega \mu} \frac{\partial E_{z}}{\partial \rho}$, so that the boundary conditions at $\rho=a$ of zero tangential electric field on the region $(\pi<\phi<2 \pi)$ and continuous field on upper part of the circular cylinder $(0<\phi<\pi)$ become:

$$
\begin{gather*}
\sum_{n=0}^{\infty} B_{n} J_{n}\left(k_{2} a\right) \cos (n \phi)+\sum_{n=1}^{\infty} C_{n} J_{n}\left(k_{2} a\right) \sin (n \phi)=0  \tag{3.229}\\
\left.\sum_{n=0}^{\infty} B_{n} J_{n}\left(k_{2} a\right) \cos (n \phi)+\sum_{n=1}^{\infty} C_{n} J_{n}\left(k_{2} a\right) \sin (n \phi)=2 \pi\right)  \tag{3.230}\\
\sum_{n=1}^{\infty}\left[4 j^{n} J_{n}\left(k_{1} a\right) \sin \left(n \phi^{i n c}\right)+A_{n} H_{n}^{(2)}\left(k_{1} a\right)\right] \sin (n \phi) \\
(0<\phi<\pi) \\
\sum_{n=0}^{\infty} B_{n} J_{n}^{\prime}\left(k_{2} a\right) \cos (n \phi) / \eta_{2}+\sum_{n=1}^{\infty} C_{n} J_{n}^{\prime}\left(k_{2} a\right) \sin (n \phi) / \eta_{2}=  \tag{3.231}\\
\sum_{n=1}^{\infty}\left[4 j^{n} J_{n}^{\prime}\left(k_{1} a\right) \sin \left(n \phi^{i n c}\right) / \eta_{1}+A_{n} H_{n}^{(2)^{\prime}}\left(k_{1} a\right)\right] \sin (n \phi) / \eta_{1} \\
(0<\phi<\pi)
\end{gather*}
$$

where $\eta=\sqrt{\frac{\mu}{\epsilon_{0} \epsilon_{r c}}}, \epsilon_{r c}=\epsilon_{r}-j \frac{\sigma}{\omega \epsilon_{0}}$. In order to find $A_{n}, B_{n}$ and $C_{n}$ it is better to define partially orthogonal integrals:

$$
\begin{gather*}
\mathbf{I}_{n m}^{c s}=\int_{0}^{\pi} \cos (n \phi) \sin (m \phi) d \phi= \begin{cases}\frac{m}{m^{2}-n^{2}}(1-\cos m \pi \cos n \pi) & m \neq n \\
0 & m=n\end{cases}  \tag{3.232}\\
\mathbf{I}_{n m}^{s s}=\int_{0}^{\pi} \sin (n \phi) \sin (m \phi) d \phi= \begin{cases}0 & m \neq n \\
\frac{\pi}{2} & m=n\end{cases}  \tag{3.233}\\
\mathbf{K}_{n m}^{c s}=\int_{\pi}^{2 \pi} \cos (n \phi) \sin (m \phi) d \phi= \begin{cases}\frac{m}{m^{2}-n^{2}}(\cos m \pi \cos n \pi-1) & m \neq n \\
0\end{cases}  \tag{3.234}\\
\mathbf{K}_{n m}^{s s}=\int_{\pi}^{2 \pi} \sin (n \phi) \sin (m \phi) d \phi= \begin{cases}0 & m \neq n \\
\frac{\pi}{2} & m=n\end{cases} \tag{3.235}
\end{gather*}
$$

By using these integrals we can now solve the problem. If we multiply both sides of Eq.(3.229), Eq.(3.230) and Eq.(3.232) by $\sin (m \phi)$ and integrate over proper boundary, ie 0 to $\pi$ or $\pi$ to $2 \pi$, we will have a set of linear equations:

$$
\begin{align*}
-A_{1} H_{1}^{(2)}\left(k_{1} a\right) \mathbf{I}_{1 m}^{s s}-A_{2} H_{2}^{(2)}\left(k_{1} a\right) \mathbf{I}_{2 m}^{s s}-\cdots-A_{M} H_{M}^{(2)}\left(k_{1} a\right) \mathbf{I}_{M m}^{s s} & + \\
B_{0} J_{0}\left(k_{2} a\right) \mathbf{I}_{0 m}^{c s}+B_{1} J_{1}\left(k_{2} a\right) \mathbf{I}_{1 m}^{c s}+\cdots+B_{M-1} J_{M-1}\left(k_{2} a\right) \mathbf{I}_{M-1 m}^{c s} & + \\
C_{1} J_{1}\left(k_{2} a\right) \mathbf{I}_{1 m}^{s s}+C_{2} J_{2}\left(k_{2} a\right) \mathbf{I}_{2 m}^{s s}+\cdots+C_{M} J_{M}\left(k_{2} a\right) \mathbf{I}_{M m}^{s s} & = \\
D_{1} J_{1}\left(k_{1} a\right) \mathbf{I}_{1 m}^{s s}+D_{2} J_{2}\left(k_{1} a\right) \mathbf{I}_{2 m}^{s s}+\cdots+D_{M} J_{M}\left(k_{1} a\right) \mathbf{I}_{M m}^{s s} & \tag{3.236}
\end{align*}
$$

where $D_{n}=4 j^{n} \sin n \phi^{i n c}$

$$
\begin{align*}
-A_{1} H_{1}^{\prime(2)}\left(k_{1} a\right) \mathbf{I}_{1 m}^{s s} / \eta_{1}-A_{2} H_{2}^{\prime(2)}\left(k_{1} a\right) \mathbf{I}_{2 m}^{s s} / \eta_{1}-\cdots-A_{M} H_{M}^{\prime(2)}\left(k_{1} a\right) \mathbf{I}_{M m}^{s s} / \eta_{1} & + \\
B_{0} J_{0}^{\prime}\left(k_{2} a\right) \mathbf{I}_{0 m}^{c s} / \eta_{2}+B_{1} J_{1}^{\prime}\left(k_{2} a\right) \mathbf{I}_{1 m}^{c s} / \eta_{2}+\cdots+B_{M-1} J_{M-1}^{\prime}\left(k_{2} a\right) \mathbf{I}_{M-1 m}^{c s} / \eta_{2} & + \\
C_{1} J_{1}^{\prime}\left(k_{2} a\right) \mathbf{I}_{1 m}^{s s} / \eta_{2}+C_{2} J_{2}^{\prime}\left(k_{2} a\right) \mathbf{I}_{2 m}^{s s} / \eta_{2}+\cdots+C_{M} J_{M}^{\prime}\left(k_{2} a\right) \mathbf{I}_{M m}^{s s} / \eta_{2} & = \\
D_{1} J_{1}^{\prime}\left(k_{1} a\right) \mathbf{I}_{1 m}^{s s} / \eta_{1}+D_{2} J_{2}^{\prime}\left(k_{1} a\right) \mathbf{I}_{2 m}^{s s} / \eta_{1}+\cdots+D_{M} J_{M}^{\prime}\left(k_{1} a\right) \mathbf{I}_{M m}^{s s} / \eta_{1} & \tag{3.237}
\end{align*}
$$

$$
\begin{array}{r}
B_{0} J_{0}\left(k_{2} a\right) \mathbf{K}_{0 m}^{c s}+B_{1} J_{1}\left(k_{2} a\right) \mathbf{K}_{1 m}^{c s}+\cdots+B_{M-1} J_{M-1}\left(k_{2} a\right) \mathbf{K}_{M-1 m}^{c s}+ \\
\quad C_{1} J_{1}\left(k_{2} a\right) \mathbf{K}_{1 m}^{s s}+C_{2} J_{2}\left(k_{2} a\right) \mathbf{K}_{2 m}^{s s}+\cdots+C_{M} J_{M}\left(k_{2} a\right) \mathbf{K}_{M m}^{s s}=0 \tag{3.238}
\end{array}
$$

If we substitute $m=1, \cdots, M$ in the above equations, we will have a square block matrix of $3 M \times 3 M$ which the $3 M$ unknowns can be calculated.

$$
\begin{gather*}
{\left[\begin{array}{lll}
\mathbf{A}_{m n}^{11} & \mathbf{A}_{m n}^{12} & \mathbf{A}_{m n}^{13} \\
\mathbf{A}_{m n}^{21} & \mathbf{A}_{m n}^{22} & \mathbf{A}_{m n}^{23} \\
\mathbf{A}_{m n}^{31} & \mathbf{A}_{m n}^{32} & \mathbf{A}_{m n}^{33}
\end{array}\right]\left[\begin{array}{l}
\mathbf{X}_{n}^{1} \\
\mathbf{X}_{n}^{2} \\
\mathbf{X}_{n}^{3}
\end{array}\right]=\left[\begin{array}{l}
\mathbf{B}_{m}^{1} \\
\mathbf{B}_{m}^{2} \\
\mathbf{B}_{m}^{3}
\end{array}\right]}  \tag{3.239}\\
\mathbf{A}_{m n}^{11}=-H_{m}^{(2)}\left(k_{1} a\right) \mathbf{I}_{m n}^{s s}, \quad \mathbf{A}_{m n}^{12}=J_{m}\left(k_{2} a\right) \mathbf{I}_{m n}^{c s}, \quad \mathbf{A}_{m n}^{13}=J_{m}\left(k_{2} a\right) \mathbf{I}_{m n}^{s s} \\
\mathbf{A}_{m n}^{21}=-H_{m}^{\prime(2)}\left(k_{1} a\right) \mathbf{I}_{m n}^{s s} / \eta_{1}, \quad \mathbf{A}_{m n}^{22}=J_{m}^{\prime}\left(k_{2} a\right) \mathbf{I}_{m n}^{c s} / \eta_{2}, \quad \mathbf{A}_{m n}^{23}=J_{m}^{\prime}\left(k_{2} a\right) \mathbf{I}_{m n}^{s s} / \eta_{2} \\
\mathbf{A}_{m n}^{31}=0, \quad \mathbf{A}_{m n}^{32}=J_{m}\left(k_{2} a\right) \mathbf{K}_{m n}^{c s}, \quad \mathbf{A}_{m n}^{33}=J_{m}\left(k_{2} a\right) \mathbf{K}_{m n}^{s s}
\end{gather*}
$$

and right hand side of the matrix will be:

$$
\mathbf{B}_{m}^{1}=\sum_{n=1}^{M} D_{n} J_{n}\left(k_{1} a\right) \mathbf{I}_{n m}^{s s}, \quad \mathbf{B}_{m}^{2}=\sum_{n=1}^{M} D_{n} J_{n}^{\prime}\left(k_{1} a\right) \mathbf{I}_{n m}^{s s} / \eta_{1}, \quad \mathbf{B}_{m}^{3}=0, \quad m=1,2, \cdots M
$$

and the unknowns are:

$$
\begin{array}{lr}
\mathbf{X}_{n}^{1}=A_{n}, & n=1,2, \cdots, M \\
\mathbf{X}_{n}^{2}=B_{n}, & n=0,1, \cdots, M-1 \\
\mathbf{X}_{n}^{3}=C_{n}, & n=1,2, \cdots, M
\end{array}
$$

### 3.14 Problems

- 1 Find the characteristic equation, $T E^{z}$ and $T M^{z}$, of a sectoral waveguide that is shown in Fig.(3.23)
- 2 Find the characteristic equation of a coaxial waveguide with inner radius of $a$ and outer radius of $b$ for $T E^{z}$ and $T M^{z}$ modes.
- 3 Find the RCS of a lossless dielectric shell with $\mu=\mu_{0}$, dielectric constant $\epsilon_{r}=4$, inner radius $r_{1}=0.25 \lambda$ and outer radius $r_{2}=0.3 \lambda$, Fig.(3.24).


Figure 3.23: Sectoral Waveguide

- 4 Prove the relation:

$$
\sum_{n=-\infty}^{\infty} j^{n} Z_{n}^{g}(x) e^{j n \phi}=\sum_{n=0}^{\infty} j^{n} \epsilon_{n} Z_{n}^{g}(x) \cos (n \phi)
$$

where $Z_{n}^{g}(x) \quad g=1,2,3,4$ is cylindrical Bessel's function and

$$
\epsilon_{n}= \begin{cases}1 & n=0 \\ 2 & n \neq 0\end{cases}
$$

- 5 Consider a lossy dielectric cylinder shell with inner radius of $\lambda / 4$ and outer radius of $\lambda / 2$. A plane wave with frequency 1 GHz and amplitude 100 Volts per meter incident on it at angle $30^{\circ}$. Calculate and draw the electric field inside dielectric along X and Y axis. $\sigma=.1[S / \mathrm{m}]$, $\epsilon_{r}=50$.
- 6 We have a conducting cylinder with radius $a=\lambda / 2$. It is coated with lossy dielectric $\sigma=.1[S / m], \epsilon_{r}=50$. and thickness of $\lambda / 20$. Calculate and draw its 2D bistatic radar cross section at frequency of 1 GHz .
- 7 A uniform plane wave with incident angle of $\phi_{0}=\pi / 6$ hits a lossy dielectric cylinder which has a hollow cylindrical shape parallel to his longitudinal axis. Find RCS and electric field distribution inside lossy


Figure 3.24: Scattering by a dielectric shell
dielectric cylinder. Confirm your results with Method of Moment MoM. $R_{1}=\lambda / 2, R_{2}=\lambda / 8$ and located at $x_{1}=R 1 / 2, y_{1}=-R_{1} / 2$ with electrical parameters of $\epsilon_{1}=5 \epsilon_{0}, \sigma=0.1 S / m$ and $\mu_{1}=\mu_{0}$.

- 8 A line source is located on axial of smaller cylinder in previous problem, find the electromagnetic fields anywhere.
- 9 Find the scattering fields from the semicircular channel in a ground plane for TE polarization.


Figure 3.25: Scattering from two eccentric cylinders

## Chapter 4

## Elliptic Cylinder

"Imagination is more important than knowledge."
Albert Einstein

### 4.1 Helmholtz Equation in Elliptic Cylindrical Coordinates

Let us first define Elliptic Cylindrical Coordinates System. The $v$ coordinates are the asymptotic angle of confocal Parabola segments symmetrical about the $x$ axis. The $u$ coordinates are confocal Ellipses centered on the origin, Fig.(4.1).

$$
\begin{align*}
x & =f \cosh u \cos v \\
y & =f \sinh u \sin v \\
z & =z \tag{4.1}
\end{align*}
$$

where $u \in[0, \infty), v \in[0,2 \pi)$, and $z \in(-\infty,+\infty)$ and on the surface of elliptic cylinder (e.g. $u=u_{0}$ )

$$
\begin{align*}
a & =f \cosh \left(u_{0}\right) \\
b & =f \sinh \left(u_{0}\right) \\
f & =\sqrt{a^{2}-b^{2}} \tag{4.2}
\end{align*}
$$



Figure 4.1: Elliptic Coordinate System

They are related to Cartesian Coordinates by

$$
\begin{align*}
\frac{x^{2}}{f^{2} \cosh ^{2} u}+\frac{y^{2}}{f^{2} \sinh ^{2} u} & =1  \tag{4.3}\\
\frac{x^{2}}{f^{2} \cos ^{2} v}-\frac{y^{2}}{f^{2} \sin ^{2} v} & =1 \tag{4.4}
\end{align*}
$$

$$
\begin{align*}
\rho & =f \sqrt{\cosh ^{2} u-\sin ^{2} v} \\
\phi & =\tan ^{-1}(\tanh u \tan v) \tag{4.5}
\end{align*}
$$

$$
\begin{align*}
\rho_{1} & =\sqrt{(x-f)^{2}+y^{2}}=f(\cosh u+\cos v) \\
\rho_{2} & =\sqrt{(x+f)^{2}+y^{2}}=f(\cosh u-\cos v) \\
\cosh u & =\frac{\rho_{1}+\rho_{2}}{2 f} \\
\cos v & =\frac{\rho_{1}-\rho_{2}}{2 f} \tag{4.6}
\end{align*}
$$

The Scale Factors are

$$
\begin{align*}
h_{1} & =f \sqrt{\cosh ^{2} u-\cos ^{2} v}  \tag{4.7}\\
h_{2} & =h_{1}  \tag{4.8}\\
h_{3} & =1 \tag{4.9}
\end{align*}
$$

and unit vector will be

$$
\begin{align*}
& \mathbf{a}_{u}=\frac{1}{\sqrt{\cosh ^{2} u-\cos ^{2} v}}\left[\mathbf{a}_{x} \sinh u \cos v+\mathbf{a}_{y} \cosh u \sin v\right] \\
& \mathbf{a}_{v}=\frac{1}{\sqrt{\cosh ^{2} u-\cos ^{2} v}}\left[\mathbf{a}_{y} \sinh u \cos v-\mathbf{a}_{x} \cosh u \sin v\right] \tag{4.10}
\end{align*}
$$

therefore $\nabla \cdot \mathbf{E}, \nabla \times \mathbf{E}$ and $\nabla \psi$ will be

$$
\begin{align*}
\nabla \cdot \mathbf{E} & =\frac{1}{h_{1}^{2}}\left[\frac{\partial}{\partial u}\left(h_{1} E_{u}\right)+\frac{\partial}{\partial v}\left(h_{1} E_{v}\right)\right]+\frac{\partial E_{z}}{\partial z} \\
\nabla \times \mathbf{E} & =\left[\frac{1}{h_{1}} \frac{\partial E_{z}}{\partial v}-\frac{\partial E_{v}}{\partial z}\right] \mathbf{a}_{u}+\left[\frac{\partial E_{u}}{\partial z}-\frac{1}{h_{1}} \frac{\partial E_{z}}{\partial u}\right] \mathbf{a}_{v} \\
& +\frac{1}{h_{1}}\left[\frac{\partial}{\partial u}\left(h_{1} E_{v}\right)-\frac{\partial}{\partial v}\left(h_{1} E_{u}\right)\right] \mathbf{a}_{z} \\
\nabla \psi & =\frac{1}{h_{1}}\left(\frac{\partial \psi}{\partial u} \mathbf{a}_{u}+\frac{\partial \psi}{\partial v} \mathbf{a}_{v}\right)+\frac{\partial \psi}{\partial z} \mathbf{a}_{z} \tag{4.11}
\end{align*}
$$

The Helmholtz differential equation is

$$
\begin{equation*}
\frac{1}{f^{2}\left(\cosh ^{2} u-\cos ^{2} v\right)}\left(\frac{\partial^{2} \psi}{\partial u^{2}}+\frac{\partial^{2} \psi}{\partial v^{2}}\right)+\frac{\partial^{2} \psi}{\partial z^{2}}+k^{2} \psi=0 \tag{4.12}
\end{equation*}
$$

Attempt separation of variables by writing $\psi(u, v, z)=U(u) V(v) Z(z)$ then the Helmholtz differential equation becomes

$$
\begin{equation*}
\frac{\partial^{2} Z}{\partial z^{2}}+k_{z}^{2} Z=0 \tag{4.13}
\end{equation*}
$$

which has the solution

$$
\begin{align*}
& Z(z)=A \cos \left(k_{z} z\right)+B \sin \left(k_{z} z\right)  \tag{4.14}\\
& \frac{\partial^{2} V}{\partial v^{2}}+[\lambda-2 q \cos (2 v)] V=0  \tag{4.15}\\
& \frac{\partial^{2} U}{\partial u^{2}}-[\lambda-2 q \cosh (2 u)] U=0 \tag{4.16}
\end{align*}
$$

where $\lambda$ and $k_{z}$ are separation constants and

$$
\begin{equation*}
q=\frac{f^{2}\left(k^{2}-k_{z}^{2}\right)}{4} \tag{4.17}
\end{equation*}
$$

### 4.1.1 Mathieu Functions

The equation Eq.(4.15) is called the Mathieu differential equation. Mathieu's equation has solution for all values of $\lambda$ but in applications we need solutions which are periodic in $v$. The periodic solution of Eq.(4.15) occur only for certain characteristic values of $\lambda$ which depend on $q$. These solutions are either even or odd functions. The $c e_{m}(v, q)$ are even functions [like $\cos (x)$ ], and $s e_{m}(v, q)$ are odd functions [like $\left.\sin (x)\right]$. These solutions are known as Mathieu functions or in applicational usage angular functions. For computation of Mathieu functions, it is advantageous and efficient to expand them in Fourier series.

$$
\begin{align*}
c e_{2 r}(v, q) & =\sum_{k=0}^{\infty} A_{2 k}^{2 r}(q) \cos 2 k v \\
c e_{2 r+1}(v, q) & =\sum_{k=0}^{\infty} A_{2 k+1}^{2 r+1}(q) \cos (2 k+1) v \\
s e_{2 r+1}(v, q) & =\sum_{k=0}^{\infty} B_{2 k+1}^{2 r+1}(q) \sin (2 k+1) v \\
s e_{2 r+2}(v, q) & =\sum_{k=0}^{\infty} B_{2 k+2}^{2 r+2}(q) \sin (2 k+2) v \tag{4.18}
\end{align*}
$$

where $r=0,1,2, \cdots$ and $A_{k}^{r}$ and $B_{k}^{r}$ are the expansion coefficients to be determined. The Mathieu functions of even order are periodic with $\pi$ and functions

### 4.1. HELMHOLTZ EQUATION IN ELLIPTIC CYLINDRICAL COORDINATES 137

of odd order are periodic with $2 \pi$. The Fourier coefficients are computed with recurrence relations, obtained by inserting Eq.(4.18) into Eq.(4.15). This also yields infinite continued fraction transcendental equations for the determination of $\lambda$ for a given parameter $q$. The calculation of $\lambda$ and of the Fourier coefficients has been derived in many papers and books [McLachlan, 1951] and [Rengarajan and Lewis,1980]. The solutions of Mathieu's equation are chosen so that

$$
\begin{align*}
& \int_{0}^{2 \pi} c e_{m}^{2}(v, q) d v=\pi \\
& \int_{0}^{2 \pi} s e_{m}^{2}(v, q) d v=\pi \tag{4.19}
\end{align*}
$$

The corresponding values of $\lambda$ are frequently written $\lambda=a_{n}(q), \quad(n=$ $0,1,2, \cdots)$ for even functions and $\lambda=b_{n}(q), \quad(n=1,2, \cdots)$ for odd functions. The Mathieu's functions are orthogonal.

$$
\begin{align*}
& \int_{0}^{2 \pi} c e_{m}(v, q) c e_{n}(v, q) d v=\pi \delta_{m n} \\
& \int_{0}^{2 \pi} s e_{m}(v, q) s e_{n}(v, q) d v=\pi \delta_{m n} \\
& \int_{0}^{2 \pi} c e_{m}(v, q) s e_{n}(v, q) d v=0 \tag{4.20}
\end{align*}
$$

The recurrence and normalization relations leads us to four different matrix [Zhang and Jin, 1996]

- For $c e_{2 r}(v, q)$ the recurrence relation:

$$
\begin{align*}
a A_{0}^{2 r}-q A_{2}^{2 r} & =0 \\
(a-4) A_{2}^{2 r}-q\left(2 A_{0}^{2 r}+A_{4}^{2 r}\right) & =0 \\
{\left[a-(2 k)^{2}\right] A_{2 k}^{2 r}-q\left(A_{2 k-2}^{2 r}+A_{2 k+2}^{2 r}\right) } & =0 \tag{4.21}
\end{align*}
$$

for $k \geq 2$ and the normalization relation:

$$
\begin{equation*}
2\left(A_{0}^{2 r}\right)^{2}+\sum_{k=1}^{\infty}\left(A_{2 k}^{2 r}\right)^{2}=1 \tag{4.22}
\end{equation*}
$$

- For $c e_{2 r+1}(v, q)$ the recurrence relation:

$$
\begin{align*}
(a-1-q) A_{1}^{2 r+1}-q A_{3}^{2 r+1} & =0 \\
{\left[a-(2 k+1)^{2}\right] A_{2 k+1}^{2 r+1}-q\left(A_{2 k+1}^{2 r+1}+A_{2 k+3}^{2 r+1}\right) } & =0 \tag{4.23}
\end{align*}
$$

for $k \geq 1$ and the normalization relation:

$$
\begin{equation*}
\sum_{k=0}^{\infty}\left(A_{2 k+1}^{2 r+1}\right)^{2}=1 \tag{4.24}
\end{equation*}
$$

- For $s e_{2 r+1}(v, q)$ the recurrence relation:

$$
\begin{align*}
(b-1+q) B_{1}^{2 r+1}-q B_{3}^{2 r+1} & =0 \\
{\left[b-(2 k+1)^{2}\right] B_{2 k+1}^{2 r+1}-q\left(B_{2 k+1}^{2 r+1}+B_{2 k+3}^{2 r+1}\right) } & =0 \tag{4.25}
\end{align*}
$$

for $k \geq 1$ and the normalization relation:

$$
\begin{equation*}
\sum_{k=0}^{\infty}\left(B_{2 k+1}^{2 r+1}\right)^{2}=1 \tag{4.26}
\end{equation*}
$$

- For $s e_{2 r+2}(v, q)$ the recurrence relation:

$$
\begin{align*}
(b-4) B_{2}^{2 r+2}-q B_{4}^{2 r+2} & =0 \\
{\left[b-(2 k+2)^{2}\right] B_{2 k+2}^{2 r+2}-q\left(B_{2 k}^{2 r+2}+B_{2 k+4}^{2 r+2}\right) } & =0 \tag{4.27}
\end{align*}
$$

for $k \geq 1$. The normalization relation:

$$
\begin{equation*}
\sum_{k=0}^{\infty}\left(B_{2 k+2}^{2 r+2}\right)^{2}=1 \tag{4.28}
\end{equation*}
$$

The characteristic values $a(q)$ and $b(q)$ can be determined either by solving an eigenvalue problem or a transcendental equation. The first recurrence relation can be written into matrix form:

$$
\left[\begin{array}{ccccc}
a & -q & 0 & & \cdots  \tag{4.29}\\
-2 q & a-4 & -q & 0 & \cdots \\
0 & -q & a-16 & 0 & \cdots \\
& & \ddots & & \\
0 & \cdots & -q & a-(2 k)^{2} & -q \cdots \\
& & \ddots & &
\end{array}\right]\left[\begin{array}{c}
A_{0}^{2 r} \\
A_{2}^{2 r} \\
A_{4}^{2 r} \\
\vdots \\
A_{2 k}^{2 r} \\
\vdots
\end{array}\right]=0
$$

or in another form as $[\mathbf{C}][\mathbf{X}]=\lambda[\mathbf{X}]$. The above matrix like Eq.(4.29) has eigenvectors and eigenvalues which gives us characteristic value for Mathieu
function. For a given value of $q$, the values of the coefficients decreases as $k$ increase. We can truncate the number of linear equations. The truncation number gives the accuracy of eigenvalues and depend on the value of $q$. In practice, we only need a finite number of eigenvalues therefore truncation number is finite. In our problem we have four matrix which each one gives the proper eigenvalues.

- For $\mathrm{Ce}_{2 r}$

$$
\left[\mathbf{C}_{\mathbf{1}}\right]=\left[\begin{array}{lllll} 
& q & & &  \tag{4.30}\\
2 q & 4 & q & & \\
& q & 16 & 0 & \\
& & \ddots & & \\
& & q & (2 k)^{2} & q \\
& & & \ddots &
\end{array}\right]
$$

- For $c e_{2 r+1}$

$$
\left[\mathbf{C}_{\mathbf{2}}\right]=\left[\begin{array}{lllll}
1+q & q & & &  \tag{4.31}\\
q & 9 & q & & \\
& q & 25 & & \\
& & \ddots & & \\
& & q & (2 k+1)^{2} & q \\
& & & \ddots &
\end{array}\right]
$$

- For $s e_{2 r+1}$

$$
\left[\mathbf{C}_{\mathbf{3}}\right]=\left[\begin{array}{lllll}
1-q & q & & &  \tag{4.32}\\
q & 9 & q & & \\
& q & 25 & & \\
& & \ddots & & \\
& & q & (2 k+1)^{2} & q \\
& & & \ddots &
\end{array}\right]
$$

- For $s e_{2 r+2}$

$$
\left[\mathbf{C}_{4}\right]=\left[\begin{array}{lllll}
4 & q & & &  \tag{4.33}\\
q & 16 & q & & \\
& q & 36 & & \\
& & \ddots & & \\
& & q & (2 k+2)^{2} & q \\
& & & \ddots &
\end{array}\right]
$$

An alternative method for calculating the characteristic values is to solve transcendental equation.

$$
\begin{equation*}
(2 r+p)^{2}+T 1+T 2-\lambda=0 \tag{4.34}
\end{equation*}
$$

where $r=0,1,2, \cdots ; p=0$ for even order and $p=1$ for odd order of Mathieu function.

$$
\begin{align*}
& T 1=-\frac{q^{2}}{(2 r+2+p)^{2}-\lambda}-\frac{q^{2}}{(2 r+4+p)^{2}-\lambda-} \cdot \frac{q^{2}}{(2 r+6+p)^{2}-\lambda-} \cdots  \tag{4.35}\\
& T 2=-\frac{q^{2}}{(2 r-2+p)^{2}-\lambda}-\frac{q^{2}}{(2 r-4+p)^{2}-\lambda-} \cdots \cdot \frac{q^{2}}{(4+p)^{2}-\lambda-q^{2} / T_{0 i}^{(4.36)}}
\end{align*}
$$

where $i=1,2,3,4$ correspond to $c e_{2 r}(v, q), c e_{2 r+1}(v, q), s e_{2 r+1}(v, q), s e_{2 r+2}(v, q)$, respectively, and

$$
\begin{align*}
& T_{01}=4-\lambda+\frac{2 q^{2}}{\lambda}=a \\
& T_{02}=1+q-\lambda=a  \tag{4.37}\\
& T_{03}=1-q-\lambda=b \\
& T_{04}=4-\lambda=b
\end{align*}
$$

The characteristic values can be determined by solving Eq.(4.34) using a numerical method such as secant method with proper estimates which in case of complex characteristic values we find it by solving one of the proper matrix.

### 4.1.2 Modified Mathieu Functions

The Eq.(4.16) is called the modified Mathieu differential equation. Solutions to Modified Mathieu's equation Eq.(4.16) has the form $M c_{n}^{(g)}(u, q), \quad(n=$ $0,1,2, \cdots)$ and $M s_{n}^{(g)}(u, q), \quad(n=1,2, \cdots)$ where $g=1,2,3$ or 4 are related to Bessel's functions $Z_{n}^{(g)}=J_{n}, Y_{n}, H_{n}^{(1)}$ or $H_{n}^{(2)}$. These solutions are known as modified Mathieu functions or in applicational usage radial functions.

$$
\begin{array}{r}
M c_{2 r}^{(g)}(u, q)=\left[c e_{2 r}(0, q)\right]^{-1} \sum_{k=0}^{\infty}(-1)^{(k+r)} A_{2 k}^{2 r}(q) . \\
Z_{2 k}^{(g)}(2 \sqrt{q} \cosh u) \tag{4.38}
\end{array}
$$

$$
\begin{array}{r}
M c_{2 r+1}^{(g)}(u, q)=\left[c e_{2 r+1}(0, q)\right]^{-1} \sum_{k=0}^{\infty}(-1)^{(k+r)} \cdot A_{2 k+1}^{2 r+1}(q) . \\
Z_{2 k+1}^{(g)}(2 \sqrt{q} \cosh u) \\
M s_{2 r+1}^{(g)}(u, q)=\left[s e_{2 r+1}^{\prime}(0, q)\right]^{-1} \tanh u \cdot \sum_{k=0}^{\infty}(-1)^{(k+r)}(2 k+1) \\
B_{2 k+1}^{2 r+1}(q) Z_{2 k+1}^{(g)}(2 \sqrt{q} \cosh u) \\
M s_{2 r+2}^{(g)}(u, q)=\left[s e_{2 r+2}^{\prime}(0, q)\right]^{-1} \tanh u \cdot \sum_{k=0}^{\infty}(-1)^{(k+r)}(2 k+2) \\
B_{2 k+2}^{2 r+2}(q) Z_{2 k+2}^{(g)}(2 \sqrt{q} \cosh u) \tag{4.41}
\end{array}
$$

### 4.2 Elliptic Waveguide

For a waveguide whose cross section is of the form of an ellipse is depicted in Fig.(4.2). For a cylindrical waveguide (i.e., of constant cross section), the z dependence of the electric and magnetic fields is simply given by $e^{-j k_{z} z}$. The solution of scalar wave equation in elliptical coordinate system will be

$$
\begin{equation*}
\nabla^{2} \psi+k^{2} \psi=0 \quad==>\quad \psi=\psi^{e}+\psi^{o} \tag{4.42}
\end{equation*}
$$

where

$$
\begin{align*}
\psi^{e} & =M c_{m}^{(1)}(u, q) c e_{m}(v, q) e^{-j k_{z} z} & & m=0,1,2, \cdots  \tag{4.43}\\
\psi^{o} & =M s_{m}^{(1)}(u, q) s e_{m}(v, q) e^{-j k_{z} z} & & m=1,2,3, \cdots
\end{align*}
$$

and $q=f^{2}\left(k^{2}-k_{z}^{2}\right) / 4$. We define two axillary vector $\mathbf{M}$ and $\mathbf{N}$ as

$$
\begin{align*}
\mathbf{M} & =\nabla \times\left(\psi \mathbf{a}_{\mathbf{z}}\right) \\
\mathbf{N} & =\frac{1}{k} \nabla \times \mathbf{M} \tag{4.44}
\end{align*}
$$



Figure 4.2: Elliptic Waveguide

If we assume $\mathbf{H}=\mathbf{M}=\nabla \times\left(\psi \mathbf{a}_{\mathbf{z}}\right)$, it means that we are going to find electromagnetic components of $T M^{z}$ modes;

$$
\begin{equation*}
\mathbf{M}=\frac{1}{h_{1}}\left[\frac{\partial \psi}{\partial v} \mathbf{a}_{u}-\frac{\partial \psi}{\partial u} \mathbf{a}_{v}\right] \tag{4.45}
\end{equation*}
$$

and from which we can determine $\mathbf{E}=\frac{1}{j \omega \epsilon} \nabla \times \mathbf{H}$.

$$
\begin{equation*}
k \mathbf{N}=\left(k^{2}-k_{z}^{2}\right) \psi \mathbf{a}_{z}+\frac{1}{h_{1}} \frac{\partial^{2} \psi}{\partial z \partial u} \mathbf{a}_{u}+\frac{1}{h_{1}} \frac{\partial^{2} \psi}{\partial z \partial v} \mathbf{a}_{v} \tag{4.46}
\end{equation*}
$$

by applying boundary condition $E_{z}=0$ at $u=u_{0}$, we have two equations, $M c_{m}^{(1)}\left(u_{0}, q\right)=0$ for even modes and $M s_{m}^{(1)}\left(u_{0}, q\right)=0$ for odd modes. For $T E^{z}$ we assume $\mathbf{E}=\mathbf{M}=\nabla \times\left(\psi \mathbf{a}_{\mathbf{z}}\right)$, therefore $\mathbf{M}=\frac{1}{h_{1}}\left[\frac{\partial \psi}{\partial v} \mathbf{a}_{u}-\frac{\partial \psi}{\partial u} \mathbf{a}_{v}\right]$ and magnetic field can be calculated by $\mathbf{H}=\frac{-1}{j \omega \mu} \nabla \times \mathbf{E}$. Applying boundary condition, $E_{v}=0$ at $u=u_{0}$, we have two equations, $M_{m}^{(1)}\left(u_{0}, q\right)=0$ for even modes and $M s_{m}^{\prime(1)}\left(u_{0}, q\right)=0$ for odd modes.

Table 4.1: CUTOFF FREQUENCY OF ELLIPTIC WAVEGUIDE IN GHz

|  |  | $\mathrm{A}=10.00 \mathrm{Cm}, \mathrm{B}=6.61 \mathrm{Cm}$ |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |
|  |  | TM-EVEN MODES |  |  |
| $\mathrm{m} / \mathrm{n}$ | 1 | 2 | 3 | 4 |
| 0 | 1.468337 | 3.680880 | 5.941186 | 8.205988 |
| 1 | 2.094981 | 4.235918 | 6.482914 | 8.742022 |
| 2 | 2.757014 | 4.817258 | 7.041880 | 9.290942 |
| 3 | 3.435940 | 5.421744 | 7.616410 | 9.851845 |


| TM-ODD MODES |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |


|  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| TE-EVEN MODES |  |  |  |  |
| $\mathrm{m} / \mathrm{n}$ | 1 | 2 | 3 | 4 |
| 0 | 2.501066 | 4.783011 | 7.055468 | 9.325514 |
| 1 | 0.890284 | 3.069237 | 5.327188 | 7.592670 |
| 2 | 1.604605 | 3.680664 | 5.892695 | 8.144793 |
| 3 | 2.289425 | 4.326062 | 6.477174 | 8.710642 |


|  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| TE-ODD MODES |  |  |  |  |
| $\mathrm{m} / \mathrm{n}$ | 1 | 2 | 3 | 4 |
| 1 | 1.300689 | 3.644210 | 5.919730 | 8.190658 |
| 2 | 1.842372 | 4.194770 | 6.459768 | 8.725789 |
| 3 | 2.423427 | 4.771852 | 7.017336 | 9.273908 |
| 4 | 3.023162 | 5.370564 | 7.590763 | 9.834119 |



Figure 4.3: Elliptic Waveguide $T M_{01}$

### 4.3 Plane Wave in Elliptic Cylinder

The incident plane wave $\psi=\exp \left\{j k\left(x \cos \phi_{0}+y \sin \phi_{0}\right)\right\}$ can be expand in elliptic cylinder. The $\psi$ may be $E_{z}$ or $H_{z}$.

$$
\begin{equation*}
\psi=\sum_{m=0}^{\infty} a_{m} c e_{m}(v, q) M c_{m}^{(1)}(u, q)+\sum_{m=1}^{\infty} b_{m} s e_{m}(v, q) M s_{m}^{(1)}(u, q) \tag{4.47}
\end{equation*}
$$

where $a_{m}=2 j^{m} c e_{m}\left(\phi_{0}, q\right)$ and $b_{m}=2 j^{m} s e_{m}\left(\phi_{0}, q\right)$

### 4.4 Line Source in Elliptic Cylinder

An infinite Electric current source $I_{0}$ which is located at $r_{0}\left(x_{0}, y_{0}\right)=r_{0}\left(u_{0}, v_{0}\right)$ can be expand in elliptic cylinder. For $u_{0}>u$

$$
\begin{align*}
E_{z}= & -\frac{I_{0}}{4} \omega \mu H_{0}^{(2)}\left(k\left|\mathbf{r}-\mathbf{r}_{0}\right|\right) \\
= & -\frac{I_{0}}{2} \omega \mu\left\{\sum_{m=0}^{\infty} a_{m}^{i} c e_{m}(v, q) M c_{m}^{(1)}(u, q)\right. \\
& \left.+\sum_{m=1}^{\infty} b_{m}^{i} s e_{m}(v, q) M s_{m}^{(1)}(u, q)\right\} \tag{4.48}
\end{align*}
$$

where for $u_{0}>u$ we have

$$
\begin{align*}
a_{m}^{i} & =c e_{m}\left(v_{0}, q\right) M c_{m}^{(4)}\left(u_{0}, q\right) \\
b_{m}^{i} & =s e_{m}\left(v_{0}, q\right) M s_{m}^{(4)}\left(u_{0}, q\right) \tag{4.49}
\end{align*}
$$

and for $u>u_{0}$

$$
\begin{align*}
E_{z}= & -\frac{I_{0}}{4} \omega \mu H_{0}^{(2)}\left(k\left|\mathbf{r}-\mathbf{r}_{0}\right|\right) \\
= & -\frac{I_{0}}{2} \omega \mu\left\{\sum_{m=0}^{\infty} a_{m}^{i} c e_{m}(v, q) M c_{m}^{(4)}(u, q)\right. \\
& \left.+\sum_{m=1}^{\infty} b_{m}^{i} s e_{m}(v, q) M s_{m}^{(4)}(u, q)\right\} \tag{4.50}
\end{align*}
$$

where for $u>u_{0}$

$$
\begin{align*}
a_{m}^{i} & =c e_{m}\left(v_{0}, q\right) M c_{m}^{(1)}\left(u_{0}, q\right) \\
b_{m}^{i} & =s e_{m}\left(v_{0}, q\right) M s_{m}^{(1)}\left(u_{0}, q\right) \tag{4.51}
\end{align*}
$$

### 4.5 Scattering by Conducting Elliptic Cylinder

Scattering of a plane wave by metallic elliptic cylinder may be TM or TE polarization.

### 4.5.1 TM-Polarization

Let a TM plane wave at angle $\phi_{0}$ incident on elliptic metallic cylinder with $u_{0}=\cosh ^{-1}\left(\frac{A}{f}\right), f=\sqrt{A^{2}-B^{2}}$, Eq.(4.18), Fig.(4.4).


Figure 4.4: Scattering TM wave by a Conducting Elliptic Cylinder

$$
E_{z}^{i}=\exp \left\{j k\left(x \cos \phi_{0}+y \sin \phi_{0}\right)\right\}
$$

or

$$
E_{z}^{i}=\sum_{m=0}^{\infty} a_{m}^{i} c e_{m}(v, q) M c_{m}^{(1)}(u, q)+\sum_{m=1}^{\infty} b_{m}^{i} s e_{m}(v, q) M s_{m}^{(1)}(u, q)
$$

where $a_{m}^{i}=2 j^{m} c e_{m}\left(\phi_{0}, q\right)$ and $b_{m}^{i}=2 j^{m} s e_{m}\left(\phi_{0}, q\right)$. The scattered wave at point $v, u$ will be

$$
\begin{align*}
E_{z}^{s} & =\sum_{m=0}^{\infty} a_{m}^{s} c e_{m}(v, q) M c_{m}^{(4)}(u, q) \\
& +\sum_{m=1}^{\infty} b_{m}^{s} s e_{m}(v, q) M s_{m}^{(4)}(u, q) \tag{4.52}
\end{align*}
$$

where $a_{m}^{s}$ and $b_{m}^{s}$ will be determined by boundary condition, i.e. $E_{z}^{i}+E_{z}^{s}$ on the surface $u_{0}$ at any angle must be zero. After some manipulation we will
have:

$$
\begin{align*}
a_{m}^{s} & =-\frac{M c_{m}^{(1)}\left(u_{0}, q\right)}{M c_{m}^{(4)}\left(u_{0}, q\right)} a_{m}^{i} \\
b_{m}^{s} & =-\frac{M s_{m}^{(1)}\left(u_{0}, q\right)}{M s_{m}^{(4)}\left(u_{0}, q\right)} b_{m}^{i} \tag{4.53}
\end{align*}
$$

The bistatic RCS of a long elliptic cylinder is

$$
\begin{equation*}
\sigma_{2 D}=\frac{4 / k f}{\sqrt{\cosh ^{2} u_{0}-\cos ^{2} \phi_{0}}}\left[\sum_{m=0}^{\infty}\left|a_{m}^{s}\right|^{2} c e_{m}\left(\phi_{0}, q\right)^{2} \sum_{m=1}^{\infty}\left|b_{m}^{s}\right|^{2} s e_{m}\left(\phi_{0}, q\right)^{2}\right] \tag{4}
\end{equation*}
$$



Figure 4.5: Scattering TM wave by a Conducting Elliptic Cylinder

### 4.5.2 TE-Polarization

Let a TE plane wave at angle $\phi_{0}$ incident on elliptic metallic cylinder with $u_{0}=\cosh ^{-1}\left(\frac{A}{f}\right), f=\sqrt{A^{2}-B^{2}}$, Eq.(4.18), Fig.(4.6).

$$
H_{z}^{i}=\exp \left\{j k\left(x \cos \phi_{0}+y \sin \phi_{0}\right)\right\}
$$



Figure 4.6: Scattering TE wave by a Conducting Elliptic Cylinder

The scattered wave at point $v, u$ will be

$$
\begin{align*}
H_{z}^{s} & =\sum_{m=0}^{\infty} a_{m}^{s} c e_{m}(v, q) M c_{m}^{(4)}(u, q) \\
& +\sum_{m=1}^{\infty} b_{m}^{s} s e_{m}(v, q) M s_{m}^{(4)}(u, q) \tag{4.55}
\end{align*}
$$

where $a_{m}^{s}$ and $b_{m}^{s}$ will be determined by boundary condition, i.e. $\frac{\partial H_{z}^{i}}{\partial u}+\frac{\partial H_{z}^{s}}{\partial u}$ on the surface $u_{0}$ at any angle must be zero. After some manipulation we will have:

$$
\begin{align*}
a_{m}^{s} & =-\frac{M c_{m}^{\prime(1)}\left(u_{0}, q\right)}{M c_{m}^{\prime(4)}\left(u_{0}, q\right)} a_{m}^{i} \\
b_{m}^{s} & =-\frac{M s_{m}^{\prime(1)}\left(u_{0}, q\right)}{M s_{m}^{\prime(4)}\left(u_{0}, q\right)} b_{m}^{i} \tag{4.56}
\end{align*}
$$

where $a_{m}^{i}=2(j)^{m} c e_{m}\left(\phi_{0}, q\right)$ and $b_{m}^{i}=2(j)^{m} s e_{m}\left(\phi_{0}, q\right)$.


Figure 4.7: Scattering TE wave by a Conducting Elliptic Cylinder

### 4.6 Scattering by PEMC Elliptic Cylinder

Solution of scalar wave equation in elliptical coordinate system will be (in two dimensional $\frac{\partial}{\partial z}=0$ ):

$$
\begin{equation*}
\nabla^{2} \psi+k^{2} \psi=0 \quad==>\quad \psi=\psi^{e}+\psi^{o} \tag{4.57}
\end{equation*}
$$

where

$$
\begin{array}{rlrl}
\psi^{e} & =M c_{m}^{(1),(4)}(u, q) c e_{m}(v, q) & & m=0,1,2, \cdots  \tag{4.58}\\
\psi^{o} & =M s_{m}^{(1),(4)}(u, q) s e_{m}(v, q) & m=1,2,3, \cdots
\end{array}
$$

and $q=(k f / 2)^{2}$. We define two axillary vector $\mathbf{M}$ and $\mathbf{N}$ as

$$
\begin{align*}
\mathbf{M} & =\nabla \times\left(\psi \mathbf{a}_{\mathbf{z}}\right) \\
\mathbf{N} & =\frac{1}{k} \nabla \times \mathbf{M} \tag{4.59}
\end{align*}
$$

after some manipolation

$$
\begin{align*}
\mathbf{M} & =\frac{1}{f \sqrt{\cosh ^{2} u-\cos ^{2} v}}\left(\frac{\partial \psi}{\partial v} \mathbf{a}_{\mathbf{u}}-\frac{\partial \psi}{\partial u} \mathbf{a}_{\mathbf{v}}\right) \\
\mathbf{N} & =k \psi \mathbf{a}_{\mathbf{z}} \tag{4.60}
\end{align*}
$$

Now we consider a $T M^{z}$ plane wave

$$
\begin{equation*}
\mathbf{E}^{\mathbf{i}}=E_{0} e^{j k\left(x \cos \phi_{0}+y \sin \phi_{0}\right)} \mathbf{a}_{\mathbf{z}} \tag{4.61}
\end{equation*}
$$

therefore we can write it as

$$
\begin{equation*}
\mathbf{E}^{\mathbf{i}}=\sum_{n=0}^{\infty} a_{n}^{e} \mathbf{N}_{n}^{e(1)}(u, q)+\sum_{n=1}^{\infty} a_{n}^{o} \mathbf{N}_{n}^{o(1)}(u, q) \tag{4.62}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{H}^{\mathbf{i}}=\frac{j}{\eta}\left[\sum_{n=0}^{\infty} a_{n}^{e} \mathbf{M}_{n}^{e(1)}(u, q)+\sum_{n=1}^{\infty} a_{n}^{o} \mathbf{M}_{n}^{o(1)}(u, q)\right] \tag{4.63}
\end{equation*}
$$

where $a_{n}^{e}=\frac{2 j^{n} E_{0}}{k} c e_{n}\left(\phi_{0}, q\right)$ and $a_{n}^{o}=\frac{2 j^{n} E_{0}}{k} s e_{n}\left(\phi_{0}, q\right)$ The scattered fields are expanded in the form

$$
\begin{align*}
\mathbf{E}^{\mathbf{s}} & =\left[\sum_{n=0}^{\infty} b_{n}^{e} \mathbf{N}_{n}^{e(4)}(u, q)+c_{n}^{e} \mathbf{M}_{n}^{e(4)}(u, q)\right.  \tag{4.64}\\
& \left.+\sum_{n=1}^{\infty} b_{n}^{o} \mathbf{N}_{n}^{o(4)}(u, q)+c_{n}^{o} \mathbf{M}_{n}^{o(4)}(u, q)\right] \\
\mathbf{H}^{\mathbf{s}} & =\frac{j}{\eta}\left[\sum_{n=0}^{\infty} b_{n}^{e} \mathbf{M}_{n}^{e(4)}(u, q)+c_{n}^{e} \mathbf{N}_{n}^{e(4)}(u, q)\right.  \tag{4.65}\\
& \left.+\sum_{n=1}^{\infty} b_{n}^{o} \mathbf{M}_{n}^{o(4)}(u, q)+c_{n}^{o} \mathbf{N}_{n}^{o(4)}(u, q)\right]
\end{align*}
$$

Since we have PEMC, there should be one more term for cross-polarized TE components. Now we can apply boundary conditions for PEMC. The tangential and normal field components have to satisfy the boundary condition at the elliptic cylinder surface.

$$
\begin{align*}
H_{t}^{i}+H_{t}^{s}+M\left(E_{t}^{i}+E_{t}^{s}\right) & =0  \tag{4.66}\\
\epsilon\left(E_{u}^{i}+E_{u}^{s}\right)-M \mu\left(H_{u}^{i}+H_{u}^{s}\right) & =0 \tag{4.67}
\end{align*}
$$

applying these boundary conditions, we obtain the following system of linear equations

$$
\begin{align*}
M c_{n}^{(4)}\left(u_{0}, q\right) b_{n}^{e}+\frac{j}{M \eta} M c_{n}^{(4)}\left(u_{0}, q\right) c_{n}^{e} & =-a_{n}^{e} M c_{n}^{(1)}\left(u_{0}, q\right)  \tag{4.68}\\
M c_{n}^{\prime(4)}\left(u_{0}, q\right) b_{n}^{e}+\frac{j}{M \eta} M{c_{n}^{\prime(4)}}_{n}^{\left(u_{0}, q\right) c_{n}^{e}} & =-a_{n}^{e} M c_{n}^{\prime(1)}\left(u_{0}, q\right) \tag{4.69}
\end{align*}
$$

and

$$
\begin{align*}
M s_{n}^{(4)}\left(u_{0}, q\right) b_{n}^{o}+\frac{j}{M \eta} M s_{n}^{(4)}\left(u_{0}, q\right) c_{n}^{o} & =-a_{n}^{o} M s_{n}^{(1)}\left(u_{0}, q\right)  \tag{4.70}\\
M s_{n}^{\prime(4)}\left(u_{0}, q\right) b_{n}^{o}+\frac{j}{M \eta} M s_{n}^{\prime(4)}\left(u_{0}, q\right) c_{n}^{o} & =-a_{n}^{o} M s_{n}^{\prime(1)}\left(u_{0}, q\right) \tag{4.71}
\end{align*}
$$

where $a$ is radius of the PEMC cylinder. By solving these equations we will have

$$
\begin{gather*}
b_{n}^{e}=-\frac{M c_{n}^{(4)}\left(u_{0}, q\right) M c_{n}^{(1)}\left(u_{0}, q\right)+M^{2} \eta^{2} M c_{n}^{\prime(4)}\left(u_{0}, q\right) M c_{n}^{(1)}\left(u_{0}, q\right)}{\left(1+M^{2} \eta^{2}\right) M c_{n}^{\prime(4)}\left(u_{0}, q\right) M c_{n}^{(4)}\left(u_{0}, q\right)} a_{n}^{e}  \tag{4.72}\\
b_{n}^{o}=-\frac{M s_{n}^{(4)}\left(u_{0}, q\right) M s_{n}^{(1)}\left(u_{0}, q\right)+M^{2} \eta^{2} M s_{n}^{\prime(4)}\left(u_{0}, q\right) M s_{n}^{(1)}\left(u_{0}, q\right)}{\left(1+M^{2} \eta^{2}\right) M s_{n}^{\prime(4)}\left(u_{0}, q\right) M s_{n}^{(4)}\left(u_{0}, q\right)} a_{n}^{o}  \tag{4.73}\\
c_{n}^{e}=\frac{2 M \eta}{\pi\left(1+M^{2} \eta^{2}\right) M c_{n}^{(4)}\left(u_{0}, q\right) M c_{n}^{\prime(4)}\left(u_{0}, q\right)} a_{n}^{e}  \tag{4.74}\\
c_{n}^{o}=\frac{2 M \eta}{\pi\left(1+M^{2} \eta^{2}\right) M s_{n}^{(4)}\left(u_{0}, q\right) M s_{n}^{\prime(4)}\left(u_{0}, q\right)} a_{n}^{o} \tag{4.75}
\end{gather*}
$$

### 4.7 Scattering by Dielectric Elliptic Cylinder

The scattering of electromagnetic wave by a elliptic dielectric cylinder is not as simple as circular dielectric cylinder, because we do not have usual mode orthogonality in elliptic cylinder. In this case we use a new method which is described in following sections.


Figure 4.8: Dielectric Elliptic Cylinder

## TM-Polarization

We have a dielectric elliptic cylinder with electrical and geometry parameters as shown in Fig.(4.8). We need the following integrals in our problems so it is better to define:

$$
\begin{align*}
\mathbf{I}_{m n}^{c c}\left(q_{i}, q_{j}\right) & =\frac{1}{\pi} \int_{0}^{2 \pi} c e_{m}\left(v, q_{i}\right) c e_{n}\left(v, q_{j}\right) d v \\
\mathbf{I}_{m n}^{c s}\left(q_{i}, q_{j}\right) & =\frac{1}{\pi} \int_{0}^{2 \pi} c e_{m}\left(v, q_{i}\right) s e_{n}\left(v, q_{j}\right) d v \\
\mathbf{I}_{m n}^{s s}\left(q_{i}, q_{j}\right) & =\frac{1}{\pi} \int_{0}^{2 \pi} s e_{m}\left(v, q_{i}\right) s e_{n}\left(v, q_{j}\right) d v \tag{4.76}
\end{align*}
$$

The result will be $\mathbf{I}_{m n}^{s c}\left(q_{i}, q_{j}\right)=\mathbf{I}_{m n}^{c s}\left(q_{i}, q_{j}\right)=0$ and

$$
\begin{equation*}
\mathbf{I}_{m n}^{c c}\left(q_{i}, q_{j}\right)=\sum_{k=0}^{\infty}\left(1+\delta_{0 k}\right) A_{k}^{m}\left(q_{i}\right) A_{k}^{n}\left(q_{j}\right) \tag{4.77}
\end{equation*}
$$

when $m$ and $n$ are even, and

$$
\begin{equation*}
\mathbf{I}_{m n}^{c c}\left(q_{i}, q_{j}\right)=\sum_{k=1}^{\infty} A_{k}^{m}\left(q_{i}\right) A_{k}^{n}\left(q_{j}\right) \tag{4.78}
\end{equation*}
$$

when $m$ and $n$ are odd. Similar

$$
\begin{equation*}
\mathbf{I}_{m n}^{s s}\left(q_{i}, q_{j}\right)=\sum_{k=2}^{\infty} B_{k}^{m}\left(q_{i}\right) B_{k}^{n}\left(q_{j}\right) \tag{4.79}
\end{equation*}
$$

when $m$ and $n$ are even, and

$$
\begin{equation*}
\mathbf{I}_{m n}^{s s}\left(q_{i}, q_{j}\right)=\sum_{k=1}^{\infty} B_{k}^{m}\left(q_{i}\right) B_{k}^{n}\left(q_{j}\right) \tag{4.80}
\end{equation*}
$$

when $m$ and $n$ are odd. The $q_{i}$ and $q_{j}$ belong to two different mediums. Let a TM plane wave at angle $\phi_{0}$ incident on elliptic dielectric cylinder with $u_{0}=\cosh ^{-1}\left(\frac{A}{f}\right), f=\sqrt{A^{2}-B^{2}}$. The incident, scattered and penetrated wave will be:

$$
E_{z}^{i}=\exp \left\{j k_{1}\left(x \cos \phi_{0}+y \sin \phi_{0}\right)\right\}
$$

or

$$
E_{z}^{i}=\sum_{m=0}^{\infty} a_{m}^{i} M c_{m}^{(1)}\left(u, q_{1}\right) c e_{m}\left(v, q_{1}\right)+\sum_{m=1}^{\infty} b_{m}^{i} M s_{m}^{(1)}\left(u, q_{1}\right) s e_{m}\left(v, q_{1}\right)
$$

where $q_{1}=\frac{f^{2} k_{1}^{2}}{4}, q_{2}=\frac{f^{2} k_{2}^{2}}{4}, k=\omega \sqrt{\mu \epsilon_{c}}$ and $\epsilon_{c}=\epsilon_{0}\left(\epsilon_{r}-j \frac{\sigma}{\omega \epsilon_{0}}\right)$. The value of $a_{m}^{i}$ and $b_{m}^{i}$ will be $a_{m}^{i}=2(j)^{m} c e_{m}\left(\phi_{0}, q_{1}\right), b_{m}^{i}=2(j)^{m} s e_{m}\left(\phi_{0}, q_{1}\right)$. The scattered and transmitted wave will be

$$
\begin{align*}
& E_{z}^{s}=\sum_{m=0}^{\infty} a_{m}^{s} M c_{m}^{(4)}\left(u, q_{1}\right) c e_{m}\left(v, q_{1}\right)+\sum_{m=1}^{\infty} b_{m}^{s} M s_{m}^{(4)}\left(u, q_{1}\right) s e_{m}\left(v, q_{1}\right)  \tag{4.81}\\
& E_{z}^{t}=\sum_{m=0}^{\infty} a_{m}^{t} M c_{m}^{(1)}\left(u, q_{2}\right) c e_{m}\left(v, q_{2}\right)+\sum_{m=1}^{\infty} b_{m}^{t} M s_{m}^{(1)}\left(u, q_{2}\right) s e_{m}\left(v, q_{2}\right) \tag{4.82}
\end{align*}
$$

where the $a_{m}^{s}, a_{m}^{t}$ and $b_{m}^{s}, b_{m}^{t}$ are unknowns and will be found by boundary conditions. At boundary $u=u_{0}$ for any angle $v$ we have $E_{z}^{i}+E_{z}^{s}=E_{z}^{t}$.

$$
\begin{align*}
\sum_{m=0}^{\infty} a_{m}^{i} M c_{m}^{(1)}\left(u_{0}, q_{1}\right) c e_{m}\left(v, q_{1}\right) & +\sum_{m=1}^{\infty} b_{m}^{i} M s_{m}^{(1)}\left(u_{0}, q_{1}\right) s e_{m}\left(v, q_{1}\right)+ \\
\sum_{m=0}^{\infty} a_{m}^{s} M c_{m}^{(4)}\left(u_{0}, q_{1}\right) c e_{m}\left(v, q_{1}\right) & +\sum_{m=1}^{\infty} b_{m}^{s} M s_{m}^{(4)}\left(u_{0}, q_{1}\right) s e_{m}\left(v, q_{1}\right)= \\
\sum_{m=0}^{\infty} a_{m}^{t} M c_{m}^{(1)}\left(u_{0}, q_{2}\right) c e_{m}\left(v, q_{2}\right) & +\sum_{m=1}^{\infty} b_{m}^{t} M s_{m}^{(1)}\left(u_{0}, q_{2}\right) s e_{m}\left(v, q_{2}\right) \tag{4.83}
\end{align*}
$$

we multiply both side of Eq.(4.83) by factor $c e_{n}\left(v, q_{1}\right) / \pi$ and integrating over $0-2 \pi$ with respect to v , there will be

$$
\begin{gather*}
a_{0}^{t} M c_{0}^{(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{0 n}^{c c}\left(q_{2}, q_{1}\right)+a_{1}^{t} M c_{1}^{(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{1 n}^{c c}\left(q_{2}, q_{1}\right)+\cdots+ \\
a_{M}^{t} M c_{M}^{(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{M n}^{c c}\left(q_{2}, q_{1}\right)-a_{n}^{s} M c_{n}^{(4)}\left(u_{0}, q_{1}\right)=a_{n}^{i} M c_{n}^{(1)}\left(u_{0}, q_{1}\right) \tag{4.84}
\end{gather*}
$$

where $M$ is maximum terms in truncation of series. Now we can apply magnetic field boundary conditions $H_{v}^{i}+H_{v}^{s}=H_{v}^{t}$, but

$$
H_{v}=\frac{1}{f \sqrt{\cosh ^{2} u-\cos ^{2} v}} \frac{1}{j \omega \mu} \frac{\partial E_{z}}{\partial u}
$$

therefore

$$
\begin{align*}
\frac{1}{\mu_{2}}\left\{a_{0}^{t} M c_{0}^{\prime(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{0 n}^{c c}\left(q_{2}, q_{1}\right)+a_{1}^{t} M c_{1}^{\prime(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{1 n}^{c c}\left(q_{2}, q_{1}\right)\right. & +\cdots+ \\
\left.a_{M}^{t} M c_{n}^{\prime(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{M n}^{c c}\left(q_{2}, q_{1}\right)\right\}-\frac{1}{\mu_{1}} a_{n}^{s} M c_{n}^{\prime(4)}\left(u_{0}, q_{1}\right) & =\frac{1}{\mu_{1}} a_{n}^{i} M c_{n}^{\prime(1)}\left(u_{0}, q_{1}\right) \tag{4.85}
\end{align*}
$$

The index $n$ in both Eq.(4.84),Eq.(4.85) can be changed from $0-M$, therefore we have $2 M+2$ unknown and $2 M+2$ linear equations which can be divide into four different submatrix.

$$
\left[\begin{array}{ll}
\mathbf{A}_{m n} & \mathbf{B}_{m n}  \tag{4.86}\\
\mathbf{C}_{m n} & \mathbf{D}_{m n}
\end{array}\right]\left[\begin{array}{l}
\mathbf{X}_{n} \\
\mathbf{Y}_{n}
\end{array}\right]=\left[\begin{array}{l}
\mathbf{G}_{m} \\
\mathbf{H}_{m}
\end{array}\right]
$$

where

$$
\begin{gathered}
\mathbf{A}_{m n}=M c_{m}^{(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{m n}^{c c}\left(q_{2}, q_{1}\right) \\
\mathbf{C}_{m n}=\frac{1}{\mu_{2}} M c_{m}^{\prime(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{m n}^{c c}\left(q_{2}, q_{1}\right) \\
\mathbf{B}_{m n}=-M c_{m}^{(4)}\left(u_{0}, q_{1}\right) \delta_{m n} \\
\mathbf{D}_{m n}=\frac{-1}{\mu_{1}} M c_{m}^{\prime(4)}\left(u_{0}, q_{1}\right) \delta_{m n} \\
\mathbf{G}_{m}=a_{n}^{i} M c_{m}^{(1)}\left(u_{0}, q_{1}\right) \\
\mathbf{H}_{m}=\frac{1}{\mu_{1}} a_{n}^{i} M c_{m}^{\prime(1)}\left(u_{0}, q_{1}\right)
\end{gathered}
$$

and the unknowns are

$$
\begin{aligned}
& \mathbf{X}_{n}=a_{n}^{t} \\
& \mathbf{Y}_{n}=a_{n}^{s}
\end{aligned}
$$

The same procedure can be applied to determine the coefficient $b_{m}^{s}, b_{m}^{t}$. The number of unknown and linear equation in this case is $2 M$.

$$
\begin{array}{r}
b_{1}^{t} M s_{1}^{(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{0, n}^{s s}\left(q_{2}, q_{1}\right)+b_{2}^{t} M s_{2}^{(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{1, n}^{s s}\left(q_{2}, q_{1}\right)+\cdots \\
+  \tag{4.87}\\
b_{M, n}^{t} M s_{M}^{(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{M, n}^{s s}\left(q_{2}, q_{1}\right)-b_{n}^{s} M s_{n}^{(4)}\left(u_{0}, q_{1}\right)=b_{n}^{i} M s_{n}^{(1)}\left(u_{0}, q_{1}\right)
\end{array}
$$

and

$$
\begin{gathered}
\frac{1}{\mu_{2}}\left\{b_{1}^{t} M s_{1}^{\prime(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{0, n}^{s s}\left(q_{2}, q_{1}\right)+b_{2}^{t} M s_{2}^{\prime(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{1, n}^{s s}\left(q_{2}, q_{1}\right)+\cdots+\right. \\
\left.b_{M, n}^{t} M s_{M}^{\prime(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{M, n}^{s s}\left(q_{2}, q_{1}\right)\right\}-\frac{1}{\mu_{1}} b_{n}^{s} M s_{n}^{\prime(4)}\left(u_{0}, q_{1}\right)=\frac{1}{\mu_{1}} b_{n}^{i} M s_{n}^{\prime(1)}\left(u_{0}, q_{1}\right)(4.88)
\end{gathered}
$$

$$
\left[\begin{array}{ll}
\mathbf{A}_{m n} & \mathbf{B}_{m n}  \tag{4.89}\\
\mathbf{C}_{m n} & \mathbf{D}_{m n}
\end{array}\right]\left[\begin{array}{l}
\mathbf{X}_{n} \\
\mathbf{Y}_{n}
\end{array}\right]=\left[\begin{array}{l}
\mathbf{G}_{m} \\
\mathbf{H}_{m}
\end{array}\right]
$$

$$
\mathbf{A}_{m n}=M s_{m}^{(1)}\left(u_{0}, q_{2}\right) I_{m n}^{s s}\left(q_{1}, q_{2}\right)
$$

$$
\mathbf{C}_{m n}=\frac{1}{\mu_{2}} M s_{m}^{\prime(1)}\left(u_{0}, q_{2}\right) I_{m n}^{s s}\left(q_{1}, q_{2}\right)
$$

$$
\mathbf{B}_{m n}=-M s_{m}^{(4)}\left(u_{0}, q_{1}\right) \delta_{m n}
$$

$$
\mathbf{D}_{m n}=\frac{-1}{\mu_{1}} M s_{m}^{\prime(4)}\left(u_{0}, q_{1}\right) \delta_{m n}
$$

$$
\mathbf{G}_{m}=b_{n}^{i} M s_{m}^{(1)}\left(u_{0}, q_{1}\right)
$$

$$
\mathbf{H}_{m}=\frac{1}{\mu_{1}} b_{n}^{i} M s_{m}^{\prime(1)}\left(u_{0}, q_{1}\right)
$$

and the unknowns are

$$
\begin{aligned}
& \mathbf{X}_{m}=b_{m}^{t} \\
& \mathbf{Y}_{m}=b_{m}^{s}
\end{aligned}
$$



Figure 4.9: Distribution of electric filed along X and Y axis, TM-Polarization

## TE-Polarization

We have a dielectric elliptic cylinder with electrical and geometry parameters as shown in Fig.(4.10). Let a $T E^{z}$ plane wave at angle $\phi_{0}$ incident on elliptic dielectric cylinder with $u_{0}=\cosh ^{-1}\left(\frac{A}{f}\right), f=\sqrt{A^{2}-B^{2}}$. The incident, scattered and penetrated wave will be:

$$
H_{z}^{i}=\exp \left\{j k_{1}\left(x \cos \phi_{0}+y \sin \phi_{0}\right)\right\}
$$

or

$$
H_{z}^{i}=\sum_{m=0}^{\infty} a_{m}^{i} M c_{m}^{(1)}\left(u, q_{1}\right) c e_{m}\left(v, q_{1}\right)+\sum_{m=1}^{\infty} b_{m}^{i} M s_{m}^{(1)}\left(u, q_{1}\right) s e_{m}\left(v, q_{1}\right)
$$

where $q_{1}=\frac{f^{2} k_{1}^{2}}{4}, q_{2}=\frac{f^{2} k_{2}^{2}}{4}, k=\omega \sqrt{\mu \epsilon_{c}}$ and $\epsilon_{c}=\epsilon_{0}\left(\epsilon_{r}-j \frac{\sigma}{\omega \epsilon_{0}}\right)$. The value of $a_{m}^{i}$ and $b_{m}^{i}$ will be $a_{m}^{i}=2(j)^{m} c e_{m}\left(\phi_{0}, q_{1}\right), b_{m}^{i}=2(j)^{m} s e_{m}\left(\phi_{0}, q_{1}\right)$. The scatter and transmitted wave will be

$$
\begin{align*}
& H_{z}^{s}=\sum_{m=0}^{\infty} a_{m}^{s} M c_{m}^{(4)}\left(u, q_{1}\right) c e_{m}\left(v, q_{1}\right)+\sum_{m=1}^{\infty} b_{m}^{s} M s_{m}^{(4)}\left(u, q_{1}\right) s e_{m}\left(v, q_{1}\right)  \tag{4.90}\\
& H_{z}^{t}=\sum_{m=0}^{\infty} a_{m}^{t} M c_{m}^{(1)}\left(u, q_{2}\right) c e_{m}\left(v, q_{2}\right)+\sum_{m=1}^{\infty} b_{m}^{t} M s_{m}^{(1)}\left(u, q_{2}\right) s e_{m}\left(v, q_{2}\right) \tag{4.91}
\end{align*}
$$



Figure 4.10: Dielectric Elliptic Cylinder
where the $a_{m}^{s}, a_{m}^{t}$ and $b_{m}^{s}, b_{m}^{t}$ are unknowns and will be found by boundary conditions. At boundary $u=u_{0}$ for any angle $v$ we have $H_{z}^{i}+H_{z}^{s}=H_{z}^{t}$.

$$
\begin{align*}
\sum_{m=0}^{\infty} a_{m}^{i} M c_{m}^{(1)}\left(u_{0}, q_{1}\right) c e_{m}\left(v, q_{1}\right) & +\sum_{m=1}^{\infty} b_{m}^{i} M s_{m}^{(1)}\left(u_{0}, q_{1}\right) s e_{m}\left(v, q_{1}\right)+ \\
\sum_{m=0}^{\infty} a_{m}^{s} M c_{m}^{(4)}\left(u_{0}, q_{1}\right) c e_{m}\left(v, q_{1}\right) & +\sum_{m=1}^{\infty} b_{m}^{s} M s_{m}^{(4)}\left(u_{0}, q_{1}\right) s e_{m}\left(v, q_{1}\right)= \\
\sum_{m=0}^{\infty} a_{m}^{t} M c_{m}^{(1)}\left(u_{0}, q_{2}\right) c e_{m}\left(v, q_{2}\right) & +\sum_{m=1}^{\infty} b_{m}^{t} M s_{m}^{(1)}\left(u_{0}, q_{2}\right) s e_{m}\left(v, q_{2}\right) \tag{4.92}
\end{align*}
$$

we multiply both side of Eq.(4.92) by factor $c e_{n}\left(v, q_{1}\right) / \pi$ and integrating over $0-2 \pi$ with respect to v , there will be

$$
\begin{gather*}
a_{0}^{t} M c_{0}^{(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{0 n}^{c c}\left(q_{2}, q_{1}\right)+a_{1}^{t} M c_{1}^{(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{1 n}^{c c}\left(q_{2}, q_{1}\right)+\cdots+ \\
a_{M}^{t} M c_{M}^{(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{M n}^{c c}\left(q_{2}, q_{1}\right)-a_{n}^{s} M c_{n}^{(4)}\left(u_{0}, q_{1}\right)=a_{n}^{i} M c_{n}^{(1)}\left(u_{0}, q_{1}\right) \tag{4.93}
\end{gather*}
$$

where $M$ is maximum terms in truncation of series. Now we can apply electric field boundary conditions $E_{v}^{i}+E_{v}^{s}=E_{v}^{t}$, but

$$
E_{v}=\frac{1}{f \sqrt{\cosh ^{2} u-\cos ^{2} v}} \frac{-1}{j \omega \epsilon} \frac{\partial H_{z}}{\partial u}
$$

therefore

$$
\begin{align*}
\frac{1}{\epsilon_{2}}\left\{a_{0}^{t} M c_{0}^{\prime(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{0 n}^{c c}\left(q_{2}, q_{1}\right)+a_{1}^{t} M c_{1}^{\prime(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{1 n}^{c c}\left(q_{2}, q_{1}\right)\right. & +\cdots+ \\
\left.a_{M}^{t} M c_{n}^{\prime(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{M n}^{c c}\left(q_{2}, q_{1}\right)\right\}-\frac{1}{\epsilon_{1}} a_{n}^{s} M c_{n}^{\prime(4)}\left(u_{0}, q_{1}\right) & =\frac{1}{\epsilon_{1}} a_{n}^{i} M c_{n}^{\prime(1)}\left(u_{0}, q_{1}\right) \tag{4.94}
\end{align*}
$$

The index $n$ in both Eq.(4.93),Eq.(4.94) can be changed from $0-M$, therefore we have $2 M+2$ unknown and $2 M+2$ linear equations which can be divide into four different submatrix.

$$
\left[\begin{array}{ll}
\mathbf{A}_{m n} & \mathbf{B}_{m n}  \tag{4.95}\\
\mathbf{C}_{m n} & \mathbf{D}_{m n}
\end{array}\right]\left[\begin{array}{l}
\mathbf{X}_{n} \\
\mathbf{Y}_{n}
\end{array}\right]=\left[\begin{array}{l}
\mathbf{G}_{m} \\
\mathbf{H}_{m}
\end{array}\right]
$$

where

$$
\begin{gathered}
\mathbf{A}_{m n}=M c_{m}^{(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{m n}^{c c}\left(q_{1}, q_{2}\right) \\
\mathbf{C}_{m n}=\frac{1}{\epsilon_{2}} M c_{m}^{\prime(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{m n}^{c c}\left(q_{1}, q_{2}\right) \\
\mathbf{B}_{m n}=-M c_{m}^{(4)}\left(u_{0}, q_{1}\right) \delta_{m n} \\
\mathbf{D}_{m n}=\frac{-1}{\epsilon_{1}} M c_{m}^{\prime(4)}\left(u_{0}, q_{1}\right) \delta_{m n} \\
\mathbf{G}_{m}=a_{m}^{i} M c_{m}^{(1)}\left(u_{0}, q_{1}\right) \\
\mathbf{H}_{m}=\frac{1}{\epsilon_{1}} a_{m}^{i} M c_{m}^{\prime(1)}\left(u_{0}, q_{1}\right)
\end{gathered}
$$

and the unknowns are

$$
\begin{aligned}
& \mathbf{X}_{n}=a_{n}^{t} \\
& \mathbf{Y}_{n}=a_{n}^{s}
\end{aligned}
$$

The same procedure can be applied to determine the coefficient $b_{m}^{s}, b_{m}^{t}$. The number of unknown and linear equation in this case is $2 M$.

$$
\begin{array}{r}
b_{1}^{t} M s_{1}^{(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{0, n}^{s s}\left(q_{2}, q_{1}\right)+b_{2}^{t} M s_{2}^{(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{1, n}^{s s}\left(q_{2}, q_{1}\right)+\cdots \\
b_{M, n}^{t} M s_{M}^{(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{M, n}^{s s}\left(q_{2}, q_{1}\right)-b_{n}^{s} M s_{n}^{(4)}\left(u_{0}, q_{1}\right)=b_{n}^{i} M s_{n}^{(1)}\left(u_{0}, q_{1}\right) \tag{4.96}
\end{array}
$$

and

$$
\begin{gathered}
\frac{1}{\epsilon_{2}}\left\{b_{1}^{t} M s_{1}^{\prime(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{0, n}^{s s}\left(q_{2}, q_{1}\right)+b_{2}^{t} M s_{2}^{\prime(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{1, n}^{s s}\left(q_{2}, q_{1}\right)+\cdots+\right. \\
\left.b_{M, n}^{t} M s_{M}^{\prime(1)}\left(u_{0}, q_{2}\right) \mathbf{I}_{M, n}^{s s}\left(q_{2}, q_{1}\right)\right\}-\frac{1}{\epsilon_{1}} b_{n}^{s} M s_{n}^{\prime(4)}\left(u_{0}, q_{1}\right)=\frac{1}{\epsilon_{1}} b_{n}^{i} M s_{n}^{\prime(1)}\left(u_{0}, q_{1}\right)(4.97)
\end{gathered}
$$

$$
\left[\begin{array}{ll}
\mathbf{A}_{m n} & \mathbf{B}_{m n}  \tag{4.98}\\
\mathbf{C}_{m n} & \mathbf{D}_{m n}
\end{array}\right]\left[\begin{array}{l}
\mathbf{X}_{n} \\
\mathbf{Y}_{n}
\end{array}\right]=\left[\begin{array}{l}
\mathbf{G}_{m} \\
\mathbf{H}_{m}
\end{array}\right]
$$

$$
\mathbf{A}_{m n}=M s_{m}^{(1)}\left(u_{0}, q_{2}\right) I_{m n}^{s s}\left(q_{1}, q_{2}\right)
$$

$$
\mathbf{C}_{m n}=\frac{1}{\epsilon_{2}} M s_{m}^{\prime(1)}\left(u_{0}, q_{2}\right) I_{m n}^{s s}\left(q_{1}, q_{2}\right)
$$

$$
\mathbf{B}_{m n}=-M s_{m}^{(4)}\left(u_{0}, q_{1}\right) \delta_{m n}
$$

$$
\mathbf{D}_{m n}=\frac{-1}{\epsilon_{1}} M s_{m}^{\prime(4)}\left(u_{0}, q_{1}\right) \delta_{m n}
$$

$$
\mathbf{G}_{m}=b_{m}^{i} M s_{m}^{(1)}\left(u_{0}, q_{1}\right)
$$

$$
\mathbf{H}_{m}=\frac{1}{\epsilon_{1}} b_{m}^{i} M s_{m}^{\prime(1)}\left(u_{0}, q_{1}\right)
$$

and the unknowns are

$$
\begin{aligned}
& \mathbf{X}_{n}=b_{n}^{t} \\
& \mathbf{Y}_{n}=b_{n}^{s}
\end{aligned}
$$

### 4.8 Multilayer Dielectric Elliptic Cylinder

In this section we will study the scattering and penetration of EM plane waves by Multilayer Lossy Dielectric Elliptic Cylinder for $T M^{z}$ polarization Fig.(4.11). It will be easy to modify the $T M^{z}$ and get results for $T E^{z}$ polarization as well. The wave that goes toward center of ellipse is called transmitted wave and the wave that goes backwards we call it scattered wave. In the $i t h$ layer we may have a current line source which is located at the point $u_{0}, v_{0}$. therefore

$$
\begin{align*}
E_{z}^{(i) t}= & \sum_{n=0}^{\infty} a t_{n}^{i} M c_{n}^{(1)}\left(u, q_{i}\right) c e_{n}\left(v, q_{i}\right)+ \\
& \sum_{n=1}^{\infty} b t_{n}^{i} M s_{n}^{(1)}\left(u, q_{i+1}\right) c e_{n}\left(v, q_{i}\right) \tag{4.99}
\end{align*}
$$



Figure 4.11: Multilayer Dielectric Elliptic Cylinder

$$
\begin{align*}
E_{z}^{(i) s}= & \sum_{n=0}^{\infty} a s_{n}^{i} M c_{n}^{(4)}\left(u, q_{i}\right) c e_{n}\left(v, q_{i}\right)+ \\
& \sum_{n=1}^{\infty} b s_{n}^{i} M s_{n}^{(4)}\left(u, q_{i}\right) c e_{n}\left(v, q_{i}\right) \tag{4.100}
\end{align*}
$$

where $q_{i}=\left(k_{i} f / 2\right)^{2}, k_{i}=\omega \sqrt{\mu_{0} \epsilon_{i}}$ and $f$ is semi confocal distance of elliptic. We can write the same equation for any layer except for first layer where $a s_{n}^{1}=b s_{n}^{1}=0$ and last layer where $a t_{n}^{N+1}=b t_{n}^{N+1}=0$

At boundary $u_{i}$ the tangential electric and magnetic wave is continuous, $H_{v}^{i t}+H_{v}^{i s}=H_{v}^{(i+1) t}+H_{v}^{(i+1) s}, E_{z}^{i t}+E_{z}^{i s}=E_{z}^{(i+1) t}+E_{z}^{(i+1) s}$ therefore

$$
\begin{aligned}
& E_{z}^{i t}+E_{z}^{i s}=E_{z}^{(i+1) t}+E_{z}^{(i+1) s} \\
& \frac{\partial E_{z}^{i t}}{\partial u}+\frac{\partial E_{z}^{i s}}{\partial u}=\frac{\partial E_{z}^{(i+1) t}}{\partial u}+\frac{\partial E_{z}^{(i+1) s}}{\partial u}
\end{aligned}
$$

by inserting Eq.(4.99) and Eq.(4.100) into Eq.(4.101) we will have

$$
\begin{align*}
\sum_{n=0}^{\infty} a t_{n}^{i+1} M c_{n}^{(1)}\left(u_{i}, q_{i+1}\right) c e_{n}\left(v, q_{i+1}\right) & + \\
\sum_{n=1}^{\infty} b t_{n}^{i+1} M s_{n}^{(1)}\left(u_{i}, q_{i+1}\right) s e_{n}\left(v, q_{i+1}\right) & + \\
\sum_{n=0}^{\infty} a s_{n}^{i+1} M c_{n}^{(4)}\left(u_{i}, q_{i+1}\right) c e_{n}\left(v, q_{i+1}\right) & + \\
\sum_{n=1}^{\infty} b s_{n}^{i+1} M s_{n}^{(4)}\left(u_{i}, q_{i+1}\right) s e_{n}\left(v, q_{i+1}\right) & = \\
\sum_{n=0}^{\infty} a t_{n}^{i} M c_{n}^{(1)}\left(u_{i}, q_{i}\right) c e_{n}\left(v, q_{i}\right) & + \\
\sum_{n=1}^{\infty} b t_{n}^{i} M s_{n}^{(1)}\left(u_{i}, q_{i}\right) s e_{n}\left(v, q_{i}\right) & + \\
\sum_{n=0}^{\infty} a s_{n}^{i} M c_{n}^{(4)}\left(u_{i}, q_{i}\right) c e_{n}\left(v, q_{i}\right) & + \\
\sum_{n=1}^{\infty} b s_{n}^{i} M s_{n}^{(4)}\left(u_{i}, q_{i}\right) s e_{n}\left(v, q_{i}\right) & + \\
\sum_{n=0}^{\infty} a I_{n}^{i} M c_{n}^{(4)}\left(u_{i}, q_{i}\right) c e_{n}\left(v, q_{i}\right) & + \\
\sum_{n=1}^{\infty} b I_{n}^{i} M s_{n}^{(4)}\left(u_{i}, q_{i}\right) s e_{n}\left(v, q_{i}\right) & \tag{4.101}
\end{align*}
$$

$$
\begin{align*}
\sum_{n=0}^{\infty} a t_{n}^{i+1} M c_{n}^{\prime(1)}\left(u_{i}, q_{i+1}\right) c e_{n}\left(v, q_{i+1}\right) & + \\
\sum_{n=1}^{\infty} b t_{n}^{i+1} M s_{n}^{\prime(1)}\left(u_{i}, q_{i+1}\right) s e_{n}\left(v, q_{i+1}\right) & + \\
\sum_{n=0}^{\infty} a s_{n}^{i+1} M c_{n}^{\prime(4)}\left(u_{i}, q_{i+1}\right) c e_{n}\left(v, q_{i+1}\right) & + \\
\sum_{n=1}^{\infty} b s_{n}^{i+1} M s_{n}^{\prime(4)}\left(u_{i}, q_{i+1}\right) s e_{n}\left(v, q_{i+1}\right) & = \\
\sum_{n=0}^{\infty} a t_{n}^{i} M c_{n}^{\prime(1)}\left(u_{i}, q_{i}\right) c e_{n}\left(v, q_{i}\right) & + \\
\sum_{n=1}^{\infty} b t_{n}^{i} M s_{n}^{\prime(1)}\left(u_{i}, q_{i}\right) s e_{n}\left(v, q_{i}\right) & + \\
\sum_{n=0}^{\infty} a s_{n}^{i} M c_{n}^{\prime(4)}\left(u_{i}, q_{i}\right) c e_{n}\left(v, q_{i}\right) & + \\
\sum_{n=1}^{\infty} b s_{n}^{i} M s_{n}^{\prime(4)}\left(u_{i}, q_{i}\right) s e_{n}\left(v, q_{i}\right) & + \\
\sum_{n=0}^{\infty} a I_{n}^{i} M c_{n}^{\prime(4)}\left(u_{i}, q_{i}\right) c e_{n}\left(v, q_{i}\right) & + \\
\sum_{n=1}^{\infty} b I_{n}^{i} M s_{n}^{\prime(4)}\left(u_{i}, q_{i}\right) s e_{n}\left(v, q_{i}\right) & \tag{4.102}
\end{align*}
$$

where the $\operatorname{prim}{ }^{\prime}$ shows derivative with respect $u$, and for the plane wave incident we have $a_{n}^{i n}=2(j)^{n} c e_{n}\left(\phi_{0}, q_{N+1}\right), b_{n}^{i n}=2(j)^{n} s e_{n}\left(\phi_{0}, q_{N+1}\right)$, and $a I_{n}^{i}=-\frac{I_{0}}{2 \omega \mu} M c_{n}^{(4)}\left(u_{0}, q_{i}\right) c e_{n}\left(v_{0}, q_{i}\right), b I_{n}^{i}=-\frac{I_{0}}{2 \omega \mu} M s_{n}^{(4)}\left(u_{0}, q_{i}\right) s e_{n}\left(v_{0}, q_{i}\right) . \quad$ In order to find the coefficient we use the C . Yeh method [64]. We multiply both side of Eq.(4.101) and Eq.(4.102) first by $c e_{m}\left(v, q_{i+1}\right) / \pi$ and then by $s e_{m}\left(v, q_{i+1}\right) / \pi$, and integrate over interval $(0,2 \pi)$, therefore for any boundary we will have four important equations. In order to have a linear system of
equation we truncate the series to maximum number $M$.

$$
\begin{align*}
a t_{m}^{i+1} M c_{m}^{(1)}\left(u_{i}, q_{i+1}\right)+a s_{m}^{i+1} M c_{m}^{(4)}\left(u_{i}, q_{i+1}\right) & = \\
\sum_{n=0}^{M} a t_{n}^{i} M c_{n}^{(1)}\left(u_{i}, q_{i}\right) \mathbf{I}_{m, n}^{c c}\left(q_{i+1}, q_{i}\right) & + \\
\sum_{n=0}^{M} a s_{n}^{i} M c_{n}^{(4)}\left(u_{i}, q_{i}\right) \mathbf{I}_{m, n}^{c c}\left(q_{i+1}, q_{i}\right) & + \\
\sum_{n=0}^{\infty} a I_{n}^{i} M c_{n}^{(4)}\left(u_{i}, q_{i}\right) \mathbf{I}_{m, n}^{c c}\left(q_{i+1}, q_{i}\right) & \tag{4.103}
\end{align*}
$$

$$
a t_{m}^{i+1} M c_{m}^{\prime(1)}\left(u_{i}, q_{i+1}\right)+a s_{m}^{i+1} M c_{m}^{\prime(4)}\left(u_{i}, q_{i+1}\right)=
$$

$$
\sum_{n=0}^{M} a t_{n}^{i} M c_{n}^{\prime(1)}\left(u_{i}, q_{i}\right) \mathbf{I}_{m, n}^{c c}\left(q_{i+1}, q_{i}\right) \quad+
$$

$$
\sum_{n=0}^{M} a s_{n}^{i} M c_{n}^{\prime(4)}\left(u_{i}, q_{i}\right) \mathbf{I}_{m, n}^{c c}\left(q_{i+1}, q_{i}\right)+
$$

$$
\begin{equation*}
\sum_{n=0}^{M} a I_{n}^{i} M c_{n}^{\prime(4)}\left(u_{i}, q_{i}\right) \mathbf{I}_{m, n}^{c c}\left(q_{i+1}, q_{i}\right) \tag{4.104}
\end{equation*}
$$

$$
b t_{m}^{i+1} M s_{m}^{(1)}\left(u_{i}, q_{i+1}\right)+b s_{m}^{i+1} M s_{m}^{(4)}\left(u_{i}, q_{i+1}\right)=
$$

$$
\sum_{n=1}^{M+1} b t_{n}^{i} M s_{n}^{(1)}\left(u_{i}, q_{i}\right) \mathbf{I}_{m, n}^{s s}\left(q_{i+1}, q_{i}\right)+
$$

$$
\begin{equation*}
\sum_{n=1}^{M+1} b s_{n}^{i} M s_{n}^{(4)}\left(u_{i}, q_{i}\right) \mathbf{I}_{m, n}^{s s}\left(q_{i+1}, q_{i}\right) \tag{4.105}
\end{equation*}
$$

$$
b t_{m}^{i+1} M s_{m}^{\prime(1)}\left(u_{i}, q_{i+1}\right)+b s_{m}^{i+1} M s_{m}^{\prime(4)}\left(u_{i}, q_{i+1}\right)=
$$

$$
\sum_{n=1}^{M+1} b t_{n}^{i} M s_{n}^{\prime(1)}\left(u_{i}, q_{i}\right) \mathbf{I}_{m, n}^{s s}\left(q_{i+1}, q_{i}\right)+
$$

$$
\begin{equation*}
\sum_{n=1}^{M+1} b s_{n}^{i} M s_{n}^{\prime(4)}\left(u_{i}, q_{i}\right) \mathbf{I}_{m, n}^{s s}\left(q_{i+1}, q_{i}\right) \tag{4.106}
\end{equation*}
$$

$$
\begin{align*}
b t_{m}^{i+1} M s_{m}^{(1)}\left(u_{i}, q_{i+1}\right)+b s_{m}^{i+1} M c_{m}^{(4)}\left(u_{i}, q_{i+1}\right) & = \\
\sum_{n=0}^{M+1} b t_{n}^{i} M s_{n}^{(1)}\left(u_{i}, q_{i}\right) \mathbf{I}_{m, n}^{s s}\left(q_{i+1}, q_{i}\right) & + \\
\sum_{n=1}^{M+1} b s_{n}^{i} M s_{n}^{(4)}\left(u_{i}, q_{i}\right) \mathbf{I}_{m, n}^{s s}\left(q_{i+1}, q_{i}\right) & + \\
\sum_{n=1}^{M+1} b I_{n}^{i} M s_{n}^{(4)}\left(u_{i}, q_{i}\right) \mathbf{I}_{m, n}^{s s}\left(q_{i+1}, q_{i}\right) &  \tag{4.107}\\
b t_{m}^{i+1} M s_{m}^{\prime(1)}\left(u_{i}, q_{i+1}\right)+b s_{m}^{i+1} M s_{m}^{\prime(4)}\left(u_{i}, q_{i+1}\right) & = \\
\sum_{n=1}^{M+1} b t_{n}^{i} M s_{n}^{\prime(1)}\left(u_{i}, q_{i}\right) \mathbf{I}_{m, n}^{s s}\left(q_{i+1}, q_{i}\right) & + \\
\sum_{n=1}^{M+1} b s_{n}^{i} M s_{n}^{\prime(4)}\left(u_{i}, q_{i}\right) \mathbf{I}_{m, n}^{s s}\left(q_{i+1}, q_{i}\right) & + \\
\sum_{n=1}^{M+1} b I_{n}^{i} M s_{n}^{\prime(4)}\left(u_{i}, q_{i}\right) \mathbf{I}_{m, n}^{s s}\left(q_{i+1}, q_{i}\right) & \tag{4.108}
\end{align*}
$$

The index $m$ also varies from 0 to $M$ or from 1 to $M+1$. Therefore after applying the same procedure for each boundary, at the end we will have two separate square matrix with $2 N(M+1) \times 2 N(M+1)$ elements, one for $a^{\prime} s$ and one for $b^{\prime} s$ coefficients. The Matrix will look like

$$
\left[\begin{array}{ccccccc}
\mathbf{S}_{w}^{N} & \mathbf{T}^{N-1} & \mathbf{S}^{N-1} & 0 & \cdots & \cdots & \cdots  \tag{4.109}\\
\mathbf{S}_{w}^{\prime N} & \mathbf{T}^{\prime N-1} & \mathbf{S}^{\prime N-1} & 0 & \cdots & \cdots & \cdots \\
0 & \mathbf{T}_{w}^{N-1} & \mathbf{S}_{w}^{N-1} & \mathbf{T}^{N-2} & \cdots & \cdots & \cdots \\
0 & \mathbf{T}_{w}^{N-1} & \mathbf{S}_{w}^{\prime N-1} & \mathbf{T}^{\prime N-2} & \cdots & \cdots & \cdots \\
& & \ddots & & & & \\
0 & \cdots & \mathbf{S}_{w}^{i} & \mathbf{T}_{w}^{i} & \mathbf{S}^{i-1} & T^{i-1} & \ldots \\
0 & \cdots & \mathbf{S}_{w}^{i} & \mathbf{T}^{\prime}{ }_{w} & \mathbf{S}^{\prime i-1} & \mathbf{T}^{i-1} & \ldots \\
& & \ddots & & & & \\
0 & \cdots & \cdots & \cdots & \mathbf{S}_{w}^{2} & \mathbf{T}_{w}^{2} & \mathbf{T}^{1} \\
0 & \cdots & \cdots & \cdots & \mathbf{S}_{w}^{\prime 2} & \mathbf{T}_{w}^{\prime 2} & \mathbf{T}^{\prime 1}
\end{array}\right]\left[\begin{array}{c}
\mathbf{a s}_{m}^{N+1} \\
\mathbf{a t}_{m}^{N} \\
\mathbf{a s}_{m}^{N} \\
\vdots \\
\mathbf{a s}_{m}^{i} \\
\mathbf{a t}_{m}^{i} \\
\vdots \\
\mathbf{a t}_{m}^{2} \\
\mathbf{a s}_{m}^{2} \\
\mathbf{a t}_{m}^{1}
\end{array}\right]=\left[\begin{array}{c}
\mathbf{a}^{1}{ }_{n} \\
0 \\
\vdots \\
\mathbf{c s}_{m} \\
\mathbf{c t}_{m} \\
\mathbf{c s}^{\prime}{ }_{m} \\
\mathbf{c t}^{\prime}{ }_{m} \\
0 \\
\vdots \\
0
\end{array}\right]
$$

The submatrix will be

$$
\begin{align*}
\mathbf{S}_{w}^{i}=M c_{m}^{(4)}\left(q_{i+1}, u_{i}\right) \delta_{m n} & \mathbf{T}_{w}^{i}=M c_{m}^{(1)}\left(q_{i+1}, u_{i}\right) \delta_{m n} \\
\mathbf{S}_{w}^{\prime i}=M c_{m}^{\prime(4)}\left(q_{i+1}, u_{i}\right) \delta_{m n} & \mathbf{T}_{w}^{\prime i}=M c_{m}^{(1)}\left(q_{i+1}, u_{i}\right) \delta_{m n} \\
\mathbf{S}^{i}=-M c_{m}^{(4)}\left(q_{i+1}, u_{i}\right) \mathbf{I}_{m, n}^{s s}\left(q_{i+1}, q_{i}\right) & \mathbf{T}^{i}=-M c_{m}^{(1)}\left(q_{i+1}, u_{i}\right) \mathbf{I}_{m, n}^{s s}\left(q_{i+1}, q_{i}\right) \\
\mathbf{S}^{\prime i}=-M c_{m}^{\prime(4)}\left(q_{i+1}, u_{i}\right) \mathbf{I}_{m, n}^{s s}\left(q_{i+1}, q_{i}\right) & \mathbf{T}^{\prime i}=-M c_{m}^{\prime(1)}\left(q_{i+1}, u_{i}\right) \mathbf{I}_{m, n}^{s s}\left(q_{i+1}, q_{i}\right) \\
\mathbf{c s}_{m}^{i}=\mathbf{S}_{m n}^{i} \cdot \mathbf{a c}_{n}^{(1)} & \mathbf{c t}_{m}^{i}=\mathbf{T}_{m n}^{i} \cdot \mathbf{a c}_{n}^{(4)} \\
\mathbf{c s}_{m}^{\prime i}=\mathbf{S}_{m n}^{\prime i} \cdot \mathbf{a c}_{n}^{(1)} & \mathbf{c t}^{\prime i}{ }_{m}=\mathbf{T}^{i \prime}{ }_{m n} \cdot \mathbf{a c}_{n}^{(4)} \\
\mathbf{a}_{m}^{1}=2(j)^{m} c e\left(\phi_{0}, q_{N+1}\right) &
\end{align*}
$$

If the line source be at layer number one, the terms $c s_{m}^{i}$ and $c s^{\prime \prime}{ }_{m}$ will be



Figure 4.12: Two layer dielectric elliptic cylinder
zero. And if the line source be at layer number $N+1$ the terms $c t_{m}^{i}$ and $c t^{\prime i}{ }_{m}$ will be zero. If we have only incident plane wave, all the line current source terms will be zero and the right hand side of the matrix Eq.(4.109)will have only two terms of the incident plane wave.

### 4.9 Scattering by a Conducting Strip

The previous section was a general problem. In special case of elliptic cylinder which looks like a strip, Fig.(4.13). If the small diameter of a conducting
elliptic cylinder goes to zero, then we will have


Figure 4.13: TE or TM wave scattering by a conducting strip


Figure 4.14: TE and TM wave scattering by a conducting strip; $d=2 \lambda$

### 4.10 Scattering from a semi-elliptical channel in ground plane

In previous section we have solved scattering of EM wave from semi-circular channel. Now we want to do the same problem for semi-elliptical case

Fig.(4.15).
A linearly polarized electromagnetic plane wave is incident at an angle $\phi_{0}$


Figure 4.15: TM
with respect to $x$-axis, on the structure shown in Fig.(4.15), in which free space region is labeled medium $I$ and dielectric region is labeled medium $I I$. For the TM case, the electric field, in each region, has only axial component $E_{z}$.
In medium $I\left(u>u_{1}, 0<v<\pi\right)$ the total field may be decomposed into three parts: the incident, specularly reflected, and scattered fields. The incident and specularly reflected fields are represented by

$$
E_{z}^{i}=\exp \left\{j k_{1}\left(x \cos \phi_{0}+y \sin \phi_{0}\right)\right\}
$$

or

$$
\begin{gather*}
E_{z}^{i}=\sum_{m=0}^{\infty} a_{m}^{i} M c_{m}^{(1)}\left(u, q_{1}\right) c e_{m}\left(v, q_{1}\right)+\sum_{m=1}^{\infty} b_{m}^{i} M s_{m}^{(1)}\left(u, q_{1}\right) s e_{m}\left(v, q_{1}\right)  \tag{4.111}\\
E_{z}^{r e f}(u, v)=-E_{z}^{i n c}(u, v) \text { with } \phi_{0} \rightarrow 2 \pi-\phi_{0} \tag{4.112}
\end{gather*}
$$

where $q_{1}=\frac{f^{2} k_{1}^{2}}{4}, q_{2}=\frac{f^{2} k_{2}^{2}}{4}, k=\omega \sqrt{\mu \epsilon_{c}}$ and $\epsilon_{c}=\epsilon_{0}\left(\epsilon_{r}-j \frac{\sigma}{\omega \epsilon_{0}}\right)$. The value of $a_{m}^{i}$ and $b_{m}^{i}$ will be $a_{m}^{i}=2(j)^{m} c e_{m}\left(\phi_{0}, q_{1}\right), b_{m}^{i}=2(j)^{m} s e_{m}\left(\phi_{0}, q_{1}\right)$.
The scattered field in medium $I$ may be given my imposing boundary condition $E_{z}^{s}=0$ on PEC substrate since $s e_{m}\left(v, q_{1}\right)=0$ at $v=0$ and $\pi$, and it is represented by

$$
\begin{equation*}
E_{z}^{s}(u, v)=\sum_{m=1}^{\infty} A_{m} M s_{m}^{(4)}\left(u, q_{1}\right) s e_{m}\left(v, q_{1}\right) \tag{4.113}
\end{equation*}
$$

The total field in medium $I$ is represented by

$$
\begin{align*}
E_{z}^{I}(u, v) & =E_{z}^{i}(u, v)+E_{z}^{s}(u, v)+E_{z}^{r}(u, v)  \tag{4.114}\\
& =\sum_{m=1}^{\infty} A_{m} M s_{m}^{(4)}\left(u, q_{1}\right) s e_{m}\left(v, q_{1}\right)+\sum_{m=1}^{\infty} a_{m}^{i r} M s_{m}^{(1)}\left(u, q_{1}\right) s e_{m}\left(v, q_{1}\right)
\end{align*}
$$

where $a_{m}^{i r}=4(j)^{m} s e_{m}\left(\phi_{0}, q_{1}\right)$.
In medium $I I\left(u<u_{1}, \pi<v<2 \pi\right)$, the transmitted electric field may also be represented as

$$
\begin{equation*}
E_{z}^{I I}(u, v)=\sum_{m=0}^{\infty} B_{m} M c_{m}^{(1)}\left(u, q_{2}\right) c e_{m}\left(v, q_{2}\right)+\sum_{m=1}^{\infty} C_{m} M s_{m}^{(1)}\left(u, q_{2}\right) s e_{m}\left(v, q_{2}\right) \tag{4.115}
\end{equation*}
$$

From Maxwell's equations, the $v$-components of the magnetic field may be represented as

$$
\begin{equation*}
H_{v}^{I, I I}=\frac{1}{f \sqrt{\cosh ^{2} u-\cos ^{2} v}} \frac{1}{j \omega \mu} \frac{\partial E_{z}^{I, I I}}{\partial u} \tag{4.116}
\end{equation*}
$$

The unknown coefficients $A_{m}, B_{m}$ and $C_{m}$ can be determined with the boundary conditions at $u=u_{1}$ of zero tangential electric field on the channel $(\pi<v<2 \pi)$ and field continuity across the aperture $(0<v<\pi)$.
Continuity of electric field on aperture $(0<v<\pi)$.

$$
\begin{align*}
& \sum_{m=1}^{\infty} A_{m} M s_{m}^{(4)}\left(u_{1}, q_{1}\right) s e_{m}\left(v, q_{1}\right)+\sum_{m=1}^{\infty} a_{m}^{i r} M s_{m}^{(1)}\left(u_{1}, q_{1}\right) s e_{m}\left(v, q_{1}\right)= \\
& \sum_{m=0}^{\infty} B_{m} M c_{m}^{(1)}\left(u_{1}, q_{2}\right) c e_{m}\left(v, q_{2}\right)+\sum_{m=1}^{\infty} C_{m} M s_{m}^{(1)}\left(u_{1}, q_{2}\right) s e_{m}\left(v, q_{2}\right) \tag{4.117}
\end{align*}
$$

Continuity of magnetic field on aperture $(0<v<\pi)$.

$$
\begin{align*}
& \frac{1}{\mu_{1}}\left[\sum_{m=1}^{\infty} A_{m} M s_{m}^{\prime(4)}\left(u_{1}, q_{1}\right) s e_{m}\left(v, q_{1}\right)+\sum_{m=1}^{\infty} a_{m}^{i r} M s_{m}^{\prime(1)}\left(u_{1}, q_{1}\right) s e_{m}\left(v, q_{1}\right)\right]= \\
& \frac{1}{\mu_{2}}\left[\sum_{m=0}^{\infty} B_{m} M c_{m}^{\prime(1)}\left(u_{1}, q_{2}\right) c e_{m}\left(v, q_{2}\right)+\sum_{m=1}^{\infty} C_{m} M c_{m}^{\prime(1)}\left(u_{1}, q_{2}\right) s e_{m}\left(v, q_{2}\right)\right](4.11 \tag{4.118}
\end{align*}
$$

and zero tangential electric field on the channel ( $\pi<v<2 \pi$ )

$$
\begin{equation*}
\sum_{m=0}^{\infty} B_{m} M c_{m}^{(1)}\left(u, q_{2}\right) c e_{m}\left(v, q_{2}\right)+\sum_{m=1}^{\infty} C_{m} M s_{m}^{(1)}\left(u, q_{2}\right) s e_{m}\left(v, q_{2}\right)=0 \tag{4.119}
\end{equation*}
$$

Since angular Mathieu functions with different parameters are not fully orthogonal, some normalization integrals may occur. Expansion of such integrals over $[0,2 \pi]$ interval in terms of Mathieu expansion coefficients were treated in [55]. In problems involving boundary conditions over half ellipse, normalization integrals occur on $[0, \pi]$ interval which can be computed by Mathieu expansion coefficients as follows

$$
\begin{gather*}
K_{m n}^{s s}\left(q_{1}, q_{2}\right)=\left\{\begin{array}{cc}
\frac{\pi}{2} \sum_{k=1,3, \ldots} B_{k}^{m}\left(q_{1}\right) B_{k}^{n}\left(q_{2}\right), & m, n \text { odd } \\
\frac{\pi}{2} \sum_{k=2,4, \ldots} B_{k}^{m}\left(q_{1}\right) B_{k}^{n}\left(q_{2}\right), & m, n \text { even; } \\
0, & \text { otherwise. }
\end{array}\right.  \tag{4.120}\\
K_{m n}^{s c}\left(q_{1}, q_{2}\right)=\left\{\begin{array}{cc}
\sum_{p=2,4, \ldots k=1,3, \ldots} \frac{2 p}{\sum_{p=1,3} \sum_{p=0,2, \ldots} \frac{2 p}{p^{2}-k^{2}} B_{p}^{m}\left(q_{1}\right) A_{k}^{n}\left(q_{2}\right),} \quad \text { modd, } n \text { even } ; \\
0, & m \text { oven, } n \text { odd } ; \\
\sum_{k}^{n}\left(q_{2}\right),
\end{array}\right. \tag{4.121}
\end{gather*}
$$

where

$$
\begin{align*}
& K_{m n}^{s c}\left(q_{1}, q_{2}\right)=\int_{0}^{\pi} s e_{m}\left(v, q_{1}\right) c e_{n}\left(v, q_{2}\right) d v  \tag{4.122}\\
& K_{m n}^{s s}\left(q_{1}, q_{2}\right)=\int_{0}^{\pi} s e_{m}\left(v, q_{1}\right) s e_{n}\left(v, q_{2}\right) d v \tag{4.123}
\end{align*}
$$

also

$$
\begin{align*}
& \int_{\pi}^{2 \pi} s e_{m}(v, q) c e_{n}(v, q) d v=-K_{m n}^{s c}(q, q)  \tag{4.124}\\
& \int_{\pi}^{2 \pi} s e_{m}(v, q) s e_{n}(v, q) d v=K_{m n}^{s s}(q, q) \tag{4.125}
\end{align*}
$$

These expressions are also valid for complex parameter case. We multiply both side of Eq.(4.117) and Eq.(4.118) by $c e_{n}\left(v, q_{1}\right)$ and integrating over $[0, \pi]$ and truncate it to maximum number of terms $M$, then we have

$$
\begin{aligned}
& -\sum_{m=1}^{M} A_{m} M s_{m}^{(4)}\left(u_{1}, q_{1}\right) K_{m n}^{s s}\left(q_{1}, q_{2}\right)+\sum_{m=0}^{M-1} B_{m} M c_{m}^{(1)}\left(u_{1}, q_{2}\right) K_{m n}^{s c}\left(q_{2}, q_{2}\right)+ \\
& \sum_{m=1}^{M} C_{m} M s_{m}^{(1)}\left(u_{1}, q_{2}\right) K_{m n}^{s s}\left(q_{2}, q_{2}\right)=\sum_{m=1}^{M} a_{m}^{i r} M s_{m}^{(1)}\left(u_{1}, q_{1}\right) K_{m n}^{s s}\left(q_{1}, q_{2}\right)(4.126)
\end{aligned}
$$

where $n=1,2, \cdots M$

$$
\begin{align*}
& -\frac{1}{\mu_{1}} \sum_{m=1}^{M} A_{m} M s_{m}^{\prime(4)}\left(u_{1}, q_{1}\right) K_{m n}^{s s}\left(q_{1}, q_{2}\right)+\frac{1}{\mu_{2}} \sum_{m=0}^{M} B_{m} M c_{m}^{\prime(1)}\left(u_{1}, q_{2}\right) K_{m n}^{s c}\left(q_{2}, q_{2}\right) \\
& +\frac{1}{\mu_{2}} \sum_{m=1}^{M} C_{m} M s_{m}^{\prime(1)}\left(u_{1}, q_{2}\right) K_{m n}^{s s}\left(q_{2}, q_{2}\right)=\frac{1}{\mu_{1}} \sum_{m=1}^{M} a_{m}^{i r} M s_{m}^{\prime(1)}\left(u_{1}, q_{1}\right) K_{m n}^{s s}\left(q_{1}, q_{2}\right) \tag{4.127}
\end{align*}
$$

We multiply both side of Eq.(4.119) by $c e_{n}\left(v, q_{2}\right)$ and integrating over $[\pi, 2 \pi]$, and using properties of Eq.(4.124)

$$
-\sum_{m=0}^{M-1} B_{m} M c_{m}^{(1)}\left(u_{1}, q_{2}\right) K_{m n}^{s c}\left(q_{2}, q_{2}\right)+\sum_{m=1}^{M} C_{m} M s_{m}^{(1)}\left(u_{1}, q_{2}\right) K_{m n}^{s s}\left(q_{2}, q_{2}\right)=0
$$

Now it is time to put Eq.(4.126), Eq.(4.127), and Eq.(4.128) into matrix form. If we substitute $n=1, \cdots, M$ in the above equations, we will have a square block matrix of $3 M \times 3 M$ which the $3 M$ unknowns can be calculated.

$$
\left[\begin{array}{lll}
\mathbf{A}_{m n}^{11} & \mathbf{A}_{m n}^{12} & \mathbf{A}_{m n}^{13}  \tag{4.128}\\
\mathbf{A}_{m n}^{21} & \mathbf{A}_{m n}^{22} & \mathbf{A}_{m n}^{23} \\
\mathbf{A}_{m n}^{31} & \mathbf{A}_{m n}^{32} & \mathbf{A}_{m n}^{33}
\end{array}\right]\left[\begin{array}{l}
\mathbf{X}_{n}^{1} \\
\mathbf{X}_{n}^{2} \\
\mathbf{X}_{n}^{3}
\end{array}\right]=\left[\begin{array}{l}
\mathbf{B}_{m}^{1} \\
\mathbf{B}_{m}^{2} \\
\mathbf{B}_{m}^{3}
\end{array}\right]
$$

where

$$
\mathbf{A}_{m n}^{11}=-M s_{m}^{(4)}\left(u_{1}, q_{1}\right) K_{m n}^{s s}\left(q_{1}, q_{2}\right), \quad \mathbf{A}_{m n}^{12}=M c_{m}^{(1)}\left(u_{1}, q_{2}\right) K_{m n}^{s c}\left(q_{2}, q_{2}\right),
$$

$$
\begin{gathered}
\mathbf{A}_{m n}^{13}=M s_{m}^{(1)}\left(u_{1}, q_{2}\right) K_{m n}^{s s}\left(q_{2}, q_{2}\right), \quad \mathbf{A}_{m n}^{21}=-\frac{1}{\mu_{1}} M s_{m}^{\prime(4)}\left(u_{1}, q_{1}\right) K_{m n}^{s s}\left(q_{1}, q_{2}\right), \\
\mathbf{A}_{m n}^{22}=\frac{1}{\mu_{2}} M c_{m}^{\prime(1)}\left(u_{1}, q_{1}\right) K_{m n}^{s c}\left(q_{2}, q_{2}\right), \quad \mathbf{A}_{m n}^{23}=\frac{1}{\mu_{2}} M s_{m}^{\prime(1)}\left(u_{1}, q_{1}\right) K_{m n}^{s s}\left(q_{2}, q_{2}\right), \\
\mathbf{A}_{m n}^{31}=0, \mathbf{A}_{m n}^{32}=-M c_{m}^{(1)}\left(u_{1}, q_{2}\right) K_{m n}^{s c}\left(q_{2}, q_{2}\right), \mathbf{A}_{m n}^{33}=M s_{m}^{(1)}\left(u_{1}, q_{2}\right) K_{m n}^{s s}\left(q_{2}, q_{2}\right) .
\end{gathered}
$$

and right hand of the above matrix:

$$
\begin{gathered}
\mathbf{B}_{m}^{1}=\sum_{n=1}^{M} a_{m}^{i r} M s_{m}^{(1)}\left(u_{1}, q_{1}\right) K_{m n}^{s s}\left(q_{1}, q_{2}\right) \\
\mathbf{B}_{m}^{2}=\frac{1}{\mu_{1}} \sum_{n=1}^{M} a_{m}^{i r} M s_{m}^{\prime(1)}\left(u_{1}, q_{1}\right) K_{m n}^{s s}\left(q_{1}, q_{2}\right), \quad \mathbf{B}_{m}^{3}=0, \quad m=1,2, \cdots M
\end{gathered}
$$

and the unknowns are

$$
\begin{array}{lr}
\mathbf{X}_{n}^{1}=A_{n}, & n=1,2, \cdots, M \\
\mathbf{X}_{n}^{2}=B_{n}, & n=0,1, \cdots, M-1 \\
\mathbf{X}_{n}^{3}=C_{n}, & n=1,2, \cdots, M
\end{array}
$$

### 4.11 EM Wave Scattering by Elliptic Chiral Cylinder

We develop an exact solution for the $T M^{z}$ polarized electromagnetic wave scattering by an elliptic chiral cylinder. Like the previous problems, the incident, transmitted and scattered waves are expressed in terms of infinite series of elliptic wave functions. The geometry of cylinder with $T M^{z}$ uniform plane wave incident on it is shown in Fig.(4.16). Let us consider a $T M^{z}$ plane wave

$$
\begin{equation*}
\mathbf{E}^{\mathbf{i}}=E_{0} e^{j k\left(x \cos \phi_{0}+y \sin \phi_{0}\right)} \mathbf{a}_{\mathbf{z}} \tag{4.129}
\end{equation*}
$$



Figure 4.16: Scattering from Elliptic Chiral Cylinder, TM-Polarization
therefore we can write it as

$$
\begin{equation*}
\mathbf{E}^{\mathbf{i}}=\sum_{n=0}^{\infty} A_{n}^{e} \mathbf{N}_{n}^{e(1)}(u, q)+\sum_{n=1}^{\infty} A_{n}^{o} \mathbf{N}_{n}^{o(1)}(u, q) \tag{4.130}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{H}^{\mathbf{i}}=\frac{j}{\eta}\left[\sum_{n=0}^{\infty} A_{n}^{e} \mathbf{M}_{n}^{e(1)}(u, q)+\sum_{n=1}^{\infty} A_{n}^{o} \mathbf{M}_{n}^{o(1)}(u, q)\right] \tag{4.131}
\end{equation*}
$$

where $A_{n}^{e}=\frac{2 j^{n} E_{0}}{k} c e_{n}\left(\phi_{0}, q\right)$ and $A_{n}^{o}=\frac{2 j^{n} E_{0}}{k} s e_{n}\left(\phi_{0}, q\right)$. The scattered field can be written as:

$$
\begin{align*}
\mathbf{E}^{\mathbf{s}} & =\left[\sum_{n=0}^{\infty} B_{n}^{e} \mathbf{N}_{n}^{e(4)}(u, q)+C_{n}^{e} \mathbf{M}_{n}^{e(4)}(u, q)\right.  \tag{4.132}\\
& \left.+\sum_{n=1}^{\infty} B_{n}^{o} \mathbf{N}_{n}^{o(4)}(u, q)+C_{n}^{o} \mathbf{M}_{n}^{o(4)}(u, q)\right]
\end{align*}
$$

$$
\begin{align*}
\mathbf{H}^{\mathrm{s}} & =\frac{j}{\eta}\left[\sum_{n=0}^{\infty} B_{n}^{e} \mathbf{M}_{n}^{e(4)}(u, q)+C_{n}^{e} \mathbf{N}_{n}^{e(4)}(u, q)\right.  \tag{4.133}\\
& \left.+\sum_{n=1}^{\infty} B_{n}^{o} \mathbf{M}_{n}^{o(4)}(u, q)+C_{n}^{o} \mathbf{N}_{n}^{o(4)}(u, q)\right]
\end{align*}
$$

and the fields inside the elliptic chiral cylinder will be: $\mathbf{E}^{t}=\mathbf{Q}_{R}+j \eta_{c} \mathbf{Q}_{L}$, $\mathbf{H}^{t}=\mathbf{Q}_{L}+\frac{j}{\eta_{c}} \mathbf{Q}_{R}$ and $\eta_{c}=\sqrt{\frac{\mu_{c}}{\epsilon_{c}}}$.

$$
\begin{align*}
\mathbf{Q}_{R} & =\sum_{n=0}^{\infty} R_{n}^{e}\left[\mathbf{M}_{n}^{e(1)}\left(u, q_{R}\right)-\mathbf{N}_{n}^{e(1)}\left(u, q_{R}\right)\right]  \tag{4.134}\\
& +\sum_{n=1}^{\infty} R_{n}^{o}\left[\mathbf{M}_{n}^{o(1)}\left(u, q_{R}\right)-\mathbf{N}_{n}^{o(1)}\left(u, q_{R}\right)\right] \\
\mathbf{Q}_{L} & =\sum_{n=0}^{\infty} L_{n}^{e}\left[\mathbf{M}_{n}^{e(1)}\left(u, q_{L}\right)+\mathbf{N}_{n}^{e(1)}\left(u, q_{L}\right)\right]  \tag{4.135}\\
& +\sum_{n=1}^{\infty} L_{n}^{o}\left[\mathbf{M}_{n}^{o(1)}\left(u, q_{L}\right)+\mathbf{N}_{n}^{o(1)}\left(u, q_{L}\right)\right]
\end{align*}
$$

where

$$
k_{R}=\frac{\omega \sqrt{\mu_{c} \epsilon_{c}}}{1+\beta \omega \sqrt{\mu_{c} \epsilon_{c}}}
$$

and

$$
k_{L}=\frac{\omega \sqrt{\mu_{c} \epsilon_{c}}}{1-\beta \omega \sqrt{\mu_{c} \epsilon_{c}}}
$$

where $q_{R}=\left(k_{R} f / 2\right)^{2}, q_{L}=\left(k_{L} f / 2\right)^{2}$ and $q=(k f / 2)^{2}$. Now we can apply boundary conditions in order to find eight array of unknown coefficients $B_{n}^{e}, B_{n}^{o}, C_{n}^{e}, C_{n}^{o}, R_{n}^{e}, R_{n}^{o}, L_{n}^{e}, L_{n}^{o}$ at $u=u_{0}$ for any values of $v$.

$$
\begin{align*}
\mathbf{a}_{z} \cdot\left[\mathbf{E}^{i}+\mathbf{E}^{s}-\mathbf{E}^{t}\right] & =0 \\
\mathbf{a}_{v} \cdot\left[\mathbf{E}^{i}+\mathbf{E}^{s}-\mathbf{E}^{t}\right] & =0 \\
\mathbf{a}_{z} \cdot\left[\mathbf{H}^{i}+\mathbf{H}^{s}-\mathbf{H}^{t}\right] & =0 \\
\mathbf{a}_{v} \cdot\left[\mathbf{H}^{i}+\mathbf{H}^{s}-\mathbf{H}^{t}\right] & =0 \tag{4.136}
\end{align*}
$$

The z-component of electric field will be:

$$
\begin{align*}
& E_{z}^{i}=k \sum_{n=0}^{\infty} A_{n}^{e} M c_{n}^{(1)}(u, q) c e_{n}(v, q)+k \sum_{n=1}^{\infty} A_{n}^{o} M s_{n}^{(1)}(u, q) s e_{n}(v, q)  \tag{4.137}\\
& E_{z}^{s}=k \sum_{n=0}^{\infty} B_{n}^{e} M c_{n}^{(4)}(u, q) c e_{n}(v, q)+k \sum_{n=1}^{\infty} B_{n}^{o} M s_{n}^{(4)}(u, q) s e_{n}(v, q)  \tag{4.138}\\
& E_{z}^{t}=k_{L} \sum_{n=0}^{\infty} L_{n}^{e} M c_{n}^{(1)}\left(u, q_{L}\right) c e_{n}\left(v, q_{L}\right)+k_{L} \sum_{n=1}^{\infty} L_{n}^{o} M s_{n}^{(1)}\left(u, q_{L}\right) s e_{n}\left(v, q_{L}\right) \\
& +j \eta_{c}\left\{k_{R} \sum_{n=0}^{\infty} R_{n}^{e} M c_{n}^{(1)}\left(u, q_{R}\right) c e_{n}\left(v, q_{R}\right)+k_{R} \sum_{n=1}^{\infty} R_{n}^{o} M s_{n}^{(1)}\left(u, q_{R}\right) s e_{n}\left(v, q_{R}\right)\right\} \tag{4.139}
\end{align*}
$$

The v-component of electric field will be:

$$
\begin{gather*}
E_{v}^{i}=0 \\
E_{v}^{s}=\sum_{n=0}^{\infty} C_{n}^{e} M c_{n}^{\prime(4)}(u, q) c e_{n}(v, q)+\sum_{n=1}^{\infty} C_{n}^{o} M s_{n}^{\prime(4)}(u, q) s e_{n}(v, q)  \tag{4.140}\\
E_{v}^{t}=-\sum_{n=0}^{\infty} L_{n}^{e} M c_{n}^{\prime(1)}\left(u, q_{L}\right) c e_{n}\left(v, q_{L}\right)-\sum_{n=1}^{\infty} L_{n}^{o} M s_{n}^{\prime(1)}\left(u, q_{L}\right) s e_{n}\left(v, q_{L}\right) \\
+\quad j \eta_{c}\left\{\sum_{n=0}^{\infty} R_{n}^{e} M c_{n}^{\prime(1)}\left(u, q_{R}\right) c e_{n}\left(v, q_{R}\right)+\sum_{n=1}^{\infty} R_{n}^{o} M s_{n}^{\prime(1)}\left(u, q_{R}\right) s e_{n}\left(v, q_{R}\right)\right\} \tag{4.141}
\end{gather*}
$$

The z-component of magnetic field will be:

$$
\begin{gather*}
H_{z}^{i}=0 \\
H_{z}^{s}=\frac{j k}{\eta}\left\{\sum_{n=0}^{\infty} C_{n}^{e} M c_{n}^{(4)}(u, q) c e_{n}(v, q)+\sum_{n=1}^{\infty} C_{n}^{o} M s_{n}^{(4)}(u, q) s e_{n}(v, q)\right\} \tag{4.142}
\end{gather*}
$$

$$
\begin{align*}
& H_{z}^{t}=\sum_{n=0}^{\infty} k_{R} R_{n}^{e} M c_{n}^{(1)}\left(u, q_{R}\right) c e_{n}\left(v, q_{R}\right)+\sum_{n=1}^{\infty} k_{R} R_{n}^{o} M s_{n}^{(1)}\left(u, q_{R}\right) s e_{n}\left(v, q_{R}\right) \\
& +\frac{j k_{L}}{\eta_{c}}\left\{\sum_{n=0}^{\infty} L_{n}^{e} M c_{n}^{(1)}\left(u, q_{L}\right) c e_{n}\left(v, q_{L}\right)+\sum_{n=1}^{\infty} L_{n}^{o} M s_{n}^{(1)}\left(u, q_{L}\right) s e_{n}\left(v, q_{L}\right)\right\} \tag{4.143}
\end{align*}
$$

The v-component of magnetic field will be:

$$
\begin{align*}
& H_{v}^{i}=\frac{j}{\eta}\left\{\sum_{n=0}^{\infty} A_{n}^{e} M c_{n}^{\prime(1)}(u, q) c e_{n}(v, q)+\sum_{n=1}^{\infty} A_{n}^{o} M s_{n}^{\prime(1)}(u, q) s e_{n}(v, q)\right\}  \tag{4.144}\\
& H_{v}^{s}=\frac{j}{\eta}\left\{\sum_{n=0}^{\infty} B_{n}^{e} M c_{n}^{\prime(4)}(u, q) c e_{n}(v, q)+\sum_{n=1}^{\infty} B_{n}^{o} M s_{n}^{\prime(4)}(u, q) s e_{n}(v, q)\right\}  \tag{4.145}\\
& H_{v}^{t}=\sum_{n=0}^{\infty} R_{n}^{e} M c_{n}^{\prime(1)}\left(u, q_{R}\right) c e_{n}\left(v, q_{L}\right)+\sum_{n=1}^{\infty} R_{n}^{o} M s_{n}^{\prime(1)}\left(u, q_{R}\right) s e_{n}\left(v, q_{R}\right) \\
& -\frac{j}{\eta_{c}}\left\{\sum_{n=0}^{\infty} L_{n}^{e} M c_{n}^{\prime(1)}\left(u, q_{L}\right) c e_{n}\left(v, q_{L}\right)+\sum_{n=1}^{\infty} L_{n}^{o} M s_{n}^{\prime(1)}\left(u, q_{L}\right) s e_{n}\left(v, q_{L}\right)\right\} \tag{4.146}
\end{align*}
$$

At boundary $u=u_{0}$ for any angle $v$ we have $E_{z}^{i}+E_{z}^{s}=E_{z}^{t}, E_{v}^{i}+E_{v}^{s}=E_{v}^{t}$, $H_{z}^{i}+H_{z}^{s}=H_{z}^{t}$ and $H_{v}^{i}+H_{v}^{s}=H_{v}^{t}$ therefore we have four equations

$$
\begin{align*}
& k \sum_{n=0}^{\infty} A_{n}^{e} M c_{n}^{(1)}\left(u_{0}, q\right) c e_{n}(v, q)+k \sum_{n=1}^{\infty} A_{n}^{o} M s_{n}^{(1)}\left(u_{0}, q\right) s e_{n}(v, q) \\
+ & k \sum_{n=0}^{\infty} B_{n}^{e} M c_{n}^{(4)}\left(u_{0}, q\right) c e_{n}(v, q)+k \sum_{n=1}^{\infty} B_{n}^{o} M s_{n}^{(4)}\left(u_{0}, q\right) s e_{n}(v, q) \\
= & k_{L} \sum_{n=0}^{\infty} L_{n}^{e} M c_{n}^{(1)}\left(u_{0}, q_{L}\right) c e_{n}\left(v, q_{L}\right)+k_{L} \sum_{n=1}^{\infty} L_{n}^{o} M s_{n}^{(1)}\left(u_{0}, q_{L}\right) s e_{n}\left(v, q_{L}\right) \\
+ & j \eta_{c}\left\{k_{R} \sum_{n=0}^{\infty} R_{n}^{e} M c_{n}^{(1)}\left(u_{0}, q_{R}\right) c e_{n}\left(v, q_{R}\right)+k_{R} \sum_{n=1}^{\infty} R_{n}^{o} M s_{n}^{(1)}\left(u_{0}, q_{R}\right) s e_{n}\left(v, q_{R}\right)\right\} \tag{4.147}
\end{align*}
$$

$$
\begin{align*}
& \sum_{n=0}^{\infty} C_{n}^{e} M c_{n}^{\prime(4)}\left(u_{0}, q\right) c e_{n}(v, q)+\sum_{n=1}^{\infty} C_{n}^{o} M s_{n}^{\prime(4)}\left(u_{0}, q\right) s e_{n}(v, q) \\
= & -\sum_{n=0}^{\infty} L_{n}^{e} M c_{n}^{\prime(1)}\left(u_{0}, q_{L}\right) c e_{n}\left(v, q_{L}\right)-\sum_{n=1}^{\infty} L_{n}^{o} M s_{n}^{\prime(1)}\left(u_{0}, q_{L}\right) s e_{n}\left(v, q_{L}\right) \\
+ & j \eta_{c}\left\{\sum_{n=0}^{\infty} R_{n}^{e} M c_{n}^{\prime(1)}\left(u_{0}, q_{R}\right) c e_{n}\left(v, q_{R}\right)+\sum_{n=1}^{\infty} R_{n}^{o} M s_{n}^{\prime(1)}\left(u_{0}, q_{R}\right) s e_{n}\left(v, q_{R}\right)\right\} \tag{4.148}
\end{align*}
$$

$$
\begin{align*}
& \frac{j k}{\eta}\left\{\sum_{n=0}^{\infty} C_{n}^{e} M c_{n}^{(4)}\left(u_{0}, q\right) c e_{n}(v, q)+\sum_{n=1}^{\infty} C_{n}^{o} M s_{n}^{(4)}\left(u_{0}, q\right) s e_{n}(v, q)\right\} \\
= & k_{R} \sum_{n=0}^{\infty} R_{n}^{e} M c_{n}^{(1)}\left(u_{0}, q_{R}\right) c e_{n}\left(v, q_{R}\right)+k_{R} \sum_{n=1}^{\infty} R_{n}^{o} M s_{n}^{(1)}\left(u_{0}, q_{R}\right) s e_{n}\left(v, q_{R}\right) \\
+ & \frac{j}{\eta_{c}}\left\{k_{L} \sum_{n=0}^{\infty} L_{n}^{e} M c_{n}^{(1)}\left(u_{0}, q_{L}\right) c e_{n}\left(v, q_{L}\right)+k_{L} \sum_{n=1}^{\infty} L_{n}^{o} M s_{n}^{(1)}\left(u_{0}, q_{L}\right) s e_{n}\left(v, q_{L}\right)\right\} \tag{4.149}
\end{align*}
$$

$$
\begin{align*}
& \frac{j}{\eta}\left\{\sum_{n=0}^{\infty} A_{n}^{e} M c_{n}^{\prime(1)}\left(u_{0}, q\right) c e_{n}(v, q)+\sum_{n=1}^{\infty} A_{n}^{o} M s_{n}^{\prime(1)}\left(u_{0}, q\right) s e_{n}(v, q)\right\} \\
+ & \frac{j}{\eta}\left\{\sum_{n=0}^{\infty} B_{n}^{e} M c_{n}^{\prime(4)}\left(u_{0}, q\right) c e_{n}(v, q)+\sum_{n=1}^{\infty} B_{n}^{o} M s_{n}^{\prime(4)}\left(u_{0}, q\right) s e_{n}(v, q)\right\} \\
= & \sum_{n=0}^{\infty} R_{n}^{e} M c_{n}^{\prime(1)}\left(u_{0}, q_{R}\right) c e_{n}\left(v, q_{L}\right)+\sum_{n=1}^{\infty} R_{n}^{o} M s_{n}^{\prime(1)}\left(u_{0}, q_{R}\right) s e_{n}\left(v, q_{R}\right) \\
- & \frac{j}{\eta_{c}}\left\{\sum_{n=0}^{\infty} L_{n}^{e} M c_{n}^{\prime(1)}\left(u_{0}, q_{L}\right) c e_{n}\left(v, q_{L}\right)+\sum_{n=1}^{\infty} L_{n}^{o} M s_{n}^{\prime(1)}\left(u_{0}, q_{L}\right) s e_{n}\left(v, q_{L}\right)\right\} \tag{4.150}
\end{align*}
$$

From boundary conditions we have four equation; Eq.(4.147),..., Eq.(4.150), with eight array unknowns $B_{n}^{e}, B_{n}^{o}, C_{n}^{e}, C_{n}^{o}, R_{n}^{e}, R_{n}^{o}, L_{n}^{e}, L_{n}^{o}$ First we multiply both side of Eq.(4.147), $\ldots$, Eq.(4.150) by $c e_{n}(v, q)$ and integrating over $[0,2 \pi]$,
and using properties of Eq.(4.76) and changing index m,n from 0 to $M$ therefore we have a big matrix consist of sixteen small matrix in the form:

$$
\left[\begin{array}{llll}
\mathbf{A}_{m n}^{11} & \mathbf{A}_{m n}^{12} & \mathbf{A}_{m n}^{13} & \mathbf{A}_{m n}^{14}  \tag{4.151}\\
\mathbf{A}_{m n}^{21} & \mathbf{A}_{m n}^{22} & \mathbf{A}_{m n}^{33} & \mathbf{A}_{m n}^{24} \\
\mathbf{A}_{m n}^{31} & \mathbf{A}_{m n}^{32} & \mathbf{A}_{m n}^{33} & \mathbf{A}_{m n}^{34} \\
\mathbf{A}_{m n}^{41} & \mathbf{A}_{m n}^{42} & \mathbf{A}_{m n}^{43} & \mathbf{A}_{m n}^{44}
\end{array}\right]\left[\begin{array}{l}
\mathbf{X}_{n}^{1} \\
\mathbf{X}_{n}^{2} \\
\mathbf{X}_{n}^{3} \\
\mathbf{X}_{n}^{4}
\end{array}\right]=\left[\begin{array}{c}
\mathbf{B}_{m}^{1} \\
\mathbf{B}_{m}^{2} \\
\mathbf{B}_{m}^{3} \\
\mathbf{B}_{m}^{4}
\end{array}\right]
$$

each small matrix $\mathbf{A}_{m n}^{11}, \mathbf{A}_{m n}^{12}, \ldots, \mathbf{A}_{m n}^{44}$ is a square matrix $(\mathrm{M}+1)$ by $(\mathrm{M}+1)$ with elements:

$$
\begin{gathered}
A_{m n}^{11}=-k M c_{n}^{(4)}\left(u_{0}, q\right) \mathbf{I}_{m n}^{c c}(q, q) \\
A_{m n}^{12}=k_{L} M c_{n}^{(1)}\left(u_{0}, q_{L}\right) \mathbf{I}_{m n}^{c c}\left(q, q_{L}\right) \\
A_{m n}^{13}=j k_{R} \eta_{c} M c_{n}^{(1)}\left(u_{0}, q_{R}\right) \mathbf{I}_{m n}^{c c}\left(q, q_{R}\right) \\
A_{m n}^{14}=0 \\
A_{m n}^{21}=0 \\
A_{m n}^{22}=-M c_{n}^{(1)}\left(u_{0}, q_{L}\right) \mathbf{I}_{m n}^{c c}\left(q, q_{L}\right) \\
A_{m n}^{23}=j \eta_{c} M c_{n}^{(1)}\left(u_{0}, q_{R}\right) \mathbf{I}_{m n}^{c c}\left(q, q_{R}\right) \\
A_{m n}^{24}=-M c_{n}^{(4)}\left(u_{0}, q\right) \mathbf{I}_{m n}^{c c}(q, q) \\
A_{m n}^{31}=0 \\
A_{m n}^{32}=\frac{j}{\eta_{c}} k_{L} M c_{n}^{(1)}\left(u_{0}, q_{L}\right) \mathbf{I}_{m n}^{c c}\left(q, q_{L}\right) \\
A_{m n}^{33}=k_{R} M c_{n}^{(1)}\left(u_{0}, q_{R}\right) \mathbf{I}_{m n}^{c c}\left(q, q_{R}\right) \\
A_{m n}^{34}=-\frac{j k}{\eta} M c_{n}^{(4)}\left(u_{0}, q\right) \mathbf{I}_{m n}^{c c}(q, q) \\
A_{m n}^{41}=-\frac{j}{\eta} M c_{n}^{\prime(4)}\left(u_{0}, q\right) \mathbf{I}_{m n}^{c c}(q, q) \\
A_{m n}^{42}=-\frac{j}{\eta_{c}} M c_{n}^{\prime(1)}\left(u_{0}, q_{L}\right) \mathbf{I}_{m n}^{c c}\left(q, q_{L}\right) \\
A_{m n}^{43}=\frac{1}{\eta_{c}} M c_{n}^{\prime(1)}\left(u_{0}, q_{R}\right) \mathbf{I}_{m n}^{c c}\left(q, q_{R}\right) \\
A_{m n}^{44}=0
\end{gathered}
$$

where $\mathbf{X}_{n}^{1}=B_{n}^{e}, \mathbf{X}_{n}^{2}=C_{n}^{e}, \mathbf{X}_{n}^{3}=L_{n}^{e}, \mathbf{X}_{n}^{4}=R_{n}^{e}$ and

$$
\begin{gathered}
\mathbf{B}_{m}^{1}=k \sum_{n=0}^{M} A_{n}^{e} M c_{n}^{(1)}\left(u_{0}, q\right) \mathbf{I}_{m n}^{c c}(q, q) \\
\mathbf{B}_{m}^{2}=\frac{j}{\eta} \sum_{n=0}^{M} A_{n}^{e} M c_{n}^{\prime(1)}\left(u_{0}, q\right) \mathbf{I}_{m n}^{c c}(q, q) \\
\mathbf{B}_{m}^{3}=\mathbf{B}_{m}^{4}=0
\end{gathered}
$$

As you have noticed these formula are for even mode, If we multiply both side of Eq.(4.147), $\ldots$, Eq.(4.150) by $s e_{n}(v, q)$ and integrating over $[0,2 \pi]$, and using properties of Eq.(4.76) and changing index m,n from 1 to $M+1$ therefore we have a big matrix consist of:

$$
\begin{gathered}
A_{m n}^{11}=-k M s_{n}^{(4)}\left(u_{0}, q\right) \mathbf{I}_{m n}^{s s}(q, q) \\
A_{m n}^{12}=k_{L} M s_{n}^{(1)}\left(u_{0}, q_{L}\right) \mathbf{I}_{m n}^{s s}\left(q, q_{L}\right) \\
A_{m n}^{13}=j k_{R} \eta_{c} M s_{n}^{(1)}\left(u_{0}, q_{R}\right) \mathbf{I}_{m n}^{s s}\left(q, q_{R}\right) \\
A_{m n}^{14}=0 \\
A_{m n}^{21}=0 \\
A_{m n}^{22}=-M s_{n}^{\prime(1)}\left(u_{0}, q_{L}\right) \mathbf{I}_{m n}^{s s}\left(q, q_{L}\right) \\
A_{m n}^{23}=j \eta_{c} M s_{n}^{\prime(1)}\left(u_{0}, q_{R}\right) \mathbf{I}_{m n}^{s s}\left(q, q_{R}\right) \\
A_{m n}^{24}=-M s_{n}^{\prime(4)}\left(u_{0}, q\right) \mathbf{I}_{m n}^{s s}(q, q) \\
A_{m n}^{31}=0 \\
A_{m n}^{32}=\frac{j}{\eta_{c}} k_{L} M s_{n}^{(1)}\left(u_{0}, q_{L}\right) \mathbf{I}_{m n}^{s s}\left(q, q_{L}\right) \\
A_{m n}^{33}=k_{R} M s_{n}^{(1)}\left(u_{0}, q_{R}\right) \mathbf{I}_{m n}^{s s}\left(q, q_{R}\right) \\
A_{m n}^{34}=-\frac{j k}{\eta} M s_{n}^{(4)}\left(u_{0}, q\right) \mathbf{I}_{m n}^{s s}(q, q) \\
A_{m n}^{41}=-\frac{j}{\eta} M s_{n}^{\prime(4)}\left(u_{0}, q\right) \mathbf{I}_{m n}^{s s}(q, q) \\
A_{m n}^{42}=-\frac{j}{\eta_{c}} M s_{n}^{\prime(1)}\left(u_{0}, q_{L}\right) \mathbf{I}_{m n}^{s s}\left(q, q_{L}\right)
\end{gathered}
$$

$$
\begin{gathered}
A_{m n}^{43}=\frac{1}{\eta_{c}} M s_{n}^{\prime(1)}\left(u_{0}, q_{R}\right) \mathbf{I}_{m n}^{s s}\left(q, q_{R}\right) \\
A_{m n}^{44}=0
\end{gathered}
$$

where $\mathbf{X}_{n}^{1}=B_{n}^{o}, \mathbf{X}_{n}^{2}=C_{n}^{o}, \mathbf{X}_{n}^{3}=L_{n}^{o}, \mathbf{X}_{n}^{1}=R_{n}^{o}$ and

$$
\begin{gathered}
\mathbf{B}_{m}^{1}=k \sum_{n=1}^{M+1} A_{n}^{o} M s_{n}^{(1)}\left(u_{0}, q\right) \mathbf{I}_{m n}^{s s}(q, q) \\
\mathbf{B}_{m}^{2}=\frac{j}{\eta} \sum_{n=1}^{M+1} A_{n}^{o} M s_{n}^{\prime(1)}\left(u_{0}, q\right) \mathbf{I}_{m n}^{s s}(q, q) \\
\mathbf{B}_{m}^{3}=\mathbf{B}_{m}^{4}=0
\end{gathered}
$$

The dimension of big matrix is $4 M+4$ by $4 M+4$.
For $T E^{z}$ polarization, we can use duality theorem.

### 4.12 Problems

- Problem 1 A uniform $T M^{z}$ plane wave incident on long conduction elliptic cylinder with $a=\lambda$ and $b=\lambda / 2$ at angle $\phi_{0}=90^{\circ}$. Calculate and draw its bistatic radar cross section and compare your result with physical optic method.
- Problem 2 A uniform $T E^{z}$ plane wave incident on long conduction elliptic cylinder with $a=\lambda$ and $b=\lambda / 2$ at angle $\phi_{0}=90^{\circ}$. Calculate and draw its bistatic radar cross section and compare your result with physical optic method.
- Problem 3 Find the scattering of EM wave by a PEMC elliptic cylinder for $T E^{z}$ polarization.
- Problem 4 A uniform $T M^{z}$ plane wave incident on a long lossy dielectric elliptic cylinder with $a=\lambda / 2, b=\lambda / 4$ and $\epsilon_{r}=5 ., \sigma=0.05$ at angle $\phi_{0}=30^{\circ}$. Calculate and draw its bistatic radar cross section and electric field along X and Y axis at frequency 1 GHz inside dielectric elliptic.
- Problem 5 A uniform $T M^{z}$ plane wave incident on a two layer lossy dielectric elliptic cylinder with parameters $a_{2}=\lambda / 2, b_{2}=\lambda / 4, u_{1}=$ $u_{2} / 2, \epsilon_{r_{1}}=25 ., \sigma_{1}=0.5[S / m], \epsilon_{r_{2}}=50 ., \sigma_{2}=1 .[S / m]$ at angle $\phi_{0}=30^{0}$. Calculate and draw its bistatic radar cross section and electric field along X and Y axis at frequency 1 GHz inside elliptic.
- Problem 6 A uniform $T M^{z}$ plane wave incident on a long dielectric elliptic cylinder with $a=\lambda / 2, b=\lambda / 4$ and $\epsilon_{r}=5 ., \sigma=0.05$ at angle $\phi_{0}=30^{\circ}$. The confocal distance of elliptic cylinder is filled with a conducting strip. Fine the normalized RCS $\sigma / \lambda$ of this configuration.
- Problem 7 Solve the above problems for $T E^{z}$ case.


## Chapter 5

## Parabolic Cylinder

"Weakness of attitude becomes weakness of character." Albert Einstein

### 5.1 Parabolic Cylindrical Coordinates

There are several different conventions for the orientation and designation of these coordinates. In this work, following Morse and Feshbach (1953), the coordinates $u, v, z$ are used. In this convention, the traces of the coordinate surfaces of the $x y$-plane are confocal parabolas with a common axis. The $u$ curves open into the negative x -axis; the $v$ curves open into the positive x -axis. The $u$ and $v$ curves intersect along the y -axis. A cylindrical parabolic coordinate system is one in which coordinates $(u, v, z)$ correspond to the Cartesian coordinates given by; Fig.(5.1). They are related to Cartesian Coordinates by

$$
\begin{align*}
x & =\frac{1}{2}\left(u^{2}-v^{2}\right) \\
y & =u v \\
z & =z \tag{5.1}
\end{align*}
$$

where $v \geq 0$ and $u$ may assume any real value, with its sign being the same as that of the $y$ coordinate. In Fig.(5.2) curves of constant $u$ and $v$ values are shown in the $z=0$ plane of cartesian coordinate system. Complete parabolic cylinders are obtained when $v$ is kept constant. For $v=0$ the parabola in the $z=0$ plane degenerates into the positive $x$ axis. Note also in the $z=0$


Figure 5.1: Parabolic cylindrical coordinates
plane that a complete family of parabolas are obtained as $v$ increases from zero to infinity. Curves with constant $u$ are semi- parabolas, in this plane, with $u$ varying from $-\infty$ to $+\infty)$. For $u=0$ the semi-parabola in the $z=0$ plane, degenerates into the negative $x$ axis.


Figure 5.2: Curves of constant values of $u$ and $v$

$$
\begin{align*}
y^{2} & =4 F(x+F) \\
x & =\frac{1}{2} u_{0}^{2}-\frac{y^{2}}{2 u_{0}^{2}} \\
x & =-\frac{1}{2} v_{0}^{2}+\frac{y^{2}}{2 v_{0}^{2}} \tag{5.2}
\end{align*}
$$

where we denote the focal length by $F, v_{0}=\sqrt{2 F}$. The parabolic cylinder coordinates $(u, v, z)$ are related to the circular cylindrical coordinates $(\rho, \phi, z)$ by

$$
\begin{align*}
& \rho=\sqrt{x^{2}+y^{2}}=\frac{1}{2}\left(u^{2}+v^{2}\right) \\
& u= \pm \sqrt{\rho+x}=\sqrt{2 \rho} \cos \frac{\phi}{2} \\
& v=\sqrt{\rho-x}=\sqrt{2 \rho} \sin \frac{\phi}{2} \tag{5.3}
\end{align*}
$$

and unit vectors will be:

$$
\begin{align*}
& \mathbf{a}_{u}=\frac{1}{\sqrt{u^{2}+v^{2}}}\left(u \mathbf{a}_{x}+v \mathbf{a}_{y}\right) \\
& \mathbf{a}_{v}=\frac{1}{\sqrt{u^{2}+v^{2}}}\left(-v \mathbf{a}_{x}+u \mathbf{a}_{y}\right) \tag{5.4}
\end{align*}
$$

The Scale Factors are:

$$
\begin{align*}
h_{1} & =\sqrt{u^{2}+v^{2}} \\
h_{2} & =h_{1} \\
h_{3} & =1 \tag{5.5}
\end{align*}
$$

therefore $\nabla \cdot \mathbf{E}, \nabla \times \mathbf{E}$ and $\nabla \psi$ will be:

$$
\begin{align*}
\nabla \cdot \mathbf{E} & =\frac{1}{h_{1}^{2}}\left[\frac{\partial}{\partial u}\left(h_{1} E_{u}\right)+\frac{\partial}{\partial v}\left(h_{1} E_{v}\right)\right]+\frac{\partial E_{z}}{\partial z} \\
\nabla \times \mathbf{E} & =\left[\frac{1}{h_{1}} \frac{\partial E_{z}}{\partial v}-\frac{\partial E_{v}}{\partial z}\right] \mathbf{a}_{u}+\left[\frac{\partial E_{u}}{\partial z}-\frac{1}{h_{1}} \frac{\partial E_{z}}{\partial u}\right] \mathbf{a}_{v} \\
& +\frac{1}{h_{1}}\left[\frac{\partial}{\partial u}\left(h_{1} E_{v}\right)-\frac{\partial}{\partial v}\left(h_{1} E_{u}\right)\right] \mathbf{a}_{z} \\
\nabla \psi & =\frac{1}{h_{1}}\left(\frac{\partial \psi}{\partial u} \mathbf{a}_{u}+\frac{\partial \psi}{\partial v} \mathbf{a}_{v}\right)+\frac{\partial \psi}{\partial z} \mathbf{a}_{z} \tag{5.6}
\end{align*}
$$

The Helmholtz differential equation is:

$$
\begin{equation*}
\frac{1}{u^{2}+v^{2}}\left(\frac{\partial^{2} \psi}{\partial u^{2}}+\frac{\partial^{2} \psi}{\partial v^{2}}\right)+\frac{\partial^{2} \psi}{\partial z^{2}}+k^{2} \psi=0 \tag{5.7}
\end{equation*}
$$

Since our problems are long cylinder, the problem is two dimensional:

$$
\begin{equation*}
\frac{1}{u^{2}+v^{2}}\left(\frac{\partial^{2} \psi}{\partial u^{2}}+\frac{\partial^{2} \psi}{\partial v^{2}}\right)+k^{2} \psi=0 \tag{5.8}
\end{equation*}
$$

Attempt separation of variables by writing $\psi(u, v)=U(u) V(v)$ then the Helmholtz differential equation becomes:

$$
\begin{align*}
& \frac{\partial^{2} U}{\partial u^{2}}+\left[k^{2} u^{2}-a\right] U=0  \tag{5.9}\\
& \frac{\partial^{2} V}{\partial v^{2}}+\left[k^{2} v^{2}+a\right] V=0 \tag{5.10}
\end{align*}
$$

where $a$ is separation constant and

$$
\begin{aligned}
& U(u)=c_{1} D_{\nu_{1}}\left(p_{1} u\right)+c_{2} D_{\nu_{2}}\left(p_{2} u\right) \\
& V(v)=c_{3} D_{\nu_{3}}\left(p_{3} v\right)+c_{4} D_{\nu_{4}}\left(p_{4} v\right)
\end{aligned}
$$

where $\nu_{1}=\frac{-j(a-j k)}{2 k}, \nu_{2}=\frac{j(a+j k)}{2 k}=\nu_{1}^{*}, \nu_{3}=\frac{-j a-k}{2 k}, \nu_{4}=\frac{j(a+j k)}{2 k}=\nu_{3}^{*}$. $p_{1}=(-1)^{3 / 4} \sqrt{2 k}, p_{2}=(-1)^{1 / 4} \sqrt{2 k}, p_{3}=(-1)^{1 / 4} \sqrt{2 k}, p_{4}=(-1)^{3 / 4} \sqrt{2 k}$, and the asterisk implies complex conjugate.
We may observe from Fig.(5.2) that the space will be divided into two region by parabolic cylinder $v=v_{0}$, the interior region, for which $v \leq v_{0}$ or $x \leq$ $-\frac{1}{2} v_{0}^{2}+\frac{y^{2}}{2 v_{0}^{2}}$ and exterior region for which $v \geq v_{0}$.
If we are solving a problem in the interior region, the required regularity of the solution in $u=v=0$ (z axis) simplifies. If we now solve the problem in the exterior region, the required regularity of the solution when $v \rightarrow \infty$ will also simplify. The general solution for interior region is:

$$
\psi=A_{m} D_{m}(p u) D_{m}\left(p^{*} v\right)
$$

and the general solution for exterior region will be

$$
\psi=A_{m} D_{m}(p u) D_{-m-1}\left(p^{*} v\right)
$$

Between the function $D_{\nu}(z)$ and $D_{-\nu-1}(z)$ there exists the following linear relation

$$
\begin{equation*}
D_{\nu}(z)=\frac{\Gamma(\nu+1)}{\sqrt{2 \pi}}\left[e^{j \pi \nu / 2} D_{-\nu-1}(j z)+e^{-j \pi \nu / 2} D_{-\nu-1}(-j z)\right] \tag{5.11}
\end{equation*}
$$

### 5.2 Plane Wave in Parabolic Cylinder

The incident plane wave $\psi=\exp \left\{j k\left(x \cos \phi_{0}+y \sin \phi_{0}\right)\right\}$ can be expand in parabolic cylinder. The $\psi$ may be $E_{z}$ or $H_{z}$ for $T M^{z}$ or $T E^{z}$ respectively. For $\left|\phi_{0}\right|<\pi / 2$ we have:

$$
\begin{equation*}
\psi=\sum_{m=0}^{\infty} \sec \left(\frac{\phi_{0}}{2}\right) \frac{\left[j \tan \left(\frac{\phi_{0}}{2}\right)\right]^{m}}{m!} D_{m}\left(p^{*} u\right) D_{m}(p v) \tag{5.12}
\end{equation*}
$$

and for $\left|\phi_{0}\right|>\pi / 2$

$$
\begin{equation*}
\psi=\sum_{m=0}^{\infty} \csc \left(\frac{\phi_{0}}{2}\right) \frac{\left[j \cot \left(\frac{\phi_{0}}{2}\right)\right]^{m}}{m!} D_{m}(p u) D_{m}\left(p^{*} v\right) \tag{5.13}
\end{equation*}
$$

where $p=\sqrt{2 j k}$ and $p^{*}=\sqrt{-2 j k^{*}}$. The first series, Eq.( 5.12 ), converges well for values of $\phi_{0}$ near zero and the second, Eq.(5.13), for values near $\pi$. When $\phi_{0}=0$ and the plane wave is moving to the left, parallel to the $x$ axis, the expression takes on the particularly simple form [11].

$$
\begin{gather*}
e^{j k x}=D_{0}\left(p^{*} u\right) D_{0}(p v) \\
e^{-j k x}=D_{0}(p u) D_{0}\left(p^{*} v\right) \tag{5.14}
\end{gather*}
$$

### 5.3 Line Source in Parabolic Cylinder

An infinite Electric current source $I_{0}$ which is located at $r_{0}\left(x_{0}, y_{0}\right)=r_{0}\left(u_{0}, v_{0}\right)$ can be expand in parabolic cylinder[11].

$$
\begin{align*}
E_{z}= & -\frac{I_{0}}{4} \omega \mu H_{0}^{(2)}\left(k\left|\mathbf{r}-\mathbf{r}_{0}\right|\right) \\
= & \frac{I_{0}}{\sqrt{2 \pi}} \omega \mu \sum_{m=0}^{\infty} \frac{j^{m}}{m!} D_{m}\left(p^{*} u_{0}\right) D_{m}\left(p^{*} u\right) \\
& \cdot \begin{cases}D_{m}(p v) D_{-m-1}\left(p^{*} v_{0}\right) ; & v_{0}>v \\
D_{m}\left(p v_{0}\right) D_{-m-1}\left(p^{*} v\right) ; & v>v_{0}\end{cases} \tag{5.15}
\end{align*}
$$

### 5.4 Plane Wave Scattering by a Conducting Parabolic Cylinder, $T M^{z}$ Polarization

Let us consider a metalic parabolic cylinder as Fig.(5.3)

$$
\begin{equation*}
x=-\frac{1}{2} v_{0}^{2}+\frac{y^{2}}{2 v_{0}^{2}} \tag{5.16}
\end{equation*}
$$

As illusttrated in Fig.(5.3), the conducting parabolic cylinder is illuminated


Figure 5.3: Conducting parabolic cylinder
by a $T M^{z}$ polarized plane wave incident from the angle $\phi_{0}$ with respect to the positive $x$ axis, and with incident electric field given by:

$$
\begin{equation*}
\mathbf{E}^{i}=e^{j k\left(x \cos \phi_{0}+y \sin \phi_{0}\right)} \mathbf{a}_{z} \tag{5.17}
\end{equation*}
$$

Since the angle of incident is between $\left|\phi_{0}\right|>\pi / 2$ therefore

$$
\begin{equation*}
E_{z}^{i}=\sum_{m=0}^{\infty} a_{m}^{i} D_{m}(p u) D_{m}\left(p^{*} v\right) \tag{5.18}
\end{equation*}
$$

where

$$
a_{m}^{i}=\csc \left(\frac{\phi_{0}}{2}\right) \frac{\left[j \cot \left(\frac{\phi_{0}}{2}\right)\right]^{m}}{m!}
$$

### 5.4. PLANE WAVE SCATTERING BY A CONDUCTING PARABOLIC CYLINDER, TM ${ }^{Z}$ POLARI

The scattered electric field which satisfies the radiation condition, is given by:

$$
\begin{equation*}
E_{z}^{s}=\sum_{m=0}^{\infty} a_{m}^{s} D_{m}(p u) D_{-m-1}(p v) \tag{5.19}
\end{equation*}
$$

where the $a_{m}^{s}$ are a series of unknown coefficients. Now we can impose boundary condition $E_{z}^{i}+E_{z}^{s}=0$ at the $v=v_{0}$, therefore

$$
\begin{equation*}
\sum_{m=0}^{\infty} a_{m}^{i} D_{m}(p u) D_{m}\left(p^{*} v_{0}\right)+a_{m}^{s} D_{m}(p u) D_{-m-1}\left(p v_{0}\right)=0 \tag{5.20}
\end{equation*}
$$

or

$$
\begin{equation*}
\sum_{m=0}^{\infty}\left[a_{m}^{i} D_{m}\left(p^{*} v_{0}\right)+a_{m}^{s} D_{-m-1}\left(p v_{0}\right)\right] D_{m}(p u)=0 \tag{5.21}
\end{equation*}
$$

we multiply both side of Eq.(5.21) by $D_{n}(p u)$ and noting that the parabolic cylinder functions satisfy the orthogonality relation

$$
\begin{equation*}
\int_{-\infty}^{\infty} D_{m}(p u) D_{n}(p u) d u=\frac{m!\sqrt{2 \pi}}{p} \delta_{m n} \tag{5.22}
\end{equation*}
$$

finally

$$
\begin{equation*}
a_{m}^{s}=-\frac{D_{m}\left(p^{*} v_{0}\right)}{D_{-m-1}\left(p v_{0}\right)} a_{m}^{i} \tag{5.23}
\end{equation*}
$$



Figure 5.4: Scattering by a conducting parabolic cylinder

### 5.5 Scattering by a Dielectric Parabolic Cylinder

Consider a parabolic cylinder as shown in Fig.(5.5)with electrical parameter $\mu, \epsilon, \sigma$ in free space. The incident $T M^{z}$ polarization electric filed will be given as


Figure 5.5: Scattering by a dielectric parabolic cylinder

$$
\begin{equation*}
E_{z}^{i}=e^{j k_{0}\left(x \cos \phi_{0}+y \sin \phi_{0}\right)}=\sum_{m=0}^{\infty} a_{m}^{i} D_{m}\left(p_{0} u\right) D_{m}\left(p_{0}^{*} v\right) \tag{5.24}
\end{equation*}
$$

where $k_{0}=2 \pi / \lambda$ is wavenumber in free space, $p_{0}=\sqrt{2 j k_{0}}, p_{0}^{*}=\sqrt{-2 j k_{0}}$ and

$$
\begin{equation*}
a_{m}^{i}=\csc \left(\frac{\phi_{0}}{2}\right) \frac{\left[j \cot \left(\frac{\phi_{0}}{2}\right)\right]^{m}}{m!} \tag{5.25}
\end{equation*}
$$

The corresponding incident magnetic field will be find by

$$
\begin{equation*}
\mathbf{H}^{i}=\frac{1}{-j \omega \mu_{0}} \nabla \times \mathbf{E}^{i}=\frac{1}{-j \omega \mu_{0} h}\left[\mathbf{a}_{u} \frac{\partial E_{z}^{i}}{\partial v}-\mathbf{a}_{v} \frac{\partial E_{z}^{i}}{\partial u}\right] \tag{5.26}
\end{equation*}
$$

where $h=\sqrt{u^{2}+v^{2}}$. The problem is to find electromagnetic scattered fields in free space and transmitted electromagnetic fields inside dielectric parabolic cylinder. The scattered electric field which satisfies the radiation condition, is given by

$$
\begin{equation*}
E_{z}^{s}=\sum_{m=0}^{\infty} a_{m}^{s} D_{m}\left(p_{0} u\right) D_{-m-1}\left(p_{0} v\right) \tag{5.27}
\end{equation*}
$$

and the corresponding scattered magnetic field will be found by

$$
\begin{equation*}
\mathbf{H}^{s}=\frac{1}{-j \omega \mu_{0}} \nabla \times \mathbf{E}^{s}=\frac{1}{-j \omega \mu_{0} h}\left[\mathbf{a}_{u} \frac{\partial E_{z}^{s}}{\partial v}-\mathbf{a}_{v} \frac{\partial E_{z}^{s}}{\partial u}\right] \tag{5.28}
\end{equation*}
$$

The electromagnetic fields inside the parabolic cylinder is

$$
\begin{equation*}
E_{z}^{t}=\sum_{m=0}^{\infty} a_{m}^{t} D_{m}(p u) D_{m}\left(p^{*} v\right) \tag{5.29}
\end{equation*}
$$

where $\epsilon_{c}=\epsilon_{0}\left(\epsilon_{r}-j \frac{\sigma}{\omega \epsilon_{0}}\right), k=\omega \sqrt{\mu \epsilon_{c}}$ is complex wavenumber in dielectric parabolic cylinder, $p=\sqrt{2 j k}, p^{*}=\sqrt{-2 j k^{*}}$ and $a_{m}^{t}$ is unknown coefficient that are found by enforcing boundary conditions. At surface of $v=v_{0}$ we have

$$
\begin{align*}
& E_{z}^{i}+E_{z}^{s}=E_{z}^{t} \\
& H_{u}^{i}+H_{u}^{s}=H_{u}^{t} \tag{5.30}
\end{align*}
$$

or

$$
\begin{align*}
& \sum_{m=0}^{\infty} a_{m}^{i} D_{m}\left(p_{0} u\right) D_{m}\left(p_{0}^{*} v_{0}\right)+\sum_{m=0}^{\infty} a_{m}^{s} D_{m}\left(p_{0} u\right) D_{-m-1}\left(p_{0} v_{0}\right)=  \tag{5.31}\\
& \sum_{m=0}^{\infty} a_{m}^{t} D_{m}(p u) D_{m}\left(p^{*} v_{0}\right) \\
& \frac{1}{\mu_{0}} \sum_{m=0}^{\infty} a_{m}^{i} D_{m}\left(p_{0} u\right) D_{m}^{\prime}\left(p_{0}^{*} v_{0}\right)+\frac{1}{\mu_{0}} \sum_{m=0}^{\infty} a_{m}^{s} D_{m}\left(p_{0} u\right) D_{-m-1}^{\prime}\left(p_{0} v_{0}\right)=  \tag{5.32}\\
& \frac{1}{\mu} \sum_{m=0}^{\infty} a_{m}^{t} D_{m}(p u) D_{m}^{\prime}\left(p^{*} v_{0}\right)
\end{align*}
$$

in which the prim implies differentiation with respect to $v$, i.e.,

$$
\begin{equation*}
D_{m}^{\prime}(p v)=\frac{\partial D_{m}(p v)}{\partial v}=\left[\frac{p^{2} v}{2} D_{m}(p v)-p D_{m+1}(p v)\right] \tag{5.33}
\end{equation*}
$$

Equation Eq.(5.30)and Eq.(5.31) represent two sets of equations in the unknown coefficients $a_{n}^{s}$ and $a_{n}^{t}$. An exact term by term solution of these equations is not possible since they contain parabolic cylinder functions with diffrent arguments., i.e., different dependence on the circumferential coordinate $u$. Thus, the eigenfunction in free space and inside cylinder are not
orthogonal. The same nonorthogonality procedure of eigenfunction modes that has periviously been used for elliptic dielectric cylinder, we will use in this problem.
Before attemting to find the unknown coefficients $a_{m}^{s}$ and $a_{m}^{t}$, we need to define this integral:

$$
\begin{equation*}
\mathbf{I}_{m n}\left(p_{i}, p_{j}\right)=\int_{-\infty}^{\infty} D_{m}\left(p_{i} u\right) D_{n}\left(p_{j} u\right) d u \tag{5.34}
\end{equation*}
$$

if $p_{i}=p_{j}=p$ we have:

$$
\begin{equation*}
\mathbf{I}_{m n}(p, p)=\frac{m!\sqrt{2 \pi}}{p} \delta_{m n} \tag{5.35}
\end{equation*}
$$

This integral has exact closed form expressions in terms of simple functions and for nonnegative integer $m$ and $n$.
For nonnegative integer orders, $D_{m}(z)$ is the product of an exponential and a Hermite polynomial:

$$
\begin{equation*}
D_{m}(z)=2^{-\frac{m}{2}} e^{-\frac{z^{2}}{4}} H_{m}\left(\frac{z}{\sqrt{2}}\right) \tag{5.36}
\end{equation*}
$$

where $H_{m}(z)$ is Hermite polynomial. Inserting Eq.(5.36) into Eq.(5.34), $I_{m n}$ can be expressed in terms of the Hermite polynomials as:

$$
\begin{equation*}
\mathbf{I}_{m n}\left(p_{i}, p_{j}\right)=2^{-\frac{m+n}{2}} \frac{2}{\sqrt{p_{i}^{2}+p_{j}^{2}}} \int_{-\infty}^{\infty} e^{-x^{2}} H_{m}(a x) H_{n}(b x) d x \tag{5.37}
\end{equation*}
$$

where $a=\frac{\sqrt{2} p_{i}}{\sqrt{p_{i}^{2}+p_{j}^{2}}}$ and $b=\frac{\sqrt{2} p_{j}}{\sqrt{p_{i}^{2}+p_{j}^{2}}}$. Let us use this short hand notation $\mathbf{I}_{m n}\left(p_{i}, p_{j}\right)=C \cdot I_{m n}(a, b)$ where the constant. $C=2^{-\frac{m+n}{2}} \frac{2 \sqrt{\pi}}{\sqrt{p_{i}^{2}+p_{j}^{2}}}$ and

$$
\begin{equation*}
I_{m n}(a, b)=\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^{2}} H_{m}(a x) H_{n}(b x) d x \tag{5.38}
\end{equation*}
$$

This integral can be calculated analytically. If $m+n$ is odd, then $I_{m n}(a, b)=0$ otherwise it has valus that we have found it by Mathematica software. Some of them are tabulated at appendix. We begin multiplying both sides of

Eq.(5.31) and Eq.(5.32) by $D_{n}\left(p_{0} u\right), n=0,1,2, \ldots, M$ and integrating from $-\infty$ to $\infty . M$ is the truncation of our series. Therefore:

$$
\begin{align*}
& \sum_{m=0}^{M} a_{m}^{i} \mathbf{I}_{m n}\left(p_{0}, p_{0}\right) D_{m}\left(p_{0}^{*} v_{0}\right)+\sum_{m=0}^{M} a_{m}^{s} \mathbf{I}_{m n}\left(p_{0}, p_{0}\right) D_{-m-1}\left(p_{0} v_{0}\right)=  \tag{5.39}\\
& \sum_{m=0}^{M} a_{m}^{t} \mathbf{I}_{m n}\left(p, p_{0}\right) D_{m}\left(p^{*} v_{0}\right) \\
& \frac{1}{\mu_{0}} \sum_{m=0}^{M} a_{m}^{i} \mathbf{I}_{m n}\left(p_{0}, p_{0}\right) D_{m}^{\prime}\left(p_{0}^{*} v_{0}\right)+\frac{1}{\mu_{0}} \sum_{m=0}^{M} a_{m}^{s} \mathbf{I}_{m n}\left(p_{0}, p_{0}\right) D_{-m-1}^{\prime}\left(p_{0} v_{0}\right)=  \tag{5.40}\\
& \frac{1}{\mu} \sum_{m=0}^{M} a_{m}^{t} \mathbf{I}_{m n}\left(p, p_{0}\right) D_{m}^{\prime}\left(p^{*} v_{0}\right)
\end{align*}
$$

the index $m$ in both Eq.(5.39),Eq.(5.40) can be changed from $0-M$, therefore we have $2 M+2$ unknown and $2 M+2$ linear equations which can be divide into four different submatrix.

$$
\left[\begin{array}{ll}
\mathbf{A}_{m n} & \mathbf{B}_{m n}  \tag{5.41}\\
\mathbf{C}_{m n} & \mathbf{D}_{m n}
\end{array}\right]\left[\begin{array}{l}
\mathbf{X}_{n} \\
\mathbf{Y}_{n}
\end{array}\right]=\left[\begin{array}{l}
\mathbf{G}_{m} \\
\mathbf{H}_{m}
\end{array}\right]
$$

where

$$
\begin{gathered}
\mathbf{A}_{m n}=D_{m}\left(p^{*} v_{0}\right) \mathbf{I}_{m n}\left(p, p_{0}\right) \\
\mathbf{C}_{m n}=\frac{1}{\mu} D_{m}^{\prime}\left(p^{*} v_{0}\right) \mathbf{I}_{m n}\left(p, p_{0}\right) \\
\mathbf{B}_{m n}=-D_{-m-1}\left(p_{0} v_{0}\right) \mathbf{I}_{m n}\left(p_{0}, p_{0}\right) \\
\mathbf{D}_{m n}=-\frac{1}{\mu_{0}} D_{-m-1}^{\prime}\left(p_{0} v_{0}\right) \mathbf{I}_{m n}\left(p_{0}, p_{0}\right) \\
\mathbf{G}_{m}=a_{m}^{i} D_{m}\left(p_{0}^{*} v_{0}\right) \mathbf{I}_{m n}\left(p_{0}, p_{0}\right) \\
\mathbf{H}_{m}=\frac{1}{\mu_{0}} a_{m}^{i} D_{m}^{\prime}\left(p_{0}^{*} v_{0}\right) \mathbf{I}_{m n}\left(p_{0}, p_{0}\right)
\end{gathered}
$$

and the unknowns are:

$$
\begin{aligned}
& \mathbf{X}_{n}=a_{n}^{t} \\
& \mathbf{Y}_{n}=a_{n}^{s}
\end{aligned}
$$

### 5.6 Scattering by a Coated Dielectric Parabolic Cylinder

In this section an eigenfunction solution to the problem of TM scattering by a material coated perfectly conducting parabolic cylinder will be presented. The surfaces of the perfect conductor and of the material coating are both parabolic cylinders with the same focal line. The parabolic cylinder is a shape of practical interest since it can model an isolated thick edge.
The basic geometry is shown in Fig.(5.6). The wavenumber in region 1 or 2

$$
\mathrm{E}_{\mathrm{z}}=\mathrm{a}_{\mathrm{z}} \mathrm{e}^{\mathrm{j} \mathrm{k}_{1}\left(\mathrm{x} \cos \varphi_{0}+\mathrm{y} \sin \varphi_{0}\right)}
$$

Figure 5.6: Geometry for a TM plane wave incident upon coated perfectly conducting parabolic cylinder
is gien by

$$
k_{1,2}=\omega \sqrt{\mu_{1,2} \epsilon_{1,2}}
$$

which will be complex if the repective region is lossy. In addition, in region 1 and 2 we define the parameter

$$
p_{1,2}=\sqrt{2 j k_{1,2}}
$$

The incident electric field can be expresses in term of the parabolic cylinder functions:

$$
\begin{equation*}
E_{z}^{i}=e^{j k_{1}\left(x \cos \phi_{0}+y \sin \phi_{0}\right)}=\sum_{m=0}^{\infty} a_{m}^{i} D_{m}\left(p_{1} u\right) D_{m}\left(p_{1}^{*} v\right) \tag{5.42}
\end{equation*}
$$

$$
\begin{equation*}
a_{m}^{i}=\csc \left(\frac{\phi_{0}}{2}\right) \frac{\left[j \cot \left(\frac{\phi_{0}}{2}\right)\right]^{m}}{m!} \tag{5.43}
\end{equation*}
$$

The corresponding incident magnetic field will be found by:

$$
\begin{equation*}
\mathbf{H}^{i}=\frac{1}{-j \omega \mu_{1}} \nabla \times \mathbf{E}^{i}=\frac{1}{-j \omega \mu_{1} h}\left[\mathbf{a}_{u} \frac{\partial E_{z}^{i}}{\partial v}-\mathbf{a}_{v} \frac{\partial E_{z}^{i}}{\partial u}\right] \tag{5.44}
\end{equation*}
$$

where $h=\sqrt{u^{2}+v^{2}}$. The scattered electric field which satisfies the radiation condition, is given by:

$$
\begin{equation*}
E_{z}^{s}=\sum_{m=0}^{\infty} a_{m}^{s} D_{m}\left(p_{1} u\right) D_{-m-1}\left(p_{1} v\right) \tag{5.45}
\end{equation*}
$$

and the corresponding scattered magnetic field will be found by:

$$
\begin{equation*}
\mathbf{H}^{s}=\frac{1}{-j \omega \mu_{1}} \nabla \times \mathbf{E}^{s}=\frac{1}{-j \omega \mu_{1} h}\left[\mathbf{a}_{u} \frac{\partial E_{z}^{s}}{\partial v}-\mathbf{a}_{v} \frac{\partial E_{z}^{s}}{\partial u}\right] \tag{5.46}
\end{equation*}
$$

The electromagnetic fields inside the parabolic cylinder is:

$$
\begin{array}{r}
E_{z}^{t}=\sum_{m=0}^{\infty} a_{m}^{t} D_{m}\left(p_{2} u\right) D_{m}\left(p_{2}^{*} v\right)+\sum_{m=0}^{\infty} b_{m}^{t} D_{m}\left(p_{2} u\right) D_{-m-1}\left(p_{2} v\right) \\
\mathbf{H}^{t}=\frac{1}{-j \omega \mu_{2}} \nabla \times \mathbf{E}^{t}=\frac{1}{-j \omega \mu_{2} h}\left[\mathbf{a}_{u} \frac{\partial E_{z}^{t}}{\partial v}-\mathbf{a}_{v} \frac{\partial E_{z}^{t}}{\partial u}\right] \tag{5.48}
\end{array}
$$

where $a_{m}^{t}$ and $b_{m}^{t}$ are unknown coefficients that are found by enforcing boundary conditions. At surface of $v=v_{1}$ we have:

$$
\begin{align*}
& E_{z}^{i}+E_{z}^{s}=E_{z}^{t}  \tag{5.49}\\
& H_{u}^{i}+H_{u}^{s}=H_{u}^{t}
\end{align*}
$$

At surface of $v=v_{2}$, conducting surface, $E_{z}^{t}=0$

$$
\begin{equation*}
E_{z}^{t}=\sum_{m=0}^{\infty} a_{m}^{t} D_{m}\left(p_{2} u\right) D_{m}\left(p_{2}^{*} v_{2}\right)+\sum_{m=0}^{\infty} b_{m}^{t} D_{m}\left(p_{2} u\right) D_{-m-1}\left(p_{2} v_{2}\right)=0 \tag{5.50}
\end{equation*}
$$

or by using the orthogonality of Eq.(5.22),

$$
\begin{equation*}
a_{m}^{t} D_{m}\left(p_{2}^{*} v_{2}\right)+b_{m}^{t} D_{-m-1}\left(p_{2} v_{2}\right)=0 \tag{5.51}
\end{equation*}
$$

$$
\begin{equation*}
b_{m}^{t}=a_{m}^{t} \frac{D_{m}\left(p_{2}^{*} v_{2}\right)}{D_{-m-1}\left(p_{2} v_{2}\right)} \tag{5.52}
\end{equation*}
$$

and from boundary conditions at $v=v_{1}$ we have:

$$
\begin{gather*}
\sum_{m=0}^{\infty} a_{m}^{i} D_{m}\left(p_{1} u\right) D_{m}\left(p_{1}^{*} v_{1}\right)+\sum_{m=0}^{\infty} a_{m}^{s} D_{m}\left(p_{1} u\right) D_{-m-1}\left(p_{1} v_{1}\right)=  \tag{5.53}\\
\sum_{m=0}^{\infty} a_{m}^{t} D_{m}\left(p_{2} u\right) D_{m}\left(p_{2}^{*} v_{1}\right)+\sum_{m=0}^{\infty} b_{m}^{t} D_{m}\left(p_{2} u\right) D_{-m-1}\left(p_{2} v_{1}\right) \\
\frac{1}{\mu_{1}}\left\{\sum_{m=0}^{\infty} a_{m}^{i} D_{m}\left(p_{1} u\right) D_{m}^{\prime}\left(p_{1}^{*} v_{1}\right)+\sum_{m=0}^{\infty} a_{m}^{s} D_{m}\left(p_{1} u\right) D_{-m-1}^{\prime}\left(p_{1} v_{1}\right)\right\}=  \tag{5.54}\\
\frac{1}{\mu_{2}}\left\{\sum_{m=0}^{\infty} a_{m}^{t} D_{m}\left(p_{2} u\right) D_{m}^{\prime}\left(p_{2}^{*} v_{1}\right)+\sum_{m=0}^{\infty} b_{m}^{t} D_{m}\left(p_{2} u\right) D_{-m-1}^{\prime}\left(p_{2} v_{1}\right)\right\}
\end{gather*}
$$

We insert Eq.(5.52) into Eq.(5.53) and Eq.(5.54), therefore:

$$
\begin{align*}
& \sum_{m=0}^{\infty} a_{m}^{i} D_{m}\left(p_{1} u\right) D_{m}\left(p_{1}^{*} v_{1}\right)+\sum_{m=0}^{\infty} a_{m}^{s} D_{m}\left(p_{1} u\right) D_{-m-1}\left(p_{1} v_{1}\right)=  \tag{5.55}\\
& \sum_{m=0}^{\infty} a_{m}^{t} D_{m}\left(p_{2} u\right) V_{m} \\
& \frac{1}{\mu_{1}}\left\{\sum_{m=0}^{\infty} a_{m}^{i} D_{m}\left(p_{1} u\right) D_{m}^{\prime}\left(p_{1}^{*} v_{1}\right)+\sum_{m=0}^{\infty} a_{m}^{s} D_{m}\left(p_{1} u\right) D_{-m-1}^{\prime}\left(p_{1} v_{1}\right)\right\}=  \tag{5.56}\\
& \frac{1}{\mu_{2}}\left\{\sum_{m=0}^{\infty} a_{m}^{t} D_{m}\left(p_{2} u\right) W_{m}\right\}
\end{align*}
$$

where

$$
\begin{align*}
& V_{m}=D_{m}\left(p_{2}^{*} v_{1}\right)+\frac{D_{m}\left(p_{2}^{*} v_{2}\right)}{D_{-m-m-1}\left(p_{2} v_{2}\right)} D_{-m-1}\left(p_{2} v_{1}\right)  \tag{5.57}\\
& W_{m}=D_{m}^{\prime}\left(p_{2}^{*} v_{1}\right)+\frac{D_{m}\left(p_{2}^{2} v_{2}\right)}{D_{-m-1}\left(p_{2} v_{2}\right)} D_{-m-1}^{\prime}\left(p_{2} v_{1}\right)
\end{align*}
$$

From Eq.(5.55), Eq.(5.56) and also Eq.(5.52) we can find the unknowns, i.e. $a_{m}^{s}, a_{m}^{t}$ and $b_{m}^{t}$.

## Chapter 6

## Conducting Wedge

> "As far as the laws of mathematics refer to reality, they are not certain, as far as they are certain, they do not refer to reality." Albert Einstein

### 6.1 Introduction

The scattering of electromagnetic wave by dielectric and conducting wedge is an exceptionally difficult problem which at present time has no known analytic solution. The wedge problem is an important one in at least two areas. The first concerns radar reflections and electromagnetic pulse response from dielectric objects which may be in free space or else buried. The second concerns the use of geometric theory of diffraction(GTD) to calculate the radiation properties of antennas and other reflectors.
In this chapter the scattering of electromagnetic waves by conducting wedge will be considered only.

### 6.2 Line Source Near a Conducting Wedge

The configuration of conducting wedge and electric line source is depicted in Fig.(6.1) It can be shown that the total electric field of line source near the


Figure 6.1: Conducting Wedge
wedge can be written as
$E_{z}^{t}=E_{z}^{i}+E_{z}^{s}= \begin{cases}\sum_{\nu} a_{\nu} J_{\nu}(k \rho) H_{\nu}^{(2)}\left(k \rho_{0}\right) \sin \left[\nu\left(\phi_{0}-\alpha\right)\right] \sin [\nu(\phi-\alpha)] & \rho \leq \rho_{0} \\ \sum_{\nu}^{\nu} a_{\nu} J_{\nu}\left(k \rho_{0}\right) H_{\nu}^{(2)}(k \rho) \sin \left[\nu\left(\phi_{0}-\alpha\right)\right] \sin [\nu(\phi-\alpha)] & \rho \geq \rho_{0}\end{cases}$
where

$$
\begin{align*}
\nu & =\frac{m \pi}{2(\pi-\alpha)} \quad m=1,2,3 \ldots \\
a_{\nu} & =-\frac{\pi \omega \mu I_{0}}{2(\pi-\alpha)} \tag{6.2}
\end{align*}
$$

and correspondent magnetic field will be calculated by

$$
\begin{align*}
H_{\rho}^{t} & =-\frac{1}{j \omega \mu} \frac{1}{\rho} \frac{\partial E_{z}^{t}}{\partial \phi} \\
H_{\phi}^{t} & =\frac{1}{j \omega \mu} \frac{\partial E_{z}^{t}}{\partial \rho} \tag{6.3}
\end{align*}
$$

### 6.3 Plane wave Scattering by Conducting Wedge $T M^{z}$ Polarization

If the line source in previous section goes to far distance, $\left(k \rho_{0} \gg 1\right.$, and $\rho_{0}>\rho$ ), we will have $T M^{z}$ plane wave, therefore the total filed will be

$$
\begin{equation*}
E_{z}^{t}=E_{0} \sum_{\nu} j^{\nu} J_{\nu}(k \rho) \sin \left[\nu\left(\phi_{0}-\alpha\right)\right] \sin [\nu(\phi-\alpha)] \tag{6.4}
\end{equation*}
$$

where we can say that $E_{z}^{i}=E_{0}^{i} \exp \left\{j k\left(x \cos \phi_{0}+y \sin \phi_{0}\right)\right\}$ incident at angle $\phi_{0}$ on a conducting wedge of interior angle $2 \alpha$. If the angle $\alpha=0$, then we will have half-plane, and the above formula will be reduced to

$$
\begin{equation*}
E_{z}^{t}=E_{0} \sum_{m=1}^{\infty} j^{m / 2} J_{m / 2}(k \rho) \sin \left(\frac{m}{2} \phi_{0}\right) \sin \left(\frac{m}{2} \phi\right) \tag{6.5}
\end{equation*}
$$

### 6.4 Plane wave Scattering by Conducting Wedge $T E^{z}$ Polarization

For $T E^{z}$ polarization, it is better to study the magnetic line-source near a conducting wedge. When the line source of Fig.(6.1)is a magnetic of current $I_{m}$, by considering boundary conditions, we find
$H_{z}^{t}=H_{z}^{i}+H_{z}^{s}= \begin{cases}\sum_{\nu} a_{\nu} J_{\nu}(k \rho) H_{\nu}^{(2)}\left(k \rho_{0}\right) \cos \left[\nu\left(\phi_{0}-\alpha\right)\right] \cos [\nu(\phi-\alpha)] & \rho \leq \rho_{0} \\ \sum_{\nu} a_{\nu} J_{\nu}\left(k \rho_{0}\right) H_{\nu}^{(2)}(k \rho) \cos \left[\nu\left(\phi_{0}-\alpha\right)\right] \cos [\nu(\phi-\alpha)] & \rho \geq \rho_{0}\end{cases}$
where

$$
\begin{align*}
\nu & =\frac{m \pi}{2(\pi-\alpha)} \quad m=0,1,2 \ldots \\
a_{\nu} & =\epsilon_{\nu}\left[\frac{\pi \omega \epsilon I_{m}}{4(\pi-\alpha)}\right] \\
\epsilon_{\nu} & = \begin{cases}1 & \nu=0 \\
2 & \nu \neq 0\end{cases} \tag{6.7}
\end{align*}
$$

and correspondent magnetic field will be calculated by

$$
\begin{align*}
E_{\rho}^{t} & =\frac{1}{j \omega \epsilon} \frac{1}{\rho} \frac{\partial H_{z}^{t}}{\partial \phi} \\
E_{\phi}^{t} & =-\frac{1}{j \omega \epsilon} \frac{\partial H_{z}^{t}}{\partial \rho} \tag{6.8}
\end{align*}
$$

If the line source in previous section goes to far distance, $\left(k \rho_{0} \gg 1\right.$, and $\rho_{0}>\rho$ ), we will have $T E^{z}$ plane wave, therefore the total filed will be

$$
\begin{equation*}
H_{z}^{t}=H_{0} \sum_{\nu} \epsilon_{\nu} j^{\nu} J_{\nu}(k \rho) \cos \left[\nu\left(\phi_{0}-\alpha\right)\right] \cos [\nu(\phi-\alpha)] \tag{6.9}
\end{equation*}
$$

where we can say that $H_{z}^{i}=H_{0}^{i} \exp \left\{j k\left(x \cos \phi_{0}+y \sin \phi_{0}\right)\right\}$ incident at angle $\phi_{0}$ on a conducting wedge of interior angle $2 \alpha$. If the angle $\alpha=0$, then we will have half-plane, and the above formula will be reduced to

$$
\begin{equation*}
H_{z}^{t}=H_{0} \sum_{m=0}^{\infty} \epsilon_{m / 2} j^{m / 2} J_{m / 2}(k \rho) \cos \left(\frac{m}{2} \phi_{0}\right) \cos \left(\frac{m}{2} \phi\right) \tag{6.10}
\end{equation*}
$$

### 6.5 Problems

- 1 A electric line source is located at $\phi_{0}=45^{\circ}$ and $\rho_{0}=2 \lambda$ of a right angle conducting corner reflector. Find the electric pattern of line source with conducting wedge theory and compare it with image theory.


Figure 6.2: Line Source near a 2D Conducting $90^{\circ}$ Corner Reflector

## Chapter 7

## Sphere

"The only real valuable thing is intuition." Albert Einstein

### 7.1 Solution of Helmholtz Equation in Spherical Coordinates

One of the common equations that we usually encounter in electromagnetic fields and waves is the scalar wave equation. The scalar wave equation which has the common form of

$$
\begin{equation*}
\nabla^{2} \Psi+k^{2} \Psi=0 \tag{7.1}
\end{equation*}
$$

where $k=\omega \sqrt{\mu \epsilon}$ is called Helmholtz Equation. In spherical coordinates as defined in appendix, the wave equation Eq.(7.1) has the form of:

$$
\begin{equation*}
\nabla^{2} \Psi+k^{2} \Psi=\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} \frac{\partial \Psi}{\partial r}\right)+\frac{1}{r^{2} \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial \Psi}{\partial \theta}\right)+\frac{1}{r^{2} \sin ^{2} \theta} \frac{\partial^{2} \Psi}{\partial \phi^{2}}+k^{2} \Psi=0 \tag{7.2}
\end{equation*}
$$

To solve the above equation, we use the separation of variables method, so we define

$$
\begin{equation*}
\Psi=R(r) T(\theta) F(\phi) \tag{7.3}
\end{equation*}
$$

by substituting $\psi$ into the Eq.(7.2) and multiplying the results by

$$
r^{2} \sin ^{2} \theta / R T F
$$

we will have

$$
\begin{equation*}
\frac{\sin ^{2} \theta}{R} \frac{d}{d r}\left(r^{2} \frac{d R}{d r}\right)+\frac{\sin \theta}{T} \frac{d}{d \theta}\left(\sin \theta \frac{d T}{d \theta}\right)+k^{2} r^{2} \sin ^{2} \theta=-\frac{1}{F} \frac{d^{2} F}{d \phi^{2}} \tag{7.4}
\end{equation*}
$$

Since the left hand side of Eq.(7.4) is independent of $\phi$, we let

$$
\begin{equation*}
-\frac{1}{F} \frac{d^{2} F}{d \phi^{2}}=m^{2} \tag{7.5}
\end{equation*}
$$

where $m$ is the first separation constant. The requirement for such condition is necessary for the physical behavior of problem. By this assumption, Eq.(7.4) reduces to

$$
\begin{equation*}
\frac{1}{R} \frac{d}{d r}\left(r^{2} \frac{d R}{d r}\right)+k^{2} r^{2}=-\frac{1}{T \sin ^{2} \theta} \frac{d}{d \theta}\left(\sin \theta \frac{d T}{d \theta}\right)+\frac{m^{2}}{\sin ^{2} \theta}=n(n+1) \tag{7.6}
\end{equation*}
$$

where $n(n+1)$ is the second separation constant. Thus the separated equations are three different ordinary differential equations:

$$
\begin{gather*}
F^{\prime \prime}+m^{2} F=0  \tag{7.7}\\
R^{\prime \prime}+\frac{2}{r} R^{\prime}+\left[k^{2}-\frac{n(n+1)}{r^{2}}\right] R=0  \tag{7.8}\\
\frac{1}{\sin \theta} \frac{d}{d \theta}\left(\sin \theta T^{\prime}\right)+\left[n(n+1)-\frac{m^{2}}{\sin ^{2} \theta}\right] T=0 \tag{7.9}
\end{gather*}
$$

### 7.1.1 Simple Harmonic Functions

The simple one, $F^{\prime \prime}+m^{2} F=0$ has the solution

$$
\begin{equation*}
F(\phi)=c_{1} e^{j m \phi} \tag{7.10}
\end{equation*}
$$

where $c_{1}$ is a constant and $m=0, \pm 1, \pm 2, \cdots$

### 7.1.2 Spherical Bessel's Functions

The second equation $R^{\prime \prime}+\frac{2}{r} R^{\prime}+\left[k^{2}-\frac{n(n+1)}{r^{2}}\right] R=0$ is called the Spherical Bessel's Differential Equation, the solution of it is called the Spherical Bessel's Functions, and it is given by

$$
\begin{equation*}
R(r)=c_{2} z_{n}^{(g)}(k r) \tag{7.11}
\end{equation*}
$$

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where $c_{2}$ is a constant and $n=0,1,2, \cdots$ and $g=1,2,3,4$ represents types of the Spherical Bessel's Functions:

$$
\begin{align*}
& z_{n}^{(1)}(k r)=j_{n}(k r) \quad \text { Spherical Bessel Function }  \tag{7.12}\\
& z_{n}^{(2)}(k r)=y_{n}(k r) \quad \text { Spherical Neumann Function }  \tag{7.13}\\
& z_{n}^{(3)}(k r)=h_{n}^{(1)}(k r) \text { Spherical Hankel Function of first kind }  \tag{7.14}\\
& z_{n}^{(4)}(k r)=h_{n}^{(2)}(k r) \text { Spherical Hankel Function of second kind } \tag{7.15}
\end{align*}
$$

Each of these functions has special properties: $g=1$ and $g=2$ indicate standing wave while $g=3$ represents an inward traveling wave and $g=4$ an outward traveling wave. We should notice that

$$
\begin{align*}
& z_{n}^{(1)}(k r)=j_{n}(k r)=\frac{h_{n}^{(1)}(k r)+h_{n}^{(2)}(k r)}{2}  \tag{7.16}\\
& z_{n}^{(2)}(k r)=y_{n}(k r)=\frac{h_{n}^{(1)}(k r)-h_{n}^{(2)}(k r)}{2 j} \tag{7.17}
\end{align*}
$$

for more properties and identities of Spherical Bessel's Functions see appendix (E) and [12]

### 7.1.3 Associated Legendre Functions

The last equation; i.e. $\frac{1}{\sin \theta} \frac{d}{d \theta}\left(\sin \theta T^{\prime}\right)+\left[n(n+1)-\frac{m^{2}}{\sin ^{2} \theta}\right] T=0$ is another famous equation which is called Legendre's Associated Differential Equation. The solution to that equation are called Associated Legendre Polynomials $P_{n}^{m}(\cos \theta)$. For more information and properties of these function see appendix (E).

### 7.1.4 Results of Helmholtz Equation

Now we have the solutions of three differential equations. After multiplying the results, the general solution is:

$$
\begin{equation*}
\Psi_{m n}^{g}(r, \theta, \phi)=\sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{m n}^{g} z_{n}^{(g)}(k r) P_{n}^{|m|}(\cos \theta) e^{j m \phi} \tag{7.18}
\end{equation*}
$$

where $A_{m n}^{g}$ is a constant.

### 7.2 Vector Wave Equation in Spherical Coordinate System

For many problems in electromagnetic wave theory, solution of vector wave equation is required. The only known general method for obtaining solution of vector wave equation is the method of applying certain vector differential operators to the scalar wave functions. The solutions are called vector wave functions. Following it will be shown that how vector wave functions are obtained from scalar wave functions and a number of spherical wave functions will be defined. It will be then possible to derive expansion of plane wave or the radiation from a Hertzian dipole in terms of these functions. This is critical step which makes it possible to generalize the Mie theory to deal with an incident field, and have realistic antenna. Now let us define vector wave functions in the spherical coordinate system. The vector wave functions are the solution of vector wave equation:

$$
\begin{equation*}
\nabla \times \nabla \times \mathbf{C}-k^{2} \mathbf{C}=0 \tag{7.19}
\end{equation*}
$$

with $k=\omega \sqrt{\mu \epsilon_{c}}, \epsilon_{c}=\epsilon_{0}\left(\epsilon_{r}-j \frac{\sigma}{\omega \epsilon_{0}}\right)$ in the case of lossy medium. The vector wave Eq.(7.19) can always be replaced by a simultaneous system of three scalar equations, but the solution of this system for any component of $\mathbf{C}$ is in most cases impractical. It is only in rectangular system that three independent equations are obtained and in that case, each one is in the form of Helmholtz Equation Eq.(7.1). Elementary solutions of the equation may be found from the following theorem which is stated without proof. If $\Psi$ satisfies the scalar wave equation $\nabla^{2} \Psi+k^{2} \Psi=0$, then the vectors $\mathbf{M}$ and $\mathbf{N}$ defined by:

$$
\begin{align*}
\mathbf{M} & =\nabla \times(\mathbf{a} \Psi)  \tag{7.20}\\
\mathbf{N} & =\frac{1}{k} \nabla \times \mathbf{M} \tag{7.21}
\end{align*}
$$

satisfy the vector wave equation and moreover, are related by:

$$
\begin{equation*}
\mathbf{M}=\frac{1}{k} \nabla \times \mathbf{N} \tag{7.22}
\end{equation*}
$$

If $\Psi_{1}$ and $\Psi_{2}$ are two solutions of the scalar wave equation, $\Psi_{1}+\Psi_{2}$ are also a solution (Linearity), and therefore $\mathbf{M}_{\mathbf{1}}, \mathbf{M}_{\mathbf{2}}, \mathbf{N}_{\mathbf{1}}$, and $\mathbf{N}_{\mathbf{2}}$ also satisfy the vector wave equation. Thus $\sum\left(a_{m n} \mathbf{M}+b_{m n} \mathbf{N}\right)$ is also a solution to the

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wave equation. a in Eq.(7.20) is a constant vector. Mie in 1908 made the remarkable discovery that in spherical coordinate system, one can also find a non constant vector $\mathbf{a}=r \mathbf{a}_{r}$, [where $\mathbf{a}_{r}$ is radial unit vector], for which $\mathbf{M}$ is also a solution of vector wave equation.
Let us define:

$$
\begin{align*}
\mathbf{L} & =\nabla \Psi  \tag{7.23}\\
\mathbf{M} & =\nabla \times\left(r \mathbf{a}_{r} \Psi\right)  \tag{7.24}\\
\mathbf{N} & =\frac{1}{k} \nabla \times \mathbf{M} \tag{7.25}
\end{align*}
$$

It can be seen that $\mathbf{L}, \mathbf{M}$ and $\mathbf{N}$ are the solution of the vector wave equation $\nabla \times \nabla \times \mathbf{C}-k^{2} \mathbf{C}=0$ and generate a complete orthogonal system. By calculating the divergence of Eq.(7.23), we find:

$$
\begin{equation*}
\nabla \cdot \mathbf{M}=\nabla \cdot \mathbf{N}=0 \tag{7.26}
\end{equation*}
$$

i.e. the function $\mathbf{M}, \mathbf{N}$ are solenoidal, but $\mathbf{L}$ is not.

$$
\begin{equation*}
\nabla \cdot \mathbf{L}=\nabla^{2} \psi=-k^{2} \psi \tag{7.27}
\end{equation*}
$$

Now we can expand any electromagnetic field in a source free region as sum of the spherical wave functions. In the spherical case, there are two types of waves: TE and TM. Where in our problem M stands for TE and $\mathbf{N}$ for TM waves. Therefore in spherical coordinates the expansion of any field is:

$$
\begin{align*}
\mathbf{E} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n} \mathbf{M}_{m n}^{g}+b_{m n} \mathbf{N}_{m n}^{g}\right)  \tag{7.28}\\
\mathbf{H} & =\frac{j}{\eta} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n} \mathbf{N}_{m n}^{g}+b_{m n} \mathbf{M}_{m n}^{g}\right) \tag{7.29}
\end{align*}
$$

where $\mathbf{M}$ and $\mathbf{N}$ are:

$$
\begin{align*}
\mathbf{M}_{m n}^{g} & =M_{m n r}^{g} \mathbf{a}_{r}+M_{m n \theta}^{g} \mathbf{a}_{\theta}+M_{m n \phi}^{g} \mathbf{a}_{\phi}  \tag{7.30}\\
\mathbf{N}_{m n}^{g} & =N_{m n r}^{g} \mathbf{a}_{r}+N_{m n \theta}^{g} \mathbf{a}_{\theta}+N_{m n \phi}^{g} \mathbf{a}_{\phi} \tag{7.31}
\end{align*}
$$

from $\mathbf{M}=\nabla \Psi \times r \mathbf{a}_{r}$ and $\mathbf{N}=\frac{1}{k} \nabla \times \mathbf{M}$, we have

$$
\begin{align*}
M_{m n r}^{g} & =0  \tag{7.32}\\
M_{m n \theta}^{g} & =\frac{j m}{\sin \theta} P_{n}^{|m|}(\cos \theta) z_{n}^{g}(k r) e^{j m \phi}  \tag{7.33}\\
M_{m n \phi}^{g} & =-\frac{d}{d \theta} P_{n}^{|m|}(\cos \theta) z_{n}^{g}(k r) e^{j m \phi} \tag{7.34}
\end{align*}
$$

$$
\begin{align*}
N_{m n r}^{g} & =\frac{n(n+1)}{k r} P_{n}^{|m|}(\cos \theta) z_{n}^{g}(k r) e^{j m \phi}  \tag{7.35}\\
N_{m n \theta}^{g} & =\frac{d}{d \theta} P_{n}^{|m|}(\cos \theta) \frac{1}{k r} \frac{d}{d r}\left[r z_{n}^{g}(k r)\right] e^{j m \phi}  \tag{7.36}\\
N_{m n \phi}^{g} & =\frac{j m}{\sin \theta} P_{n}^{|m|}(\cos \theta) \frac{1}{k r} \frac{d}{d r}\left[r z_{n}^{g}(k r)\right] e^{j m \phi} \tag{7.37}
\end{align*}
$$

and the orthogonalities of $\mathbf{M}$ and $\mathbf{N}$ vectors will be

$$
\begin{gather*}
\int_{0}^{2 \pi} d \phi \int_{0}^{\pi} d \theta \sin \theta \mathbf{M}_{m n} \cdot \mathbf{M}_{m^{\prime} n^{\prime}} \sin \theta=\delta_{m m^{\prime}} \delta_{m n^{\prime}} \frac{4 \pi n(n+1)}{2 n+1} \frac{(n+|m|)!}{(n-|m|)!}\left\{z_{n}^{(g)}(k r)\right\}^{2}  \tag{7.38}\\
\int_{0}^{2 \pi} d \phi \int_{0}^{\pi} d \theta \sin \theta \mathbf{N}_{m n} \cdot \mathbf{N}_{m^{\prime} n^{\prime}} \sin \theta=  \tag{7.39}\\
\delta_{m m^{\prime}} \delta_{n n^{\prime}} \frac{4 \pi n(n+1)}{(2 n+1) r^{2}} \frac{(n+|m|)!}{(n-|m|)!}\left[n(n+1)\left\{z_{n}^{(g)}(k r)\right\}^{2}+\left\{\frac{d}{d r}\left[r z_{n}^{(g)}(k r)\right]\right\}^{2}\right] \\
\quad \int_{0}^{2 \pi} d \phi \int_{0}^{\pi} d \theta \sin \theta \mathbf{M}_{m n} \cdot \mathbf{N}_{m^{\prime} n^{\prime}} \sin \theta=0 \tag{7.40}
\end{gather*}
$$

where $m, m^{\prime}, n, n^{\prime}$ are integers
We can rewrite $\mathbf{M}$ and $\mathbf{N}$ in other format

$$
\begin{align*}
& \mathbf{M}=\mathbf{M}^{e}+j \mathbf{M}^{o}=\mathbf{M}_{o}^{e}  \tag{7.41}\\
& \mathbf{N}=\mathbf{N}^{e}+j \mathbf{N}^{o}=\mathbf{N}_{o}^{e} \tag{7.42}
\end{align*}
$$

where

$$
\begin{align*}
& \mathbf{M}_{o}^{e}=\mp \frac{m \mathbf{a}_{\theta}}{\sin \theta} z_{n}^{g}(k r) P_{n}^{m}(\cos \theta){ }_{\cos }^{\sin }(m \phi)  \tag{7.43}\\
& -\mathbf{a}_{\phi} z_{n}^{g}(k r) \frac{\partial}{\partial \theta} P_{n}^{m}(\cos \theta) \underset{\sin }{\cos }(m \phi) \\
& \mathbf{N}_{o}^{e}=\mathbf{a}_{r} \frac{n(n+1)}{k r} z_{n}^{g}(k r) P_{n}^{m}(\cos \theta) \underset{\sin }{\cos }(m \phi)  \tag{7.44}\\
& +\mathbf{a}_{\theta} \frac{1}{k r} \frac{\partial}{\partial r}\left[r z_{n}^{g}(k r)\right] \frac{\partial}{\partial \theta} P_{n}^{m}(\cos \theta){ }_{\sin }^{\cos }(m \phi) \\
& \mp \frac{m \mathbf{a}_{\phi}}{\sin \theta} \frac{1}{k r} \frac{\partial}{\partial r}\left[r z_{n}^{g}(k r)\right] P_{n}^{m}(\cos \theta) \frac{\sin }{\cos }(m \phi)
\end{align*}
$$

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and the orthogonality of M and N vectors will be

$$
\begin{equation*}
\int_{0}^{2 \pi} d \phi \int_{0}^{\pi} d \theta \mathbf{M}_{l m n} \cdot \mathbf{N}_{l^{\prime} m^{\prime} n^{\prime}} \sin \theta=0 \tag{7.45}
\end{equation*}
$$

where $l$ or $l^{\prime}$ are even or odd, and $m, m^{\prime}, n, n^{\prime}$ are integers

$$
\begin{align*}
& \int_{0}^{2 \pi} d \phi \int_{0}^{\pi} d \theta \mathbf{M}_{l m n} \cdot \mathbf{M}_{l^{\prime} m^{\prime} n^{\prime}} \sin \theta \\
& = \begin{cases}0 & l \neq l^{\prime} \text { or } m \neq m^{\prime} \text { or } n \neq n^{\prime} \\
\frac{4 \pi}{\epsilon_{m}} \frac{n(n+1)}{(2 n+1)} \frac{(n+m)!}{(n-m)!}\left\{z_{n}(k r)\right\}^{2} & l=l^{\prime}, m=m^{\prime} \text { and } n=n^{\prime}\end{cases}  \tag{7.46}\\
& \int_{0}^{2 \pi} d \phi \int_{0}^{\pi} d \theta \mathbf{N}_{l m n} \cdot \mathbf{N}_{l^{\prime} m^{\prime} n^{\prime}} \sin \theta \\
& =\left\{\begin{array}{l}
0 \quad l \neq l^{\prime} \text { or } m \neq m^{\prime} \text { or } n \neq n^{\prime} \\
\frac{4 \pi}{\epsilon_{m}} \frac{n(n+1)}{(2 n+1)^{2}} \frac{(n+m)!}{(n-m)!}\left[(n+1)\left\{z_{n-1}(k r)\right\}^{2}+n\left\{z_{n+1}(k r)\right\}^{2}\right] \\
l=l^{\prime}, m=m^{\prime} \text { and } n=n^{\prime}
\end{array}\right. \tag{7.47}
\end{align*}
$$

### 7.2.1 Spherical Cavity Resonators

Suppose we have a spherical cavity with radius $a$. Now we want to find the resonant frequency of this cavity. We have two type modes $T E^{r}$ and $T M^{r}$. If one takes $\mathbf{E}=\mathbf{M}=\nabla \times\left(\psi r \mathbf{a}_{r}\right)$, for $T E^{r}$ case, we have

$$
\left.\left.\begin{array}{rl}
\mathbf{E}= & \bar{c}\left[\mp \frac{m \mathbf{a}_{\theta}}{\sin \theta} j_{n}(k r) P_{n}^{m}(\cos \theta)\right.  \tag{7.48}\\
& -\mathbf{a}_{\phi} j_{n}(k r) \frac{\partial}{\partial \theta} P_{n}^{m}(\cos \theta) \\
\cos \\
\sin \\
\cos \\
\sin
\end{array} m \phi\right)\right]
$$

where $\bar{c}$ is amplitude and the corresponding $\mathbf{H}$ will be fund by $\mathbf{H}=\frac{-1}{j \omega \mu} \nabla \times$ $\mathbf{E}=\frac{k}{-j \omega \mu} \mathbf{N}$. We should mention that $P_{n}^{m}(\cos \theta)=0$ if $m>n$ therefore $m$ is always $m \leq n$ therefor the starting integer is $m=1, n=1$ why $=$ ?.

In some literature, for simplicity, they define $\hat{z}_{n}^{g}(x)=x z_{n}^{g}(x)$ as a Riccati or schelkunoff Bessel functions.

$$
\left.\begin{array}{rl}
\mathbf{H}= & \frac{j \bar{c}}{\eta}\left[\mathbf{a}_{r} \frac{n(n+1)}{k r} j_{n}(k r) P_{n}^{m}(\cos \theta)\right.  \tag{7.49}\\
& \begin{array}{l}
\cos \\
\sin
\end{array}(m \phi) \\
& +\mathbf{a}_{\theta} \frac{1}{k r} \frac{\partial}{\partial r}\left[r j_{n}(k r)\right] \frac{\partial}{\partial \theta} P_{n}^{m}(\cos \theta) \\
& \mp \frac{m \mathbf{a}_{\phi}}{\sin \theta} \frac{1}{\sin } \frac{\partial}{\partial r}[m \phi) \\
\left.\sin j_{n}(k r)\right] P_{n}^{m}(\cos \theta) & \sin \\
\cos
\end{array}(m \phi)\right] .
$$

The boundary conditions that must be satisfied are

$$
\begin{align*}
& E_{\theta}(r=a, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2 \pi)=0  \tag{7.50}\\
& E_{\phi}(r=a, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2 \pi)=0
\end{align*}
$$

Either condition gives us the same results.

$$
\begin{equation*}
j_{n}(k a)=0 \Rightarrow k a=\alpha_{n p} \tag{7.51}
\end{equation*}
$$

where the $\alpha_{n p}$ is the $p$ th roots of spherical bessel function $j_{n}(k a)$. which is tabulated in Tabel[G.3]. The resonant frequency can be written as

$$
f_{r}\left[T E_{m n p}^{r}\right]=\frac{\alpha_{n p}}{2 \pi a \sqrt{\mu \epsilon}} \quad \begin{align*}
& m=0,1,2, \cdots \leq n  \tag{7.52}\\
& n=1,2,3, \cdots \\
& \\
& p=1,2,3, \cdots
\end{align*}
$$

Following the same procedure as before, $\mathbf{H}=\mathbf{M}=\nabla \times\left(\psi r \mathbf{a}_{r}\right)$ for $T M^{r}$, it can be shown that

$$
\left.\left.\begin{array}{rl}
\mathbf{H}= & \bar{c}\left[\mp \frac{m \mathbf{a}_{\theta}}{\sin \theta} j_{n}(k r) P_{n}^{m}(\cos \theta)\right.  \tag{7.53}\\
& \begin{array}{l}
\sin \\
\cos
\end{array}(m \phi) \\
& -\mathbf{a}_{\phi} j_{n}(k r) \frac{\partial}{\partial \theta} P_{n}^{m}(\cos \theta) \\
\sin \\
\cos \\
\sin
\end{array} m \phi\right)\right] \quad \$
$$

and the corresponding $\mathbf{E}$ will be fund by $\mathbf{E}=\frac{1}{j \omega \epsilon} \nabla \times \mathbf{H}=\frac{k}{j \omega \epsilon} \mathbf{N}$.

$$
\left.\begin{array}{rl}
\mathbf{E}= & -j \bar{c} \eta\left[\mathbf{a}_{r} \frac{n(n+1)}{k r} j_{n}(k r) P_{n}^{m}(\cos \theta)\right.  \tag{7.54}\\
& +\mathbf{a}_{\theta} \frac{1}{k r} \frac{\partial}{\sin }\left[r j_{n}(k r)\right] \frac{\partial}{\partial \theta} P_{n}^{m}(\cos \theta) \\
& \mp \frac{m \mathbf{a}_{\phi}}{\sin \theta} \frac{1}{k r} \frac{\partial}{\partial r}\left[r j_{n}(k r)\right] P_{n}^{m}(\cos \theta) \\
\sin \\
\sin
\end{array}(m \phi)\right]
$$

Applying boundary condition $E_{\theta}=0$ at the surface of sphere, it leads to

$$
\begin{equation*}
E_{\theta}(r=a, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2 \pi)=0 \Rightarrow\left[\frac{1}{k r} \frac{\partial}{\partial r}\left[r j_{n}(k r)\right]\right]_{r=a}=0 \tag{7.55}
\end{equation*}
$$

or

$$
\begin{equation*}
\hat{J}_{n}^{\prime}(k a)=0 \Rightarrow k a=\alpha_{n p}^{\prime} \tag{7.56}
\end{equation*}
$$

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where $\alpha_{n p}^{\prime}$ represents the $p$ th zeros of the derivative of the Riccati Bessel function of order n , which is tabulated in Tabel[G.5]. The resonant frequency can be written as

$$
f_{r}\left[T M_{m n p}^{r}\right]=\frac{\alpha_{n p}^{\prime}}{2 \pi a \sqrt{\mu \epsilon}} \quad \begin{align*}
& m=0,1,2, \cdots \leq n  \tag{7.57}\\
& n=1,2,3, \cdots \\
& p=1,2,3, \cdots
\end{align*}
$$

### 7.2.2 What is a Schumann Resonance?

Believe it or not, the Earth behaves like an enormous electric circuit. The atmosphere is actually a weak conductor and if there were no sources of charge, its existing electric charge would diffuse away in about 10 minutes. There is a 'cavity 'defined by the surface of the Earth and the inner edge of the ionosphere 55 kilometers up. At any moment, the total charge residing in this cavity is 500,000 Coulombs. There is a vertical current flow between the ground and the ionosphere of $1-3 \times 10^{-12}$ Amperes per square meter. The resistance of the atmosphere is 200 Ohms. The voltage potential is 200,000 Volts. There are about 1000 lightning storms at any given moment worldwide. Each produces 0.5 to 1 Ampere and these collectively account for the measured current flow in the Earth's 'electromagnetic' cavity.
The Schumann Resonances are quasi standing wave electromagnetic waves that exist in this cavity. Like waves on a spring, they are not present all the time, but have to be 'excited' to be observed. They are not caused by anything internal to the Earth, its crust or its core. They seem to be related to electrical activity in the atmosphere, particularly during times of intense lightning activity.
They occur at several frequencies between 6 and 50 cycles per second; specifically $7.8,14,20,26,33,39$ and 45 Hertz, with a daily variation of about $\pm 0.5$ Hertz. So long as the properties of Earth's electromagnetic cavity remains about the same, these frequencies remain the same. Presumably there is some change due to the solar sunspot cycle as the Earth's ionosphere changes in response to the 11-year cycle of solar activity. Schumann resonances are most easily seen between 2000 and 2200 UT.
Given that the earth's atmosphere carries a charge, a current and a voltage, it is not surprising to find such electromagnetic waves. The resonant properties of this terrestrial cavity were first predicted by the German physicist W. O. Schumann between 1952 and 1957, and first detected by Schumann and Konig in 1954. The first spectral representation of this phenomenon was
prepared by Balser and Wagner in 1960. Much of the research in the last 20 years has been conducted by the Department of the Navy ${ }^{1}$ who investigate Extremely Low Frequency communication with submarines.
Schumann resonances in the Earth-ionosphere cavity are excited by the radial E-field component of lightning discharges (the frequency component of EM waves produced by lightning at these Schumann resonance frequencies). Lightning discharges (anywhere on Earth) contain a wide spectrum of frequencies of EM radiation.

Schumann resonances were first definitively observed in 1960. (M. Balser and C.A. Wagner, Nature 188, 638 (1960)).
Nikola Tesla may have observed them before 1900!!! (Before the ionosphere was known to even exist!!!) He also estimated the lowest modal frequency to be $f 01 \approx 6 H z!!!$

- On July 9, 1962, a nuclear explosion (EMP) detonated at high altitude ( 400 km ) over Johnson Island in the Pacific \{Test Shot: Starfish Prime, Operation Dominic I\}.
- Measurably affected the Earths ionosphere and radiation belts on a worldwide scale.
- Sudden decrease of $\sim 3-5 \%$ in Schumann frequencies, therefore increase in height of ionosphere.
- Change in height of ionosphere: $\Delta h=h^{\prime}-h \simeq(0.03-0.05) R_{0} \approx$ $400-600 \mathrm{~km}$ !!!
- Height changes decayed away after several hours.
- Artificial radiation belts lasted several years.
- Note that number of lightning strikes, (e.g. in tropics) is strongly correlated to average temperature.
- Scientists have used Schumann resonances and monthly mean magnetic field strengths to monitor lightning rates and thus monitor monthly temperatures.
they all correlate very well.
- Monitoring Schumann Resonances rise Global Thermometer and it is useful for Global Warming studies.

[^0]
### 7.3 Hertzian Dipole Expansion

The electromagnetic radiation of a Hertzian dipole can be found analytically. Let the dipole moment be $\mathbf{P}_{\mathbf{o}}=p_{o} \mathbf{a}_{d}$ where $p_{o}=\frac{I \Delta l}{j \omega}$ and $I$ is the dipole current, $\Delta l$ the length of the dipole and finally $\mathbf{a}_{d}$ is a unit vector showing the orientation of the dipole at the point $r_{0}, \theta_{0}, \phi_{0}$, can be found analytically [9]. The result for $r<r_{0}$ is:

$$
\begin{align*}
\mathbf{E} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left[a_{m n}^{(4)} \mathbf{M}_{m n}^{(1)}(r, \theta, \phi)+b_{m n}^{(4)} \mathbf{N}_{m n}^{(1)}(r, \theta, \phi)\right]  \tag{7.58}\\
\mathbf{H} & =\frac{j}{\eta} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left[a_{m n}^{(4)} \mathbf{N}_{m n}^{(1)}(r, \theta, \phi)+b_{m n}^{(4)} \mathbf{M}_{m n}^{(1)}(r, \theta, \phi)\right] \tag{7.59}
\end{align*}
$$

and for $r>r_{0}$, the superscripts (4) and (1) must be changed to (1) and (4), respectively. therefore:

$$
\begin{align*}
\mathbf{E} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left[a_{m n}^{(1)} \mathbf{M}_{m n}^{(4)}(r, \theta, \phi)+b_{m n}^{(1)} \mathbf{N}_{m n}^{(4)}(r, \theta, \phi)\right]  \tag{7.60}\\
\mathbf{H} & =\frac{j}{\eta} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left[a_{m n}^{(1)} \mathbf{N}_{m n}^{(4)}(r, \theta, \phi)+b_{m n}^{(1)} \mathbf{M}_{m n}^{(4)}(r, \theta, \phi)\right] \tag{7.61}
\end{align*}
$$

The values of $a_{m n}^{(1),(4)}$ and $b_{m n}^{(1),(4)}$ are:

$$
\begin{align*}
a_{m n}^{(1),(4)} & =\left(\frac{-j k^{3} p_{0}}{4 \pi \epsilon}\right) \frac{(n-|m|)!}{(n+|m|)!} \frac{2 n+1}{n(n+1)} \mathbf{a}_{d} \cdot \mathbf{M}_{-m, n}^{(1),(4)}\left(r_{0}, \theta_{0}, \phi_{0}\right)  \tag{7.62}\\
b_{m n}^{(1),(4)} & =\left(\frac{-j k^{3} p_{0}}{4 \pi \epsilon}\right) \frac{(n-|m|)!}{(n+|m|)!} \frac{2 n+1}{n(n+1)} \mathbf{a}_{d} \cdot \mathbf{N}_{-m, n}^{(1),(4)}\left(r_{0}, \theta_{0}, \phi_{0}\right) \tag{7.63}
\end{align*}
$$

### 7.4 Plane Wave Expansion

Consider a dielectric sphere with radius $a$ illuminated by a uniform plane wave propagating in the $z$ direction and polarized in the $x$ direction:

$$
\begin{align*}
\mathbf{E}^{i} & =E_{0} e^{-j k z} \mathbf{a}_{x}  \tag{7.64}\\
\mathbf{H}^{i} & =\frac{E_{0}}{\eta} e^{-j k z} \mathbf{a}_{y} \tag{7.65}
\end{align*}
$$

where $\eta$ is intrinsic impedance of the medium. In the plane wave case, the field expansions are derived from equation Eq. $(7.43,7.44)$ with $m= \pm 1$. The expansion of such a plane wave in spherical coordinates is:

$$
\begin{align*}
\mathbf{E}^{i} & =E_{0} e^{-j k z} \mathbf{a}_{x}=E_{0} \sum_{n=1}^{\infty} \alpha_{n}\left[\mathbf{M}_{o 1 n}^{1}(k r)+j \mathbf{N}_{e 1 n}^{1}(k r)\right]  \tag{7.66}\\
\mathbf{H}^{i} & =\frac{E_{0}}{\eta} e^{-j k z} \mathbf{a}_{y}=\frac{j E_{0}}{\eta} \sum_{n=1}^{\infty} \alpha_{n}\left[\mathbf{N}_{o 1 n}^{1}(k r)+j \mathbf{M}_{e 1 n}^{1}(k r)\right]
\end{align*}
$$

where $\alpha_{n}=(-j)^{n} \frac{2 n+1}{n(n+1)}$

### 7.5 Dipole Antenna Near a Dielectric Sphere

Suppose that we have a Hertzian dipole antenna located at point $r_{0}, \theta_{0}, \phi_{0}$ in the front of a dielectric sphere with radius $a$. We are asked to find the electromagnetic radiation of this antenna at every point of space, including inside dielectric sphere. The application of this arrangement is the mutual interaction of the human head and the mobile antenna. According to Fig.(7.1), for $r<r_{0}$ we can write:

$$
\begin{align*}
\mathbf{E}^{i} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left[A_{i}^{(4)} \mathbf{M}^{(1)}(r, \theta, \phi)+B_{i}^{(4)} \mathbf{N}^{(1)}(r, \theta, \phi)\right]  \tag{7.67}\\
\mathbf{H}^{i} & =\frac{j}{\eta_{1}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left[A_{i}^{(4)} \mathbf{N}^{(1)}(r, \theta, \phi)+B_{i}^{(4)} \mathbf{M}^{(1)}(r, \theta, \phi)\right] \tag{7.68}
\end{align*}
$$

and for $r>r_{0}$ we will have:

$$
\begin{align*}
\mathbf{E}^{i} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left[A_{i}^{(1)} \mathbf{M}^{(4)}(r, \theta, \phi)+B_{i}^{(1)} \mathbf{N}^{(4)}(r, \theta, \phi)\right]  \tag{7.69}\\
\mathbf{H}^{i} & =\frac{j}{\eta_{1}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left[A_{i}^{(1)} \mathbf{N}^{(4)}(r, \theta, \phi)+B_{i}^{(1)} \mathbf{M}^{(4)}(r, \theta, \phi)\right] \tag{7.70}
\end{align*}
$$

The transmitted and scattered fields will be:

$$
\begin{align*}
\mathbf{E}^{t} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left[a_{m n}^{t} \mathbf{M}^{(1)}(r, \theta, \phi)+b_{m n}^{t} \mathbf{N}^{(1)}(r, \theta, \phi)\right]  \tag{7.71}\\
\mathbf{H}^{t} & =\frac{j}{\eta_{2}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left[a_{m n}^{t} \mathbf{N}^{(1)}(r, \theta, \phi)+b_{m n}^{t} \mathbf{M}^{(1)}(r, \theta, \phi)\right] \tag{7.72}
\end{align*}
$$



Figure 7.1: Dipole antenna near a dielectric sphere

$$
\begin{align*}
\mathbf{E}^{s} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left[a_{m n}^{s} \mathbf{M}^{(4)}(r, \theta, \phi)+b_{m n}^{s} \mathbf{N}^{(4)}(r, \theta, \phi)\right]  \tag{7.73}\\
\mathbf{H}^{s} & =\frac{j}{\eta_{1}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left[a_{m n}^{s} \mathbf{N}^{(4)}(r, \theta, \phi)+b_{m n}^{s} \mathbf{M}^{(4)}(r, \theta, \phi)\right] \tag{7.74}
\end{align*}
$$

where $k_{1}=\omega \sqrt{\mu_{1} \epsilon_{1}}, k_{2}=\omega \sqrt{\mu_{2} \epsilon_{2}}$. If the dielectric sphere have loss $\sigma_{2}, \epsilon_{2}$ will be complex $\epsilon_{r 2 c}=\epsilon_{r 2}-j \frac{\sigma_{2}}{\omega \epsilon_{0}}$. The $A_{i}^{(1),(4)}$ and $B_{i}^{(1),(4)}$ are known from antenna position and direction, and $a_{m n}^{t}, a_{m n}^{s}, b_{m n}^{t}$ and $b_{m n}^{s}$ are unknowns. By applying boundary conditions at the surface of sphere, $E_{\theta}^{s}+E_{\theta}^{i}=E_{\theta}^{t}$ and $H_{\theta}^{s}+H_{\theta}^{i}=H_{\theta}^{t}$. Let us assume that $\rho_{1}=k_{1} a, \rho_{2}=k_{2} a$ for TE and TM modes. And using Riccati-Bessel as new notation, $\hat{b}^{\prime}{ }_{n}(z)=\frac{d}{d z}\left[z b_{n}(z)\right]$ we will have:

$$
\begin{align*}
A_{i} j_{n}\left(\rho_{1}\right)+a_{m n}^{s} h_{n}^{(2)}\left(\rho_{1}\right) & =a_{m n}^{t} j_{n}\left(\rho_{2}\right)  \tag{7.75}\\
\frac{A_{i}}{\rho_{1} \eta_{1}} \hat{J}_{n}^{\prime}\left(\rho_{1}\right)+\frac{a_{m n}^{s}}{\rho_{1} \eta_{1}} \hat{H}_{n}^{(2)}\left(\rho_{1}\right) & =\frac{a_{m n}^{t}}{\rho_{2} \eta_{2}} \hat{J}_{n}^{\prime}\left(\rho_{2}\right) \tag{7.76}
\end{align*}
$$

$$
\begin{align*}
\frac{B_{i}}{\rho_{1}} \hat{J}_{n}^{\prime}\left(\rho_{1}\right)+\frac{b_{m n}^{s}}{\rho_{1}} \hat{H}_{n}^{\prime(2)}\left(\rho_{1}\right) & =\frac{b_{m n}^{t}}{\rho_{2}} \hat{J}_{n}^{\prime}\left(\rho_{2}\right)  \tag{7.77}\\
\frac{B_{i}}{\eta_{1}} j_{n}\left(\rho_{1}\right)+\frac{b_{m n}^{s}}{\eta_{1}} h_{n}^{(2)}\left(\rho_{1}\right) & =\frac{b_{m n}^{t}}{\eta_{2}} j_{n}\left(\rho_{2}\right) \tag{7.78}
\end{align*}
$$

with $\frac{\rho_{1} \eta_{1}}{\rho_{2} \eta_{2}}=\frac{\mu_{1}}{\mu_{2}}$ and $j_{n}(z) h_{n}^{(2)}(z)-j^{\prime}{ }_{n}(z) h_{n}^{(2)}(z)=-\frac{j}{z^{2}}$ the four unknowns will be:

$$
\begin{gather*}
a_{m n}^{s}=\frac{\frac{\mu_{2}}{\mu_{1}} \hat{J}_{n}^{\prime}\left(\rho_{1}\right) j_{n}\left(\rho_{2}\right)-j_{n}\left(\rho_{1}\right) \hat{J}_{n}^{\prime}{ }_{n}\left(\rho_{2}\right)}{\hat{J}_{n}^{\prime}\left(\rho_{2}\right) h_{n}^{(2)}\left(\rho_{1}\right)-\frac{\mu_{1}}{\mu_{2}} j_{n}\left(\rho_{2}\right) \hat{H}_{n}^{\prime(2)}\left(\rho_{1}\right)} A_{i}  \tag{7.79}\\
b_{m n}^{s}=\frac{\left(\frac{\mu_{1}}{\mu_{2}}\right)\left(\frac{k_{2}}{k_{1}}\right)^{2} \hat{J}_{n}^{\prime}\left(\rho_{1}\right) j_{n}\left(\rho_{2}\right)-j_{n}\left(\rho_{1}\right) \hat{J}_{n}^{\prime}\left(\rho_{2}\right)}{\hat{J}_{n}^{\prime}\left(\rho_{2}\right) h_{n}^{(2)}\left(\rho_{1}\right)-\left(\frac{\mu_{1}}{\mu_{2}}\right)\left(\frac{k_{2}}{k_{1}}\right)^{2} j_{n}\left(\rho_{2}\right) \hat{H}_{n}^{(2)}\left(\rho_{1}\right)} B_{i}  \tag{7.80}\\
a_{m n}^{t}=\frac{-j / \rho_{1}}{j_{n}\left(\rho_{2}\right) \hat{H}_{n}^{\prime(2)}\left(\rho_{1}\right)-\frac{\mu_{1}}{\mu_{2}} \hat{J}_{n}^{\prime}\left(\rho_{2}\right) h_{n}^{(2)}\left(\rho_{1}\right)} a_{m n}^{i}  \tag{7.81}\\
b_{m n}^{t}=\frac{j / \rho_{1}}{\frac{k_{1}}{k_{2}} \hat{J}_{n}^{\prime}\left(\rho_{2}\right) h_{n}^{(2)}\left(\rho_{1}\right)-\frac{k_{2} \mu_{1}}{k_{1} \mu_{2}} j_{n}\left(\rho_{2}\right) \hat{H}_{n}^{(2)}\left(\rho_{1}\right)} B_{i} \tag{7.82}
\end{gather*}
$$

### 7.6 Dipole Antenna Inside a Dielectric Sphere

The dielectric resonators are traditionally used in filter and oscillator applications, however, recently it is found that dielectric resonators (DR) of different shapes can be designed to be efficient radiators at microwave frequencies. The shape of DR can be cylindrical, rectangular, or hemispherical. Hemispherical DR is chosen because of its simple interface with free space which results in the simplicity and much accuracies. We can calculate the fields of antenna inside and outside of a dielectric sphere Fig.(7.5) as:

$$
\begin{align*}
\mathbf{E}^{i} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(A_{m n}^{(1)} \mathbf{M}^{(4)}(r, \theta, \phi)+B_{m n}^{(1)} \mathbf{N}^{(4)}(r, \theta, \phi)\right)  \tag{7.83}\\
\mathbf{H}^{i} & =\frac{j}{\eta_{2}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(A_{m n}^{(1)} \mathbf{N}^{(4)}(r, \theta, \phi)+B_{m n}^{(1)} \mathbf{M}^{(4)}(r, \theta, \phi)\right) \tag{7.84}
\end{align*}
$$



Figure 7.2: Perturbation input impedance of a $\lambda / 2$ dipole antenna near a conducting sphere
and the fields radiated outside and reflected inside will be

$$
\begin{align*}
\mathbf{E}^{s 2} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(c_{m n} \mathbf{M}^{(1)}(r, \theta, \phi)+d_{m n} \mathbf{N}^{(1)}(r, \theta, \phi)\right)  \tag{7.85}\\
\mathbf{H}^{s 2} & =\frac{j}{\eta_{2}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(c_{m n} \mathbf{N}^{(1)}(r, \theta, \phi)+d_{m n} \mathbf{M}^{(1)}(r, \theta, \phi)\right)  \tag{7.86}\\
\mathbf{E}^{s 1} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n} \mathbf{M}^{(4)}(r, \theta, \phi)+b_{m n} \mathbf{N}^{(4)}(r, \theta, \phi)\right)  \tag{7.87}\\
\mathbf{H}^{s 1} & =\frac{j}{\eta_{1}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n} \mathbf{N}^{(4)}(r, \theta, \phi)+b_{m n} \mathbf{M}^{(4)}(r, \theta, \phi)\right) \tag{7.88}
\end{align*}
$$

By applying the boundary conditions we have

$$
\begin{align*}
& A_{m n}^{(1)} h_{n}^{(2)}\left(\rho_{2}\right)+c_{m n} j_{n}\left(\rho_{2}\right)=a_{m n} h_{n}^{(2)}\left(\rho_{1}\right)  \tag{7.89}\\
& \frac{A_{m n}^{(1)} \hat{H}_{n}^{\prime(2)}}{\rho_{2}}\left(\rho_{2}\right)  \tag{7.90}\\
& \rho_{2} \eta_{2} \frac{c_{m n} \hat{J}_{n}^{\prime}\left(\rho_{2}\right)}{\rho_{2} \eta_{2}}
\end{align*}=\frac{a_{m n} \hat{H}_{n}^{\prime(2)}\left(\rho_{1}\right)}{\rho_{1} \eta_{1}}, ~ l
$$



Figure 7.3: Perturbation input impedance of a $\lambda / 2$ dipole antenna near a dielectric sphere

$$
\begin{align*}
\frac{B_{m n}^{(1)} \hat{H}^{\prime}{ }_{n}^{(2)}\left(\rho_{2}\right)}{\rho_{2}}+\frac{d_{m n} \hat{J}_{n}^{\prime}\left(\rho_{2}\right)}{\rho_{2}} & =\frac{b_{m n} \hat{H}_{n}^{\prime(2)}\left(\rho_{1}\right)}{\rho_{1}}  \tag{7.91}\\
\frac{B_{m n}^{(1)} h_{n}^{(2)}\left(\rho_{2}\right)}{\eta_{2}}+\frac{d_{m n} j_{n}\left(\rho_{2}\right)}{\eta_{2}} & =\frac{b_{m n} h_{n}^{(2)}\left(\rho_{1}\right)}{\eta_{1}} \tag{7.92}
\end{align*}
$$

and therefore,

$$
\begin{gather*}
a_{m n}=\frac{\hat{J}_{n}^{\prime}\left(\rho_{2}\right) h_{n}^{(2)}\left(\rho_{2}\right)-j_{n}\left(\rho_{2}\right) \hat{H}_{n}^{(2)}\left(\rho_{2}\right)}{\hat{J}_{n}^{\prime}\left(\rho_{2}\right) h_{n}^{(2)}\left(\rho_{1}\right)-\frac{\mu_{2}}{\mu_{1}} j_{n}\left(\rho_{2}\right) \hat{H}_{n}^{(2)}\left(\rho_{1}\right)} A_{m n}^{(1)}  \tag{7.93}\\
b_{m n}=\frac{\hat{J}_{n}^{\prime}\left(\rho_{2}\right) h_{n}^{(2)}\left(\rho_{2}\right)-j_{n}\left(\rho_{2}\right) \hat{H}_{n}^{(2)}\left(\rho_{2}\right)}{\frac{k_{1} \mu_{2}}{k_{2} \mu_{1}} \hat{J}_{n}^{\prime}\left(\rho_{2}\right) h_{n}^{(2)}\left(\rho_{1}\right)-\frac{k_{2}}{k_{1}} j_{n}\left(\rho_{2}\right) \hat{H}_{n}^{(2)}\left(\rho_{1}\right)} B_{m n}^{(1)} \tag{7.94}
\end{gather*}
$$

Equation Eq.(7.93) and Eq.(7.94) can be simplified, because

$$
\begin{equation*}
h_{n}^{\prime(2)}(z) j_{n}(z)-j_{n}^{\prime}(z) h_{n}^{(2)}(z)=-\frac{j}{z^{2}} \tag{7.95}
\end{equation*}
$$



Figure 7.4: Pattern of a $\lambda / 2$ dipole antenna near a conducting sphere

Therefore:

$$
\begin{align*}
a_{m n} & =\frac{\frac{j}{\rho_{2}}}{\hat{J}_{n}^{\prime}\left(\rho_{2}\right) h_{n}^{(2)}\left(\rho_{1}\right)-\frac{\mu_{2}}{\mu_{1}} j_{n}\left(\rho_{2}\right) \hat{H}_{n}^{(2)}\left(\rho_{1}\right)} A_{m n}^{(1)}  \tag{7.96}\\
b_{m n} & =\frac{\frac{j}{\rho_{2}}}{\frac{k_{1} \mu_{2}}{k_{2} \mu_{1}} \hat{J}_{n}^{\prime}\left(\rho_{2}\right) h_{n}^{(2)}\left(\rho_{1}\right)-\frac{k_{2}}{k_{1}} j_{n}\left(\rho_{2}\right) \hat{H}_{n}^{(2)}\left(\rho_{1}\right)} B_{m n}^{(1)}  \tag{7.97}\\
c_{m n} & =\frac{\frac{\mu_{2}}{\mu_{1}} \hat{H}_{n}^{(2)}\left(\rho_{1}\right) h_{n}^{(2)}\left(\rho_{2}\right)-\hat{H}_{n}^{\prime 2}\left(\rho_{2}\right) h_{n}^{(2)}\left(\rho_{1}\right)}{\hat{J}_{n}^{\prime}\left(\rho_{2}\right) h_{n}^{(2)}\left(\rho_{1}\right)-\frac{\mu_{2}}{\mu_{1}} j_{n}\left(\rho_{2}\right) \hat{H}_{n}^{\prime}{ }_{n}^{(2)}\left(\rho_{1}\right)} A_{m n}^{(1)}  \tag{7.98}\\
d_{m n} & =\frac{\left(\frac{k_{2}}{k_{1}}\right)^{2} \hat{H}_{n}^{(2)}\left(\rho_{1}\right) h_{n}^{(2)}\left(\rho_{2}\right)-\left(\frac{\mu_{2}}{\mu_{1}}\right) \hat{H}_{n}^{\prime}{ }_{n}^{(2)}\left(\rho_{2}\right) h_{n}^{(2)}\left(\rho_{1}\right)}{\left(\frac{\mu_{2}}{\mu_{1}}\right) h_{n}^{(2)}\left(\rho_{1}\right) \hat{J}_{n}^{\prime}\left(\rho_{1}\right)-\left(\frac{k_{2}}{k_{1}}\right)^{2} \hat{H}_{n}^{(2)}\left(\rho_{1}\right) j_{n}\left(\rho_{2}\right)} B_{m n}^{(1)} \tag{7.99}
\end{align*}
$$

### 7.6.1 Small Dipole at Center of Dielectric Sphere

In the above section if the dipole was at center of dielectric sphere, we would be faced with singularity. Now we want to look at this problem. suppose we have a Hertzian dipole with length $\Delta l$ and current $I_{0}$ oriented in $z$ direction


Figure 7.5: Dipole Antenna Inside a Dielectric Sphere
at center of a dielectric sphere with radius $a$ Fig.(7.6). We want to find $\mathbf{E}$ or $\mathbf{H}$ fields at any point, inside or outside of dielectric sphere. Inside the dielectric sphere we have two types of fields; incident and reflected waves but outside the only transmitted wave. The electric and magnetic of Hertzian antenna will be:

$$
\begin{gather*}
\mathbf{H}=\frac{I_{0} \Delta l e^{-j \beta_{1} r}}{4 \pi r}\left(j \beta_{1}+\frac{1}{r}\right) \sin \theta \mathbf{a}_{\phi} \quad 0<r<a \\
\mathbf{E}=\frac{j \omega \mu_{1} I_{0} \Delta l e^{-j \beta_{1} r}}{4 \pi r}\left[\mathbf{a}_{r}\left(\frac{1}{\beta_{1}^{2} r^{2}}-\frac{j}{\beta_{1} r}\right) 2 \cos \theta+\mathbf{a}_{\theta}\left(\frac{1}{\beta_{1}^{2} r^{2}}-\frac{j}{\beta_{1} r}-1\right) \sin \theta\right] \tag{7.101}
\end{gather*}
$$

These can be written as

$$
\begin{align*}
\mathbf{H}^{i} & =\frac{-j \beta_{1}^{2} I_{0} \Delta l}{4 \pi} h_{1}^{(2)}\left(\beta_{1} r\right) P_{1}^{1}(\cos \theta) \mathbf{a}_{\phi} \\
\mathbf{E}^{i} & =\frac{1}{j \omega \epsilon_{1}} \nabla \times \mathbf{H}^{i} \tag{7.102}
\end{align*}
$$



Figure 7.6: Small Dipole at Center of Dielectric Sphere
We can denote $A^{i}=\frac{-j \beta_{1}^{2} I_{0} \Delta l}{4 \pi}$. The reflected wave from dielectric boundary will be:

$$
\begin{equation*}
\mathbf{H}^{r}=A^{r} j_{1}\left(\beta_{1} r\right) P_{1}^{1}(\cos \theta) \mathbf{a}_{\phi} \tag{7.103}
\end{equation*}
$$

and the corresponding electric field:

$$
\begin{equation*}
\mathbf{E}^{r}=\frac{\nabla \times \mathbf{H}^{r}}{j \omega \epsilon_{1}} \tag{7.104}
\end{equation*}
$$

and the transmitted wave:

$$
\begin{equation*}
\mathbf{H}^{t}=A^{t} h_{1}^{(2)}\left(\beta_{2} r\right) P_{1}^{1}(\cos \theta) \mathbf{a}_{\phi} \tag{7.105}
\end{equation*}
$$

and the corresponding electric field:

$$
\begin{equation*}
\mathbf{E}^{t}=\frac{\nabla \times \mathbf{H}^{t}}{j \omega \epsilon_{2}} \tag{7.106}
\end{equation*}
$$

where $\beta=\omega \sqrt{\mu \epsilon}$ and $E_{r}=\frac{1}{r \sin \theta}\left[\frac{\partial}{\partial \theta}\left(H_{\phi} \sin \theta\right)\right] \frac{1}{j \omega \epsilon}$ and $E_{\theta}=\frac{1}{j \omega \epsilon}\left[\frac{-1}{r} \frac{\partial}{\partial r}\left(r H_{\phi}\right)\right]$. In our problem $A^{r}$ and $A^{t}$ are unknowns and can be find by applying boundary conditions.

$$
\begin{equation*}
H_{\phi}^{i}+H_{\phi}^{r}=H_{\phi}^{t} \quad E_{\theta}^{i}+E_{\theta}^{r}=E_{\theta}^{t} \tag{7.107}
\end{equation*}
$$

From these equations we find:

$$
\begin{align*}
A^{i} h_{1}^{(2)}\left(\beta_{1} a\right)+A^{r} j_{1}\left(\beta_{1} a\right) & =A^{t} h_{1}^{(2)}\left(\beta_{2} a\right)  \tag{7.108}\\
\frac{A^{i}}{\epsilon_{r 1}} \hat{h}_{1}^{(2)^{\prime}}\left(\beta_{1} a\right)+\frac{A^{r}}{\epsilon_{r 1}} \hat{j}_{1}^{\prime}\left(\beta_{1} a\right) & =\frac{A^{t}}{\epsilon_{r 2}} \hat{h}_{1}^{(2)^{\prime}}\left(\beta_{2} a\right) \tag{7.109}
\end{align*}
$$

from the above equations we can find two unknowns.

### 7.6.2 Antenna Inside a Multilayer Dielectric Sphere

The antenna located inside the region 3, and we want to find the pattern of antenna far from sphere, Fig.(7.7), The radiation fields of antenna will be

$$
\begin{align*}
\mathbf{E}^{i} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(A_{m n}^{(1)} \mathbf{M}^{(4)}(r, \theta, \phi)+B_{m n}^{(1)} \mathbf{N}^{(4)}(r, \theta, \phi)\right)  \tag{7.110}\\
\mathbf{H}^{i} & =\frac{j}{\eta_{3}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(A_{m n}^{(1)} \mathbf{N}^{(4)}(r, \theta, \phi)+B_{m n}^{(1)} \mathbf{M}^{(4)}(r, \theta, \phi)\right) \tag{7.111}
\end{align*}
$$

If you want to find the fields in region 3 only, the condition that it is $r>r_{0}$ or $r<r_{0}$ should be used. The scattered fields inside region 3:

$$
\begin{align*}
\mathbf{E}^{(3)} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n}^{(3)} \mathbf{M}^{(1)}(r, \theta, \phi)+b_{m n}^{(3)} \mathbf{N}^{(1)}(r, \theta, \phi)\right)  \tag{7.112}\\
\mathbf{H}^{(3)} & =\frac{j}{\eta_{3}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n}^{(3)} \mathbf{N}^{(1)}(r, \theta, \phi)+b_{m n}^{(3)} \mathbf{M}^{(1)}(r, \theta, \phi)\right) \tag{7.113}
\end{align*}
$$

The transmitted and reflected fields in region 2 will be;

$$
\begin{align*}
\mathbf{E}^{2 t} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n}^{2 t} \mathbf{M}^{(4)}(r, \theta, \phi)+b_{m n}^{2 t} \mathbf{N}^{(4)}(r, \theta, \phi)\right)  \tag{7.114}\\
\mathbf{H}^{2 t} & =\frac{j}{\eta_{2}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n}^{2 t} \mathbf{N}^{(4)}(r, \theta, \phi)+b_{m n}^{2 t} \mathbf{M}^{(4)}(r, \theta, \phi)\right)  \tag{7.115}\\
\mathbf{E}^{2 r} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n}^{2 r} \mathbf{M}^{(1)}(r, \theta, \phi)+b_{m n}^{2 r} \mathbf{N}^{(1)}(r, \theta, \phi)\right)  \tag{7.116}\\
\mathbf{H}^{2 r} & =\frac{j}{\eta_{2}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n}^{2 r} \mathbf{N}^{(1)}(r, \theta, \phi)+b_{m n}^{2 r} \mathbf{M}^{(1)}(r, \theta, \phi)\right) \tag{7.117}
\end{align*}
$$



Figure 7.7: Antenna inside a multilayer dielectric sphere

And finally the fields in region 1 are:

$$
\begin{align*}
\mathbf{E}^{(1)} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n}^{(1)} \mathbf{M}^{(4)}(r, \theta, \phi)+b_{m n}^{(1)} \mathbf{N}^{(4)}(r, \theta, \phi)\right)  \tag{7.118}\\
\mathbf{H}^{(1)} & =\frac{j}{\eta_{1}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n}^{(1)} \mathbf{N}^{(4)}(r, \theta, \phi)+b_{m n}^{(1)} \mathbf{M}^{(4)}(r, \theta, \phi)\right) \tag{7.119}
\end{align*}
$$

Now we use boundary conditions at $r=R_{3}$ and $r=R_{2}$ for TE and TM modes. Again we assume that $\rho_{3}=k_{3} R_{3}, \rho_{23}=k_{2} R_{3}, \rho_{2}=k_{2} R_{2}, \rho_{12}=k_{1} R_{2}$

$$
\begin{equation*}
A_{m n}^{(1)} h_{n}^{(2)}\left(\rho_{3}\right)+a_{m n}^{(3)} j_{n}\left(\rho_{3}\right)=a_{m n}^{2 t} h_{n}^{(2)}\left(\rho_{23}\right)+a_{m n}^{2 r} j_{n}\left(\rho_{23}\right) \tag{7.120}
\end{equation*}
$$

$$
\begin{gather*}
\frac{A_{m n}^{(1)}}{\rho_{3} \eta_{3}} \hat{h}_{n}^{(2)}\left(\rho_{3}\right)+\frac{a_{m n}^{(3)}}{\rho_{3} \eta_{3}} \hat{j}_{n}^{\prime}\left(\rho_{3}\right)=\frac{a_{m n}^{2 t}}{\rho_{23} \eta_{2}} \hat{h}_{n}^{(2)}\left(\rho_{23}\right)+\frac{a_{m n}^{2 r}}{\rho_{23} \eta_{2}} \hat{j}^{\prime}{ }_{n}\left(\rho_{23}\right)  \tag{7.121}\\
a_{m n}^{2 t} h_{n}^{(2)}\left(\rho_{2}\right)+a_{m n}^{2 r} j_{n}\left(\rho_{2}\right)=a_{m n}^{1} h_{n}^{(2)}\left(\rho_{12}\right) \tag{7.122}
\end{gather*}
$$

$$
\begin{equation*}
\frac{a_{m n}^{2 t}}{\rho_{2} \eta_{2}} \hat{h}_{n}^{\prime(2)}\left(\rho_{2}\right)+\frac{a_{m n}^{2 r}}{\rho_{2} \eta_{2}} \hat{j}_{n}^{\prime}\left(\rho_{2}\right)=\frac{a_{m n}^{(1)}}{\rho_{12} \eta_{1}} \hat{h}_{n}^{(2)}\left(\rho_{12}\right) \tag{7.123}
\end{equation*}
$$

From the four above equations, we can find four unknowns $a_{m n}^{(3)}, a_{m n}^{2 t}, a_{m n}^{2 r}$ and $a_{m n}^{(1)} . A_{m n}^{(1)}$ is known from antenna position and configuration. And for TM case

$$
\begin{gather*}
\frac{B_{m n}^{1}}{\eta_{3}} h_{n}^{(2)}\left(\rho_{3}\right)+\frac{b_{m n}^{3}}{\eta_{3}} j_{n}\left(\rho_{3}\right)=\frac{b_{m n}^{2 t}}{\eta_{2}} h_{n}^{(2)}\left(\rho_{23}\right)+\frac{b_{m n}^{2 r}}{\eta_{2}} j_{n}\left(\rho_{23}\right)  \tag{7.124}\\
\frac{B_{m n}^{1}}{\rho_{3}} \hat{h}_{n}^{\prime(2)}\left(\rho_{3}\right)+\frac{b_{m n}^{3}}{\rho_{3}} \hat{j}_{n}^{\prime}\left(\rho_{3}\right)=\frac{b_{m n}^{2 t}}{\rho_{23}} \hat{h}_{n}^{(2)}\left(\rho_{23}\right)+\frac{b_{m n}^{2 r}}{\rho_{23}} \hat{j}_{n}^{\prime}\left(\rho_{23}\right)  \tag{7.125}\\
\frac{b_{m n}^{2 t}}{\eta_{2}} h_{n}^{(2)}\left(\rho_{2}\right)+\frac{b_{m n}^{2 r}}{\eta_{2}} j_{n}\left(\rho_{2}\right)=\frac{b_{m n}^{1}}{\eta_{1}} h_{n}^{(2)}\left(\rho_{12}\right)  \tag{7.126}\\
\frac{b_{m n}^{2 t}}{\rho_{2}} \hat{h}_{n}^{(2)}\left(\rho_{2}\right)+\frac{b_{m n}^{2 r}}{\rho_{2}} \hat{j}_{n}^{\prime}\left(\rho_{2}\right)=\frac{b_{m n}^{1}}{\rho_{12}} \hat{h}_{n}^{(2)}\left(\rho_{12}\right) \tag{7.127}
\end{gather*}
$$

From the four above equations, we can find four unknowns $b_{m n}^{3}, b_{m n}^{2 t}, b_{m n}^{2 r}$ and $b_{m n}^{1} . B_{m n}^{1}$ is known from antenna position and configuration.
Now let us look at multilayer case, a dipole antenna located at layer numbered $l$ and $r>r_{0}$ :

$$
\begin{align*}
\mathbf{E}^{i} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(A_{m n}^{(1)} \mathbf{M}^{(4)}+B_{m n}^{(1)} \mathbf{N}^{(4)}\right)  \tag{7.128}\\
\mathbf{H}^{i} & =\frac{j}{\eta_{l}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(A_{m n}^{(1)} \mathbf{N}^{(4)}+B_{m n}^{(1)} \mathbf{M}^{(4)}\right)
\end{align*}
$$

for $r<r_{0}$ change: $(1) \Rightarrow(4)$, and $(4) \Rightarrow(1)$. The transmitted and scattered wave in layer $l$ and $l \neq N$ or $l \neq 1$ will be:

$$
\begin{align*}
\mathbf{E}^{l t} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n}^{l t} \mathbf{M}^{(1)}+b_{m n}^{l t} \mathbf{N}^{(1)}\right)  \tag{7.129}\\
\mathbf{H}^{l t} & =\frac{j}{\eta_{l}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n}^{l t} \mathbf{N}^{(1)}+b_{m n}^{l t} \mathbf{M}^{(1)}\right)
\end{align*}
$$

$$
\begin{align*}
\mathbf{E}^{l s} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n}^{l s} \mathbf{M}^{(4)}+b_{m n}^{l s} \mathbf{N}^{(4)}\right)  \tag{7.130}\\
\mathbf{H}^{l s} & =\frac{j}{\eta_{l}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n}^{l s} \mathbf{N}^{(4)}+b_{m n}^{l s} \mathbf{M}^{(4)}\right)
\end{align*}
$$

In layer $\# 1$ :

$$
\begin{align*}
\mathbf{E}^{(1)} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n}^{(1)} \mathbf{M}^{(4)}+b_{m n}^{(1)} \mathbf{N}^{(4)}\right)  \tag{7.131}\\
\mathbf{H}^{(1)} & =\frac{j}{\eta_{1}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n}^{(4)} \mathbf{N}^{(4)}+b_{m n}^{(1)} \mathbf{M}^{(4)}\right)
\end{align*}
$$

In layer $\# N$ :

$$
\begin{align*}
\mathbf{E}^{(N)} & =\sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n}^{(1)} \mathbf{M}^{(1)}+b_{m n}^{(1)} \mathbf{N}^{(1)}\right)  \tag{7.132}\\
\mathbf{H}^{(N)} & =\frac{j}{\eta_{N}} \sum_{n=1}^{\infty} \sum_{m=-n}^{m=n}\left(a_{m n}^{(1)} \mathbf{N}^{(1)}+b_{m n}^{(1)} \mathbf{M}^{(1)}\right)
\end{align*}
$$

and boundary conditions for TE and TM modes:

- $\mathbf{M} \Rightarrow T E$
- $\mathbf{N} \Rightarrow T M$
- $l=2,3, \cdots, N$ and $r=r_{l}$ For $\mathbf{E} \& \mathbf{H}$ all coefficients can be found $\left\{a_{m n}^{(l)} \& b_{m n}^{l}\right\},\left\{a_{m n}^{(l)} \& b_{m n}^{(l)}\right\},\left\{a_{m n}^{N} \& b_{m n}^{N}\right\}$
- Matrix Form of Coefficients:
- Sub-Matrix: TE case

$$
\begin{equation*}
\mathbf{T}_{l}^{T E} \times \mathbf{A}=\mathbf{S}_{A} \tag{7.133}
\end{equation*}
$$

- $\mathbf{T}_{l}^{T E}$ Stands for:

$$
\left[\begin{array}{cccc}
-h_{n}^{(2)}\left(k_{l-1} r_{l}\right) & -j_{n}\left(k_{l-1} r_{l}\right) & h_{n}^{(2)}\left(k_{l} r_{l}\right) & j_{n}\left(k_{l} r_{l}\right)  \tag{7.134}\\
\frac{-\hat{h}_{n}^{\prime 2}\left(k_{l-1} r_{l}\right)}{k_{l-1} \eta_{l-1}} & \frac{-\hat{j}_{n}^{\prime}\left(k_{l-1} r_{l}\right)}{k_{l-1} \eta_{l-1}} & \frac{\hat{h}_{n}^{\prime 2}\left(k_{l} r_{l}\right)}{k_{l} \eta_{l}} & \frac{\hat{j}^{\prime}\left(k_{l} r_{l}\right)}{k_{l} \eta_{l}}
\end{array}\right]
$$

- A Stands for: $\left[a_{m n}^{(l-1) s}, a_{m n}^{(l-1) t}, a_{m n}^{(l) s}, a_{m n}^{(l) t}\right]^{T}$
- $\mathbf{S}_{\mathbf{A}}$ Stands for: $[0,0]^{T}$, if there is no source
- Sub-Matrix: TM case

$$
\begin{equation*}
\mathbf{T}_{l}^{T M} \times \mathbf{B}=\mathbf{S}_{B} \tag{7.135}
\end{equation*}
$$

- $\mathbf{T}_{l}^{T M}$ Stands for:

$$
\left[\begin{array}{cccc}
\frac{-h_{n}^{(2)}\left(k_{l-1} r_{l}\right)}{\eta_{l-1}} & \frac{-j_{n}\left(k_{l-1} r_{l}\right)}{\eta_{l-1}} & \frac{h_{n}^{(2)}\left(k_{l} r_{l}\right)}{\eta_{l}} & \frac{j_{n}\left(k_{l} r_{l}\right)}{k_{l} \eta_{l}}  \tag{7.136}\\
\frac{-\hat{h}_{n}^{(2)}\left(k_{l-1} r_{l}\right)}{k_{l-1}} & \frac{-\hat{j}_{n}^{\prime}\left(k_{l-1} r_{l}\right)}{k_{l-1}} & \frac{\hat{h}_{n}^{(2)}\left(k_{l} r_{l}\right)}{k_{l}} & \frac{-\hat{j}_{n}^{\prime}\left(k_{l} r_{l}\right)}{k_{l}}
\end{array}\right]
$$

- B Stands for: $\left[b_{m n}^{(l-1) s}, b_{m n}^{(l-1) t}, b_{m n}^{(l) s}, b_{m n}^{(l) t}\right]^{T}$
- $\mathbf{S}_{\mathbf{B}}$ Stands for: $[0,0]^{T}$, if there is no source
- If there is antenna in layer $\# l$ : TE case
for $r=r_{l}$

$$
\left[\begin{array}{l}
0  \tag{7.137}\\
0
\end{array}\right] \Rightarrow\left[\begin{array}{c}
-A_{m n}^{i(1)} h_{n}^{(2)}\left(k_{l} r_{l}\right) \\
-A_{m n}^{i(1)} \frac{\hat{h}_{n}^{\prime(2)}\left(k_{l} r_{l}\right)}{\eta_{l} k_{l}}
\end{array}\right]=\mathbf{S}_{A}
$$

for $r=r_{l+1}$

$$
\left[\begin{array}{l}
0  \tag{7.138}\\
0
\end{array}\right] \Rightarrow\left[\begin{array}{c}
A_{m n}^{i(4)} j_{n}\left(k_{l} r_{l+1}\right) \\
A_{m n}^{i(4)} \frac{\hat{j}_{n}^{\prime}\left(k_{l} l_{l+1}\right)}{\eta_{l} k_{l}}
\end{array}\right]=\mathbf{S}_{A}
$$

- Four terms for antenna coefficient in layer $\# l$

From $2 \times(l-2)+1$ to $2 \times(l-2)+4$ in TE mode

- If there is antenna in layer $\# l$ : TM case
for $r=r_{l}$

$$
\left[\begin{array}{l}
0  \tag{7.139}\\
0
\end{array}\right] \Rightarrow\left[\begin{array}{l}
-B_{m n}^{i(1)} \frac{h_{n}^{(2)}\left(k_{l} r_{l}\right)}{} \\
-B_{m n}^{i(1)} \frac{\hat{h}_{n}^{\prime(2)}\left(k_{l} r_{l}\right)}{k_{l}}
\end{array}\right]=\mathbf{S}_{B}
$$

for $r=r_{l+1}$

$$
\left[\begin{array}{l}
0  \tag{7.140}\\
0
\end{array}\right] \Rightarrow\left[\begin{array}{c}
B_{m n}^{i(4)} \frac{j_{n}\left(k_{l} r_{l+1}\right)}{\left(l_{l}\right.} \\
B_{m n}^{i(4)} \frac{\hat{j}_{n}^{\prime}{ }_{n}\left(k_{l} l_{l+1}\right)}{k_{l}}
\end{array}\right]=\mathbf{S}_{B}
$$

- Four terms for antenna coefficient in layer $\# l$

From $2 \times(l-2)+1$ to $2 \times(l-2)+4$ in $\mathbf{T M}$ mode

- Total Matrix of Coefficient
$\mathbf{T}^{T E}(2 N-2) \times(2 N-2)$

$$
\left[\begin{array}{cccc}
\mathbf{T}_{2}^{T E} & 0 & \cdots & 0  \tag{7.141}\\
0 & \mathbf{T}_{3}^{T E} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \mathbf{T}_{l}^{T E} & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \mathbf{T}_{N}^{T E}
\end{array}\right]\left[\begin{array}{c}
\mathbf{A}_{2} \\
\\
\vdots \\
\\
\mathbf{A}_{N}
\end{array}\right]=\left[\begin{array}{c}
0 \\
\vdots \\
\vdots \\
\mathbf{S}_{A} \\
\vdots \\
0
\end{array}\right]
$$

- From solution of above matrix we find $\left\{a_{m n}^{1 s} \& a_{m n}^{1 t}\right\}, \cdots\left\{a_{m n}^{N s} \& a_{m n}^{N t}\right\}$
- Total Matrix of Coefficient
$\mathbf{T}^{T M}(2 N-2) \times(2 N-2)$

$$
\left[\begin{array}{cccc}
\mathbf{T}_{2}^{T M} & 0 & \cdots & 0  \tag{7.142}\\
0 & \mathbf{T}_{3}^{T M} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \mathbf{T}_{l}^{T M} & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \mathbf{T}_{N}^{T M}
\end{array}\right]\left[\begin{array}{c}
\mathbf{B}_{2} \\
\vdots \\
\vdots \\
\mathbf{B}_{N}
\end{array}\right]=\left[\begin{array}{c}
0 \\
\mathbf{B}_{l} \\
\vdots \\
\mathbf{S}_{B} \\
\vdots \\
0
\end{array}\right]
$$

- From solution of above matrix we find $\left\{b_{m n}^{1 s} \& b_{m n}^{1 t}\right\}, \cdots\left\{b_{m n}^{N s} \& b_{m n}^{N t}\right\}$


### 7.6.3 Scattering by a Dielectric Sphere, Mie Theory

German physisist born in 1868 in Rostock and died in 1957 in Freiburg im Breisgau in Germany. He studied natural science and mathematics in Rostock and Heidelberg. He obtained his doctorate in 1891 in Heidelberg. From 1892 to 1902 he was an assistent at the Physics Institute at the Technical University of Karlsruhe. In 1897 he obtained his Habilitation for theoretical physics. In 1902 he became a special professor (Extraordinarius) at the University of Greifswald where he wrote his famous paper on particle light scattering. In 1917 he became Professor at the University of Halle and in 1924 he joint the University of Freiburg. Mie theory provides rigorous solutions for EM wave scattering by an isotropic sphere embedded in a homogeneous medium. Extensions of Mie theory include solutions for core or shell spheres and gradient-index spheres. Although these theories are restricted to the case of a perfect sphere, the results have provided insight into the scattering and absorption properties for a wide variety of pigment systems, including non-spherical pigments.

Let a plane wave be incident on a dielectric sphere of radius $a$, as shown in Fig.(7.9).


Figure 7.8: Gustav Mie ( 1868 - 1957 )

$$
\begin{align*}
\mathbf{E}^{i} & =E_{0} e^{-j k_{1} z} \mathbf{a}_{x}=\sum_{n=1}^{\infty} a_{n}^{i}\left[\mathbf{M}_{o 1 n}^{(1)}\left(k_{1} r\right)+j \mathbf{N}_{e 1 n}^{(1)}\left(k_{1} r\right)\right]  \tag{7.143}\\
\mathbf{H}^{i} & =\frac{E_{0}}{\eta_{1}} e^{-j k_{1} z} \mathbf{a}_{y}=\frac{j}{\eta_{1}} \sum_{n=1}^{\infty} a_{n}^{i}\left[\mathbf{N}_{o 1 n}^{(1)}\left(k_{1} r\right)+j \mathbf{M}_{e 1 n}^{(1)}\left(k_{1} r\right)\right]
\end{align*}
$$

where $a_{n}^{i}=(-j)^{n} \frac{2 n+1}{n(n+1)} E_{0}$. The scattered and transmitted waves will be:

$$
\begin{align*}
\mathbf{E}^{s} & =\sum_{n=1}^{\infty}\left[a_{n}^{s} \mathbf{M}_{o 1 n}^{(4)}\left(k_{1} r\right)+j b_{n}^{s} \mathbf{N}_{e 1 n}^{(4)}\left(k_{1} r\right)\right]  \tag{7.144}\\
\mathbf{H}^{s} & =\frac{j}{\eta_{1}} \sum_{n=1}^{\infty}\left[a_{n}^{s} \mathbf{N}_{o 1 n}^{(4)}\left(k_{1} r\right)+j b_{n}^{s} \mathbf{M}_{e 1 n}^{(4)}\left(k_{1} r\right)\right] \\
\mathbf{E}^{t} & =\sum_{n=1}^{\infty}\left[a_{n}^{t} \mathbf{M}_{o 1 n}^{(1)}\left(k_{2} r\right)+j b_{n}^{t} \mathbf{N}_{e 1 n}^{(1)}\left(k_{2} r\right)\right]  \tag{7.145}\\
\mathbf{H}^{t} & =\frac{j}{\eta_{2}} \sum_{n=1}^{\infty}\left[a_{n}^{t} \mathbf{N}_{o 1 n}^{(1)}\left(k_{2} r\right)+j b_{n}^{t} \mathbf{M}_{e 1 n}^{(1)}\left(k_{2} r\right)\right]
\end{align*}
$$



Figure 7.9: Scattering of Plane Wave by a Dielectric Sphere

At the boundary $r=a$, we have $E_{\theta}^{i}+E_{\theta}^{s}=E_{\theta}^{t}$ and also $H_{\theta}^{i}+H_{\theta}^{s}=H_{\theta}^{t}$. The same result will be found for $E_{\phi}$ component.

$$
\begin{align*}
a_{n}^{i} j_{n}\left(\rho_{1}\right)+a_{n}^{s} h_{n}^{(2)}\left(\rho_{1}\right) & =a_{n}^{t} j_{n}\left(\rho_{2}\right)  \tag{7.146}\\
\frac{a_{n}^{i}}{\rho_{1} \eta_{1}} \hat{j}_{n}^{\prime}\left(\rho_{1}\right)+\frac{a_{n}^{s}}{\rho_{1} \eta_{1}} \hat{h}_{n}^{(2)}\left(\rho_{1}\right) & =\frac{a_{n}^{t}}{\rho_{2} \eta_{2}} \hat{j}_{n}^{\prime}\left(\rho_{2}\right)  \tag{7.147}\\
\frac{a_{n}^{i}}{\rho_{1}} \hat{j}_{n}^{\prime}\left(\rho_{1}\right)+\frac{b_{n}^{s}}{\rho_{1}} \hat{h}_{n}^{(2)}\left(\rho_{1}\right) & =\frac{b_{n}^{t}}{\rho_{2}} \hat{j}_{n}^{\prime}\left(\rho_{2}\right)  \tag{7.148}\\
\frac{a_{n}^{i}}{\eta_{1}} j_{n}\left(\rho_{1}\right)+\frac{b_{n}^{s}}{\eta_{1}} h_{n}^{(2)}\left(\rho_{1}\right) & =\frac{b_{n}^{t}}{\eta_{2}} j_{n}\left(\rho_{2}\right) \tag{7.149}
\end{align*}
$$

where $\rho_{1}=k_{1} a, \rho_{2}=k_{2} a$ and $k=\omega \sqrt{\mu \epsilon}$.

$$
\begin{align*}
& {\left[\begin{array}{cc}
-h_{n}^{(2)}\left(\rho_{1}\right) & j_{n}\left(\rho_{2}\right) \\
\frac{-1}{\mu_{1}} \hat{h}_{n}^{(2)} & \left(\rho_{1}\right) \\
\frac{1}{\mu_{2}} \hat{j}^{\prime} \\
{ }_{n} & \left(\rho_{2}\right)
\end{array}\right]\left[\begin{array}{c}
a_{n}^{s} \\
a_{n}^{t}
\end{array}\right]=\left[\begin{array}{c}
a_{n}^{i} j_{n}\left(\rho_{1}\right) \\
\frac{a_{n}^{i}}{\mu_{1}} \hat{j}^{\prime} \\
{ }_{n} \\
\left(\rho_{1}\right)
\end{array}\right]}  \tag{7.150}\\
& {\left[\begin{array}{cc}
-\frac{1}{k_{1}} \hat{h}_{n}^{\prime}(2) & \left(\rho_{1}\right) \\
\frac{1}{k_{2}} \hat{j}^{\prime}{ }_{n}\left(\rho_{2}\right) \\
\frac{-1}{\eta_{1}} h_{n}^{(2)}\left(\rho_{1}\right) & \frac{1}{\eta_{2}} j_{n}\left(\rho_{2}\right)
\end{array}\right]\left[\begin{array}{c}
b_{n}^{s} \\
b_{n}^{t}
\end{array}\right]=\left[\begin{array}{c}
\frac{a_{n}^{i}}{k_{1}} \hat{j}_{n}^{\prime}\left(\rho_{1}\right) \\
\frac{a_{n}^{i}}{\eta_{1}} j_{n}\left(\rho_{1}\right)
\end{array}\right]} \tag{7.151}
\end{align*}
$$

If $\sigma_{2}=\infty$ or in other word, sphere is a perfect conductor, we will have:

$$
\begin{align*}
a_{n}^{s} & =-\frac{j_{n}\left(k_{2} a\right)}{h_{n}^{(2)}\left(k_{2} a\right)} a_{n}^{i}  \tag{7.152}\\
b_{n}^{s} & =-\frac{\hat{j}_{n}^{\prime}\left(k_{2} a\right)}{\hat{h}_{n}^{(2)}\left(k_{2} a\right)} a_{n}^{i}
\end{align*}
$$

### 7.7 Scattering by a PEMC Sphere

The expansion of incident plane wave will be

$$
\begin{align*}
\mathbf{E}^{i} & =E_{0} e^{-j k z} \mathbf{a}_{x}=E_{0} \sum_{n=1}^{\infty} \alpha_{n}\left[\mathbf{M}_{o 1 n}^{(1)}(k r)+j \mathbf{N}_{e 1 n}^{(1)}(k r)\right]  \tag{7.153}\\
\mathbf{H}^{i} & =\frac{E_{0}}{\eta_{2}} e^{-j k z} \mathbf{a}_{y}=\frac{j E_{0}}{\eta} \sum_{n=1}^{\infty} \alpha_{n}\left[\mathbf{N}_{o 1 n}^{(1)}(k r)+j \mathbf{M}_{e 1 n}^{(1)}(k r)\right]
\end{align*}
$$

where $\alpha_{n}=(-j)^{n} \frac{2 n+1}{n(n+1)}$. The scattered and transmitted waves will be

$$
\begin{align*}
\mathbf{E}^{s} & =E_{0} \sum_{n=1}^{\infty} \alpha_{n}\left[a_{n}^{s} \mathbf{M}_{o 1 n}^{(4)}(k r)+c_{n}^{s} \mathbf{M}_{e 1 n}^{(4)}(k r)+j b_{n}^{s} \mathbf{N}_{e 1 n}^{(4)}(k r)+j d_{n}^{s} \mathbf{N}_{o 1 n}^{(4)}(k r)\right] \\
\mathbf{H}^{s} & =\frac{j E_{0}}{\eta} \sum_{n=1}^{\infty} \alpha_{n}\left[a_{n}^{s} \mathbf{N}_{o 1 n}^{(4)}(k r)+c_{n}^{s} \mathbf{N}_{e 1 n}^{(4)}(k r)+j b_{n}^{s} \mathbf{M}_{e 1 n}^{(4)}(k r)+j d_{n}^{s} \mathbf{M}_{o 1 n}^{(4)}(k r)\right] \tag{7.154}
\end{align*}
$$

Unlike the case of standard Mie theory, in which only the coefficient $a_{n}$ and $b_{n}$ are needed in the scattered field expansion, here, due to the mixing of $\mathbf{E}$ and $\mathbf{H}$ in the boundary conditions, the coefficient $c_{n}$ and $d_{n}$ have to be added. These new coefficients represent the cross-polarized components of the scattered field. By applying boundary condition at the surface of PEMC sphere with radius $r=a$, we will have

$$
\begin{align*}
b_{n} h_{n}^{(2)}(k a)-c_{n} M \eta h_{n}^{(2)}(k a) & =-j_{n}(k a)  \tag{7.155}\\
b_{n} M \eta \hat{h}_{n}^{(2)}(k a)+c_{n}{\hat{h_{n}^{\prime}}}_{n}^{(2)}(k a) & =-M \eta \hat{j}_{n}^{\prime}(k a)  \tag{7.156}\\
a_{n} M \eta h_{n}^{(2)}(k a)-d_{n} h_{n}^{(2)}(k a) & =-M \eta j_{n}(k a)  \tag{7.157}\\
a_{n}{\hat{h_{n}^{\prime}}}_{n}^{(2)}(k a)+d_{n} M \eta{\hat{h^{\prime}}}_{n}^{(2)}(k a) & =-\hat{j}_{n}^{\prime}(k a) \tag{7.158}
\end{align*}
$$

solving these equations we obtain

$$
\begin{gather*}
a_{n}=-\frac{h_{n}^{(2)}(k a) \hat{j}_{n}^{\prime}(k a)+M^{2} \eta j_{n}(k a){\hat{h_{n}^{\prime}}}_{n}^{(2)}(k a)}{\left(1+M^{2} \eta^{2}\right) h_{n}^{(2)}(k a) \hat{h}_{n}^{(2)}(k a)}  \tag{7.159}\\
b_{n}=-\frac{\hat{h}_{n}^{\prime(2)}(k a) j_{n}(k a)+M^{2} \eta \hat{j}^{\prime}{ }_{n}(k a) h_{n}^{(2)}(k a)}{\left(1+M^{2} \eta^{2}\right) h_{n}^{(2)}(k a){\hat{h_{n}^{\prime}}}_{n}^{(2)}(k a)}  \tag{7.160}\\
c_{n}=d_{n}=\frac{-j M \eta}{k a\left(1+M^{2} \eta^{2}\right) h_{n}^{(2)}(k a) \hat{h}_{n}^{(2)}(k a)} \tag{7.161}
\end{gather*}
$$

Note that in the limiting cases of a PEC sphere $(M=\infty)$ and a PMC sphere $(M=0)$ the cross-polarization coefficients $c_{n}$ and $d_{n}$ vanish.

### 7.8 Scattering by a DNG Sphere

### 7.9 Scattering by a Chiral Sphere

Electromagnetic waves propagation in chiral and bi-isotropic media has recently been modeled by various numerical techniques in various studies. In most of these studies, the validity of the developed techniques was verified by comparing the numerical results to the results of one-dimensional and twodimensional problems that have known, exact solutions. For the techniques for solving three dimensional problems, plane-wave scattering from a chiral sphere was the benchmark. The exact analytical solution of the scattering by a chiral sphere has been introduced by Bohren [60], and a detailed analysis of the solution was given by Worasawate [61]. This formulation has been used for verification of the scattering from arbitrary shaped three-dimensional chiral objects using a Method of Moments analysis [62] and a Finite- Difference Time-Domain analysis [63].
The spherical vector wave functions, $\mathbf{M}_{\mathbf{o} m n}^{\mathbf{e}}(k r)$ and $\mathbf{N}_{\mathbf{o} m n}^{\mathbf{e}}(k r)$, required for
the representation of the fields for chiral sphere is the same as 7.43-7.44.

$$
\begin{align*}
& \mathbf{M}_{o}^{e}=\mp \frac{m \mathbf{a}_{\theta}}{\sin \theta} z_{n}^{g}(k r) P_{n}^{m}(\cos \theta) \sin _{\cos }^{\sin }(m \phi)  \tag{7.162}\\
& -\mathbf{a}_{\phi} z_{n}^{g}(k r) \frac{\partial}{\partial \theta} P_{n}^{m}(\cos \theta) \underset{\sin }{\cos }(m \phi) \\
& \mathbf{N}_{o}^{e}=\mathbf{a}_{r} \frac{n(n+1)}{k r} z_{n}^{g}(k r) P_{n}^{m}(\cos \theta){ }_{\sin }^{\cos }(m \phi)  \tag{7.163}\\
& +\mathbf{a}_{\theta} \frac{1}{k r} \frac{\partial}{\partial r}\left[r z_{n}^{g}(k r)\right] \frac{\partial}{\partial \theta} P_{n}^{m}(\cos \theta) \underset{\sin }{\cos }(m \phi) \\
& \mp \frac{m \mathbf{a}_{\phi}}{\sin \theta} \frac{1}{k r} \frac{\partial}{\partial r}\left[r z_{n}^{g}(k r)\right] P_{n}^{m}(\cos \theta) \underset{\cos }{\sin }(m \phi)
\end{align*}
$$

Suppose we have a homogeneous chiral sphere with radius $a$ with electrical parameters $\left(\mu_{2}, \epsilon_{2}, \beta\right)$ which have located in a homogeneous linear media $\left(\mu_{1}, \epsilon_{1}\right)$. The sphere is illuminated with a linearly polarized plane wave as

$$
\begin{align*}
& \mathbf{E}^{i n c}=\mathbf{a}_{x} E_{0} e^{-j k_{1} z}=\mathbf{a}_{x} E_{0} e^{-j k_{1} r \cos \theta}  \tag{7.164}\\
& \mathbf{H}^{i n c}=\mathbf{a}_{y} \frac{E_{0}}{\eta_{1}} e^{-j k_{1} z}=\mathbf{a}_{y} \frac{E_{0}}{\eta_{1}} e^{-j k_{1} r \cos \theta} \tag{7.165}
\end{align*}
$$

this wave can be represented in terms of the spherical vector wave functions in order to apply the appropriate boundary conditions.

$$
\begin{align*}
\mathbf{E}^{i} & =E_{0} e^{-j k_{1} z} \mathbf{a}_{x}=\sum_{n=1}^{\infty} a_{n}^{i}\left[\mathbf{M}_{o 1 n}^{(1)}\left(k_{1} r\right)+j \mathbf{N}_{e 1 n}^{(1)}\left(k_{1} r\right)\right]  \tag{7.166}\\
\mathbf{H}^{i} & =\frac{E_{0}}{\eta_{1}} e^{-j k_{1} z} \mathbf{a}_{y}=\frac{j}{\eta_{1}} \sum_{n=1}^{\infty} a_{n}^{i}\left[\mathbf{N}_{o 1 n}^{(1)}\left(k_{1} r\right)+j \mathbf{M}_{e 1 n}^{(1)}\left(k_{1} r\right)\right]
\end{align*}
$$

where $a_{n}^{i}=(-j)^{n} \frac{2 n+1}{n(n+1)} E_{0}$. The scattered-field vectors, $\mathbf{E}^{s}$ and $\mathbf{H}^{s}$, are given by

$$
\begin{align*}
\mathbf{E}^{s}= & \sum_{n=1}^{\infty}\left\{\left[a_{n}^{s} \mathbf{M}_{e 1 n}^{(4)}\left(k_{1} r\right)+b_{n}^{s} \mathbf{M}_{o 1 n}^{(4)}\left(k_{1} r\right)\right]\right.  \tag{7.167}\\
& \left.+j\left[c_{n}^{s} \mathbf{N}_{e 1 n}^{(4)}\left(k_{1} r\right)+d_{n}^{s} \mathbf{N}_{o 1 n}^{(4)}\left(k_{1} r\right)\right]\right\}
\end{align*}
$$

$$
\begin{align*}
\mathbf{H}^{s}= & \frac{j}{\eta_{1}} \sum_{n=1}^{\infty}\left\{\left[a_{n}^{s} \mathbf{N}_{e 1 n}^{(4)}\left(k_{1} r\right)+b_{n}^{s} \mathbf{N}_{o 1 n}^{(4)}\left(k_{1} r\right)\right]\right.  \tag{7.168}\\
& \left.+j\left[c_{n}^{s} \mathbf{M}_{e 1 n}^{(4)}\left(k_{1} r\right)+d_{n}^{s} \mathbf{M}_{o 1 n}^{(4)}\left(k_{1} r\right)\right]\right\}
\end{align*}
$$

and the fields inside the chiral sphere, $\mathbf{E}^{t}$ and $\mathbf{H}^{t}$, are given by

$$
\begin{align*}
\mathbf{E}^{t} & =\mathbf{Q}_{R}+j \eta_{2} \mathbf{Q}_{L}  \tag{7.169}\\
\mathbf{H}^{t} & =\mathbf{Q}_{L}+\frac{j}{\eta_{2}} \mathbf{Q}_{R}
\end{align*}
$$

where

$$
\begin{align*}
& \mathbf{Q}_{R}= \sum_{n=1}^{\infty}\left\{a_{n}^{t}\left[\mathbf{M}_{e 1 n}^{(1)}\left(k_{R} r\right)+\mathbf{N}_{e 1 n}^{(1)}\left(k_{R} r\right)\right]\right.  \tag{7.170}\\
&\left.+b_{n}^{t}\left[\mathbf{M}_{o 1 n}^{(1)}\left(k_{R} r\right)+\mathbf{N}_{o 1 n}^{(1)}\left(k_{R} r\right)\right]\right\} \\
& \mathbf{Q}_{L}= \sum_{n=1}^{\infty}\left\{c_{n}^{t}\left[\mathbf{M}_{e 1 n}^{(1)}\left(k_{L} r\right)-\mathbf{N}_{e 1 n}^{(1)}\left(k_{L} r\right)\right]\right.  \tag{7.171}\\
&\left.+d_{n}^{t}\left[\mathbf{M}_{o 1 n}^{(1)}\left(k_{L} r\right)-\mathbf{N}_{o 1 n}^{(1)}\left(k_{L} r\right)\right]\right\} \\
& k_{R}=\frac{\omega \sqrt{\mu_{2} \epsilon_{2}}}{1+\beta \omega \sqrt{\mu_{2} \epsilon_{2}}} \\
& k_{L}=\frac{\omega \sqrt{\mu_{2} \epsilon_{2}}}{1-\beta \omega \sqrt{\mu_{2} \epsilon_{2}}}
\end{align*}
$$

The scattered and internal electromagnetic fields of a chiral sphere of radius $r=a$ can be obtained using Equations 7.166-7.171. By applying boundary conditions at $r=a$ :

$$
\begin{gathered}
\mathbf{a}_{\phi} \cdot\left[\mathbf{E}^{i}+\mathbf{E}^{s}-\mathbf{E}^{t}\right]=0 \\
\mathbf{a}_{\phi} \cdot\left[\mathbf{H}^{i}+\mathbf{H}^{s}-\mathbf{H}^{t}\right]=0
\end{gathered}
$$

These equations are used to construct a set of simultaneous equations to solve for the eight unknown coefficients $a_{n}^{s}, b_{n}^{s}, c_{n}^{s}, d_{n}^{s}, a_{n}^{t}, b_{n}^{t}, c_{n}^{t}$ and $d_{n}^{t}$.

$$
\begin{align*}
a_{n}^{i} j_{n}\left(k_{1} a\right)+b_{n}^{s} h_{n}^{(2)}\left(k_{1} a\right) & =b_{n}^{t} j_{n}\left(k_{R} a\right)+j \eta_{2} d_{n}^{t} j_{n}\left(k_{L} a\right)  \tag{7.172}\\
+a_{n}^{s} h_{n}^{(2)}\left(k_{1} a\right) & =a_{n}^{t} j_{n}\left(k_{R} a\right)+j \eta_{2} c_{n}^{t} j_{n}\left(k_{L} a\right) \\
+\frac{j d_{n}^{s}}{k_{1}} \hat{h}_{n}^{(2)}\left(k_{1} a\right) & =\frac{b_{n}^{t}}{k_{R}} \hat{j}_{n}^{\prime}\left(k_{R} a\right)-\frac{j \eta_{2} d_{n}^{t}}{k_{L}} \hat{j}^{\prime}\left(k_{L} a\right) \\
\frac{j a_{n}^{i}}{k_{1}} \hat{j}^{\prime}{ }_{n}\left(k_{1} a\right)+\frac{j c_{n}^{s}}{k_{1}} \hat{h}_{n}^{(2)}\left(k_{1} a\right) & =\frac{a_{n}^{t}}{k_{R}} \hat{j}_{n}^{\prime}\left(k_{R} a\right)-\frac{j \eta_{2} c_{n}^{t}}{k_{L}} \hat{j}^{\prime}{ }_{n}\left(k_{L} a\right) \\
-\frac{d_{n}^{s}}{\eta_{1}} h_{n}^{(2)}\left(k_{1} a\right) & =d_{n}^{t} j_{n}\left(k_{L} a\right)+\frac{j}{\eta_{2}} b_{n}^{t} j_{n}\left(k_{R} a\right) \\
-\frac{a_{n}^{i}}{\eta_{1}} j_{n}\left(k_{1} a\right)-\frac{c_{n}^{s}}{\eta_{1}} h_{n}^{(2)}\left(k_{1} a\right) & =c_{n}^{t} j_{n}\left(k_{L} a\right)+\frac{j}{\eta_{2}} a_{n}^{t} j_{n}\left(k_{R} a\right) \\
\frac{j a_{n}^{i}}{\eta_{1} k_{1}} \hat{j}_{n}^{\prime}\left(k_{1} a\right)+\frac{j b_{n}^{s}}{\eta_{1} k_{1}} \hat{h}_{n}^{(2)}\left(k_{1} a\right) & =-\frac{d_{n}^{t}}{k_{L}} \hat{j}_{n}^{\prime}\left(k_{L} a\right)+\frac{j b_{n}^{t}}{\eta_{2} k_{R}} \hat{j}_{n}^{\prime}\left(k_{R} a\right) \\
+\frac{j a_{n}^{s}}{\eta_{1} k_{1}^{\prime}} \hat{h}_{n}^{(2)}\left(k_{1} a\right) & =-\frac{c_{n}^{t}}{k_{L}} \hat{j}_{n}^{\prime}\left(k_{L} a\right)+\frac{j a_{n}^{t}}{\eta_{2} k_{R}} \hat{j}_{n}^{\prime}\left(k_{R} a\right)
\end{align*}
$$

### 7.10 Problems

- 1 We have a partially dielectric filled spherical cavity as shown in Fig.(7.10). Find the relations (characteristic equations) that the roots of it will give us the resonant frequencies of TE and TM modes of this cavity.
- 2 The Earths surface and the Earths ionosphere behave as a spherical resonant cavity with the Earths surface approximately as the inner spherical surface,(Earths mean equatorial radius): $r=6378 \mathrm{~km}$, the height h (above the surface of the Earth) of ionosphere is: $h=100 \mathrm{~km}$. Find the resonant frequency of this cavity.
- 3 Derive formula for E and H fields anywhere in the space, if the antenna is located near a conducting sphere.
- 4 Center of a half wavelength dipole antenna is located at the point $x=13[\mathrm{~cm}], y=0$ and $z=0$, directed in $\mathbf{a}_{z}$ near a conducting sphere


Figure 7.10: partially dielectric filled spherical cavity
with radius $a=10[\mathrm{~cm}]$. The working frequency is $\mathrm{f}=1 \mathrm{GHz}$ and driven current $I_{0}=1 \mathrm{~A}$.
a) Find the E-Plane and H-Plane pattern of antenna.
b) Find the input impedance of antenna (the antenna is assumed to be ideal)
c) What will happen when the antenna is moved away from the sphere. Draw the input impedance against the distance from the sphere.
Hint: make a half wave dipole with current distribution $I(z)=I_{0} \cos (\beta z)$ as N Hertzian dipoles with current $I_{n}=I_{0} \cos \left(\beta z_{n}\right)$.

- 5 Center of half wavelength dipole antenna is located at point $x=$ $13[c m], y=0$ and $z=0$, directed along $\mathbf{a}_{z}$ near a lossy dielectric sphere with electrical parameters: $\sigma=1.23$ and $\epsilon_{r}=55.0$, with radius $a=10[\mathrm{~cm}]$. The working frequency is $\mathrm{f}=1 \mathrm{GHz}$ and driven current $I_{0}=1 \mathrm{~A}$.
a) Find electric field intensity along $x, y, z$ axes inside the sphere.
b) Find the E-Plane and H-Plane pattern of the antenna.
c) Find the input impedance of the antenna (the antenna is assumed to be ideal)
d) What will happen when the antenna is moved away from the sphere. Draw the input impedance against the distance from the sphere.
- 6 Write a general FORTRAN code that computes the pattern of a thin wire dipole antenna which is located in any position in a multilayer spherical dielectric.
- 7 The center of a half wavelength thin dipole antenna which carry one Ampere current and oriented along z is at center of dielectric sphere with radius half wavelength and dielectric constant 9. Find E and H field at $\mathrm{r}=1 \mathrm{~km}, \phi=\pi / 4$ and $\theta=\pi / 4$.
- 8 Find the normalized bistatic rcs $\frac{\sigma}{\pi a^{2}}$ of a dielectric sphere with radius $a=\lambda / 2$ and $\epsilon_{r}=4$. in zox and zoy plane.
- 9 Find the normalized bistatic $\operatorname{rcs} \frac{\sigma}{\pi a^{2}}$ of a conducting sphere with radius $a=\lambda / 2$ in zox and zoy plane.
- 10 Write a general FORTRAN or MATLAB code that computes the rcs of a multilayer spherical dielectric.


## Chapter 8

## Spheroid

"Do not worry about your difficulties in Mathematics. I can assure you mine are still greater."
Albert Einstein

### 8.1 Spheroidal Coordinate systems

A spheroid is a quadric surface in three dimensions obtained by rotating an ellipse about one of its principal axes. If the ellipse is rotated about its major axis, the surface is called a prolate spheroid (similar to the shape of a rugby ball). If the minor axis is chosen, the surface is called an oblate spheroid (similar to the shape of the planet Earth). The sphere is a special case of the spheroid in which the generating ellipse is a circle. A spheroid is a special case of an ellipsoid where two of the three major axes are equal.

### 8.1.1 Prolate Spheroidal Coordinates

The prolate and oblate spheroids are used to construct the prolate and oblate coordinate systems, usually denoted with the symbols $\xi, \eta, \phi$. For both systems, the azimuthal coordinate $\phi$ is defined as the angle between a plane passing through a point and the $z$-axis and the $x z$ plane, measured from the positive $x$-axis. For the prolate spheroidal coordinates, the surfaces of constant $\xi$ and $\eta$ are prolate spheroids and hyperboloids of two sheets, while for oblate spheroidal coordinates they are oblate spheroids and hyperboloids of one sheet. The prolate spheroidal coordinate system $\zeta, \eta, \phi$ is related to


Figure 8.1: Prolate and Oblate Spheroid
cartesian coordinates $x, y, z$ by the transformation

$$
\begin{align*}
x & =f\left(\left(1-\eta^{2}\right)\left(\xi^{2}-1\right)\right)^{1 / 2} \cos \phi  \tag{8.1}\\
y & =f\left(\left(1-\eta^{2}\right)\left(\xi^{2}-1\right)\right)^{1 / 2} \sin \phi \\
z & =f \xi \eta
\end{align*}
$$

with the ranges of the coordinates being

$$
\begin{equation*}
-1 \leq \eta \leq 1, \quad 1 \leq \xi<\infty, \quad 0 \leq \phi<2 \pi \tag{8.2}
\end{equation*}
$$

For oblate spheroidal coordinates the corresponding transformation is:

$$
\begin{align*}
& x=f\left(\left(1-\eta^{2}\right)\left(\xi^{2}+1\right)\right)^{1 / 2} \cos \phi  \tag{8.3}\\
& y=f\left(\left(1-\eta^{2}\right)\left(\xi^{2}+1\right)\right)^{1 / 2} \sin \phi \\
& z=f \xi \eta \\
&-1 \leq \eta \leq 1, \quad 0 \leq \xi<\infty, \quad 0 \leq \phi<2 \pi \tag{8.4}
\end{align*}
$$

The prolate coordinates $\xi$ and $\eta$ also bear a simple relationship to the focal distances $r_{A}$ and $r_{B}$ : they are given by:

$$
\begin{equation*}
\xi=\frac{r_{A}+r_{B}}{2 f}, \quad \eta=\frac{r_{A}-r_{B}}{2 f} \tag{8.5}
\end{equation*}
$$

Such a simple interpretation does not exist for oblate coordinates, because in that case $r_{A}$ and $r_{B}$ are not well defined quantities.

$$
\begin{align*}
& r_{A}^{2}=x^{2}+y^{2}+(z+f)^{2}  \tag{8.6}\\
& r_{B}^{2}=x^{2}+y^{2}+(z-f)^{2}
\end{align*}
$$



Figure 8.2: Prolate spheroidal coordinates

### 8.2 Relation with the Spherical Coordinate Systems

The transformation between the unit vectors in the spherical and Spheroidal Coordinate Systems is:

$$
\begin{align*}
\mathbf{a}_{r} & =\xi\left[\frac{\xi^{2}-1}{\left(\xi^{2}-\eta^{2}\right)\left(\xi^{2}+\eta^{2}-1\right)}\right]^{1 / 2} \mathbf{a}_{\xi}  \tag{8.7}\\
& +\eta\left[\frac{1-\eta^{2}}{\left(\xi^{2}-\eta^{2}\right)\left(\xi^{2}+\eta^{2}-1\right)}\right]^{1 / 2} \mathbf{a}_{\eta} \\
\mathbf{a}_{\theta} & =\eta\left[\frac{1-\eta^{2}}{\left(\xi^{2}-\eta^{2}\right)\left(\xi^{2}+\eta^{2}-1\right)}\right]^{1 / 2} \mathbf{a}_{\xi} \\
& -\xi\left[\frac{\xi^{2}-1}{\left(\xi^{2}-\eta^{2}\right)\left(\xi^{2}+\eta^{2}-1\right)}\right]^{1 / 2} \mathbf{a}_{\eta} \\
\mathbf{a}_{\phi} & =\mathbf{a}_{\phi}
\end{align*}
$$



Figure 8.3: Prolate
and

$$
\begin{align*}
r & =f\left(\xi^{2}+\eta^{2}-1\right)^{1 / 2}  \tag{8.8}\\
\sin \theta & =\left[\frac{\left(\xi^{2}-1\right)\left(1-\eta^{2}\right)}{\left(\xi^{2}+\eta^{2}-1\right)}\right]^{1 / 2} \\
\cos \theta & =\frac{\xi \eta}{\left(\xi^{2}+\eta^{2}-1\right)^{1 / 2}}
\end{align*}
$$

### 8.2.1 The Scalar Wave Equation and Spheroidal Harmonics

The prolate and oblate coordinates are two of the eleven coordinate systems for which the (scalar) wave equation,

$$
\begin{equation*}
\nabla^{2} \psi+k^{2} \psi=0 \tag{8.9}
\end{equation*}
$$

separates into three ordinary differential equations. From standard vector analysis, the metric coefficients $h_{\xi}, h_{\eta}$ and $h_{\phi}$ can be derived using (Eq8.1)


Figure 8.4: Oblate Spheroidal Coordinates
and (Eq8.3); they are

$$
\begin{align*}
h_{\xi} & =f\left(\frac{\xi^{2}-\eta^{2}}{\xi^{2}-1}\right)^{1 / 2}  \tag{8.10}\\
h_{\eta} & =f\left(\frac{\xi^{2}-\eta^{2}}{1-\eta^{2}}\right)^{1 / 2} \\
h_{\phi} & =f\left(\left(1-\eta^{2}\right)\left(\xi^{2}-1\right)\right)^{1 / 2}
\end{align*}
$$

for prolate coordinates, and

$$
\begin{align*}
h_{\xi} & =f\left(\frac{\xi^{2}+\eta^{2}}{\xi^{2}+1}\right)^{1 / 2}  \tag{8.11}\\
h_{\eta} & =f\left(\frac{\xi^{2}+\eta^{2}}{1-\eta^{2}}\right)^{1 / 2} \\
h_{\phi} & =f\left(\left(1-\eta^{2}\right)\left(\xi^{2}+1\right)\right)^{1 / 2}
\end{align*}
$$

for oblate coordinates. With these coefficients, the Cartesian distance element becomes:

$$
\begin{equation*}
d x^{2}+d y^{2}+d z^{2}=h_{\xi}^{2} d \xi^{2}+h_{\eta}^{2} d \eta^{2}+h_{\phi}^{2} d \phi^{2} \tag{8.12}
\end{equation*}
$$

Using the formula for the Laplacian, $\nabla^{2}$, in orthogonal curvilinear coordinates (see [18]), the wave equation (Eq8.9) becomes:
$\frac{\partial}{\partial \eta}\left(\left(1-\eta^{2}\right) \frac{\partial \psi}{\partial \eta}\right)+\frac{\partial}{\partial \xi}\left(\left(\xi^{2}-1\right) \frac{\partial \psi}{\partial \xi}\right)+\frac{\xi^{2}-\eta^{2}}{\left(\xi^{2}-1\right)\left(1-\eta^{2}\right)} \frac{\partial^{2} \psi}{\partial \phi^{2}}+c^{2}\left(\xi^{2}-\eta^{2}\right) \psi=0$
for prolate coordinates, and

$$
\begin{equation*}
\frac{\partial}{\partial \eta}\left(\left(1-\eta^{2}\right) \frac{\partial \psi}{\partial \eta}\right)+\frac{\partial}{\partial \xi}\left(\left(\xi^{2}+1\right) \frac{\partial \psi}{\partial \xi}\right)+\frac{\xi^{2}+\eta^{2}}{\left(\xi^{2}+1\right)\left(1-\eta^{2}\right)} \frac{\partial^{2} \psi}{\partial \phi^{2}}+c^{2}\left(\xi^{2}+\eta^{2}\right) \psi=0 \tag{8.14}
\end{equation*}
$$

for oblate coordinates. In these equations we have defined $c=k f$, which will henceforth be known as the oblateness parameter. Note that Eq.(8.14) can be obtained from Eq.(8.13)) with the substitutions

$$
\begin{equation*}
c \longrightarrow \pm j c, \quad \xi \longrightarrow \pm j \xi \tag{8.15}
\end{equation*}
$$

where the $\pm$ signs are independent. It thus follows from this fact that the oblate spheroidal solutions will be related to the prolate via the same substitution. Our focus will henceforth be on the prolate functions, with the understanding that the oblate ones can be readily obtained from these using Eq.(8.15).

### 8.3 Helmholtz Equation and Spheroidal Functions

We take the time dependence to be $e^{j \omega t}$, therefore the Helmholtz differential equation $\left(\nabla^{2}+k^{2}\right) \psi=0$ in prolate spheroidal coordinates is:
$\frac{\partial}{\partial \xi}\left[\left(\xi^{2}-1\right) \frac{\partial \psi}{\partial \xi}\right]+\frac{\partial}{\partial \eta}\left[\left(1-\eta^{2}\right) \frac{\partial \psi}{\partial \eta}\right]+\frac{\xi^{2}-\eta^{2}}{\left(\xi^{2}-1\right)\left(1-\eta^{2}\right)} \frac{\partial^{2} \psi}{\partial \phi^{2}}+c^{2}\left(\xi^{2}-\eta^{2}\right) \psi=0$
and in oblate spheroidal coordinates, Helmholtz Equation may be obtained from Eq.(8.16) by the transformations:

$$
\begin{equation*}
\xi \rightarrow \pm j \xi, c \rightarrow \mp j c \tag{8.17}
\end{equation*}
$$

With these coordinate systems, the Helmholtz scalar wave equation becomes separable. The solutions of the wave are expressed by the scalar wave func-
tions equation

$$
\begin{equation*}
\psi_{m n}=S_{m n}(\eta, c) R_{m n}(\xi, c) \underset{\sin }{\cos }(m \phi) \tag{8.18}
\end{equation*}
$$

then the "radial solution" $R_{m n}(\xi, c)$ and "angular solution" $S_{m n}(\eta, c)$ satisfy the differential equations:

$$
\begin{align*}
& \frac{\partial}{\partial \xi}\left[\left(\xi^{2}-1\right) \frac{\partial}{\partial \xi} R_{m n}(\xi, c)\right]-\left(\lambda_{m n}-c^{2} \xi^{2}+\frac{m^{2}}{\xi^{2}-1}\right) R_{m n}(\xi, c)=0  \tag{8.19}\\
& \frac{\partial}{\partial \eta}\left[\left(1-\eta^{2}\right) \frac{\partial}{\partial \eta} S_{m n}(\eta, c)\right]+\left(\lambda_{m n}-c^{2} \eta^{2}-\frac{m^{2}}{1-\eta^{2}}\right) S_{m n}(\eta, c)=0 \tag{8.20}
\end{align*}
$$

where the separation constants (or eigenvalues) $\lambda_{m n}$ are to be determined so that $R_{m n}(\xi, c)$ and $S_{m n}(\eta, c)$ are finite at $\xi= \pm 1$ and $\eta= \pm 1$ respectively. The prolate angular functions $S_{m n}^{(1)}(\eta, c), S_{m n}^{(2)}(\eta, c)$ can be expand in term of associated Legendre functions of the first $P_{n}^{m}(\eta)$ and second $Q_{n}^{m}(\eta)$ kinds respectively.

$$
\begin{align*}
& S_{m n}^{(1)}(\eta, c)=\sum_{k=0,1}^{\infty} d_{k}^{m n}(c) P_{m+k}^{m}(\eta)  \tag{8.21}\\
& S_{m n}^{(2)}(\eta, c)=\sum_{k=0,1}^{\infty} d_{k}^{m n}(c) Q_{m+k}^{m}(\eta) \tag{8.22}
\end{align*}
$$

where the prime ( ${ }^{\prime}$ ) denotes that the summation is over only even values of $k$ when $n-m$ is even, and over only odd values of $k$ when $n-m$ is odd. Substitution of Eq.(8.21)and Eq.(8.22) in Eq.(8.20), with the subsequent use of the associated Legendre differential equation and of the recursion formula for the associated Legendre functions, yields the following recursion formula for the coefficients $d_{k}^{m n}$ :

$$
\begin{gather*}
\frac{(2 m+k+2)(2 m+k+1) c^{2}}{(2 m+2 k+3)(2 m+2 k+5)} d_{k+2}^{m n}(c)+ \\
{\left[(m+k)(m+k+1)-\lambda_{m n}(c)+\frac{2(m+k)(m+k+1)-2 m^{2}-1}{(2 m+2 k-1)(2 m+2 k+3)} c^{2}\right] d_{k}^{m n}+}  \tag{8.23}\\
\frac{k(k-1) c^{2}}{(2 m+2 k-3)(2 m+2 k-1)} d_{k-2}^{m n}(c)=0,(k \geq 0)
\end{gather*}
$$

we can define:

$$
\begin{gather*}
\alpha_{k}=\frac{(2 m+k+2)(2 m+k+1)}{(2 m+2 k+3)(2 m+2 k+5)} c^{2} \\
\beta_{k}=\left[(m+k)(m+k+1)+\frac{2(m+k)(m+k+1)-2 m^{2}-1}{(2 m+2 k-1)(2 m+2 k+3)} c^{2}\right]  \tag{8.24}\\
\gamma_{k}=\frac{k(k-1)}{(2 m+2 k-3)(2 m+2 k-1)} c^{2}
\end{gather*}
$$

Another important property of angle function is the orthogonality in the interval $-1 \geq \eta \geq 1$ which results from the theory of Sturm-Liouville differential equations. Thus

$$
\begin{equation*}
\int_{-1}^{1} S_{m n}(\eta) S_{m n^{\prime}}(\eta) d \eta=\delta_{n n^{\prime}} N_{m n} \tag{8.25}
\end{equation*}
$$

where $\delta_{n n^{\prime}}$ is the Kroneker delta function and $N_{m n}$

$$
\begin{equation*}
N_{m n}=2 \sum_{k=0,1}^{\infty} \frac{(k+2 m)!\left(d_{k}^{m n}\right)^{2}}{(2 k+2 m+1) k!} \tag{8.26}
\end{equation*}
$$

is the normalization constant.
The radial function can be expressed as an expansion of spherical Bessel functions as [12].

$$
\begin{equation*}
R_{m n}^{(g)}=\frac{\left(\frac{\xi^{2}-1}{\xi^{2}}\right)^{m / 2}}{\sum_{r=0,1}^{\infty} d_{r}^{m n} \frac{(2 m+r)!}{r!}} \sum_{r=0,1}^{\infty} d_{r}^{m n} \frac{(2 m+r)!}{r!} j^{r+m-n} z_{m+r}^{(g)}(c \xi) \tag{8.27}
\end{equation*}
$$

where $z_{m+r}^{(g)}(c \xi)$ represents the spherical Bessel functions $j_{m+r}(c \xi), y_{m+r}(c \xi)$, $h_{m+r}^{(1)}(c \xi)$, and $h_{m+r}^{(2)}(c \xi)$, for $g=1 ; 2 ; 3 ; 4$ respectively [12].

### 8.4 Computation of Eigenvalues

In order to find the expansion coefficients, we have to find the eigenvalues $\lambda_{m n}$ first. The Eq.(8.23) defines a set of homogeneous equations for $d_{k}^{m n}(c)$. To have nontrivial solution, the determinant must be zero, from which we can determine the eigenvalues $\lambda_{m n}(c)$, where parameter $c$ may be real or complex. This is equivalent to solving tridiagonal eigenvalue problem given by:

$$
\left[\begin{array}{ccccc}
\beta_{0} & \alpha_{0} & 0 & & \cdots  \tag{8.28}\\
\gamma_{2} & \beta_{2} & \alpha_{2} & 0 & \cdots \\
& & \ddots & & \\
0 & \cdots & \gamma_{2 k} & \beta_{2 k} & \alpha_{2 k} \cdots \\
& & \ddots & &
\end{array}\right]\left[\begin{array}{c}
d_{0}^{m n}(c) \\
d_{2}^{m n}(c) \\
\vdots \\
d_{2 k}^{m n}(c) \\
\vdots
\end{array}\right]=\lambda_{m n}\left[\begin{array}{c}
d_{0}^{m n}(c) \\
d_{2}^{m n}(c) \\
\vdots \\
d_{2 k}^{m n}(c) \\
\vdots
\end{array}\right]
$$

where $(n-m)$ are even. and

$$
\left[\begin{array}{ccccc}
\beta_{1} & \alpha_{1} & 0 & & \cdots  \tag{8.29}\\
\gamma_{3} & \beta_{3} & \alpha_{3} & 0 & \cdots \\
& & \ddots & & \\
0 & \cdots & \gamma_{2 k+1} & \beta_{2 k+1} & \alpha_{2 k+1} \cdots \\
& & \ddots & &
\end{array}\right]\left[\begin{array}{c}
d_{1}^{m n}(c) \\
d_{3}^{m n}(c) \\
\vdots \\
d_{2 k+1}^{m n}(c) \\
\vdots
\end{array}\right]=\lambda_{m n}\left[\begin{array}{c}
d_{1}^{m n}(c) \\
d_{3}^{m n}(c) \\
\vdots \\
d_{2 k+1}^{m n}(c) \\
\vdots
\end{array}\right]
$$

where $(n-m)$ are odd. The matrix Eq.(8.28) or Eq.(8.29) has eigenvectors and eigenvalues which gives us eigenvalues for spheroidal wave functions. For a given value of $c$, the value of the coefficients decreases as $k$ increase. We can truncate the number of linear equations. The truncation number gives the accuracy of eigenvalues and depend on the value of $c$. In practice, we only need a finite number of eigenvalues therefore truncation number is finite. In our problem we have four matrix which each one gives the proper eigenvalues. When the $|c|>500$ is very large, we can use asymptotic expansion [13] for $\lambda_{m n}, m=0,1,2 \cdots$ and $n=m, m+1, \cdots$.

An alternative method for calculating the characteristic values is to solve transcendental equation derived from Eq.(8.23). Dividing Eq.(8.23) by $d_{k}^{m n}$, we obtain:

$$
\begin{equation*}
\alpha_{k} \frac{d_{k+2}^{m n}(c)}{d_{k}^{m n}(c)}+\beta_{k}-\lambda_{m n}(c)+\gamma_{k} \frac{d_{k-2}^{m n}(c)}{d_{k}^{m n}(c)}=0 \tag{8.30}
\end{equation*}
$$

denoting

$$
\begin{equation*}
N_{k}^{m}=-\alpha_{k-2} \frac{d_{k}^{m n}(c)}{d_{k-2}^{m n}(c)} \quad \gamma_{k}^{m}=\beta_{k} \quad \beta_{k}^{m}=\gamma_{k} \alpha_{k-2} \tag{8.31}
\end{equation*}
$$

we can write Eq.(8.30) as

$$
\begin{equation*}
N_{k+2}^{m}=\gamma_{k}^{m}-\lambda_{m n}(c)-\frac{\beta_{k}^{m}}{N_{k}^{m}} \quad(k \geq 2) \tag{8.32}
\end{equation*}
$$

where $N_{2}^{m}=\gamma_{0}^{m}-\lambda_{m n}(c)$ and $N_{3}^{m}=\gamma_{1}^{m}-\lambda_{m n}(c)$, on the other hand, we can rewrite Eq.(8.32) as

$$
\begin{equation*}
N_{k}^{m}=\frac{\beta_{k}^{m}}{\gamma_{k}^{m}-\lambda_{m n}-N_{k+2}^{m}} \tag{8.33}
\end{equation*}
$$

by letting $k \rightarrow k+2$ in Eq.(8.33), we can obtain the expression for $N_{k+2}^{m}$ in terms of $N_{k+4}^{m}$. A series of repeated substitutions similar to one described
above yields another continued fraction for $N_{k+2}^{m}$ as

$$
\begin{equation*}
N_{k+2}^{m}=\frac{\beta_{k+2}^{m}}{\gamma_{k+2}^{m}-\lambda_{m n}-} \frac{\beta_{k+4}^{m}}{\gamma_{k+4}^{m}-\lambda_{m n}-} \frac{\beta_{k+6}^{m}}{\gamma_{k+6}^{m}-\lambda_{m n}-} \cdots \tag{8.34}
\end{equation*}
$$

by letting $k=n-m$ in Eq.(8.32) and Eq.(8.34) and denoting
$U_{1}\left(\lambda_{m n}\right)=\gamma_{n-m}^{m}-\lambda_{m n}-\frac{\beta_{n-m}^{m}}{\gamma_{n-m-2}^{m}-\lambda_{m n}-} \frac{\beta_{n-m-2}^{m}}{\gamma_{n-m-4}^{m}-\lambda_{m n}-} \frac{\beta_{n-m-4}^{m}}{\gamma_{n-m-6}^{m}-\lambda_{m n}-} \cdots$
and

$$
\begin{equation*}
U_{2}\left(\lambda_{m n}\right)=-\frac{\beta_{n-m+2}^{m}}{\gamma_{n-m+2}^{m}-\lambda_{m n}-} \frac{\beta_{n-m+4}^{m}}{\gamma_{n-m+4}^{m}-\lambda_{m n}-} \frac{\beta_{n-m+6}^{m}}{\gamma_{n-m+6}^{m}-\lambda_{m n}-} \cdots \tag{8.36}
\end{equation*}
$$

we obtain the transcendental equation for $\lambda_{m n}$ as

$$
\begin{equation*}
U_{1}\left(\lambda_{m n}\right)+U_{2}\left(\lambda_{m n}\right)=0 \tag{8.37}
\end{equation*}
$$

The characteristic values can be determined by solving Eq.(8.37) using a numerical method such as secant method with proper estimates which in case of complex characteristic values we find it by solving one the proper matrix.

### 8.5 Spheroidal Vector Wave Functions

By the application of vector differential operators to the scalar spheroidal wave function given in Eq.(8.18), the vector spheroidal wave functions M and $\mathbf{N}$, [64], are defined as :

$$
\begin{align*}
\mathbf{M}_{m n} & =\nabla \psi_{m n} \times \mathbf{a}  \tag{8.38}\\
\mathbf{N}_{m n} & =\frac{1}{k} \nabla \times \mathbf{M}_{m n}
\end{align*}
$$

where $\mathbf{a}$ is either an arbitrary constant unit vector or the position vector $\mathbf{r}$. None of the coordinate unit vectors $\xi, \eta$, or $\phi$ in the spheroidal coordinate systems has the properties required for $\mathbf{a}$. Instead, the Cartesian unit vectors can efficiently be used, because the transformation from the Cartesian system to the spheroidal systems is relatively simpler. The three Cartesian
unit vectors $\mathbf{a}_{x}, \mathbf{a}_{y}, \mathbf{a}_{z}$ and the radial vector $\mathbf{a}_{r}$ generate the following prolate spheroidal vector wave functions $\mathbf{M}(c ; \xi, \eta, \phi), \mathbf{N}(c ; \xi, \eta, \phi)$ :

$$
\begin{align*}
& \mathbf{M}_{\substack{p m n \\
o}}^{p(g)}(c ; \xi, \eta, \phi)=\nabla \psi_{\substack{e_{o n n} \\
o}}^{(g)}(c ; \xi, \eta, \phi) \times \mathbf{a}_{p}  \tag{8.39}\\
& \mathbf{a}_{p}=\mathbf{a}_{x}, \mathbf{a}_{y}, \mathbf{a}_{z}, \quad p=x, y, z \\
& \mathbf{M}_{\substack{r \\
o \\
j \\
r(g)}}(c ; \xi, \eta, \phi)=\nabla \psi_{\substack{(g) \\
o m n}}^{(g)}(c ; \xi, \eta, \phi) \times \mathbf{r}  \tag{8.40}\\
& \mathbf{N}_{\substack{e m n \\
o \\
p(g)}}^{\rho_{m}}(c ; \xi, \eta, \phi)=\frac{1}{k} \nabla \times \mathbf{M}_{\substack{e \\
o \\
\\
p(g n}}^{p(c)}(c ; \xi, \eta, \phi) \\
& \mathbf{N}_{\substack{e m n \\
e \\
r(g)}}^{\substack{m}}(c ; \xi, \eta, \phi)=\frac{1}{k} \nabla \times \mathbf{M}_{\substack{e \\
e_{m n} \\
r(g)}}(c ; \xi, \eta, \phi)
\end{align*}
$$

in which e and o refer to the even and odd functions, respectively, and $g$ indicates the kind of function, $g=1,2,3,4$.
Explicit expressions for these vector spheroidal wave functions are available
 dependence of various components is simply given by the product of $\cos \phi$ or $\sin \phi$ with either $\cos m \phi$ or $\sin m \phi$. It is convenient therefore to define the following additional vector wave functions:
where the components with an index $m+1$ have a $\phi$ dependence of either $\cos (m+1) \phi$ or $\sin (m+1) \phi$, whereas those with an index $m-1$ have a $\phi$ dependence of either $\cos (m-1) \phi$ or $\sin (m-1) \phi$. The - and + signs in Eq.(8.41)-Eq.(8.43), and the + and - signs in Eq.(8.42)-Eq.(8.44), on the right-hand sides, are associated with the even and odd vector wave functions, respectively.

### 8.6 Expressions of the Spheroidal Vector Wave Functions

In all pairs of signs in the following expressions, the upper sign pertains to the oblate functions and the lower one to the prolate functions.

$$
\begin{aligned}
& \star \mathbf{M}_{\substack{e m n \\
o x(g)}}^{o_{m}}(c ; \xi, \eta, \phi) \\
& M_{\substack{e \\
o \\
o n n \eta}}^{x(g)}=-\frac{\left(\xi^{2} \pm 1\right)^{1 / 2}}{f\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}}\left[S_{m n} \frac{d}{d \xi} R_{m n}^{(g)} \sin \phi_{\sin }^{\cos } m \phi-\frac{m \xi}{\xi^{2} \pm 1} S_{m n} R_{m n}^{(g)} \cos \phi \underset{(-1) \cos }{\sin } m \phi\right]
\end{aligned}
$$

$$
\begin{aligned}
& M_{\substack{e m n \phi \\
o \\
o \\
o(g)}} \frac{1}{f\left(\xi^{2} \pm \eta^{2}\right)}\left[\xi\left(1-\eta^{2}\right) \frac{d}{d \eta} S_{m n} R_{m n}^{(g)}+\eta\left(\xi^{2} \pm 1\right) S_{m n} \frac{d}{d \xi} R_{m n}^{(g)}\right] \cos \phi{ }_{\sin }^{\cos } m \phi
\end{aligned}
$$

The expressions of the components are obtained from those of $\mathbf{M}_{\substack{e \\ e \\ e \\{ }_{e}(g)}}(c ; \xi, \eta, \phi)$ by replacing the factor $\cos \phi$ and $\sin \phi$ by the $\sin \phi$ and $-\cos \phi$, respectively.

$$
\star \mathbf{M}_{\substack{e m n \\ z(g)}}^{\mathrm{o}^{z}}(c ;, \eta, \phi)
$$

$$
\left.\begin{array}{c}
M_{\substack{e \\
o \\
o}}^{z(g)}=\frac{m \eta}{f\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}\left(1-\eta^{2}\right)^{1 / 2}} S_{m n} R_{m n}^{(g)} \stackrel{\sin }{(-1) \cos } m \phi \\
M_{\substack{e \\
o \\
o}}^{z(g)} \\
M_{\substack{e \\
o} m n \phi}^{z(g)}=\frac{m \xi}{f\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}\left(\xi^{2} \pm 1\right)^{1 / 2}} S_{m n} R_{m n}^{(g)}(-1) \sin \cos m \phi \\
f\left(\xi^{2} \pm \eta^{2}\right)
\end{array} \eta \frac{\left(1-\eta^{2}\right)^{1 / 2}\left(\xi^{2} \pm 1\right)^{1 / 2}}{d \eta} S_{m n} R_{m n}^{(g)}-\xi S_{m n} \frac{d}{d \xi} R_{m n}^{(g)}\right]{ }_{\sin }^{\cos } m \phi \quad .
$$

$\star \mathbf{M}_{\substack{e m n \\ o \\ r(g)}}^{m_{m}}(c ; \xi, \eta, \phi)$

$$
\begin{gathered}
M_{\substack{e \\
o \\
r}}^{r(g)}=\frac{m \xi}{\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}\left(1-\eta^{2}\right)^{1 / 2}} S_{m n} R_{m n}^{(g)} \sin (-1) \cos m \phi \\
M_{\substack{e \\
o \\
o}}^{r(g)}=\frac{( \pm) m \eta}{\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}\left(\xi^{2} \pm 1\right)^{1 / 2}} S_{m n} R_{m n}^{(g)} \sin (-1) \cos m \phi \\
M_{\substack{e m n \phi \\
o}}^{r(g)}=\frac{\left(1-\eta^{2}\right)^{1 / 2}\left(\xi^{2} \pm 1\right)^{1 / 2}}{\left(\xi^{2} \pm \eta^{2}\right)}\left[\xi \frac{d}{d \eta} S_{m n} R_{m n}^{(g)} \pm \eta S_{m n} \frac{d}{d \xi} R_{m n}^{(g)}\right]{ }_{\sin }^{\cos } m \phi
\end{gathered}
$$

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$\star \mathbf{M}_{e_{o m-1, n}^{-(g)}}^{-(c ; \xi, \eta, \phi)}$

$$
\star \mathbf{N}_{\substack{e_{o n n}}}^{x(g)}(c ; \xi, \eta, \phi)
$$

$$
\begin{aligned}
& N_{\substack{e m n \eta \\
o(g)}}=\frac{1}{k f^{2}\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}}\left\{\left[\eta S_{m n} \frac{\partial}{\partial \xi}\left(\frac{\left(\xi^{2} \pm 1\right)^{3 / 2}}{\left(\xi^{2} \pm \eta^{2}\right)} \frac{d}{d \xi} R_{m n}^{(g)}\right)-\frac{1}{\left(\xi^{2} \pm 1\right)^{1 / 2}} \frac{d}{d \eta} S_{m n} R_{m n}^{(g)}\right.\right. \\
& \left.+\left(1-\eta^{2}\right) \frac{d}{d \eta} S_{m n} \frac{\partial}{\partial \xi}\left(\frac{\xi\left(\xi^{2} \pm 1\right)^{1 / 2}}{\xi^{2} \pm \eta^{2}} R_{m n}^{(g)}\right)-\frac{m^{2} \eta S_{m n} R_{m n}^{(g)}}{\left(1-\eta^{2}\right)\left(\xi^{2} \pm 1\right)^{1 / 2}}\right] \cos \phi{ }_{\sin }^{\cos } m \phi \\
& \left.+\frac{m}{\left(\xi^{2} \pm 1\right)^{1 / 2}}\left[\frac{d}{d \eta} S_{m n}+\frac{\eta}{1-\eta^{2}} S_{m n}\right] \cdot R_{m n}^{(g)} \sin \phi \begin{array}{c}
\sin \\
(-1) \cos
\end{array} m\right\} \\
& N_{\substack{e \\
o \\
o(g n \xi}}^{x(g)}=-\frac{1}{k f^{2}\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}}\left\{\left[\xi \frac{\partial}{\partial \eta}\left(\frac{\left(1-\eta^{2}\right)^{3 / 2}}{\left(\xi^{2} \pm \eta^{2}\right)} \frac{d}{d \eta} S_{m n}\right) R_{m n}^{(g)}+\frac{1}{\left(1-\eta^{2}\right)^{1 / 2}} S_{m n} \frac{d}{d \xi} R_{m n}^{(g)}\right.\right. \\
& \left.+\left(\xi^{2} \pm 1\right) \frac{\partial}{\partial \eta}\left(\frac{\eta\left(1-\eta^{2}\right)^{1 / 2}}{\xi^{2} \pm \eta^{2}} S_{m n}\right) \frac{d}{d \xi} R_{m n}^{(g)}-\frac{m^{2} \xi S_{m n} R_{m n}^{(g)}}{\left(1-\eta^{2}\right)^{1 / 2}\left(\xi^{2} \pm 1\right)}\right] \cos \phi{ }_{\sin }^{\cos } m \phi \\
& +\frac{m}{\left(1-\eta^{2}\right)^{1 / 2}} S_{m n}\left[\frac{\xi}{\xi^{2} \pm 1} R_{m n}^{(g)}-\frac{d}{d \xi} R_{m n}^{(g)}\right] \cdot \sin \phi \quad \begin{array}{c}
\sin \\
(-1) \cos
\end{array} m \phi
\end{aligned}
$$

$$
\begin{aligned}
& M_{\substack{e \\
o \\
-(g) \\
-(g) n \eta}}=\frac{\left(\xi^{2} \pm 1\right)^{1 / 2}}{d\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}}\left[S_{m n} \frac{d}{d \xi} R_{m n}^{(g)}+\frac{m \xi}{\xi^{2} \pm 1} S_{m n} R_{m n}^{(g)}\right] \sin _{(-1) \cos }(m-1) \phi \\
& M_{\substack{e \\
o \\
o}}^{-(g)}=\frac{\left(1-\eta^{2}\right)^{1 / 2}}{d\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}}\left[\frac{d}{d \eta} S_{m n} R_{m n}^{(g)}-\frac{m \eta}{1-\eta^{2}} S_{m n} R_{m n}^{(g)}\right] \underset{\cos }{(-1) \sin }(m-1) \phi \\
& M_{\substack{e \\
o \\
m-1, n \phi}}^{-(g)}=\frac{1}{d\left(\xi^{2} \pm \eta^{2}\right)}\left[\xi\left(1-\eta^{2}\right) \frac{d}{d \eta} S_{m n} R_{m n}^{(g)}+\eta\left(\xi^{2} \pm 1\right) S_{m n} \frac{d}{d \xi} R_{m n}^{(g)}\right]{ }_{\sin }^{\cos }(m-1) \phi
\end{aligned}
$$

$$
\begin{aligned}
& \star \mathbf{M}_{\substack{e m+1, n \\
o(g)}}^{+}(c ;, \eta, \phi) \\
& M_{e_{o m+1, n \eta}^{+(g)}}^{o+}=\frac{\left(\xi^{2} \pm 1\right)^{1 / 2}}{d\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}}\left[S_{m n} \frac{d}{d \xi} R_{m n}^{(g)}-\frac{m \xi}{\xi^{2} \pm 1} S_{m n} R_{m n}^{(g)}\right] \underset{\cos }{(-1) \sin }(m+1) \phi \\
& M_{\substack{e m+1, n \xi}}^{+(g)}=\frac{\left(1-\eta^{2}\right)^{1 / 2}}{d\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}}\left[\frac{d}{d \eta} S_{m n} R_{m n}^{(g)}+\frac{m \eta}{1-\eta^{2}} S_{m n} R_{m n}^{(g)}\right] \sin _{(-1) \cos }(m+1) \phi \\
& M_{\substack{e m+1, n \phi \\
+(g)}} \frac{1}{d\left(\xi^{2} \pm \eta^{2}\right)}\left[\xi\left(1-\eta^{2}\right) \frac{d}{d \eta} S_{m n} R_{m n}^{(g)}+\eta\left(\xi^{2} \pm 1\right) S_{m n} \frac{d}{d \xi} R_{m n}^{(g)}\right]{ }_{\sin }^{\cos }(m+1) \phi
\end{aligned}
$$

$$
\begin{aligned}
& \substack{N_{e}^{e}(g n) \\
o} \frac{\left(1-\eta^{2}\right)^{1 / 2}\left(\xi^{2} \pm 1\right)^{1 / 2}}{k f^{2}\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}}\left\{\left[\frac{1}{\left(\xi^{2} \pm 1\right)^{1 / 2}} \frac{d}{d \eta}\left(\left(1-\eta^{2}\right)^{1 / 2} \frac{d}{d \eta} S_{m n}\right) R_{m n}^{(g)}\right.\right. \\
&\left.+\frac{1}{\left(1-\eta^{2}\right)^{1 / 2}} S_{m n} \frac{d}{d \xi}\left(\left(\xi^{2} \pm 1\right)^{1 / 2} \frac{d}{d \xi} R_{m n}^{(g)}\right)\right] \sin \phi \sin _{\cos }^{m \phi} \\
& m\left[\frac{1}{\left(\xi^{2} \pm 1\right)^{1 / 2}} \frac{d}{d \eta}\left(\frac{\eta}{\left(1-\eta^{2}\right)^{1 / 2}} S_{m n}\right) R_{m n}^{(g)}\right. \\
&\left.\quad-\frac{1}{\left(1-\eta^{2}\right)^{1 / 2}} S_{m n} \frac{d}{d \xi}\left(\frac{\xi}{\left(\xi^{2} \pm 1\right)^{1 / 2}} R_{m n}^{(g)}\right) \cos \phi \quad \sin (-1) \cos m \phi\right\} \\
& \star \mathbf{N}_{\substack{e \\
o \\
e \\
e n n}}^{y(c)}(c ; \xi, \eta, \phi)
\end{aligned}
$$

The expressions of the components are obtained from those of $\mathbf{N}_{\substack{x \\ \hline \\ e_{m n}}}^{x(g)}(c ; \xi, \eta, \phi)$ by replacing the factor $\cos \phi$ and $\sin \phi$ by the $\sin \phi$ and $-\cos \phi$, respectively. $\star \mathbf{N}_{\substack{e \\ o \\ m n}}^{z(g)}(c ; \xi, \eta, \phi)$

$$
\begin{aligned}
& N_{\substack{e \\
o \\
o \\
z(g)}}^{\sigma_{n}}=\frac{\left(1-\eta^{2}\right)^{1 / 2}}{k f^{2}\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}}\left[\eta \frac{d}{d \eta} S_{m n} \frac{\partial}{\partial \xi}\left(\frac{\xi^{2} \pm 1}{\xi^{2} \pm \eta^{2}} R_{m n}^{(g)}\right)\right. \\
& \left.-S_{m n} \frac{\partial}{\partial \xi}\left(\frac{\xi\left(\xi^{2} \pm 1\right)}{\xi^{2} \pm \eta^{2}} \frac{d}{d \xi} R_{m n}^{(g)}\right)+\frac{m^{2} \xi S_{m n} R_{m n}^{(g)}}{\left(1-\eta^{2}\right)\left(\xi^{2} \pm 1\right)}\right]{ }_{\sin }^{\cos } m \phi \\
& N_{\substack{e m n \xi \\
e_{m}(g)}}^{\underset{k f^{2}\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}}{\left(\xi^{2} \pm 1\right)^{1 / 2}}\left[\xi \frac{\partial}{\partial \eta}\left(\frac{1-\eta^{2}}{\xi^{2} \pm \eta^{2}} S_{m n}\right) \frac{d}{d \xi} R_{m n}^{(g)}, ~(\eta)\right.} \\
& -\frac{\partial}{\partial \eta}\left(\frac{\eta\left(1-\eta^{2}\right)}{\xi^{2} \pm \eta^{2}} \frac{d}{d \eta} S_{m n}\right) R_{m n}^{(g)} \\
& \left.+\frac{m^{2} \eta S_{m n} R_{m n}^{(g)}}{\left(1-\eta^{2}\right)\left(\xi^{2} \pm 1\right)}\right]{ }_{\sin }^{\cos } m \phi \\
& N_{\substack{e \\
o m n \phi}}^{z(g)}=\frac{m\left(1-\eta^{2}\right)^{1 / 2}\left(\xi^{2} \pm 1\right)^{1 / 2}}{k f^{2}\left(\xi^{2} \pm \eta^{2}\right)}\left[\frac{\xi}{\left(\xi^{2} \pm 1\right)} \frac{d}{d \eta} S_{m n} R_{m n}^{(g)}\right. \\
& \left.+\frac{\eta}{\left(1-\eta^{2}\right)} S_{m n} \frac{d}{d \xi} R_{m n}^{(g)}\right] \underset{\cos }{(-1) \sin } m \phi
\end{aligned}
$$

$\star \mathbf{N}_{\substack{r(g) \\ o m n}}^{r(c ; \xi, \eta, \phi)}$

$$
N_{\substack{r(g) \\ o \\ o}}^{r(g n \eta}=\frac{\left(1-\eta^{2}\right)^{1 / 2}}{k f\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}}\left[\frac{d}{d \eta} S_{m n} \frac{\partial}{\partial \xi}\left(\frac{\xi\left(\xi^{2} \pm 1\right)}{\xi^{2} \pm \eta^{2}} R_{m n}^{(g)}\right)\right.
$$

$$
\begin{aligned}
& \left. \pm \eta S_{m n} \frac{\partial}{\partial \xi}\left(\frac{\xi^{2} \pm 1}{\xi^{2} \pm \eta^{2}} \frac{d}{d \xi} R_{m n}^{(g)}\right) \mp \frac{m^{2} \eta S_{m n} R_{m n}^{(g)}}{\left(1-\eta^{2}\right)\left(\xi^{2} \pm 1\right)}\right]{ }_{\sin }^{\cos } m \phi \\
& \underset{\substack{e_{o m n} \\
N_{m}^{r(g)}}}{\substack{\left(\xi^{2} \pm 1\right)^{1 / 2}}}\left[( \pm) \frac{\partial}{\partial \eta\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}}\left(\frac{\eta\left(1-\eta^{2}\right)}{\xi^{2} \pm \eta^{2}} S_{m n}\right) \frac{d}{d \xi} R_{m n}^{(g)}\right. \\
& \left.+\xi \frac{\partial}{\partial \eta}\left(\frac{1-\eta^{2}}{\xi^{2} \pm \eta^{2}} \frac{d}{d \eta} S_{m n}\right) R_{m n}^{(g)}-\frac{m^{2} \xi S_{m n} R_{m n}^{(g)}}{\left(1-\eta^{2}\right)\left(\xi^{2} \pm 1\right)}\right]{ }_{\sin }^{\cos } m \phi \\
& N_{\substack{e \\
o m n \phi}}^{r(g)}=\frac{m\left(1-\eta^{2}\right)^{1 / 2}\left(\xi^{2} \pm 1\right)^{1 / 2}}{k f\left(\xi^{2} \pm \eta^{2}\right)}\left[( \pm) \frac{1}{\left(\xi^{2} \pm 1\right)} \frac{d}{d \eta}\left(\eta S_{m n}\right) R_{m n}^{(g)}\right. \\
& \left.-\frac{1}{\left(1-\eta^{2}\right)} S_{m n} \frac{d}{d \xi}\left(\xi R_{m n}^{(g)}\right)\right] \begin{array}{c}
\sin \\
(-1) \cos
\end{array} m \phi \\
& \star \mathbf{N}_{\substack{e \\
o \\
\hline \\
+(g) \\
\hline}}(c ; \xi, \eta, \phi) \\
& N_{\substack{e \\
o m+1, n \eta}}^{+(g)}=\frac{2}{k d^{2}\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}}\left[\eta S_{m n} \frac{\partial}{\partial \xi}\left(\frac{\left(\xi^{2} \pm 1\right)^{3 / 2}}{\left(\xi^{2} \pm \eta^{2}\right)} \frac{d}{d \xi} R_{m n}^{(g)}\right)\right. \\
& +\left(1-\eta^{2}\right) \frac{d}{d \eta} S_{m n} \frac{\partial}{\partial \xi}\left(\frac{\xi\left(\xi^{2} \pm 1\right)^{1 / 2}}{\xi^{2} \pm \eta^{2}} R_{m n}^{(g)}\right) \\
& \left.-\frac{m+1}{\left(\xi^{2} \pm 1\right)^{1 / 2}} \frac{d}{d \eta} S_{m n} R_{m n}^{(g)}-\frac{m(m+1) \eta}{\left(1-\eta^{2}\right)\left(\xi^{2} \pm 1\right)^{1 / 2}} S_{m n} R_{m n}^{(g)}\right]{ }_{\sin }^{\cos }(m+1) \phi \\
& N_{e_{m}+1, n \xi}^{+(g)}=-\frac{2}{k d^{2}\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}}\left[\xi \frac{\partial}{\partial \eta}\left(\frac{\left(1-\eta^{2}\right)^{3 / 2}}{\left(\xi^{2} \pm \eta^{2}\right)} \frac{d}{d \eta} S_{m n}\right) R_{m n}^{(g)}\right. \\
& +\left(\xi^{2} \pm 1\right) \frac{\partial}{\partial \eta}\left(\frac{\eta\left(1-\eta^{2}\right)^{1 / 2}}{\xi^{2} \pm \eta^{2}} S_{m n}\right) \frac{d}{d \xi} R_{m n}^{(g)} \\
& \left.+\frac{m+1}{\left(1-\eta^{2}\right)^{1 / 2}} S_{m n} \frac{d}{d \xi} R_{m n}^{(g)}-\frac{m(m+1) \xi}{\left(1-\eta^{2}\right)^{1 / 2}\left(\xi^{2} \pm 1\right)} S_{m n} R_{m n}^{(g)}\right] \cdot{ }_{\sin }^{\cos }(m+1) \phi \\
& N_{\substack{e m+1, n \phi \\
e_{m} \\
+(g)}} \frac{2\left(1-\eta^{2}\right)^{1 / 2}\left(\xi^{2} \pm 1\right)^{1 / 2}}{k d^{2}\left(\xi^{2} \pm \eta^{2}\right)}\left[\frac{1}{\left(\xi^{2} \pm 1\right)^{1 / 2}} \frac{d}{d \eta}\left(\left(1-\eta^{2}\right)^{1 / 2} \frac{d}{d \eta} S_{m n}\right) R_{m n}^{(g)}\right. \\
& +\frac{1}{\left(1-\eta^{2}\right)^{1 / 2}} S_{m n} \frac{d}{d \xi}\left(\left(\xi^{2} \pm 1\right)^{1 / 2} \frac{d}{d \xi} R_{m n}^{(g)}\right) \\
& +\frac{m}{\left(\xi^{2} \pm 1\right)^{1 / 2}} \frac{d}{d \eta}\left(\frac{\eta}{\left(1-\eta^{2}\right)^{1 / 2}} S_{m n}\right) R_{m n}^{(g)}
\end{aligned}
$$

$$
-\frac{m}{\left(1-\eta^{2}\right)^{1 / 2}} S_{m n} \frac{d}{d \xi}\left(\frac{\xi}{\left(\xi^{2} \pm 1\right)^{1 / 2}} R_{m n}^{(g)}\right) \underset{(-1) \cos }{\sin }(m+1) \phi
$$



$$
\begin{aligned}
& \underset{\substack{e_{m-1, n \eta}^{-(g)} \\
N_{m}}}{k d^{2}\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}}\left[\eta S_{m n} \frac{\partial}{\partial \xi}\left(\frac{\left(\xi^{2} \pm 1\right)^{3 / 2}}{\left(\xi^{2} \pm \eta^{2}\right)} \frac{d}{d \xi} R_{m n}^{(g)}\right)\right. \\
& +\left(1-\eta^{2}\right) \frac{d}{d \eta} S_{m n} \frac{\partial}{\partial \xi}\left(\frac{\xi\left(\xi^{2} \pm 1\right)^{1 / 2}}{\xi^{2} \pm \eta^{2}} R_{m n}^{(g)}\right) \\
& \left.+\frac{m-1}{\left(\xi^{2} \pm 1\right)^{1 / 2}} \frac{d}{d \eta} S_{m n} R_{m n}^{(g)}-\frac{m(m-1) \eta}{\left(1-\eta^{2}\right)\left(\xi^{2} \pm 1\right)^{1 / 2}} S_{m n} R_{m n}^{(g)}\right]{ }_{\sin }^{\cos }(m-1) \phi \\
& N_{\substack{e \\
o \\
o \\
-(g) \\
-(g) n \xi}}=-\frac{2}{k d^{2}\left(\xi^{2} \pm \eta^{2}\right)^{1 / 2}}\left[\xi \frac{\partial}{\partial \eta}\left(\frac{\left(1-\eta^{2}\right)^{3 / 2}}{\left(\xi^{2} \pm \eta^{2}\right)} \frac{d}{d \eta} S_{m n}\right) R_{m n}^{(g)}\right. \\
& +\left(\xi^{2} \pm 1\right) \frac{\partial}{\partial \eta}\left(\frac{\eta\left(1-\eta^{2}\right)^{1 / 2}}{\xi^{2} \pm \eta^{2}} S_{m n}\right) \frac{d}{d \xi} R_{m n}^{(g)} \\
& \left.-\frac{m-1}{\left(1-\eta^{2}\right)^{1 / 2}} S_{m n} \frac{d}{d \xi} R_{m n}^{(g)}-\frac{m(m-1) \xi}{\left(1-\eta^{2}\right)^{1 / 2}\left(\xi^{2} \pm 1\right)} S_{m n} R_{m n}^{(g)}\right] \cdot{ }_{\sin }^{\cos }(m-1) \phi \\
& N_{e_{m}-1, n \phi}^{-(g)}=\frac{2\left(1-\eta^{2}\right)^{1 / 2}\left(\xi^{2} \pm 1\right)^{1 / 2}}{k d^{2}\left(\xi^{2} \pm \eta^{2}\right)}\left[\frac{1}{\left(\xi^{2} \pm 1\right)^{1 / 2}} \frac{d}{d \eta}\left(\left(1-\eta^{2}\right)^{1 / 2} \frac{d}{d \eta} S_{m n}\right) R_{m n}^{(g)}\right. \\
& +\frac{1}{\left(1-\eta^{2}\right)^{1 / 2}} S_{m n} \frac{d}{d \xi}\left(\left(\xi^{2} \pm 1\right)^{1 / 2} \frac{d}{d \xi} R_{m n}^{(g)}\right) \\
& -\frac{m}{\left(\xi^{2} \pm 1\right)^{1 / 2}} \frac{d}{d \eta}\left(\frac{\eta}{\left(1-\eta^{2}\right)^{1 / 2}} S_{m n}\right) R_{m n}^{(g)} \\
& \left.+\frac{m}{\left(1-\eta^{2}\right)^{1 / 2}} S_{m n} \frac{d}{d \xi}\left(\frac{\xi}{\left(\xi^{2} \pm 1\right)^{1 / 2}} R_{m n}^{(g)}\right)\right] \underset{\cos }{(-1) \sin }(m-1) \phi
\end{aligned}
$$

### 8.7 Expansions of the Green's Functions

The Green's function $G\left(r, r^{\prime}\right)=\frac{\exp \left(-j k\left|r-r^{\prime}\right|\right)}{4 \pi\left|r-r^{\prime}\right|}$ can be expanded in terms of spheroidal wave functions.

$$
\begin{align*}
\frac{\exp \left(-j k\left|r-r^{\prime}\right|\right)}{4 \pi\left|r-r^{\prime}\right|}= & \frac{-j k}{2 \pi} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{2-\delta_{0 m}}{N_{m n}} S_{m n}(c, \eta) S_{m n}\left(c, \eta^{\prime}\right)  \tag{8.45}\\
& \cos m\left(\phi-\phi^{\prime}\right) \begin{cases}R_{m n}^{(1)}(c, \xi) R_{m n}^{(4)}\left(c, \xi^{\prime}\right) & \xi \leq \xi^{\prime} \\
R_{m n}^{(4)}(c, \xi) R_{m n}^{(1)}\left(c, \xi^{\prime}\right) & \xi \geq \xi^{\prime}\end{cases}
\end{align*}
$$

### 8.8 Expansions of the Radiation from Hertzian Dipole

The radiation fields of a dipole which is located at the point $\mathbf{r}_{0}$ in free space can be derived with the aid of magnetic vector potential $\mathbf{A}$ and electric vector potential $\mathbf{F}$ or from Hertz Vectors $\boldsymbol{\Pi}^{e}$ and $\boldsymbol{\Pi}^{m}$ at observation point $\mathbf{r}$.

$$
\begin{gather*}
\mathbf{E}=\nabla \times \nabla \times \boldsymbol{\Pi}^{e}, \quad \mathbf{H}=j \omega \epsilon_{0} \nabla \times \boldsymbol{\Pi}^{e}  \tag{8.46}\\
\mathbf{E}=-j \omega \mu_{0} \nabla \times \boldsymbol{\Pi}^{m}, \quad \mathbf{H}=\nabla \times \nabla \times \boldsymbol{\Pi}^{m} \tag{8.47}
\end{gather*}
$$

where $\boldsymbol{\Pi}^{e}=\frac{1}{j \omega \mu_{0} \epsilon_{0}} \mathbf{A}, \boldsymbol{\Pi}^{m}=\frac{1}{j \omega \mu_{0} \epsilon_{0}} \mathbf{F}$ here the first set of formulas pertain to an electric dipole and the second set to a magnetic dipole. The rectangular components of the Hertz vectors are defined by

$$
\begin{equation*}
\Pi_{\times}^{e}=\frac{e^{-j k\left|r-r_{0}\right|}}{4 \pi \epsilon_{0}\left|r-r_{0}\right|} p_{\times}, \quad \Pi_{\times}^{m}=\frac{e^{-j k\left|r-r_{0}\right|}}{4 \pi\left|r-r_{0}\right|} m_{\times}, \quad(\times=x, y, z) \tag{8.48}
\end{equation*}
$$

where $p_{\times}$and $m_{\times}$are the components of the electric and magnetic dipole moments along the $\times$ - axis. But the radial Hertz vector associated with an electric dipole of the moment $p_{r}$ or $m_{r}$ in radial direction is

$$
\begin{equation*}
\Pi_{r}^{e}=\frac{r e^{-j k\left|r-r_{0}\right|}}{4 \pi \epsilon_{0}\left|r-r_{0}\right|} p_{r}, \quad \Pi_{r}^{m}=\frac{r e^{-j k\left|r-r_{0}\right|}}{4 \pi\left|r-r_{0}\right|} m_{r} \tag{8.49}
\end{equation*}
$$

Let us consider an electric dipole oriented in the z-direction, $\mathbf{a}_{z}$, and lying at the point $r_{0}, \theta_{0}, 0$, with spheroidal coordinates $\eta_{0}, \xi_{0}, 0$. The electromagnetic
field of this dipole can be expanded in the spheroidal coordinate systems when observation point is $\xi \leq \xi_{0}$ :

$$
\begin{align*}
& \mathbf{E}=\left(\omega \mu I_{0} \Delta l\right) \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a_{m n}^{(4)} \mathbf{N}_{e m n}^{z(1)}(c ; \eta, \xi, \phi)  \tag{8.50}\\
& \mathbf{H}=\left(j k I_{0} \Delta l\right) \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a_{m n}^{(4)} \mathbf{M}_{e m n}^{z(1)}(c ; \eta, \xi, \phi) \tag{8.51}
\end{align*}
$$

and for $\xi_{0} \leq \xi$ the fields are:

$$
\begin{align*}
& \mathbf{E}=\left(\omega \mu I_{0} \Delta l\right) \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a_{m n}^{(1)} \mathbf{N}_{e m n}^{z(4)}(c ; \eta, \xi, \phi)  \tag{8.52}\\
& \mathbf{H}=\left(j k I_{0} \Delta l\right) \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a_{m n}^{(1)} \mathbf{M}_{e m n}^{z(4)}(c ; \eta, \xi, \phi) \tag{8.53}
\end{align*}
$$

In these equations $k$ is the wave number of the medium, $I_{0}$ antenna current, $\Delta l$ antenna length and $a_{m n}^{(1),(4)}\left(c ; \eta_{0}, \xi_{0}, 0\right)$ are:

$$
\begin{equation*}
a_{m n}^{(1),(4)}\left(c ; \eta_{0}, \xi_{0}, 0\right)=\frac{\left(2-\delta_{0 m}\right)}{2 \pi N_{m n}} S_{m n}\left(c, \eta_{0}\right) R_{m n}^{(4),(1)}\left(c, \xi_{0}\right) \tag{8.54}
\end{equation*}
$$

The components of $\mathbf{M}_{e m n}^{z(g)}$ will be:

$$
\begin{gather*}
M_{e m n \eta}^{z(g)}=\frac{m \eta}{f \sqrt{\left(\xi^{2}-\eta^{2}\right)\left(1-\eta^{2}\right)}} S_{m n} R_{m n}^{(g)} \sin m \phi  \tag{8.55}\\
M_{e m n \xi}^{z(g)}=\frac{-m \xi}{f \sqrt{\left(\xi^{2}-\eta^{2}\right)\left(\xi^{2}-1\right)}} S_{m n} R_{m n}^{(g)} \sin m \phi  \tag{8.56}\\
M_{e m n \phi}^{z(g)}=\frac{\sqrt{\left(1-\eta^{2}\right)\left(\xi^{2}-1\right)}}{f\left(\xi^{2}-\eta^{2}\right)}\left[\eta \frac{d}{d \eta} S_{m n} R_{m n}^{(g)}-\xi S_{m n} \frac{d}{d \xi} R_{m n}^{(g)}\right] \cos m \phi \tag{8.57}
\end{gather*}
$$

and the $\mathbf{N}_{e m n}^{z(g)}$ component can be written as:

$$
\begin{align*}
N_{e m n \eta}^{z(g)} & =\frac{1}{k f^{2}} \sqrt{\frac{1-\eta^{2}}{\xi^{2}-\eta^{2}}}\left[\eta \frac{d}{d \eta} S_{m n} \frac{\partial}{\partial \xi}\left(\frac{\xi^{2}-1}{\xi^{2}-\eta^{2}} R_{m n}^{(g)}\right)\right.  \tag{8.58}\\
& -S_{m n} \frac{\partial}{\partial \xi}\left(\frac{\xi\left(\xi^{2}-1\right)}{\xi^{2}-\eta^{2}} \frac{d}{d \xi} R_{m n}^{(g)}\right) \\
& \left.+\frac{m^{2} \xi}{\left(1-\eta^{2}\right)\left(\xi^{2}-1\right)} S_{m n} R_{m n}^{(g)}\right] \cos m \phi
\end{align*}
$$

$$
\begin{align*}
N_{e m n \xi}^{z(g)} & =\frac{1}{k f^{2}} \sqrt{\frac{\xi^{2}-1}{\xi^{2}-\eta^{2}}}\left[\xi \frac{\partial}{\partial \eta}\left(\frac{1-\eta^{2}}{\xi^{2}-\eta^{2}} S_{m n}\right) \frac{d}{d \xi} R_{m n}^{(g)}\right.  \tag{8.59}\\
& -\frac{\partial}{\partial \eta}\left(\frac{\eta\left(1-\eta^{2}\right)}{\xi^{2}-\eta^{2}} \frac{d}{d \eta} S_{m n}\right) R_{m n}^{(g)} \\
& \left.+\frac{m^{2} \eta}{\left(1-\eta^{2}\right)\left(\xi^{2}-1\right)} S_{m n} R_{m n}^{(g)}\right] \cos m \phi \\
N_{e m n \phi}^{z(g)} & =-\frac{m}{k f^{2}} \frac{\sqrt{\left(1-\eta^{2}\right)\left(\xi^{2}-1\right)}}{\left(\xi^{2}-\eta^{2}\right)}\left[\frac{\xi}{\xi^{2}-1} \frac{d}{d \eta} S_{m n} R_{m n}^{(g)}\right.  \tag{8.60}\\
& \left.+\frac{\eta}{1-\eta^{2}} S_{m n} \frac{d}{d \xi} R_{m n}^{(g)}\right] \sin m \phi
\end{align*}
$$

### 8.9 Useful Spheroidal Integral

Because the Associated Legendre polynomials form a complete orthogonal system over the interval-1,1, any function may be expanded in terms of them as

$$
\begin{align*}
f(x) & =\sum_{n=0}^{\infty} a_{n} P_{n}^{m}(x)  \tag{8.61}\\
a_{n} & =\frac{(2 n+1)(n-m)!}{2(n+m)!} \int_{-1}^{1} f(x) P_{n}^{m}(x) d x
\end{align*}
$$

In the following two equation $n, n^{\prime} \geq m \geq 0$ and $k>m-n$ :

$$
\begin{gather*}
\int_{-1}^{1} S_{n}^{m}(c, \eta) S_{n^{\prime}}^{m}(c, \eta) d \eta=\frac{2}{2 n+1} \frac{(n+m)!}{(n-m)!} \delta_{n, n^{\prime}},  \tag{8.62}\\
\int_{-1}^{1} S_{n}^{m}(c, \eta) P_{n+k}^{m}(\eta) d \eta=\frac{2}{2 n+2 k+1} \frac{(n+m+k)!}{(n-m+k)!} d_{k}^{m n} \delta_{0, k \bmod 2}  \tag{8.63}\\
=\frac{\int_{-1}^{1}\left(1-x^{2}\right)^{\rho-1} P_{\nu}^{m}(x) d x}{\Gamma\left(1+\frac{1}{2}(\nu-m)\right) \Gamma\left(\rho-\frac{1}{2} \nu\right) \Gamma\left(\frac{1}{2}-\frac{1}{2}(\nu+m)\right) \Gamma\left(1+\rho+\frac{1}{2} \nu\right)} \tag{8.64}
\end{gather*}
$$

provided that $2 \Re(\rho)>|\Re(m)|$.
In order to find the unknown coefficients, the following formula are needed. For $m \geq 1$,

$$
\begin{align*}
& \sum_{t=0}^{\infty} I_{t, 1}^{m n}(c) \cdot P_{m-1+t}^{m-1}(\eta)=\left(1-\eta^{2}\right)^{-1 / 2} S_{n}^{m}(c, \eta)  \tag{8.65}\\
& I_{t, 1}^{m n}=\left\{\begin{array}{lc}
(2 t+2 m-1) \sum_{r=t}^{\infty^{\prime}} d_{r}^{m n}(c) & (n-m)+t \text { even } \\
0 & (n-m)+t \text { odd }
\end{array}\right.  \tag{8.66}\\
& \sum_{t=0}^{\infty} I_{t, 2}^{m n}(c) \cdot P_{m-1+t}^{m-1}(\eta)=\left(1-\eta^{2}\right)^{1 / 2} S_{n}^{m}(c, \eta)  \tag{8.67}\\
& I_{t, 2}^{m n}(c)= \begin{cases}\frac{(t+2 m-1)(t+2 m)}{2 t+2 m+1} d_{t}^{m n}-\frac{t(t-1)}{2 t+2 m-3} d_{t-2}^{m n} & (n-m)+t \text { even } \\
0 & (n-m)+t \text { odd }\end{cases}  \tag{8.68}\\
& \sum_{t=0}^{\infty} I_{t, 3}^{m n}(c) \cdot P_{m-1+t}^{m-1}(\eta)=\left(1-\eta^{2}\right)^{3 / 2} S_{n}^{m}(c, \eta) \tag{8.69}
\end{align*}
$$

for odd $(n-m)+t$ the $I_{t, 3}^{m n}=0$ but for even $(n-m)+t$ we have

$$
\begin{align*}
I_{t, 3}^{m n}(c) & =\frac{(t+2 m-1)(t+2 m)(t+2 m+1)(t+2 m+2)}{(2 t+2 m+1)(2 t+2 m+3)}  \tag{8.70}\\
& \times\left[\frac{d_{t}^{m n}}{2 t+2 m+1}-\frac{d_{t+2}^{m n}}{2 t+2 m+5}\right] \\
& -\frac{2 t(t-1)(t+2 m)(t+2 m-1)}{(2 t+2 m-3)(2 t+2 m+1)} \cdot\left[\frac{d_{t-2}^{m n}}{2 t+2 m-3}\right. \\
& \left.-\frac{d_{t}^{m n}}{2 t+2 m+1}\right]+\frac{t(t-1)(t-2)(t-3)}{(2 t+2 m-3)(2 t+2 m-5)} \\
& \times\left[\frac{d_{t-4}^{m n}}{2 t+2 m-7}-\frac{d_{t-2}^{m n}}{2 t+2 m-3}\right]
\end{align*}
$$

$$
\begin{equation*}
\sum_{t=0}^{\infty} I_{t, 4}^{m n}(c) \cdot P_{m-1+t}^{m-1}(\eta)=\left(1-\eta^{2}\right)^{5 / 2} S_{n}^{m}(c, \eta) \tag{8.71}
\end{equation*}
$$

for odd $(n-m)+t$ the $I_{t, 4}^{m n}=0$ but for even $(n-m)+t$ we have

$$
\begin{align*}
I_{t, 4}^{m n}(c) & =I_{t, 3}^{m n}-\left\{\frac{(t+2 m-1)(t+2 m)(t+2 m+1)(t+2 m+2)}{(2 t+2 m+5)(2 t+2 m+1)}\right.  \tag{8.72}\\
& \times \frac{t+2 m+3}{2 t+2 m+3}\left[\frac{(t+1) d_{t}^{m n}}{(2 t+2 m+1)(2 t+2 m+3)}\right. \\
& \left.+\frac{(2 m+1) d_{t+2}^{m n}}{(2 t+2 m+3)(2 t+2 m+7)}-\frac{(2 m+t+4) d_{t+4}^{m n}}{(2 t+2 m+7)(2 t+2 m+9)}\right] \\
& -\frac{t(t-2 m)(t+2 m-1)(t+2 m)(t+2 m+1)}{(2 t+2 m-3)(2 t+2 m+1)(2 t+2 m+3)} \\
& \times\left[\frac{(t-1) d_{t-2}^{m n}}{(2 t+2 m-3)(2 t+2 m-1)}+\frac{(2 m+1) d_{t}^{m n}}{(2 t+2 m-1)(2 t+2 m+3)}\right. \\
& \left.-\frac{(t+2 m+2) d_{t+2}^{m n}}{(2 t+2 m+3)(2 t+2 m+5)}\right] \\
& -\frac{t(t-1)(t-2)(t+2 m-1)(t+4 m-1)}{(2 t+2 m-5)(2 t+2 m-3)(2 t+2 m+1)} \\
& \times\left[\frac{(t-3) d_{t-4}^{m n}}{(2 t+2 m-7)(2 t+2 m-5)}+\frac{(2 m+1) d_{t-2}^{m n}}{(2 t+2 m-5)(2 t+2 m-1)}\right. \\
& \left.-\frac{(2 m+t) d_{t}^{m n}}{(2 t+2 m-1)(2 t+2 m+1)}\right] \\
& +\frac{t(t-1)(t-2)(t-3)(t-4)}{(2 t+2 m-7)(2 t+2 m-5)(2 t+2 m-3)} \\
& \times\left[\frac{(t-5) d_{t-6}^{m n}}{(2 t+2 m-11)(2 t+2 m-9)}+\frac{(2 m+1) d_{t-4}^{m n}}{(2 t+2 m-9)(2 t+2 m-5)}\right. \\
& \left.\left.-\frac{(2 m+t-2) d_{t-2}^{m n}}{(2 t+2 m-5)(2 t+2 m-3)}\right]\right\} \\
& \sum_{t=0}^{\infty} I_{t, 5}^{m n}(c) \cdot P_{m-1+t}^{m-1}(\eta)=\eta\left(1-\eta^{2}\right)^{-1 / 2} S_{n}^{m}(c, \eta)
\end{align*}
$$

for even $(n-m)+t$ the $I_{t, 5}^{m n}=0$ but for odd $(n-m)+t$ we have

$$
\begin{gather*}
I_{t, 5}^{m n}(c)=t d_{t-1}^{m n}+(2 t+2 m-1) \sum_{r=t}^{\infty} d_{r}^{m n}(c)  \tag{8.74}\\
\sum_{t=0}^{\infty} I_{t, 6}^{m n}(c) \cdot P_{m-1+t}^{m-1}(\eta)=\eta\left(1-\eta^{2}\right)^{1 / 2} S_{n}^{m}(c, \eta) \tag{8.75}
\end{gather*}
$$

for even $(n-m)+t$ the $I_{t, 6}^{m n}=0$ but for odd $(n-m)+t$ we have

$$
\begin{align*}
I_{t, 6}^{m n}(c)= & \frac{(t+2 m-1)(t+2 m)}{2 t+2 m+1} \cdot\left[\frac{(t+2 m+1) d_{t+1}^{m n}}{2 t+2 m+3}+\frac{t \cdot d_{t-1}^{m n}}{2 t+2 m-1}\right]  \tag{8.76}\\
- & \frac{t(t-1)}{2 t+2 m-3} \cdot\left[\frac{(t+2 m-1) d_{t-1}^{m n}}{2 t+2 m-1}+\frac{(t-2) d_{t-3}^{m n}}{2 t+2 m-5}\right] \\
& \sum_{t=0}^{\infty} I_{t, 7}^{m n}(c) \cdot P_{m-1+t}^{m-1}(\eta)=\eta\left(1-\eta^{2}\right)^{3 / 2} S_{n}^{m}(c, \eta) \tag{8.77}
\end{align*}
$$

for even $(n-m)+t$ the $I_{t, 7}^{m n}=0$ but for $\operatorname{odd}(n-m)+t$ we have:

$$
\begin{align*}
I_{t, 7}^{m n}(c) & =\frac{(t+2 m-1)(t+2 m)(t+2 m+1)(t+2 m+2)(t+2 m+3)}{(2 t+2 m+1)(2 t+2 m+3)(2 t+2 m+5)}  \tag{8.78}\\
& \times\left[\frac{d_{t+1}^{m n}}{2 t+2 m+3}-\frac{d_{t+3}^{m n}}{2 t+2 m+7}\right] \\
& -\frac{t(t-2 m)(t+2 m)(t+2 m)(t+2 m+1)}{(2 t+2 m-3)(2 t+2 m+1)(2 t+2 m+3)} \\
& \times\left[\frac{d_{t-1}^{m n}}{2 t+2 m-1}-\frac{d_{t+1}^{m n}}{2 t+2 m+3}\right] \\
& -\frac{t(t-1)(t-2)(t+2 m-1)(t+4 m-1)}{(2 t+2 m-5)(2 t+2 m-3)(2 t+2 m+1)} \cdot\left[\frac{d_{t-3}^{m n}}{2 t+2 m-5}\right. \\
& \left.-\frac{d_{t-1}^{m n}}{2 t+2 m-1}\right]+\frac{t(t-1)(t-2)(t-3)(t-4)}{(2 t+2 m-7)(2 t+2 m-5)(2 t+2 m-3)} \\
& \times\left[\frac{d_{t-5}^{m n}}{2 t+2 m-9}-\frac{d_{t-3}^{m n}}{2 t+2 m-5}\right] \\
& \sum_{t=0}^{\infty} I_{t, 8}^{m n}(c) \cdot P_{m-1+t}^{m-1}(\eta)=\left(1-\eta^{2}\right)^{1 / 2} \frac{d S_{n}^{m}(c, \eta)}{d \eta} \tag{8.79}
\end{align*}
$$

for even $(n-m)+t$ the $I_{t, 8}^{m n}=0$ but for $o d d(n-m)+t$ we have:

$$
\begin{equation*}
I_{t, 8}^{m n}(c)=-t(t+m-1) d_{t-1}^{m n}+m(2 t+2 m-1) \sum_{r=t+1}^{\infty} d_{r}^{m n}(c) \tag{8.80}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{t=0}^{\infty} I_{t, 10}^{m n}(c) \cdot P_{m-1+t}^{m-1}(\eta)=\left(1-\eta^{2}\right)^{5 / 2} \frac{d S_{n}^{m}(c, \eta)}{d \eta} \tag{8.81}
\end{equation*}
$$

for even $(n-m)+t$ the $I_{t, 10}^{m n}=0$ but for $o d d(n-m)+t$ we have:

$$
\begin{align*}
& I_{t, 10}^{m n}(c)=I_{t, 9}^{m n}(c)-\left\{\frac{(t+2 m-1)(t+2 m)(t+2 m+1)}{(2 t+2 m+1)(2 t+2 m+3)(2 t+2 m+5)}\right.  \tag{8.82}\\
& \times\left[\frac{(t+1)(t+2 m+2)(t+2 m+3)}{2(2 t+2 m+3)} d_{t+1}^{m n}\right. \\
&+\frac{(t+2)(t+3)(t+2 m+1)}{2(2 t+2 m+3)} d_{t+1}^{m n} \\
&\left.+\frac{(t+2 m+2)(t+2 m+3)(t+2 m+4)}{2 t+2 m+7} d_{t+3}^{m n}\right] \\
&-\frac{t(t-2 m)(t+2 m-1)}{(2 t+2 m-3)(2 t+2 m+1)(2 t+2 m+3)} \\
& \times\left[\frac{t(t+1)(t+2 m-1)+(t-1)(t+2 m)(t+2 m+1)}{2(2 t+2 m-1)} d_{t-1}^{m n}\right. \\
&\left.\left.+\frac{(t+2 m)(t+2 m+1)(t+m+2)}{(2 t+2 m+3)} d_{t+1}^{m n}\right]\right\} \\
&+\frac{m(t+2)(t+2 m-1)(t+2 m)(t+2 m+1)}{(2 t+2 m+1)(2 t+2 m+3)^{2}} d_{t+1}^{m n} \\
&+\frac{t(t-1)(t-2)(t+4 m-1)(t+2 m-1)(t+m)}{(t+2 m-5)(2 t+2 m-3)(2 t+2 m-1)(2 t+2 m+1)} d_{t-1}^{m n} \\
&-\frac{m t(t+2 m-1)}{2 t+2 m+3} \cdot\left[\frac{t+1}{2 t+2 m+1}\right. \\
&+\frac{t(t-1)(t-2)(t-3)(t+4 m-1)(t+2 m-1)}{2(2 t+2 m-3)(2 t+2 m+1)(2 t+2 m-5)^{2}} d_{t-3}^{m n} \\
&-\frac{t(t-1)(t-2)}{(2 t+2 m-3)(2 t+2 m-5)^{2}} \cdot\left[\frac{(t-3)(t-4)(t+m-2)}{2 t+2 m-7}+m(t-2)\right] d_{t-3}^{m n} \\
&+\frac{t(t-1)(t-2)(t+2 m-3)}{(2 t+2 m-5)(2 t+2 m-3)(2 t+2 m+1)} \\
& \times\left[\frac{(t-2)(t-1)+4 m(2 m-1)+(t-2 m-1)(2 m+1)}{2(2 t+2 m-5)} d_{t-3}^{m n}\right. \\
&+2 m-9)(2 t+2 m-7)(2 t+2 m-5)(2 t+2 m-3) \\
& d_{t-5}^{m n}
\end{align*}
$$

$$
\begin{equation*}
\sum_{t=0}^{\infty} I_{t, 11}^{m n}(c) \cdot P_{m-1+t}^{m-1}(\eta)=\eta\left(1-\eta^{2}\right)^{1 / 2} \frac{d S_{n}^{m}(c, \eta)}{d \eta} \tag{8.83}
\end{equation*}
$$

for odd $(n-m)+t$ the $I_{t, 11}^{m n}=0$ but for even $(n-m)+t$ we have:

$$
\begin{align*}
I_{t, 11}^{m n}(c)= & -\frac{t(t-1)(t+m-2)}{2 t+2 m-3} d_{t-2}^{m n}+m(2 t+2 m-1) \sum_{r=t+2}^{\infty} d_{r}^{m n}(c)  \tag{8.84}\\
- & {\left[\frac{t(t-1)}{2}+\frac{(t+2 m)(t+2 m-1)}{2(2 t+2 m+1)}\right] d_{t}^{m n} } \\
& \sum_{t=0}^{\infty} I_{t, 12}^{m n}(c) \cdot P_{m-1+t}^{m-1}(\eta)=\eta\left(1-\eta^{2}\right)^{3 / 2} \frac{d S_{n}^{m}(c, \eta)}{d \eta} \tag{8.85}
\end{align*}
$$

for odd $(n-m)+t$ the $I_{t, 12}^{m n}=0$ but for even $(n-m)+t$ we have:

$$
\begin{aligned}
I_{t, 12}^{m n}(c) & =\frac{(t+2 m-1)(t+2 m)}{(2 t+2 m+1)(2 t+2 m+3)} \cdot\left[\left(\frac{t(t+2 m+1)(t+2 m+2)}{2(2 t+2 m+1)} .86\right)\right. \\
& \left.+\frac{(t+1)(t+2)(t+2 m)}{2(2 t+2 m+1)}\right) d_{t}^{m n} \\
& \left.+\frac{(t+m+3)(t+2 m+1)(t+2 m+2)}{2 t+2 m+5} d_{t+2}^{m n}\right] \\
& -\frac{2 t(t-1)}{(2 t+2 m-3)(2 t+2 m+1)} \cdot\left[\left(\frac{(t-2)(t+2 m-1)(t+2 m)}{2(2 t+2 m-3)}\right.\right. \\
& \left.+\frac{t(t-1)(t+2 m-2)}{2(2 t+2 m-3)}\right) d_{t-2}^{m n} \\
& -\frac{m(t+1)(t+2 m-1)(t+2 m)}{(2 t+2 m-5)(2 t+2 m-3)^{2}} d_{t}^{m n} \\
& -\frac{m t(t-1)}{(2 t+2 m+1)} \cdot\left[1-\frac{2(t-1)(2 t+2 m-1)}{(2 t+2 m-3)^{2}}\right] d_{t-2}^{m n} \\
& +\frac{t(t-1)(t-2)(t-3)(t+m-4)}{(2 t+2 m-7)(2 t+2 m-5)(2 t+2 m-3)} d_{t-4}^{m n}
\end{aligned}
$$

For $m=0$ we need special formulas

$$
\begin{equation*}
\sum_{t=0}^{\infty} I_{t, 2}^{0 n}(c) \cdot P_{1+t}^{1}(\eta)=\left(1-\eta^{2}\right)^{1 / 2} S_{n}^{0}(c, \eta) \tag{8.87}
\end{equation*}
$$

for odd $(n+t)$ the $I_{t, 2}^{0 n}=0$ but for even $(n+t)$ we have:

$$
\begin{gather*}
I_{t, 2}^{0 n}(c)=\frac{d_{t}^{0 n}}{2 t+1}-\frac{d_{t+2}^{0 n}}{2 t+5}  \tag{8.88}\\
\sum_{t=0}^{\infty} I_{t, 3}^{0 n}(c) \cdot P_{1+t}^{1}(\eta)=\left(1-\eta^{2}\right)^{3 / 2} S_{n}^{0}(c, \eta) \tag{8.89}
\end{gather*}
$$

for odd $(n+t)$ the $I_{t, 3}^{0 n}=0$ but for even $(n+t)$ we have

$$
\begin{align*}
I_{t, 3}^{0 n}(c)= & \frac{(t+3)(t+4)}{2 t+5}\left[\frac{d_{t}^{0 n}}{(2 t+1)(2 t+3)}-\frac{2 d_{t+2}^{0 n}}{(2 t+3)(2 t+7)}\right.  \tag{8.90}\\
+ & \left.\frac{d_{t+4}^{0 n}}{(2 t+7)(2 t+9)}\right]-\frac{t(t-1)}{2 t+1} \\
\times & {\left[\frac{d_{t-2}^{0 n}}{(2 t-3)(2 t-1)}-\frac{2 d_{t}^{0 n}}{(2 t-1)(2 t+3)}+\frac{d_{t+2}^{0 n}}{(2 t+3)(2 t+5)}\right] } \\
& \sum_{t=0}^{\infty} I_{t, 4}^{0 n}(c) \cdot P_{1+t}^{1}(\eta)=\left(1-\eta^{2}\right)^{5 / 2} S_{n}^{0}(c, \eta) \tag{8.91}
\end{align*}
$$

for odd $(n+t)$ the $I_{t, 4}^{0 n}=0$ but for even $(n+t)$ we have

$$
\begin{align*}
I_{t, 4}^{0 n}(c) & =I_{t, 3}^{0 n}(c)-\frac{(t+3)(t+4)(t+5)(t+6)}{(2 t+5)(2 t+7)(2 t+9)}  \tag{8.92}\\
& \times\left[\frac{d_{t+2}^{00}}{(2 t+5)(2 t+7)}-\frac{2 d_{t+4}^{0 n}}{(2 t+7)(2 t+11)}+\frac{d_{t+6}^{0 n}}{(2 t+11)(2 t+13)}\right] \\
& -\frac{(t+3)(t+4)}{2 t+5} \cdot\left[\frac{(t+1)(t+5)}{(2 t+5)(2 t+7)}+\frac{3 t}{(2 t+1)(2 t+5)}\right] \\
& \times\left[\frac{d_{t}^{0 n}}{(2 t+1)(2 t+3)}-\frac{2 d_{t+2}^{0 n}}{(2 t+3)(2 t+7)}+\frac{d_{t+4}^{0 n}}{(2 t+7)(2 t+9)}\right] \\
& -\frac{t(t-1)}{2 t+1} \cdot\left[\frac{3(t+3)}{(2 t+1)(2 t+5)}-\frac{(t+2)(t-2)}{(2 t+1)(2 t-1)}\right] \\
& \times\left[\frac{d_{t-2}^{00}}{(2 t-3)(2 t-1)}-\frac{2 d_{t}^{0 n}}{(2 t+3)(2 t-1)}+\frac{d_{t+2}^{0 n}}{(2 t+3)(2 t+5)}\right] \\
& +\frac{t(t-1)(t-2)(t-3)}{(2 t+1)(2 t-1)(2 t-3)} \\
& \times\left[\frac{d_{t-4}^{0 n}}{(2 t-7)(2 t-5)}-\frac{2 d_{t-2}^{0 n}}{(2 t-1)(2 t-5)}+\frac{d_{t}^{0 n}}{(2 t+1)(2 t-1)}\right]
\end{align*}
$$

$$
\begin{equation*}
\sum_{t=0}^{\infty} I_{t, 6}^{0 n}(c) \cdot P_{1+t}^{1}(\eta)=\eta\left(1-\eta^{2}\right)^{1 / 2} S_{n}^{0}(c, \eta) \tag{8.93}
\end{equation*}
$$

for even $(n+t)$ the $I_{t, 6}^{0 n}=0$ but for odd $(n+t)$ we have

$$
\begin{gather*}
I_{t, 6}^{0 n}(c)=\frac{t+3}{2 t+5}\left[\frac{d_{t+1}^{0 n}}{2 t+3}-\frac{d_{t+3}^{0 n}}{2 t+7}\right]+\frac{t}{2 t+1}\left[\frac{d_{t-1}^{0 n}}{2 t-1}-\frac{d_{t+1}^{0 n}}{2 t+3}\right]  \tag{8.94}\\
\sum_{t=0}^{\infty} I_{t, 7}^{0 n}(c) \cdot P_{1+t}^{1}(\eta)=\eta\left(1-\eta^{2}\right)^{3 / 2} S_{n}^{0}(c, \eta) \tag{8.95}
\end{gather*}
$$

for even $(n+t)$ the $I_{t, 7}^{0 n}=0$ but for odd $(n+t$ we have

$$
\begin{align*}
I_{t, 7}^{0 n}(c)= & \frac{(t+3)(t+4)(t+5)}{(2 t+5)(2 t+7)}  \tag{8.96}\\
\times & {\left[\frac{d_{t+1}^{0 n}}{(2 t+3)(2 t+5)}-\frac{2 d_{t+3}^{0 n}}{(2 t+5)(2 t+9)}+\frac{d_{t+5}^{0 n}}{(2 t+9)(2 t+11)}\right] } \\
+ & \frac{3 t(t+3)}{(2 t+1)(2 t+5)} \\
\times & {\left[\frac{d_{t-1}^{0 n}}{(2 t-1)(2 t+1)}-\frac{2 d_{t+1}^{0 n}}{(2 t+1)(2 t+5)}+\frac{d_{t+3}^{0 n}}{(2 t+5)(2 t+7)}\right] } \\
- & \frac{t(t-1)(t-2)}{(2 t-1)(2 t+1)} \\
\times & {\left[\frac{d_{t-3}^{0 n}}{(2 t-5)(2 t-3)}-\frac{2 d_{t-1}^{0 n}}{(2 t-3)(2 t+1)}+\frac{d_{t+1}^{0 n}}{(2 t+1)(2 t+3)}\right] } \\
& \sum_{t=0}^{\infty} I_{t, 8}^{0 n}(c) \cdot P_{1+t}^{1}(\eta)=\left(1-\eta^{2}\right)^{1 / 2} \frac{d S_{n}^{0}(c, \eta)}{d \eta} \tag{8.97}
\end{align*}
$$

for even $(n+t)$ the $I_{t, 8}^{0 n}=0$ but for odd $(n+t)$ we have

$$
\begin{align*}
I_{t, 8}^{0 n}(c) & =d_{t+1}^{0 n}  \tag{8.98}\\
\sum_{t=0}^{\infty} I_{t, 9}^{0 n}(c) \cdot P_{1+t}^{1}(\eta) & =\left(1-\eta^{2}\right)^{3 / 2} \frac{d S_{n}^{0}(c, \eta)}{d \eta} \tag{8.99}
\end{align*}
$$

for even $(n+t)$ the $I_{t, 9}^{0 n}=0$ but for odd $(n+t)$ we have

$$
\begin{align*}
& I_{t, 9}^{0 n}(c)=\frac{(t+3)(t+4)}{2 t+5}\left[\frac{d_{t+1}^{0 n}}{2 t+3}-\frac{d_{t+3}^{0 n}}{2 t+7}\right]  \tag{8.100}\\
&-\frac{t(t-1)}{2 t+1}\left[\frac{d_{t-1}^{0 n}}{2 t-1}-\frac{d_{t+1}^{0 n}}{2 t+3}\right] \\
& \sum_{t=0}^{\infty} I_{t, 10}^{0 n}(c) \cdot P_{1+t}^{1}(\eta)=\left(1-\eta^{2}\right)^{5 / 2} \frac{d S_{n}^{0}(c, \eta)}{d \eta} \tag{8.101}
\end{align*}
$$

for even $(n+t)$ the $I_{t, 10}^{0 n}=0$ but for odd $(n+t)$ we have

$$
\begin{align*}
I_{t, 10}^{0 n}(c) & =I_{t, 9}^{0 n}(c)-\frac{(t+3)(t+4)(t+5)}{(2 t+5)(2 t+7)} \cdot\left[\frac{(t+1) d_{t+1}^{0 n}}{(2 t+3)(2 t+5)}\right.  \tag{8.102}\\
& \left.+\frac{1}{2 t+7}\left(\frac{t+6}{2 t+9}-\frac{t+1}{2 t+5}\right) d_{t+3}^{0 n}-\frac{(t+6) d_{t+5}^{0 n}}{(2 t+11)(2 t+9)}\right] \\
& -\frac{t(t+3)}{2 t+3}\left(\frac{t+4}{2 t+5}-\frac{t-1}{2 t+1}\right) \cdot\left[\frac{(t-1) d_{t-1}^{0 n}}{(2 t-1)(2 t+1)}\right. \\
& \left.+\frac{1}{2 t+3}\left(\frac{t+4}{2 t+5}-\frac{t-1}{2 t+1}\right) d_{t+1}^{0 n}-\frac{(t+4) d_{t+3}^{0 n}}{(2 t+7)(2 t+5)}\right] \\
& +\frac{t(t-1)(t-2)}{(2 t+1)(2 t-1)} \cdot\left[\frac{(t-3) d_{t-3}^{0 n}}{(2 t-5)(2 t-3)}\right. \\
& \left.+\frac{1}{2 t-1}\left(\frac{t+2}{2 t+1}-\frac{t-3}{2 t-3}\right) d_{t-1}^{0 n}-\frac{(t+2) d_{t+1}^{0 n}}{(2 t+3)(2 t+1)}\right] \\
& \sum_{t=0}^{\infty} I_{t, 11}^{0 n}(c) \cdot P_{1+t}^{1}(\eta)=\eta\left(1-\eta^{2}\right)^{1 / 2} \frac{d S_{n}^{0}(c, \eta)}{d \eta} \tag{8.103}
\end{align*}
$$

for odd $\left(n+t\right.$ the $I_{t, 11}^{m n}=0$ but for even $(n+t)$ we have

$$
\begin{gather*}
I_{t, 11}^{m n}=\frac{t+3}{2 t+5} d_{t+2}^{0 n}+\frac{t}{2 t+1} d_{t}^{0 n}  \tag{8.104}\\
\sum_{t=0}^{\infty} I_{t, 12}^{0 n}(c) \cdot P_{1+t}^{1}(\eta)=\eta\left(1-\eta^{2}\right)^{3 / 2} \frac{d S_{n}^{0}(c, \eta)}{d \eta} \tag{8.105}
\end{gather*}
$$

for odd $(n+t)$ the $I_{t, 12}^{0 n}=0$ but for even $(n+t)$ we have

$$
\begin{align*}
I_{t, 12}^{0 n}(c) & =\frac{(t+3)(t+4)(t+5)}{(2 t+5)(2 t+7)} \cdot\left(\frac{d_{t+2}^{0 n}}{2 t+5}-\frac{d_{t+4}^{0 n}}{2 t+9}\right)  \tag{8.106}\\
& +\frac{3 t(t+3)}{(2 t+1)(2 t+5)} \cdot\left(\frac{d_{t}^{0 n}}{2 t+1}-\frac{d_{t+2}^{0 n}}{2 t+5}\right) \\
& -\frac{t(t-1)(t-2)}{(2 t-1)(2 t+1)} \cdot\left(\frac{d_{t-2}^{0 n}}{2 t-3}-\frac{d_{t}^{0 n}}{2 t+1}\right)
\end{align*}
$$

### 8.10 Scattering From a Conducting Spheroid

Figure (8.5) defines the geometry of the problem that we will consider. A plane wave incident in a xz-plane with an angle $\theta_{0}$ between positive z axis and the direction of incidence. For oblique incidence, the polarized incident


Figure 8.5: Geometry of the Conducting Prolate Spheroid
wave is resolved into two components as shown in Fig.(8.5): the TE mode for which the electric vector of incident wave is per perpendicular to the incident plane, and $T M$ mode for which the magnetic vector field is perpendicular to the incident plane. For $\theta_{0} \neq 0$, we will consider two case separately.

For $\theta_{0}=0$, the parallel incident, due to symmetry of configuration, we have identical results for both cases.

$$
\begin{align*}
\mathbf{E}_{T E}^{i} & =\mathbf{a}_{y} e^{-j k\left(x \sin \theta_{0}+z \cos \theta_{0}\right)}  \tag{8.107}\\
\mathbf{H}_{T E}^{i} & =\frac{1}{Z}\left(\mathbf{a}_{x} \cos \theta_{0}-\mathbf{a}_{z} \sin \theta_{0}\right) \mathbf{a}_{y} e^{-j k\left(x \sin \theta_{0}+z \cos \theta_{0}\right)}
\end{align*}
$$

where $Z=\sqrt{\mu / \epsilon}$ is impedance of medium.

$$
\begin{gather*}
\mathbf{E}_{T E}^{i}=\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{i}\left(c, \theta_{0}\right) \mathbf{M}_{e, m n}^{r(1)}(c, \mathbf{r})+b_{m n}^{i}\left(c, \theta_{0}\right) \mathbf{N}_{o, m n}^{r(1)}(c, \mathbf{r})\right]  \tag{8.108}\\
\mathbf{H}_{T E}^{i}=\frac{j}{Z} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{i}\left(c, \theta_{0}\right) \mathbf{N}_{e, m n}^{r(1)}(c, \mathbf{r})+b_{m n}^{i}\left(c, \theta_{0}\right) \mathbf{M}_{o, m n}^{r(1)}(c, \mathbf{r})\right]
\end{gather*}
$$

where

$$
\begin{gather*}
a_{m n}^{i}\left(c, \theta_{0}\right)=\frac{2\left(2-\delta_{0 m}\right) j^{n}}{N_{m n}(c)} \sum_{k=0,1}^{\infty} \alpha_{k}^{m n}\left(c, \theta_{0}\right)  \tag{8.109}\\
b_{m n}^{i}\left(c, \theta_{0}\right)=\frac{4 m j^{n-1}}{N_{m n}(c)} \sum_{k=0,1}^{\prime^{\prime}} \beta_{k}^{m n}\left(c, \theta_{0}\right) \\
\alpha_{k}^{m n}\left(c, \theta_{0}\right)=\frac{d_{k}^{m n}(c)}{(k+m)(k+m+1)} \frac{d}{d \theta_{0}} P_{m+k}^{m}\left(\cos \theta_{0}\right)  \tag{8.110}\\
\beta_{k}^{m n}\left(c, \theta_{0}\right)=\frac{d_{k}^{m n}(c)}{(k+m)(k+m+1)} \frac{P_{m+k}^{m}\left(\cos \theta_{0}\right)}{\sin \theta_{0}} \\
\mathbf{E}_{T E}^{s}=\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{s} \mathbf{M}_{e, m n}^{r(4)}(c, \mathbf{r})+b_{m n}^{s} \mathbf{N}_{o, m n}^{r(4)}(c, \mathbf{r})\right]  \tag{8.111}\\
\mathbf{H}_{T E}^{s}=\frac{j}{Z} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{s} \mathbf{N}_{e, m n}^{r(4)}(c, \mathbf{r})+b_{m n}^{s} \mathbf{M}_{o, m n}^{r(4)}(c, \mathbf{r})\right]
\end{gather*}
$$

The $a_{m n}^{s}$ and $b_{m n}^{s}$ are unknown and can be determined by applying boundary conditions. At $\xi=\xi_{0}$ the tangential electric fields on conducting spheroid should be vanished. Therefore

$$
\begin{align*}
\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{i} M_{e, m n \eta}^{r(1)}\left(c ; \eta, \xi_{0}, \phi\right)\right. & +b_{m n}^{i} N_{o, m n \eta}^{r(1)}\left(c ; \eta, \xi_{0}, \phi\right)+  \tag{8.112}\\
a_{m n}^{s} M_{e, m n \eta}^{r(4)}\left(c ; \eta, \xi_{0}, \phi\right) & \left.+b_{m n}^{s} N_{o, m n \eta}^{r(4)}\left(c ; \eta, \xi_{0}, \phi\right)\right]=0
\end{align*}
$$

$$
\begin{align*}
\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{i} M_{e, m n \phi}^{r(1)}\left(c ; \eta, \xi_{0}, \phi\right)\right. & +b_{m n}^{i} N_{o, m n \phi}^{r(1)}\left(c ; \eta, \xi_{0}, \phi\right)+  \tag{8.113}\\
a_{m n}^{s} M_{e, m n \phi}^{r(4)}\left(c ; \eta, \xi_{0}, \phi\right) & \left.+b_{m n}^{s} N_{o, m n \phi}^{r(4)}\left(c ; \eta, \xi_{0}, \phi\right)\right]=0
\end{align*}
$$

from Eq.(8.112) and Eq.(8.113) we can find unknowns $a_{m n}^{s}$ and $b_{m n}^{s}$.
Because of the orthogonality of the trigonometric functions $\cos m \phi$ and $\sin m \phi$, in each expansion, the coefficients of the same $\phi$-dependent trigonometric function must be equal, component by component; the equalities must hold for each corresponding term in the summation over $m$. For the summation over $n$, however, the individual terms in the series cannot be matched term by term. This is the cause of difficulty in determining the unknown coefficients. We do the same procedure as we did in scattering by elliptic cylinder. The method used is as follows: the equation that stands for continuity of $\eta$ components, Eq.(8.112), are multiplied by $\left(\xi_{0}^{2}-\eta^{2}\right)^{5 / 2}=\left[\left(\xi_{0}^{2}-1\right)+\left(1-\eta^{2}\right)\right]^{5 / 2}$, and the equations for $\phi$-components, Eq. (8.113) , by $\left(\xi_{0}^{2}-1\right)^{-1 / 2}\left(\xi_{0}^{2}-\eta^{2}\right)$, where these multipliers are positive in full range of $\eta$; then are factors that are function of $\eta$ are replaced by the series of associated Legendre functions of the first kind, which are orthogonal functions in the interval $-1 \leq \eta \leq 1$. For the $m \geq 1$, we expand them in terms of $P_{m-1+t}^{m-1}(\eta)$, but for $m=0$ we should expand them by the functions $P_{1+t}^{1}(\eta)$ as we did in previous section. By using the useful integrals of previous section in boundary condition equations, Eq.(8.112) and Eq.(8.113), we will reach:

$$
\begin{equation*}
\sum_{n=0}^{\infty} A_{n t}^{i} a_{m n}^{i}+B_{n t}^{i} b_{m n}^{i}+A_{n t}^{s} a_{m n}^{s}+B_{n t}^{s} b_{m n}^{s}=0 \tag{8.114}
\end{equation*}
$$

for $\eta$-components.

$$
\begin{equation*}
\sum_{n=0}^{\infty} C_{n t}^{i} a_{m n}^{i}+D_{n t}^{i} b_{m n}^{i}+C_{n t}^{s} a_{m n}^{s}+D_{n t}^{s} b_{m n}^{s}=0 \tag{8.115}
\end{equation*}
$$

for $\phi$-components. We truncate the series to Max and index $t$ in both Eq.(8.114),Eq.(8.115) can be changed from $0-\operatorname{Max}$, therefore we have $2 M a x+2$ unknown and $2 M a x+2$ linear equations which can be divide into four different submatrix.

$$
\left[\begin{array}{ll}
\mathbf{A}_{n t}^{(4)} & \mathbf{B}_{n t}^{(4)}  \tag{8.116}\\
\mathbf{C}_{n t}^{(4)} & \mathbf{D}_{n t}^{(4)}
\end{array}\right]\left[\begin{array}{l}
\mathbf{a}_{t}^{s} \\
\mathbf{b}_{t}^{s}
\end{array}\right]+\left[\begin{array}{ll}
\mathbf{A}_{n t}^{(1)} & \mathbf{B}_{n t}^{(1)} \\
\mathbf{C}_{n t}^{(1)} & \mathbf{D}_{n t}^{(1)}
\end{array}\right]\left[\begin{array}{l}
\mathbf{a}_{t}^{i} \\
\mathbf{b}_{t}^{i}
\end{array}\right]=0
$$

with the incident matrix and known coefficients

$$
-\left[\begin{array}{ll}
\mathbf{A}_{n t}^{(1)} & \mathbf{B}_{n t}^{(1)}  \tag{8.117}\\
\mathbf{C}_{n t}^{(1)} & \mathbf{D}_{n t}^{(1)}
\end{array}\right]\left[\begin{array}{c}
\mathbf{a}_{t}^{i} \\
\mathbf{b}_{t}^{i}
\end{array}\right]=\left[\begin{array}{c}
\mathbf{G}_{n}^{i} \\
\mathbf{H}_{n}^{i}
\end{array}\right]
$$

where

$$
\begin{align*}
& U_{m n \eta}^{r(g), t}=m \xi_{0}\left[\left(\xi_{0}^{2}-1\right)^{2} I_{t, 1}^{m n}(c)+2\left(\xi_{0}^{2}-1\right) I_{t, 2}^{m n}(c)+I_{t, 3}^{m n}(c)\right] R_{m n}^{(g)}(c)  \tag{8.118}\\
& V_{m n \eta}^{r(g), t}=\frac{1}{k f}\left\{\left(\xi_{0}^{2}-1\right)^{2}\left[R_{m n}^{(g)}+\xi_{0} \frac{d}{d \xi_{0}} R_{m n}^{(g)}\right] I_{t, 8}^{m n}(c)\right.  \tag{8.119}\\
&\left.+\left[\left(3 \xi_{0}^{2}-1\right) R_{m n}^{(g)}+\xi_{0}\left(\xi_{0}^{2}-1\right) \frac{d}{d \xi_{0}} R_{m n}^{(g)}\right] I_{t, 9}^{m n}(c)\right\} \\
&-\frac{1}{k f}\left\{\left[2 \xi_{0} \frac{d}{d \xi_{0}} R_{m n}^{(g)}+\left(\xi_{0}^{2}-1\right) \frac{d^{2}}{d \xi_{0}^{2}} R_{m n}^{(g)}\right] I_{t, 7}^{m n}(c)\right. \\
&\left.+\left[\left(\xi_{0}^{2}-1\right)^{2} \frac{d^{2}}{d \xi_{0}^{2}} R_{m n}^{(g)}\right] I_{t, 6}^{m n}(c)\right\} \\
&+\frac{m^{2}}{k f}\left[\left(\xi_{0}^{2}-1\right)^{3 / 2} I_{t, 1}^{m n}(c)+\left(\xi_{0}^{2}-1\right)^{-1 / 2} I_{t, 7}^{m n}(c)\right. \\
&\left.+2\left(\xi_{0}^{2}-1\right)^{1 / 2} I_{t, 6}^{m n}(c)\right] R_{m n}^{(g)} \\
& U_{n t \phi}^{r(g)}=\xi_{0} I_{t, 8}^{m n}(c) R_{m n}^{(g)}(c)-I_{t, 6}^{m n}(c) \frac{d}{d \xi_{0}} R_{m n}^{(g)}  \tag{8.120}\\
& V_{n t \phi}^{r(g)}=\frac{m}{k f\left(\xi_{0}^{2}-1\right)}\left\{I_{t, 2}^{m n}(c) R_{m n}^{(g)}(c)+I_{t, 11}^{m n}(c) R_{m n}^{(g)}(c)\right.  \tag{8.121}\\
&\left.-\left(\xi_{0}^{2}-1\right) I_{t, 1}^{m n}(c)\left[R_{m n}^{(g)}+\xi_{0} \frac{d}{d \xi_{0}} R_{m n}^{(g)}\right]\right\}
\end{align*}
$$

We can rewrite the Eq.(8.115) in a general form of linear system of equations $\mathbf{A} \cdot \mathbf{X}=\mathbf{B}$ and find the unknowns.
Now let us do the same procedure for TM polarization.

$$
\begin{align*}
\mathbf{E}_{T M}^{i} & =\left(\mathbf{a}_{x} \cos \theta_{0}-\mathbf{a}_{z} \sin \theta_{0}\right) \mathbf{a}_{y} e^{-j k\left(x \sin \theta_{0}+z \cos \theta_{0}\right)}  \tag{8.122}\\
\mathbf{H}_{T M}^{i} & =\frac{\mathbf{a}_{y}}{Z_{0}} e^{-j k\left(x \sin \theta_{0}+z \cos \theta_{0}\right)}
\end{align*}
$$

for $\theta_{0}=0$ we have $E_{x}=Z_{0} H_{y}=e^{-j k z}$.
The expansions of the incident and scattered electric fields for the case of transverse magnetic $T M$ ) polarization can be written in terms of vector spheroidal wave functions as[64]

$$
\begin{equation*}
\mathbf{E}_{T M}^{i}=-j \sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{i}\left(c, \theta_{0}\right) \mathbf{N}_{e, m n}^{r(1)}(c, \mathbf{r})+b_{m n}^{i}\left(c, \theta_{0}\right) \mathbf{M}_{o, m n}^{r(1)}(c, \mathbf{r})\right] \tag{8.123}
\end{equation*}
$$

and corresponding magnetic filed can be found by $\nabla \times \mathbf{E}=-\frac{1}{j \omega \mu} \mathbf{H}$

$$
\begin{equation*}
\mathbf{E}_{T M}^{s}=\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{s} \mathbf{N}_{e, m n}^{r(4)}(c, \mathbf{r})+b_{m n}^{s} \mathbf{M}_{o, m n}^{r(4)}(c, \mathbf{r})\right] \tag{8.124}
\end{equation*}
$$

by the same procedure that we did for TE polarization we can find the unknown coefficients.

### 8.11 Scattering From a PEMC Spheroid

In this section, we consider the scattering of electromagnetic waves from a perfect electromagnetic conducting spheroid, when it is excited by a plane wave of arbitrary polarization and angle of incidence. We express the incident and scattered electromagnetic fields in terms of vector spheroidal wave functions and then imposing the appropriate boundary conditions on the surface of the spheroid[64].
A perfect electromagnetic conducting (PEMC) medium can be considered as a generalized form of a perfect electric conducting (PEC) medium and a perfect magnetic conducting (PMC) medium, in which certain linear combinations of electromagnetic fields become extinct. Since the tangential components of the electric $\mathbf{E}$ ) and magnetic $\mathbf{H}$ ) fields and the normal components of the electric flux density $\mathbf{D}$ ) and the magnetic flux density $\mathbf{B}$ ) are continuous across any interface, using the boundary conditions at the surface of a PEC object and those at the surface of a PMC object and the fact that a PEMC medium is a generalization of a PEC and a PMC medium, the boundary conditions to be satisfied on the surface of a corresponding PEMC object can be written as[57].

$$
\begin{equation*}
\mathbf{n} \times(\mathbf{H}+\mathcal{M} \mathbf{E}) \quad \mathbf{n} \cdot(\mathbf{D}-\mathcal{M} \mathbf{B}) \tag{8.125}
\end{equation*}
$$

where $\mathcal{M}$ is defined as the PEMC admittance, and $\mathbf{n}$ is the unit normal to the boundary.
Consider an arbitrarily polarized, monochromatic uniform plane electromagnetic wave with an electric field intensity of unit amplitude incident on a PEMC spheroid located in free space, with its center at the origin of a Cartesian coordinate system and its axis of symmetry along the $z$ axis of this coordinate system, as shown in Fig.(8.6). Without any loss of generality, the $y=0$ plane can be assumed to be the plane of incidence. The incident


Figure 8.6: Geometry of the PEMC spheroid
and scattered electric fields in the case of transverse electric TE polarization can be expanded in terms of vector spheroidal wave functions $\mathbf{M}$ and $\mathbf{N}$ as [64]

$$
\begin{align*}
& \mathbf{E}_{T E}^{i}=\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{i}\left(c, \theta_{0}\right) \mathbf{M}_{e, m n}^{r(1)}(c, \mathbf{r})+b_{m n}^{i}\left(c, \theta_{0}\right) \mathbf{N}_{o, m n}^{r(1)}(c, \mathbf{r})\right]  \tag{8.126}\\
& \mathbf{E}_{T E}^{s}=\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{s} \mathbf{M}_{e, m n}^{r(4)}(c, \mathbf{r})+b_{m n}^{s} \mathbf{N}_{o, m n}^{r(4)}(c, \mathbf{r})\right.  \tag{8.127}\\
&\left.+c_{m n}^{s} \mathbf{M}_{o, m n}^{r(4)}(c, \mathbf{r})+d_{m n}^{s} \mathbf{N}_{e, m n}^{r(4)}(c, \mathbf{r})\right]
\end{align*}
$$

where $c=k f$, with $k$ being the wavenumber and $f$ the semiinterfocal distance of the spheroid, $\mathbf{r}$ denotes the spheroidal coordinate triad $(\xi, \eta, \phi)$, and $\theta_{0}$ is the angle of incidence measured from the $z$ axis. The expansion coefficients of Eq.(8.127) are unknown, and those of Eq.(8.126) are given by

$$
\begin{align*}
& a_{m n}^{i}\left(c, \theta_{0}\right)=\frac{2\left(2-\delta_{0 m}\right) j^{n}}{N_{m n}(c)} \sum_{k=0,1}^{\infty} \alpha_{k}^{m n}\left(c, \theta_{0}\right)  \tag{8.128}\\
& b_{m n}^{i}\left(c, \theta_{0}\right)=\frac{4 m j^{n-1}}{N_{m n}(c)} \sum_{k=0,1}^{\infty} \beta_{k}^{m n}\left(c, \theta_{0}\right)
\end{align*}
$$

where $N_{m n}(c)$ is the normalization constant of the spheroidal angle function $S_{m n}\left(c, \cos \theta_{0}\right), \delta_{0 m}$ is the Kronecker delta function, and

$$
\begin{align*}
\alpha_{k}^{m n}\left(c, \theta_{0}\right) & =\frac{d_{k}^{m n}(c)}{(k+m)(k+m+1)} \frac{d}{d \theta_{0}} P_{m+k}^{m}\left(\cos \theta_{0}\right)  \tag{8.129}\\
\beta_{k}^{m n}\left(c, \theta_{0}\right) & =\frac{d_{k}^{m n}(c)}{(k+m)(k+m+1)} \frac{P_{m+k}^{m}\left(\cos \theta_{0}\right)}{\sin \theta_{0}}
\end{align*}
$$

in which the prime over the summation sign indicating that the summation is over only even(odd) values of $k$ when $n-m$ is even(odd), $d_{k}^{m n}(c)$ are the spheroidal expansion coefficients, and $P_{m+k}^{m}$ are the associated Legendre functions of the first kind. The vector spheroidal wave functions $\mathbf{M}$ and $\mathbf{N}$ are defined as

$$
\begin{equation*}
\mathbf{M}_{e_{o n n}^{r}}^{r(g)}(c, \mathbf{r})=\nabla \psi_{e_{o m n}}^{(g)}(c, r) \times \mathbf{a}_{r} \tag{8.130}
\end{equation*}
$$

where $\mathbf{a}_{r}$ is the unit position vector, and

$$
\begin{equation*}
\psi_{\substack{(g) \\ o}}^{(g)}(c, r)=S_{m n}(c, \eta) R_{m n}^{(g)}(c, \xi)_{\sin }^{\cos }(m \phi) \tag{8.131}
\end{equation*}
$$

with $S_{m n}(c, \eta)$ and $R_{m n}^{(g)}(c, \xi)$ being the spheroidal angle function and the spheroidal radial function of the gth kind, respectively [65]. The vector spheroidal wave functions $\mathbf{N}$ are obtained from $\mathbf{M}$ as

$$
\begin{equation*}
\underset{\substack{\mathbf{N}_{o} \\ r(g n}}{r(c,}(c, \mathbf{r})=\frac{1}{k} \nabla \times \mathbf{M}_{\substack{e \\ o \\ r}}^{r(g)}(c, \mathbf{r}) \tag{8.132}
\end{equation*}
$$

The expansions corresponding to the incident and scattered magnetic fields $H_{T E}^{i}, H_{T E}^{s}$ are obtained from Eq.(8.126) and Eq.(8.127), respectively, using

Maxwells equations. This corresponds to interchanging $\mathbf{M}$ and $\mathbf{N}$ on the right-hand sides of Eq.(8.126) and Eq.(8.127), and multiplying each equation by $j / Z_{0}$, where $Z_{0}$ is the wave impedance in free space.

$$
\begin{align*}
\mathbf{H}_{T E}^{i}= & \frac{j}{Z_{0}} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{i}\left(c, \theta_{0}\right) \mathbf{N}_{e, m n}^{r(1)}(c, \mathbf{r})+b_{m n}^{i}\left(c, \theta_{0}\right) \mathbf{M}_{o, m n}^{r(1)}(c, \mathbf{r})\right]  \tag{8.133}\\
\mathbf{H}_{T E}^{s} & =\frac{j}{Z_{0}} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{s} \mathbf{N}_{e, m n}^{r(4)}(c, \mathbf{r})+b_{m n}^{s} \mathbf{M}_{o, m n}^{r(4)}(c, \mathbf{r})\right.  \tag{8.134}\\
& \left.+c_{m n}^{s} \mathbf{N}_{o, m n}^{r(4)}(c, \mathbf{r})+d_{m n}^{s} \mathbf{M}_{e, m n}^{r(4)}(c, \mathbf{r})\right]
\end{align*}
$$

The expansions of the incident and scattered electric fields for the case of transverse magnetic $T M$ ) polarization can be written in terms of vector spheroidal wave functions as[64]

$$
\begin{align*}
\mathbf{E}_{T M}^{i}=-j \sum_{m=0}^{\infty} & \sum_{n=m}^{\infty}\left[a_{m n}^{i}\left(c, \theta_{0}\right) \mathbf{N}_{e, m n}^{r(1)}(c, \mathbf{r})+b_{m n}^{i}\left(c, \theta_{0}\right) \mathbf{M}_{o, m n}^{r(1)}(c, \mathbf{r})\right]  \tag{8.135}\\
\mathbf{E}_{T M}^{s}= & \sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{s} \mathbf{N}_{e, m n}^{r(4)}(c, \mathbf{r})+b_{m n}^{s} \mathbf{M}_{o, m n}^{r(4)}(c, \mathbf{r})\right.  \tag{8.136}\\
& \left.+c_{m n}^{s} \mathbf{N}_{o, m n}^{r(4)}(c, \mathbf{r})+d_{m n}^{s} \mathbf{M}_{e, m n}^{r(4)}(c, \mathbf{r})\right]
\end{align*}
$$

The expansions corresponding to the incident and scattered magnetic fields $\mathbf{H}_{T M}^{i}, \mathbf{H}_{T M}^{s}$ are obtained from Eq.(8.135) and Eq.(8.136), respectively, using Maxwells equations.

$$
\begin{align*}
\mathbf{H}_{T M}^{i}= & \frac{-1}{Z_{0}} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}\left(c, \theta_{0}\right) \mathbf{M}_{e, m n}^{r(1)}(c, \mathbf{r})+b_{m n}\left(c, \theta_{0}\right) \mathbf{N}_{o, m n}^{r(1)}(c, \mathbf{r})\right]  \tag{8.137}\\
\mathbf{H}_{T M}^{s} & =\frac{j}{Z_{0}} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{s} \mathbf{M}_{e, m n}^{r(4)}(c, \mathbf{r})+b_{m n}^{s} \mathbf{N}_{o, m n}^{r(4)}(c, \mathbf{r})\right.  \tag{8.138}\\
& \left.+c_{m n}^{s} \mathbf{M}_{o, m n}^{r(4)}(c, \mathbf{r})+d_{m n}^{s} \mathbf{N}_{e, m n}^{r(4)}(c, \mathbf{r})\right]
\end{align*}
$$

Imposing the boundary conditions specified by Eq.(8.125) on the surface $\xi=\xi_{0}$ of the PEMC spheroid yields

$$
\begin{equation*}
\mathbf{a}_{\xi} \times\left.\left[\mathbf{H}_{p}^{i}+\mathbf{H}_{p}^{s}+\mathcal{M}\left(\mathbf{E}_{p}^{i}+\mathbf{E}_{p}^{i}\right)\right]\right|_{\xi=\xi_{0}}=0 \tag{8.139}
\end{equation*}
$$

where $p$ is either $T E$ or $T M$, and $\mathbf{a}_{\xi}$ is the unit vector normal to the boundary $\xi=\xi_{0}$ denoting the surface of the spheroid. Equation Eq.(8.139) can now be rewritten in the form

$$
\begin{align*}
& \mathbf{a}_{\xi} \times\left.\left[\mathbf{H}_{p \eta}^{i}+\mathbf{H}_{p \eta}^{s}+\mathcal{M}\left(\mathbf{E}_{p \eta}^{i}+\mathbf{E}_{p \eta}^{i}\right)\right]\right|_{\xi=\xi_{0}}=0  \tag{8.140}\\
& \mathbf{a}_{\xi} \times\left.\left[\mathbf{H}_{p \phi}^{i}+\mathbf{H}_{p \phi}^{s}+\mathcal{M}\left(\mathbf{E}_{p \phi}^{i}+\mathbf{E}_{p \phi}^{i}\right)\right]\right|_{\xi=\xi_{0}}=0
\end{align*}
$$

The unknown expansion coefficients are evaluated by solving a set of linear equations obtained by substituting for the different components of the $\mathbf{E}$ and $\mathbf{H}$ fields in Eq.(8.140) in terms of spheroidal wave function expansions, and then using the orthogonality properties of the trigonometric functions and the spheroidal angle functions as in [64].

### 8.12 Scattering by Circular Metallic Disks

Exact solution to problems of scattering of plane electromagnetic waves by perfectly conducting bodies of finite dimensions are limited. From view point of electromagnetic, spheroid has special shape. The limiting case of oblate spheroid will be circular disk. The incident electric field is given by


Figure 8.7: Scattering of plane wave by circular conducting disk

$$
\begin{equation*}
\mathbf{E}^{i}=E_{0}\left(-\cos \theta_{0} \cos \alpha \mathbf{a}_{x}+\sin \alpha \mathbf{a}_{y}-\sin \theta_{0} \cos \alpha \mathbf{a}_{z}\right) e^{-j \mathbf{k}^{i} \cdot \mathbf{r}} \tag{8.141}
\end{equation*}
$$

where

$$
\begin{align*}
\mathbf{k}^{i} & =k\left(-\sin \theta_{0} \mathbf{a}_{x}+\cos \theta_{0} \mathbf{a}_{z}\right)  \tag{8.142}\\
k & =\omega \sqrt{\mu_{0} \epsilon_{0}}
\end{align*}
$$

The scattered electric field intensity at an arbitrary point $r, \theta, \phi$ on the far field sphere may be expressed as

$$
\begin{equation*}
\mathbf{E}^{s}=\frac{j E_{0}}{k r}\left(\frac{\cos \alpha}{\cos \theta_{0}} \mathbf{E}_{\|}+\sin \alpha \mathbf{E}_{\perp}\right) e^{-j \mathbf{k}^{i} \cdot \mathbf{r}} \tag{8.143}
\end{equation*}
$$

It is convenient to express the scattered filed $\mathbf{E}^{s}$ in terms of a normalized field $\mathbf{E}_{n}^{s}$ such that

$$
\begin{equation*}
\mathbf{E}^{s}=\frac{a E_{0}}{2 r} \mathbf{E}_{n}^{s} e^{-j \mathbf{k}^{i} \cdot \mathbf{r}} \tag{8.144}
\end{equation*}
$$

This choice of normalized yields a particularly simple form for the normalized radar cross section, i.e.,

$$
\begin{equation*}
\frac{\sigma}{\pi a^{2}}=\left|\mathbf{E}_{n}^{s}\right|^{2} \tag{8.145}
\end{equation*}
$$

The normalized electric field intensity in this case is given by

$$
\begin{equation*}
\mathbf{E}_{n}^{s}=\frac{2 j}{k a}\left(\frac{\cos \alpha}{\cos \theta_{0}} \mathbf{E}_{\|}+\sin \alpha \mathbf{E}_{\perp}\right) \tag{8.146}
\end{equation*}
$$

where

$$
\begin{align*}
E_{\| \theta}= & \cos \theta \sum_{m=0}^{\infty}\left\{-2\left(2-\delta_{0, m}\right) \cos (m \phi) \cos \phi\right.  \tag{8.147}\\
& \cdot Y_{m}\left(\cos \theta, c, \cos \theta_{0}\right)+j^{-m}\left[U_{m+1} \cos (m+1) \phi\right. \\
- & \left.\left.\left(1+\delta_{m, 1}\right) U_{m-1} \cos (m-1) \phi\right] Y_{m}(\cos \theta, c, 0)\right\} \\
E_{\| \phi}= & \sum_{m=0}^{\infty}\left\{-2\left(2-\delta_{0, m}\right) \cos (m \phi) \sin \phi\right.  \tag{8.148}\\
& \cdot Y_{m}\left(\cos \theta, c, \cos \theta_{0}\right)+j^{-m}\left[U_{m+1} \sin (m+1) \phi\right. \\
+ & \left.\left.\left(1+\delta_{m, 1}\right) U_{m-1} \sin (m-1) \phi\right] Y_{m}(\cos \theta, c, 0)\right\}
\end{align*}
$$

$$
\begin{align*}
E_{\perp \theta}= & \cos \theta \sum_{m=0}^{\infty}\left\{2\left(2-\delta_{0, m}\right) \cos (m \phi) \sin \phi\right.  \tag{8.149}\\
& \cdot Y_{m}\left(\cos \theta, c, \cos \theta_{0}\right)-j^{-m}\left[X_{m+1} \sin (m+1) \phi\right. \\
& \left.\left.-X_{m-1} \sin (m-1) \phi\right] Y_{m}(\cos \theta, c, 0)\right\} \\
E_{\perp \phi}= & \sum_{m=0}^{\infty}\left\{-2\left(2-\delta_{0, m}\right) \cos (m \phi) \cos \phi\right.  \tag{8.150}\\
& \cdot Y_{m}\left(\cos \theta, c, \cos \theta_{0}\right)+j^{-m}\left[X_{m+1} \cos (m+1) \phi\right. \\
+ & \left.\left.\left(1-\delta_{m, 1}\right) X_{m-1} \cos (m-1) \phi\right] Y_{m}(\cos \theta, c, 0)\right\}
\end{align*}
$$

and $c=k a$.
The function $Y_{m}$ are given in terms of the spheroidal radial functions $R_{m n}^{(g)}(-j c, j 0)$ and the spheroidal angular functions $S_{m n}(-j c, \cos \theta)$ by

$$
\begin{align*}
Y_{m}\left(\cos \theta, c, \cos \theta_{0}\right)= & \sum_{\substack{n=m \\
\text { aven } \\
\text { even }}}^{\infty} \frac{(-1)^{n}}{N_{m n}(-j c)} \cdot \frac{R_{m n}^{(1)}(-j c, j 0)}{R_{m n}^{(4)}(-j c, j 0)}  \tag{8.151}\\
& \cdot S_{m n}\left(-j c, \cos \theta_{0}\right) S_{m n}(-j c, \cos \theta)
\end{align*}
$$

The prim on the summation symbol emphasizes the fact that the summation over n proceeds by increments of two as a consequence of the condition that $n-m$ is even. The $N_{m n}$ is normalization function which we defined previously. The $U_{m}$ and $X_{m}$ functions are given by

$$
\begin{array}{rlr}
U_{m} & =2 j^{m-1} \frac{W_{m-1}+W_{m+1}}{\psi_{m-1}+\psi_{m+1}} & m \geq 1 \\
U_{0} & =-j \frac{W_{1}}{\psi_{1}} & \\
U_{m} & =0 \quad m<0 & \\
X_{m} & =2 j^{m-1} \frac{W_{m-1}-W_{m+1}}{\psi_{m-1}+\psi_{m+1}} & m \geq 1  \tag{8.153}\\
X_{m} & =0 \quad m<0 &
\end{array}
$$

Finally, the $W_{m}$ and $\psi_{m}$ functions are given in terms of the spheroidal functions by

$$
\begin{gather*}
W_{m}=\sum_{\substack{n=m \\
n=m \\
e v e n}}^{\infty} \frac{(j)^{n}}{N_{m n}(-j c)} \cdot \frac{S_{m n}\left(-j c, \cos \theta_{0}\right) S_{m n}(-j c, 0)}{R_{m n}^{(4)}(-j c, j 0)}  \tag{8.154}\\
\psi_{m}=\sum_{\substack{n=m \\
n=m \\
\text { even }}}^{\infty} \frac{(j)^{n}}{N_{m n}(-j c)} \cdot \frac{\left[S_{m n}(-j c, 0)\right]^{2}}{R_{m n}^{(4)}(-j c, j 0)} \tag{8.155}
\end{gather*}
$$



Figure 8.8: RCS of conducting disk

### 8.13 EM Scattering by Dielectric Spheroid

Figure (8.9) defines the geometry of the problem that we will consider. A plane wave incident in a xz-plane with an angle $\theta_{0}$ between positive z axis and the direction of incidence. For oblique incidence, the polarized incident wave is resolved into two components as shown in Fig.(8.9): the TE mode for which the electric vector of incident wave is per perpendicular to the incident plane, and $T M$ mode for which the magnetic vector field is perpendicular


Figure 8.9: Geometry of the dielectric prolate spheroid
to the incident plane. For $\theta_{0} \neq 0$, we will consider two case separately. For $\theta_{0}=0$, the parallel incident, due to symmetry of configuration, we have identical results for both cases.

$$
\begin{align*}
\mathbf{E}_{T E}^{i} & =\mathbf{a}_{y} e^{-j k_{0}\left(x \sin \theta_{0}+z \cos \theta_{0}\right)}  \tag{8.156}\\
\mathbf{H}_{T E}^{i} & =\frac{1}{Z_{0}}\left(\mathbf{a}_{x} \cos \theta_{0}-\mathbf{a}_{z} \sin \theta_{0}\right) \mathbf{a}_{y} e^{-j k_{0}\left(x \sin \theta_{0}+z \cos \theta_{0}\right)}
\end{align*}
$$

where $Z_{0}=\sqrt{\mu_{0} / \epsilon_{0}}$ is impedance of free space.

$$
\begin{gather*}
\mathbf{E}_{T E}^{i}=\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{i}\left(c_{0}, \theta_{0}\right) \mathbf{M}_{e, m n}^{r(1)}\left(c_{0}, \mathbf{r}\right)+b_{m n}^{i}\left(c_{0}, \theta_{0}\right) \mathbf{N}_{o, m n}^{r(1)}\left(c_{0}, \mathbf{r}\right)\right] 8 \\
\mathbf{H}_{T E}^{i}=\frac{j}{Z_{0}} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{i}\left(c_{0}, \theta_{0}\right) \mathbf{N}_{e, m n}^{r(1)}\left(c_{0}, \mathbf{r}\right)+b_{m n}^{i}\left(c_{0}, \theta_{0}\right) \mathbf{M}_{o, m n}^{r(1)}\left(c_{0}, \mathbf{r}\right)\right]
\end{gather*}
$$

where $c_{0}=k_{0} f$ and

$$
\begin{align*}
a_{m n}^{i}\left(c_{0}, \theta_{0}\right) & =\frac{2\left(2-\delta_{0 m}\right) j^{n}}{N_{m n}\left(c_{0}\right)} \sum_{k=0,1}^{\infty} \alpha_{k}^{m n}\left(c_{0}, \theta_{0}\right)  \tag{8.158}\\
b_{m n}^{i}\left(c_{0}, \theta_{0}\right) & =\frac{4 m j^{n-1}}{N_{m n}\left(c_{0}\right)} \sum_{k=0,1}^{\infty} \beta_{k}^{m n}\left(c_{0}, \theta_{0}\right)
\end{align*}
$$

$$
\begin{gather*}
\alpha_{k}^{m n}\left(c_{0}, \theta_{0}\right)=\frac{d_{k}^{m n}\left(c_{0}\right)}{(k+m)(k+m+1)} \frac{d}{d \theta_{0}} P_{m+k}^{m}\left(\cos \theta_{0}\right)  \tag{8.159}\\
\beta_{k}^{m n}\left(c_{0}, \theta_{0}\right)=\frac{d_{k}^{m n}\left(c_{0}\right)}{(k+m)(k+m+1)} \frac{P_{m+k}^{m}\left(\cos \theta_{0}\right)}{\sin \theta_{0}} \\
\mathbf{E}_{T E}^{s}=\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{s} \mathbf{M}_{e, m n}^{r(4)}\left(c_{0}, \mathbf{r}\right)+b_{m n}^{s} \mathbf{N}_{o, m n}^{r(4)}\left(c_{0}, \mathbf{r}\right)\right]  \tag{8.160}\\
\mathbf{H}_{T E}^{s}=\frac{j}{Z_{0}} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{s} \mathbf{N}_{e, m n}^{r(4)}\left(c_{0}, \mathbf{r}\right)+b_{m n}^{s} \mathbf{M}_{o, m n}^{r(4)}\left(c_{0}, \mathbf{r}\right)\right] \\
\mathbf{E}_{T E}^{t}=\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{t} \mathbf{M}_{e, m n}^{r(1)}(c, \mathbf{r})+b_{m n}^{t} \mathbf{N}_{o, m n}^{r(1)}(c, \mathbf{r})\right]  \tag{8.161}\\
\mathbf{H}_{T E}^{t}=\frac{j}{Z} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{t} \mathbf{N}_{e, m n}^{r(1)}(c, \mathbf{r})+b_{m n}^{t} \mathbf{M}_{o, m n}^{r(1)}(c, \mathbf{r})\right]
\end{gather*}
$$

where $Z=\sqrt{\mu / \epsilon}$ is impedance of dielectric spheroid and $c=k f, k=\omega \sqrt{\mu \epsilon}$.
The $a_{m n}^{s}, b_{m n}^{s}, a_{m n}^{t}$ and $b_{m n}^{t}$ are unknown and can be determined by applying boundary conditions. At $\xi=\xi_{0}$ the tangential electric and magnetic fields on the dielectric spheroid are continous. Therefore

$$
\begin{align*}
\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{i} M_{e, m n \eta}^{r(1)}\left(c_{0} ; \eta, \xi_{0}, \phi\right)\right. & +b_{m n}^{i} N_{o, m n \eta}^{r(1)}\left(c_{0} ; \eta, \xi_{0}, \phi\right)+  \tag{8.162}\\
a_{m n}^{s} M_{e, m n \eta}^{r(4)}\left(c_{0} ; \eta, \xi_{0}, \phi\right) & +b_{m n}^{s} N_{o, m n \eta}^{r(4)}\left(c_{0} ; \eta, \xi_{0}, \phi\right)- \\
a_{m n}^{t} M_{e, m n \eta}^{r(4)}\left(c ; \eta, \xi_{0}, \phi\right) & \left.-b_{m n}^{t} N_{o, m n \eta}^{r(4)}\left(c ; \eta, \xi_{0}, \phi\right)\right]=0 \\
\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[a_{m n}^{i} M_{e, m n \phi}^{r(1)}\left(c_{0} ; \eta, \xi_{0}, \phi\right)\right. & +b_{m n}^{i} N_{o, m n \phi}^{r(1)}\left(c_{0} ; \eta, \xi_{0}, \phi\right)+  \tag{8.163}\\
a_{m n}^{s} M_{e, m n \phi}^{r(4)}\left(c_{0} ; \eta, \xi_{0}, \phi\right) & +b_{m n}^{s} N_{o, m n \phi}^{r(4)}\left(c_{0} ; \eta, \xi_{0}, \phi\right)- \\
a_{m n}^{t} M_{e, m n \phi}^{r(1)}\left(c ; \eta, \xi_{0}, \phi\right) & \left.-b_{m n}^{t} N_{o, m n \phi}^{r(1)}\left(c ; \eta, \xi_{0}, \phi\right)\right]=0
\end{align*}
$$

$$
\begin{align*}
\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[\frac{1}{Z_{0}} a_{m n}^{i} N_{e, m n \eta}^{r(1)}\left(c_{0} ; \eta, \xi_{0}, \phi\right)\right. & +\frac{1}{Z_{0}} b_{m n}^{i} M_{o, m n \eta}^{r(1)}\left(c_{0} ; \eta, \xi_{0}, \phi\right)+(8  \tag{8.164}\\
\frac{1}{Z_{0}} a_{m n}^{s} N_{e, m n \eta}^{r(4)}\left(c_{0} ; \eta, \xi_{0}, \phi\right) & +\frac{1}{Z_{0}} b_{m n}^{s} M_{o, m n \eta}^{r(4)}\left(c_{0} ; \eta, \xi_{0}, \phi\right)+ \\
\frac{1}{Z} a_{m n}^{t} N_{e, m n \eta}^{r(4)}\left(c ; \eta, \xi_{0}, \phi\right) & \left.+\frac{1}{Z} b_{m n}^{t} M_{o, m n \eta}^{r(4)}\left(c ; \eta, \xi_{0}, \phi\right)\right]=0 \\
\sum_{m=0}^{\infty} \sum_{n=m}^{\infty}\left[\frac{1}{Z_{0}} a_{m n}^{i} N_{e, m n \phi}^{r(1)}\left(c_{0} ; \eta, \xi_{0}, \phi\right)\right. & +\frac{1}{Z_{0}} b_{m n}^{i} M_{o, m n \phi}^{r(1)}\left(c_{0} ; \eta, \xi_{0}, \phi\right)+(8  \tag{8.165}\\
\frac{1}{Z_{0}} a_{m n}^{s} N_{e, m n \phi}^{r(4)}\left(c_{0} ; \eta, \xi_{0}, \phi\right) & +\frac{1}{Z_{0}} b_{m n}^{s} M_{o, m n \phi}^{r(4)}\left(c_{0} ; \eta, \xi_{0}, \phi\right)+ \\
\frac{1}{Z} a_{m n}^{t} N_{e, m n \phi}^{r(1)}\left(c ; \eta, \xi_{0}, \phi\right) & \left.+\frac{1}{Z} b_{m n}^{t} M_{o, m n \phi}^{r(1)}\left(c ; \eta, \xi_{0}, \phi\right)\right]=0
\end{align*}
$$

Like the scattering by conducting spheroid, the equation that stand for continuity of $\eta$ components, Eq.(8.162)and Eq.(8.164), are multiplied by $\left(\xi_{0}^{2}-\eta^{2}\right)^{5 / 2}=\left[\left(\xi_{0}^{2}-1\right)+\left(1-\eta^{2}\right)\right]^{5 / 2}$, and the equations for $\phi$-components, Eq.(8.163) and Eq.(8.165) by $\left(\xi_{0}^{2}-1\right)^{-1 / 2}\left(\xi_{0}^{2}-\eta^{2}\right)$. We also truncate the series to Max term and index $t$ in all Eq.(8.162),...Eq.(8.165) can be changed from $0-M a x$, therefore we have $4 M a x+4$ unknown and $4 M a x+4$ linear equations which can be divide into sixteen different submatrix.
For $E_{\eta}$-components:

$$
\begin{equation*}
\sum_{n=0}^{\infty} A_{n t}^{i} a_{m n}^{i}+B_{n t}^{i} b_{m n}^{i}+A_{n t}^{s} a_{m n}^{s}+B_{n t}^{s} b_{m n}^{s}=A_{n t}^{t} a_{m n}^{t}+B_{n t}^{t} b_{m n}^{t} \tag{8.166}
\end{equation*}
$$

For $H_{\eta}$-components:

$$
\begin{equation*}
\sum_{n=0}^{\infty} A_{n t}^{i} a_{m n}^{i}+B_{n t}^{i} b_{m n}^{i}+A_{n t}^{s} a_{m n}^{s}+B_{n t}^{s} b_{m n}^{s}=A_{n t}^{t} a_{m n}^{t}+B_{n t}^{t} b_{m n}^{t} \tag{8.167}
\end{equation*}
$$

For $E_{\phi}$-components:

$$
\begin{equation*}
\sum_{n=0}^{\infty} C_{n t}^{i} a_{m n}^{i}+D_{n t}^{i} b_{m n}^{i}+C_{n t}^{s} a_{m n}^{s}+D_{n t}^{s} b_{m n}^{s}=C_{n t}^{t} a_{m n}^{t}+D_{n t}^{s} b_{m n}^{t} \tag{8.168}
\end{equation*}
$$

or the $\left[\mathbf{B}_{n}^{1}, \mathbf{B}_{n}^{2}, \mathbf{B}_{n}^{3}, \mathbf{B}_{n}^{4}\right]^{T}$ can be be found by

### 8.14 Problems

- 1 Find the RCS of conducting oblate spheroid for TE polarization.
- 2 Find the RCS of a circular metallic disk for TE polarization.
- 3 A conducting prolate spheroid with semi major axis $A=\lambda / 4$ and semi minor axis $B=\lambda / 8$ is given. plot the rcs of it as a function of $\theta$ when $\phi=0$ and $\phi=\pi / 2$


## Chapter 9

## Mesh Generation

" God does not care about our mathematical difficulties. He integrates empirically."
Albert Einstein

### 9.1 Introduction

Grid generation is an integral part of the majority of numerical methods for the prediction of charge and potential. In the more general case, it involves the discretization of the domain of interest in a big number of predefined, simple-shaped, sub-domains, where the governing equations can be expressed in more manageable terms. These sub-domains can be triangles, quadri-laterals, etc. for two-dimensional fields, or tetrahedrals, pyramids, cubes, etc, for three-dimensional domains. One particular type of grid, that presents us with significant advantages in terms of flexibility, adaptivity and generality is the so-called unstructured type. Such grids can in principal describe efficiently even the most complex geometries, are easier to construct than their structured counterparts and allow for direct and accurate control to their characteristics. The subject has been investigated extensively and thus numerous techniques and algorithms are available in the literature, but also as commercial and public domain software implementations.

Meshing can be defined as the process of breaking up a physical domain into smaller sub-domains (elements) in order to facilitate the numerical solution of a partial differential equation. While meshing can be used for a wide variety of applications, the principal application of interest is the finite
element method. Surface domains may be subdivided into triangle or quadrilateral shapes, while volumes may be subdivided primarily into tetrahedra or hexahedra shapes. The shape and distribution of the elements is ideally defined by automatic meshing algorithms.

The finite element method in recent decades has become a mainstay for industrial engineering design and analysis. Increasingly larger and more complex designs are being simulated using the finite element method. With its increasing popularity comes the incentive to improve automatic meshing algorithms.

At the inception of the finite element method, most users were satisfied to simulate vastly simplified forms of their final design utilizing only tens or hundreds of elements. Painstaking preprocessing was required to subdivide domains into usable elements. Market forces have now pushed meshing technology to a point where users now expect to mesh complex domains with thousands or millions of elements with no more interactions than the push of a button.

Consumers of finite element technology such as aerospace and automotive industries have immediate needs to shorten design cycles and overall time to market. Improving the robustness, speed and quality of automatic meshers, while only a small part of the entire process, can translate into increased revenue and competitive advantage.

While there is certainly the incentive from a market-based perspective to improve finite element meshing technology, opinions on the specifics of what should be improved are diverse. Amongst users of finite element technology their has long been a debate as to what shape of element produces the most accurate result. There is the often-held position that quadrilateral and hexahedral shaped elements have superior performance to triangle and tetrahedral shaped elements when comparing an equivalent number of degrees of freedom. Use of hex elements can also vastly reduce the number of elements and consequently analysis and post-processing times. In addition, hex and quadrilateral elements are more suited for non-linear analysis as well as situations where alignment of elements is important to the physics of the problem, such as in computational fluid dynamics or simulation of composite materials.

The automatic mesh generation problem is that of attempting to define a set of nodes and elements in order to best describe a geometric domain, subject to various element size and shape criteria. Geometry is most often composed of vertices, curves, surfaces and solids as described by a CAD or
solids modeling package.
Many applications, use a "bottom-up" approach to mesh generation. Vertices are first meshed, followed by curves, then surfaces and finally solids. The input for the subsequent meshing operation is the result of the previous lower dimension meshing operation.

For example, nodes are first placed at all vertices of the geometry. Nodes are then distributed along geometric curves. The result of the curve meshing process provides input to a surface meshing algorithm, where a set of curves define a closed set of surface loops. Decomposing the surface into well-shaped triangles or quadrilaterals is the next phase of the meshing process. Finally, if a solid model is provided as the geometric domain, a set of meshed areas defining a closed volume is provided as input to a volume mesher for automatic formation of tetrahedra, hexahedra or mixed element types.

The journal articles referenced in this web site all are relevant to the process of mesh generation. Although many authors will take radically different approaches, the ultimate goal is to provide a mesh that can be used to solve a partial differential equation. This field is relatively new. Most of the papers published in this area have been within the past five to ten years. Many questions have been solved in this time, but there are still a significant number of problems to be addressed.

Like any science, mesh generation has its jargon. Paul Heckbert has compiled a brief glossary of words you may want to be familiar with when viewing this web site.
"http://www.cs.cmu.edu/ ph/heckbert.html"

### 9.2 Classification of Mesh Generation

One of the important part in numerical electromagnetic computation is mesh generation. A mesh is a discretization of a geometric domain into simple elements, for example a partition of a polygon into small triangles. Meshes find use in computer graphics, geographic information systems, and finite element methods. Although different applications have different requirements, it is generally true that a good mesh will have small elements for detail, large elements for efficiency, and "nicely shaped" elements for accuracy. The first task in numerical computation is discretization. It should mention that we make just simple mesh for our simple computation. If you want more accrued and sophisticated mesh see references. We have three type of discretization:
a) volume discretization.
b) surface discretization.
c) line discretization.

It may be classified in other way:
a) One Dimensional Mesh Generation or (1D)
b) Two Dimensional Mesh Generation or (2D)
c) Three Dimensional Mesh Generation or 3DV,3DS; (3DV: meshing a volume); (3DS: meshing the surface of a volume) Here we show some of mesh for example.

### 9.3 Tools

Any work done, some instruments has been used. In our task we need
a) Personal Computer with at least 16 MB RAM, 200 MHz speed.
b) FORTRAN77 compiler
c) MATHEMATICA package
d) MATLAB package

MATLAB or MATHEMATICA will be used to show our results. So they are used as graphical environment.

You may use other language such as Pascal or C, we prefer FORTRAN77 because you can change it other language easily. Are you ready? let us go.

### 9.4 1D Mesh Generation

We start with very simple and primitive problem. This type of mesh generation is not a well classical design, but is useful for our educational goal. Suppose we have a straight line, and want to divide into N section. Each section of the line is called an element or cell. Each element has two terminals. We call those terminals as nodes. Each element has a common node with adjacent element. In our computer program we usually need the coordinates of each elements; or the distance between elements. In order to limit our memory requirment, we define an integer array pointer $\mathrm{NV}(2, \mathrm{I})$
$\mathrm{NV}(2, \mathrm{I})=$ array pointer(terminals of each element)
$\mathrm{X}(\mathrm{j}), \mathrm{Y}(\mathrm{j}), \mathrm{Z}(\mathrm{j})=$ coordinates of each nodes.
$\mathrm{NP}=$ total number of nodes.
$\mathrm{NC}=$ total number of cells or elements.

For example $\mathrm{NV}(1,6)=6$ means that node number one (or in another word first terminal) of element number six is equal to 6 and $\operatorname{NV}(2,6)=7$ means that the second terminal of element number six is 7 . The first or second X coordinates of element numbered j can be found from $\mathrm{X}(\mathrm{NV}(1, \mathrm{j})), \mathrm{X}(\mathrm{NV}(2, \mathrm{j}))$. Example:
We have a segment line denoted by two points: $(3,7)$ and $(-5,-4)$ in $\mathrm{X}-\mathrm{Y}$ plane. It is asked to divide it into 5 uniform elements. Write a FORTRAN code that gives the coordinate of each elements.

In order to implement the described problem, a code in FORTRAN was developed. The program was designed to calculate: NP (number of nodes); X and Y coordinates of each node, NC(number of element). Program LINE1D.FOR do this job. There is two subroutines in this code. One of the important routine that we use in all of the mesh generation code is $\operatorname{TESTP}(\ldots)$. The main task of this subroutine is to check whether the produced node is previously created and marked or not. If it is new, the TESTP accept it. The pointer ID may be 0 or 1. If it is old and periviously generated, ID will be zero and return. Otherwise, it is new node and ID will be one. Therefore this subroutine will accept it and fill the coordinate $\mathrm{X}, \mathrm{Y},(\mathrm{Z}$ in 3D) array. ID is also used for POINTER NV( $\mathrm{j}, \mathrm{NC})$, to show the terminals. In all of our mesh generation codes we use this type of routine. You may have better idea about mesh generation. Just do it and compare your results.

We use the MATHEMATICA to show the results. In order to show the divisions on the line, it is better to make a file and load it in MATHEMATICA environment. As you see in LINE1D.FOR, the results is saved in file s.p in directory $c: \backslash$ save $\backslash$ shape. It is better to make a small file in your MATHEMATICA directory. I assigned PLT.M name for it. By calling or in other word loading that file, the graphic will be shown. Therefore
a) Make a file plt.m
b) Write on that file
sur:= <<c:\save\shape\s.p
Show[Graphics3D[sur],Boxed $->$ False]
c)In MATHEMATICA write $\ll$ plt.m it means that load plt.m

Notice: if the file s.p is in another directory, write your own PATH.

### 9.4.1 Glossary of Mesh Generation

NODE: a vertex in the mesh.

ELEMENT : a polygonal or polyhedral piece in the mesh; most often a triangle, quadrilateral, tetrahedron, or hexahedron

TRIANGULATION: given a set of points, connect them into a mesh of triangles MESH GENERATION: a (typically 2-D or 3-D) domain, generate nodes and triangulate them to create a mesh

ANISOTROPIC: not the same in all directions anisotropic mesh generation uses stretched elements; desired edge length is a function of orientation; e.g. doesn't strive for equilateral triangles graded mesh generation uses elements that vary in size as a function of position

STRUCTURED MESH: all elements and nodes have the same topology (i.e. same number of neighbors)

UNSTRUCTRUCTURED MESH: elements and nodes can have different topology; e.g. an arbitrary triangular subdivision

ADAPTIVE MESHING : iteratively regenerate mesh and solve Finite Element problem, hopefully improving the mesh and the accuracy of the solution on each iteration


Figure 9.1: 3DS Mesh Generation: a missile made by triangular


Figure 9.2: 3DS Mesh Generation: a missile made by triangular


Figure 9.3: 3DS Mesh Generation: surface made by triangular


Figure 9.4: 3DV Mesh Generation made by tetrahedrons


Figure 9.5: 3DV Mesh Generation made by tetrahedrons


Figure 9.6: 3DV Mesh Generation made by cubes


Figure 9.7: 3DS Mesh Generation made by triangles


Figure 9.8: 3DV Mesh Generation made by tetrahedrons


Figure 9.9: 3DV: a cube made by 5 tetrahedrons


Figure 9.10: 2D Mesh Generation made by triangles


Figure 9.11: 1D Mesh Generation made by lines

## Chapter 10

## Finite Difference Method

"Peace cannot be kept by force. It can only be achieved by understanding."
Albert Einstein

### 10.1 Introduction

Many engineering problems in electromagnetics may be formulated in terms of partial differential equations. This may include electrostatic problems formulated in terms of Laplace's Equation $\nabla^{2} \Phi=0$ and Poisson's Equation $\nabla^{2} \Phi=-q / \epsilon$ equation respectively, and dynamic fields problems formulated in term of Helmholtz's Equation $\left(\nabla^{2}+\beta^{2}\right) \mathbf{E}=j \omega \mathbf{J}$ where $\beta$ is the complex wave number, and the current source may or may not be zero. In electromagnetics scattering and radiation, $\mathbf{J} \neq \mathbf{0}$, and $\beta$ is known. In waveguide and cavity resonator problems, $\mathbf{J}=\mathbf{0}$ and the $\beta$ is unknown or in other word, cutoff or resonant frequency will be unknown.

### 10.2 Numerical Differentiation

We can define three type approximation for first derivative

$$
\begin{array}{ll}
f^{\prime}(x) \approx \frac{f(x+\Delta x)-f(x)}{} \quad \text { Forward Difference } \\
f^{\prime}(x) \approx \frac{f(x)-\dot{f}(x-\Delta x)}{\Delta x} & \text { Backward Difference }  \tag{10.1}\\
f^{\prime}(x) \approx \frac{f(x+\Delta x)-f(x-\Delta x)}{2 \Delta x} & \text { Central Difference }
\end{array}
$$

The Backward Difference (BD) and Forward Difference (FD) have truncation error of $O(\Delta x)$ and Central Difference (CD) has truncation error of $O(\Delta x)^{2}$ (why ?)

### 10.3 Numerical Differentiation Formula

For simplicity, we use new notation: $x=i \Delta x$ therefore Eq.(10.1) will be in this form

$$
\begin{gather*}
f^{\prime}(x) \approx \frac{f_{i+1}-f_{i}}{f^{x}} \quad \text { Forward Difference } \\
f^{\prime}(x) \approx \frac{f_{i} f_{i-1}}{\Delta x} \quad \text { Backward Difference }  \tag{10.2}\\
f^{\prime}(x) \approx \frac{f_{i+1}-f_{i-1}}{2 \Delta x} \quad \text { Central Difference } \\
f^{\prime}(x) \approx \frac{-f_{i+2}+8 f_{i+1}-8 f_{i-1}+f_{i-2}}{12 \Delta x}+O(\Delta x)^{2}  \tag{10.3}\\
f^{\prime}(x) \approx \frac{9}{8} \frac{f_{i+1 / 2}-f_{i-1 / 2}}{\Delta x}-\frac{1}{24} \frac{f_{i+3 / 2}-f_{i-3 / 2}}{\Delta x}+O(\Delta x)^{4}  \tag{10.4}\\
f^{\prime \prime}(x) \approx \frac{f_{i+1}-2 f_{i}+f_{i-1}}{(\Delta x)^{2}}+O(\Delta x)^{2}  \tag{10.5}\\
f^{\prime \prime}(x) \approx \frac{f_{i+2}-2 f_{i+1}+f_{i}}{(\Delta x)^{2}}+O(\Delta x)^{2}  \tag{10.6}\\
f^{\prime \prime}(x) \approx \frac{f_{i}-2 f_{i-1}+f_{i-2}}{(\Delta x)^{2}}+O(\Delta x)^{2}  \tag{10.7}\\
f^{\prime \prime}(x) \approx \frac{-f_{i+2}+16 f_{i+1}-30 f_{i}+16 f_{i-1}-f_{i-2}}{12(\Delta x)^{2}}+O(\Delta x)^{4} \tag{10.8}
\end{gather*}
$$

In the following $\Delta x=\Delta y=h$

$$
\begin{align*}
\frac{\partial^{2} f(x, y)}{\partial x^{2}} \approx & \frac{1}{3 h^{2}}\left(f_{i+1, j+1}-2 f_{i, j+1}+f_{i-1, j+1}+f_{i+1, j}-2 f_{i, j}+f_{i-1, j}\right. \\
& \left.+f_{i+1, j-1}-2 f_{i, j-1}+f_{i-1, j-1}\right)+O\left(h^{2}\right) \tag{10.9}
\end{align*}
$$

$$
\begin{equation*}
\frac{\partial^{2} f(x, y)}{\partial x \partial y} \approx \frac{1}{4 h^{2}}\left(f_{i+1, j+1}-f_{i+1, j-1}-f_{i-1, j+1}+f_{i-1, j-1}\right)+O\left(h^{2}\right) \tag{10.10}
\end{equation*}
$$

$$
\begin{align*}
\frac{\partial^{2} f(x, y)}{\partial x \partial y} \approx & \frac{-1}{2 h^{2}}\left(f_{i+1, j}+f_{i-1, j}+f_{i, j+1}+f_{i, j-1}\right. \\
& \left.-2 f_{i, j}-f_{i+1, j+1}-f_{i-1, j-1}\right)+O\left(h^{2}\right) \tag{10.11}
\end{align*}
$$

$$
\begin{gather*}
\frac{\partial^{2} f(x, y)}{\partial x^{4}} \approx \frac{1}{h^{4}}\left(f_{i+2, j}-4 f_{i+1, j}+6 f_{i, j}-4 f_{i-1, j}+2 f_{i-2, j}\right)+O\left(h^{2}\right)  \tag{10.12}\\
\frac{\partial^{4} f(x, y)}{\partial x^{2} \partial y^{2}} \approx \\
\begin{array}{c}
1 \\
\\
\left.\quad-2 f_{i-1, j}^{4}-f_{i, j+1}-2 f_{i, j-1}+4 f_{i, j}\right)+O\left(h^{2}\right)
\end{array} \tag{10.13}
\end{gather*}
$$



Figure 10.1: Equal Mesh Arms

$$
\begin{equation*}
\frac{\partial^{2} f(x, y)}{\partial x^{2}}+\frac{\partial^{2} f(x, y)}{\partial y^{2}} \approx \frac{1}{h^{2}}\left(f_{i+1, j}+f_{i, j+1}+f_{i-1, j}+f_{i, j-1}-4 f_{i, j}\right)+O\left(h^{2}\right) \tag{10.14}
\end{equation*}
$$

by using Laplacian notation $\frac{\partial^{2} f(x, y)}{\partial x^{2}}+\frac{\partial^{2} f(x, y)}{\partial y^{2}}=\nabla^{2} f(x, y)$

$$
\begin{align*}
\nabla^{2} f(x, y) \approx & \frac{1}{12 h^{2}}\left\{-60 f_{i, j}+16\left(f_{i, j+1}+f_{i-1, j}+f_{i, j-1}\right)\right. \\
& \left.-\left(f_{i+2, j}+f_{i, j+2}+f_{i-2, j}+f_{i, j-2}\right)\right\}+O\left(h^{4}\right) \tag{10.15}
\end{align*}
$$

## - Unequal Mesh Arms

In certain applications, the boundary of the domain of interest, in which a solution for $V$ is desired, does not fit the regular mesh structure required by equal arm five point cell formula. In this case, a very fine mesh may be utilized to minimize distortion of the shape of the boundary or different finite difference formula for five point cell with unequal arms should be used. Fig.(10.2).


Figure 10.2: Unequal Mesh Arms

$$
\begin{align*}
V_{i, j} \approx & \frac{V_{i+1, j}}{\left(1+\frac{h_{1}}{h_{3}}\right)\left(1+\frac{h_{1} h_{3}}{h_{4} h_{2}}\right)}+\frac{V_{i, j+1}}{\left(1+\frac{h_{2}}{h_{4}}\right)\left(1+\frac{h_{2} h_{4}}{h_{1} h_{3}}\right)}+ \\
& \frac{V_{i-1, j}}{\left(1+\frac{h_{3}}{h_{1}}\right)\left(1+\frac{h_{3} h_{1}}{h_{2} h_{4}}\right)}+\frac{V_{i, j-1}}{\left(1+\frac{h_{4}}{h_{2}}\right)\left(1+\frac{h_{4} h_{2}}{h_{3} h_{1}}\right)} \tag{10.16}
\end{align*}
$$

## - FD in Cylindrical Coordinates

Laplace's equation in cylindrical coordinates can be written as

$$
\begin{equation*}
\nabla^{2} V=\frac{\partial^{2} V}{\partial \rho^{2}}+\frac{1}{\rho} \frac{\partial V}{\partial \rho}+\frac{1}{\rho^{2}} \frac{\partial^{2} V}{\partial \phi^{2}}+\frac{\partial^{2} V}{\partial z^{2}} \tag{10.17}
\end{equation*}
$$

Let us assume two dimension Laplace's equation $z=$ constant

$$
\begin{equation*}
\frac{\partial^{2} V}{\partial \rho^{2}}+\frac{1}{\rho} \frac{\partial V}{\partial \rho}+\frac{1}{\rho^{2}} \frac{\partial^{2} V}{\partial \phi^{2}}=0 \tag{10.18}
\end{equation*}
$$

with $\phi=j \delta \phi,(j=0,1,2, \cdots)$, and $r=i h,(i=1,2, \cdots)$ we will have

$$
\begin{align*}
& \frac{V_{i+1, j}-2 V_{i, j}+V_{i-1, j}}{h^{2}}+\frac{1}{i h} \frac{V_{i+1, j}-V_{i-1, j}}{2 h} \\
& +\frac{1}{(i h)^{2}} \frac{V_{i, j+1}-2 V_{i, j}+V_{i, j-1}}{(\delta \phi)^{2}}=0 \tag{10.19}
\end{align*}
$$

we can rearrange it

$$
\begin{align*}
& \left(1-\frac{1}{2 i}\right) V_{i-1, j}+\left(1+\frac{1}{2 i}\right) V_{i+1, j}-2\left[1+\frac{1}{(i \delta \phi)^{2}}\right] V_{i, j} \\
& +\frac{1}{(i \delta \phi)^{2}} V_{i, j-1}+\frac{1}{(i \delta \phi)^{2}} V_{i, j+1}=0 \tag{10.20}
\end{align*}
$$

### 10.4 FD and Boundary Conditions

In EM problems we generally face with boundary, so we should implement boundary condition in our computation and simulation. There are three type of boundary condition:
Dirichlet Boundary Condition where $\psi$ is given at the boundary
Neumann Boundary Condition where $\frac{\partial \psi}{\partial n}$ is specified at the boundary In order to program Neumann boundary condition, $A D D$ an extra row of points outside of the boundary $\psi_{n+1}$, and solve for them using $\psi_{n+1}=\frac{\partial \psi}{\partial n}+$ $\psi_{n-1}$ which the term $\frac{\partial \psi}{\partial n}$ is known, Fig.(10.3). In electrostatic we have


Figure 10.3: Neumann Boundary Condition

$$
\begin{align*}
\frac{\partial V}{\partial n} & =\nabla V \cdot \mathbf{n} \\
\mathbf{E} & =-\nabla V \tag{10.21}
\end{align*}
$$

and the third type is combination of Neumann and Dirichlet boundary condition which is called Mixed Boundary Condition.

### 10.4.1 Symmetry

Let us look at the case shown in Fig.(10.4). The line ab is symmetric line. The node $(i, j)$ lies on symmetric line. Due to symmetry fictitious node $i+1, j$ is placed on the symmetric position of point $(i-1, j)$, in such a way that $\psi(i-1, j)=\psi(i+1, j)$, therefore for $\frac{\partial^{2} \psi(x, y)}{\partial x^{2}}+\frac{\partial^{2} \psi(x, y)}{\partial y^{2}}=F(x, y)$ with $h=\Delta x=\Delta y$


Figure 10.4: Symmetry Conditions

$$
\begin{equation*}
\psi(i, j)=\frac{1}{4}\left(2 \psi(i-1, j)+\psi(i, j+1)+\psi(i, j-1)-h^{2} F(i, j)\right) \tag{10.22}
\end{equation*}
$$

If the line of symmetry is diagonal as shown in Fig.(10.4), then

$$
\begin{equation*}
\psi(i, j)=\frac{1}{4}\left[2 \psi(i-1, j)+2 \psi(i, j-1)-h^{2} F(i, j)\right] \tag{10.23}
\end{equation*}
$$



Figure 10.5: Dielectric Boundary Condition

### 10.4.2 Dielectric Boundary Condition

Suppose we have two media with dielectric permittivity $\epsilon_{1}$ and $\epsilon_{2}$, Fig.(10.5). And we want to apply Laplace or Poisson's equation. What should we do at the interface of two media? From Gauss's law we have

$$
\begin{align*}
\oint_{s} \mathbf{D}(\mathbf{r}, t) \cdot d \mathbf{S} & =\int_{v} \rho(\mathbf{r}, t) d V \\
\int_{v} \nabla \cdot \mathbf{D} d V & =\int_{v} \rho d V \tag{10.24}
\end{align*}
$$

According to Fig.(10.5) we can rewrite Eq.(10.24) as

$$
\begin{equation*}
\oint_{c} \mathbf{D} \cdot d \mathbf{L}=\oint_{c} \epsilon \mathbf{E} \cdot d \mathbf{L}=Q_{e n c}=0 \tag{10.25}
\end{equation*}
$$

At the interface we don't have free charge. In static case we have $\mathbf{E}=-\nabla V$ and substituting in Eq.(10.25) gives

$$
\begin{equation*}
\oint_{c} \epsilon \nabla V \cdot d \mathbf{L}=\oint_{c} \epsilon \frac{\partial V}{\partial n} d l=Q_{e n c}=0 \tag{10.26}
\end{equation*}
$$

where $\frac{\partial V}{\partial n}$ shows the derivative of $V$ normal to the contour $c$. According to Fig.(10.5) and Eq.(10.26) we will have

$$
\begin{aligned}
& \oint_{c} \epsilon \frac{\partial V}{\partial n} d l=\left(\frac{V_{i, j+1}-V_{i, j}}{h}\right) h \epsilon_{1}+\left(\frac{V_{i, j-1}-V_{i, j}}{h}\right) h \epsilon_{2}+ \\
& \left(\frac{V_{i+1, j}-V_{i, j}}{h}\right)\left(\frac{h}{2} \epsilon_{1}+\frac{h}{2} \epsilon_{2}\right)+\left(\frac{V_{i-1, j}-V_{i, j}}{h}\right)\left(\frac{h}{2} \epsilon_{1}+\frac{h}{2} \epsilon_{2}\right)=0(10.27)
\end{aligned}
$$

Rearranging terms and dividing by $\epsilon_{0}$ :

$$
\begin{align*}
& \epsilon_{r 1} V_{i, j+1}+\epsilon_{r 2} V_{i, j-1}+\frac{\epsilon_{r 1}+\epsilon_{r 2}}{2} V_{i-1, j}+ \\
& \frac{\epsilon_{r 1}+\epsilon_{r 2}}{2} V_{i+1, j}-4 \frac{\epsilon_{r 1}+\epsilon_{r 2}}{2} V_{i, j}=0 \tag{10.28}
\end{align*}
$$

or

$$
\begin{equation*}
V_{i, j}=\frac{\epsilon_{r 1}}{2\left(\epsilon_{r 1}+\epsilon_{r 2}\right)} V_{i, j+1}+\frac{\epsilon_{r 2}}{2\left(\epsilon_{r 1}+\epsilon_{r 2}\right)} V_{i, j-1}+\frac{1}{4} V_{i-1, j}+\frac{1}{4} V_{i+1, j} \tag{10.29}
\end{equation*}
$$

The values are averaged at interfaces. This is not surprising, but we did not originally assume averaging. If $\epsilon_{r 1}=\epsilon_{r 2}$ we will have the same equation as before.

### 10.5 Laplace Equation and Finite Difference Method

Application of numerical differentiation can be used for Laplace and Poisson's Equation. Consider the Laplace equation in two dimension:

$$
\begin{equation*}
\frac{\partial^{2} V(x, y)}{\partial x^{2}}+\frac{\partial^{2} V(x, y)}{\partial y^{2}}=0 \tag{10.30}
\end{equation*}
$$

in rectangular domain described by x and y , we discretize the domain of our problem and use 5 -point computational molecules or Equal Arm Star, Fig10.1, $\Delta x=\Delta y=h$ we will have

$$
\begin{equation*}
\nabla^{2} V(x, y) \approx \frac{1}{h^{2}}\left(V_{i+1, j}+V_{i, j+1}+V_{i-1, j}+V_{i, j-1}-4 V_{i, j}\right) \tag{10.31}
\end{equation*}
$$

## - Example

Suppose we have a $3 \mathrm{~cm} \times 4 \mathrm{~cm}$ rectangle subject to the boundary conditions:

$$
\begin{align*}
V(0, y) & =10 \text { volts } \\
V(4, y) & =0 \text { volts } \\
V(x, 0) & =20 \text { volts } \\
V(x, 3) & =40 \text { volts } \tag{10.32}
\end{align*}
$$

let us lay out coarse mesh; Fig.(10.1) and $h=1$ Applying difference equation at each node $1 \cdots 6$ we will have;
$V(I-1, J)-4 V(I, J)+V(I+1, J)+V(I, J-1)+V(I, J+1)=0$

$$
\begin{align*}
& -4 V_{1}+10+V_{3}+40+V_{2}=0 \\
& -4 V_{2}+20+V_{1}+20+V_{4}=0 \\
& -4 V_{3}+40+V_{1}+V_{4}+V_{5}=0 \\
& -4 V_{4}+20+V_{2}+V_{3}+V_{6}=0 \\
& -4 V_{5}+40+V_{3}+V_{6}+0.0=0 \\
& -4 V_{6}+20+V_{4}+V_{5}+0.0=0 \tag{10.33}
\end{align*}
$$

or in matrix form

$$
\left[\begin{array}{cccccc}
-4 & 1 & 1 & 0 & 0 & 0  \tag{10.34}\\
1 & -4 & 0 & 1 & 0 & 0 \\
1 & 0 & -4 & 1 & 1 & 0 \\
0 & 1 & 1 & -4 & 0 & 1 \\
0 & 0 & 1 & 0 & -4 & 1 \\
0 & 0 & 0 & 1 & 1 & -4
\end{array}\right]\left[\begin{array}{c}
V_{1} \\
V_{2} \\
V_{3} \\
V_{4} \\
V_{5} \\
V_{6}
\end{array}\right]=\left[\begin{array}{l}
-50 \\
-30 \\
-40 \\
-20 \\
-40 \\
-20
\end{array}\right]
$$

### 10.6 Solution Methods of Difference Equation

There are two main methods for solving the system of equation:
a) Direct Method (Band Matrix Method)
b) Iterative Method


Figure 10.6: Application of Laplace's Equation

### 10.6.1 Band Matrix Method

Provided the boundary conditions are linear, as we got, the resulting system of equations may be written as

$$
\begin{equation*}
\mathrm{A} \cdot \mathrm{X}=\mathrm{B} \tag{10.35}
\end{equation*}
$$

The structure of the system ensures $\mathbf{A}$ is relatively sparse matrix, and $\mathbf{X}$ is column vector: values of free nodes (unknown) and $\mathbf{B}$ is also column vector (known). By using Gauss elimination method or LU Decomposition we can find the unknowns. Careful choice of the order the matrix element may help reduce the size of this matrix.
Because of the wide spread need to solve Laplace's Equations, specialist solvers have been developed for this problem. One of the best of these is Hockney's method. For more information you can see literatures. For the above example, code is written and Gauss elimination is used so the result would be

$$
\begin{align*}
& V_{1}=23.56108 \text { volts } \\
& V_{2}=18.34369 \text { volts } \\
& V_{3}=25.90062 \text { volts } \\
& V_{4}=19.81366 \text { volts } \\
& V_{5}=20.22774 \text { volts } \\
& V_{6}=15.01035 \text { volts } \tag{10.36}
\end{align*}
$$

### 10.6.2 Iterative Methods

An alternative to direct solution of finite difference equation is an iterative numerical solution. These iterative methods are often referred to as relaxation methods as an initial gauss at the solution is allowed to relax towards the true solution, reducing the error as it dose so. There are a variety of approaches with differing complexity and speed:
a) Jacobi
b) Gauss-Seidel
c) Successive Over Relaxation (SOR)

## - Jacobi

The Jacobi is the simplest type of relaxation method. Let us consider the special case when $\Delta x=\Delta y$. To find the solution for a 2D Laplace equation, we use the following algorithm:

1) Initialize some value for $V_{i, j}^{k}$ (Initial Guess).
2) Apply the boundary conditions.
3) For each internal mesh point set

$$
\begin{equation*}
V_{i, j}^{(k+1)}=\left(V_{i+1, j}^{(k)}+V_{i-1, j}^{(k)}+V_{i, j+1}^{(k)}+V_{i, j-1}^{(k)}\right) / 4 \tag{10.37}
\end{equation*}
$$

4) Replace old solution $V_{i, j}^{(k)}$ with new $V_{i, j}^{(k+1)}$.
5) If solution dose not satisfy tolerance, repeat from step 2 .

## - Gauss-Seidel

The Gauss-Seidel Iteration is very similar to Jacobi Iteration, the only difference being that the new estimate $V_{i, j}^{(k+1)}$ is returned to solution $V_{i, j}^{(k)}$ as soon as it is completed, allowing it to be used immediately rather than deferring its use to the next iteration. The advantages of this are:
a) Less memory required (there is no need to store $V_{i, j}^{(k+1)}$ )
b) Faster convergence

## - Successive over relaxation (SOR)

Suppose we want to solve $\mathbf{A} \cdot \mathbf{X}=\mathbf{B}$. We can write $\mathbf{A}=\mathbf{I}-\mathbf{C}$, therefore

$$
\begin{equation*}
(\mathbf{I}-\mathbf{C}) \cdot \mathbf{X}=\mathbf{B} \quad \Rightarrow \quad \mathbf{X}=(\mathbf{I}-\mathbf{C})^{-\mathbf{1}} \cdot \mathbf{B} \tag{10.38}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathbf{X}=(\mathbf{I}-\mathbf{C})^{-1} \mathbf{B}=\left(\mathbf{I}+\mathbf{C}+\mathbf{C}^{2}+\mathbf{C}^{3}+\cdots\right) \cdot \mathbf{B} \tag{10.39}
\end{equation*}
$$

If we look carefully to Eq.(10.39); it tell us that we can get the result by iteration. Therefore

$$
\begin{equation*}
\mathbf{X}^{(k+1)}=\mathbf{C} \cdot \mathbf{X}^{(k)}+\mathbf{B} \tag{10.40}
\end{equation*}
$$

provided that iteration be convergence. Now let us find the condition of convergence. From mathematical view point the Eq.(10.40) is

$$
\begin{equation*}
\mathbf{X}=\sum_{n=0}^{\infty} \mathbf{C}^{n} \cdot \mathbf{B}=\left(\mathbf{I}+\mathbf{C}+\mathbf{C}^{2}+\mathbf{C}^{3}+\cdots\right) \cdot \mathbf{B} \tag{10.41}
\end{equation*}
$$

This iteration converge if and only if that the absolute value of greatest eigenvalue of matrix $\mathbf{C}$ be less than one. This can be estimated very easily by finding greatest eigenvalue of matrix $\mathbf{C}$ by power method.
It may be the greatest eigenvalue of the matrix $\mathbf{C}$ be nearly one. In this case the iterative method will be slow and time consuming. Is there any rule that can be apply in order to increase the speed and get the result? Yes, Mathematician tell us that we can do something. This method is called Over-Relaxation. First, Instead of relation $\mathbf{X}^{k+1}=\mathbf{C} \cdot \mathbf{X}^{(k)}+\mathbf{B}$ we use $\mathbf{U}^{(k+1)}=\mathbf{C} \cdot \mathbf{X}^{(k)}+\mathbf{B}$ then the value of $\mathbf{U}^{(k+1)}-\mathbf{X}^{(k)}$ will be multiply by $\omega$ and will be added to $\mathbf{X}^{(k)}$ therefore

$$
\begin{equation*}
\mathbf{X}^{(k+1)}=\mathbf{X}^{(k)}+\omega\left(\mathbf{U}^{(k+1)}-\mathbf{X}^{(k)}\right) \tag{10.42}
\end{equation*}
$$

where $\omega$ is called Over Relaxation Factor and is greater than one. If $\omega$ is less than one, it is called Under Relaxation Factor. Usually $1<\omega<2$ can be determined by trial and error. So

$$
\begin{gather*}
\mathbf{X}^{(k+1)}=[\mathbf{I}-\omega(\mathbf{I}-\mathbf{C})] \cdot \mathbf{X}^{(k)}+\omega \mathbf{B}  \tag{10.43}\\
\mathbf{X}^{(k+1)}=[\mathbf{I}-\omega \mathbf{A}] \cdot \mathbf{X}^{(k)}+\omega \mathbf{B} \tag{10.44}
\end{gather*}
$$

If $\left\{\lambda_{i}\right\}$ be the eigenvalues of matrix $\mathbf{C}$, then the eigenvalues of $\mathbf{C}^{\prime}=\mathbf{I}-$ $\omega(\mathbf{I}-\mathbf{C})$ will be $\left\{1-\omega\left(1-\lambda_{i}\right)\right\}$.
To apply the method of SOR:
a) define residual $R^{(k)}$ at node $V_{i, j}$ at k iteration

$$
\begin{equation*}
R=V_{i+1, j}+V_{i-1, j}+V_{i, j+1}+V_{i, j-1}-4 V_{i, j} \tag{10.45}
\end{equation*}
$$

b) use iteration with relaxation factor $\omega$

$$
\begin{equation*}
V_{i, j}^{(k+1)}=V_{i, j}^{(k)}+\frac{\omega}{4} R^{(k)} \tag{10.46}
\end{equation*}
$$

or simply

$$
\begin{equation*}
V_{i, j}^{(k+1)}=(1-\omega) V_{i, j}^{(k)}+\frac{\omega}{4}\left(V_{i+1, j}^{(k)}+V_{i-1, j}^{(k)}+V_{i, j+1}^{(k)}+V_{i, j-1}^{(k)}\right) \tag{10.47}
\end{equation*}
$$

In rectangular shape region the optimum over relaxation factor is given by:

$$
\begin{align*}
t & =\cos \left(\frac{\pi}{N_{x}}\right)+\cos \left(\frac{\pi}{N_{y}}\right) \\
\omega & =\frac{8-\sqrt{64-16 t^{2}}}{t^{2}} \tag{10.48}
\end{align*}
$$

In the case of multiple dielectric, Fig.(10.5)

$$
\begin{align*}
& \epsilon_{r 1} V_{i, j+1}+\epsilon_{r 2} V_{i, j-1}+\left(\frac{\epsilon_{r 1}+\epsilon_{r 2}}{2}\right) V_{i-1, j}+\left(\frac{\epsilon_{r 1}+\epsilon_{r 2}}{2}\right) V_{i+1, j} \\
& -4\left(\frac{\epsilon_{r 1}+\epsilon_{r 2}}{2}\right) V_{i, j}=-\frac{\rho_{v} h^{2}}{2 \epsilon_{0}} \tag{10.49}
\end{align*}
$$

perfect dielectric have no free charges, so $\rho_{v}=0$. Now we can use SOR

$$
\begin{gather*}
R_{i, j}^{(k)}=\left(\frac{2 \epsilon_{r 1}}{\epsilon_{r 1}+\epsilon_{r 2}}\right) V_{i, j+1}^{(k)}+\left(\frac{2 \epsilon_{r 1}}{\epsilon_{r 1}+\epsilon_{r 2}}\right) V_{i, j-1}^{(k)}+V_{i+1, j}^{(k)}+V_{i-1, j}^{(k)} \\
-4 V_{i, j}^{(k)}+\frac{\rho_{v}}{\epsilon_{0}\left(\epsilon_{r 1}+\epsilon_{r 2}\right)}  \tag{10.50}\\
V_{i, j}^{(k+1)}=V_{i, j}^{(k)}+\frac{\omega}{4} R_{i, j}^{(k)} \tag{10.51}
\end{gather*}
$$

### 10.7 Application of Finite Difference

Where there is a differential equation, in most cases FD method is applicable. So in EM problem.

### 10.7.1 Microstrip Transmission Line

The formulated difference equation with its special case described in the previous section may be applied to variety of engineering problems. Fig.(10.7) was actually produced from paper [1] where the authors used a variational technique to calculate the characteristic impedance and velocity of propagation in microstrips or partially filled coaxial transmission lines. In this kind of general problem, the student will use difference equations, developed for both equal and unequal arms, at an interface between difference dielectrics, and may even utilize derivative boundary condition if symmetry is considered in the solution. For simplicity, the microstrip structures with open boundaries may be treated with Dirichlet boundary conditions at artificial boundaries placed sufficiently far. Clearly, there is a tradeoff for how far such boundary should be. Through several trial, students will quickly learn that excessively far boundaries require very large matrices when a mesh or reasonable size is used while very close boundaries are inaccurate unless a reasonable value of $V$ (instead of the assumed zero value) is known on these boundaries(which is not the case). Hence, a sufficiently far boundary to justify the assumption of $V=0$ on it while maintaining a reasonably size matrix is desired. The


Figure 10.7: Transmission Line
only remaining question to be expected from students is how to relate calculated values of the node potential to the engineering quantities of interest, such as the characteristic impedance $Z_{0}$ and the velocity of propagation $V_{p}$
in the guiding structures of Fig.(10.7). Assuming TEM mode of propagation in these structures, these values are given by

$$
\begin{equation*}
V_{p}=\frac{1}{\sqrt{L C}}, \quad Z_{0}=\sqrt{\frac{L}{C}} \tag{10.52}
\end{equation*}
$$

where $L$ and $C$ are the inductance and capacitance per unit length, respectively. If the dielectric loading is assumed to have no effect on value of $L$, then

$$
\begin{gather*}
Z_{0}=\frac{\sqrt{L C_{0}}}{\sqrt{C C_{0}}}=\frac{1}{V_{0} \sqrt{C C_{0}}}  \tag{10.53}\\
V_{p}=\frac{1}{\sqrt{L C}}=V_{0} \sqrt{\frac{C_{0}}{C}} \tag{10.54}
\end{gather*}
$$

where $V_{0}=\frac{1}{\sqrt{L C_{0}}} \approx 3 \times 10^{8} \mathrm{~m} / \mathrm{s}$.
$C_{0}=$ Capacitance per unit length of transmission line without dielectric.
$C=$ Capacitance per unit length of transmission line with dielectric.
Calculations of capacitance from the obtained potential distribution may be done through Gauss's law: hence,

$$
\begin{align*}
C & =\frac{q}{V} \\
q & =\oint_{s} \epsilon \mathbf{E} \cdot d \mathbf{S}=-\oint_{c} \epsilon \nabla \psi \cdot d \mathbf{C}=-\oint_{c} \epsilon \frac{d \psi}{d n} d c \tag{10.55}
\end{align*}
$$

where the two-dimensional closed surface S in Eq.(10.55) was replaced by the closed contour C. This results in a charge $q$ in columbus per unit length. Evaluating Eq.(10.55) by utilizing the discrete node values of $V$ (see Fig.(10.8) and using numerical differentiation and integration:

$$
\begin{equation*}
\frac{d \psi}{d n}=\left(\psi_{n+1}+\psi_{n}\right) / d n \tag{10.56}
\end{equation*}
$$

trapezoidal integration $=h\left[f(a) / 2+f(b) / 2+\sum(f)\right]:$

$$
\begin{equation*}
\frac{d V}{d n}=\frac{\left(V_{j, 2}-V_{j, 1}\right)}{h_{x}} \tag{10.57}
\end{equation*}
$$



Figure 10.8: Transmission Line

$$
\begin{align*}
\oint_{c 1} \epsilon \frac{\partial \psi}{\partial n} \cdot d l_{1}= & h_{y}\left\{\epsilon_{1}\left[\frac{\psi_{21}-\psi_{11}}{h_{x}}+\frac{\psi_{22}-\psi_{12}}{h_{x}}+\frac{\psi_{23}-\psi_{13}}{h_{x}}\right]\right. \\
& +\left(\frac{\epsilon_{1}+\epsilon_{2}}{2}\right)\left[\frac{\psi_{24}-\psi_{14}}{h_{x}}\right] \\
& \left.+\epsilon_{2}\left[\frac{\psi_{25}-\psi_{15}}{h_{x}}+\frac{\psi_{26}-\psi_{16}}{h_{x}}\right]\right\} \tag{10.58}
\end{align*}
$$

where $V_{0}$ is the initially assumed potential difference between the center conductor and the ground.
The procedure in Eq.(10.58) is then repeated with the transmission line completely field with air. (i.e., the dielectric is removed) to calculate $C_{0}$. Values of $C$ and $C_{0}$ are used to calculate the characteristic impedance $Z_{0}$ and the velocity of propagation as given in Eq.(10.52).

### 10.8 Application to Eigenvalue Problems

In the previous section, attention was focused on solving partial differential equations where the scalar function $\psi$ was the only unknown. In eigenvalue problems, including Helmholtz equations $\left(\nabla^{2}+\beta^{2}\right) \psi=0, \psi$ is not the only unknown, but instead both $\beta$ and $\psi$ are to be determined. For each value of the eigenvalue $\beta_{i}$ there is a solution for $\psi_{i}$ that represents the corresponding
eigenfunction. In waveguide problems, for example, there is a $\psi_{i}$ distribution (field configuration of a propagating mode) for each value of the cutoff wave number $\beta_{i}$. In cavity resonator $\beta_{i}$ gives the resonant frequency of cavity and $\psi_{i}$ field configuration inside cavity. In such problems, by discretizing the cross section of the waveguide or cavity by suitable square mesh and applying the finite difference representation of the Helmholtz equation at each node, we obtain the following matrix equation

$$
\begin{equation*}
(\mathbf{A}-\lambda \mathbf{I}) \Psi=0 \tag{10.59}
\end{equation*}
$$

where $\mathbf{A}$ is the coefficient matrix that results from applying the difference equation at each node, $\lambda=\left(4-h^{2} \beta^{2}\right)$ is the unknown eigenvalues, and $\mathbf{I}$ is identity matrix. In Eq.(10.59), both the eigenvalue $\lambda$ and the eigenvector $\Psi$ are unknowns and must be determined. There are several ways of determining $\lambda$ 's and the corresponding value of $\psi$ 's. The following is summary of these options:

1) First, Eq.(10.59) can be satisfied only if $\operatorname{det}[\mathbf{A}-\lambda \mathbf{I}]=0$. Hence calculating $\operatorname{det}[\mathbf{A}-\lambda \mathbf{I}]=0$ will result in a polynomial in $\lambda$, which can then be solved for the various eigenvalues $\lambda$ 's. For each of these eigenvalues, the corresponding eigenfunction $\Psi$ may be obtain from Eq.(10.59).
2) The second alternative is to use the power method for solving eigenvalue problems. In this iterative method we search for an eigenfunction $\Psi$ that satisfies the following equation:

$$
\begin{equation*}
\mathbf{A} \Psi=\lambda \Psi \tag{10.60}
\end{equation*}
$$

i.e., when $\mathbf{A}$ is multiplied by $\Psi$, the result will be constant multiplied by the same $\Psi$. Hence, the iterative procedure starts by assuming the vector $\Psi$ (contains the value of $\psi$ at the various nodes) and through the repeated multiplications of $\Psi$ with $\mathbf{A}$, the solution will converge to a vector $\Psi$ that satisfies Eq.(10.60) multiplied by $\lambda^{m}$ where $m$ is the number of repeated multiplications needed for the solution to converge. Detailed examination of this method will prove its convergence for an arbitrary choice of the initial assumption of the vector $\Psi$ as described elsewhere. The power method, however, provides the eigenvector $\Psi$ with largest eigenvalue $\lambda$. Since, in waveguide problems we are interested in solution with the smallest eigenvalue, i.e., modes of lowest cutoff frequencies, the power method should be applied on $\mathbf{A}^{-1}$, the inverse of matrix $\mathbf{A}$. From Eq.(10.60)we have

$$
\begin{equation*}
\mathbf{A}^{-1} \mathbf{A} \Psi=\lambda \mathbf{A}^{-1} \Psi \tag{10.61}
\end{equation*}
$$

and since $\mathbf{A}^{-1} \mathbf{A}=\mathbf{I}$ we obtain

$$
\begin{equation*}
\mathbf{A}^{-1} \Psi=\frac{1}{\lambda} \Psi \tag{10.62}
\end{equation*}
$$

Hence, applying the power method on $\mathbf{A}^{-1}$ results in the $\Psi$ solution with largest value which corresponds to the smallest eigenvalue $\lambda$.
3) The other available option is to use is to use the double iterative procedure based on Liebmann's method. The iterative procedure starts with assuming a value for the eigenvalue $\lambda=4-h^{2} \beta^{2}$. The potential $\psi_{i, j}^{k-1}$ at the $(i, j)$ th mode in the $(k-1)$ th iteration is obtained from its known value in the $(k)$ th iteration by

$$
\begin{equation*}
\psi_{i, j}^{(k+1)}=\psi_{i, j}^{(k)}+\frac{\omega R_{i, j}}{\left(4-h^{2} \beta^{2}\right)} \tag{10.63}
\end{equation*}
$$

where $\omega$ is the acceleration factor $1<\omega<2$ that may be used to speed up the convergence of the solution and $R_{i, j}$ is the residual at $(i, j)$ th node calculated from

$$
\begin{equation*}
R_{i, j}=\psi_{i, j+1}+\psi_{i, j-1}+\psi_{i+1, j}+\psi_{i-1, j}-\left(4-h^{2} \beta^{2}\right) \psi_{i, j} \tag{10.64}
\end{equation*}
$$

After a few iterations using Eq.(10.63) to improve the initial assumption for the $\psi$ 's values, the value of the eigenvalue $\lambda=4-h^{2} \beta^{2}$ should be updated using Rayleigh formula

$$
\begin{equation*}
\beta^{2}=\frac{\iint_{s} \psi \nabla^{2} \psi d s}{\iint_{s} \psi^{2} d s} \tag{10.65}
\end{equation*}
$$

Replacing $\nabla^{2} \psi$ in Eq.(10.65) by its difference equation and carrying out the integration in Eq.(10.65) using the discrete values of $\psi$, we obtain

$$
\begin{equation*}
\beta^{2}=\frac{\sum_{i=1}^{N} \sum_{j=1}^{M} \psi_{i, j}\left[\psi_{i+1, j}+\psi_{i-1, j}+\psi_{i, j+1}+\psi_{i, j-1}-4 \psi_{i, j}\right]}{h^{2} \sum_{i=1}^{N} \sum_{j=1}^{M} \psi_{i, j}^{2}} \tag{10.66}
\end{equation*}
$$

where the summation is carried out over all points in the domain of interest. The iteration procedure involves carrying out Eq.(10.63) for a few iterations and then updating the eigenvalue using Eq.(10.66). The $\psi$ distribution from Eq.(10.63) should continue until convergent solution is obtained.

## - 1D Cavity Resonator

Let us find the resonant frequency of one simple cavity resonator by FD method. Suppose we have two parallel perfect conducting plate, separated by distance $d$. The resonant frequency of such one dimensional cavity analytically can be found by wave equation and boundary conditions. Fig.(10.9)


Figure 10.9:

$$
\begin{align*}
E_{x} & =A e^{-j \beta z}+B e^{j \beta z} \\
H_{y} & =\frac{A e^{-j \beta z}-B e^{j \beta z}}{\eta} \tag{10.67}
\end{align*}
$$

applying boundary condition; $E_{x}=0$ at $z=0$ and $z=d$, we will have $-2 j A \sin \beta d=0$ which gives the resonant frequency of 1D cavity resonator

$$
\begin{equation*}
f_{r}=\frac{m}{2 d \sqrt{\mu \epsilon}} \quad m=1,2, \cdots \tag{10.68}
\end{equation*}
$$

Now let us find the resonant frequency by using Eq.(10.8).

$$
\begin{equation*}
16\left(\psi_{i+1}+\psi_{i-1}\right)-\left(\psi_{i+2}+\psi_{i-2}\right)-\left(30-12 h^{2} \beta^{2}\right) \psi_{i}=0 \tag{10.69}
\end{equation*}
$$

according to Fig.(10.9) we will have four linear equations, where $\lambda=(30-$ $12 h^{2} \beta^{2}$ ) and $\psi=E x$

$$
\begin{align*}
-\lambda \psi_{2}+16 \psi_{3}-\psi_{4} & =0 \\
16 \psi_{2}-\lambda \psi_{3}+16 \psi_{4}-\psi_{5} & =0 \\
-\psi_{2}+16 \psi_{3}-\lambda \psi_{4}+16 \psi_{5} & =0 \\
-\psi_{3}+16 \psi_{4}-\lambda \psi_{5} & =0 \tag{10.70}
\end{align*}
$$

In matrix form

$$
\left[\begin{array}{cccc}
-\lambda & 16 & -1 & 0  \tag{10.71}\\
16 & -\lambda & 16 & -1 \\
-1 & 16 & -\lambda & 16 \\
0 & -1 & 16 & -\lambda
\end{array}\right]\left[\begin{array}{l}
\psi_{2} \\
\psi_{3} \\
\psi_{4} \\
\psi_{5}
\end{array}\right]=0
$$

this matrix, Eq.(10.71) will give us the following eigenvalues $\lambda \approx-9.0,25.0,-26.788,10.788$
If $\mathrm{d}=3$ centimeter, from relation $f_{r}=\frac{C}{2 \pi h} \sqrt{\frac{30-\lambda}{12}}$ we will find $f_{r} \approx 5.136,10.07,14.34,17.31[G H z]$.
The three modes of this type of cavity is plotted in Fig.(10.10).


Figure 10.10: Modes of 1D cavity Resonator

## - FD Frequency Domain Method Applied to Waveguide

As we have seen finite difference frequency domain method has been used to solve Laplace's and Poisson's equation. This can also be used to evaluate the scalar wave equation (Helmholtz Equation). Consider a waveguide, uniform along the z direction, with an arbitrarily shape cross section located in $x-y$ plane. Let the waveguide be loaded homogeneously with isotropic lossless material. Assume that modes vary as $e^{j\left(\omega t-\beta_{z} z\right)}$ where $\beta_{z}$ is propagation constant ( $\mathrm{rad} / \mathrm{sec}$ ).
All TE and TM mode fields may be derived from the scalar potential function which satisfies the two dimensional Helmholtz equation

$$
\begin{align*}
& \nabla_{t}^{2} \psi+\beta_{c}^{2} \psi=0 \\
& \psi_{i+1, j}+\psi_{i-1, j}+\psi_{i, j+1}+\psi_{i, j-1}-\left(4-h^{2} \beta_{c}^{2}\right) \psi_{i, j}=0 \tag{10.72}
\end{align*}
$$

where $\psi$ may be $\psi=E_{z}$ for TM modes and $\psi=H_{z}$ for TE modes and $\beta_{c}$ is cutoff wave number. For any given mode these constants are related through

$$
\begin{equation*}
\beta_{z}^{2}=\omega^{2} \mu \epsilon-\beta_{c}^{2} \tag{10.73}
\end{equation*}
$$

In rectangular waveguide we have:

$$
\begin{equation*}
\beta_{z}=\sqrt{\omega^{2} \mu \epsilon-\left(\frac{m \pi}{a}\right)^{2}-\left(\frac{n \pi}{b}\right)^{2}} \tag{10.74}
\end{equation*}
$$

where $a \& b$ are sides of rectangular waveguide and $\mu, \epsilon$ are respectively, the permeability and permittivity of isotropic medium filling the guide. Waveguides have different modes. Multiple modes can exist in a waveguide at a given time. The higher the frequency, the more modes can propagate. A waveguide's mode is determined by two things:
a) The frequency
b) The feed system

For TE Mode case $m=0,1,2 \cdots$ and $n=0,1,2, \cdots$ and for TM Mode case $m=1,2,3 \cdots$ and $n=1,2,3 \cdots$. When $\beta_{z}=0$ we can find cut off frequency of waveguides.

The field distribution in the waveguide can often be found as an analytical solution of the Helmholtz equation applying the boundary conditions. But if it cannot be found analytically, FD frequency domain is good method for such cases.
The potential function is subject to certain boundary conditions. For TM
modes $\psi=E z=0$ at a perfectly conducting surface. This is Dirichlet condition. For TE modes the $H_{z}$ is tangential to the boundaries, hence $E_{\text {tan }}$ should be zero at boundaries, therefore the Neumann boundary condition is that $\frac{\partial H_{z}}{\partial n}=0$ along such a surface.
Once solution of Eq.(10.72) are obtained subject to boundary conditions, the electric and magnetic fields may be derived from the following equations:

$$
\begin{align*}
T M \text { modes: } \mathbf{E}_{t} & =-\frac{j \beta_{z}}{\beta_{c}^{2}} \nabla_{t} E_{z} \\
\mathbf{H}_{t} & =-\frac{j \omega \epsilon}{\beta_{c}^{2}} \mathbf{a}_{z} \times \nabla_{t} E_{z} \\
H_{z} & =0 \\
T E \text { modes: } \mathbf{H}_{t} & =-\frac{j \beta_{z}}{\beta_{c}^{2}} \nabla_{t} E_{z} \\
\mathbf{E}_{t} & =\frac{j \omega \mu}{\beta_{c}^{2}} \mathbf{a}_{z} \times \nabla_{t} E_{z} \\
E_{z} & =0 \tag{10.75}
\end{align*}
$$

The steps to apply the FD method to Helmholtz Equation:

1. Divide the cross section of waveguide ( $\mathrm{x}-\mathrm{y}$ ) plane into mesh
2. Apply 5 -point difference formula Eq.(10.72).

## - Band Matrix Method

To apply the FD method, we discretize the cross section of the waveguide by suitable square mesh.

$$
\begin{equation*}
\psi_{i+1, j}+\psi_{i-1, j}+\psi_{i, j+1}+\psi_{i, j-1}-\left(4-h^{2} \beta_{c}^{2}\right) \psi_{i, j}=0 \tag{10.76}
\end{equation*}
$$

By applying Eq.(10.76) to all mesh nodes and using proper boundary condition, we will face with eigenvalue problem of the form:

$$
\begin{equation*}
(\mathbf{A}-\lambda \mathbf{I}) \psi=0 \tag{10.77}
\end{equation*}
$$

where $\mathbf{A}$ is a band matrix and $\lambda=\left(4-h^{2} \beta_{c}^{2}\right)$ is the eigenvalues of matrix $\mathbf{A}$. We may use power method to find the cutoff frequency of waveguide. This method is memory consuming and accuracy deteriorates rapidly for higher modes.

- Derive SOR

$$
\begin{gather*}
R_{i, j}=\frac{\psi_{i+1, j}+\psi_{i-1, j}+\psi_{i, j+1}+\psi_{i, j-1}-\left(4-h^{2} \beta_{z}^{2}\right) \psi_{i, j}}{\left(4-h^{2} \beta_{c}^{2}\right)}  \tag{10.78}\\
\psi_{i, j}^{(k+1)}=\psi_{i, j}^{(k)}+\omega R^{(k)} \tag{10.79}
\end{gather*}
$$

We GUESS $\beta_{z}$, and iterate 3 or 4 times, then apply the Rayleigh formula to find $\beta_{z}$ :

$$
\begin{equation*}
\beta_{c}^{2}=\frac{\int_{s} \psi \nabla^{2} \psi d S}{\int_{s} \psi^{2} d S} \tag{10.80}
\end{equation*}
$$

In FD format

$$
\begin{equation*}
\beta_{c}^{2}=\frac{\sum_{i=1} \sum_{j=1} \psi_{i, j}\left[\psi_{i+1, j}+\psi_{i-1, j}+\psi_{i, j+1}+\psi_{i, j-1}-4 \psi_{i, j}\right]}{h^{2} \sum_{i=1} \sum_{j=1} \psi_{i, j}^{2}} \tag{10.81}
\end{equation*}
$$

use this new value of $\beta_{c}$ and continue updating $\psi$.

### 10.9 Finite Difference Time Domain

The finite difference time domain (FDTD) method is one of the most widely used computational methods in electromagnetics. There are a number of reasons for this:
It is easy to understand, easy to implement in software, and since it is a timedomain technique it can cover a wide frequency range with a single simulation run. The FDTD method belongs in the general class of differential time domain numerical modeling method. Using FDTD Maxwell's equations are solved directly in time domain via finite difference and time stepping. The basic approach is relatively easy to understand and is an alternative to the more usual frequency-domain approaches.

## - How does FDTD work?

When Maxwell's differential form equations are examined, it can be seen that the time derivative of the $\mathbf{E}$ field is dependent on the curl of $\mathbf{H}$ field. This can be simplified to state that the change in the $\mathbf{E}$ (the time derivative) is
dependent on the change in the $\mathbf{H}$ field across space( the Curl). This results in the basic FDTD equation that the new value of the $\mathbf{E}$ field is dependent on the old value of $\mathbf{E}$ field (hence the difference in time) and the difference in the old value of the $\mathbf{H}$ field on either side of the $\mathbf{E}$ field point in space. Naturally this is a simple description, which has omitted constants, etc. But the overall effect is as described.
The $\mathbf{H}$ field is found in the same manner. The new value of $\mathbf{H}$ field is dependent on the old value of the $\mathbf{H}$ field(hence the difference in time), and also dependent on the difference in the $\mathbf{E}$ on either side of the $\mathbf{H}$ field point. This description holds true for 1D, 2D and 3D FDTD techniques. When multiple dimension are considered, the difference in space must be considered in all appropriate dimensions.

## - Using FDTD

In order to use FDTD a computational domain must be established. The computational domain is simply the "space" where the simulation will be performed. The $\mathbf{E}$ and $\mathbf{H}$ fields will be determined at every point within the computational domain. The material of each cell within the computational domain must be specified. Typically, the material will be either free space (air), metal (perfect electrical conductor(PEC)), or dielectric, any material can be used, as the permeability, permittivity, and conductivity can be specified.
Once the computational domain and the grid material is established, a source is specified. The source can be an impinging plane wave, a current on a wire, or an electric field between metal plates (basically a voltage between the two plates), depending on the type of situation to be modeled.
Since the $\mathbf{E}$ and $\mathbf{H}$ fields are determined directly, the output of the simulation is usually the $\mathbf{E}$ or $\mathbf{H}$ field at a point or a series of point within the computational domain.

## - What are the strengths of the FDTD Technique?

Every modeling technique has some strengths and some weaknesses. some types of models were a given technique will excel and some types of models were the same technique will have difficulty (if it is even possible to use) performing rapidly and accurately.
FDTD is a very versatile modeling technique. It is a very intuitive technique,
sources can easily understand how to use it, and know what to expect from a given model.
FDTD is a time domain technique, and when a time domain pulse (such as Gaussian pulse) is used as a source pulse, then a wide frequency range is solved with only one simulation. This is extremely useful in applications where resonant frequencies are not known exactly, or anytime that a broadband result is desired.
Since FDTD is a time domain technique which finds the $\mathbf{E}, \mathbf{H}$ fields everywhere in the computational domain, it lends itself to providing animation displays (movies) of the $\mathbf{E}, \mathbf{H}$ filed movement throughout the model. This type display is extremely useful to understanding exactly what is going on the model, and help insure that the model is working correctly.
FDTD allows user to specify the material at all points within the computational domain. All materials are possible and dielectrics, magnetic material, etc, can be simply modeled without the need to resort to 'work arounds' or 'tricks' to model these materials.
FDTD allows the effects of apertures to be determined directly. Shielding effects can be found, and the fields both inside and outside a the structure can be found directly.
FDTD provides the $\mathbf{E}$ and $\mathbf{H}$ fields directly. Since most EMI/EMC modeling application are interested in the $\mathbf{E}, \mathbf{H}$ fields, it is best that no conversions must be made after the simulation has run to get these values.
Since the computational domain must end at some point (or we would be modeling the entire universe !) a boundary must be established. FDTD has a number of good absorbing boundary condition to chose from ( and some that not quite so good). The absorbing boundary condition (ABC) simulates the effect of free space beyond the boundary forever.

## -What are the weaknesses of FDTD Technique?

Since FDTD requires that the entire computational domain be grided, and these grids must be small compared to the smallest wavelength and smaller than the smallest feature in the model, very large computational domain can be developed, which result in very long solution times. Model with long, thin features, (like wires) are difficult to model in FDTD because of the excessively large computational domain required.
FDTD finds the $\mathbf{E}$ and $\mathbf{H}$ fields directly everywhere in the computational domain. If the field values at some distance (like 10 meter away) are desired,
it is likely that this distance will force the computational domain to be excessively large. Far field extensions are available for FDTD, but require some amount of post processing.

### 10.9.1 One-Dimensional FDTD Formulation

In order to reduce the complexity of programming and display FDTD computation, we formulate the difference equation in one dimension. We assume that $\mathbf{E}$ and $\mathbf{H}$ are depend on time $t$ and space $z$ ie. $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$, therefore from Maxwell's Equation for source free region lossy medium but nonhomogeneous, we have

$$
\begin{align*}
\frac{\partial E_{x}}{\partial t} & =-\frac{1}{\epsilon(z)}\left[\frac{\partial H_{y}}{\partial z}-\sigma(z) E_{x}\right] \\
\frac{\partial H_{y}}{\partial t} & =-\frac{1}{\mu(z)} \frac{\partial E_{x}}{\partial z} \tag{10.82}
\end{align*}
$$

Now let us use finite difference method instead of a direct analytical which is usually done. Since the two equations are valid for every values of $z$ and $t$, we assume that $E_{x} \& H_{y}$ are continuous with respect to time and space. Let $t=n \Delta t$ and $z=i \Delta z$ and following Yee's notation

$$
\begin{align*}
& \frac{\partial E_{x}(z, t)}{\partial t} \simeq \frac{E_{x}[i \Delta z,(n+1) \Delta t]-E_{x}[i \Delta z, n \Delta t]}{\Delta t}=\frac{E_{x}^{n+1}(i)-E_{x}^{n}(i)}{\Delta t}(10.83) \\
& \quad \frac{\partial H_{y}\left(z+\frac{\Delta z}{2}, t\right)}{\partial t} \simeq \\
& \frac{H_{y}[(i+1 / 2) \Delta z,(n+1 / 2) \Delta t]-H_{y}[(i+1 / 2) \Delta z,(n-1 / 2) \Delta t]}{\Delta t}= \\
& \quad \frac{H_{y}^{n+1 / 2}(i+1 / 2)-H_{y}^{n-1 / 2}(i+1 / 2)}{\Delta t} \tag{10.84}
\end{align*}
$$

$$
\frac{\partial E_{x}(z, t)}{\partial z} \simeq \frac{E_{x}[(i+1) \Delta z, n \Delta t]-E_{x}[i \Delta z, n \Delta t]}{\Delta z}
$$

$$
\begin{equation*}
=\frac{E_{x}^{n}(i+1)-E_{x}^{n}(\bar{i})}{\Delta z} \tag{10.85}
\end{equation*}
$$

$$
\begin{align*}
\frac{\partial H_{y}(z, t)}{\partial z} & \simeq \frac{H_{y}[(i+1 / 2) \Delta z,(n+1 / 2) \Delta t]-H_{y}[(i-1 / 2) \Delta z,(n+1 / 2) \Delta t]}{\Delta z} \\
& =\frac{H_{y}^{n+1 / 2}(i+1 / 2)-H_{y}^{n+1 / 2}(i-1 / 2)}{\Delta z} \tag{10.86}
\end{align*}
$$

Using this type of notation, we can write Eq.(10.82) in finite difference approximations:

$$
\begin{align*}
& \frac{H_{y}^{n+1 / 2}(i+1 / 2)-H_{y}^{n-1 / 2}(i+1 / 2)}{\Delta t}= \\
& -\frac{1}{\mu(i+1 / 2)} \frac{E_{x}^{n}(i+1)-E_{x}^{n}(i)}{\Delta z} \tag{10.87}
\end{align*}
$$

or simply

$$
\begin{equation*}
H_{y}^{n+1 / 2}(i+1 / 2)=H_{y}^{n-1 / 2}(i+1 / 2)-R\left[E_{x}^{n}(i+1)-E_{x}^{n}(i)\right] \tag{10.88}
\end{equation*}
$$

where $R=\frac{\Delta t}{\mu(i+1 / 2) \Delta z}$. We do the same job to the other equation

$$
\begin{align*}
& \frac{E_{x}^{n+1}(i)-E_{x}^{n}(i)}{\Delta t}=-\frac{1}{\epsilon(i)}\left[H_{y}^{n+1 / 2}(i+1 / 2)\right. \\
& \left.-H_{y}^{n+1 / 2}(i-1 / 2)\right] / \Delta z-\frac{\sigma(i)}{\epsilon(i)} E_{x}^{n+1 / 2}(i) \tag{10.89}
\end{align*}
$$

the term $E_{x}^{n+1 / 2}(i)$ can be treated to be an average over time increments (i+1) and i;

$$
\begin{equation*}
E_{x}^{n+1 / 2}(i)=\frac{E_{x}^{n+1}(i)+E_{x}^{n}(i)}{2} \tag{10.90}
\end{equation*}
$$

Now rewriting it with new parameters we will have

$$
\begin{equation*}
E_{x}^{n+1}(i)=C_{a} E_{x}^{n}(i)-C_{b}\left[H_{y}^{n+1 / 2}(i+1 / 2)-H_{y}^{n+1 / 2}(i-1 / 2)\right] \tag{10.91}
\end{equation*}
$$

where

$$
\begin{align*}
C_{a} & =\frac{2 \epsilon(i)-\sigma(i) \Delta t}{2 \epsilon(i)+\sigma(i) \Delta t}  \tag{10.92}\\
C_{b} & =\frac{2 \frac{\Delta t}{\Delta z}}{2 \epsilon(i)+\sigma(i) \Delta t} \tag{10.93}
\end{align*}
$$

Equation Eq.(10.88) and Eq.(10.91) are useful for solution by iteration of each equation alternatively, or in another word time stepping algorithms for calculation of electric and magnetic field components. The update new value of a field component at any layer points (or cell) depends only on its value in the previous time step and the previous values of the components of the other field at the adjacent spatial points. Hence, at any given time step, the computation of a field component will proceed one point at a time.
The finite difference time domain numerical approach just discussed is quite useful for studying one dimensional time dependent electric and magnetic fields. The medium can have any conductivity, permittivity and permeability characteristics. Even the case of a one dimensional inhomogeneous medium, in the form of layered medium, can be conveniently modeled by just specifying medium parameters at appropriate spatial points. In this approach, there is no restriction on the selection of the type of time dependent excitation. One can discuss different pulse shapes relative to their frequency content.

### 10.9.2 Excitation pulse

In a transient analysis one is generally interested in determining the scattered response over a particular bandwidth of interest. To do this the input source must incorporate all of the frequencies of interest. This can be achieved by using a pulsed source, as opposed to a sinusoidal source. Generally there are two types of input pulses used in the FDTD, the first is a raised cosine, and the second is a Gaussian pulse.

## \& Raised Cosine Pulse

A raised cosine pulse consists of a single cycle of a cosine wave :

$$
E_{i}(t)=\left\{\begin{array}{l}
1-\cos \left(2 \pi F_{\max } t\right), \quad \text { if } \quad 0<t<1 / F_{\max }  \tag{10.94}\\
0 \quad \text { elsewhere }
\end{array}\right.
$$

We can define the impulse width as that where the function value is half its maximum, thus the width of a raised cosine is $1 / 2 F_{\text {max }}$, The spectrum of this pulse can be calculated by taking Fourier transform, the result is:

$$
\begin{equation*}
E(f)=\int_{-\infty}^{\infty} E_{i}(t) e^{-j 2 \pi f t} d t=\int_{0}^{1 / F_{\max }}\left[1-\cos \left(2 \pi F_{\max } t\right)\right] e^{-j 2 \pi f t} d t \tag{10.95}
\end{equation*}
$$

or after many manipulation

$$
\begin{equation*}
E(f)=\frac{e^{j \pi f / F_{\max }}}{F_{\max }\left[1-\left(\frac{f}{F_{\max }}\right)^{2}\right]} \frac{\sin \left(\pi \frac{f}{F_{\max }}\right)}{\left(\pi \frac{f}{F_{\max }}\right)} \tag{10.96}
\end{equation*}
$$

This function has a shape similar to $\operatorname{sinc}(x)$, or $\sin x / x$, but it decreases faster. In fact, the spectrum given by Eq.(10.95) has a maximum value at $f=0$ and half maximum value at $f=F_{\max }$. The first zero is located at $f=2 F_{\max }$. Therefore we can take $2 F_{\max }$ as the effective bandwidth of the raised cosine. As a result, the scattering field data, as determined from the FDTD from using a raised cosine pulse as incident wave, includes at least the scattering information in the frequency range of $\left(0,2 F_{\max }\right)$. The wavelength corresponding to $F_{\max }$ is $\lambda_{\min }=\frac{c}{F_{\max }}$. If the space increment is set as $\delta=\frac{\lambda_{\text {min }}}{20}$ and $c \Delta t=\delta / 2$, then we have $\delta=\frac{c}{20 F_{\text {max }}}$ and $\Delta t=\frac{1}{40 F_{\text {max }}}$, that is, the raised cosine is sampled 40 times over the duration of the raised cosine pulse.

## \& Gaussian Pulse

A traditional choice is the Gaussian pulse. We consider Gaussian pulse electric field in the form of:

$$
\begin{equation*}
E_{g}=E_{0} e^{-\frac{\left(t-t_{0}\right)^{2}}{2 \sigma^{2}}} \tag{10.97}
\end{equation*}
$$

where $E_{0}$ is amplitude, $t_{0}$ center of the pulse and $\sigma$ is related to pulse width. This pulse will be excited at point $z=z_{0}$. To illustrate the basic concept of the numerical simulation of wave, the propagation of a Gaussian pulse and half sinusoidal time pulse in many different medium ( lossless free space, lossy or layered) will be considered. The source generator develops a half sinusoidal pulse given by

$$
\begin{equation*}
E_{g}=E_{0} \sin (2 \pi f n \Delta t) \delta(i-1) \quad n=1,2, \cdots, N \tag{10.98}
\end{equation*}
$$

### 10.9.3 1D Helmholtz Wave Equation

Let us assume a one dimensional wave equation

$$
\begin{equation*}
\frac{\partial^{2} \psi(z, t)}{\partial z^{2}}=\frac{1}{C^{2}} \frac{\partial^{2} \psi(z, t)}{\partial t^{2}} \tag{10.99}
\end{equation*}
$$

that hitting boundary. We want to write a FORTRAN code

$$
\begin{equation*}
\psi^{n+1}(i)=\left(\frac{C \Delta t}{\Delta z}\right)^{2}\left[\psi^{n}(i+1)-2 \psi^{n}(i)+\psi^{n}(i-1)\right]+2 \psi^{n}(i)-\psi^{n-1}(i) \tag{10.100}
\end{equation*}
$$

if $\tau=\left(\frac{C \Delta t}{\Delta z}\right)=1$ then

$$
\begin{equation*}
\psi^{n+1}(i)=\psi^{n}(i+1)+\psi^{n}(i-1)-\psi^{n-1}(i) \tag{10.101}
\end{equation*}
$$

The FORTRAN CODE WAV1.FOR is written for simple one dimensional wave equation.

### 10.9.4 Stability in 1D FDTD

The stability and accuracy is another important issue which should be considered next. The choice of space increment $\Delta z$ and time increment $\Delta t$ is dictated by the reasons of accuracy and algorithm stability, respectively. Let us consider 1D wave equation

$$
\begin{equation*}
\frac{\partial^{2} \psi(z, t)}{\partial z^{2}}=\frac{1}{C^{2}} \frac{\partial^{2} \psi(z, t)}{\partial t^{2}} \tag{10.102}
\end{equation*}
$$

We approximate the above equation by central differences

$$
\begin{equation*}
\frac{\psi^{n+1}(i)-2 \psi^{n}(i)+\psi^{n-1}(i)}{(C \Delta t)^{2}}=\frac{\psi^{n}(i+1)-2 \psi^{n}(i)+\psi^{n}(i-1)}{(\Delta z)^{2}} \tag{10.103}
\end{equation*}
$$

we look for a solution of a wave form which has the properties of oscillation, attenuation and propagation.

$$
\begin{equation*}
\psi^{n}(i)=\left.\alpha^{n} e^{-j k z}\right|_{z=i \Delta z} \tag{10.104}
\end{equation*}
$$

this wave is stable if $|\alpha| \leq 1$ and unstable if $|\alpha|>1$. We plug solution into second order difference equation

$$
\begin{gather*}
\frac{\alpha^{n+1} e^{-j k i \Delta z}-2 \alpha^{n} e^{-j k i \Delta z}+\alpha^{n-1} e^{-j k i \Delta z}}{(C \Delta t)^{2}}= \\
\frac{\alpha^{n} e^{-j k(i+1) \Delta z}-2 \alpha^{n} e^{-j k i \Delta z}+\alpha^{n} e^{-j k(i-1) \Delta z}}{(\Delta z)^{2}}= \tag{10.105}
\end{gather*}
$$

By simplifying the equation (10.105) we will have

$$
\begin{equation*}
\frac{\alpha-2+\alpha-1}{(C \Delta t)^{2}}=\frac{e^{-j k \Delta z}-2+e^{j k \Delta z}}{(\Delta z)^{2}} \tag{10.106}
\end{equation*}
$$

This will give us

$$
\begin{equation*}
\alpha^{2}-2 A \alpha+1=0 \tag{10.107}
\end{equation*}
$$

where

$$
\begin{equation*}
A=1-2\left(\frac{C \Delta t}{\Delta z}\right)^{2} \sin ^{2}\left(\frac{k \Delta z}{2}\right) \tag{10.108}
\end{equation*}
$$

To ensure the stability of the computed fields $\Delta t$ is chosen to satisfy the inequality for the one dimensional layer model as

$$
\begin{equation*}
C_{\max } \Delta t \leq \Delta z \tag{10.109}
\end{equation*}
$$

where $C_{\text {max }}$ is maximum wave velocity within the model. This shows that in order to ensure stability, the spatial discretization and the time step cannot be increased beyond a certain limit. This means that a wave traveling from a neighboring unit cell must not pass though the next cell within the time step chosen. A conservative choice is

$$
\begin{equation*}
C_{\max } \cdot \Delta t=0.5 \cdot \Delta z \tag{10.110}
\end{equation*}
$$

The stability limit can be derived for higher dimensional cases. In the 3D case, we will have

$$
\begin{equation*}
C \Delta t \leq \frac{1}{\sqrt{\frac{1}{\Delta x^{2}}+\frac{1}{\Delta y^{2}}+\frac{1}{\Delta z^{2}}}} \tag{10.111}
\end{equation*}
$$

### 10.10 Absorbing Boundary Conditions

It is clear that we cannot simulate the propagation of the wave indefinitely, and we need to terminate somehow the FDTD grid. The problem does not exist in the case of spatially limited structure, like waveguide or cavity, where we need to mode a region that trap the field inside. In most of the problems, however, we need to simulate open space regions. In these cases, simulation region must be limited, so we need to find a way to simulate the open space. These boundary conditions are called Radiation Boundary Conditions or Absorbing Boundary Conditions (ABC). Such ABCs have been
the subject of investigations over many years. In the following sections, we briefly review the most common ABCs, paying special attention to perfectly matched layer. Next, we list the traditional local ABCs used in FDTD, and highlight some of their properties. The listing follows the guidelines given in, [28] chapter 7, in condensed form. The so called global ABCs never enjoyed much popularity in FDTD, because they involve very expensive integration of fields during each time step.
The most commonly used grid truncation techniques for open-region FDTD modeling problems are the Mur absorbing boundary condition (ABC), the Liao ABC, and various perfectly matched layer (PML) formulations. The Mur and Liao techniques are simpler than PML. However, PML can provide orders-of-magnitude lower reflections. The PML concept was introduced by J.-P. Berenger in a seminal 1994 paper in the Journal of Computational Physics. Since 1994, Berenger's original split-field implementation has been modified and extended to the uniaxial PML (UPML), the convolutional PML (CPML), and the higher-order PML. The latter two PML formulations have increased ability to absorb evanescent waves, and therefore can in principle be placed closer to a simulated scattering or radiating structure than Berenger's original formulation

### 10.10.1 Bayliss-Turkel ABC

The Bayliss-Turkel ABC [51] can be applied most naturally in spherical coordinates. Technically, it is a differential operator that annihilates a number of terms in a series expansion of the outgoing field. The form of the annihilator resembles that of the Sommerfeld radiation boundary condition, but the Bayliss-Turkel operators are enhanced to annihilate not only $1 / r$ terms, but also higher-order terms. The order of the operator can be taken as a parameter. Usually a second-order operator is used, providing remainder terms of the order $1 / r^{5}$.
With moderate modifications, the Bayliss-Turkel operator can be applied also in cylindrical coordinates, but unfortunately the application in Cartesian coordinates is not feasible. The practical realizations are reported to provide reflections on the order of -40 dB [28].

### 10.10.2 Engquist-Majda operator and Mur ABC

Engquist-Majda developed a famous pseudodiffrential operator allowing wave propagate in only one direction [28]. Theoretically, the ABC based on Engquist-Majda operator is perfect for dispersionless media. The difficulty lies in the realization of the pseudodifferential operator. Many subsequent practical ABCs involve different approximations of the operator

$$
\begin{equation*}
\sqrt{1-S^{2}}, \quad \text { where } \quad S=c \frac{\partial_{s}}{\partial_{t}} \tag{10.112}
\end{equation*}
$$

Approximating Eq.(10.112) by polynomials of $S$ results in realizable algorithms. The simplest approximations are based on Taylor expansion, and efficiently implemented in FDTD by Mur [29]. The second-order Mur is a notable simple ABC and a preferred choice if extremely good absorption is not crucial.
In 1D formulation, we can observe that if we use $d t=\frac{d z}{2 C}$, since the field travel at the speed of $C$, in one time step the field will travel only half a cell. This means that to entirely cross one cell two time step are necessary. The absorbing boundary conditions for 1D case can be therefore expressed by:

$$
\begin{array}{ll}
\psi^{n+1 / 2}(1)=\psi^{n-2+1 / 2}(2) & \text { for left side of the mesh }  \tag{10.113}\\
\psi^{n+1 / 2}(N)=\psi^{n-2+1 / 2}(N-1) & \text { for right side of the mesh }
\end{array}
$$

Now let us see what will we get for 2D problems. The Eq.(10.99) can be written in 2D as:

$$
\begin{equation*}
\psi_{x x}+\psi_{y y}-\psi_{t t} / C^{2}=0 \tag{10.114}
\end{equation*}
$$

where $\psi_{x x}=\frac{\partial^{2} \psi(x, y, t)}{\partial x^{2}}$ and using $L$ as operator notation, 2 D wave equation will be $L \psi=0$ where operator $L=D_{x}^{2}+D_{y}^{2}-D_{t}^{2}$ and $D_{x}^{2}=\frac{\partial^{2}}{\partial x^{2}}, D_{y}^{2}=\frac{\partial^{2}}{\partial y^{2}}, D_{t}^{2}=$ $\frac{1}{C^{2}} \frac{\partial^{2}}{\partial t^{2}}$. The $L \psi$ can be factored into forward and backward traveling wave ie:

$$
\begin{equation*}
L \psi=L^{+} L^{-} \psi=0 \tag{10.115}
\end{equation*}
$$

where

$$
\begin{align*}
L^{-} & =D_{x}-D_{t} \sqrt{1-S^{2}} \\
L^{+} & =D_{x}+D_{t} \sqrt{1-S^{2}} \\
S & =\frac{D_{y}}{D_{t}} \tag{10.116}
\end{align*}
$$

Enquist and Majda show that a x-traveling wave will be absorbed if $L^{-} \psi=0$ at $\mathrm{x}=0$,. This is $E X A C T$ and no approximation are made.
If you find $\psi$ such that $L^{-} \psi=0$, then the wave will be perfectly absorbed. None will reflect. For discrete programming (such as FDTD), approximation of $L^{-}$are made:
First Order:(Using Taylor Expansion)

$$
\begin{equation*}
\sqrt{1-S^{2}} \approx 1 \tag{10.117}
\end{equation*}
$$

therefore

$$
\begin{equation*}
L^{-} \approx \frac{\partial}{\partial x}-\frac{1}{C} \frac{\partial}{\partial t} \tag{10.118}
\end{equation*}
$$

or

$$
\begin{gather*}
\left.\left(\frac{\partial}{\partial x}-\frac{1}{C} \frac{\partial}{\partial t}\right) \psi\right|_{x=0}=0  \tag{10.119}\\
\Gamma=\frac{\cos (\theta)-1}{\cos (\theta)+1} \tag{10.120}
\end{gather*}
$$

which is $\Gamma=0 \%$ for normal incident $\theta=90$ wave and $\Gamma \approx 17 \%$ for $45^{\circ}$ incident wave Fig.(10.11).
Second Order:


Figure 10.11:

$$
\begin{equation*}
\sqrt{1-S^{2}} \approx 1-\frac{S^{2}}{2} \tag{10.121}
\end{equation*}
$$

therefore

$$
\begin{equation*}
L^{-} \approx \frac{\partial}{\partial x}-\frac{1}{C} \frac{\partial}{\partial t}\left(1-\frac{1}{2}\left(\frac{\frac{\partial}{\partial y}}{\frac{1}{C} \frac{\partial}{\partial t}}\right)^{2}\right) \tag{10.122}
\end{equation*}
$$

multiply $L^{-}$by $\frac{1}{C} \frac{\partial}{\partial t}$ we will have

$$
\begin{gather*}
L^{-} \approx \frac{1}{C^{2}} \frac{\partial^{2}}{\partial x \partial t}-\frac{1}{C^{2}} \frac{\partial^{2}}{\partial t^{2}}+\frac{1}{2} \frac{\partial^{2}}{\partial y^{2}}  \tag{10.123}\\
\left.\left(\frac{1}{C^{2}} \frac{\partial^{2}}{\partial x \partial t}-\frac{1}{C^{2}} \frac{\partial^{2}}{\partial t^{2}}+\frac{1}{2} \frac{\partial^{2}}{\partial y^{2}}\right) \psi\right|_{x=0}=0 \tag{10.124}
\end{gather*}
$$

In this case $\Gamma=0 \%$ for normal incident $\theta=90$ wave and $\Gamma \approx 3 \%$ for $45^{\circ}$ incident wave. Both the $1^{\text {st }}$ and $2^{\text {nd }}$ order are called "MUR" boundary conditions approximations, and are therefor it is not perfect. The amount of error depends on:

- The angle of incidence.
- The cell size and frequency.

Steps of finite difference and absorbing boundary condition for applying to FDTD:
a) Apply numerical differentiation to derivatives.
b) Apply Boundary condition.

$$
\begin{equation*}
\psi^{n+1}(i)=\psi^{n}(i-1)+\left(\frac{\tau-1}{\tau+1}\right)\left(\psi^{n+1}(i-1)-\psi^{n}(i)\right) \tag{10.125}
\end{equation*}
$$

where $\tau=\frac{C \Delta t}{\Delta z}$

## Solved problem

Derive Eq.(10.125).
Solution:

$$
\begin{aligned}
& \frac{\partial \psi(z, t)}{\partial z}=\frac{\psi^{n+1}(i)-\psi^{n+1}(i-1)}{\Delta z} \\
& \frac{\partial \psi(z, t)}{\partial z}=\frac{\psi^{n}(i)-\psi^{n}(i-1)}{\Delta z} \\
& \frac{\partial \psi(z, t)}{\partial t}=\frac{\psi^{n+1}(i)-\psi^{n}(i)}{\Delta t} \\
& \frac{\partial \psi(z, t)}{\partial t}=\frac{\psi^{n+1}(i-1)-\psi^{n}(i-1)}{\Delta t}
\end{aligned}
$$

We use the average of above relation to $\left(\frac{\partial}{\partial z}+\frac{1}{C} \frac{\partial}{\partial t}\right) \psi=0$ for a wave traveling from LEFT to RIGHT and have absorbing boundary at cell $i$ Eq.(10.125).

For a wave traveling from RIGHT to LEFT we should use absorbing boundary condition $\left(\frac{\partial}{\partial z}-\frac{1}{C} \frac{\partial}{\partial t}\right) \psi=0$

$$
\begin{equation*}
\psi^{n+1}(i-1)=\psi^{n}(i)+\left(\frac{\tau-1}{\tau+1}\right)\left(\psi^{n+1}(i)-\psi^{n}(i-1)\right) \tag{10.126}
\end{equation*}
$$

Let at the domain edge (say $i=1$ and $i=M$ ), we should know $\psi(M+1)$ and $\psi(0)$ which is outside the domain for the computation. we may either set this value to zero, in which case we will get reflections back into the domain off the ends, or we may implement absorbing boundary condition. The absorbing boundary conditions can be implemented as:

$$
\begin{align*}
\psi^{n+1}(M+1) & =\psi^{n}(M)+\left(\frac{\tau-1}{\tau+1}\right)\left(\psi^{n+1}(M)-\psi^{n}(M+1)\right) \\
\psi^{n+1}(0) & =\psi^{n}(1)+\left(\frac{\tau-1}{\tau+1}\right)\left(\psi^{n+1}(1)-\psi^{n}(0)\right) \tag{10.127}
\end{align*}
$$

In 3 D the ABC will be

$$
\begin{array}{ll}
\left(\partial_{x}-c^{-1} \partial_{t}\right) \psi(x, y, z, t)=0, & x=x_{\min } \\
\left(\partial_{x}+c^{-1} \partial_{t}\right) \psi(x, y, z, t)=0, & x=x_{\max } \\
\left(\partial_{y}-c^{-1} \partial_{t}\right) \psi(x, y, z, t)=0, & y=y_{\min } \\
\left(\partial_{y}+c^{-1} \partial_{t}\right) \psi(x, y, z, t)=0, & y=y_{\max } \\
\left(\partial_{z}-c^{-1} \partial_{t}\right) \psi(x, y, z, t)=0, & z=z_{\min } \\
\left(\partial_{z}+c^{-1} \partial_{t}\right) \psi(x, y, z, t)=0, & z=z_{\max } \tag{10.128}
\end{array}
$$

### 10.10.3 Liao extrapolation

Another proposition for ABC is given by Liao et al. [30]. As interpreted in [28], the Liao ABC can best be understood as a simple extrapolation of the fields inside the computational domain to the outer boundary. In the extrapolation, several time levels and several spatial points inside the computational domain are involved to efficiently perform the extrapolation. The extrapolated value is then used in normal manner in update equations.

The reflectivity of a three-time level Liao ABC is reported to be much less than any of the computationally comparable versions of the other ABCs discussed so far, and about 20 dB less than second-order Mur[28].

### 10.10.4 The Perfectly Matched Layer Boundary Condition for FDTD

Recently, Jean Pierre Berenger published a novel approach called Perfectly Matched Layer (PML) [31] [32]that yields improved performance compared to the earlier techniques. Typically MUR's first or second order Absorbing Boundary Conditions have been used with acceptable results. However, for the cases of multilayered dielectric materials extending to the ABC, the results were not very useful as significant reflections were corrupting the results. The Berengers PML ABC is based on an artificial absorbing layer surrounding the simulation region. This layer then absorbes any outgoing wave in a similar fashion to the Mur's ABC's discussed previously.
The crux of the PML definition is that the Yee cells in the PML region are split into 12 components instead of the usual six components.
The governing equations are similar to the standard FDTD equations, however, there exists several extra parameters. For the case of Ex, the electric field vector propagating in the Y direction, the terms $\sigma_{y}$, and $\sigma_{y}^{m}$ are conductivity terms depending on the depth in the PML material. An important point to note is that the conductivity values for the electric field, and magnetic field are offset by half a space step.
In order to efficiently implement the PML Boundary condition for FDTD, several points must be taken into consideration. Firstly is that for the three dimensional case, the FDTD region consists of 6 field components, whereas the PML region has 12 firld components, both E , and H fields are split.
The most common temptation is to allocate a huge block of 12 field components, and use that - however, it wastes a LOT of memory. An efficient scheme, as Berenger suggested, is to divide the simulation regions, between FDTD and PML. This enables optimal usage of memory for FDTD with PML, including using double precision field components for the PML regions, and single precision field components for the standard FDTD region. The detailed description of the PML theory is given in [32]. We point out here the aspects of importance regarding FDTD implementation and numerical computation. The PML formulation of Maxwell's equations can be written
as:

$$
\begin{array}{ll}
\epsilon_{0} \frac{\partial E_{x y}}{\partial t}+\sigma_{y} E_{x y}=\frac{\partial\left(H_{z x}+H_{z y}\right)}{\partial y} & \mu_{0} \frac{\partial H_{x y}}{\partial t}+\sigma_{y}^{*} H_{x y}=-\frac{\partial\left(E_{z x}+E_{z y}\right)}{\partial y} \\
\epsilon_{0} \frac{\partial E_{x z}}{\partial t}+\sigma_{z} E_{x z}=-\frac{\partial\left(H_{y x}+H_{y z}\right)}{\partial z} & \mu_{0} \frac{\partial H_{x z}}{\partial t}+\sigma_{z}^{*} H_{x z}=\frac{\partial\left(E_{y x}+E_{y z}\right)}{\partial z} \\
\epsilon_{0} \frac{\partial E_{y z}}{\partial t}+\sigma_{z} E_{y z}=\frac{\partial\left(H_{x y}+H_{x z}\right)}{\partial z} & \mu_{0} \frac{\partial H_{y z}}{\partial t}+\sigma_{z}^{*} H_{y z}=-\frac{\partial\left(E_{x y}+E_{x z}\right)}{\partial z} \\
\epsilon_{0} \frac{\partial E_{y x}}{\partial t}+\sigma_{x} E_{y x}=-\frac{\partial\left(H_{z x}+H_{z y}\right)}{\partial x} & \mu_{0} \frac{\partial H_{y x}}{\partial t}+\sigma_{x}^{*} H_{y x}=\frac{\partial\left(E_{z x}+E_{z y}\right)}{\partial x} \\
\epsilon_{0} \frac{\partial E_{y x}}{\partial t}+\sigma_{x} E_{x z}=\frac{\partial\left(H_{y x}+H_{y z}\right)}{\partial x} & \mu_{0} \frac{\partial H_{z x}}{\partial t}+\sigma_{x}^{*} H_{z x}=-\frac{\partial\left(E_{y x}+E_{y z}\right)}{\partial x} \\
\epsilon_{0} \frac{\partial E_{z y}}{\partial t}+\sigma_{y} E_{z y}=-\frac{\partial\left(H_{x y}+H_{y z}\right)}{\partial y} & \mu_{0} \frac{\partial H_{z y}}{\partial t}+\sigma_{y}^{*} H_{z y}=\frac{\partial\left(E_{x y}+E_{x z}\right)}{\partial y} \tag{10.129}
\end{array}
$$

where $\sigma_{i}$ and $\sigma_{i}^{*}$ denote electric and magnetic conductivities. Each field component is split into two subcomponents. If the condition

$$
\begin{equation*}
\frac{\sigma_{i}}{\epsilon_{0}}=\frac{\sigma_{i}^{*}}{\mu_{0}} \tag{10.130}
\end{equation*}
$$

is satisfied the characteristic impedance of the lossy free space medium equals that of lossless vacuum and no reflection occurs for a plane wave propagating in normal direction across a vacuum-medium interface.
One should note that the perfectly match interface exist only for continuum space. For discretize space where $\mathbf{E}$ and $\mathbf{H}$ field components do not lie in the same plane (Yee Cell), the interface produces reflections [32]. This is due to the fact that in numerical process there is no equal absorption for the electric and magnetic fields because of their position in the FDTD elementary cell. The reflection error can be minimized by choosing a certain profile of conductivity depending on the distance from interface. Berenger found that the parabolic profile yields optimum absorption. In practical computations, the PML medium consists of several layers. Eq.(10.129)and Eq.(10.130) shows that the PML formulation corresponds to that of a physical medium. If $\sigma_{x}=\sigma_{y}=\sigma_{z}=\sigma_{x}^{*}=\sigma_{y}^{*}=\sigma_{z}^{*}=0$, Eq.(10.129)and Eq.(10.130) reduce to Maxwell's equations in free space. Hence, the implementation in FDTD does not involve a special treatment.

For application of PML in dielectric media, the conductivities are chosen such that the phase velocity of the propagating wave inside PML medium
does not change when it crosses the interface between two different dielectric materials. This leads to the condition

$$
\begin{equation*}
\frac{\sigma_{i}}{\epsilon_{0}}=\frac{\sigma_{i r 1}}{\epsilon_{0} \epsilon_{r 1}}=\frac{\sigma_{i r 2}}{\epsilon_{0} \epsilon_{r 2}}=\cdots \tag{10.131}
\end{equation*}
$$

where $\sigma_{i r 1}$ denotes the electric conductivity in the dielectric medium with $\epsilon_{r 1}$. The corresponding magnetic conductivity can be calculated from Eq.(10.131). The electric conductivity of first layer $\sigma_{i}(0)$ defines the cutoff frequency of the PML medium [32]. Frequency lower than

$$
\begin{equation*}
f_{c}=\frac{\sigma_{i}(0)}{2 \pi \epsilon_{0}} \tag{10.132}
\end{equation*}
$$

are reflected.

## Original Formulation

In 2D TE case, the equations governing the original PML are:

$$
\begin{gather*}
\epsilon_{0} \frac{\partial E_{x}}{\partial t}+\sigma_{y} E_{x}=\frac{\partial\left(H_{z x}+H_{z y}\right)}{\partial y}  \tag{10.133}\\
\epsilon_{0} \frac{\partial E_{y}}{\partial t}+\sigma_{x} E_{y}=-\frac{\partial\left(H_{z x}+H_{z y}\right)}{\partial x}  \tag{10.134}\\
\mu_{0} \frac{\partial H_{z x}}{\partial t}+\sigma_{y}^{*} H_{z x}=-\frac{\partial E_{y}}{\partial x}  \tag{10.135}\\
\mu_{0} \frac{\partial H_{z y}}{\partial t}+\sigma_{y}^{*} H_{z y}=\frac{\partial E_{x}}{\partial y} \tag{10.136}
\end{gather*}
$$

Here, the $H_{y}$ component has been split: $H_{z}=H_{z} z x+H_{z y}$. The cornerstone of PML is that $\sigma_{x}^{*}$ and $\sigma_{y}^{*}$ can be chosen independently. If they were constrained to be equal, equations Eq.(10.135) and Eq.(10.136) could be merged, resulting ordinary Maxwell's equations with magnetic conductivity. Note that Eq.(10.133) and Eq.(10.136) allow direct implementation with FDTD. A PML can conveniently be parameterized by the set $\left(\sigma_{x}, \sigma_{x}^{*}, \sigma_{y}, \sigma_{y}^{*}\right)$. Vacuum is a special case, namely $(0,0,0,0)$. Berenger showed in [31] that a

PML is perfectly matched to vacuum at an interface normal to x -direction, i.e. producing no reflection from the interface at any angle of incidence and any frequency, provided it is of the form $\left(\sigma_{x}, \sigma_{x}^{*}, 0,0\right)$, and the matching condition

$$
\begin{equation*}
\frac{\sigma_{x}}{\epsilon_{0}}=\frac{\sigma_{x}^{*}}{\mu_{0}} \tag{10.137}
\end{equation*}
$$

is fulfilled. At an interface normal to y-direction, matching is obtained by a medium $\left(0,0, \sigma_{y}, \sigma_{y}^{*}\right)$, provided the conductivities again satisfy a similar condition than Eq.(10.137). Berenger's original paper involved only 2D case; PML was soon generalized into 3D [52].

### 10.11 Dispersion

When we simulate a pulse wave that propagates in a medium, we see that the shape of pulse changes as it moves along a path. We can name this effects dispersion. In other word, wave propagation velocity in numerical simulation may vary with frequency, direction of propagation, and distance. It happens in frequency dependent materials. This error in differential forms happens because of error in numerical differentiation. If $d z \approx \lambda / 10$ gives less than $10 \%$ error. You will observe this effect in your simulation.

### 10.12 Problem

The purpose of this exercise is to understand and use finite difference approximation in transient analysis of dielectric slabs.
Consider Maxwell's equations in time domain, namely

$$
\begin{align*}
\nabla \times \mathbf{E}(\mathbf{r}, t) & =-\mu \frac{\partial \mathbf{H}(\mathbf{r}, t)}{\partial t} \\
\nabla \times \mathbf{H}(\mathbf{r}, t) & =\epsilon \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} \tag{10.138}
\end{align*}
$$

1) Show that in one dimension in a charge free medium, the scalar wave equation derived from Eq.(10.138) may be expressed as

$$
\begin{equation*}
\frac{\partial^{2} E_{x}(z, t)}{\partial t^{2}}=c^{2} \frac{\partial^{2} E_{x}(z, t)}{\partial z^{2}} \tag{10.139}
\end{equation*}
$$

2) Use the difference Eq.(10.101) approximations, write a computer program to implement the finite difference equation. At $z=0$, enforce a source condition in time such that

$$
E_{x}(0, t)=e^{-\left(t-t_{0}\right)^{2} / 2 \sigma^{2}}
$$

where $t_{0}$ is chosen such that the pulse is essentially zero at $t=0$. The parameter $\sigma$ controls the width of Gaussian pulse, and therefore should be set reasonably small (order of $10-20$ spatial cell wide). Set up your program so that in the center of the domain you have a dielectric slab whose thickness is several times the spatial step size $\Delta z$.
3) Run your program and monitor the output for the pulse hitting the dielectric slab. Also run the code such that $\frac{c \Delta t}{\Delta z} \leq 1$ or $\frac{c \Delta t}{\Delta z} \gg 1$
4) Use absorbing boundary conditions at both ends of spatial domain.

### 10.12.1 Two Dimensional FDTD

We consider a scattering problem in two dimensions. we assume that the field components do not depend on the z coordinate of a point $\frac{\partial}{\partial z}=0$. Furthermore, we take $\epsilon$ and $\mu$ be constants and $J=0$. The only source of our problem is then an incident wave. This incident wave will be scattered after it encounters the obstacle. The obstacle will be of few wavelengths in its linear dimension. Further simplification can be obtained if we observe the fact that in cylindrical coordinates we can decompose any electromagnetic field into transverse electric and transverse magnetic fields if $\epsilon$ and $\mu$ are constants. The two modes of electromagnetic wave are characterized by 1) Transverse Electric (TE)

$$
\begin{align*}
H_{x} & =H_{y}=0, \quad E_{z}=0 \\
\frac{\partial H_{z}}{\partial t} & =\frac{1}{\mu}\left(\frac{\partial E_{x}}{\partial y}-\frac{\partial E_{y}}{\partial x}\right) \\
\frac{\partial E_{x}}{\partial t} & =\frac{1}{\epsilon}\left(\frac{\partial H_{z}}{\partial y}-\sigma E_{x}\right) \\
\frac{\partial E_{y}}{\partial t} & =\frac{1}{\epsilon}\left(-\frac{\partial H_{z}}{\partial x}-\sigma E_{y}\right) \tag{10.140}
\end{align*}
$$

2) Transverse Magnetic (TM)

$$
\begin{align*}
E_{x} & =E_{y}=0, \quad H_{z}=0 \\
\frac{\partial H_{x}}{\partial t} & =\frac{1}{\mu}\left(-\frac{\partial E_{z}}{\partial y}\right) \\
\frac{\partial H_{y}}{\partial t} & =\frac{1}{\mu}\left(\frac{\partial E_{z}}{\partial x}\right) \\
\frac{\partial E_{z}}{\partial t} & =\frac{1}{\epsilon}\left(\frac{\partial H_{y}}{\partial x}-\frac{\partial H_{x}}{\partial y}-\sigma E_{z}\right) \tag{10.141}
\end{align*}
$$

Let in our problem conductivity be zero $\sigma=0$ and C be a perfect conducting boundary curve. We approximate it by a polygon whose sides are parallel to the coordinate axes. if the grid dimensions are small compared to the wavelength, we expect the approximation to yield meaningful results.

Letting

$$
\begin{equation*}
\tau=c t=\frac{1}{\sqrt{\mu \epsilon}} t \tag{10.142}
\end{equation*}
$$

and

$$
\begin{equation*}
\eta=\sqrt{\frac{\mu}{\epsilon}}=376.7 \tag{10.143}
\end{equation*}
$$

we can write the finite difference equation for TE and TM waves.
TE waves:

$$
\begin{align*}
& H_{z}^{n+1 / 2}(i+1 / 2, j+1 / 2)=H_{z}^{n-1 / 2}(i+1 / 2, j+1 / 2) \\
- & \frac{1}{\eta} \frac{\Delta \tau}{\Delta x}\left[E_{y}^{n}(i+1, j+1 / 2)-E_{y}^{n}(i, j+1 / 2)\right] \\
+ & \frac{1}{\eta} \frac{\Delta \tau}{\Delta y}\left[E_{x}^{n}(i+1 / 2, j+1)-E_{x}^{n}(i+1 / 2, j)\right]  \tag{10.144}\\
E_{x}^{n+1}(i+1 / 2, j)= & E_{x}^{n}(i+1 / 2, j) \\
+ & \frac{1}{\eta} \frac{\Delta \tau}{\Delta y}\left[H_{z}^{n+1 / 2}(i+1 / 2, j+1 / 2)\right. \\
- & \left.H_{z}^{n+1 / 2}(i+1 / 2, j-1 / 2)\right]  \tag{10.145}\\
E_{y}^{n+1}(i, j+1 / 2)= & -\frac{1}{\eta} \frac{\Delta \tau}{\Delta x}\left[H_{z}^{n+1 / 2}(i+1 / 2, j+1 / 2)\right. \\
& \left.-H_{z}^{n+1 / 2}(i-1 / 2, j+1 / 2)\right] \tag{10.146}
\end{align*}
$$

TM waves:

$$
\begin{align*}
& E_{z}^{n+1}(i, j)=E_{z}^{n}(i, j) \\
+ & \eta \frac{\Delta \tau}{\Delta x}\left[H_{y}^{n+1 / 2}(i+1 / 2, j)-H_{y}^{n+1 / 2}(i-1 / 2, j)\right] \\
- & \eta \frac{\Delta \tau}{\Delta y}\left[H_{x}^{n+1 / 2}(i, j+1 / 2)-H_{x}^{n+1 / 2}(i, j-1 / 2)\right]  \tag{10.147}\\
& H_{x}^{n+1 / 2}(i, j+1 / 2)=H_{x}^{n-1 / 2}(i, j+1 / 2)  \tag{10.148}\\
& \quad-\frac{1}{\eta} \frac{\Delta \tau}{\Delta y}\left[E_{z}^{n}(i, j+1)-E_{z}^{n}(i, j)\right] \\
& H_{y}^{n+1 / 2}(i+1 / 2, j)=H_{y}^{n-1 / 2}(i+1 / 2, j)  \tag{10.149}\\
& +\frac{1}{\eta} \frac{\Delta \tau}{\Delta x}\left[E_{z}^{n}(i+1, j)-E_{z}^{n}(i, j)\right]
\end{align*}
$$

## - Numerical Computations for TM waves

For further numerical discussion we shall limit ourselves to TM waves. In this case we use the finite difference equations Eq.(10.147), Eq.(10.148), Eq.(10.149). The value for $E_{x}^{0}(i, j), H_{y}^{1 / 2}(i+1 / 2, j)$, and $H_{x}^{1 / 2}(i, j-1 / 2)$ are obtained from the incident wave. We choose $t$ such that when $t=0$ the incident wave has not countered obstacle. Subsequent values are evaluated from the finite difference equations Eq.(10.148), Eq.(10.149). The boundary condition is approximated by putting the boundary value of $E_{z}^{n}(i, j)$ equal to zero for any n.

To be specific, we shall consider the diffraction of an incident TM wave by a perfect conducting square. The dimensions of the obstacle, as well as the profile of incident wave, are shown in Fig.(10.12). Let the incident wave be plane, with its profile being a half sine wave. The width of the incident wave is taken to be $\alpha$ units and the square has sides of length $4 \alpha$ unit. The incident wave will have only an $E_{y}$ component and an $H_{y}$ component. We choose

$$
\begin{equation*}
\Delta x=\Delta y=\alpha / 8 \tag{10.150}
\end{equation*}
$$

and

$$
\begin{equation*}
\Delta \tau=c \Delta t=\frac{1}{2} \Delta x=\alpha / 16 \tag{10.151}
\end{equation*}
$$

A finite difference scheme over the whole $x-y$ plane is impractical; we therefore have to limit the extent of our calculation region. we assume that at time $\mathrm{t}=0$, the left traveling plane wave is "near the obstacle. For a restricted


Figure 10.12: Scattering from Conducting Square by TM wave
period of time, we can therefore replace the original problems by equivalent problem shown in Fig(??).

The input data are taken from the incident wave with

$$
\begin{gather*}
E_{z}(x, y, t)=\sin \left[\frac{(x-50 \alpha+c t) \pi}{8 \alpha}\right] \\
0 \leq x-50 \alpha+c t \leq 8 \alpha \\
H_{y}(x, y, t)=\frac{1}{\eta} E_{x}(x, y, t) \tag{10.152}
\end{gather*}
$$

Numerical results are presented for the TM waves discussed above. To gain some idea of the accuracy of the finite difference equation, we have used the Eq.(10.148), Eq.(10.149) with the initial $E_{z}$ being a half sine wave for the case of no obstacle. We note that the outer boundary conditions will not affect this incident wave as there is no $H_{x}$ component in the incident wave. Ninety-five time cycles were run with the finite difference system Eq.(10.148), Eq.(10.149), and the machine output is shown in Fig.(10.13). The oscillation and the widening of the initial pulse is due to the imperfection of finite difference system. Fig.(10.14) shows the value of $E_{z}$ of the TM wave as a function of the horizontal grid coordinate $i$ for a fixed vertical grid coordinate $j=30$. At the end of five time cycle, the wave just hits the obstacle. The line $j=30$ does not meet the obstacle, but is "sufficiently" near the obstacle to be affected by a "partially reelected" wave. There is also a partially transmitted wave. The phase of the reflected wave is opposite that of the incident wave,
as required by the boundary condition of the obstacle. There should also be a decrease in wave amplitude due to cylindrical divergence, but the calculation was not carried far enough to show this effect. Fig.(??) shows the value of $E_{z}$ of the TM wave as a function of the horizontal grid coordinate $i$ for a fixed vertical grid coordinate $j=50$. This line $(j=50)$ meets the obstacle, and hence we expect a reflected wave going to the right. These expectations are borne out in Fig.(??). After the reflected wave from the object meets the right boundary, the wave is reflected again. This effect is shown for the time cycle 75,85 and 95 .

Fig.(??) is for $j=65$. This line forms part of the boundary of the obstacle. Because of the required boundary condition, $E_{z}$ is zero on boundary point. To the right of the obstacle there is a "partially" reflected wave of about half the amplitude of a fully reflected wave. To the left of the obstacle there is a "transmitted" wave after 85 time cycles.

When $\sigma \neq 0$ the general FDTD formula for $T E^{z}$ and $T M^{z}$ polarization will be

- $T M^{z}$ :

$$
\begin{gather*}
\frac{\partial E_{z}}{\partial y}=-\mu \frac{\partial H_{x}}{\partial t}-\sigma^{(m)} H_{x} \\
\frac{\partial E_{z}}{\partial x}=\mu \frac{\partial H_{y}}{\partial t}-\sigma^{(m)} H_{y} \\
\frac{\partial H_{y}}{\partial x}-\frac{\partial H_{x}}{\partial y}=\epsilon \frac{\partial E_{z}}{\partial t}+\sigma^{(e)} E_{z}  \tag{10.153}\\
H_{x}^{n+\frac{1}{2}}\left(i, j+\frac{1}{2}\right)=C_{a}\left(i, j+\frac{1}{2}\right) H_{x}^{n-\frac{1}{2}}\left(i, j+\frac{1}{2}\right) \\
+C_{b}\left(i, j+\frac{1}{2}\right)\left[\frac{E_{z}^{n}(i, j)-E_{z}^{n}(i, j+1)}{\Delta y}\right]  \tag{10.154}\\
H_{y}^{n+\frac{1}{2}}\left(i+\frac{1}{2}, j\right)=C_{a}\left(i+\frac{1}{2}, j\right) H_{y}^{n-\frac{1}{2}}\left(i+\frac{1}{2}, j\right) \\
+  \tag{10.155}\\
C_{b}\left(i+\frac{1}{2}, j\right)\left[\frac{E_{z}^{n}(i+1, j)-E_{z}^{n}(i, j)}{\Delta x}\right]
\end{gather*}
$$

$$
\begin{align*}
E_{z}^{n+1}(i, j) & =C_{c}(i, j) E_{z}^{n}(i, j) \\
& +C_{d}(i, j)\left[\frac{H_{y}^{n+\frac{1}{2}}\left(i+\frac{1}{2}, j\right)-H_{y}^{n+\frac{1}{2}}\left(i-\frac{1}{2}, j\right)}{\Delta x}\right] \\
& +C_{d}(i, j)\left[\frac{H_{x}^{n+\frac{1}{2}}\left(i, j-\frac{1}{2}\right)-H_{x}^{n+\frac{1}{2}}\left(i, j+\frac{1}{2}\right)}{\Delta y}\right] \tag{10.156}
\end{align*}
$$

where the cell media coefficients are given by

$$
\begin{gather*}
R_{b}(i, j)=\frac{\Delta t}{\mu(i, j)}  \tag{10.157}\\
C_{a}(i, j)=\frac{1-\frac{\sigma^{(m)}(i, j) \Delta t}{2 \mu(i, j)}}{1+\frac{\sigma^{(m)}(i, j) \Delta t}{2 \mu(i, j)}}  \tag{10.158}\\
C_{b}(i, j)=\frac{R_{b}(i, j)}{1+\frac{\sigma^{(m)}(i, j) \Delta t}{2 \mu(i, j)}} \tag{10.159}
\end{gather*}
$$

and

$$
\begin{gather*}
R_{d}(i, j)=\frac{\Delta t}{\epsilon(i, j)}  \tag{10.160}\\
C_{c}(i, j)=\frac{1-\frac{\sigma^{(e)}(i, j) \Delta t}{2 \epsilon(i, j)}}{1+\frac{\sigma^{(e)}(i, j) \Delta t}{2 \epsilon(i, j)}}  \tag{10.161}\\
C_{d}(i, j)=\frac{\left.R_{d} i, j\right)}{1+\frac{\sigma^{(e)}(i, j) \Delta t}{2 \epsilon(i, j)}} \tag{10.162}
\end{gather*}
$$

- $T E^{z}$ :

$$
\begin{align*}
\frac{\partial H_{z}}{\partial y} & =\epsilon \frac{\partial E_{x}}{\partial t}+\sigma^{(e)} E_{x} \\
-\frac{\partial H_{z}}{\partial x} & =\epsilon \frac{\partial E_{y}}{\partial t}+\sigma^{(e)} E_{y} \\
\frac{\partial E_{y}}{\partial x}-\frac{\partial E_{x}}{\partial y} & =-\mu \frac{\partial H_{z}}{\partial t}-\sigma^{(m)} H_{z} \tag{10.163}
\end{align*}
$$

$$
\begin{align*}
E_{x}^{n+\frac{1}{2}}\left(i, j+\frac{1}{2}\right) & =C_{c}\left(i, j+\frac{1}{2}\right) E_{x}^{n-\frac{1}{2}}\left(i, j+\frac{1}{2}\right) \\
& +C_{d}\left(i, j+\frac{1}{2}\right)\left[\frac{H_{z}^{n}(i, j+1)-H_{z}^{n}(i, j)}{\Delta y}\right]  \tag{10.164}\\
E_{y}^{n+\frac{1}{2}}\left(i+\frac{1}{2}, j\right) & =C_{c}\left(i+\frac{1}{2}, j\right) E_{y}^{n-\frac{1}{2}}\left(i+\frac{1}{2}, j\right) \\
& +C_{d}\left(i+\frac{1}{2}, j\right)\left[\frac{H_{z}^{n}(i, j)-H_{z}^{n}(i+1, j)}{\Delta x}\right]  \tag{10.165}\\
H_{z}^{n+1}(i, j)= & C_{a}(i, j) H_{z}^{n}(i, j) \\
& +C_{b}(i, j)\left[\frac{E_{y}^{n+\frac{1}{2}}\left(i-\frac{1}{2}, j\right)-E_{y}^{n+\frac{1}{2}}\left(i+\frac{1}{2}, j\right)}{\Delta x}\right] \\
& +C_{b}(i, j)\left[\frac{E_{x}^{n+\frac{1}{2}}\left(i, j+\frac{1}{2}\right)-E_{x}^{n+\frac{1}{2}}\left(i, j-\frac{1}{2}\right)}{\Delta y}\right] \tag{10.166}
\end{align*}
$$

### 10.12.2 Three Dimensional FDTD

using the MKS system of units, and assuming that the dielectric parameters $\mu, \epsilon$ and $\sigma$ are independent of time, the following system of scalar equations is equivalent to Maxwell's equations in the rectangular coordinate system.

$$
\begin{align*}
\frac{\partial H_{x}}{\partial t} & =\frac{1}{\mu}\left(\frac{\partial E_{y}}{\partial z}-\frac{\partial E_{x}}{\partial y}\right) \\
\frac{\partial H_{y}}{\partial t} & =\frac{1}{\mu}\left(\frac{\partial E_{z}}{\partial x}-\frac{\partial E_{x}}{\partial z}\right) \\
\frac{\partial H_{z}}{\partial t} & =\frac{1}{\mu}\left(\frac{\partial E_{x}}{\partial y}-\frac{\partial E_{y}}{\partial x}\right) \\
\frac{\partial E_{x}}{\partial t} & =\frac{1}{\epsilon}\left(\frac{\partial H_{x}}{\partial y}-\frac{\partial H_{y}}{\partial z}-\sigma E_{x}\right) \\
\frac{\partial E_{y}}{\partial t} & =\frac{1}{\epsilon}\left(\frac{\partial H_{x}}{\partial z}-\frac{\partial H_{z}}{\partial x}-\sigma E_{y}\right) \\
\frac{\partial E_{z}}{\partial t} & =\frac{1}{\epsilon}\left(\frac{\partial H_{y}}{\partial x}-\frac{\partial H_{x}}{\partial y}-\sigma E_{z}\right) \tag{10.167}
\end{align*}
$$

Yee introduces a set of finite difference equations for the system of Eq.(10.167). Following Yee's notation, we denote a space lattice point as

$$
\begin{equation*}
(i, j, k)=(i \Delta x, j \Delta y, k \Delta z) \tag{10.168}
\end{equation*}
$$

where $\delta=\Delta x=\Delta y=\Delta z$ is the space increment, and $\delta t$ is the time increment. Yee uses finite difference expressions for the space and time derivatives that are both simply programmed and second order accurate in $\delta$ and in $\delta t$, respectively,

$$
\begin{equation*}
\frac{\partial F^{n}(i, j, k)}{\partial x}=\frac{F^{n}(i+1 / 2, j, k)-F^{n}(i-1 / 2, j, k)}{\delta t}+O\left(\delta^{2}\right) \tag{10.169}
\end{equation*}
$$

$$
\begin{equation*}
\frac{\partial F^{n}(i, j, k)}{\partial t}=\frac{F^{n+1 / 2}(i, j, k)-F^{n-1 / 2}(i, j, k)}{\delta t}+O\left(\delta t^{2}\right) \tag{10.170}
\end{equation*}
$$

To achieve the accuracy of Eq.(10.169), and to realize all of the space derivatives of Eq.(10.167), Yee positions the components of $\mathbf{E}$ and $\mathbf{H}$ about a unit cell of the lattice as shown in Fig(??). To achieve the accuracy of Eq.(10.170), he evaluates $\mathbf{E}$ and $\mathbf{H}$ at alternate half time steps. The result of these assumptions is the following system of finite difference equations for the system of Eq.(10.167).

$$
\begin{align*}
& H_{x}^{n+\frac{1}{2}}\left(i, j+\frac{1}{2}, k+\frac{1}{2}\right)= \\
& H_{x}^{n-\frac{1}{2}}\left(i, j+\frac{1}{2}, k+\frac{1}{2}\right)+\frac{\delta t}{\mu\left(i, j+\frac{1}{2}, k+\frac{1}{2}\right) \delta}  \tag{10.171}\\
& \cdot\left[\begin{array}{l}
E_{y}^{n}\left(i, j+\frac{1}{2}, k+1\right)-E_{y}^{n}\left(i, j+\frac{1}{2}, k\right)+ \\
E_{z}^{n}\left(i, j, k+\frac{1}{2}\right)-E_{z}^{n}\left(i, j+1, k+\frac{1}{2}\right)
\end{array}\right]
\end{align*}
$$

$$
\begin{align*}
& H_{y}^{n+\frac{1}{2}}\left(i+\frac{1}{2}, j, k+\frac{1}{2}\right)= \\
& H_{y}^{n-\frac{1}{2}}\left(i+\frac{1}{2}, j, k+\frac{1}{2}\right)+\frac{\delta t}{\mu\left(i+\frac{1}{2}, j, k+\frac{1}{2}\right) \delta}  \tag{10.172}\\
& \cdot\left[\begin{array}{l}
E_{z}^{n}\left(i+1, j, k+\frac{1}{2}\right)-E_{z}^{n}\left(i, j, k+\frac{1}{2}\right)+ \\
E_{x}^{n}\left(i+\frac{1}{2}, j, k\right)-E_{x}^{n}\left(i+\frac{1}{2}, j, k+1\right)
\end{array}\right]
\end{align*}
$$

$$
\begin{align*}
& H_{z}^{n+\frac{1}{2}}\left(i+\frac{1}{2}, j+\frac{1}{2}, k\right)=  \tag{10.173}\\
& H_{z}^{n-\frac{1}{2}}\left(i+\frac{1}{2}, j+\frac{1}{2}, k\right)+\frac{\delta t}{\mu\left(i+\frac{1}{2}, j+\frac{1}{2}, k\right) \delta} \\
& \cdot\left[\begin{array}{l}
E_{x}^{n}\left(i+\frac{1}{2}, j+1, k\right)-E_{x}^{n}\left(i+\frac{1}{2}, j, k\right)+ \\
E_{y}^{n}\left(i, j+\frac{1}{2}, k\right)-E_{y}^{n}\left(i+1, j+\frac{1}{2}, k\right)
\end{array}\right]
\end{align*}
$$

$$
\begin{aligned}
& \begin{array}{|l|}
E_{z}^{n+1}\left(i+\frac{1}{2}, j, k\right)= \\
{\left[1-\frac{\sigma\left(i+\frac{1}{2}, j, k\right) \delta t}{\epsilon\left(i+\frac{1}{2}, j, k\right)}\right] E_{z}^{n}\left(i+\frac{1}{2}, j, k\right)+\frac{\delta t}{\epsilon\left(i+\frac{1}{2}, j, k\right) \delta}} \\
\cdot\left[\begin{array}{c}
H_{z}^{n+\frac{1}{2}}\left(i+\frac{1}{2}, j+\frac{1}{2}, k\right)-H_{z}^{n+\frac{1}{2}}\left(i+\frac{1}{2}, j-\frac{1}{2}, k\right)+ \\
H_{y}^{n+\frac{1}{2}}\left(i+\frac{1}{2}, j, k-\frac{1}{2}\right)-H_{y}^{n+\frac{1}{2}}\left(i+\frac{1}{2}, j, k+\frac{1}{2}\right)
\end{array}\right]
\end{array} \\
& \begin{array}{l}
E_{y}^{n+1}\left(i, j+\frac{1}{2}, k\right)= \\
{\left[\begin{array}{l}
\left.1-\frac{\sigma\left(i, j+\frac{1}{2}, k\right) \delta t}{\epsilon\left(i, j+\frac{1}{2}, k\right)}\right] E_{y}^{n}\left(i, j+\frac{1}{2}, k\right)+\frac{\delta t}{\epsilon\left(i, j+\frac{1}{2}, k\right) \delta} \\
\cdot\left[\begin{array}{c}
H_{x}^{n+\frac{1}{2}}\left(i, j+\frac{1}{2}, k+\frac{1}{2}\right)-H_{x}^{n+\frac{1}{2}}\left(i, j+\frac{1}{2}, k-\frac{1}{2}\right)+ \\
H_{z}^{n+\frac{1}{2}}\left(i-\frac{1}{2}, j+\frac{1}{2}, k\right)-H_{z}^{n+\frac{1}{2}}\left(i+\frac{1}{2}, j+\frac{1}{2}, k\right)
\end{array}\right]
\end{array}\right.}
\end{array} \\
& \begin{array}{|l|}
\hline E_{z}^{n+1}\left(i, j, k+\frac{1}{2}\right)= \\
{\left[1-\frac{\sigma\left(i, j, k+\frac{1}{2}\right) \delta t}{\epsilon\left(i, j, k+\frac{1}{2}\right)}\right] E_{y}^{n}\left(i, j+\frac{1}{2}, k\right)+\frac{\delta t}{\epsilon\left(i, j, k+\frac{1}{2}\right) \delta}} \\
\cdot\left[\begin{array}{c}
H_{y}^{n+\frac{1}{2}}\left(i+\frac{1}{2}, j, k+\frac{1}{2}\right)-H_{x}^{n+\frac{1}{2}}\left(i-\frac{1}{2}, j, k+\frac{1}{2}\right)+ \\
H_{x}^{n+\frac{1}{2}}\left(i, j-\frac{1}{2}, k+\frac{1}{2}\right)-H_{z}^{n+\frac{1}{2}}\left(i, j+\frac{1}{2}, k+\frac{1}{2}\right)
\end{array}\right]
\end{array}
\end{aligned}
$$

With the system of Eq.(10.171)-Eq.(10.176), the new value of a field vector component at any lattice point depends only on its previous value and on the previous values of the components of other vector at adjacent points. Therefore, at any given time steps, the computation of a field vector may proceed one point at a time. Computer storage must be provided for 11 quantities at each unit cell of the lattice: the 6 field vector components, the values of $\epsilon$ and $\sigma$, and maximum $\left|E_{x}\right|,\left|E_{y}\right|$, and $\left|E_{z}\right|$ achieved during the final half-wave cycle of time stepping.

## - Far Field Transformation

FDTD can only be used in limited space around the calculating subject. In many problems, e.g., in scattering problems, the far field is needed. A near field to far field transform fulfills this purpose. The near field to far field transform is based on Huygens's surface equivalence theorem. A Huygens's surface is set around the scatterer. The equivalent currents equal to $\mathbf{J}=\mathbf{n} \times \mathbf{H}$ and $\mathbf{M}=\mathbf{E} \times \mathbf{n}$, where $\mathbf{n}$ is the local surface unit normal, $\mathbf{H}$ and $\mathbf{E}$ are the magnetic and electric fields at the surface. $\mathbf{E}$ and $\mathbf{H}$ are calculated in the FDTD. With the equivalent currents, the far field is obtained using
the Green's function.
The FDTD method computes the fields in a region around the objects that lies in the near-field. To determine the far-field scattering or radiation pattern, the near-field data can be transformed to the far-field by weighting it with the free space Green's function and integrating over a surface, S, surrounding the objects. The near field electric and magnetic fields calculated in the FDTD program are time domain quantities. To implement the far field transform, these values must be converted to frequency domain values using a discrete Fourier transform, which is implemented by computing the and quadrature components separately.

$$
\begin{aligned}
& \mathbf{E}_{r}=E_{s} \sin (\omega t) \\
& \mathbf{E}_{i}=E_{s} \cos (\omega t)
\end{aligned}
$$

where $E_{s}$ represents any time domain field component on the surface S at $t=$ $n \Delta t$. When a sinusoidal plane wave source is used they are only calculated at time steps during the last period of the sinusoidal wave. Thesin and cos quadrature components are transformed to complex phasor quantities using

$$
\begin{equation*}
\mathbf{E}=\sqrt{\mathbf{E}_{r}^{2}+\mathbf{E}_{i}^{2}} e^{-j \arctan \left(\frac{\mathbf{E}_{i}}{\mathbf{E}_{r}}\right)} \tag{10.177}
\end{equation*}
$$

Once the near-field time-domain values are converted to frequency domain values, the equivalent electric and magnetic surface current densities, $\mathbf{J}_{s}$ and $\mathbf{M}_{s}$, are defined on the surface S as:

$$
\begin{align*}
\mathbf{J}_{s}\left(\mathbf{r}^{\prime}\right) & =\mathbf{n} \times \mathbf{H} \\
\mathbf{M}_{s}\left(\mathbf{r}^{\prime}\right) & =-\mathbf{n} \times \mathbf{E} \tag{10.178}
\end{align*}
$$

where $\mathbf{r}^{\prime}$ is a point on $S$ and is a unit vector normal to $S$, where the origin of the far field transform is located in the center of the grid. The $\mathbf{E}$ and $\mathbf{H}$ quantities in Eq.(10.178) refer to the complex phasor values computed with Eq.(10.177). The vector potentials $\mathbf{A}$ and $\mathbf{F}$ are computed by numerically integrating $\mathbf{J}_{s}$ and $\mathbf{M}_{s}$ over the surface S ,

$$
\begin{align*}
& \mathbf{A}=\frac{\mu_{0} e^{-j \beta r}}{4 \pi r} \iint_{s} \mathbf{J}_{s} e^{j \beta r^{\prime} \cos \psi} d s^{\prime}  \tag{10.179}\\
& \mathbf{F}=\frac{\epsilon_{0} e^{-J \beta r}}{4 \pi r} \iint_{s} \mathbf{M}_{s} e^{j \beta r^{\prime} \cos \psi} d s^{\prime}
\end{align*}
$$

where $\mathbf{r}$ is a point in the far field and $\psi$ is the angle between $\mathbf{r}$ and $\mathbf{r}^{\prime}$. The term $r \cos \psi$ in the exponent is written as:

$$
\begin{equation*}
r^{\prime} \cos \psi=\left(x^{\prime} \cos \phi+y^{\prime} \cos \phi\right)+z^{\prime} \cos \theta \tag{10.180}
\end{equation*}
$$

The far field electric and magnetic field components, in spherical coordinates, are then given by:

$$
\begin{align*}
E_{\theta} & =-j \omega\left(A_{\theta}+\eta F_{\phi}\right)  \tag{10.181}\\
E_{\phi} & =-j \omega\left(A_{\phi}-\eta F_{\theta}\right)  \tag{10.182}\\
H_{\theta} & =\frac{j \omega}{\eta}\left(A_{\theta}-\eta F_{\phi}\right)  \tag{10.183}\\
H_{\phi} & =-\frac{j \omega}{\eta}\left(A_{\phi}+\eta F_{\theta}\right) \tag{10.184}
\end{align*}
$$

where $\eta=\sqrt{\frac{\mu}{\epsilon}}$ and

$$
\begin{align*}
A_{\theta} & =A_{x} \cos \theta \cos \phi+A_{y} \cos \theta \sin \phi-A_{z} \sin \theta  \tag{10.185}\\
A_{\phi} & =-A_{x} \sin \phi+A_{y} \cos \phi  \tag{10.186}\\
A_{\theta} & =F_{x} \cos \theta \cos \phi+F_{y} \cos \theta \sin \phi-F_{z} \sin \theta  \tag{10.187}\\
F_{\phi} & =-F_{x} \sin \phi+F_{y} \cos \phi \tag{10.188}
\end{align*}
$$

The far field scattering pattern $F_{s}(\theta, \phi)$ is defined by:

$$
\begin{equation*}
F_{s}(\theta, \phi)=\frac{1}{2} \Re\left(E_{\theta} H_{\phi}^{*}\right)+\frac{1}{2} \Re\left(-E_{\phi} H_{\theta}^{*}\right) \tag{10.189}
\end{equation*}
$$

where $\theta$ and $\phi$ are the angles measured from the z and x axes, respectively, in spherical coordinates. Physically this represents the scattered intensity at any point in the far field. Other parameters such as the anisotropy and scattering cross-section can be computed from the scattering pattern, $F_{s}(\theta, \phi)$.

In two dimensional case the formula for far fields will be

$$
\begin{align*}
E_{\phi}(\rho, \phi) & =-j \omega \mu A_{\phi}-j \beta F_{z}  \tag{10.190}\\
E_{z}(\rho, \phi) & =-j \omega \mu A_{z}-j \beta F_{\phi}
\end{align*}
$$

where

$$
\begin{align*}
\mathbf{A}(\rho, \phi) & \left.=\frac{e^{-j \beta \rho}}{\sqrt{8 j \beta \pi \rho}} \int_{S} \mathbf{J}_{s}\left(\rho^{\prime}\right) e^{-j \beta \rho^{\prime} \cos \left(\phi-\phi^{\prime}\right)}\right) d s^{\prime}  \tag{10.191}\\
\mathbf{F}(\rho, \phi) & \left.=\frac{e^{-j \beta \rho}}{\sqrt{8 j \beta \pi \rho}} \int_{S} \mathbf{M}_{s}\left(\rho^{\prime}\right) e^{-j \beta \rho^{\prime} \cos \left(\phi-\phi^{\prime}\right)}\right) d s^{\prime}
\end{align*}
$$

## - Source In Scattering Problem

A number of different sources can be used to find the scattered fields including a Gaussian pulse plane wave, sinusoidal pulse, and sinusoidal plane wave. The incident fields described by Gaussian pulse are

$$
\begin{equation*}
\mathbf{E}_{i n c}=\mathbf{E}_{0} e-\frac{\left(r-r_{0}\right)^{2}}{w^{2}} \tag{10.192}
\end{equation*}
$$

This source however is not a single frequency, or monochromatic plane wave. A single frequency source is modeled with either a sinusoidal pulse or a continuous wave source. In either case the incident field is of the form

$$
\begin{equation*}
\mathbf{E}_{i n c}=\mathbf{E}_{0} \sin (\omega t) \tag{10.193}
\end{equation*}
$$

In the case of a pulse, the incident field is turned off after a specified number of time steps

### 10.13 Problems

- 1 Write a program to show the truncation error of three different type derivatives Eq.(10.1) for function $f(x)=3 x^{3}+4 x^{2}-5 x+1$ at point $x=2.5$.
- 2 Write the matrix for the Fig.(10.6); a: without using symmetry, b: with symmetry.
- 3 Write a program to find the $\nabla^{2} f(x, y)$ for function $f(x, y)=\frac{x+y+3}{x^{2}+y^{2}}$ at origin. Compare your numeric result with analytic result
- 4 Write an FD code to compute the cutoff frequencies of the first ten TM and TE modes of a rectangular waveguide with dimensions 1 cm x 2 cm . Compare your results to the analytical cutoff frequencies. How rapidly do the FD results approach the exact results as the number of grid points increases?
- 5 Fine the capacitance per meter of a air filled coaxial cable with inner radius of $r 1=.005 m$ and uter radius of $r_{2}=.01 \mathrm{~m}$. Compare your result with analytical formula $C=\frac{2 \pi \epsilon_{0}}{\ln \left(\frac{\epsilon_{2}}{r_{1}}\right)}$


Figure 10.13: $E_{z}$ of the $T M$ wave in the absence of the obstacle for various time cycle


Figure 10.14: $E_{z}$ of the $T M$ wave in the presence of the obstacle for various time cycle when $\mathrm{j}=30$


Figure 10.15: Yee's Cell

## Chapter 11

## Functional Methods

"God does not care about our mathematical difficulties. He integrates empirically."
Albert Einstein

### 11.1 Introduction

Many problems in electrical and mechanical engineering may be formulated by Functonal Methods. The Functonal Methods is a mathematical base for Method of Moment (MoM) and Finite Element Method (FEM). In order to understand these methods and related ones, we first review math terms in linear algebra, then functional or variational will be defined, and finally approximation methods will be used for solution of integral or differential equations.


### 11.2 Review of Linear Algebra

### 11.2.1 Linear Space

Let $\mathbf{x}_{\mathbf{1}}, \mathbf{x}_{\mathbf{2}}, \mathbf{x}_{\mathbf{3}}, \cdots$ be elements of a set $\mathcal{S}$. These elements are called vectors. Let $\alpha_{1}, \alpha_{2}, \alpha_{3}, \cdots$ be the elements of the field of numbers $\mathbf{F}$. Those belong to field of real $\mathbf{R}$ or complex $\mathbf{C}$ numbers. The set $\mathcal{S}$ is a linear space if the following addition and multiplication rule can be applied.

- Addition

1) $\left(x_{1}+x_{2}\right)+x_{3}=x_{1}+\left(x_{2}+x_{3}\right)$
2) There exists a zero vector $\mathbf{0}$ such that $\mathbf{x}_{\mathbf{1}}+\mathbf{0}=\mathbf{0}+\mathbf{x}_{\mathbf{1}}$
3) For every $\mathbf{x}_{\mathbf{1}} \in \mathcal{S}$, there exists $-\mathbf{x}_{\mathbf{1}} \in \mathcal{S}$ such that $\mathbf{x}_{\mathbf{1}}+\left(-\mathbf{x}_{\mathbf{1}}\right)=\left(-\mathbf{x}_{\mathbf{1}}\right)+$ $\mathrm{x}_{1}=0$.
4) $x_{1}+x_{2}=x_{2}+x_{1}$

- Multiplication

1) $\alpha_{1}\left(\alpha_{2} \mathbf{x}_{\mathbf{1}}\right)=\left(\alpha_{1} \alpha_{2}\right) \mathbf{x}_{\mathbf{1}}$
2) $1 x_{1}=x_{1}$
3) $\alpha_{1}\left(\mathbf{x}_{\mathbf{1}}+\mathbf{x}_{\mathbf{2}}\right)=\alpha_{1} \mathbf{x}_{\mathbf{1}}+\alpha_{1} \mathbf{x}_{\mathbf{2}}$
4) $\left(\alpha_{1}+\alpha_{2}\right) \mathbf{x}_{1}=\alpha_{1} \mathbf{x}_{1}+\alpha_{2} \mathbf{x}_{1}$

Sometimes in mathematics we deal with linearly dependent vectors. Let $\mathbf{x}_{\mathbf{1}}, \mathbf{x}_{\mathbf{2}}, \mathbf{x}_{\mathbf{3}}, \cdots$ be elements of a set vectors in $\mathcal{S}$. The vectors are linearly dependent if there exists $\alpha_{k} \in \mathbf{F}$, for $k=1,2, \cdots n$, not all zero, such that

$$
\begin{equation*}
\sum_{k=1}^{n} \alpha_{k} \mathbf{x}_{\mathbf{k}}=\mathbf{0} \tag{11.1}
\end{equation*}
$$

In another word, if the only way to satisfy Eq.(11.1) is $\alpha_{k} k=1,2, \cdots n$, then the elements $\mathbf{x}_{\mathbf{k}}, k=1,2, \cdots n$ are linearly independent. The sum

$$
\begin{equation*}
\sum_{k=1}^{n} \alpha_{k} \mathbf{x}_{\mathbf{k}} \tag{11.2}
\end{equation*}
$$

is called a linear combination of vectors $x_{k}$

### 11.2.2 Inner Product Space

A linear space $\mathcal{S}$ is a complex inner product space, if for every ordered pair $(\mathbf{x}, \mathbf{y})$ of vectors in $\mathcal{S}$, there exists a unique scalar in $\mathbf{C}$, symbolized $\langle\mathbf{x}, \mathbf{y}\rangle$, such that:

1) $\langle\mathbf{x}, \mathbf{y}\rangle=\langle\mathbf{y}, \mathbf{x}\rangle^{*} \quad\left[{ }^{*}=\right.$ complex conjugate $]$
2) $\langle\mathbf{x}+\mathbf{y}, \mathbf{z}\rangle=\langle\mathbf{x}, \mathbf{z}\rangle+\langle\mathbf{y}, \mathbf{z}\rangle$
3) $\langle\alpha \mathbf{x}, \mathbf{y}\rangle=\alpha\langle\mathbf{y}, \mathbf{x}\rangle \alpha \in \mathbf{C}$
4) $\langle\mathbf{x}, \mathbf{y}\rangle \geq \mathbf{0}$, with equality if and only if $\mathbf{x}=\mathbf{0}$

If $\langle\mathbf{x}, \mathbf{y}\rangle=\mathbf{0}$ the vectors $\mathbf{x}, \mathbf{y}$ are called orthogonal. In this book we define inner (dot or scalar) product of any two elements $\langle\mathbf{f}, \mathbf{g}\rangle$ as

$$
\begin{equation*}
\langle\mathbf{f}, \mathbf{g}\rangle=\int_{\Omega} \mathbf{f}(\mathbf{r}) \mathbf{g}(\mathbf{r})^{*} d \Omega \tag{11.3}
\end{equation*}
$$

### 11.2.3 Normed Linear Space

A linear space $\mathcal{S}$ is a normed linear space if, for every vector $\mathrm{x} \in \mathcal{S}$, there is assigned a unique number $\|\mathbf{x}\| \in \mathbf{R}$ such that the following rules apply:

1) $\|\mathbf{x}\| \geq 0$ with equality if and only if $\mathbf{x}=\mathbf{0}$
2) $\|\alpha \mathbf{x}\|=|\alpha|\|\mathbf{x}\|, \quad \alpha \in \mathbf{F}$
3) $\left\|\mathbf{x}_{1}+\mathbf{x}_{\mathbf{2}}\right\| \leq\left\|\mathrm{x}_{\mathbf{1}}\right\|+\left\|\mathrm{x}_{\mathbf{2}}\right\|$

Although there are many possible definitions of norms, we exclusively use the norm induced by the inner product, defined by

$$
\begin{equation*}
\|\mathbf{x}\|=\sqrt{\langle\mathbf{x}, \mathbf{x}\rangle} \tag{11.4}
\end{equation*}
$$

## - Hilbert Space

A linear space is a Hilbert space if it is complete in norm induced by inner product.

### 11.2.4 Definition and Properties of Operators

An operator represents the relationship between two functions as

$$
\begin{equation*}
\mathcal{L}(\mathbf{f})=\mathrm{g} \tag{11.5}
\end{equation*}
$$

The properties of an operator determine the methods used for numerical solution of the operator equations.

## - Linear Operator

The operator $\mathcal{L}$ is linear if

$$
\begin{equation*}
\mathcal{L}\left(\mathbf{f}_{1}+\mathbf{f}_{\mathbf{2}}\right)=\mathcal{L}\left(\mathbf{f}_{1}\right)+\mathcal{L}\left(\mathbf{f}_{2}\right) \tag{11.6}
\end{equation*}
$$

$$
\begin{equation*}
\mathcal{L}(c \mathbf{f})=c \mathcal{L}(\mathbf{f}) \tag{11.7}
\end{equation*}
$$

where $c$ is a constant.

## - Symmetric Operator

The $\mathcal{L}$ is a symmetric operator if

$$
\begin{equation*}
\left\langle\mathcal{L} \mathrm{f}_{1}, \mathrm{f}_{\mathbf{2}}\right\rangle=\left\langle\mathbf{f}_{1}, \mathcal{L} \mathrm{f}_{\mathbf{2}}\right\rangle \tag{11.8}
\end{equation*}
$$

where $\mathbf{f}_{\mathbf{1}}, \mathbf{f}_{\mathbf{2}}$ are any two vectors in space $\mathcal{L}$.

## - Positive definite Operator

If

$$
\begin{equation*}
\langle\mathcal{L} \mathbf{f}, \mathbf{f}\rangle>0 \tag{11.9}
\end{equation*}
$$

for all $\mathbf{f} \neq \mathbf{0}$ in its domain, $\mathcal{L}$ is a positive definite operator.

## - Self-adjoint Operator

The adjoint of operator $\mathcal{L}$ is the operator $\mathcal{L}^{*}$ such that

$$
\begin{equation*}
\left\langle\mathcal{L} \mathbf{f}_{1}, \mathbf{f}_{2}\right\rangle=\left\langle\mathbf{f}_{1}, \mathcal{L}^{*} \mathbf{f}_{\mathbf{2}}\right\rangle \tag{11.10}
\end{equation*}
$$

where the domain of $\mathcal{L}^{*}$ is same as $\mathcal{L}$. If $\mathcal{L}=\mathcal{L}^{*}$, the operator $\mathcal{L}$ is called self-adjoint.
$\star$ A self-adjoint operator is symmetric.
$\star$ The operators having even power such as $\nabla^{2}$ and $\nabla^{2}+\frac{\partial^{2}}{\partial t^{2}}$ are self-adjoint, but operator with the odd power are not.
$\star$ If the kernel of an integral equation is symmetric, then the integral operator is self-adjoint. (we will talk about kernel later on)

## - Eigen Value of an Operator

If there is a number $\lambda$ and vector (or element) $\mathbf{x} \neq 0$ such that

$$
\begin{equation*}
\mathcal{L} \mathrm{x}=\lambda \mathrm{x} \tag{11.11}
\end{equation*}
$$

where $\lambda$ and $\mathbf{x}$ are called eigen value and eigen vector of operator $\mathcal{L}$, respectively.

### 11.2.5 The relation between the properties of operators and solution of the operator equations

The purpose of various numerical methods that are used to solve the electromagnetic field problems is to convert an operator equation into a matrix equation. Both the approximation approach for the formulation and the solution methods for resulting matrix equation dependent on the properties of the differential and integral operators.

1) If $\mathcal{L}$ is symmetric positive definite, then the operator equation $\mathcal{L}(\mathbf{f})=\mathrm{g}$ has only one stable solution.
2) If $\mathcal{L}$ is self-adjoint positive definite operator in Hilbert space, then the solution of Eq.(11.5) can be approximated by the associated problem which minimizes the quadratic functional $I(\mathbf{f})$. (we call function of function as functional)

$$
\begin{equation*}
I(\mathbf{f})=\langle\mathcal{L} \mathbf{f}, \mathbf{f}\rangle-\langle\mathbf{f}, \mathbf{g}\rangle-\langle\mathbf{g}, \mathbf{f}\rangle \tag{11.12}
\end{equation*}
$$

3) Self-adjoint operators are symmetric and generate a symmetric matrix which has real eigenvalues.
4) If the inverse operator $\mathcal{L}^{-1}$ exists, the solution of the original operator equation is unique.

### 11.3 Linear Operators and Quadratic Forms.

The solution, $\mathbf{f}$, of a self-adjoint linear differential equation

$$
\begin{equation*}
\mathcal{L} \mathrm{f}=\mathrm{g} \quad \text { in } \Omega \tag{11.13}
\end{equation*}
$$

corresponds to a stationary point for the quadratic form

$$
\begin{equation*}
I(\mathbf{f})=\langle\mathcal{L} \mathbf{f}, \mathbf{f}\rangle-2\langle\mathbf{f}, \mathbf{g}\rangle \tag{11.14}
\end{equation*}
$$

## - Proof:

$I(\mathbf{f})$ is stationary if for all functions $\mathbf{w}: \frac{\partial}{\partial \alpha}\{I(\mathbf{f}+\alpha \mathbf{w})\}_{\alpha=0}=0$

$$
\begin{equation*}
I(\mathbf{f}+\alpha \mathbf{w})=\langle\mathbf{f}+\alpha \mathbf{w}, \mathcal{L}(\mathbf{f}+\alpha \mathbf{w})\rangle-2\langle\mathbf{f}+\alpha \mathbf{w}, \mathbf{g}\rangle \tag{11.15}
\end{equation*}
$$

or

$$
\begin{equation*}
I(\mathbf{f}+\alpha \mathbf{w})=\langle\mathbf{f}, \mathcal{L} \mathbf{f}\rangle-2\langle\mathbf{f}, \mathbf{g}\rangle+\alpha\{\langle\mathbf{w}, \mathcal{L} \mathbf{f}\rangle+\langle\mathbf{f}, \mathcal{L} \mathbf{w}\rangle-2\langle\mathbf{w}, \mathbf{g}\rangle\}+O\left(\alpha^{2}\right) \tag{11.16}
\end{equation*}
$$

Thus $I$ stationary $\Longleftrightarrow\langle\mathbf{w}, \mathcal{L} \mathbf{f}+\langle\mathbf{f}, \mathcal{L} \mathbf{w}\rangle-2\langle\mathbf{w}, \mathbf{g}\rangle=0$ for all $\mathbf{w}$. Because $\mathcal{L}$ is self adjoint, $\langle\mathbf{f}, \mathcal{L} \mathbf{w}\rangle=\langle\mathbf{w}, \mathcal{L} \mathbf{f}$,$\rangle so the condition for I$ stationary becomes $\langle\mathbf{w}, \mathcal{L} \mathbf{f}\rangle-\langle\mathbf{w}, \mathbf{g}\rangle=\langle\mathbf{w}, \mathcal{L} \mathbf{f}-\mathbf{g}\rangle=$,0 . Thus, for every admissible variation $\mathbf{w}$ we have

$$
\begin{equation*}
\langle\mathbf{w}, \mathcal{L} \mathbf{f}-\mathbf{g}\rangle=\int_{\Omega} \mathbf{w}(\mathcal{L} \mathbf{f}-\mathbf{g}) d \Omega=0 \tag{11.17}
\end{equation*}
$$

Since $\mathbf{w}$ is arbitrary, this requires that the residual $\mathbf{r}=\mathcal{L} \mathbf{f}-\mathbf{g}$ vanish everywhere in $\Omega$, that is, the differential equation Eq.(11.5) is satisfied.

The proof shows that we can solve the differential equation Eq.(11.5) by finding the function $\mathbf{f}$ that makes $I(\mathbf{f})$ stationary. In physical contexts, $I$ is often represents the energy and the solution of the differential equation is the one that minimize the energy.

We can interpret Eq.(11.17) as saying that the residual weighted by an arbitrary function $\mathbf{w}$ in the region $\Omega$ is zero.

### 11.4 Rayleigh-Ritz Method

In electrical engineering, energy is minimum if the system is stable. The Rayleigh-Ritz method is based on the above principle. It is a variational method in which the boundary value problems as expressed by $\mathcal{L}(\mathbf{f})=\mathrm{g}$ is formulated by variational expression, referred to as functional $I(\mathbf{f})$. Thus, the advantage of the variational formulation is that it makes it possible to find approximate solutions. The Rayleigh-Ritz method consists of:

- Approximation of $\mathbf{f}$ by an expansion in a finite set of basis or trial functions $\psi_{j}(\mathbf{r}), j=1,2, \cdots, N$

$$
\begin{equation*}
\mathbf{f}(\mathbf{r}) \approx \tilde{\mathbf{f}}(\mathbf{r})=\sum_{j=1}^{N} a_{j} \psi_{j}(\mathbf{r}) \tag{11.18}
\end{equation*}
$$

- Evaluation of the quadratic variational form $I(\mathbf{f})$ as a function of the expansion coefficients

$$
\begin{array}{r}
I\left(a_{1}, a_{2}, \cdots, a_{N}\right)=I(\tilde{\mathbf{f}})=\langle\tilde{\mathbf{f}}, \mathcal{L} \tilde{\mathbf{f}}\rangle-2\langle\tilde{\mathbf{f}}, \mathbf{g}\rangle= \\
\sum_{i} \sum_{j} a_{i} a_{j}\left\langle\psi_{i}(\mathbf{r}), \mathcal{L} \psi_{j}(\mathbf{r})\right\rangle-2 \sum_{i} a_{i}\left\langle\psi_{i}(\mathbf{r}), \mathbf{g}\right\rangle \tag{11.20}
\end{array}
$$

or

$$
\begin{equation*}
I\left(a_{1}, a_{2}, \cdots, a_{N}\right)=\sum_{i} \sum_{j} L_{i j} a_{i} a_{j}-2 \sum_{i} g_{i} a_{i} \tag{11.21}
\end{equation*}
$$

where $L_{i j}=\left\langle\psi_{i}(\mathbf{r}), \mathcal{L} \psi_{j}(\mathbf{r})\right\rangle$ and $g_{i}=\left\langle\psi_{i}(\mathbf{r}), \mathbf{g}\right\rangle$. Note that since the operator $\mathcal{L}$ is self-adjoint, the matrix $L$ is symmetric $L i j=L_{j i}$.

- Determination of the expansion coefficient $a_{i}$ by recalling that $I$ is stationary with respect to all coefficients:

$$
\begin{equation*}
\frac{\partial I}{\partial a_{k}}=\sum_{j} L_{k j}+\sum_{i} L_{i k} a_{i}-2 g_{k}=2 \sum_{i} L_{k i} a_{i}-2 g_{k}=0 \tag{11.22}
\end{equation*}
$$

or simply

$$
\begin{equation*}
\sum_{i} L_{k i} a_{i}=g_{k} \tag{11.23}
\end{equation*}
$$

Eq.(11.23) is linear symmetric $N \times N$ system for the expansion coefficients. In matrix form we will have:

$$
\left[\begin{array}{ccc}
\left\langle\psi_{1}, \mathcal{L}\left(\psi_{1}\right)\right\rangle\left\langle\psi_{1}, \mathcal{L}\left(\psi_{2}\right)\right\rangle & \cdots\left\langle\psi_{1}, \mathcal{L}\left(\psi_{N}\right)\right\rangle  \tag{11.24}\\
\left\langle\psi_{2}, \mathcal{L}\left(\psi_{1}\right)\right\rangle & \left\langle\psi_{2}, \mathcal{L}\left(\psi_{2}\right)\right\rangle & \cdots\left\langle\left\langle\psi_{2}, \mathcal{L}\left(\psi_{N}\right)\right\rangle\right. \\
\vdots & & \\
\left\langle\psi_{N}, \mathcal{L}\left(\psi_{1}\right)\right\rangle & \left\langle\psi_{N}, \mathcal{L}\left(\psi_{2}\right)\right\rangle & \cdots\left\langle\psi_{N}, \mathcal{L}\left(\psi_{N}\right)\right\rangle
\end{array}\right]\left[\begin{array}{c}
a_{1} \\
a_{2} \\
\vdots \\
a_{N}
\end{array}\right]=\left[\begin{array}{c}
\left\langle\psi_{1}, g\right\rangle \\
\left\langle\psi_{2}, g\right\rangle \\
\vdots \\
\left\langle\psi_{N}, g\right\rangle
\end{array}\right]
$$

### 11.5 Weighted Residual

Having defined a $\langle\cdot, \cdot\rangle$ and norm $\|\cdot\|$, the calculation essentially amounts to minimization of a residual vector

$$
\begin{equation*}
\mathcal{L}(\mathbf{f})=\mathrm{g} \tag{11.25}
\end{equation*}
$$

Let us approximate the function $\mathbf{f}$ as

$$
\begin{equation*}
\tilde{f}=\sum_{j=1}^{N} \alpha_{j} \cdot f_{j} \tag{11.26}
\end{equation*}
$$

where $u_{j}$ are the chosen base functions, and $\alpha_{j}$ is unknown coefficient which should be determined. A set of equations for the coefficients $\alpha_{j}$ are the obtained by taking the inner product of Eq.(11.26) with a set of weighting or testing functions $\mathbf{w}:\left[w_{i}\right]=\left[w_{1}, w_{2}, \cdots w_{N}\right]$. In another word, If we denote $\mathbf{r}=$
$\mathcal{L}(\tilde{\mathbf{f}}-\mathbf{f})$ as residual, we should minimize this residue by weighting function $\mathbf{w}$, as

$$
\begin{equation*}
\langle\mathbf{r}, \mathbf{w}\rangle=0 \tag{11.27}
\end{equation*}
$$

According to Eq.(11.25) we will have :

$$
\begin{equation*}
\langle\mathcal{L}(\tilde{\mathbf{f}})-\mathbf{g}, \mathbf{w}\rangle=0 \tag{11.28}
\end{equation*}
$$

or

$$
\begin{equation*}
\langle\mathcal{L}(\tilde{\mathbf{f}}), \mathbf{w}\rangle=\langle\mathbf{g}, \mathbf{w}\rangle \tag{11.29}
\end{equation*}
$$

we can rewrite the Eq.(11.29) as

$$
\begin{equation*}
<w_{i}, \mathcal{L}(f)>=<w_{i}, g>\quad i=1,2, \cdots, N \tag{11.30}
\end{equation*}
$$

Due to the linearity of operator $\mathcal{L}(\cdot)$, if we substitute the expansion of function $f$ in Eq.(11.30, we have:

$$
\begin{equation*}
\sum_{j=1}^{N} \alpha_{j}<w_{i}, \mathcal{L}\left(f_{j}\right)>=<w_{i}, g>\quad i=1,2, \cdots, N \tag{11.31}
\end{equation*}
$$

by expanding the Eq.(11.31, we have:

$$
\begin{equation*}
\alpha_{1}\left\langle w_{i}, \mathcal{L}\left(f_{1}\right)\right\rangle+\alpha_{2}\left\langle w_{i}, \mathcal{L}\left(f_{2}\right)\right\rangle+\cdots+\alpha_{N}\left\langle w_{i}, \mathcal{L}\left(f_{N}\right)\right\rangle=\left\langle w_{i}, g\right\rangle \tag{11.32}
\end{equation*}
$$

$i=1,2, \cdots, N$ or

$$
\left[\begin{array}{ccc}
\left\langle w_{1}, \mathcal{L}\left(f_{1}\right)\right\rangle\left\langle w_{1}, \mathcal{L}\left(f_{2}\right)\right\rangle & \cdots\left\langle w_{1}, \mathcal{L}\left(f_{N}\right)\right\rangle  \tag{11.33}\\
\left\langle w_{2}, \mathcal{L}\left(f_{1}\right)\right\rangle & \left\langle w_{2}, \mathcal{L}\left(f_{2}\right)\right\rangle & \cdots\left\langle w_{2}, \mathcal{L}\left(f_{N}\right)\right\rangle \\
\vdots & & \\
\left\langle w_{N}, \mathcal{L}\left(f_{1}\right)\right\rangle & \left\langle w_{N}, \mathcal{L}\left(f_{2}\right)\right\rangle & \cdots\left\langle w_{N}, \mathcal{L}\left(f_{N}\right)\right\rangle
\end{array}\right]\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{N}
\end{array}\right]=\left[\begin{array}{c}
\left\langle w_{1}, g\right\rangle \\
\left\langle w_{2}, g\right\rangle \\
\vdots \\
\left\langle w_{N}, g\right\rangle
\end{array}\right]
$$

This set of equations can be written in the matrix form:

$$
\begin{equation*}
[A][X]=[B] \tag{11.34}
\end{equation*}
$$

where $[A]$ is an $N \times N$ matrix, $[X]$ and $[B]$ are vectors with $N$ elements, so $A_{i j}=\left\langle w_{i}, \mathcal{L}\left(f_{j}\right)\right\rangle, X_{j}=\alpha_{j}$ and $B_{i}=\left\langle w_{i}, g\right\rangle$, for $i=1,2, \cdots, N$ and $j=1,2, \cdots, N$. The solution is then

$$
\begin{equation*}
[X]=[A]^{-1}[B] \tag{11.35}
\end{equation*}
$$

The solution of Eq.(11.35) may be exact or approximate, depending upon the choice of the base function $\left[f_{i}\right]$ and weighting function $\left[w_{i}\right]$.

### 11.6 Galerkin's Method

In the Galerkin's method the weighting function and base function are the same, therefore we will have the matrix equation

$$
\left[\begin{array}{ccc}
\left\langle f_{1}, \mathcal{L}\left(f_{1}\right)\right\rangle\left\langle f_{1}, \mathcal{L}\left(f_{2}\right)\right\rangle & \cdots\left\langle f_{1}, \mathcal{L}\left(f_{N}\right)\right\rangle  \tag{11.36}\\
\left\langle f_{2}, \mathcal{L}\left(f_{1}\right)\right\rangle & \left\langle f_{2}, \mathcal{L}\left(f_{2}\right)\right\rangle & \cdots\left\langle f_{2}, \mathcal{L}\left(f_{N}\right)\right\rangle \\
\vdots & & \\
\left\langle f_{N}, \mathcal{L}\left(f_{1}\right)\right\rangle & \left\langle f_{N}, \mathcal{L}\left(f_{2}\right)\right\rangle & \cdots\left\langle f_{N}, \mathcal{L}\left(f_{N}\right)\right\rangle
\end{array}\right]\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{N}
\end{array}\right]=\left[\begin{array}{c}
\left\langle f_{1}, g\right\rangle \\
\left\langle f_{2}, g\right\rangle \\
\vdots \\
\left\langle f_{N}, g\right\rangle
\end{array}\right]
$$

### 11.7 Least Square Method

In the least square method, our interest is the square of the error over the domain of the problem

$$
\begin{equation*}
I=\int_{\Omega} r^{2} d \Omega \tag{11.37}
\end{equation*}
$$

next we compute the derivatives

$$
\begin{equation*}
\frac{\partial I}{\partial \alpha_{i}}=2 \int_{\Omega} r \frac{\partial r}{\partial \alpha_{i}} d \Omega, \quad i=1,2, \cdots, N \tag{11.38}
\end{equation*}
$$

It implies from Eq.(11.38) that

$$
\begin{equation*}
\frac{\partial I}{\partial \alpha_{i}}=0 \quad i=1,2, \cdots, N \tag{11.39}
\end{equation*}
$$

Therefore $I$ is stationary and the square of the error attain its minimum. In explicit form the linear system will be

$$
\frac{\left[\begin{array}{ccc}
\left\langle\mathcal{L}\left(f_{1}\right), \mathcal{L}\left(f_{1}\right)\right\rangle & \left\langle\mathcal{L}\left(f_{1}\right), \mathcal{L}\left(f_{2}\right)\right\rangle & \cdots\left\langle\mathcal{L}\left(f_{1}\right), \mathcal{L}\left(f_{N}\right)\right\rangle \\
\left\langle\mathcal{L}\left(f_{2}\right), \mathcal{L}\left(f_{1}\right)\right\rangle & \left\langle\mathcal{L}\left(f_{2}\right), \mathcal{L}\left(f_{2}\right)\right\rangle & \cdots\left\langle\mathcal{L}\left(f_{2}\right), \mathcal{L}\left(f_{N}\right)\right\rangle \\
\vdots & \\
\left\langle\mathcal{L}\left(f_{N}\right), \mathcal{L}\left(f_{1}\right)\right\rangle & \left\langle\mathcal{L}\left(f_{N}\right), \mathcal{L}\left(f_{2}\right)\right\rangle & \cdots\left\langle\mathcal{L}\left(f_{N}\right), \mathcal{L}\left(f_{N}\right)\right\rangle
\end{array}\right]\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{N}
\end{array}\right]=\left[\begin{array}{c}
\left\langle\mathcal{L}\left(f_{1}\right), g\right\rangle \\
\left\langle\mathcal{L}\left(f_{2}\right), g\right\rangle \\
\vdots \\
\left\langle\mathcal{L}\left(f_{N}\right), g\right\rangle
\end{array}\right]}{(11.40)}
$$

### 11.8 Entire-Domain and Sub-Domain Method

The choice of base (trial) and weighting (testing) functions is very important factor in solution of electromagnetic problems. It is proved that the base and weighting functions do not have to be complete in the domain of the operator, so we can have two kinds of expansion functions: subdomain and entire domain expansion functions. In boundary element method, the boundary is subdivided so the subdomain expansion function should be used. In following we formulate common expansion functions.

## - Point Collocation Method

In the point matching or collocation method, the weighting function $\mathbf{w}$ is such that $w_{i}=\delta\left(\mathbf{r}-\mathbf{r}_{\mathbf{i}}\right)$, where the fixed points $\mathbf{r}_{i} \in \Omega,(i=1,2, \cdots, N)$ are called collocation points. Here Dirac's delta functions $\delta\left(\mathbf{r}-\mathbf{r}_{\mathbf{i}}\right)$ are defined as:

$$
\delta\left(\mathbf{r}-\mathbf{r}_{i}\right)= \begin{cases}\infty, & \text { if } \quad \mathbf{r}=\mathbf{r}_{\mathbf{i}}  \tag{11.41}\\ 0 & \text { elsewhere }\end{cases}
$$

Inserting Eq.(11.41) in Eq.(11.33), the linear system takes the form

$$
\left[\begin{array}{cccc}
\mathcal{L}\left[f_{1}\left(\mathbf{r}_{1}\right)\right] & \mathcal{L}\left[f_{2}\left(\mathbf{r}_{1}\right)\right] & \cdots \mathcal{L}\left[f_{N}\left(\mathbf{r}_{1}\right)\right]  \tag{11.42}\\
\mathcal{L}\left[f_{1}\left(\mathbf{r}_{2}\right)\right] & \mathcal{L}\left[f_{2}\left(\mathbf{r}_{2}\right)\right] & \cdots \mathcal{L}\left[f_{N}\left(\mathbf{r}_{2}\right)\right] \\
\vdots & & \\
\mathcal{L}\left[f_{1}\left(\mathbf{r}_{N}\right)\right] & \mathcal{L}\left[f_{2}\left(\mathbf{r}_{N}\right)\right] & \cdots \mathcal{L}\left[f_{N}\left(\mathbf{r}_{N}\right)\right]
\end{array}\right]\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{N}
\end{array}\right]=\left[\begin{array}{c}
g\left(\mathbf{r}_{1}\right) \\
g\left(\mathbf{r}_{2}\right) \\
\vdots \\
g\left(\mathbf{r}_{N}\right)
\end{array}\right]
$$

## - Pulse Functions

The pulse functions $\left\{P_{n}(\mathbf{x}), n=1,2, \cdots, N\right\}$ is defined as

$$
P_{n}(\mathrm{x})= \begin{cases}1 & \mathrm{x} \in \Delta \mathrm{x}_{n}  \tag{11.43}\\ 0 & \text { else }\end{cases}
$$

Let the weight function one dimensional problem be pulse function, such that

$$
\left[\begin{array}{cccc}
\int_{x_{1}}^{x_{2}} \mathcal{L} f_{1}(x) d x & \int_{x_{1}}^{x_{2}} \mathcal{L} f_{2}(x) d x & \cdots & \int_{x_{1}}^{x_{2}} \mathcal{L} f_{N}(x) d x  \tag{11.44}\\
\int_{x_{2}}^{x_{3}} \mathcal{L} f_{1}(x) d x & \int_{x_{2}}^{x_{3}} \mathcal{L} f_{2}(x) d x & \cdots & \int_{x_{2}}^{x_{3}} \mathcal{L} f_{N}(x) d x \\
\vdots & & \\
\int_{x_{N-1}}^{x_{N}} \mathcal{L} f_{1}(x) d x & \int_{x_{N-1}}^{x_{N}} \mathcal{L} f_{2}(x) d x & \cdots \int_{x_{N}}^{x_{N+1}} \mathcal{L} f_{N+1}(x) d x
\end{array}\right]\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{N}
\end{array}\right]=\left[\begin{array}{c}
\int_{x_{1}}^{x_{2}} g d x \\
\int_{x_{2}}^{x_{3}} g d x \\
\vdots \\
\int_{x_{N}}^{x_{N+1}} g d x
\end{array}\right]
$$

## - Triangle Functions

The triangle functions $\left\{T_{n}(\mathbf{x}), n=1,2, \cdots, N\right\}$ is defined as

$$
\begin{align*}
& T_{1}(x)= \begin{cases}1-\frac{x-x_{1}}{\Delta_{1}} & x_{1} \leq x \leq x_{2} \\
0 & \text { otherwise }\end{cases} \\
& T_{n}(x)=\left\{\begin{array}{cc}
1-\frac{x_{n}-x}{\Delta_{n-1}} & x_{n-1} \leq x \leq x_{n} \\
1-\frac{x-x_{n}}{\Delta_{n}} & x_{n} \leq x \leq x_{n+1} \\
0 & \text { otherwise }
\end{array}\right.  \tag{11.45}\\
& T_{N}(x)=\left\{\begin{array}{cc}
1-\frac{x_{N}-x}{\Delta_{N-1}} & x_{N-1} \leq x \leq x_{N} \\
0 & \text { otherwise }
\end{array}\right.
\end{align*}
$$

where $\Delta_{n}=x_{n+1}-x_{n}$ for $n=1,2, \cdots, N-1$. The $T_{1}(x)$ starts at the beginning of the region at $x=x_{1}$ and $T_{N}(x)$ ends at the end of the region at $x=x_{N}$. Thus $T_{1}(x)$ and $T_{N}(x)$ are only half triangles. For more information see [?]

## - Piecewise Sinusoidal Function

The piecewise sinusoidal function $S_{n}(\mathbf{x})$ is defined as

$$
S_{n}(x)=\left\{\begin{array}{l}
\frac{\sin \left(k\left(\Delta-\left|x-x_{n}\right|\right)\right)}{\sin (k \Delta)}, \quad\left|x-x_{n}\right| \leq \Delta  \tag{11.46}\\
0 \quad \text { otherwise }
\end{array}\right.
$$

where $k$ is the wave number. The piecewise sinusoidal function Eq.(11.46) is often used in the analysis of wire antenna.

### 11.9 Eigenvalue Problems

The classical mathematical eigenvalue problem in electromagnetic is defined as the solution of the following equation:

$$
\begin{equation*}
\mathcal{L} \mathbf{f}=\lambda \mathbf{f} \tag{11.47}
\end{equation*}
$$

where the $\mathbf{f}$ may be the modes of a cavity resonator or a waveguide and $\lambda$ may be related to resonant frequency of a cavity or cutoff frequency of a waveguide. With the method of weighted residual we solve and change the problem as

$$
\begin{equation*}
\mathbf{A} \cdot \mathbf{X}=\lambda \mathbf{B} \cdot \mathbf{X} \tag{11.48}
\end{equation*}
$$

or simply

$$
\begin{equation*}
\mathbf{C} \cdot \mathbf{X}=\lambda \mathbf{X} \tag{11.49}
\end{equation*}
$$

where $\mathbf{C}=\mathbf{B}^{-\mathbf{1}} \cdot \mathbf{A}$, and the eigenvalues and corresponding eigenvectors should be determined. If we denote $\mathbf{r}=(\mathcal{L}-\lambda)(\tilde{\mathbf{f}}-\mathbf{f})$ as residual, we should minimize this residue by weighting function $\mathbf{w}$, as

$$
\begin{equation*}
\langle\mathbf{r}, \mathbf{w}\rangle=0 \tag{11.50}
\end{equation*}
$$

According to Eq.(11.47) we will have :

$$
\begin{equation*}
\langle\mathcal{L}(\tilde{\mathbf{f}})-\lambda \tilde{\mathbf{f}}, \mathbf{w}\rangle=0 \tag{11.51}
\end{equation*}
$$

or

$$
\begin{equation*}
\langle\mathcal{L}(\tilde{\mathbf{f}}), \mathbf{w}\rangle=\lambda\langle\tilde{\mathbf{f}}, \mathbf{w}\rangle \tag{11.52}
\end{equation*}
$$

or

$$
\begin{equation*}
\sum_{j=1}^{N} \alpha_{j}<w_{i}, \mathcal{L}\left(f_{j}\right)>=\lambda \sum_{j=1}^{N} \alpha_{j}<w_{i}, f_{j}>\quad i=1,2, \cdots, N \tag{11.53}
\end{equation*}
$$

by expanding the Eq.(11.53), we have:

$$
\begin{align*}
& \alpha_{1}\left\langle w_{i}, \mathcal{L}\left(f_{1}\right)\right\rangle+\alpha_{2}\left\langle w_{i}, \mathcal{L}\left(f_{2}\right)\right\rangle+\cdots+\alpha_{N}\left\langle w_{i}, \mathcal{L}\left(f_{N}\right)\right\rangle= \\
& \lambda\left\{\alpha_{1}\left\langle w_{i}, f_{1}\right\rangle+\alpha_{2}\left\langle w_{i}, f_{2}\right\rangle+\cdots+\alpha_{N}\left\langle w_{i}, f_{N}\right\rangle\right\} \tag{11.54}
\end{align*}
$$

$i=1,2, \cdots, N$ or

$$
\begin{array}{r}
{\left[\begin{array}{cc}
\left\langle w_{1}, \mathcal{L}\left(f_{1}\right)\right\rangle\left\langle w_{1}, \mathcal{L}\left(f_{2}\right)\right\rangle & \cdots\left\langle w_{1}, \mathcal{L}\left(f_{N}\right)\right\rangle \\
\left\langle w_{2}, \mathcal{L}\left(f_{1}\right)\right\rangle\left\langle w_{2}, \mathcal{L}\left(f_{2}\right)\right\rangle & \cdots\left\langle w_{2}, \mathcal{L}\left(f_{N}\right)\right\rangle \\
\vdots \\
\left\langle w_{N}, \mathcal{L}\left(f_{1}\right)\right\rangle\left\langle w_{N}, \mathcal{L}\left(f_{2}\right)\right\rangle & \cdots\left\langle w_{N}, \mathcal{L}\left(f_{N}\right)\right\rangle
\end{array}\right]\left[\begin{array}{c}
\alpha_{1} \\
\alpha_{2} \\
\vdots \\
\alpha_{N}
\end{array}\right]}
\end{array}=-
$$

This set of equations can be written in the matrix form:

$$
\begin{equation*}
[A][X]=\lambda[B][X] \tag{11.56}
\end{equation*}
$$

where $[A]$ and $[B]$ are an $N \times N$ matrix, $[X]$ vector with $N$ elements, so $A_{i j}=$ $\left\langle w_{i}, \mathcal{L}\left(f_{j}\right)\right\rangle, B_{i j}=\left\langle w_{i}, f_{j}\right\rangle, X_{j}=\alpha_{j}$, for $i=1,2, \cdots, N$ and $j=1,2, \cdots, N$. The solution of Eq.(11.56) give us the eigenvalues and eigenvectors of the matrix $\mathbf{C}=\mathbf{B}^{-1} \cdot \mathbf{A}$.

## Chapter 12

## Method of Moments

" The secret to creativity is knowing how to hide your sources." Albert Einstein

### 12.1 Introduction

In previous chapter we introduce the main mathematical structure for solution $\mathcal{L}(\mathbf{f})=\mathbf{g}$ type problem. In this chapter we will work on special type of operators and usually have special name which we call it Method of Moments. The operators may be in integral of integro-differential form.

### 12.2 What is the Method of Moments?

One of the best known numerical methods in computational electromagnetic is the Method of Moments or MoM which become popular by Harrington [7] in 1967. The MoM is based on residual method for electromagnetic Integral or Integro-differential equation. The MoM can be applied to time-harmonic problems, once the integral equation governing the electromagnetic phenomena is known. The good example is the calculation of the current distribution on a dipole antenna. According to electric or magnetic excitation, there are two types of Integral Equation, Electric Field Integral Equation or briefly (EFIE) and Magnetic Field Integral Equation or briefly (MFIE). We shall derive these equation in the following section.

### 12.3 How does MoM work?

The MoM technique requires that the entire structure to be modeled be broken down into wires and/or metal plates (in the case of metallic structure). or small cubic box in the case of dielectric structure or combination of both. Each wire is subdivided to a number of wire segments which must be small compared to the frequencies wavelength (so that the assumption of a constant value of current across that wire segment is valid). Each metal plate is subdivided into a number of surface patches, which must be small compared to the wavelength (again so the assumption of constant current is valid). In case of dielectric structures, cubic box must be small enough, in order to get better and accurate results.

Once the model is defined, a source is imposed (a plane wave approaching, or a voltage source on one of the wire segments). The MoM technique is to determine the current on every wire segment and surface patch due to the source and all the other currents (or the other wire segments and surface patches). Once these currents are known, then the E field at any point in space is determined from the sum of all the contributions from all the wire segments and surface patches.

### 12.4 What are the strengths of the MoM Technique?

Every modeling technique has some strengths and some weaknesses. Some types of models were a given technique will excel and some types of models were the same technique will have difficulty (if it is even possible to use) performing rapidly and accurately.
MoM is a very versatile modeling technique. It is also a very intuitive technique, so users can easily understand how to use it, and know what to expect from a given model. Users can picture the RF currents on a structure and understand how they would lead to a E/H field. MoM models only the structure, and not the space around it. Therefore, long wires are easily modeled using MoM.
Since MoM is a frequency domain technique, it can solve problems very quickly, if only one frequency is desired. If multiple frequencies are desired, then the simulation will take longer, but still solutions are often available is a short amount of time.

### 12.5 The weaknesses of the MoM Technique?

Although MoM is very easy to use for wires and metal plates, it is very difficult to use for dielectric and special magnetic materials. Special solution techniques do exist to allow dielectric in a MoM solution, but these are not widely implemented and care must be taken when they are used.
MoM assumes the current on a wire segment, or on a surface patch to be the same throughout the conductors depth. Therefore, using MoM to determine the effect of an aperture with fields both inside and outside is difficult.
MoM is a frequency domain technique, therefore, if a wide frequency range is desired in the solution, the simulation must be run a number of times. If the frequency step size is not sufficiently small, important effects (e.g. resonance) may be over looked.

### 12.6 Basic solution of method of moments

1) Derive an appropriate integral equation (IE) governing your electromagnetic problem
2) Convert the IE into a matrix equation using the method of weighted rsiduals.
3 ) Evaluate impedance or admittance immittance matrix and excitation vector elements.
3) Solve the linear system and obtain your unknowns.
we encounter two kinds of integral equation in electromagnetism.
4) Fredholm Integral Equation of First Kind

$$
\begin{equation*}
-\int_{V} G\left(\mathbf{r}, \mathbf{r}^{\prime}\right) f\left(\mathbf{r}^{\prime}\right) d \mathbf{r}^{\prime}=g(\mathbf{r}) \tag{12.1}
\end{equation*}
$$

2) Fredholm Integral Equation of Second Kind

$$
\begin{equation*}
h(\mathbf{r}) f(\mathbf{r})-\int_{V} G\left(\mathbf{r}, \mathbf{r}^{\prime}\right) f\left(\mathbf{r}^{\prime}\right) d \mathbf{r}^{\prime}=g(\mathbf{r}) \tag{12.2}
\end{equation*}
$$

where $G\left(\mathbf{r}, \mathbf{r}^{\prime}\right)$ is the kernel of the integral equation, in general case it is dyadic. The kernel of integral equation is Green's Function. Physically, a Green's function represents the field arising from an impulse source. A Green's function may be viewed as the electromagnetic impulse response of
the physical system under consideration. The $g(\mathbf{r})$ is the known excitation. The excitation is applied field; i.e., the field that excites in the structure for which the Green's function has been obtained. $h(\mathbf{r})$ is the immittance function, it is often a surface impedance or admittance. V is the region over which integral equation is applicable and $f(\mathbf{r})$ is the unknown function to be determined. It is usually an electric or magnetic charge density or an equivalent electric or magnetic current density. It should be noted that if eigenfunction of Green's function are used as base functions, then immittance matrix is sparse due to the orthogonality of these eigenfunctions.
Let us start with simple one dimensional MoM problem.

- Example (1): Charge Q[C] will be put on a piece of wire with radius $a$ and length $L$, find charge distribution density on the wire.


Figure 12.1: Charge distribution on a rod

- Solution: It is clear that charge density on each end is maximum and in middle is minimum. Now we want to find this fact by mathematics. The surface charge density $\sigma(z)$ is depend on z so the governing Integral Equation will be:

$$
\begin{equation*}
V=\frac{1}{4 \pi \epsilon_{0}} \iint \frac{\sigma\left(z^{\prime}\right) a d \phi^{\prime} d z^{\prime}}{\sqrt{\left(x-x^{\prime}\right)^{2}+\left(y-y^{\prime}\right)^{2}+\left(z-z^{\prime}\right)^{2}}} \tag{12.3}
\end{equation*}
$$

we take observation point on $x=0, y=0, z$ and $x^{\prime 2}+y^{\prime 2}=a^{2}$ so Eq.(12.3) will be reduce to

$$
\begin{equation*}
V=\frac{a}{2 \epsilon_{0}} \int \frac{\sigma\left(z^{\prime}\right) d z^{\prime}}{\sqrt{\left(a^{2}+\left(z-z^{\prime}\right)^{2}\right.}} \tag{12.4}
\end{equation*}
$$

Let us divide the total length of rod into N section or cell and each cell have length $d$, and take $\sigma(z)=\sum_{n=1}^{n=N} \alpha_{n} \sigma_{n}(z)$ and use weighting function as $w_{m}=\delta\left(z-z_{m}\right)$ (point matching) and base function in the form of

$$
\sigma_{n}(z)= \begin{cases}1 & \text { if } z_{n}-d / 2<z<z_{n}+d / 2  \tag{12.5}\\ 0 & \text { otherwise }\end{cases}
$$

therefore the $Z_{m n}=\left\langle\mathcal{L} f_{n}, w_{m}\right\rangle$ will look like

$$
\begin{equation*}
Z_{m n}=\frac{a}{2 \epsilon_{0}} \int_{0}^{L} \int_{z_{n}-d / 2}^{z_{n}+d / 2} \frac{\delta\left(z-z_{m}\right) d z d z^{\prime}}{\sqrt{a^{2}+\left(z-z^{\prime}\right)^{2}}} \tag{12.6}
\end{equation*}
$$

after using delta function properties

$$
\begin{equation*}
Z_{m n}=\frac{a}{2 \epsilon_{0}} \int_{z_{n}-d / 2}^{z_{n}+d / 2} \frac{d z^{\prime}}{\sqrt{a^{2}+\left(z_{m}-z^{\prime}\right)^{2}}} \tag{12.7}
\end{equation*}
$$

by change of variables we can find the elements of matrix $Z_{m n}$ with the help of following integral formula very easily

$$
\begin{equation*}
\int \frac{d u}{\sqrt{a^{2}+u^{2}}}=\ln \left[u+\sqrt{a^{2}+u^{2}}\right] \tag{12.8}
\end{equation*}
$$

In our problem $V=g$ is known and since it is a metallic wire it is equipotential, so we assume that it is one volt $V=1$. The value of $V_{m}=\left\langle v, w_{m}\right\rangle$ will be $V_{m}=1$. The unknowns $\alpha_{n}=I_{n}$ will be found by solution of this linear equation $\left[Z_{m n}\right] \cdot\left[I_{n}\right]=\left[V_{m}\right]$. We may attack the problem by other weighting and base functions. Suppose weighting and base function be the same

$$
\sigma_{n}(z)= \begin{cases}1 & \text { if } z_{n}-d / 2<z<z_{n}+d / 2  \tag{12.9}\\ 0 & \text { otherwise }\end{cases}
$$

In another word Galerkin method with pulse function as base function. In that condition the only change that we have is $Z_{m n}$.

$$
\begin{equation*}
Z_{m n}=\frac{a}{2 \epsilon_{0}} \int_{z_{m}-d / 2}^{z_{m}+d / 2} \int_{z_{n}-d / 2}^{z_{n}+d / 2} \frac{d z d z^{\prime}}{\sqrt{a^{2}+\left(z-z^{\prime}\right)^{2}}} \tag{12.10}
\end{equation*}
$$

and the result of this double integral will be

$$
\begin{equation*}
\iint \frac{d U d V}{\sqrt{a^{2}+X^{2}}}=\sqrt{a^{2}+X^{2}}-X \ln \left(X+\sqrt{a^{2}+X^{2}}\right) ; X=U-V \tag{12.11}
\end{equation*}
$$

The FORTRAN Code rod.for rod1.for is written and you can check the results.

- Exercise: Write a program for previous example when base function is triangular function.
- Example (2): We have a square metallic sheet, and we put Q[C] charge on it. find charge distribution on that square and capacitance of it.
- Solution: The integral equation of our problem is

$$
\begin{equation*}
V=\frac{1}{4 \pi \epsilon_{0}} \iint \frac{\sigma\left(x^{\prime} y^{\prime}\right) d x^{\prime} d y^{\prime}}{\sqrt{\left(x-x^{\prime}\right)^{2}+\left(y-y^{\prime}\right)^{2}}} \tag{12.12}
\end{equation*}
$$

In our problem $V=g$ is known and since it is a metallic conductor it is equipotential, so we assume it is one volt $V=1$. We divide the square into small square or another word N cell. Each cell has side $d$. First let use weighting function as $w_{m}(x, y)=\delta\left(x-x_{m}\right) \delta\left(y-y_{m}\right)$ (point matching); and base function as (Pulse function)

$$
\sigma_{n}(x, y)= \begin{cases}1 & \text { if } x_{n}-d / 2<x<x_{n}+d / 2, y_{n}-d / 2<y<y_{n}+d / 2  \tag{12.13}\\ 0 & \text { otherwise }\end{cases}
$$

The elements of our matrix will be $Z_{m n}=\left\langle\mathcal{L} f_{n}, w_{m}\right\rangle$, therefore we will have four integral (why ?)

$$
\begin{equation*}
Z_{m n}=\frac{1}{4 \pi \epsilon_{0}} \int_{x_{n}-d / 2}^{x_{n}+d / 2} \int_{y_{n}-d / 2}^{y_{n}+d / 2}\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\delta\left(x-x_{m}\right)\left(y-y_{m}\right) d x d y}{\sqrt{\left(x-x^{\prime}\right)^{2}+\left(y-y^{\prime}\right)^{2}}}\right] d x^{\prime} d y^{\prime} \tag{12.14}
\end{equation*}
$$

according to properties of delta function we will have:

$$
\begin{equation*}
Z_{m n}=\frac{1}{4 \pi \epsilon_{0}} \int_{x_{n}-d / 2}^{x_{n}+d / 2} \int_{y_{n}-d / 2}^{y_{n}+d / 2} \frac{d x^{\prime} d y^{\prime}}{\sqrt{\left(x_{m}-x^{\prime}\right)^{2}+\left(y_{m}-y^{\prime}\right)^{2}}} \tag{12.15}
\end{equation*}
$$

by changing $U=x_{m}-x^{\prime}, V=y_{m}-y^{\prime}, d U=-d x^{\prime}, d V=-d y^{\prime}$ and $U 1=x_{m}-$ $\left(x_{n}-d / 2\right), U 2=x_{m}-\left(x_{n}+d / 2\right), V 1=y_{m}-\left(y_{n}-d / 2\right), V 2=y_{m}-\left(y_{n}+d / 2\right)$ therefore we will have:

$$
\begin{equation*}
Z_{m n}=\frac{1}{4 \pi \epsilon_{0}} \int_{U 1}^{U 2} \int_{V 1}^{V 2} \frac{d U \cdot d V}{\sqrt{U^{2}+V^{2}}} \tag{12.16}
\end{equation*}
$$

so the result will be:

$$
\begin{align*}
\left(4 \pi \epsilon_{0}\right) Z_{m n}= & V 1 \ln \left(U 1+\sqrt{U 1^{2}+V 1^{2}}\right)+U 1 \ln \left(V 1+\sqrt{U 1^{2}+V 1^{2}}\right)- \\
& V 1 \ln \left(U 2+\sqrt{U 2^{2}+V 1^{2}}\right)-U 2 \ln \left(V 1+\sqrt{U 2^{2}+V 1^{2}}\right)- \\
& V 2 \ln \left(U 1+\sqrt{U 1^{2}+V 2^{2}}\right)-U 1 \ln \left(V 2+\sqrt{U 1^{2}+V 2^{2}}\right)+ \\
& V 2 \ln \left(U 2+\sqrt{U 2^{2}+V 2^{2}}\right)+U 2 \ln \left(V 2+\sqrt{U 2^{2}+V 2^{2}}\right)(12 \tag{12.17}
\end{align*}
$$

Let us find the $\left\langle g, w_{m}\right\rangle$, in our problem $g=V$ and $w_{m}$ is delta function as mentioned above. The vector $\left\langle g, w_{m}\right\rangle=1$., so the matrix $[A][X]=[B]$ is ready for solution. There are many ways for finding $[X]$ and we usually use Gauss elimination method. Fig.(12.2) shows distribution of charge on a circular conductive disk and it is compared with exact method which is given on appendix.

- Example (3): Charge Distribution on Microstrip. We will assume that


Figure 12.2: Charge distribution on a circular conducting disk
entire upper half space is filled with the same material, Fig[??]. Science we have perfect ground, we employ the image theory method to create an equivalent problem for upper half space. The potential at point $r$ in homogeneous
space in 2D problem is given by

$$
\begin{equation*}
V(r)=\frac{1}{2 \pi \epsilon} \int_{c^{\prime}} \rho\left(r^{\prime}\right) \ln \left(\frac{1}{r-r^{\prime}}\right) d l^{\prime} \tag{12.18}
\end{equation*}
$$

or

$$
\begin{equation*}
V(r)=-\frac{1}{2 \pi \epsilon} \int_{c^{\prime}} \rho\left(r^{\prime}\right) \ln \left(\sqrt{\left(x-x^{\prime}\right)^{2}+\left(y-y^{\prime}\right)^{2}}\right) d l^{\prime} \tag{12.19}
\end{equation*}
$$

Note that we must integrate along all contours $C^{\prime}$ that we have surface charge. Hence $C^{\prime}$ includes both the strip and its image. Now, we will apply the boundary condition that on the top strip

$$
\begin{equation*}
V(r)=V_{0} \quad \forall r \in\{\text { upper strip }\} \tag{12.20}
\end{equation*}
$$

Using Eq.(12.20) in Eq.(12.19) and accounting for both the $+\rho_{l}$ and $-\rho_{l}$ charge distribution on the strips then

$$
\begin{align*}
V_{0} & =-\frac{1}{2 \pi \epsilon}\left\{\int_{\text {top }} \rho\left(r^{\prime}\right) \ln \left(\sqrt{\left(x-x^{\prime}\right)^{2}+(d-d)^{2}}\right) d l^{\prime}\right.  \tag{12.21}\\
& \left.+\int_{\text {bottom }}-\rho\left(r^{\prime}\right) \ln \left(\sqrt{\left(x-x^{\prime}\right)^{2}+(d+d)^{2}}\right) d l^{\prime}\right\}
\end{align*}
$$

or

$$
\begin{equation*}
V_{0}=-\frac{1}{2 \pi \epsilon} \int_{0}^{w} \rho\left(r^{\prime}\right)\left[\ln \left(\left|x-x^{\prime}\right|\right)-\ln \left(\sqrt{\left(x-x^{\prime}\right)^{2}+4 d^{2}}\right)\right] d x^{\prime} \tag{12.22}
\end{equation*}
$$

This is the integral equation for the line charge density on the strip. Now let us use pulse expansion as a base function for charge density and delta function as a weighting functions (Point Matching Method). Therefore

$$
\begin{equation*}
\rho\left(r^{\prime}\right)=\sum_{n=1}^{N} \alpha_{n} P_{n}\left(x^{\prime} ; x_{n-1}, x_{n}\right) \tag{12.23}
\end{equation*}
$$

After using point matching and pulse expansion we will have a set of linear equations.

$$
\begin{equation*}
V_{0}=-\frac{1}{2 \pi \epsilon} \sum_{n=1}^{N} \alpha_{n} \int_{x_{n-1}}^{x_{n}} G\left(\left|x_{m}-x^{\prime}\right|\right) d x^{\prime} \quad m=1, \cdots, N \tag{12.24}
\end{equation*}
$$

where $G\left(\left|x-x^{\prime}\right|\right)=\ln \left(\left|x-x^{\prime}\right|\right)-\ln \left(\sqrt{\left(x-x^{\prime}\right)^{2}+4 d^{2}}\right)$ and the above equation is a matrix equation of the form:

$$
\begin{equation*}
\underbrace{\left[V_{m}\right]}_{N \times 1}=[\underbrace{\left.Z_{m n}\right]}_{N \times N} \cdot[\underbrace{\alpha_{n}}_{N \times 1}] \tag{12.25}
\end{equation*}
$$

where $V_{m}=V_{0}$ and

$$
\begin{equation*}
Z_{m n}=-\frac{1}{2 \pi \epsilon} \int_{x_{n-1}}^{x_{n}} G\left(\left|x_{m}-x^{\prime}\right|\right) d x^{\prime} \quad m \neq n \tag{12.26}
\end{equation*}
$$

As before, the process for obtaining a numerical solution from Eq.(12.25) is to fill $[V]$ and $[Z]$, then solve this system of equations for the line charge density coefficients $\alpha$. In particular, for $[V]$ choose $\mathrm{v}=1$ volt, and compute Eq.(12.26)analytically, if possible, or use numerical integration. In this particular problem, we are able to evaluate Eq.(12.26) analytically science a simple antiderivative of the integrand is available.
In order to evaluate Eq.(12.26) analytically, we will begin with a segment of width $\Delta$ located at the origin that is supporting a uniform line charge density $\rho_{l}$ : Fig[??]. The electrostatic potential at point $r$ produced by this pulse of line charge density is:

$$
\begin{equation*}
V(r)=-\frac{1}{2 \pi \epsilon} \int_{-\Delta / 2}^{\Delta / 2} \ln \left[\sqrt{\left(x-x^{\prime}\right)^{2}+y^{2}}\right] d x^{\prime} \tag{12.27}
\end{equation*}
$$

where $r$ does not lie anywhere on this strip, the potential is

$$
\begin{align*}
V(r) & =-\frac{1}{2 \pi \epsilon}\left\{(x+\Delta / 2) \ln \left[(x+\Delta / 2)^{2}+y^{2}\right]\right.  \tag{12.28}\\
& -(x-\Delta / 2) \ln \left[(x-\Delta / 2)^{2}+y^{2}\right]-2 \Delta \\
& \left.+2 y\left[\tan ^{-1}\left(\frac{x+\Delta / 2}{y}\right)-\tan ^{-1}\left(\frac{x-\Delta / 2}{y}\right)\right]\right\}
\end{align*}
$$

Using Eq.(12.28) in Eq.(12.26), it can be shown that for $m \neq n$ :

$$
\begin{align*}
Z_{m n} & =-\frac{1}{4 \pi \epsilon}\left\{\left(\Delta_{m n}+\Delta / 2\right) \ln \left[\frac{\left(\Delta_{m n}+\Delta / 2\right)^{2}}{\left(\Delta_{m n}+\Delta / 2\right)^{2}+4 d^{2}}\right]\right.  \tag{12.29}\\
& -\left(\Delta_{m n}-\Delta / 2\right) \ln \left[\frac{\left(\Delta_{m n}-\Delta / 2\right)^{2}}{\left(\Delta_{m n}-\Delta / 2\right)^{2}+4 d^{2}}\right] \\
& \left.-4 d\left[\tan ^{-1}\left(\frac{\Delta_{m n}+\Delta / 2}{2 d}\right)-\tan ^{-1}\left(\frac{\Delta_{m n}-\Delta / 2}{2 d}\right)\right]\right\}
\end{align*}
$$

where $\Delta_{m n}=x_{m}-x_{n}$. For self cell evaluations, $Z_{m m}$, the observation point will be located at the center of segment in the chosen point matching scheme. Referring to the Fig[??], if $x=y=0$, it can be shown that $V(0)=\frac{\rho_{l}}{2 \pi \epsilon}[1-$ $\ln (\Delta / 2)]$, Consequently, from this result it can be shown that for $m=n$ we have:

$$
\begin{align*}
Z_{m m} & =\frac{\Delta}{2 \pi \epsilon}[1-\ln (\Delta / 2)]  \tag{12.30}\\
& +\frac{1}{4 \pi \epsilon}\left\{\Delta \ln \left[(\Delta / 2)^{2}+4 d^{2}\right]-2 \Delta+8 d \tan ^{-1}\left(\frac{\Delta}{4 d}\right)\right\}
\end{align*}
$$

### 12.7 Electric Field Integral Equation, EFIE

The integro-differential equation for the current distribution based on the electric field operator is called the electric field integral equation (EFIE). In this section, we are interested in the electric field operator equation. The method of moments is applied to the electric field boundary value equation to obtain a set of linear equations for the induced electric surface current on the surface of a scatterer or antenna. We shall give a derivation of the operator equation in this Section, and we will concentrate on evaluating the so called impedance matrix.

Let $S$ denote the surface of a perfectly conducting scatterer with unit normal vector $\mathbf{n}$ may be either open or closed. The incident electric field $\mathbf{E}^{\text {inc }}$ is due to an impressed source in the absence of the scatterer. The boundary condition is such that the sum of the incident, $\mathbf{E}^{i}$, and the scattered, $\mathbf{E}^{s}$, electric fields has no tangential component on the perfectly conducting body surface, i.e.,

$$
\begin{equation*}
\mathbf{E}_{\text {tan }}^{i}+\mathbf{E}_{\text {tan }}^{s}=0 \quad \text { on } \mathrm{S} \tag{12.31}
\end{equation*}
$$

where the subscript "tan" denotes the components tangential to the surface $S$. $\mathbf{J}$ is the electric current which is induced on the surface due to the incident field. If $S$ is open, we regard $\mathbf{J}$ as the vector sum of the currents on opposite sides of $S$.

The scattered electric field can be represented by the vector potential and the scalar potential which are produced by the surface current, as below:

$$
\begin{equation*}
\mathbf{E}^{s}(\mathbf{J})=-j \omega \mathbf{A}-\nabla \phi \tag{12.32}
\end{equation*}
$$

The magnetic vector potential, A, and the electric scalar potential $\phi$ are given by [8]:

$$
\begin{align*}
& \mathbf{A}(\mathbf{r})=\mu_{0} \int_{S} \mathbf{J}\left(\mathbf{r}^{\prime}\right) \frac{e^{-j \beta\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}}{4 \pi\left|\mathbf{r}-\mathbf{r}^{\prime}\right|} d \mathbf{s}  \tag{12.33}\\
& \phi(\mathbf{r})=\frac{1}{\epsilon_{0}} \int_{S} \sigma\left(\mathbf{r}^{\prime}\right) \frac{e^{-j \beta\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}}{4 \pi\left|\mathbf{r}-\mathbf{r}^{\prime}\right|} d \mathbf{s} \tag{12.34}
\end{align*}
$$

An $e^{j \omega t}$ time dependence is assumed and is suppressed, and $\beta=\omega \sqrt{\mu_{0} \epsilon_{0}}=$ $\frac{2 \pi}{\lambda}$, where $\lambda$ is the wavelength. The permeability and permittivity of the surrounding medium are $\mu_{0}$ and $\epsilon_{0}$, respectively, and $\mathbf{r}$ and $\mathbf{r}^{\prime}$ are the arbitrarily located observation point and source point, respectively. The surface charge density $\sigma$ is related to the surface divergence of $\mathbf{J}$ through the equation of continuity,

$$
\begin{equation*}
\nabla_{s} \cdot \mathbf{J}=-j \omega \sigma \tag{12.35}
\end{equation*}
$$

where $\nabla_{s}$ is the surface divergence operator. Substituting Eq.(12.32) into Eq.(12.31) an integro-differential equation for $\mathbf{J}$ is given by

$$
\begin{equation*}
(j \omega \mathbf{A}+\nabla \phi)_{\tan }=\mathbf{E}_{t a n}^{i} \quad \mathrm{r} \text { on } \mathrm{S} \tag{12.36}
\end{equation*}
$$

With $\mathbf{A}$ and $\phi$ given by Eq.(12.33) and Eq.(12.34), Eq.(12.36) is called electric field integral equation (EFIE).

### 12.7.1 TM-wave Scattering from Conducting Cylinder with Arbitrary Cross Section

Let us find the cross section of a metalic cylinder with method of moments. A TM wave incident normally to a cylinder.

$$
\begin{align*}
\mathbf{E}^{i} & =\mathbf{a}_{z} e^{j k\left(x \cos \phi_{0}+y \sin \phi_{0}\right)}  \tag{12.37}\\
\mathbf{H}^{i} & =\frac{1}{-j \omega \mu} \nabla \times \mathbf{E}^{i} \tag{12.38}
\end{align*}
$$

The incident wave produce current density $J_{z}$ witch in turn will produce scattered field $E_{z}^{s}$.

$$
\begin{align*}
\mathbf{E}^{s} & =\mathbf{a}_{z} \frac{-k \eta}{4} \int_{c} J_{z}(c) H_{0}^{(2)}(k R) d c  \tag{12.39}\\
\mathbf{H}^{s} & =\frac{1}{-j \omega \mu} \nabla \times \mathbf{E}^{s} \tag{12.40}
\end{align*}
$$

where $R=\sqrt{\left[x(c)-x^{\prime}(c)\right]^{2}+\left[y(c)-y^{\prime}(c)\right]^{2}}$ and $c$ is a parameter on contour of cylinder. On the surface of cylinder $E_{z}^{i}+E_{z}^{s}=0$ therefore we will have the following integral equation on cylinder.

$$
\begin{equation*}
E_{z}^{i}=\frac{k \eta}{4} \int_{c} J_{z}(c) H_{0}^{(2)}(k R) d c \tag{12.41}
\end{equation*}
$$

Let us use point matching with pulse as a base functions.

$$
\begin{equation*}
J_{z}=\sum_{n=1}^{N} J_{n} P_{n}(c) \tag{12.42}
\end{equation*}
$$

where

$$
P_{n}(c)= \begin{cases}1 & \text { if } \mathrm{c} \in \text { to cell } \mathrm{n}  \tag{12.43}\\ 0 & \text { otherwise }\end{cases}
$$

If we divide the contour of cylinder with N cells, and coordinate $x_{i}, y_{i}$ be the center of cell $i=1,2, \cdots N$, then by using MoM formula, we will have a $N \times N$ linear system of equation $\mathbf{A X}=\mathbf{B}$.

$$
\begin{align*}
& B_{m}=E_{z}^{i}\left(x_{m}, y_{m}\right) \quad m=1,2, \cdots, N \\
& A_{m n}=\frac{k \eta}{4} H_{0}^{(2)}\left(k R_{m n}\right) \Delta C_{n} \quad m \neq n  \tag{12.44}\\
& X_{n}=J_{n} \quad n=1,2, \cdots, N \quad \text { unknowns }
\end{align*}
$$

where $R_{m n}=\sqrt{\left(x_{m}-x_{n}\right)^{2}+\left(y_{m}-y_{n}\right)^{2}}$ and $\Delta C_{n}$ is length of cell $n$. It is obvious that if $m=n$ the $R_{m n}=0$,or Hankel function will be infinite or in another word we will have singularity! In order to remove the singularity, the expansion of Hankel function with small argument will be used. $H_{0}^{(2)}(x) \approx$ $1-j \frac{2}{\pi} \ln \left(\frac{\gamma x}{2}\right)$ where $\gamma=1.781072418$. If we assume that the cell length $\Delta C_{n}$ is small and flat, then the $A_{m m}$ will be

$$
\begin{equation*}
A_{m m}=\frac{k \eta \Delta C_{m}}{4}\left\{1-j \frac{2}{\pi}\left[\ln \left(\frac{\gamma k \Delta C_{m}}{4}\right)-1\right]\right\} \tag{12.45}
\end{equation*}
$$

After finding the $J_{n}$ we can calculate the scattering electromagnetic fields at any points in $x y$ plane. E

$$
\begin{equation*}
E_{z}^{s}=-\sum_{n=1}^{N} J_{n} \frac{k \eta \Delta C_{n}}{4} H_{0}^{(2)}(k R) \tag{12.46}
\end{equation*}
$$

where $R=\sqrt{\left(x-x_{n}\right)^{2}+\left(y-y_{n}\right)^{2}} ; x, y$ is coordinates of observation point and $x_{n}, y_{n}$ are source points.
There is another method to remove the singularity. Instead of expanding Hankel function of small argument, we can use the exact formula.

$$
\begin{align*}
I & =\int_{x_{1}, y_{1}}^{x_{2}, y_{2}} H_{0}^{(2)}(k R) d c \\
& =\Delta C H_{0}^{(2)}\left(\frac{k \Delta C}{2}\right) \\
& -\frac{\pi \Delta C}{2}\left\{H_{0}^{(2)}(k \Delta C / 2) H_{1}(k \Delta C / 2)+H_{1}^{(2)}(k \Delta C / 2) H_{0}(k \Delta C / 2)\right\} \tag{12.47}
\end{align*}
$$

where $\Delta C=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}}$ and $H_{0}, H_{1}$ are Struve Function of zero and first order respectively.

### 12.8 Magnetic Field Integral Equation, MFIE

In this section the magnetic field integral equation is derived for conducting scatterer. It is well known that the MFIE applies only to closed bodies, so that throughout this section we assume that the object has no boundary edges. Let $S$ be the surface of that perfectly conducting scatterer with unit normal vector $\mathbf{n}$. Suppose that the incident magnetic field $\mathbf{H}^{i}$ is due to an impressed source in the absence of scatterer. The scatterer is in a homogeneous space with electrical parameters $\left(\mu_{0}, \epsilon_{0}\right)$, where $\mu_{0}$ is the inductivity or permeability and $\epsilon_{0}$ is the capacitivity or permittivity. The result of enforcing the boundary condition on the magnetic field is given by:

$$
\begin{equation*}
\mathbf{n} \times\left(\mathbf{H}^{i}+\mathbf{H}^{s}\right)=0 \quad \text { on } S^{-} \tag{12.48}
\end{equation*}
$$

where $\mathbf{n}$ is an outward unit normal vector on $S, S^{-}$is the surface is just inside of $S$, and $\mathbf{H}^{i}$ and $\mathbf{H}^{s}$ are the incident and scattered magnetic fields, respectively. The tangential component of the scattered magnetic field can be expressed as a limit for observation points $\mathbf{r}$ not on an edge, (see [4]).

$$
\begin{equation*}
\mathbf{n} \times \mathbf{H}^{s}=\lim _{\mathbf{r} \rightarrow S} \mathbf{n} \times \nabla \times \mathbf{A}=-\frac{\mathbf{J}_{s}}{2}+\mathbf{n} \times \iint_{S} \mathbf{J}_{s}\left(r^{\prime}\right) \times \nabla^{\prime} G\left(\mathbf{r}, \mathbf{r}^{\prime}\right) d \mathbf{s}^{\prime} \tag{12.49}
\end{equation*}
$$

The $\mathbf{J}_{s}$ is the induced electric surface current on $S$, and the integral on the right hand side in Eq.(12.49), with the field point exactly on $S$, is interpreted
as the Cauchy principal value. $G\left(\mathbf{r}, \mathbf{r}^{\prime}\right)$ is free space Green's function, and $\nabla^{\prime}$ is the gradient operator on the primed coordinates. The Green's function and its gradient are given below as

$$
\begin{equation*}
G=\frac{e^{-j \beta R}}{4 \pi R} \tag{12.50}
\end{equation*}
$$

and

$$
\begin{equation*}
\nabla^{\prime} G=\left(\mathbf{r}-\mathbf{r}^{\prime}\right) \frac{(1+j \beta R) e^{-j \beta R}}{4 \pi R^{3}} \tag{12.51}
\end{equation*}
$$

where $\beta$ is the wave number, $R$ is the distance between the source point and the observation point which is $R=\left|\mathbf{r}-\mathbf{r}^{\prime}\right|$, and $\mathbf{r}$ approaches $S$ from the interior. Substituting Eq.(12.49) into Eq.(12.48), we obtain the magnetic field integral equation:

$$
\begin{equation*}
\frac{\mathbf{J}_{s}}{2}-\mathbf{n} \times \iint_{S} \mathbf{J}_{s}\left(r^{\prime}\right) \times \nabla^{\prime} G\left(\mathbf{r}, \mathbf{r}^{\prime}\right) d \mathbf{s}^{\prime}=\mathbf{n} \times \mathbf{H}^{i} \tag{12.52}
\end{equation*}
$$

In order to apply the method of moments, the surface of the scatterer is decomposed into a set of triangular patches using a parametric description of the surface. The procedure of the parametric surface model generation has been described in mesh generation chapter. The next step after surface modeling is to define a set of basis functions which are used to approximate the surface current. As long as $S$ is a closed and smooth surface, the MFIE yields good results even with a simple modeling in which the current on each patch is approximated by a constant pulse function and a MoM solution is pursued by point matching at the center of each patch. Here, the electric surface current can be approximated as a linear combination of the expansion functions with a set of unknown coefficients.

$$
\begin{equation*}
\mathbf{J}_{s}(\mathbf{r})=\sum_{n=1}^{N} \mathbf{J}_{n}(\mathbf{r}) \tag{12.53}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{J}_{n}(\mathbf{r})=\mathbf{I}_{n}^{1} \mathbf{P}_{n}^{1}(\mathbf{r})+\mathbf{I}_{n}^{2} \mathbf{P}_{n}^{2}(\mathbf{r}) \tag{12.54}
\end{equation*}
$$

and

$$
\mathbf{P}_{n}^{i}(\mathbf{r})= \begin{cases}\mathbf{a}_{n}^{i} & \text { for } \mathbf{r} \in \Delta \mathbf{S}_{n}  \tag{12.55}\\ 0 & \text { elsewhere }\end{cases}
$$

for $i=1,2 ; n=1,2, \cdots N$ and $\Delta \mathbf{S}_{n}$ is the nth of the N triangular patches on which the unknown currents are to be determined. The unit vectors $\mathbf{a}_{n}^{i}$ are
two arbitrarily chosen orthogonal unit vectors on the plane of $\Delta \mathbf{S}_{n}$ as shown in Fig.(12.3). A simple way to select a consistent orientation for $\mathbf{a}_{n}^{1}$ is to let it be perpendicular to the x or z coordinate, that is, by defining


Figure 12.3: A Triangular Patch

$$
\mathbf{a}_{n}^{1}= \begin{cases}\mathbf{n}_{n} \times \mathbf{a}_{z} & \text { if } \mathbf{n}_{n} \times \mathbf{a}_{z} \neq 0  \tag{12.56}\\ \mathbf{n}_{n} \times \mathbf{a}_{x} & \text { otherwise }\end{cases}
$$

where the $\mathbf{n}_{n}$ is the unit vector normal to the surface $\Delta \mathbf{S}_{n}$ and pointed outward to the exterior of the closed surface $S$. Next we define

$$
\begin{equation*}
\mathbf{a}_{n}^{2}=\mathbf{n}_{n} \times \mathbf{a}_{n}^{1} \tag{12.57}
\end{equation*}
$$

Another way to define these unit vectors is to choose them to be in phase with the incident wave, more closely approximating the phase of $\mathbf{J}_{s}$. Thus we define

$$
\begin{equation*}
\mathbf{a}_{n}^{1}=\frac{\mathbf{n} \times \mathbf{H}^{i}\left(\mathbf{r}_{n}\right)}{\left|\mathbf{n} \times \mathbf{H}^{i}\left(\mathbf{r}_{n}\right)\right|} \tag{12.58}
\end{equation*}
$$

and choose $\mathbf{a}_{n}^{2}$ according to Eq.(12.57) where $\mathbf{r}_{n}$ is the position vector from the origin to the center of the patch $\Delta S_{n}$.
To obtain the outward unit normal vector on the patch, we note that for the patch in Fig.(12.3),

$$
\begin{equation*}
\mathbf{n}_{n}=\frac{\left(\mathbf{r}_{r}-\mathbf{r}_{q}\right) \times\left(\mathbf{r}_{p}-\mathbf{r}_{r}\right)}{\left|\left(\mathbf{r}_{r}-\mathbf{r}_{q}\right) \times\left(\mathbf{r}_{p}-\mathbf{r}_{r}\right)\right|} \tag{12.59}
\end{equation*}
$$

By choosing weighting function $\mathbf{W}_{m}^{i}$ as

$$
\begin{equation*}
\mathbf{W}_{m}^{i}=\delta\left(\mathbf{r}-\mathbf{r}_{m}\right) \mathbf{P}_{m}^{i}(\mathbf{r}) \quad i=1,2 ; \quad m=1,2, \cdots, N \tag{12.60}
\end{equation*}
$$

and applying method of MoM, we have

$$
\begin{equation*}
\sum_{n=1}^{N} \sum_{j=1}^{2} Z_{m n}^{i j} I_{n}^{j}=V_{m}^{i} \quad i=1,2 ; m=1,2, \cdots, N \tag{12.61}
\end{equation*}
$$

After a long manipulations we have

$$
\begin{equation*}
v_{m}^{i}=\mathbf{a}_{m}^{i} \cdot 2 \mathbf{n}_{m} \times \mathbf{H}^{i}\left(\mathbf{r}_{m}\right) \tag{12.62}
\end{equation*}
$$

and

$$
\begin{equation*}
Z_{m n}^{i j}=\delta_{m}^{n} \delta_{i}^{j}-\frac{1+j \beta R_{m n}}{2 n R_{m n}^{2}} e^{-j \beta R_{m n}} \Delta S_{n}\left[\mathbf{a}_{m}^{i} \cdot \mathbf{n}_{m} \times\left(\mathbf{a}_{n}^{j} \times \mathbf{R}_{m n}\right)\right]\left(1-\delta_{m}^{n}\right) \tag{12.63}
\end{equation*}
$$

After finding currents components we would like to calculate the far fields.


Figure 12.4: Scattering by a conducting Sphere

### 12.9 Pocklington Integral Equation

Let us assume that we have a straight wire antenna with length $L=2 H$ and radius $a$ which is small compared to wavelength $\lambda . \frac{a}{L}$ do not need to be small. Axial current component $I(z)$ is the only significant component, Fig.(12.5). Modeling the current distribution is infinitely thin sheet of current forming


Figure 12.5: A Dipole Antenna
a tube of radius $a$. We use EFIE in order to analyze the problem.
step 1: Find the operator $\mathcal{L}$
$\mathbf{E}=\mathbf{E}^{i}+\mathbf{E}^{s}$ must be zero at the surface of wire, but $\mathbf{E}^{s}=-j \omega \mathbf{A}-\nabla \Phi$ where $\mathbf{A}=\mu \oint \int_{s} \mathbf{J} \frac{e^{-j k R}}{4 \pi R} d s$ and $\Phi=\frac{1}{\epsilon} \iint_{s} \sigma \frac{e^{-j k R}}{4 \pi R} d s$. Let us apply boundary conditions, i.e. total electric field must vanish on the surface $\mathbf{E}_{\text {tan }}^{t}\left(r_{s}\right)=0$. therefore $\mathbf{E}_{\text {tan }}^{i}\left(r_{s}\right)=-\mathbf{E}_{\text {tan }}^{s}\left(r_{s}\right)$. It is possible to show that the scattered electric field can be expressed as:

$$
\begin{equation*}
E_{z}^{s c a t}=-j \omega A_{z}-\frac{\partial \Phi}{\partial z}=\frac{-j}{\omega \mu \epsilon}\left(k^{2} A_{z}+\frac{\partial^{2} A_{z}}{\partial z^{2}}\right) \tag{12.64}
\end{equation*}
$$

where the Lorentz condition $\frac{\partial A_{z}}{\partial z}=-j \omega \mu \epsilon \Phi$ have been applied. We also know that:

$$
\begin{align*}
A_{z} & =\frac{\mu}{4 \pi} \iint_{s} J_{z} \frac{e^{-j k R}}{R} d s \\
& =\frac{\mu}{4 \pi} \int_{0}^{L} \oint_{0}^{2 \pi} J_{z} \frac{e^{-j k R}}{R} a d \phi^{\prime} d z^{\prime} \\
& =\mu \int_{0}^{L} I_{z}\left(z^{\prime}\right) G\left(z, z^{\prime}\right) d z^{\prime} \tag{12.65}
\end{align*}
$$

Therefore, the EFIE becomes:

$$
\begin{equation*}
\int_{0}^{L} I_{z}\left(z^{\prime}\right)\left[\left(\frac{\partial^{2}}{\partial z^{2}}+k^{2}\right) G\left(z, z^{\prime}\right)\right] d z^{\prime}=-j \omega \epsilon E_{z}^{i n c}(\rho=a) \tag{12.66}
\end{equation*}
$$

where $G\left(z, z^{\prime}\right)=\frac{e^{-j k R}}{4 \pi R}$ with $R=\sqrt{a^{2}+\left(z-z^{\prime}\right)}, \frac{\partial G\left(z, z^{\prime}\right)}{\partial z}=-\frac{\partial G\left(z, z^{\prime}\right)}{\partial z^{\prime}}, \frac{\partial^{2} G\left(z, z^{\prime}\right)}{\partial z^{2}}=$ $\frac{\partial^{2} G\left(z, z^{\prime}\right)}{\partial z^{\prime 2}}$. Therefore the equation Eq.(12.66) is known as Pocklington's Integral Equation. We can rewrite it in other form

$$
\begin{equation*}
\frac{j \omega \mu}{4 \pi}\left[\left(1+\frac{1}{k^{2}} \frac{\partial^{2}}{\partial z^{\prime 2}}\right)\right] \int_{-h}^{h} I_{z}\left(z^{\prime}\right) G\left(z, z^{\prime}\right) d z=E_{z}^{i n c}(z) \tag{12.67}
\end{equation*}
$$

Now we are going $t$ to find the current distribution on a wire antenna. We break up the wire into N segments, and expand $\mathbf{J}(z)$ in series:

$$
\begin{gather*}
\boxed{\mathbf{J}(z) \sum_{n=1}^{N} a_{n} \mathbf{J}_{n}(z)}  \tag{12.68}\\
\mathbf{J}_{n}(z)= \begin{cases}\frac{\sin \left\{k_{0}\left(z-z_{n-1}\right)\right\}}{\sin \left\{k_{0}\left(z_{n}-z_{n-1}\right)\right\}} & \left(z_{n-1}<z<z_{n}\right) \\
\frac{\sin \left\{k_{0}\left(z_{n+1}-z\right)\right\}}{\sin \left\{k_{0}\left(z_{n+1}-z_{n}\right)\right\}} & \left(z_{n}<z<z_{n+1}\right) \\
0 & \text { elsewhere }\end{cases}  \tag{12.69}\\
\frac{j \omega \mu}{4 \pi} \sum_{n=1}^{N} a_{n}\left[\left(1+\frac{1}{k^{2}} \frac{\partial^{2}}{\partial z^{\prime 2}}\right)\right] \int_{z_{n-1}}^{z_{n+1}} \mathbf{J}_{n}\left(z^{\prime}\right) G\left(z, z^{\prime}\right) d z=E_{z}^{i n c}(z)  \tag{12.70}\\
\hline
\end{gather*}
$$



Figure 12.6: Current distribution on a wire antenna
or

$$
\begin{equation*}
\frac{j \omega \mu}{4 \pi} \sum_{n=1}^{N} a_{n}\left[\left(1+\frac{1}{k^{2}} \frac{\partial^{2}}{\partial z^{\prime 2}}\right)\right] \int_{z_{n-1}}^{z_{n+1}} \mathbf{J}_{n}\left(z^{\prime}\right) G\left(z, z^{\prime}\right) d z=E_{z}^{i n c}(z) \tag{12.71}
\end{equation*}
$$

and by using weighting function we will have

$$
\begin{aligned}
\frac{j \omega \mu}{4 \pi} \sum_{n=1}^{N} a_{n} \int_{z_{m-1}}^{z_{m+1}} \mathbf{J}_{m}(z)\left[\left(1+\frac{1}{k^{2}} \frac{\partial^{2}}{\partial z^{\prime 2}}\right)\right] & \int_{z_{n-1}}^{z_{n+1}} \mathbf{J}_{n}\left(z^{\prime}\right) G\left(z, z^{\prime}\right) d z d z^{\prime}(12.72) \\
& =\int_{z_{m-1}}^{z_{m+1}} \mathbf{J}_{m}(z) E_{z}^{i n c}(z) d z
\end{aligned}
$$

To rewrite it in matrix form:

$$
\begin{equation*}
\left[Z_{m n}\right]\left[a_{n}\right]=\left[V_{m}\right] \tag{12.73}
\end{equation*}
$$

where

$$
\begin{equation*}
Z_{m n}=\int_{z_{m-1}}^{z_{m+1}} \mathbf{J}_{m}(z) E_{n}(z) d z \tag{12.74}
\end{equation*}
$$

and

$$
\begin{equation*}
E_{n}=\frac{j \omega \mu}{4 \pi}\left[\left(1+\frac{1}{k^{2}} \frac{\partial^{2}}{\partial z^{2}}\right)\right] \int_{z_{n-1}}^{z_{n+1}} \mathbf{J}_{n}\left(z^{\prime}\right) G\left(z, z^{\prime}\right) d z^{\prime} \tag{12.75}
\end{equation*}
$$

Piecewise Sinusoidal(PWS) basis function are used since they closely resemble the current distribution on a thin wire. PWS basis functions start to


Figure 12.7: Piecewise sinusoidal basis function
become straight lines as the sample interval becomes small with respect to a wavelength. By summing the contributions of basis functions where they overlap (which occurs in every segment except the first and last), a continuous function can be created. Now let us apply Basis and Weighting Functions to our Pocklington's Integral Equation. The Galerkin approach using PWS basis functions is the most commonly used approach in electromagnetic. The $E_{n}\left(z^{\prime}\right)$ can be calculated for PWS as

$$
\begin{align*}
& E_{n}\left(z^{\prime}\right)=j 30\left[\frac{1}{\sin \left\{k_{0}\left(z_{n}-z_{n-1}\right)\right\}}\left\{\cos \left(k_{0}\left(z_{n}-z_{n-1}\right)\right) G\left(z^{\prime}, z_{n}\right)-G\left(z^{\prime}, z_{n-1}\right)\right\}\right. \\
& \left.+\frac{1}{\sin \left\{k_{0}\left(z_{n+1}-z_{n}\right)\right\}}\left\{\cos \left(k_{0}\left(z_{n+1}-z_{n}\right)\right) G\left(z^{\prime}, z_{n}\right)-G\left(z^{\prime}, z_{n+1}\right)\right\}\right] \tag{12.76}
\end{align*}
$$

and

$$
\begin{equation*}
Z_{m n}=\int_{z_{m-1}}^{z_{m}} \frac{\sin \left\{k_{0}\left(z^{\prime}-z_{m-1}\right)\right\}}{\sin \left\{k_{0}\left(z_{m}-z_{m-1}\right)\right\}} E_{n}\left(z^{\prime}\right) d z^{\prime}+\int_{z_{m}}^{z_{m+1}} \frac{\sin \left\{k_{0}\left(z_{m+1}-z^{\prime}\right)\right\}}{\sin \left\{k_{0}\left(z_{m+1}-z_{m}\right)\right\}} E_{n}\left(z^{\prime}\right) d z^{\prime} \tag{12.77}
\end{equation*}
$$

$$
\begin{equation*}
V_{m}=-\int_{z_{m-1}}^{z_{m}} \frac{\sin \left\{k_{0}\left(z^{\prime}-z_{m-1}\right)\right\}}{\sin \left\{k_{0}\left(z_{m}-z_{m-1}\right)\right\}} E_{z}^{i} d z^{\prime}-\int_{z_{m}}^{z_{m+1}} \frac{\sin \left\{k_{0}\left(z_{m+1}-z^{\prime}\right)\right\}}{\sin \left\{k_{0}\left(z_{m+1}-z_{m}\right)\right\}} E_{z}^{i} d z^{\prime} \tag{12.78}
\end{equation*}
$$

### 12.9.1 1D Cavity Resonator

Suppose two large flat mirror separated by distance $d$. We want to find the first two lowest resonant frequency of this 1 D cavity by analytic and point matching MoM with a base functions; $f_{1}=z\left(1-\frac{z}{d}\right)$ and $f_{2}=z\left[1-\left(\frac{z}{d}\right)^{2}\right]$.


Figure 12.8: 1D Cavity resonator
We select $w_{1}=\delta\left(z-\frac{d}{3}\right)$ and $w_{2}=\delta\left(z-\frac{2 d}{3}\right)$. The wave equation will be

$$
\frac{\partial^{2} E_{x}}{\partial z^{2}}+k^{2} E_{x}=0
$$

The solution will be $E x=A e^{-j k z}+B e^{+j k z}$ and by applying boundary condition at $z=0$ and $z=d$ that $E_{x}=0$.

$$
-2 j A \sin (k d)=0
$$

therefore $k d=m \pi ; \quad m=1,2, \ldots$ or the resonant frequencies or eigenvalues will be $f_{r}=\frac{m c}{2 d}$ and eigenfunction $E_{x}=-2 j A \sin \left(\frac{m \pi}{d} z\right)$. Now let us use MoM for this problem.

$$
\tilde{f}=\alpha_{1} f_{1}+\alpha_{2} f_{2}
$$

The $A_{m n}=\left\langle\mathcal{L} f_{n}, w_{m}\right\rangle$

$$
A_{11}=\int_{0}^{d} \frac{2}{d} \delta\left(z-\frac{d}{3}\right) d z=\frac{2}{d}
$$

$$
\begin{aligned}
& A_{12}=\int_{0}^{d} \frac{6}{d^{2}} z \delta\left(z-\frac{d}{3}\right) d z=\frac{2}{d} \\
& A_{21}=\int_{0}^{d} \frac{2}{d} \delta\left(z-\frac{2 d}{3}\right) d z=\frac{2}{d} \\
& A_{22}=\int_{0}^{d} \frac{6}{d^{2}} z \delta\left(z-\frac{2 d}{3}\right) d z=\frac{4}{d}
\end{aligned}
$$

and the $B_{m n}=\left\langle f_{n}, w_{m}\right\rangle$

$$
\begin{gathered}
B_{11}=\int_{0}^{d} z\left(1-\frac{z}{d}\right) \delta\left(z-\frac{d}{3}\right) d z=\frac{2 d}{9} \\
B_{12}=\int_{0}^{d} z\left[1-\left(\frac{z}{d}\right)^{2}\right] \delta\left(z-\frac{d}{3}\right) d z=\frac{8 d}{27} \\
B_{21}=\int_{0}^{d} z\left(1-\frac{z}{d}\right) \delta\left(z-\frac{2 d}{3}\right) d z=\frac{2 d}{9} \\
B_{22}=\int_{0}^{d} z\left[1-\left(\frac{z}{d}\right)^{2}\right] \delta\left(z-\frac{2 d}{3}\right) d z=\frac{10 d}{27}
\end{gathered}
$$

after calculating the eigenvalues, the approximate resonant frequency will be $f_{r}=\frac{3 c}{2 \pi d}$ and $f_{r}=\frac{3 \sqrt{3} c}{2 \pi d}$. If $d=1$ meter, the exact first and second resonant frequency would be 150 MHz and 300 MHz respectively but the Method of Moments will give us 143.31 MHz and 248.22 MHz . Now let us use $w_{n}=f_{n}$ Galerkin Method. In this case the $(k d)^{2}=10$ and 42, therefore the resonant frequency would be $f_{r}=\frac{c \sqrt{10}}{2 \pi d}$ and $f_{r}=\frac{c \sqrt{42}}{2 \pi d}$ respectively. If $d=1$ meter, the first and second resonant frequency will be $f_{r}=150.98 \mathrm{MHz}$ and $f_{r}=$ 309.43 MHz , respectively.

### 12.10 Problems

- 1 A TM plane wave $\mathbf{E}=\mathbf{a}_{z} e^{-j \beta x}$ is incident on a perfect electric conductor (pec) with radius $a=\lambda$. Find normalized bistatic RCS of this cylinder by method of moments and physical optic method and compare your results with rigorous solution.
- 2 A plane wave with frequency of $1 G H z$ is incident on a perfectly conducting sphere with radius $\lambda / 2$. Find the RCS of the sphere by MFIE and compare your results with Mie method.


Figure 12.9: Modes in 1D cavity resonator by MoM, $d=3 \mathrm{Cm}$

## Chapter 13

## Finite Element Method (FEM)

"I want to know God's thoughts; the rest are details." Albert Einstein

### 13.1 Introduction

The Finite Element Method (FEM) is one the most important numerical method that has been used for boundary value problems. The finite element method was first outlined in 1942 by Courant [48]. When preparing the text of his 1942 address to the American Mathematical Society for publication, he added a two-page appendix on the use of variational methods in potential theory, following the principles already described by Lord Rayleigh. Choosing a piecewise linear approximation on a set of triangles, which he called elements, he constructed two-dimensional examples, so marking the birth of the finite element method. Its application to electronic engineering only began in 1969, when a finite element solution of the classical waveguide mode problem was published in a special issue of the Italian journal Alta Frequenza [49]. Other articles soon followed, on magnetic fields in nonlinear materials, on dielectric loaded waveguide, and other well-known problems. Finite elements were soon applied also to integral operators in both electrostatic and antenna problems. In the 1980s this method developed rapidly.
In this introductory course, we will start with theoretical foundations and algorithm implementations for one-dimensional problems so that we can learn the essential tools that carry over to higher dimensions. The student will be acquainted with basic role Rayleigh-Ritz Method which we discussed in gen-
eral as Functional Method in variational concepts. Originally in RayleighRitz method they use entire domain base functions but in FEM they use sub-domain base functions. The basic idea of FEM is to divide the solution domain into a number of small sub-domain region which each one is called element. In one dimensional case, it is segment line, in 2D it may be triangle, rectangle or mutigonal shapes. In 3D it may be pyramid, cubic or other shapes.
There are five essential steps in Finite Element Method
(1) Discretizing the domain of problem
(2) Selection of interpolation functions
(3) Derivation of element characteristic matrices and vectors
(4) Assembling of characteristic matrices and vectors
(5) Solution of the system of equations $(\mathbf{A X}=\mathbf{B})$


Figure 13.1: One, two and three-dimensional mesh generation

### 13.2 One Dimensional FEM

In this section we will consider one dimensional Laplace, Poisson and eigenvalue problem in the form of 1D cavity resonator. In each case these problem which will be given, the reader should verify their results with analytical methods.

### 13.2.1 1D Laplace Equation

1D Laplace Problem: Implement the one-dimensional finite-element solution of the Laplace equation if :
a- $L=4 m$
b- $V_{1}=-1 \mathrm{~V}$
c- $V_{2}=+1 \mathrm{~V}$
d- $\epsilon_{r}(x)=2+\sin (\pi x / 2)$ (x-in meter)
and compare your results with exact solution.
Solution We have faced with a electrostatic problem with nonhomogeneous media. We discrete the length of 1D problem into sub-domain or element, Fig.(13.2). In that figure, $e$ shows the element's numbering, $i$ shows the global nodes numbering and $j$ shows the local nodes numbering. Therefore for each element we have two local numbers, we give number 1 to the left node and 2 to right node. As shown in Fig.(13.2), local numbering is give for element $e=2$. This element also have global numbering. $i=2$ for first node and $i=3$ for second node. We may divide the length $L=4 m$ into N elements. This was mesh generation for our 1D problem.
Now we should proceed in order to find potential distribution along 1D problem with boundary conditions as it is mentioned.


Figure 13.2: 1D global and local Numbering

$$
\begin{equation*}
V(x)=\sum_{e=1}^{N} V_{e}(x) \tag{13.1}
\end{equation*}
$$

The simple way to approximate the $V^{e}(x)$ within an element is polynomial approximation, ie;

$$
\begin{equation*}
V_{e}(x)=a+b x \tag{13.2}
\end{equation*}
$$

and field in each element:

$$
\begin{equation*}
\mathbf{E}_{e}=-\nabla V_{e}=-b \mathbf{a}_{\mathbf{x}} \tag{13.3}
\end{equation*}
$$

Let the potential at nodes $x_{1}^{e}$, and $x_{2}^{e}$ for element $e$ be $V_{1}^{e}, V_{2}^{e}$. therefore

$$
\left[\begin{array}{c}
V_{1}^{e}  \tag{13.4}\\
V_{2}^{e}
\end{array}\right]=\left[\begin{array}{ll}
1 & x_{1}^{e} \\
1 & x_{2}^{e}
\end{array}\right]\left[\begin{array}{l}
a \\
b
\end{array}\right]
$$

from these equation we can find $a$ and $b$ in terms of potentials and positions. Or the potential at each element will be

$$
\begin{equation*}
V_{e}(x)=\sum_{i=1}^{2} \alpha_{i}(x) V_{i}^{e} \tag{13.5}
\end{equation*}
$$

where

$$
\begin{align*}
\alpha_{1}(x) & =\frac{x_{2}^{e}-x}{x_{2}^{e}-x_{1}^{e}} \\
\alpha_{2}(x) & =\frac{x-x_{1}^{e}}{x_{2}^{e}-x_{1}^{e}} \tag{13.6}
\end{align*}
$$

The $\alpha_{1}(x)$ and $\alpha_{2}(x)$ are called the element shape functions. Now we should find, energy of each element.

$$
\begin{equation*}
W_{e}=\frac{1}{2} \int_{x_{1}^{e}}^{x_{2}^{e}} \epsilon(x)|\mathbf{E}|^{2} d x=\frac{1}{2} \int_{x_{1}^{e}}^{x_{2}^{e}} \epsilon(x)\left|\frac{d V_{e}(x)}{d x}\right|^{2} d x \tag{13.7}
\end{equation*}
$$

Substituting Eq.(13.5) into Eq.(13.7), we have

$$
\begin{equation*}
W_{e}=\frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} V_{i}^{e}\left[\int_{x_{1}^{e}}^{x_{2}^{e}} \epsilon(x)\left(\frac{d \alpha_{i}(x)}{d x}\right)\left(\frac{d \alpha_{j}(x)}{d x}\right) d x\right] V_{j}^{e} \tag{13.8}
\end{equation*}
$$

or we may define a 2 by 2 matrix as

$$
\begin{equation*}
C_{i j}^{e}=\int_{x_{1}^{e}}^{x_{2}^{e}} \epsilon(x)\left(\frac{d \alpha_{i}(x)}{d x}\right)\left(\frac{d \alpha_{j}(x)}{d x}\right) d x \tag{13.9}
\end{equation*}
$$

which the energy of our element in matrix form will be

$$
\begin{equation*}
W_{e}=\frac{1}{2}\left[V_{e}\right]^{t}\left[C_{e}\right]\left[V_{e}\right] \tag{13.10}
\end{equation*}
$$

which is similar to $W=\frac{1}{2} C V^{2}$ storing energy in capacitor. In Eq.(13.9) superscript $t$ stand for transpose of the matrix.

$$
\left[V_{e}\right]=\left[\begin{array}{c}
V_{1}^{e}  \tag{13.11}\\
V_{2}^{e}
\end{array}\right]
$$

$$
\left[C_{e}\right]=\left[\begin{array}{cc}
C_{11}^{e} & C_{12}^{e}  \tag{13.12}\\
C_{21}^{e} & C_{22}^{e}
\end{array}\right]
$$

which is usually called stiffness matrix that comes from mechanical engineering and its value will be

$$
\left[C_{e}\right]=\frac{\epsilon}{x_{2}^{e}-x_{1}^{e}}\left[\begin{array}{ll}
+1 & -1  \tag{13.13}\\
-1 & +1
\end{array}\right]
$$

if $\epsilon$ be constant, otherwise we should proceed the integration.
Up to here, we have calculated the energy of only one element and total energy will be the assembling energy of all the elements:

$$
\begin{equation*}
W=\sum_{e=1}^{N} W_{e}=\frac{1}{2}[\mathbf{V}]^{t}[\mathbf{C}][\mathbf{V}] \tag{13.14}
\end{equation*}
$$

where

$$
[\mathbf{V}]=\left[\begin{array}{c}
V_{1}  \tag{13.15}\\
V_{2} \\
\vdots \\
V_{M}
\end{array}\right]
$$

where $M$ is the number of nodes, $N$ is the number of elements and [ $\mathbf{C}]$ is called Global Stiffness Matrix. We can rewrite the total energy of the system as:

$$
W=\frac{1}{2}\left[\begin{array}{ll}
\mathbf{V}_{f} & \mathbf{V}_{p}
\end{array}\right]^{t}\left[\begin{array}{ll}
\mathbf{C}_{f f} & \mathbf{C}_{f p}  \tag{13.16}\\
\mathbf{C}_{p f} & \mathbf{C}_{p p}
\end{array}\right]\left[\begin{array}{l}
\mathbf{V}_{f} \\
\mathbf{V}_{p}
\end{array}\right]
$$

where the vector $\mathbf{V}_{f}$ shows the free or unknown potentials, and $\mathbf{V}_{p}$ stands for prescribed or known potentials. The matrix [C] divided into four block capacitor matrix.
If we differentiate energy with respect to free potentials, we will obtain

$$
\left[\begin{array}{ll}
\mathbf{C}_{f f} & \mathbf{C}_{f p}
\end{array}\right]\left[\begin{array}{l}
\mathbf{V}_{f}  \tag{13.17}\\
\mathbf{V}_{p}
\end{array}\right]=0
$$

or simply

$$
\begin{equation*}
\left[\mathbf{C}_{f f}\right]\left[\mathbf{V}_{f}\right]=-\left[\mathbf{C}_{f p}\right]\left[\mathbf{V}_{p}\right] \tag{13.18}
\end{equation*}
$$

which looks like $\mathbf{A} \cdot \mathbf{X}=\mathbf{B}$ and solution of it is straight forward. As we know from numerical analysis, there are two methods for solution of $\mathbf{A} \cdot \mathbf{X}=\mathbf{B}$
equation: Direct and Indirect. Gauss Elimination method is a direct method but Iteration Method, is called as indirect method. Let us find a way for indirect method. In order to find a way for obtaining the unknown potentials, let us return to Eq.(13.14). If we differentiate energy with respect to unknown potential nodes and put them equal zero, we will have $K=N-2$ equations. One of the unknown potential or free node, say $V_{k}$, will be

$$
\begin{equation*}
V_{k}=-\frac{1}{C_{k k}}\left[\sum_{i=1, i \neq k}^{K} V_{i} C_{k i}\right] \tag{13.19}
\end{equation*}
$$

This means that the nodes which is connected directly to node $k$ contributes to potential $V_{k}$. We guess some value for each free potentials, then by iteration we reach the exact value.
Exact solution: Let also find exact solution for the above problem. We start by Gauss's law $\nabla \cdot \mathbf{D}=\rho$ and electrostatic case, $\mathbf{E}=-\nabla V$ in inhomogeneous $\epsilon(x)$ media.

$$
\begin{equation*}
\nabla \cdot(\epsilon \mathbf{E})=\rho=\nabla \epsilon \cdot \mathbf{E}+\epsilon \nabla \cdot \mathbf{E} \tag{13.20}
\end{equation*}
$$

or

$$
\begin{equation*}
\nabla \epsilon \cdot \nabla V+\epsilon \nabla^{2} V=-\rho \tag{13.21}
\end{equation*}
$$

In our 1D problem, the differential equation will be $\epsilon^{\prime} V^{\prime}+\epsilon V^{\prime \prime}=0$.

### 13.2.2 1D Poisson Equation

1D Poisson Problem: Implement the one-dimensional finite-element solution of the Poisson equation if :
a- $L=4 m$
b- $V_{1}=-1 \mathrm{~V}$
c- $V_{2}=+1 \mathrm{~V}$
d- $\epsilon_{r}(x)=2+\sin (\pi x / 2)$ (x-in meter)
e- $\rho=1$ (C/m)
and compare your results with exact solution.
Solution Like the previous Laplace problem, we divide the domain into elements, and denote for each node an unknown potential. we also know each node's charges. We approximate the potential and charge distribution on our


Figure 13.3: 1D Laplace Equation

1D problem as:

$$
\begin{align*}
V_{e}(x) & =\sum_{i=1}^{2} \alpha_{i}(x) V_{i}^{e}  \tag{13.22}\\
\rho_{e}(x) & =\sum_{i=1}^{2} \alpha_{i}(x) \rho_{i}^{e} \tag{13.23}
\end{align*}
$$

and energy associated to an element will be:

$$
\begin{equation*}
W_{e}=\frac{1}{2} \int_{x_{1}^{e}}^{x_{2}^{e}}\left[\epsilon(x)\left|\frac{d V_{e}(x)}{d x}\right|^{2}-2 \rho_{e}(x) V_{e}(x)\right] d x \tag{13.24}
\end{equation*}
$$

or

$$
\begin{align*}
W_{e} & =\frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} V_{i}^{e}\left[\int_{x_{1}^{e}}^{x_{2}^{e}} \epsilon(x)\left(\frac{d \alpha_{i}(x)}{d x}\right)\left(\frac{d \alpha_{j}(x)}{d x}\right) d x\right] V_{j}^{e}  \tag{13.25}\\
& -\sum_{i=1}^{2} \sum_{j=1}^{2} V_{i}^{e}\left[\int_{x_{1}^{e}}^{x_{2}^{e}} \alpha_{i}(x) \alpha_{j}(x) d x\right] \rho_{j}^{e} \tag{13.26}
\end{align*}
$$

By introducing new notation:

$$
\begin{equation*}
T_{i j}^{e}=\int_{x_{1}^{e}}^{x_{2}^{e}} \alpha_{i}(x) \alpha_{j}(x) d x \tag{13.27}
\end{equation*}
$$

The above equation can be written in matrix form:

$$
\begin{equation*}
W_{e}=\frac{1}{2}\left[V_{e}\right]^{t}\left[C_{e}\right]\left[V_{e}\right]-\left[V_{e}\right]^{t}\left[T_{e}\right]\left[\rho_{e}\right] \tag{13.28}
\end{equation*}
$$

where $\left[\rho_{e}\right]=\left[\begin{array}{ll}\rho_{1}^{e} & \left.\rho_{2}^{e}\right]^{t} \text {, and }\left[C_{e}\right] \text { is the same as Eq.(13.12) and Eq.(13.13). }\end{array}\right.$

$$
\left[T_{e}\right]=\left[\begin{array}{ll}
T_{11}^{e} & T_{12}^{e}  \tag{13.29}\\
T_{21}^{e} & T_{22}^{e}
\end{array}\right]
$$

After some simple manipulation

$$
T_{i j}= \begin{cases}\frac{x_{2}^{e}-x_{1}^{e}}{x_{2}^{3}-x_{1}^{e}} & \text { If } i=j  \tag{13.30}\\ \frac{\text { If } i \neq j}{}\end{cases}
$$

The total energy of the system will be

$$
\begin{equation*}
W=\sum_{e=1}^{N} W_{e}=\frac{1}{2}[\mathbf{V}]^{t}[\mathbf{C}][\mathbf{V}]-[\mathbf{V}]^{t}[\mathbf{T}][\mathbf{R}] \tag{13.31}
\end{equation*}
$$

where $[\mathbf{V}],[\mathbf{C}]$ are the same as before, but the new nodal charge vector $[\mathbf{R}]$ will be

$$
[\mathbf{R}]=\left[\begin{array}{c}
\rho_{1}  \tag{13.32}\\
\rho_{2} \\
\vdots \\
\rho_{M}
\end{array}\right]
$$

We can rewrite the total energy of the system as:
$W=\frac{1}{2}\left[\begin{array}{ll}\mathbf{V}_{f} & \mathbf{V}_{p}\end{array}\right]^{t}\left[\begin{array}{ll}\mathbf{C}_{f f} & \mathbf{C}_{f p} \\ \mathbf{C}_{p f} & \mathbf{C}_{p p}\end{array}\right]\left[\begin{array}{l}\mathbf{V}_{f} \\ \mathbf{V}_{p}\end{array}\right]-\left[\begin{array}{ll}\mathbf{V}_{f} & \mathbf{V}_{p}\end{array}\right]^{t}\left[\begin{array}{ll}\mathbf{T}_{f f} & \mathbf{T}_{f p} \\ \mathbf{T}_{p f} & \mathbf{T}_{p p}\end{array}\right]\left[\begin{array}{l}\mathbf{R}_{f} \\ \mathbf{R}_{p}\end{array}\right]$
where the vector $\mathbf{V}_{f}$ shows the free or known potentials, and $\mathbf{V}_{p}$ stands for prescribed or unknown potentials. The $[\mathbf{C}]$ and $[\mathbf{T}]$ divided into four block
matrix. Minimization of energy with respect to free potentials, $\frac{\partial W}{\partial V_{f}}=0$, will use system of linear equations:

$$
\begin{equation*}
\left[\mathbf{C}_{f f}\right]\left[\mathbf{V}_{f}\right]=-\left[\mathbf{C}_{f p}\right]\left[\mathbf{V}_{p}\right]+\left[\mathbf{T}_{f f}\right]\left[\mathbf{R}_{f}\right]+\left[\mathbf{T}_{f p}\right]\left[\mathbf{R}_{p}\right] \tag{13.34}
\end{equation*}
$$

which looks like $\mathbf{A} \cdot \mathbf{X}=\mathbf{B}$ that can be solved by regular methods. Iteration method is another choice for obtaining unknown potentials. We do the job as before, but in this case we have more terms.

$$
\begin{equation*}
V_{k}=\frac{1}{C_{k k}}\left[-\sum_{i=1, i \neq k}^{K} V_{i} C_{k i}+\sum_{i=1}^{K} T_{k i} R_{i}\right] \tag{13.35}
\end{equation*}
$$

we know the potential of boundaries, also we may guess some value for each free node, then by iteration we reach the exact value. Solution CODE for these two problems will be in FEM1D directory.


Figure 13.4: 1D Poisson Equation

### 13.2.3 1D Cavity Resonator

1D CavityResonator Problem: Suppose we have 1D cavity resonator, like two large and long mirror, Fig.(13.5), which is separated by distance $d$, Find
a- Resonant frequencies of this cavity
b- Draw first, second and third modes of it and confirm your results with exact solution.
Solution: We should find two things, first; eigenvalues which gives us resonant frequencies of the cavity, second; eigenfunctions which gives us the modes of cavity. The general quadratic functional for Helmholtz's equation $\nabla^{2} \psi+k^{2} \psi=g$ will be


Figure 13.5: 1D Cavity Resonator

$$
\begin{equation*}
I(\mathbf{f})=\langle\mathcal{L} \mathbf{f}, \mathbf{f}\rangle-2\langle\mathbf{f}, \mathbf{g}\rangle=\frac{1}{2} \int\left[|\nabla \psi|^{2}-k^{2} \psi^{2}+2 \psi g\right] d v \tag{13.36}
\end{equation*}
$$

which in our 1D problem $g=0$ and $\psi(z)=E_{x}(z)$. We divide the domain of the problem into elements and for each element we devote a shape function

$$
\begin{equation*}
\psi_{e}(z)=\sum_{i=1}^{2} \alpha_{i}(z) \psi_{i}^{e} \tag{13.37}
\end{equation*}
$$

We do the same procedure as we have done for pervious problem,

$$
\begin{equation*}
I_{e}\left(\psi_{e}\right)=\frac{1}{2}\left[\psi_{e}\right]^{t}\left[C_{e}\right]\left[\psi_{e}\right]-\frac{k^{2}}{2}\left[\psi_{e}\right]^{t}\left[T_{e}\right]\left[\psi_{e}\right] \tag{13.38}
\end{equation*}
$$

where $\left[\psi_{e}\right]=\left[\begin{array}{ll}\psi_{1}^{e} & \psi_{2}^{e}\end{array}\right]^{t}$

$$
\begin{equation*}
C_{i j}^{e}=\int_{x_{1}^{e}}^{x_{2}^{e}}\left(\frac{d \alpha_{i}(x)}{d x}\right)\left(\frac{d \alpha_{j}(x)}{d x}\right) d x \tag{13.40}
\end{equation*}
$$

and $T_{e}$ the same value as before. The global functional will be

$$
\begin{equation*}
I(\psi)=\sum_{e=1}^{N} I_{e}\left(\psi_{e}\right) \tag{13.41}
\end{equation*}
$$

or

$$
\begin{equation*}
I(\psi)=\frac{1}{2}[\boldsymbol{\Psi}]^{t}[\mathbf{C}][\boldsymbol{\Psi}]-\frac{k^{2}}{2}[\boldsymbol{\Psi}]^{t}[\mathbf{T}][\mathbf{\Psi}] \tag{13.42}
\end{equation*}
$$

where

$$
[\Psi]=\left[\begin{array}{c}
\psi_{1}  \tag{13.43}\\
\psi_{2} \\
\vdots \\
\psi_{M}
\end{array}\right]
$$

Boundary conditions tells us that tangential component of electric fields on mirror vanishes; $\psi_{1}=E_{x}(0)=0$ and $\psi_{M}=E_{x}(d)=0$ The $[\mathbf{C}]$ and $[\mathbf{T}]$ are global matrices and consisting of local matrices $\left[C_{e}\right]$ and $\left[T_{e}\right]$, respectively. We reform these equation as free and prescribe node fields, the same procedure as before. Therefore

We take derivative and setting it to be equal to zero.

$$
\begin{equation*}
\frac{\partial I(\psi)}{\partial \Psi_{f}}=0 \tag{13.45}
\end{equation*}
$$

which gives us

$$
\left[\begin{array}{ll}
\mathbf{C}_{f f} & \mathbf{C}_{f p}
\end{array}\right]\left[\begin{array}{l}
\mathbf{\Psi}_{f}  \tag{13.46}\\
\mathbf{\Psi}_{p}
\end{array}\right]-k^{2}\left[\begin{array}{ll}
\mathbf{T}_{f f} & \mathbf{T}_{f p}
\end{array}\right]\left[\begin{array}{l}
\mathbf{\Psi}_{f} \\
\mathbf{\Psi}_{p}
\end{array}\right]=0
$$

by applying boundary conditions, $\boldsymbol{\Psi}_{p}=0$, we will have

$$
\begin{equation*}
\left[\mathbf{C}_{f f}\right]\left[\mathbf{\Psi}_{f}\right]=k^{2}\left[\mathbf{T}_{f f}\right]\left[\mathbf{\Psi}_{f}\right] \tag{13.47}
\end{equation*}
$$

or

$$
\begin{equation*}
\left[\mathbf{T}_{f f}^{-1} \cdot \mathbf{C}_{f f}\right]\left[\mathbf{\Psi}_{f}\right]=k^{2}\left[\mathbf{\Psi}_{f}\right] \tag{13.48}
\end{equation*}
$$

which looks like equation of $\mathbf{A} \cdot \mathbf{X}=\lambda \mathbf{X}$, general eigenvalue problem. eigenvalues of this equation, $\lambda$, give us resonant frequency of 1D cavity resonator, and eigenvectors, $\mathbf{X}$, of it can be used to show the modes of cavity.


Figure 13.6: Modes of 1D Cavity Resonator by FEM, $d=6 \mathrm{Cm}$

## \& Problems

- 1-If you were using second order linear elements $V_{e}(x)=a+b x+c x^{2}$, what should be changed in the above formula and what would be the results.
- 2- In 1D cavity resonator $d=5 \mathrm{Cm}$ and filled equally by three different dielectric $\epsilon_{r 1}=3, \epsilon_{r 2}=7, \epsilon_{r 3}=1.5$. Find the dominant resonant frequency and mode of the cavity. Confirm your results by exact values.


### 13.3 Two Dimensional FEM

Two dimensional finite element analysis can be performed with either quadrilateral or triangular elements. In electromagnetics, triangular elements are typically used as triangles can be easily approximate objects with arbitrarily shapes. A triangular element or cell in a finite element mesh consists of nodes or points that define the vertices of each triangle. It also consists of edges that make up sides of each element. Mesh generation plays an important


Figure 13.7: Element $e$ consists of nodes $i, j$ and $k$ oriented in a counter clockwise order. Element $e$ also consists of edges $s i, s j$ and $s k$ which are opposite nodes $i, j$ and $k$, respectively.
role in finite element method. There are many commercial and free mesh generation packages available. For our educational problems, we should do the job ourself. Let us start with an electrostatic problem.

2D Laplace Problem: Fined the capacitance of a air filled coaxial cable with inner radius of $R_{1}=1 \mathrm{Cm}$ and outer radius of $R_{2}=2 \mathrm{Cm}$. Draw distribution of potential and Compare your results with exact values. Inner
potential $1 V$ and outer potential grounded.

- 1- MeshGeneration or Discretization is the first step. The cross section of the coaxial cable is divided into small triangles in $x, y$ plane, so each triangle is called an element. In Fig.(9.1), the local numbering and global numbering are shown for element number 3. The numbers inside the triangle vertex is local numbering and outside of it are global. Mesh generation of coaxial cable is also shown in Fig.(13.8).


Figure 13.8: 2D-Mesh Generation of a Coaxial Cable

- 2- Base Function Suppose we have an element and every node has a potential. How the potential can be found inside this triangle approximately? The same as one dimensional, interpolation is a choice.

$$
\begin{equation*}
V_{e}(x, y)=a+b x+c y \tag{13.49}
\end{equation*}
$$

Let potential at local nodes $\left(x_{i}, y_{i}\right) ; i=1,2,3$ of element number $e$ be $V_{i}^{e}$. therefore

$$
\begin{align*}
V_{1}^{e}\left(x_{1}, y_{1}\right) & =a+b x_{1}+c y_{1} \\
V_{2}^{e}\left(x_{2}, y_{2}\right) & =a+b x_{2}+c y_{2} \\
V_{3}^{e}\left(x_{3}, y_{3}\right) & =a+b x_{3}+c y_{3} \tag{13.50}
\end{align*}
$$

or

$$
\left[\begin{array}{c}
V_{1}^{e}  \tag{13.51}\\
V_{2}^{e} \\
V_{2}^{e}
\end{array}\right]=\left[\begin{array}{lll}
1 & x_{1} & y_{1} \\
1 & x_{2} & y_{2} \\
1 & x_{3} & y_{3}
\end{array}\right]\left[\begin{array}{l}
a \\
b \\
c
\end{array}\right]
$$

These equations can be rewrite them as in 1D case:

$$
\begin{gather*}
V_{e}(x, y)=\sum_{i=1}^{3} \alpha_{i}(x, y) V_{i}^{e}  \tag{13.52}\\
\alpha_{1}(x, y)=\frac{1}{2 A}\left[\left(x_{2} y_{3}-x_{3} y_{2}\right)+\left(y_{2}-y_{3}\right) x+\left(x_{3}-x_{2}\right) y\right] \\
\alpha_{2}(x, y)=\frac{1}{2 A}\left[\left(x_{3} y_{1}-x_{1} y_{3}\right)+\left(y_{3}-y_{1}\right) x+\left(x_{1}-x_{3}\right) y\right] \\
\alpha_{3}(x, y)=\frac{1}{2 A}\left[\left(x_{1} y_{2}-x_{2} y_{1}\right)+\left(y_{1}-y_{2}\right) x+\left(x_{2}-x_{1}\right) y\right](1
\end{gather*}
$$

where $A$ is the area of element $e$,

$$
\begin{equation*}
A=\frac{1}{2}\left[\left(x_{2}-x_{1}\right)\left(y_{3}-y_{1}\right)-\left(x_{3}-x_{1}\right)\left(y_{2}-y_{1}\right)\right] \tag{13.54}
\end{equation*}
$$

and $\alpha_{i}(x, y) ; i=1,2,3$ are called the element shape functions. Now we should find, energy of each element, $\mathbf{E}=-\nabla V=-\left(b \mathbf{a}_{x}+c \mathbf{a}_{y}\right)$.

$$
\begin{equation*}
W_{e}=\frac{1}{2} \int \epsilon(x, y)|\mathbf{E}|^{2} d s=\frac{1}{2} \int \epsilon(x, y)|\nabla V|^{2} d s \tag{13.55}
\end{equation*}
$$

Similar to 1D case, we can rewrite the Eq.(13.55) as

$$
\begin{equation*}
W_{e}=\frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} V_{i}^{e}\left[\int \epsilon(x, y) \nabla \alpha_{i}(x, y) \cdot \nabla \alpha_{j}(x, y) d s\right] V_{j}^{e} \tag{13.56}
\end{equation*}
$$

Defining a $3 \times 3$ matrix as

$$
\begin{gather*}
C_{i j}^{e}=\int \epsilon(x, y) \nabla \alpha_{i}(x, y) \cdot \nabla \alpha_{j}(x, y) d s  \tag{13.57}\\
W_{e}=\frac{1}{2}\left[V_{e}\right]^{t}\left[C_{e}\right]\left[V_{e}\right] \tag{13.58}
\end{gather*}
$$

which is similar to $W=\frac{1}{2} C V^{2}$ storing energy in a capacitor. Vector $\left[V_{e}\right]$ and matrix $\left[C_{e}\right]$ will be:

$$
\begin{gather*}
{\left[V_{e}\right]=\left[\begin{array}{c}
V_{1}^{e} \\
V_{2}^{e} \\
V_{3}^{e}
\end{array}\right]}  \tag{13.59}\\
{\left[C_{e}\right]=\left[\begin{array}{ccc}
C_{11}^{e} & C_{12}^{e} & C_{13}^{e} \\
C_{21}^{e} & C_{22}^{e} & C_{23}^{e} \\
C_{31}^{e} & C_{32}^{e} & C_{33}^{e}
\end{array}\right]} \tag{13.60}
\end{gather*}
$$

Now let us assembly the energy of all elements, to find total energy of system or in another word, to find a function of function.

$$
\begin{equation*}
W=\sum_{e=1}^{N} W_{e}=\frac{1}{2}[V]^{t}[C][V] \tag{13.61}
\end{equation*}
$$

where

$$
[\mathbf{V}]=\left[\begin{array}{c}
V_{1}  \tag{13.62}\\
V_{2} \\
\vdots \\
V_{M}
\end{array}\right]
$$

Some values of vector $[V]$ is given at boundaries which we called them prescribe potentials, and some unknowns, free potential, which we should find them. The $M$ is the number of nodes, $N$ is the number of elements and $[\mathbf{C}]$ is called Global Stiffness Matrix. We can rewrite the total energy of the system as:

$$
W=\frac{1}{2}\left[\begin{array}{ll}
\mathbf{V}_{f} & \mathbf{V}_{p}
\end{array}\right]^{t}\left[\begin{array}{ll}
\mathbf{C}_{f f} & \mathbf{C}_{f p}  \tag{13.63}\\
\mathbf{C}_{p f} & \mathbf{C}_{p p}
\end{array}\right]\left[\begin{array}{l}
\mathbf{V}_{f} \\
\mathbf{V}_{p}
\end{array}\right]
$$

The matrix $[\mathbf{C}]$ divided into four block capacitor matrix. If we differentiate energy with respect to free potentials, we will obtain

$$
\left[\begin{array}{ll}
\mathbf{C}_{f f} & \mathbf{C}_{f p}
\end{array}\right]\left[\begin{array}{l}
\mathbf{V}_{f}  \tag{13.64}\\
\mathbf{V}_{p}
\end{array}\right]=0
$$

or simply

$$
\begin{equation*}
\left[\mathbf{C}_{f f}\right]\left[\mathbf{V}_{f}\right]=-\left[\mathbf{C}_{f p}\right]\left[\mathbf{V}_{p}\right] \tag{13.65}
\end{equation*}
$$

which looks like $\mathbf{A} \cdot \mathbf{X}=\mathbf{B}$ and solution of it is straight forward. As we know from numerical analysis, there are two methods for solution of $\mathbf{A} \cdot \mathbf{X}=\mathbf{B}$ equation: Direct and Indirect. Gauss Elimination method is a direct method but Iteration Method, is called as indirect method. Let us find a way for indirect method. In order to find a way to find the unknown potentials, let us return to Eq.(13.61). If we differentiate energy with respect to unknown potential nodes and put them equal to zero, we will have $F$ equations, ( $F=$ number of free nodes). One of the unknown potential or free node, say $V_{k}$, will be

$$
\begin{equation*}
V_{k}=-\frac{1}{C_{k k}}\left[\sum_{i=1, i \neq k}^{K} V_{i} C_{k i}\right] \tag{13.66}
\end{equation*}
$$

This means that the nodes which is connected directly to node $k$, contributes to potential $V_{k}$. We guess some value for each free potentials, then by iteration we reach the exact value.

By those method we can find the potentials inside the coaxial problem and as long as the triangle be small, we will get more accurate results. The capacitance can be calculated by $W=\frac{1}{2} C V_{d}^{2}=\frac{1}{2}[V]^{t}[C][V]$. where, $V_{d}$ is potential difference between inner and outer of coaxial cable. Sice we can calculate $W$, thus $C=2 * W / V_{d}^{2}$. The capacitance of unit length for a coaxial capacitor can be determined exactly by:

$$
C=\frac{2 \pi \epsilon_{0}}{\ln \left(R_{2} / R_{1}\right)}
$$

### 13.4 Waveguide Modes

The scalar wave equation for a homogeneous isotropic medium is chosen. The equation is written as:

$$
\begin{equation*}
\nabla^{2} \psi+k^{2} \psi=0 \tag{13.67}
\end{equation*}
$$

where $k^{2}$ is eigenvalue. The FEM solves the Eq.(13.67) by minimization of a corresponding functional given by:

$$
\begin{equation*}
I(\psi)=\frac{1}{2} \int_{s}\left[\left(\frac{\partial \psi}{\partial x}\right)^{2}+\left(\frac{\partial \psi}{\partial y}\right)^{2}-k^{2} \psi^{2}\right] d s \tag{13.68}
\end{equation*}
$$

where $S$ represent the area cross section area of the waveguide. $k^{2}=\epsilon_{r} k_{0}^{2}-k_{z}^{2}$, and $k_{0}=\frac{2 \pi}{\lambda_{0}}$,free space wave number, and $k_{z}$ wave number of waveguide. When $k_{z}=0$, we can find the cutoff frequency of waveguide. $\epsilon_{r}$ is relative permittivity of waveguide. The preceding functional automatically satisfies the wave equation and Neumann boundary condition. Like the previous example, the cross section area of waveguide should be divided into small triangles. Hence we can write:

$$
\begin{equation*}
I(\psi)=\sum_{e=1}^{N} \frac{1}{2} \int_{s}\left[\left(\frac{\partial \psi_{e}}{\partial x}\right)^{2}+\left(\frac{\partial \psi_{e}}{\partial y}\right)^{2}-k^{2} \psi_{e}^{2}\right] d S \tag{13.69}
\end{equation*}
$$

where

$$
\begin{equation*}
\psi_{e}(x, y)=a+b x+c y \tag{13.70}
\end{equation*}
$$

for simplicity we considered linear terms. Using the linear approximation in Eq.(13.70), we can write

$$
\begin{equation*}
\psi_{1}^{e}=a+b x_{1}+c y_{1}, \quad \psi_{2}^{e}=a+b x_{2}+c y_{2}, \quad \psi_{3}^{e}=a+b x_{3}+c y_{3} \tag{13.71}
\end{equation*}
$$

or like previous work, we can write

$$
\begin{equation*}
\psi_{e}(x, y)=\sum_{i=1}^{3} \alpha_{i}(x, y) \psi_{i}^{e} \tag{13.72}
\end{equation*}
$$

and $\alpha_{i}(x, y)$ is the same as before; Eq.(13.53). Rewriting $I(\psi)$ in matrix form as we have done it in 2D case;

$$
\begin{equation*}
I_{e}\left(\psi_{e}\right)=\frac{1}{2}\left[\psi_{e}\right]^{t}\left[C_{e}\right]\left[\psi_{e}\right]-\frac{k^{2}}{2}\left[\psi_{e}\right]^{t}\left[T_{e}\right]\left[\psi_{e}\right] \tag{13.73}
\end{equation*}
$$

where $\left[\psi_{e}\right]=\left[\begin{array}{lll}\psi_{1}^{e} & \psi_{2}^{e} & \psi_{3}^{e}\end{array}\right]^{t}$

$$
\begin{equation*}
C_{i j}^{e}=\int_{s} \nabla \alpha_{i} \cdot \nabla \alpha_{j} d S \tag{13.75}
\end{equation*}
$$

and $T_{e}$

$$
\begin{equation*}
T_{i j}^{e}=\int_{s} \alpha_{i} \alpha_{j} d S \tag{13.76}
\end{equation*}
$$

The global functional will be

$$
\begin{equation*}
I(\psi)=\sum_{e=1}^{N} I_{e}\left(\psi_{e}\right) \tag{13.77}
\end{equation*}
$$

or

$$
\begin{equation*}
I(\psi)=\frac{1}{2}[\mathbf{\Psi}]^{t}[\mathbf{C}][\mathbf{\Psi}]-\frac{k^{2}}{2}[\mathbf{\Psi}]^{t}[\mathbf{T}][\mathbf{\Psi}] \tag{13.78}
\end{equation*}
$$

where

$$
[\boldsymbol{\Psi}]=\left[\begin{array}{c}
\psi_{1}  \tag{13.79}\\
\psi_{2} \\
\vdots \\
\psi_{M}
\end{array}\right]
$$

For $T M^{z}$ case, Eq.(13.78) can be rewritten as:
$I(\psi)=\frac{1}{2}\left[\begin{array}{ll}\boldsymbol{\Psi}_{f} & \boldsymbol{\Psi}_{p}\end{array}\right]^{t}\left[\begin{array}{l}\mathbf{C}_{f f} \mathbf{C}_{f p} \\ \mathbf{C}_{p f} \\ \mathbf{C}_{p p}\end{array}\right]\left[\begin{array}{c}\boldsymbol{\Psi}_{f} \\ \boldsymbol{\Psi}_{p}\end{array}\right]-\frac{k^{2}}{2}\left[\boldsymbol{\Psi}_{f} \boldsymbol{\Psi}_{p}\right]^{t}\left[\begin{array}{l}\mathbf{T}_{f f} \mathbf{T}_{f p} \\ \mathbf{T}_{p f} \\ \mathbf{T}_{p p}\end{array}\right]\left[\begin{array}{c}\boldsymbol{\Psi}_{f} \\ \boldsymbol{\Psi}_{p}\end{array}\right]$
We take derivative and setting it to be equal to zero.

$$
\left[\begin{array}{ll}
\mathbf{C}_{f f} & \mathbf{C}_{f p}
\end{array}\right]\left[\begin{array}{l}
\mathbf{\Psi}_{f}  \tag{13.81}\\
\boldsymbol{\Psi}_{p}
\end{array}\right]-k^{2}\left[\begin{array}{ll}
\mathbf{T}_{f f} & \mathbf{T}_{f p}
\end{array}\right]\left[\begin{array}{l}
\mathbf{\Psi}_{f} \\
\mathbf{\Psi}_{p}
\end{array}\right]=0
$$

by applying boundary conditions, $\Psi_{p}=0$, we will have

$$
\begin{equation*}
\left[\mathbf{C}_{f f}\right]\left[\mathbf{\Psi}_{f}\right]=k^{2}\left[\mathbf{T}_{f f}\right]\left[\mathbf{\Psi}_{f}\right] \tag{13.82}
\end{equation*}
$$

For $T E^{z}$ modes, the $\Psi$ represent the axial magnetic field, $H_{z}$, and the boundary condition is the Neumann condition, $\partial \psi / \partial n=0$, where $n$ is the normal to perfectly conducting boundary. In FEM, this is a natural boundary condition and need not be imposed.

$$
\begin{equation*}
[\mathbf{C}][\mathbf{\Psi}]=k^{2}[\mathbf{T}][\mathbf{\Psi}] \tag{13.83}
\end{equation*}
$$

The $\Psi$ may be $E_{z}$ for $T M^{z}$ modes or $H_{z}$ for $T E^{z}$ modes. Suppose we have calculated the $\psi_{i}^{e}$ for each element, which may be for $T E^{z}$ or $T M^{z}$ modes.

How can we find other components $E_{x}, E_{y}, H_{x}, H_{y}$ ? From waveguide section, for $T E^{z}$ we have:

$$
\begin{array}{ll}
E_{x}=\frac{\partial \psi}{\partial y}, & E_{y}=-\frac{\partial \psi}{\partial x} \\
H_{x}=\frac{-1}{j \omega \mu} \frac{\partial^{2} \psi}{\partial x \partial z}, & H_{y}=\frac{-1}{j \omega \mu} \frac{\partial^{2} \psi}{\partial y \partial z}  \tag{13.84}\\
H_{z}=\frac{k_{z}^{2}-\epsilon_{r} k_{0}^{2}}{j \omega \mu} \psi &
\end{array}
$$

and for $T M^{z}$ :

$$
\begin{array}{lr}
H_{x}=\frac{\partial \psi}{\partial y}, & H_{y}=-\frac{\partial \psi}{\partial x} \\
E_{x}=\frac{1}{j \omega \epsilon} \frac{\partial^{2} \psi}{\partial x \partial z}, & E_{y}=\frac{1}{j \omega \epsilon} \frac{\partial^{2} \psi}{\partial y \partial z}  \tag{13.85}\\
E_{z}=\frac{k_{z}^{2}-\epsilon_{r} k_{0}^{2}}{j \omega \epsilon} \psi &
\end{array}
$$

hence; $\frac{\partial \psi_{e}(x, y)}{\partial x}, \frac{\partial \psi_{e}(x, y)}{\partial y}, \frac{\partial^{2} \psi}{\partial x \partial z}$ and $\frac{\partial^{2} \psi}{\partial y \partial z}$. should be calculated.

$$
\begin{align*}
& \frac{\partial \psi_{e}(x, y)}{\partial x}=\frac{1}{2 A}\left[\left(y_{2}-y_{3}\right) \psi_{1}^{e}+\left(y_{3}-y_{1}\right) \psi_{2}^{e}+\left(y_{1}-y_{2}\right) \psi_{3}^{e}\right]  \tag{13.86}\\
& \frac{\partial \psi_{e}(x, y)}{\partial y}=\frac{1}{2 A}\left[\left(x_{3}-x_{2}\right) \psi_{1}^{e}+\left(x_{1}-x_{3}\right) \psi_{2}^{e}+\left(x_{2}-x_{1}\right) \psi_{3}^{e}\right] \tag{13.87}
\end{align*}
$$

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## Appendix A

## Fourier Transform

## A. 1 Introduction

The Fourier and Laplace transform are widely used in solving problems in science and engineering. The Fourier transform is used in linear system analysis, antenna design, optical random process modeling, probability theory, quantum physics, and boundary value problems.

In this section will review the Fourier transform formula. We have four types of Fourier Transform.

| Time Domain | Frequency Domain |
| :---: | :---: |
| Continuous \& Nonperiodic | Continuous \& Nonperiodic |
| $x(t)$ | $X(f)$ |

$$
\begin{equation*}
x(t)=\int_{-\infty}^{\infty} X(f) e^{j 2 \pi f t} d f \Longleftrightarrow X(f)=\int_{-\infty}^{\infty} x(t) e^{-j 2 \pi f t} d t \tag{A.1}
\end{equation*}
$$

| Time Domain | Frequency Domain |
| :---: | :---: |
| Continuous \& Periodic | Discrete \& Nonperiodic |
| $x(t)=x\left(t+T_{p}\right)$ | $X\left(m f_{0}\right), \quad f_{0}=\frac{1}{T_{p}}$ |

$$
\begin{equation*}
x(t)=\sum_{m=-\infty}^{\infty} X\left(m f_{0}\right) e^{j 2 \pi m f_{0} t} \Longleftrightarrow X\left(m f_{0}\right)=\frac{1}{T_{p}} \int_{0}^{T_{p}} x(t) e^{-j 2 \pi m f_{0} t} d t \tag{A.2}
\end{equation*}
$$

| Time Domain | Frequency Domain |
| :---: | :---: |
| Discrete \& Nonperiodic | Continuous \& Periodic |
| $x(n T), \quad T=\frac{1}{f_{p}}$ | $X(f)=X\left(f+f_{p}\right)$ |

where $T$ is sampling rate and related to maximum frequency of the signal $f_{\text {max }}$ as $T \leq \frac{1}{2 f_{\max }}$.

$$
\begin{equation*}
x(n T)=\frac{1}{f_{p}} \int_{0}^{f_{p}} X(f) e^{j 2 \pi f n T} d f \Longleftrightarrow X(f)=\sum_{n=-\infty}^{\infty} x(n T) e^{-j 2 \pi f n T} \tag{A.3}
\end{equation*}
$$

| Time Domain | Frequency Domain |
| :---: | :---: |
| Discrete \& Periodic | Discrete \& Periodic |
| $x(n T)=x(n T+N T), T=\frac{1}{N f_{0}}$ | $X\left(m f_{0}\right)=X\left(m f_{0}+N f_{0}\right), f_{0}=\frac{1}{N T}$ |

$$
\begin{equation*}
x(n T)=\sum_{m=0}^{N-1} X\left(m f_{0}\right) e^{\frac{j 2 \pi}{N} m n} \Longleftrightarrow X\left(m f_{0}\right)=\frac{1}{N} \sum_{n=0}^{N-1} x(n T) e^{\frac{-j 2 \pi}{N} m n} \tag{A.4}
\end{equation*}
$$

The last one which is called Discrete Fourier Transform (DFT), can be represented in matrix form. Since a digital computer works only with discrete data, numerical computation of the Fourier transform of $x(t)$ requires discrete sample values of $x(t)$, which we will call it $x(n T)$. In addition, a computer can compute $X(f)$ only at discrete values of $f$, that is, it can provide discrete samples of the transform, $X\left(m f_{0}\right)$
$\left[\begin{array}{c}X(0) \\ X\left(f_{0}\right) \\ \vdots \\ X\left[(N-1) f_{0}\right]\end{array}\right]=\frac{1}{N}\left[\begin{array}{cccc}1 & 1 & \cdots & 1 \\ 1 & W & \cdots & W^{(N-1)} \\ \vdots & \cdots & \cdots & \cdots \\ 1 & W^{(N-1)} & \cdots & W^{(N-1)^{2}}\end{array}\right]\left[\begin{array}{c}x(0) \\ x(T) \\ \vdots \\ x[(N-1) T]\end{array}\right]$
where $W=e^{-\frac{j 2 \pi}{N}}$, and $T$ is the sampling time interval $T \leq \frac{1}{2 f_{\max }}$ and $f_{\max }$ is the maximum frequency that is needed for $x(t)$ processing. If we need more resolution in spectrum of $x(t)$ i.e. $f_{0}=\frac{1}{N T}$ we should increase the period $T_{p}=N T$, or in another word increasing dimension of matrix $N \times N$. Fast Fourier Transform or (FFT) can be used for the computation of $X(0), X\left(f_{0}\right), \cdots X\left[(N-1) f_{0}\right]$. The FFT algorithm developed by Tukey and Cooley in 1965, [25] which reduces the number of computation from something on the order of $N^{2}$ to $N \log N$. The algorithm is simplified if N is chosen to be a power of 2 , but it is not a requirement.

## Appendix B

## Matrix

## B. 1 Basic Definitions

The system of simultaneous linear algebraic equations involving real or complex numbers:

$$
\begin{gather*}
y_{1}=a_{11} x_{1}+a_{12} x_{2}+\cdots+a_{1 N} x_{N} \\
y_{2}=a_{21} x_{1}+a_{22} x_{2}+\cdots+a_{2 N} x_{N}  \tag{B.1}\\
\cdot \\
\cdot \\
y_{M}=a_{M 1} x_{1}+a_{M 2} x_{2}+\cdot+a_{M N} x_{N}
\end{gather*}
$$

The system in summation notation:

$$
\begin{equation*}
y_{m}=\sum_{n=1}^{N} a_{m n} x_{n} \quad 1 \leq m \leq M \tag{B.2}
\end{equation*}
$$

As we will see, this equation can be expressed as a matrix times a column vector.

$$
\mathbf{A}=\left[a_{m n}\right]=\left[\begin{array}{cccc}
a_{11} & a_{12} & \ldots & a_{1 N}  \tag{B.3}\\
a_{21} & a_{22} & \ldots & a_{2 N} \\
\cdot & \cdot & \ldots & \cdot \\
\cdot & \cdot & \ldots & \cdot \\
\cdot & \cdot & \ldots & \cdot \\
a_{M 1} & a_{M 2} & \ldots & a_{M N}
\end{array}\right]
$$

Column vector: a single columned matrix:

$$
\mathbf{C}=\left[\begin{array}{l}
c_{1}  \tag{B.4}\\
c_{2} \\
\cdot \\
\cdot \\
\cdot \\
c_{N}
\end{array}\right]
$$

Row vector: a single rowed matrix:

$$
\mathbf{r}=\left[\begin{array}{llll}
r_{1} & r_{2} & \ldots & r_{N} \tag{B.5}
\end{array}\right]
$$

Equality: $\mathrm{A}=\mathrm{B}$ means that all of the elements in the two matrices are equal, $a_{m n}=b_{m n}$
Special Matrices

- The Null or Zero matrix: 0 all elements are zero
- The Kronecker delta: $\delta_{m n}= \begin{cases}1 & m=n \\ 0 & m \neq n\end{cases}$
- Diagonal matrices:

$$
\mathbf{D}=\left[\begin{array}{cccc}
d_{11} & 0 & \ldots & 0  \tag{B.6}\\
0 & d_{22} & \ldots & 0 \\
. & & . & \ldots \\
0 & 0 & \ldots & d_{M N}
\end{array}\right]
$$

- The Identity matrix: $I_{m n}=\delta_{m n}$

$$
\mathbf{I}=\left[\begin{array}{cccc}
1 & 0 & \ldots & 0  \tag{B.7}\\
0 & 1 & \ldots & 0 \\
. & & . & \ldots \\
0 & 0 & \ldots & 1
\end{array}\right]
$$

Matrix Operations
Unary: Operations on a single matrix

- Scalar multiplication: $\mathbf{A}=\lambda \mathbf{B}, a_{m n}=\lambda b_{m n}$
- Transpose: $\mathbf{A}^{T} . \quad\left[a_{m n}\right]^{T}=a_{n m}$ Note that

$$
\mathbf{C}^{T}=\left[\begin{array}{l}
c_{1}  \tag{B.8}\\
c_{2} \\
\cdot \\
\cdot \\
\cdot \\
c_{N}
\end{array}\right]^{T}=\left[\begin{array}{llll}
c_{1} & c_{2} & \ldots & c_{N}
\end{array}\right]
$$

- Complex conjugate: $\mathbf{A}^{*} . \quad\left[a_{m n}\right]^{*}=a_{m n}^{*}$
- Hermitian conjugate or adjoint: $\mathbf{A}^{\dagger} . \quad\left[a_{m n}\right]^{\dagger}=a_{n m}^{*}$
- Inverse: $\mathbf{A}^{-1} . \quad \mathbf{A}^{-1} \mathbf{A}=\mathbf{A A}^{-1}=\mathbf{I}$
- Trace: $\operatorname{Tr} \mathbf{A}=\sum_{m=1}^{m=M} a_{m m}$. Square matrices only.

Binary: Operations that combine matrices

- Addition: $\mathbf{C}=\mathbf{A}+\mathbf{B} \cdot c_{m n}=a_{m n}+b_{m n}$
- Subtraction: $\mathbf{C}=\mathbf{A}-\mathbf{B} \cdot c_{m n}=a_{m n}-b_{m n}$
- Array multiplication: $\mathbf{C}=\mathbf{A} \circ \mathbf{B} \cdot c_{m n}=a_{m n} b_{m n}$ Not standard, but useful.
- Matrix multiplication $\mathbf{C}=\mathbf{A B} . \quad c_{m n}=\sum_{k=1}^{k=K} a_{m k} b_{k n} \quad 1 \leq m \leq M$ $1 \leq n \leq N, \mathbf{A}$ is M by K and B is K by N .

$$
\left[\begin{array}{cccc}
c_{11} & c_{12} & \ldots & c_{1 N}  \tag{B.9}\\
c_{21} & c_{22} & \ldots & c_{2 N} \\
\cdot & \cdot & \ldots & \cdot \\
\cdot & \cdot & \ldots & \cdot \\
\cdot & \cdot & \ldots & \cdot \\
c_{M 1} & c_{M 2} & \ldots & \cdot \\
c_{M N}
\end{array}\right]=\left[\begin{array}{cccc}
a_{11} & a_{12} & \ldots & a_{1 K} \\
a_{21} & a_{22} & \ldots & a_{2 K} \\
\cdot & \cdot & \ldots & \cdot \\
\cdot & \cdot & \ldots & \cdot \\
\cdot & \cdot & \ldots & \cdot \\
\cdot & \cdot & \ldots & \cdot \\
a_{M 1} & a_{M 2} & \ldots & a_{M K}
\end{array}\right]\left[\begin{array}{cccc}
b_{11} & b_{12} & \ldots & b_{1 N} \\
a_{21} & a_{22} & \ldots & b_{2 N} \\
\cdot & \cdot & \ldots & \cdot \\
\cdot & \cdot & \ldots & \cdot \\
\cdot & \cdot & \ldots & \cdot \\
b_{K 1} & b_{K 2} & \ldots & b_{K N}
\end{array}\right]
$$

- A matrix left (pre)-multiplying a column vector: $\mathbf{Y}=\mathbf{A X}, \quad y_{m}=$

$$
\begin{array}{rl}
\sum_{n=1}^{N} a_{m n} x_{n} & 1 \leq m \leq M \\
& {\left[\begin{array}{l}
y_{1} \\
y_{2} \\
\cdot \\
\cdot \\
\cdot \\
y_{M}
\end{array}\right]=\left[\begin{array}{cccc}
a_{11} & a_{12} & \ldots & a_{1 N} \\
a_{21} & a_{22} & \ldots & a_{2 N} \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \ldots & \cdot \\
a_{M 1} & a_{M 2} & \ldots & a_{M N}
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
\cdot \\
\cdot \\
\cdot \\
x_{N}
\end{array}\right]} \tag{B.10}
\end{array}
$$

- A matrix right (post)-multiplying a row vector: $\mathbf{Y}=\mathbf{X A}, \quad y_{n}=$ $\sum_{m=1}^{M} x_{m} a_{m n} \quad 1 \leq n \leq N$

$$
\left[\begin{array}{llll}
y_{1} & y_{2} & \ldots & y_{M}
\end{array}\right]=\left[\begin{array}{llll}
x_{1} & x_{2} & \ldots & x_{N}
\end{array}\right]\left[\begin{array}{llll}
a_{11} & a_{12} & \ldots & a_{1 N}  \tag{B}\\
a_{21} & a_{22} & \ldots & a_{2 N} \\
\cdot & \cdot & \ldots & \cdot \\
\cdot & \cdot & \ldots & \cdot \\
\cdot & \cdot & \ldots & \cdot \\
a_{M 1} & a_{M 2} & \ldots & a_{M N}
\end{array}\right]
$$

- Important properties of matrix multiplication
- Matrix multiplication is not commutative
- The $m n^{\text {th }}$ element of the result is the sum of products between elements from the $m^{t h}$ row of $\mathbf{A}$ and the $n^{t h}$ column of $\mathbf{B}$.
- For pre-multiplication of $\mathbf{x}$ by $\mathbf{A}, \mathbf{x}$ can be organized into columns which are related to the columns in the result through matrix column-vector products. The column structure is preserved.

$$
\left[\binom{y_{11}}{y_{21}}\binom{y_{21}}{y_{22}}\right]=\left[\begin{array}{ll}
a_{11} & a_{12}  \tag{B.12}\\
a_{12} & a_{22}
\end{array}\right]\left[\binom{x_{11}}{x_{21}}\binom{x_{21}}{x_{22}}\right]
$$

where

$$
\binom{y_{11}}{y_{21}}=\left[\begin{array}{ll}
a_{11} & a_{12} \\
a_{12} & a_{22}
\end{array}\right]\binom{x_{11}}{x_{21}}
$$

and

$$
\binom{y_{12}}{y_{22}}=\left[\begin{array}{ll}
a_{11} & a_{12} \\
a_{12} & a_{22}
\end{array}\right]\binom{x_{12}}{x_{22}}
$$

- For post-multiplication of $\mathbf{x}$ by $\mathbf{A}, \mathbf{x}$ can be organized into rows which are related to the rows in the result through row-vector products matrix. The row structure is preserved.

$$
\left.\left[\begin{array}{l}
\left(\begin{array}{ll}
y_{11} & y_{12}
\end{array}\right)  \tag{B.13}\\
\left(y_{21}\right.
\end{array} y_{22}\right)\right]=\left[\begin{array}{ll}
\left(x_{11}\right. & x_{12}
\end{array}\right)\left[\begin{array}{ll}
a_{11} & a_{12} \\
x_{21} & x_{22}
\end{array}\right]
$$

where

$$
\left(\begin{array}{ll}
y_{11} & y_{12}
\end{array}\right)=\left(\begin{array}{ll}
x_{11} & x_{12}
\end{array}\right)\left[\begin{array}{ll}
a_{11} & a_{12} \\
a_{12} & a_{22}
\end{array}\right]
$$

and

$$
\left(\begin{array}{ll}
y_{21} & y_{22}
\end{array}\right)=\left(\begin{array}{ll}
x_{21} & x_{22}
\end{array}\right)\left[\begin{array}{ll}
a_{11} & a_{12} \\
a_{12} & a_{22}
\end{array}\right]
$$

More Special Matrices

- Real: $\quad \mathbf{A}^{*}=\mathbf{A}$
- Symmetric: $\quad \mathbf{A}^{T}=\mathbf{A}$
- Skew-symmetric or anti-symmetric: $\quad \mathbf{A}^{T}=-\mathbf{A}$
- Hermitian: $\mathbf{A}^{\dagger}=\mathbf{A}$
- Orthogonal: $\quad \mathbf{A}^{-1}=\mathbf{A}^{T}$
- Unitary: $\quad \mathbf{A}^{-1}=\mathbf{A}^{\dagger}$
- Idempotent: $\mathbf{A}^{2}=\mathbf{A}$
- Nilpotent: $\quad \mathbf{A}^{m}=\mathbf{0}$ for some integer m .

Example: Left-multiplication of a 2 -element column vector by a 2 x 2 matrix
Let $\mathbf{A}=\left[\begin{array}{ll}a_{11} & a_{12} \\ a_{12} & a_{22}\end{array}\right], \mathbf{b}=\left[\begin{array}{l}b_{1} \\ b_{2}\end{array}\right]$ and $\mathbf{c}=\left[\begin{array}{l}c_{1} \\ c_{2}\end{array}\right]$
We have, for left multiplication, $\mathbf{c}=\mathbf{A b}$ and that $\mathbf{c}=\left[\begin{array}{l}c_{1} \\ c_{2}\end{array}\right]=\left[\begin{array}{l}a_{11} b_{1}+a_{12} b_{2} \\ a_{12} b_{1}+a_{22} b_{2}\end{array}\right]$.
$\underline{\text { Inverse of a square non-singular matrix: } A^{-1} A=A^{-1}=I}$

- 1. The following is a formula for calculating the inverse using the cofactors of the matrix. This method has fallen out of favor in recent years and has been replaced by the Gauss-Jordan method described later in this section. For actually computing, the Gaussian elimination method of solution, described later in this lecture, is preferred for its efficiency.

$$
\begin{equation*}
\left(\mathbf{A}^{-1}\right)_{m n}=\frac{\operatorname{cofactor}\left(a_{n m}\right)}{\operatorname{det}(\mathbf{A})}=\frac{\text { cofactor }\left[\left(\mathbf{A}^{T}\right)_{m n}\right]}{\operatorname{det}(\mathbf{A})} \tag{B.14}
\end{equation*}
$$

Note: it is the cofactor transpose that is used in the inverse,

$$
\begin{aligned}
& \operatorname{cofactor}\left(a_{n m}\right)=(-1)^{m+n} \cdot\binom{\text { The determinant of a matrix with the }}{m^{t h} \text { row and then } n^{t h} \text { column removed }} \\
& |\mathbf{A}|=\operatorname{det}(\mathbf{A})=\sum_{m} a_{m n_{0}} \operatorname{cofactor}\left(a_{m n_{0}}\right) \text { for any } n_{0} \\
& =\sum_{n} a_{m_{0} n} \operatorname{cofactor}\left(a_{m_{0} n}\right) \text { for any } m_{0}
\end{aligned}
$$

Singular matrices: $\operatorname{det}(\mathbf{A})=0$

- 2. Some definitions (see for example Howard Eves Elementary Matrix Theory Dover, 1980,N.Y.)
- Partitioning a matrix. Partitioning of a matrix is just the grouping together of rows and/or columns. Vertical and horizontal lines between the rows and columns are used to denote the partitions. The elements in the various partitions can be grouped into submatrices in the matrices as is demonstrated in the appendix on block matrices

$$
\left[\begin{array}{ccc}
a_{11} & a_{12} & a_{13}  \tag{B.15}\\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right]=\left[\begin{array}{ll|l}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
\hline a_{31} & a_{32} & a_{33}
\end{array}\right]
$$

Augmenting a matrix: Augmenting a matrix is adding rows or columns.

- Elementary Row Operations
* $o(\mathbf{m}, \mathbf{n})$ : interchange rows m and n .
* $o(c \mathbf{n}$ :multiply row n by scalar c .
* $o(\mathbf{m}+c \mathbf{n})$ : multiply the row n by a constant c , then add it to the row $m . m \neq n$
- Row equivalence: Matrix $\mathbf{B}$ is row-equivalent to matrix $\mathbf{A}$ if and only if $\mathbf{B}$ is obtainable from $\mathbf{A}$ by finitely many elementary row operations.
- 3. To invert the matrix A using the Gauss-Jordan method:

$$
\left[\begin{array}{ccc}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right]
$$

First augment the matrix with the identity matrix:

$$
[\mathbf{A} \mid \mathbf{I}]=\left[\begin{array}{lll|lll}
a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\
a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\
a_{31} & a_{32} & a_{33} & 0 & 0 & 1
\end{array}\right]
$$

Then, perform elementary row operations on the augmented matrix with the goal of converting the left partition into the identity matrix. When this goal is reached, the right partition will contain the desired inverse. It helps to keep track of the operations performed. The end result will be

$$
\left[\mathbf{I} \mid \mathbf{A}^{-1}\right]=\left[\begin{array}{lll|lll}
1 & 0 & 0 & a_{11}^{-1} & a_{12}^{-1} & a_{13}^{-1} \\
0 & 1 & 0 & a_{21}^{-1} & a_{22}^{-1} & a_{23}^{-1} \\
0 & 0 & 1 & a_{31}^{-1} & a_{32}^{-1} & a_{33}^{-1}
\end{array}\right]
$$

Example: the first step in the inversion
$\left[\begin{array}{lll|lll}a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1\end{array}\right] \xrightarrow[o\left(\frac{1}{a_{11}} \mathbf{I}\right)]{ }\left[\begin{array}{ccc|ccc}1 & a_{12} / a_{11} & a_{13} / a_{11} & 1 / a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1\end{array}\right]$

- 4. Inverses of the elementary row operations
- a. $o(\mathbf{m}, \mathbf{n})$ interchange rows $m$ and $n$.
- b. $o\left(c^{-1} \mathbf{n}\right)$ multiply row n by scalar $1 / c$.
- c. $o(\mathbf{m}-c \mathbf{n})$ multiply the row $n$ by the constant $c$ then subtract the result from row m. $m \neq n$
- 5. Elementary Column operations
- a. $o^{\prime}(\mathbf{m}, \mathbf{n})$ interchange columns $m$ and $n$.
- b. $o^{\prime}(c \mathbf{n})$ multiply column $n$ by a scalar $c$.
- c. $o^{\prime}(\mathbf{m}+c \mathbf{n})$ multiply the column $n$ by the constant $c$ then add the result to column $m . m \neq n$
- 6. Column equivalence: Matrix $\mathbf{B}$ is column-equivalent to matrix $\mathbf{A}$ if and only if $\mathbf{B}$ is obtainable from $\mathbf{A}$ by finitely many elementary column operations.
- 7. Inverses of the elementary column operations
- a. $o^{\prime}(\mathbf{m}, \mathbf{n})$ interchange columns $m$ and $n$.
- b. $o^{\prime}\left(c^{-1} \mathbf{n}\right)$ multiply column $n$ by a scalar $1 / c$.
- c. $o^{\prime}(\mathbf{m}-c \mathbf{n})$ multiply the column $n$ by the constant $c$ then subtract the result from column $m . m \neq n$


## Triangular Matrices

Triangular matrices have non-zero elements only on the diagonal and either above the diagonal or below the diagonal. There may be any number of zeros on the diagonal.

1. Upper triangular These matrices have non-zero elements only on the diagonal and above the diagonal. As an example, consider

$$
\mathbf{A}=\left[\begin{array}{ccc}
a_{11} & a_{12} & a_{13} \\
0 & a_{22} & a_{23} \\
0 & 0 & a_{33}
\end{array}\right]
$$

2. Lower triangular The matrices have non-zero elements only on the diagonal and above the diagonal. As an example consider

$$
\mathbf{A}=\left[\begin{array}{ccc}
a_{11} & 0 & 0 \\
a_{21} & a_{22} & 0 \\
a_{31} & a_{32} & a_{33}
\end{array}\right]
$$

## Echelon form

The echelon form of a matrix is a matrix arrived at through the application of elementary row and column operations to the original matrix. In the resulting matrix, the first non-zero element of each row is one. It is positioned one element to the right of the first non-zero element in the row above it. (Note Kreyszig's definition does not require the first non-zero element in each row to be one.)
The echelon form distinguishes itself from the upper-triangular matrix in that the latter is only for square matrices whereas non-square matrices may be in the echelon form. Also, there is no constraint on the first non-zero element in any row.
The result is upper triangular, through there may exist a row below which all elements are zero, but above which each row has at least one non-zero element. For example, the following matrix is in echelon form,

$$
\mathbf{A}=\left[\begin{array}{ccc}
a_{11} & a_{12} & a_{13} \\
0 & 0 & a_{23} \\
0 & 0 & 0
\end{array}\right]
$$

## Gaussian elimination

Another way of solving a set of linear equations is Gaussian elimination. The original matrix is first augmented with the knowns in the equation set. Elementary row operations are then applied to the augmented matrix with the goal of converting the left partition into an upper triangular matrix. The last variable is solved for, and then its value is substituted into the previous row to get the next to last variable. The substitutions continue row by row until all of the variables are determined.
For example let our set of linear equations be represented by: $\mathbf{y}=\mathbf{A x}$ or

$$
\left[\begin{array}{l}
y_{1}  \tag{B.16}\\
y_{2}
\end{array}\right]=\left[\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]
$$

We augment A to get: $\left[\begin{array}{ll}a_{11} & a_{12} \\ a_{21} & a_{22}\end{array}\right]\left[\begin{array}{l}y_{1} \\ y_{2}\end{array}\right]$.
We then apply an elementary row operation

$$
\left[\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right]\left[\begin{array}{l}
y_{1} \\
y_{2}
\end{array}\right] \frac{o\left(2-\frac{a_{21}}{a_{11}} 1\right)}{\rightarrow}\left[\begin{array}{cl|c}
a_{11} & a_{12} & y_{1} \\
0 & a_{22}-\frac{a_{21}}{a_{11}} a_{12} & y_{2}-\frac{a_{21}}{a_{11}} y_{1}
\end{array}\right]
$$

The second row then corresponds to the equation

$$
\left(a_{22}-\frac{a_{21}}{a_{11}} a_{12}\right) x_{2}=y_{2}-\frac{a_{21}}{a_{11}} y_{1}
$$

We solve for $x_{2}$ and substitute the result into the equation represented by the first row,

$$
a_{11} x_{1}+a_{12} x_{2}=y_{1}
$$

We then solve for $x_{1}$. If we succeed in bringing the matrix into echelon form, but there is one or more rows of zeros, then the original matrix was singular. As a matter of terminology, the process of subtracting one equation from those below it in order to eliminate the left-most variable is known as pivoting. The equation that is being subtracted is called the pivot equation. Its left-most element is called the pivot.

## Matrix division

Given a matrix and its inverse, we have that

$$
\mathbf{A}^{-1} \mathbf{A}=\mathbf{A A}^{-1}=\mathbf{I}
$$

Can we define a division operation? The first attempt at such a definition might be $\mathbf{B A}^{-1}=\mathbf{B} / \mathbf{A}$. This is sometimes known as right division. The second attempt might be $\mathbf{A}^{-1} \mathbf{B}=\mathbf{B} \backslash \mathbf{A}$ sometimes called left division. Both of these operations are used. They do not necessarily give the same result. The reason is that matrix multiplication is not commutative.

## Block matrices

The idea of partitioning matrices can be carried a step further. After a matrix has been partitioned into sub-matrices, the sub-matrices can be named. As an example lets take a $4 \times 4$ matrix and name it $\mathbf{A}$. One particular partitioning is

$$
\mathbf{A}=\left[\begin{array}{c|ccc}
a_{11} & a_{12} & a_{13} & a_{14} \\
\hline a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{array}\right]
$$

Lets name the sub-matrices

$$
\mathbf{A}_{11}=\left[a_{11}\right] \quad \mathbf{A}_{12}=\left[\begin{array}{lll}
a_{12} & a_{13} & a_{14}
\end{array}\right] \quad \mathbf{A}_{21}=\left[\begin{array}{c}
a_{21} \\
a_{31} \\
a_{41}
\end{array}\right] \quad \mathbf{A}_{33}=\left[\begin{array}{lll}
a_{22} & a_{23} & a_{24} \\
a_{32} & a_{33} & a_{34} \\
a_{42} & a_{43} & a_{44}
\end{array}\right]
$$

We can write $\mathbf{A}$ in terms of its sub-matrices as

$$
\mathbf{A}=\left[\begin{array}{ll}
\mathbf{A}_{11} & \mathbf{A}_{12} \\
\mathbf{A}_{21} & \mathbf{A}_{22}
\end{array}\right]
$$

The expression on the right is called a block matrix. It is a matrix whose elements are themselves matrices. Block matrices have a fascinating property, above and beyond being a handy organizational tool. With some limitations, they obey the same operational rules as simple matrices. Let $\mathbf{B}$ be a block matrix representing a simple matrix of the same dimensions as $\mathbf{A}$.

$$
\mathbf{B}=\left[\begin{array}{ll}
\mathbf{B}_{11} & \mathbf{B}_{12} \\
\mathbf{B}_{21} & \mathbf{B}_{22}
\end{array}\right]
$$

Addition and subtraction carry over because

$$
\mathbf{A} \pm \mathbf{B}=\left[\begin{array}{ll}
\mathbf{A}_{11} & \mathbf{A}_{12} \\
\mathbf{A}_{21} & \mathbf{A}_{22}
\end{array}\right] \pm\left[\begin{array}{ll}
\mathbf{B}_{11} & \mathbf{B}_{12} \\
\mathbf{B}_{21} & \mathbf{B}_{22}
\end{array}\right]=\left[\begin{array}{ll}
\mathbf{A}_{11} \pm \mathbf{B}_{11} & \mathbf{A}_{12} \pm \mathbf{B}_{12} \\
\mathbf{A}_{21} \pm \mathbf{B}_{21} & \mathbf{A}_{22} \pm \mathbf{B}_{22}
\end{array}\right]
$$

The most interesting question is whether matrix multiplication carries over. The answer is yes, under certain circumstances. We can see what these circumstances are by writing out the general definition of matrix multiplication.

$$
(\mathbf{A B})_{m n}=\sum_{k=1}^{K} \mathbf{A}_{m k} \mathbf{B}_{k n}
$$

The number of columns in all of the $\mathbf{A}_{m k}$ must be he same as the number of rows in all of the $\mathbf{B}_{k n}$. This condition is fulfilled if $\mathbf{A}$ and $\mathbf{B}$ are partitioned such that the grouping of the columns in $\mathbf{A}$ is the same as the grouping of the rows of $\mathbf{B}$.

## Translating to homogeneous coordinates

The systems we have been considering are governed by the equation

$$
\mathbf{y}_{0}=\mathbf{A}_{0} \mathbf{x}_{0}
$$

This equation is a homogeneous linear equation. The systems that are described by this equation are linear systems. There are systems that are governed by inhomogeneous linear equations.

$$
\mathbf{y}=\mathbf{A x}+\mathrm{b}
$$

For those systems, the methods that we have been studying do not apply. Fortunately, there is a straightforward method for converting the inhomogeneous equations to homogeneous equations. The tactic is to augment the
matrix and the vectors. Using a $3 \times 3$ matrix for this example we start with the matrix and vectors

$$
\mathbf{A}=\left[\begin{array}{lll}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right] \quad \mathbf{b}=\left[\begin{array}{l}
b_{1} \\
b_{2} \\
b_{3}
\end{array}\right] \quad \mathbf{x}=\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right] \quad \mathbf{y}=\left[\begin{array}{l}
y_{1} \\
y_{2} \\
y_{3}
\end{array}\right]
$$

Next we augment the matrix and input vector to be

$$
\mathbf{A}^{\prime}=\left[\begin{array}{lll|l}
a_{11} & a_{12} & a_{13} & b_{1} \\
a_{21} & a_{22} & a_{23} & b_{2} \\
a_{31} & a_{32} & a_{33} & b_{3} \\
\hline 0 & 0 & 0 & 1
\end{array}\right] \quad \mathbf{x}^{\prime}=\left[\begin{array}{c}
x_{1} \\
x_{2} \\
x_{3} \\
1
\end{array}\right]
$$

The output of the equation $\mathbf{y}^{\prime}=\mathbf{A}^{\prime} \mathbf{x}^{\prime}$ is

$$
\mathbf{y}^{\prime}=\left[\begin{array}{l}
y_{1} \\
y_{2} \\
y_{3} \\
1
\end{array}\right]
$$

Surprisingly enough, the first three elements of the output vector are exactly what we desire. As an additional bonus, the fourth element is exactly what is needed to make $\mathbf{y}^{\prime}$ suitable for being operated upon by another matrix in the format of $\mathbf{A}^{\prime}$.

## Appendix C

## VECTOR ANALYSIS

In this appendix relationships, identities, theorems and transformation of a vector in some coordinate systems will be shown.

## C. 1 Rectangular Coordinate System

The coordinate vectors $\mathbf{a}_{x}, \mathbf{a}_{y}, \mathbf{a}_{z}$ form an orthogonal system of unit vectors in a Cartesian coordinate system. Element of volume is $d V=d x d y d z$ and Element of length is $d \mathbf{L}=d x \mathbf{a}_{x}+d y \mathbf{a}_{y}+d z \mathbf{a}_{z}$, and $|d L|=\sqrt{d x^{2}+d y^{2}+d z^{2}}$.

$$
\begin{equation*}
\mathbf{A}=A_{x} \mathbf{a}_{x}+A_{y} \mathbf{a}_{y}+A_{z} \mathbf{a}_{z} \tag{C.1}
\end{equation*}
$$

$$
\begin{equation*}
|\mathbf{A}|=\sqrt{A_{x}^{2}+A_{y}^{2}+A_{z}^{2}} \tag{C.2}
\end{equation*}
$$

$$
\begin{gather*}
\mathbf{A} \cdot \mathbf{B}=A_{x} B_{x}+A_{y} B_{y}+A_{z} B_{z}  \tag{C.3}\\
\mathbf{a}_{x} \times \mathbf{a}_{y}=\mathbf{a}_{z}, \quad \mathbf{a}_{y} \times \mathbf{a}_{z}=\mathbf{a}_{x}, \quad \mathbf{a}_{z} \times \mathbf{a}_{x}=\mathbf{a}_{y} \tag{C.4}
\end{gather*}
$$

$\mathbf{A} \times \mathbf{B}=\left(A_{y} B_{z}-A_{z} B_{y}\right) \mathbf{a}_{x}+\left(A_{z} B_{x}-A_{x} B_{z}\right) \mathbf{a}_{y}+\left(A_{x} B_{y}-A_{y} B_{x}\right) \mathbf{a}_{z}$

$$
\begin{align*}
& \nabla \psi=\mathbf{a}_{x} \frac{\partial \psi}{\partial x}+\mathbf{a}_{y} \frac{\partial \psi}{\partial y}+\mathbf{a}_{z} \frac{\partial \psi}{\partial z}  \tag{C.6}\\
& \nabla \cdot \mathbf{A}=\frac{\partial A_{x}}{\partial x}+\frac{\partial A_{y}}{\partial y}+\frac{\partial A_{z}}{\partial z}
\end{align*}
$$

$$
\begin{gather*}
\nabla \times \mathbf{A}=\mathbf{a}_{x}\left(\frac{\partial A_{z}}{\partial y}-\frac{\partial A_{y}}{\partial z}\right)+\mathbf{a}_{y}\left(\frac{\partial A_{x}}{\partial z}-\frac{\partial A_{z}}{\partial x}\right)+\mathbf{a}_{z}\left(\frac{\partial A_{y}}{\partial x}-\frac{\partial A_{x}}{\partial y}\right)  \tag{C.8}\\
\nabla \cdot \nabla \psi=\nabla^{2} \psi=\frac{\partial^{2} \psi}{\partial x^{2}}+\frac{\partial^{2} \psi}{\partial y^{2}}+\frac{\partial^{2} \psi}{\partial z^{2}}  \tag{C.9}\\
\nabla^{2} \mathbf{A}=\nabla(\nabla \cdot \mathbf{A})-\nabla \times \nabla \times \mathbf{A}  \tag{C.10}\\
\nabla \times \nabla \times\left(\mathbf{a}_{x} \times \mathbf{A}\right)=-\mathbf{a}_{x} \times \nabla(\nabla \cdot \mathbf{A})-\nabla \times\left(\frac{\partial \mathbf{A}}{\partial x}\right)  \tag{C.11}\\
\nabla\left(\nabla \cdot\left(\mathbf{a}_{x} \times \mathbf{A}\right)\right)=-\mathbf{a}_{x} \times \nabla \times \nabla \times \mathbf{A}-\nabla \times\left(\frac{\partial \mathbf{A}}{\partial x}\right)  \tag{C.12}\\
\nabla \nabla \cdot\left(\psi \mathbf{a}_{x}\right)=\mathbf{a}_{x} \nabla^{2} \psi  \tag{C.13}\\
\nabla \cdot\left(\mathbf{a}_{x} \times \mathbf{A}\right)=-\mathbf{a}_{x} \cdot \nabla \times \mathbf{A}  \tag{C.14}\\
\nabla\left(\mathbf{a}_{x} \cdot \mathbf{A}\right)=\mathbf{a}_{x} \times \nabla \times \mathbf{A}+\frac{\partial \mathbf{A}}{\partial x}  \tag{C.15}\\
\nabla\left(\mathbf{a}_{x} \times \mathbf{A}\right)=\mathbf{a}_{x} \nabla \cdot \mathbf{A}-\frac{\partial \mathbf{A}}{\partial x}  \tag{C.16}\\
\nabla^{2} \mathbf{A}=\mathbf{a}_{x} \nabla^{2} A_{x}+\mathbf{a}_{y} \nabla^{2} A_{y}+\mathbf{a}_{z} \nabla^{2} A_{z} \tag{C.17}
\end{gather*}
$$

$$
\nabla(\nabla \cdot \mathbf{A})=\mathbf{a}_{x}\left(\frac{\partial^{2} A_{x}}{\partial x^{2}}+\frac{\partial^{2} A_{y}}{\partial x \partial y}+\frac{\partial^{2} A_{z}}{\partial x \partial z}\right)
$$

$$
\begin{equation*}
+\mathbf{a}_{y}\left(\frac{\partial^{2} A_{x}}{\partial x \partial y}+\frac{\partial^{2} A_{y}}{\partial y^{2}}+\frac{\partial^{2} A_{z}}{\partial y \partial z}\right) \tag{C.18}
\end{equation*}
$$

$$
+\mathbf{a}_{z}\left(\frac{\partial^{2} A_{x}}{\partial x \partial z}+\frac{\partial^{2} A_{y}}{\partial y \partial z}+\frac{\partial^{2} A_{z}}{\partial z^{2}}\right)
$$

$$
\begin{gather*}
\nabla \times \nabla \times \mathbf{A}= \\
\mathbf{a}_{x}\left(-\frac{\partial^{2} A_{x}}{\partial y^{2}}-\frac{\partial^{2} A_{x}}{\partial z^{2}}+\frac{\partial^{2} A_{y}}{\partial x \partial y}+\frac{\partial^{2} A_{z}}{\partial x \partial z}\right)  \tag{C.19}\\
+\mathbf{a}_{y}\left(-\frac{\partial^{2} A_{y}}{\partial x^{2}}-\frac{\partial^{2} A_{y}}{\partial z^{2}}+\frac{\partial^{2} A_{x}}{\partial x \partial y}+\frac{\partial^{2} A_{z}}{\partial y \partial z}\right) \\
+\mathbf{a}_{z}\left(-\frac{\partial^{2} A_{z}}{\partial x^{2}}-\frac{\partial^{2} A_{z}}{\partial y^{2}}+\frac{\partial^{2} A_{x}}{\partial x \partial z}+\frac{\partial^{2} A_{y}}{\partial y \partial z}\right)  \tag{C.20}\\
\nabla^{2}\left(\psi \mathbf{a}_{x}\right)=\mathbf{a}_{x} \nabla^{2} \psi  \tag{C.21}\\
\nabla^{2}\left(\mathbf{a}_{x} \times \mathbf{A}\right)=\mathbf{a}_{x} \times \nabla^{2} \mathbf{A}
\end{gather*}
$$

## C. 2 General Curvilinear Coordinates

One of the main advantages of the vector calculus is the possibility to formulate physical laws without using some particular coordinate system. If one has to solve a special case of electromagnetic field defined by given boundary conditions it is necessary to decompose corresponding vector equations into coordinate components and to solve them in that coordinate system. The choice of a suitable coordinate system can significantly simplify the solution. In the opposite the case construction of a suitable analytical solution is rather tedious. The appropriate coordinate system is chosen in such a way that its coordinate surfaces, obtained by putting the particular coordinates equal to constants, coincide with the boundary surfaces of the region in which the solution is to be seek.
In this respect the definition and utilization of curvilinear coordinate system is one of the important topics that are necessary to cope with for a proper understanding of the formulation and solution of electromagnetic problems. Let us consider three independent, unambiguous and smooth functions $f_{1}(x, y, z)$, $f_{2}(x, y, z), f_{3}(x, y, z)$, of the three independent space variables $x, y, z$ in the cartesian coordinate system $(x, y, z)$. Setting these functions equal to constant parameters $u_{1}, u_{2}, u_{3}$ defines three surfaces, that can be labeled by these numbers, see Fig.(C.1). Common intersection of the surfaces $u_{1}=$ const $_{1}, u_{2}=$ const $_{2}, u_{3}=$ const $_{3}$ defines one point in the space to which a set of three unique numbers $\left(u_{1}, u_{2}, u_{3}\right)$ can be assigned. These numbers


Figure C.1: General Curvilinear Coordinates
are called curvilinear coordinates of that point, see Fig.(C.1). The set of equations

$$
\begin{align*}
& u_{1}=f_{1}(x, y, z) \\
& u_{2}=f_{2}(x, y, z)  \tag{C.22}\\
& u_{3}=f_{2}(x, y, z)
\end{align*}
$$

can be solved and the solution can be written in the form

$$
\begin{align*}
& x=g_{1}\left(u_{1}, u_{2}, u_{3}\right) \\
& y=g_{2}\left(u_{1}, u_{2}, u_{3}\right)  \tag{C.23}\\
& z=g_{3}\left(u_{1}, u_{2}, u_{3}\right)
\end{align*}
$$

It defines the position of the point $A$ in the cartesian system $(x, y, z)$ using coordinates $\left(u_{1}, u_{2}, u_{3}\right)$, where $\mathbf{r}=x \mathbf{a}_{x}+y \mathbf{a}_{y}+z \mathbf{a}_{z}=g_{1}\left(u_{1}, u_{2}, u_{3}\right) \mathbf{a}_{x}+$ $g_{2}\left(u_{1}, u_{2}, u_{3}\right) \mathbf{a}_{y}+g_{3}\left(u_{1}, u_{2}, u_{3}\right) \mathbf{a}_{z}$ is the position vector of the point $A$ and $\mathbf{a}_{x}, \mathbf{a}_{y}, \mathbf{a}_{z}$ are the unit vectors aligned in the space with the coordinate axes of the cartesian system, see Fig.(C.1). An elementary displacement of the point A in the space can be described by the differential formula

$$
\begin{equation*}
d \mathbf{r}=\frac{\partial \mathbf{r}}{\partial u_{1}} d u_{1}+\frac{\partial \mathbf{r}}{\partial u_{2}} d u_{2}+\frac{\partial \mathbf{r}}{\partial u_{3}} d u_{3} \tag{C.24}
\end{equation*}
$$

The small change along each coordinate curve is tangential to the curve. Define

$$
\begin{equation*}
\mathbf{a}^{i}=\frac{\partial \mathbf{r}}{\partial u_{i}}, \quad i=1,2,3 \tag{C.25}
\end{equation*}
$$

$\mathbf{a}^{1}, \mathbf{a}^{2}, \mathbf{a}^{3}$ are called the basic vectors of the general curvilinear coordinate system at point $A\left(u_{1}, u_{2}, u_{3}\right)$. It is to be noted that the absolute values of $\mathbf{a}^{1}, \mathbf{a}^{2}, \mathbf{a}^{3}$ are not equal to 1 , they are generally not unit vectors. If $\mathbf{a}^{1} \perp \mathbf{a}^{2} \perp \mathbf{a}^{3}$ we have the orthogonal curvilinear coordinate system. In the following we shall consider only orthogonal coordinate systems. The relation Eq.(C.24) can be rewritten into the form

$$
\begin{equation*}
d \mathbf{r}=h_{1}\left(u_{1}, u_{2}, u_{3}\right) d u_{1} \mathbf{a}_{1}+h_{2}\left(u_{1}, u_{2}, u_{3}\right) d u_{2} \mathbf{a}_{2}+h_{3}\left(u_{1}, u_{2}, u_{3}\right) d u_{3} \mathbf{a}_{3} \tag{C.26}
\end{equation*}
$$

where we have put $\mathbf{a}^{i}=h_{i}\left(u_{1}, u_{2}, u_{3}\right) \mathbf{a}_{i}$ where $i=1,2,3$. The functions $h_{1}, h_{2}, h_{3}$ are usually called the metric coefficients. The physical meaning of these coefficients can be understood when defining the length elements along the particular directions given by vectors $\mathbf{a}_{1}, \mathbf{a}_{2}, \mathbf{a}_{3}$ respectively. Consequently the change of the curvilinear coordinate $d u_{i}$ can be transformed into corresponding displacement in space by multiplying it by $h_{i}$, corresponding metric coefficient see Fig.(C.2). Similarly one can define elementary coordinate surface corresponding with the elementary changes of a couple of coordinates. It is explicitly defined by the

$$
\begin{equation*}
d \mathbf{S}_{i}=d \mathbf{s}_{j} \times d \mathbf{s}_{k} \tag{C.27}
\end{equation*}
$$

Using expressions for elementary displacements $d \mathbf{s}_{j}, d \mathbf{s}_{k}$ from Eq.(C.27) it is possible to write

$$
\begin{equation*}
d \mathbf{s}=d \mathbf{r}=h_{1} d u_{1} \mathbf{a}_{1}+h_{2} d u_{2} \mathbf{a}_{2}+h_{3} d u_{3} \mathbf{a}_{3} \tag{C.28}
\end{equation*}
$$

In Eq.(C.28) the relations $\mathbf{a}_{1} \times \mathbf{a}_{2}=\mathbf{a}_{3}, \mathbf{a}_{2} \times \mathbf{a}_{3}=\mathbf{a}_{1}$ and $\mathbf{a}_{3} \times \mathbf{a}_{1}=\mathbf{a}_{2}$ were used. According to Eq.(C.28) a general elementary surface $d \mathbf{S}$ is composed of the three elementary surfaces $d \mathbf{S}_{1}, d \mathbf{S}_{2}, d \mathbf{S}_{3}$ oriented along the unit vectors $\mathbf{a}_{1}, \mathbf{a}_{2}, \mathbf{a}_{3}$ see Fig.(C.2). Elementary volume element $d V$ can be described by

$$
\begin{equation*}
d V=d \mathbf{s}_{i} \cdot d \mathbf{S}_{i}=h_{1} h_{2} h_{3} d u_{1} d u_{2} d u_{3} \tag{C.29}
\end{equation*}
$$

## C.2.1 Gradient

Gradient of a scalar function is a vector the direction of which points the maximum growth of the function and its magnitude is equal to the derivative


Figure C.2: Definition of the elementary displacement $d \mathbf{s}$, surface $d \mathbf{S}$ and volume $d V$, respectively
of that function along that direction. To obtain the formula for the gradient of a scalar function in curvilinear coordinate system $\left(u_{1}, u_{2}, u_{3}\right)$ let us consider the function $\psi\left(u_{1}, u_{2}, u_{3}\right)$. For the differential of the function one can write

$$
\begin{equation*}
d \psi=\frac{\partial \psi}{\partial u_{1}} d u_{1}+\frac{\partial \psi}{\partial u_{2}} d u_{2}+\frac{\partial \psi}{\partial u_{3}} d u_{3} \tag{C.30}
\end{equation*}
$$

we know that $d \mathbf{s}=h_{1} d u_{1} \mathbf{a}_{1}+h_{2} d u_{2} \mathbf{a}_{2}+h_{3} d u_{3} \mathbf{a}_{3}$, therefore we define $\nabla \psi=$ $\frac{1}{h_{1}} \frac{\partial \psi}{\partial u_{1}} \mathbf{a}_{1}+\frac{1}{h_{2}} \frac{\partial \psi}{\partial u_{2}} \mathbf{a}_{2}+\frac{1}{h_{3}} \frac{\partial \psi}{\partial u_{3}} \mathbf{a}_{3}$, so

$$
\begin{equation*}
d \psi=\nabla \psi \cdot d \mathbf{s} \tag{C.31}
\end{equation*}
$$

The $\operatorname{grad} \psi=\nabla \psi$ is defined as gradient of the function $\psi$.

## C.2.2 Divergence

The divergence of a vector function $\mathbf{A}\left(u_{1}, u_{2}, u_{3}\right)$ is defined by the formula

$$
\begin{equation*}
\operatorname{div} \mathbf{A}=\nabla \cdot \mathbf{A}=\lim _{\Delta V \rightarrow 0} \frac{\oint \mathbf{A} \cdot d \mathbf{S}}{\Delta V} \tag{C.32}
\end{equation*}
$$

The volume of the considered region is expected to be shrunk to a very small elementary volume. In this case one can speak about the volume density of the outflow of the vector as a function of point position in the space.
To obtain the formula for divergence in curvilinear coordinate system we use the relations for elementary surface and volume element. After some mathematical manipulations, we have

$$
\begin{equation*}
\operatorname{div} \mathbf{A}=\nabla \cdot \mathbf{A}=\frac{1}{h_{1} h_{2} h_{3}}\left[\frac{\partial\left(A_{1} h_{2} h_{3}\right)}{\partial u_{1}}+\frac{\partial\left(A_{2} h_{3} h_{1}\right)}{\partial u_{2}}+\frac{\partial\left(A_{3} h_{1} h_{2}\right)}{\partial u_{3}}\right] \tag{C.33}
\end{equation*}
$$

## C.2.3 Curl

The third and last of the special differential operator analysis which is frequently used for the characterizing of a special physical vector fields is curloperator. The curl of a vector field, denoted $\operatorname{curl} \mathbf{A}$ or $\nabla \times \mathbf{A}$ (the notation used in this work), is defined as the vector field having magnitude equal to the maximum "circulation" at each point and to be oriented perpendicularly to this plane of circulation for each point. More precisely, the magnitude of $\nabla \times \mathbf{A}$ is the limiting value of circulation per unit area. Written explicitly

$$
\begin{equation*}
(\nabla \times \mathbf{A}) \cdot \mathbf{n}=\lim _{\Delta S \rightarrow 0} \frac{\oint_{C} \mathbf{A} \cdot d \mathbf{S}}{\Delta S} \tag{C.34}
\end{equation*}
$$

where the right side is a line integral around an infinitesimal region of area $S$ that is allowed to shrink to zero via a limiting process and $\mathbf{n}$ is the unit normal vector to this region. If $\nabla \times \mathbf{A}=0$, then the field is said to be an irrotational field. The symbol $\nabla$ is variously known as "nabla" or "del."
The curl can be similarly defined in arbitrary orthogonal curvilinear coordinates using $\mathbf{A}=A_{1} \mathbf{a}_{1}+A_{2} \mathbf{a}_{2}+A_{3} \mathbf{a}_{3}$. In an orthogonal curvilinear system is given by

$$
\begin{align*}
\nabla \times \mathbf{A} & =\frac{1}{h_{2} h_{3}}\left[\frac{\partial\left(h_{3} A_{3}\right)}{\partial u_{2}}-\frac{\partial\left(h_{2} A_{2}\right)}{\partial u_{3}}\right] \mathbf{a}_{1}  \tag{C.35}\\
& +\frac{1}{h_{3} h_{1}}\left[\frac{\partial\left(h_{1} A_{1}\right)}{\partial u_{3}}-\frac{\partial\left(h_{3} A_{3}\right)}{\partial u_{1}}\right] \mathbf{a}_{2} \\
& +\frac{1}{h_{1} h_{2}}\left[\frac{\partial\left(h_{2} A_{2}\right)}{\partial u_{1}}-\frac{\partial\left(h_{1} A_{1}\right)}{\partial u_{2}}\right] \mathbf{a}_{3}
\end{align*}
$$

## C. 3 Cylindrical Coordinate System

The coordinate vectors $\mathbf{a}_{\rho}, \mathbf{a}_{\phi}, \mathbf{a}_{z}$ form an orthogonal system of unit vectors in a cylindrical coordinate system. Element of volume is $d V=\rho d \rho d \phi d z$; element of length is $d \mathbf{L}=d \rho \mathbf{a}_{\rho}+\rho d \phi \mathbf{a}_{\phi}+d z \mathbf{a}_{z},|d L|=\sqrt{d \rho^{2}+\rho^{2} d \phi^{2}+d z^{2}}$

$$
\begin{gather*}
\mathbf{A}=A_{\rho} \mathbf{a}_{\rho}+A_{\phi} \mathbf{a}_{\phi}+A_{z} \mathbf{a}_{z}  \tag{C.36}\\
|A|=\sqrt{A_{\rho}^{2}+A_{\phi}^{2}+A_{z}{ }^{2}}  \tag{C.37}\\
\mathbf{A} \cdot \mathbf{B}=A_{\rho} B_{\rho}+A_{\phi} B_{\phi}+A_{z} B_{z}  \tag{C.38}\\
\mathbf{A} \times \mathbf{B}=\left(A_{\phi} B_{z}-A_{z} B_{\phi}\right) \mathbf{a}_{\rho}+\left(A_{z} B_{\rho}-A_{\rho} B_{z}\right) \mathbf{a}_{\phi}+\left(A_{\rho} B_{\phi}-A_{\phi} B_{\rho}\right) \mathbf{a}_{z}  \tag{C.39}\\
\mathbf{a}_{\rho} \times \mathbf{a}_{\phi}=\mathbf{a}_{z}, \quad \mathbf{a}_{\phi} \times \mathbf{a}_{z}=\mathbf{a}_{\rho}, \quad \mathbf{a}_{z} \times \mathbf{a}_{\rho}=\mathbf{a}_{\phi}  \tag{C.40}\\
\frac{d \mathbf{a}_{\rho}}{d \phi}=\mathbf{a}_{\phi}, \quad \frac{d \mathbf{a}_{\phi}}{d \phi}=-\mathbf{a}_{\rho}  \tag{C.41}\\
\nabla \psi=\mathbf{a}_{\rho} \frac{\partial \psi}{\partial \rho}+\mathbf{a}_{\phi} \frac{1}{\rho} \frac{\partial \psi}{\partial \phi}+\mathbf{a}_{z} \frac{\partial \psi}{\partial z}  \tag{C.42}\\
\nabla \cdot \mathbf{A}=\frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho A_{\rho}\right)+\frac{1}{\rho} \frac{\partial A_{\phi}}{\partial \phi}+\frac{\partial A_{z}}{\partial z}  \tag{C.43}\\
\nabla \cdot \mathbf{a}_{\rho}=\frac{1}{\rho} ; \quad \nabla \cdot \mathbf{a}_{\phi}=\nabla \cdot \mathbf{a}_{z}=0  \tag{C.44}\\
\nabla \times \mathbf{A}=\mathbf{a}_{\rho}\left(\frac{1}{\rho} \frac{\partial A_{z}}{\partial \phi}-\frac{\partial A_{\phi}}{\partial z}\right)+\mathbf{a}_{\phi}\left(\frac{\partial A_{\rho}}{\partial z}-\frac{\partial A_{z}}{\partial \rho}\right)+\mathbf{a}_{z}\left(\frac{1}{\rho} \frac{\partial\left(\rho A_{\phi}\right)}{\partial \rho}-\frac{1}{\rho} \frac{\partial A_{\rho}}{\partial \phi}\right)  \tag{C.45}\\
\nabla \times \mathbf{a}_{\phi}=\frac{\mathbf{a}_{z}}{\rho} ; \quad \nabla \times \mathbf{a}_{\rho}=\nabla \times \mathbf{a}_{z}=0  \tag{C.46}\\
\psi=\frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right)+\frac{1}{\rho^{2}} \frac{\partial^{2} \psi}{\partial \phi^{2}}+\frac{\partial^{2} \psi}{\partial z^{2}} \tag{C.47}
\end{gather*}
$$

$$
\begin{align*}
& \nabla^{2} \mathbf{A}=\mathbf{a}_{\rho}\left(\frac{\partial^{2} A_{\rho}}{\partial \rho^{2}}+\frac{1}{\rho} \frac{\partial A_{\rho}}{\partial \rho}-\frac{A_{\rho}}{\rho^{2}}+\frac{1}{\rho^{2}} \frac{\partial^{2} A_{\rho}}{\partial \phi^{2}}-\frac{2}{\rho^{2}} \frac{\partial A_{\phi}}{\partial \phi}+\frac{\partial^{2} A_{\rho}}{\partial z^{2}}\right) \\
& \quad+\mathbf{a}_{\phi}\left(\frac{\partial^{2} A_{\phi}}{\partial \rho^{2}}+\frac{1}{\rho} \frac{\partial A_{\phi}}{\partial \rho}-\frac{A_{\phi}}{\rho^{2}}+\frac{1}{\rho^{2}} \frac{\partial^{2} A_{\phi}}{\partial \phi^{2}}-\frac{2}{\rho^{2}} \frac{\partial A_{\rho}}{\partial \phi}+\frac{\partial^{2} A_{\phi}}{\partial z^{2}}\right) \\
& \quad+\mathbf{a}_{z}\left(\frac{\partial^{2} A_{z}}{\partial \rho^{2}}+\frac{1}{\rho} \frac{\partial A_{z}}{\partial \rho}+\frac{1}{\rho^{2}} \frac{\partial^{2} A_{z}}{\partial \phi^{2}}+\frac{\partial^{2} A_{z}}{\partial z^{2}}\right) \tag{C.48}
\end{align*}
$$

Note:

$$
\begin{gather*}
\nabla^{2} \mathbf{A} \neq \mathbf{a}_{\rho} \nabla^{2} A_{\rho}+\mathbf{a}_{\phi} \nabla^{2} A_{\phi}+\mathbf{a}_{z} \nabla^{2} A_{z}  \tag{С.49}\\
\nabla \nabla \cdot \mathbf{A}=\mathbf{a}_{\rho}\left(\frac{\partial^{2} A_{\rho}}{\partial \rho^{2}}+\frac{\partial^{2} A_{z}}{\partial \rho \partial z}+\frac{1}{\rho} \frac{\partial^{2} A_{\phi}}{\partial \rho \partial \phi}+\frac{1}{\rho} \frac{\partial A_{\rho}}{\partial \rho}-\frac{1}{\rho^{2}} \frac{\partial A_{\phi}}{\partial \phi}-\frac{A_{\rho}}{\rho^{2}}\right) \\
+\mathbf{a}_{\phi}\left(\frac{1}{\rho} \frac{\partial^{2} A_{z}}{\partial \phi \partial z}+\frac{1}{\rho^{2}} \frac{\partial^{2} A_{\phi}}{\partial \phi^{2}}+\frac{1}{\rho} \frac{\partial^{2} A_{\rho}}{\partial \rho \partial \phi}+\frac{1}{\rho^{2}} \frac{\partial A_{\rho}}{\partial \phi}\right) \\
+\mathbf{a}_{z}\left(\frac{\partial^{2} A_{z}}{\partial z^{2}}+\frac{1}{\rho} \frac{\partial^{2} A_{\phi}}{\partial \phi \partial z}+\frac{\partial^{2} A_{\rho}}{\partial \rho \partial z}+\frac{1}{\rho} \frac{\partial A_{\rho}}{\partial z}\right)  \tag{C.50}\\
\nabla \times \nabla \times \mathbf{A}=\mathbf{a}_{\rho}\left(-\frac{1}{\rho^{2}} \frac{\partial^{2} A_{\rho}}{\partial \phi^{2}}-\frac{\partial^{2} A_{\rho}}{\partial z^{2}}+\frac{\partial^{2} A_{z}}{\partial \rho \partial z}+\frac{1}{\rho} \frac{\partial^{2} A_{\phi}}{\partial \rho \partial \phi}+\frac{1}{\rho^{2}} \frac{\partial A_{\phi}}{\partial \phi}\right) \\
+\mathbf{a}_{\phi}\left(-\frac{\partial^{2} A_{\phi}}{\partial z^{2}}+\frac{1}{\rho} \frac{\partial^{2} A_{z}}{\partial \phi \partial z}-\frac{\partial^{2} A_{\phi}}{\partial \rho^{2}}-\frac{1}{\rho} \frac{\partial A_{\phi}}{\partial \rho}+\frac{A_{\phi}}{\rho^{2}}-\frac{1}{\rho^{2}} \frac{\partial A_{\rho}}{\partial \phi}+\frac{1}{\rho} \frac{\partial^{2} A_{\rho}}{\partial \phi \partial \rho}\right) \\
+\mathbf{a}_{z}\left(-\frac{\partial^{2} A_{z}}{\partial \rho^{2}}-\frac{1}{\rho^{2}} \frac{\partial^{2} A_{z}}{\partial \phi^{2}}+\frac{\partial^{2} A_{\rho}}{\partial \rho \partial z}+\frac{1}{\rho} \frac{\partial A_{\phi}}{\partial \phi \partial z}+\frac{1}{\rho} \frac{\partial A_{\rho}}{\partial z}-\frac{1}{\rho} \frac{\partial A_{z}}{\partial \rho}\right) \quad(\mathrm{C} .51) \tag{C.51}
\end{gather*}
$$

## C. 4 Spherical Coordinate System

The coordinate vectors $\mathbf{a}_{r}, \mathbf{a}_{\theta}, \mathbf{a}_{\phi}$ form an orthogonal system of unit vectors in a spherical coordinate system. Element of volume is $d V=r^{2} \sin \theta d r d \theta d \phi$. Element of length is $d \mathbf{L}=d r \mathbf{a}_{r}+r d \theta \mathbf{a}_{\theta}+r \sin \theta d \phi \mathbf{a}_{\phi}$, and $|d L|=\sqrt{d r^{2}+r^{2} d \theta^{2}+r^{2} \sin ^{2} \theta d \phi^{2}}$

$$
\begin{equation*}
\mathbf{A}=A_{r} \mathbf{a}_{r}+A_{\theta} \mathbf{a}_{\theta}+A_{\phi} \mathbf{a}_{\phi} \tag{C.52}
\end{equation*}
$$

$$
\begin{equation*}
|\mathbf{A}|=\sqrt{A_{r}^{2}+A_{\theta}^{2}+A_{\phi}^{2}} \tag{C.53}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{A} \cdot \mathbf{B}=A_{r} B_{r}+A_{\theta} B_{\theta}+A_{\phi} B_{\phi} \tag{C.54}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{a}_{r} \times \mathbf{a}_{\theta}=\mathbf{a}_{\phi}, \quad \mathbf{a}_{\theta} \times \mathbf{a}_{\phi}=\mathbf{a}_{r}, \quad \mathbf{a}_{\phi} \times \mathbf{a}_{r}=\mathbf{a}_{\theta} \tag{C.55}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{A} \times \mathbf{B}=\left(A_{\theta} B_{\phi}-A_{\phi} B_{\theta}\right) \mathbf{a}_{r}+\left(A_{\phi} B_{r}-A_{r} B_{\phi}\right) \mathbf{a}_{\theta}+\left(A_{r} B_{\theta}-A_{\theta} B_{r}\right) \mathbf{a}_{\phi} \tag{C.56}
\end{equation*}
$$

$$
\begin{gather*}
\frac{d \mathbf{a}_{r}}{d \theta}=\mathbf{a}_{\theta}, \quad \frac{d \mathbf{a}_{\theta}}{d \theta}=-\mathbf{a}_{r}, \quad \frac{d \mathbf{a}_{r}}{d \phi}=\sin \theta \mathbf{a}_{\phi}, \quad \frac{d \mathbf{a}_{\theta}}{d \phi}=\cos \phi \mathbf{a}_{\phi}  \tag{C.57}\\
\mathbf{a}_{r} \times \mathbf{a}_{x}=\mathbf{a}_{\phi} \cos \theta \cos \phi+\mathbf{a}_{\theta} \sin \phi \\
\mathbf{a}_{r} \times \mathbf{a}_{y}=\mathbf{a}_{\phi} \cos \theta \sin \phi-\mathbf{a}_{\theta} \cos \phi  \tag{C.58}\\
\mathbf{a}_{r} \times \mathbf{a}_{z}=-\mathbf{a}_{\phi} \sin \theta \\
\frac{\partial \mathbf{a}_{r}}{\partial \phi}=\sin \theta \mathbf{a}_{\phi} ; \quad \frac{\partial \mathbf{a}_{r}}{\partial \theta}=\mathbf{a}_{\theta} ; \\
\frac{\partial \mathbf{a}_{\theta}}{\partial \theta}=-\mathbf{a}_{r} ; \quad \frac{\partial \mathbf{a}_{\theta}}{\partial \phi}=\cos \theta \mathbf{a}_{\phi} ;  \tag{C.59}\\
\nabla \psi=\mathbf{a}_{r} \frac{\partial \psi}{\partial r}+\mathbf{a}_{\theta} \frac{1}{r} \frac{\partial \psi}{\partial \theta}+\mathbf{a}_{\phi} \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi} \tag{C.60}
\end{gather*}
$$

$$
\begin{equation*}
\nabla \cdot \mathbf{A}=\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} A_{r}\right)+\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta A_{\theta}\right)+\frac{1}{r \sin \theta} \frac{\partial A_{\phi}}{\partial \phi} \tag{C.61}
\end{equation*}
$$

$$
\begin{gather*}
\nabla \times \mathbf{A}=\frac{\mathbf{a}_{r}}{r \sin \theta}\left[\frac{\partial}{\partial \theta}\left(A_{\phi} \sin \theta\right)-\frac{\partial A_{\theta}}{\partial \phi}\right] \\
+\frac{\mathbf{a}_{\theta}}{r}\left[\frac{1}{\sin \theta} \frac{\partial A_{r}}{\partial \phi}-\frac{\partial}{\partial r}\left(r A_{\phi}\right)\right] \\
+\frac{\mathbf{a}_{\phi}}{r}\left[\frac{\partial}{\partial r}\left(r A_{\theta}\right)-\frac{\partial A_{r}}{\partial \theta}\right]  \tag{C.62}\\
\nabla^{2} \psi=\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} \frac{\partial \psi}{\partial r}\right)+\frac{1}{r^{2} \sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial \psi}{\partial \theta}\right)+\frac{1}{r^{2} \sin ^{2} \theta} \frac{\partial^{2} \psi}{\partial \phi^{2}} \tag{C.63}
\end{gather*}
$$

$$
\begin{align*}
& \nabla^{2} \mathbf{A}=\mathbf{a}_{r}\left(\frac{\partial^{2} A_{r}}{\partial r^{2}}+\frac{2}{r} \frac{\partial A_{r}}{\partial r}-\frac{1}{r^{2}} A_{r}+\frac{1}{r^{2}} \frac{\partial^{2} A_{\theta}}{\partial \theta^{2}}+\frac{\cot \theta}{r^{2}} \frac{\partial A_{r}}{\partial \theta}+\right. \\
&\left.\frac{1}{r^{2} \sin ^{2} \theta} \frac{\partial^{2} A_{r}}{\partial \phi^{2}}-\frac{2}{r^{2}} \frac{\partial A_{\theta}}{\partial \theta}-\frac{2 \cot \theta}{r^{2}} A_{\theta}-\frac{2}{r^{2} \sin \theta} \frac{\partial A_{\phi}}{\partial \phi}\right) \\
& \quad+\mathbf{a}_{\theta}\left(\frac{\partial^{2} A_{\theta}}{\partial r^{2}}+\frac{2}{r} \frac{\partial A_{\theta}}{\partial r}-\frac{A_{\theta}}{r^{2} \sin ^{2} \theta}+\frac{1}{r^{2}} \frac{\partial^{2} A_{\theta}}{\partial \theta^{2}}+\frac{\cot \theta}{r^{2}} \frac{\partial A_{\theta}}{\partial \theta}\right. \\
&\left.\quad+\frac{1}{r^{2} \sin ^{2} \theta} \frac{\partial^{2} A_{\theta}}{\partial \phi^{2}}+\frac{2}{r^{2}} \frac{\partial A_{r}}{\partial \theta}-\frac{2 \cot \theta}{r^{2} \sin \theta} \frac{\partial A_{\phi}}{\partial \phi}\right) \\
& \quad+\mathbf{a}_{\phi}\left(\frac{\partial^{2} A_{\phi}}{\partial r^{2}}+\frac{2}{r} \frac{\partial A_{\phi}}{\partial r}-\frac{A_{\phi}}{r^{2} \sin ^{2} \theta}+\frac{1}{r^{2}} \frac{\partial^{2} A_{\phi}}{\partial \theta^{2}}\right. \\
&\left.\quad+\frac{\cot \theta}{r^{2}} \frac{\partial A_{\phi}}{\partial \theta}+\frac{1}{r^{2} \sin ^{2} \theta} \frac{\partial^{2} A_{\phi}}{\partial \phi^{2}}+\frac{2}{r^{2} \sin \theta} \frac{\partial A_{r}}{\partial \phi}+\frac{2 \cot \theta}{r^{2} \sin \theta} \frac{\partial A_{\theta}}{\partial \phi}\right) \tag{C.64}
\end{align*}
$$

Note:

$$
\begin{equation*}
\nabla^{2} \mathbf{A} \neq \mathbf{a}_{r} \nabla^{2} A_{r}+\mathbf{a}_{\theta} \nabla^{2} A_{\theta}+\mathbf{a}_{\phi} \nabla^{2} A_{\phi} \tag{C.65}
\end{equation*}
$$

## C.4.1 Rectangular to Cylindrical and Spherical Transformation

Let point $x, y, z$ be in rectangular, and the same point in cylindrical be $\rho, \phi, z$ and in spherical be $r, \theta, \phi$, thus

$$
\begin{align*}
x & =\rho \cos \phi \\
y & =\rho \sin \phi  \tag{C.66}\\
z & =z \\
x & =r \sin \theta \cos \phi \\
y & =r \sin \theta \sin \phi  \tag{C.67}\\
z & =r \cos \theta \\
\frac{\partial}{\partial x}=\cos \phi \frac{\partial}{\partial \rho}-\frac{\sin \phi}{\rho} \frac{\partial}{\partial \phi} ; & \frac{\partial}{\partial y}=\sin \phi \frac{\partial}{\partial \rho}+\frac{\cos \phi}{\rho} \frac{\partial}{\partial \phi} \tag{C.68}
\end{align*}
$$

$$
\begin{align*}
\frac{\partial}{\partial x} & =\sin \theta \cos \phi \frac{\partial}{\partial r}+\frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta}-\frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \\
\frac{\partial}{\partial y} & =\sin \theta \sin \phi \frac{\partial}{\partial r}+\frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta}+\frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \\
\frac{\partial}{\partial z} & =\cos \theta \frac{\partial}{\partial r}-\frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \tag{C.69}
\end{align*}
$$

$$
\begin{align*}
& \mathbf{a}_{x}=\mathbf{a}_{\rho} \cos \phi-\mathbf{a}_{\phi} \sin \phi \\
& \mathbf{a}_{y}=\mathbf{a}_{\rho} \sin \phi+\mathbf{a}_{\phi} \cos \phi  \tag{С.70}\\
& \mathbf{a}_{z}=\mathbf{a}_{z}
\end{align*}
$$

$\mathbf{a}_{x}=\mathbf{a}_{r} \sin \theta \cos \phi+\mathbf{a}_{\theta} \cos \theta \cos \phi-\mathbf{a}_{\phi} \sin \phi$
$\mathbf{a}_{y}=\mathbf{a}_{r} \sin \theta \sin \phi+\mathbf{a}_{\theta} \cos \theta \sin \phi+\mathbf{a}_{\phi} \cos \phi$
$\mathbf{a}_{z}=\mathbf{a}_{r} \cos \theta-\mathbf{a}_{\theta} \sin \theta$

$$
\begin{align*}
& A_{x}=A_{\rho} \cos \phi-A_{\phi} \sin \phi \\
& A_{y}=A_{\rho} \sin \phi+A_{\phi} \cos \phi  \tag{C.72}\\
& A_{z}=A_{z}
\end{align*}
$$

$$
\left[\begin{array}{c}
A_{x}  \tag{C.73}\\
A_{y} \\
A_{z}
\end{array}\right]=\left[\begin{array}{ccc}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{c}
A_{\rho} \\
A_{\phi} \\
A_{z}
\end{array}\right]
$$

$A_{x}=A_{r} \sin \theta \cos \phi+A_{\theta} \cos \theta \cos \phi-A_{\phi} \sin \phi$
$A_{y}=A_{r} \sin \theta \sin \phi+A_{\theta} \cos \theta \sin \phi+A_{\phi} \cos \phi$ $A_{z}=A_{r} \cos \theta-A_{\theta} \sin \theta$

$$
\left[\begin{array}{c}
A_{x}  \tag{C.75}\\
A_{y} \\
A_{z}
\end{array}\right]=\left[\begin{array}{ccc}
\sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\
\sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\
\cos \theta & -\sin \theta & 0
\end{array}\right]\left[\begin{array}{c}
A_{r} \\
A_{\theta} \\
A_{\phi}
\end{array}\right]
$$

## C.4.2 Cylindrical to Rectangular and Spherical Transformation

Let point $x, y, z$ be in rectangular, and the same point in cylindrical be $\rho, \phi, z$ thus

$$
\begin{align*}
& \begin{array}{c}
\rho=\sqrt{x^{2}+y^{2}} \\
\phi=\sin ^{-1} \frac{y}{\sqrt{x^{2}+y^{2}}}=\cos ^{-1} \frac{y}{\sqrt{x^{2}+y^{2}}}=\tan ^{-1} \frac{y}{x}
\end{array} \\
& \mathbf{a}_{\rho}=\mathbf{a}_{x} \cos \phi+\mathbf{a}_{y} \sin \phi \\
& \mathbf{a}_{\phi}=-\mathbf{a}_{x} \sin \phi+\mathbf{a}_{y} \cos \phi  \tag{C.76}\\
& \mathbf{a}_{z}=\mathbf{a}_{z} \\
& \mathbf{a}_{\rho}=\mathbf{a}_{r} \sin \theta+\mathbf{a}_{\theta} \cos \theta \\
& \mathbf{a}_{z}=\mathbf{a}_{r} \cos \theta-\mathbf{a}_{z} \sin \theta  \tag{C.77}\\
& \mathbf{a}_{\phi}=\mathbf{a}_{\phi} \\
& \frac{\partial}{\partial \rho}=\frac{x}{\sqrt{x^{2}+y^{2}}} \frac{\partial}{\partial x}+\frac{y}{\sqrt{x^{2}+y^{2}}} \frac{\partial}{\partial y} ; \quad \frac{\partial}{\partial \phi}=x \frac{\partial}{\partial y}-y \frac{\partial}{\partial x}  \tag{C.78}\\
& A_{\rho}=A_{x} \cos \phi+A_{y} \sin \phi \\
& A_{\phi}=-A_{x} \sin \phi+A_{y} \cos \phi  \tag{C.79}\\
& A_{z}=A_{z} \\
& {\left[\begin{array}{c}
A_{\rho} \\
A_{\phi} \\
A_{z}
\end{array}\right]=\left[\begin{array}{ccc}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{c}
A_{x} \\
A_{y} \\
A_{z}
\end{array}\right]}  \tag{C.80}\\
& {\left[\begin{array}{c}
A_{\rho} \\
A_{z}
\end{array}\right]=\left[\begin{array}{cc}
\sin \theta & \cos \theta \\
\cos \theta & -\sin \theta
\end{array}\right]\left[\begin{array}{c}
A_{r} \\
A_{\theta}
\end{array}\right]} \tag{C.81}
\end{align*}
$$

## C.4.3 Spherical to Rectangular and Cylindrical Transformation

Let point $x, y, z$ be in rectangular, and the same point in spherical be $r, \theta, \phi$ and in cylindrical be $\rho, \phi, z$, ; thus

$$
\begin{aligned}
r & =\sqrt{x^{2}+y^{2}+z^{2}} \\
\theta=\sin ^{-1} \frac{\sqrt{x^{2}+y^{2}}}{\sqrt{x^{2}+y^{2}+z^{2}}} & =\cos ^{-1} \frac{z}{\sqrt{x^{2}+y^{2}+z^{2}}}=\tan ^{-1} \frac{\sqrt{x^{2}+y^{2}}}{z} \\
\phi & =\tan ^{-1} \frac{y}{x}
\end{aligned}
$$

and the same point in cylindrical:

$$
\begin{align*}
& r=\sqrt{\rho^{2}+z^{2}}  \tag{C.83}\\
& \theta=\sin ^{-1} \frac{\rho}{\sqrt{\rho^{2}+z^{2}}}=\cos ^{-1} \frac{z}{\sqrt{\rho^{2}+z^{2}}}=\tan ^{-1} \frac{\rho}{z} \\
& \phi=\phi \\
& \frac{\partial}{\partial r}=\frac{x}{\sqrt{x^{2}+y^{2}+z^{2}}} \frac{\partial}{\partial x}+\frac{y}{\sqrt{x^{2}+y^{2}+z^{2}}} \frac{\partial}{\partial y}+\frac{z}{\sqrt{x^{2}+y^{2}+z^{2}}} \frac{\partial}{\partial z} \\
& \frac{\partial}{\partial \theta}=\frac{x z}{\sqrt{x^{2}+y^{2}}} \frac{\partial}{\partial x}+\frac{y z}{\sqrt{x^{2}+y^{2}}} \frac{\partial}{\partial y}-\sqrt{x^{2}+y^{2}} \frac{\partial}{\partial z} \\
& \frac{\partial}{\partial \phi}=x \frac{\partial}{\partial y}-y \frac{\partial}{\partial x}  \tag{C.84}\\
& \mathbf{a}_{r}=\mathbf{a}_{r} \sin \theta+\mathbf{a}_{z} \cos \theta \\
& \mathbf{a}_{\theta}=\mathbf{a}_{\rho} \cos \theta-\mathbf{a}_{z} \sin \theta  \tag{C.85}\\
& \mathbf{a}_{\phi}=\mathbf{a}_{\phi}
\end{align*}
$$

$$
\begin{align*}
\mathbf{a}_{r} & =\mathbf{a}_{x} \sin \theta \cos \phi+\mathbf{a}_{y} \sin \theta \sin \phi+\mathbf{a}_{z} \cos \theta \\
\mathbf{a}_{\theta} & =\mathbf{a}_{x} \cos \theta \cos \phi+\mathbf{a}_{y} \cos \theta \sin \phi-\mathbf{a}_{z} \sin \theta  \tag{C.86}\\
\mathbf{a}_{\phi} & =-\mathbf{a}_{x} \sin \phi+\mathbf{a}_{y} \cos \phi \\
A_{r} & =A_{x} \sin \theta \cos \phi+A_{y} \sin \theta \sin \phi+A_{z} \cos \theta \\
A_{\theta} & =A_{x} \cos \theta \cos \phi+A_{y} \cos \theta \sin \phi-A_{z} \sin \theta  \tag{C.87}\\
A_{\phi} & =-A_{x} \sin \phi+A_{y} \cos \phi
\end{align*}
$$

$$
\left[\begin{array}{c}
A_{r}  \tag{C.88}\\
A_{\theta} \\
A_{\phi}
\end{array}\right]=\left[\begin{array}{ccc}
\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\
\cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\
-\sin \phi & \cos \phi & 0
\end{array}\right]\left[\begin{array}{c}
A_{x} \\
A_{y} \\
A_{z}
\end{array}\right]
$$

## C. 5 Addition and Multiplication of Vectors

$$
\begin{align*}
\mathbf{A} \cdot \mathbf{A} & =|\mathbf{A}|^{2}  \tag{C.89}\\
\mathbf{A} \cdot \mathbf{A}^{*} & =|\mathbf{A}|^{2} \\
\mathbf{A}+\mathbf{B} & =\mathbf{B}+\mathbf{A} \\
\mathbf{A} \cdot \mathbf{B} & =\mathbf{B} \cdot \mathbf{A} \\
\mathbf{A} \times \mathbf{B} & =-\mathbf{B} \times \mathbf{A} \\
(\mathbf{A}+\mathbf{B}) \cdot \mathbf{C} & =\mathbf{A} \cdot \mathbf{C}+\mathbf{B} \cdot \mathbf{C} \\
(\mathbf{A}+\mathbf{B}) \times \mathbf{C} & =\mathbf{A} \times \mathbf{C}+\mathbf{B} \times \mathbf{C} \\
\mathbf{A} \cdot(\mathbf{B} \times \mathbf{C}) & =\mathbf{B} \cdot(\mathbf{C} \times \mathbf{A})=\mathbf{C} \cdot(\mathbf{A} \times \mathbf{B}) \\
\mathbf{A} \times(\mathbf{B} \times \mathbf{C}) & =\mathbf{B}(\mathbf{A} \cdot \mathbf{C})-\mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \\
(\mathbf{A} \times \mathbf{B}) \cdot(\mathbf{C} \times \mathbf{D}) & =(\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D})-(\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \\
(\mathbf{A} \times \mathbf{B}) \times(\mathbf{C} \times \mathbf{D}) & =(\mathbf{A} \times \mathbf{B} \cdot \mathbf{D}) \mathbf{C}-(\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}) \mathbf{D}
\end{align*}
$$

## C. 6 Differentiation of Vector Fields

$$
\begin{align*}
\nabla \cdot \nabla \times \mathbf{A} & =0  \tag{C.90}\\
\nabla \times \nabla \psi & =0 \\
\nabla(\psi+\phi) & =\nabla \psi+\nabla \phi \\
\nabla(\psi \phi) & =\phi \nabla \psi+\psi \nabla \phi \\
\nabla\left(\frac{\phi}{\psi}\right) & =\frac{\psi \nabla \phi-\phi \nabla \psi}{\psi^{2}} \\
\nabla \times(\mathbf{A}+\mathbf{B}) & =\nabla \times \mathbf{A}+\nabla \times \mathbf{B} \\
\nabla \cdot(\mathbf{A}+\mathbf{B}) & =\nabla \cdot \mathbf{A}+\nabla \cdot \mathbf{B} \\
\nabla \cdot(\psi \mathbf{A}) & =\nabla \psi \cdot \mathbf{A}+\psi \nabla \cdot \mathbf{A} \\
\nabla \cdot\left(\frac{\mathbf{A}}{\psi}\right) & =\frac{\psi \nabla \cdot \mathbf{A}-\mathbf{A} \cdot \nabla \psi}{\psi^{2}} \\
\nabla \times(\psi \mathbf{A}) & =\nabla \psi \times \mathbf{A}+\psi \nabla \times \mathbf{A} \\
\nabla \times\left(\frac{\mathbf{A}}{\psi}\right) & =\frac{\psi \nabla \times \mathbf{A}+\mathbf{A} \times \nabla \psi}{\psi^{2}} \\
\nabla \cdot(\mathbf{A} \times \mathbf{B}) & =\mathbf{B} \cdot \nabla \times \mathbf{A}-\mathbf{A} \cdot \nabla \times \mathbf{B} \\
\nabla \times \nabla \times \mathbf{A} & =\nabla(\nabla \cdot \mathbf{A})-\nabla^{2} \mathbf{A} \\
\nabla \times(\mathbf{A} \times \mathbf{B}) & =\mathbf{A} \nabla \cdot \mathbf{B}-\mathbf{B} \nabla \cdot \mathbf{A}+(\mathbf{B} \cdot \nabla) \mathbf{A}-(\mathbf{A} \cdot \nabla) \mathbf{B}
\end{align*}
$$

## C. 7 Coordinate Differentials Formula

$$
\begin{align*}
\frac{\partial}{\partial x} & =\mathbf{a}_{x} \cdot \nabla & & \frac{\partial}{\partial y}=\mathbf{a}_{y} \cdot \nabla \\
\frac{1}{h_{n}} \frac{\partial}{\partial n} & =\mathbf{a}_{n} \cdot \nabla & & \frac{1}{h_{t}} \frac{\partial}{\partial t}=\mathbf{a}_{t} \cdot \nabla \tag{C.91}
\end{align*}
$$

## C. 8 Integration

In the following equations, $\psi, \phi$ are scalar fields and $\mathbf{A}, \mathbf{B}$ are vector fields. If region $V$ is bounded by the closed surface $S$, then

$$
\begin{equation*}
\oint_{S} \mathbf{A} \cdot d \mathbf{S}=\int_{V}(\nabla \cdot \mathbf{A}) d V \quad \text { Divergence Theorem } \tag{C.92}
\end{equation*}
$$

$$
\begin{gather*}
\int_{V}[(\nabla \times \mathbf{A}) \cdot(\nabla \times \mathbf{B})-\mathbf{A} \cdot(\nabla \times \nabla \times \mathbf{B})] d V=\oint_{S}(\mathbf{A} \times \nabla \times \mathbf{B}) \cdot d \mathbf{S}(\mathrm{C} .93) \\
\int_{V}\left(\phi \nabla^{2} \psi-\psi \nabla^{2} \phi\right) d V=\oint_{S}(\phi \nabla \psi-\psi \nabla \phi) \cdot d \mathbf{S} \quad \text { Green's Identity }(\mathrm{C} .94) \\
\int_{V}\left(\phi \nabla^{2} \psi+\nabla \psi \cdot \nabla \phi\right) d V=\oint_{S}(\phi \nabla \psi) \cdot d \mathbf{S}  \tag{C.95}\\
\int_{V}(\nabla \phi) \cdot(\nabla \times \mathbf{A}) d V=\oint_{S} \phi(\nabla \times \mathbf{A}) \cdot d \mathbf{S}=\oint_{S}(\mathbf{A} \times \nabla \phi) \cdot d \mathbf{S} \tag{C.96}
\end{gather*}
$$

Also, If the surface $S$ spans the closed contour $C$, then

$$
\begin{gather*}
\oint_{C}(\nabla \psi) \cdot d \mathbf{L}=0  \tag{C.97}\\
\oint_{S}(\nabla \times \mathbf{A}) \cdot d \mathbf{S}=0  \tag{C.98}\\
\oint_{C} \mathbf{A} \cdot d \mathbf{L}=\int_{S}(\nabla \times \mathbf{A}) \cdot d \mathbf{S} \quad \text { Stokes }^{\prime} \text { Theorem } \tag{C.99}
\end{gather*}
$$

## C. 9 The Dirac Delta Function

The Dirac $\delta$-function answers the need to describe quantities that exist only at a point, along a line, or on a sheet; in other words, quantities that do not extend over all dimensions.

$$
\begin{gather*}
\delta(a t)=\frac{1}{|a|} \delta(t)  \tag{C.100}\\
\delta(t)=\delta(-t)  \tag{C.101}\\
\delta^{\prime}(t)=-\delta^{\prime}(-t)  \tag{C.102}\\
\delta\left(t^{2}-a^{2}\right)=\frac{1}{2 a}[\delta(t+a)+\delta(t-a)]  \tag{C.103}\\
f(t) \delta\left(t-t_{0}\right)=f\left(t_{0}\right) \delta\left(t-t_{0}\right) \tag{C.104}
\end{gather*}
$$

$$
\begin{align*}
& \int f(t) \delta\left(t-t_{0}\right) d t=f\left(t_{0}\right)  \tag{C.105}\\
& \int f(t) \delta^{(n)}\left(t-t_{0}\right) d t=(-1)^{n} f\left(t_{0}\right)  \tag{C.106}\\
& \delta[f(t)]=\sum_{i=1}^{N} \frac{\delta\left(t-t_{i}\right)}{\left|d f\left(t_{i}\right) / d t\right|} \quad \text { where } \quad f\left(t_{i}\right)=0, i=1,2,3 \cdots N  \tag{C.107}\\
& \int g(t) \delta[f(t)] d t=\sum_{i=1}^{N} \frac{g\left(t_{i}\right)}{\left|d f\left(t_{i}\right) / d t\right|}  \tag{C.108}\\
& \delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right) \quad(3 \mathrm{D} \text { delta function) } \\
& =\delta\left(x-x^{\prime}\right) \delta\left(y-y^{\prime}\right) \delta\left(z-z^{\prime}\right)(\text { Cartesian }) \\
& =\frac{\delta\left(\rho-\rho^{\prime}\right)}{\rho} \delta \phi-\phi^{\prime} \delta\left(z-z^{\prime}\right)(\text { Cylindrical }) \\
& =\frac{\delta\left(r-r^{\prime}\right)}{r r^{\prime}} \frac{\delta\left(\theta-\theta^{\prime}\right)}{\sin \theta} \delta\left(\phi-\phi^{\prime}\right)(\text { spherical }) \\
& =\frac{\delta\left(r-r^{\prime}\right)}{r r^{\prime}} \delta\left(\cos \theta-\cos \theta^{\prime}\right) \delta\left(\phi-\phi^{\prime}\right)(\text { spherical }) \tag{C.109}
\end{align*}
$$

## C. 10 Useful Formulas

In EM computation, we usually face with mathematical manipulation with distance source and observation point. The following formula may be useful in those computations:

$$
\begin{gather*}
\mathbf{R}=\left(x-x^{\prime}\right) \mathbf{a}_{x}+\left(y-y^{\prime}\right) \mathbf{a}_{y}+\left(z-z^{\prime}\right) \mathbf{a}_{z}  \tag{C.110}\\
R=\left|\mathbf{r}-\mathbf{r}^{\prime}\right|=\sqrt{\left(x-x^{\prime}\right)^{2}+\left(y-y^{\prime}\right)^{2}+\left(z-z^{\prime}\right)^{2}}  \tag{C.111}\\
\mathbf{a}_{R}=\frac{\mathbf{R}}{R}=\frac{\left(x-x^{\prime}\right)}{R} \mathbf{a}_{x}+\frac{\left(y-y^{\prime}\right)}{R} \mathbf{a}_{y}+\frac{\left(z-z^{\prime}\right)}{R} \mathbf{a}_{z}  \tag{C.112}\\
\nabla R=\mathbf{a}_{R} \tag{C.113}
\end{gather*}
$$

$$
\begin{gather*}
\frac{\partial R}{\partial x}=\frac{x-x^{\prime}}{R}=-\frac{\partial R}{\partial x^{\prime}}  \tag{C.114}\\
\nabla F(R)=-\nabla^{\prime} F(R)=F^{\prime}(R) \mathbf{a}_{R}  \tag{C.115}\\
\nabla \cdot\left[F(R) \mathbf{a}_{R}\right]=-\nabla^{\prime} \cdot\left[F(R) \mathbf{a}_{R}\right]=\frac{2 F(R)}{R}+F^{\prime}(R)  \tag{C.116}\\
\nabla \cdot \mathbf{a}_{R}=\frac{2}{R}  \tag{C.117}\\
\nabla \times\left[F(R) \mathbf{a}_{R}\right]=0  \tag{C.118}\\
\nabla^{2} F(R)=\nabla^{\prime 2} F(R)=\frac{2 F^{\prime}(R)}{R}+F^{\prime}(R)  \tag{C.119}\\
\nabla^{2} R^{n}=n(n+1) R^{n-2}  \tag{C.120}\\
\nabla^{2} \frac{1}{R}=-4 \pi \delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right)  \tag{C.121}\\
\left(\nabla^{2}+\beta^{2}\right)\left(\frac{e^{-j \beta R}}{R}\right)=-4 \pi \delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right)  \tag{C.122}\\
\nabla\left(\frac{e^{-j \beta R}}{R}\right)=-\left(\frac{1+j \beta R}{R}\right)\left(\frac{e^{-j \beta R}}{R}\right) \mathbf{a}_{R}  \tag{C.123}\\
\nabla \cdot(R \mathbf{B})=\mathbf{a}_{R} \cdot \mathbf{B} ; \quad \nabla \times(R \mathbf{B})=\mathbf{a}_{R} \times \mathbf{B} ; \tag{C.124}
\end{gather*}
$$

where $\mathbf{B}$ is constant vector.

$$
\begin{gather*}
\nabla F(t-R / c)=-\frac{F^{\prime}(t-R / c)}{c R} \mathbf{a}_{R}=-\nabla^{\prime} F(t-R / c)  \tag{C.125}\\
\nabla \frac{F(t-R / c)}{R}=\left[-\frac{F^{\prime}(t-R / c)}{c R}+\frac{F(t-R / c)}{R^{2}}\right] \mathbf{a}_{R}=-\nabla^{\prime} \frac{F(t-R / c)}{R}  \tag{C.126}\\
\nabla \cdot \mathbf{J}(t-R / c)=-\frac{1}{c} \mathbf{a}_{R} \cdot \mathbf{J}^{\prime}(t-R / c) \tag{C.127}
\end{gather*}
$$

$$
\begin{gather*}
\nabla \times \mathbf{J}(t-R / c)=-\frac{1}{c} \mathbf{a}_{R} \times \mathbf{J}^{\prime}(t-R / c)  \tag{C.128}\\
\nabla \cdot \frac{\mathbf{J}(t-R / c)}{R}=-\left[\frac{\mathbf{J}(t-R / c)}{R^{2}}+\frac{\mathbf{J}^{\prime}(t-R / c)}{c R}\right] \cdot \mathbf{a}_{R}  \tag{C.129}\\
\nabla \times \frac{\mathbf{J}(t-R / c)}{R}=-\frac{1}{R^{2}} \mathbf{a}_{R} \times \mathbf{J}(t-R / c)-\frac{1}{c R} \mathbf{a}_{R} \times \mathbf{J}^{\prime}(t-R / c)  \tag{C.130}\\
\nabla^{2} F(t-R / c)=-\frac{2}{c R} F^{\prime}(t-R / c)+\frac{F^{\prime \prime}(t-R / c)}{c^{2}}  \tag{C.131}\\
\nabla^{2} \mathbf{J}(t-R / c)=-\frac{2}{c R} \mathbf{J}^{\prime}(t-R / c)+\frac{\mathbf{J}^{\prime \prime}(t-R / c)}{c^{2}}  \tag{C.132}\\
\nabla^{2} \frac{\mathbf{J}(t-R / c)}{R}=\frac{\mathbf{J}^{\prime \prime}(t-R / c)}{c^{2} R} \tag{C.133}
\end{gather*}
$$

Identities involving the plane-wave function
$\mathbf{E}$ is a constant vector, $k=|\mathbf{k}|$.

$$
\begin{gather*}
\nabla\left(e^{-j \mathbf{k} \cdot \mathbf{r}}\right)=-j \mathbf{k} e^{-j \mathbf{k} \cdot \mathbf{r}}  \tag{C.134}\\
\nabla \cdot\left(\mathbf{E} e^{-j \mathbf{k} \cdot \mathbf{r}}\right)=-j \mathbf{k} \cdot \mathbf{E} e^{-j \mathbf{k} \cdot \mathbf{r}}  \tag{C.135}\\
\nabla \times\left(\mathbf{E} e^{-j \mathbf{k} \cdot \mathbf{r}}\right)=-j \mathbf{k} \times \mathbf{E} e^{-j \mathbf{k} \cdot \mathbf{r}}  \tag{C.136}\\
\nabla^{2}\left(\mathbf{E} e^{-j \mathbf{k} \cdot \mathbf{r}}\right)=-j k^{2} \mathbf{E} e^{-j \mathbf{k} \cdot \mathbf{r}} \tag{C.137}
\end{gather*}
$$

Identities involving the transverse/longitudinal decomposition $\mathbf{n}$ is a constant unit vector, $A_{n} \equiv \mathbf{n} \cdot \mathbf{A}, \frac{\partial}{\partial n} \equiv \mathbf{n} \cdot \nabla, \mathbf{A}_{t} \equiv \mathbf{A}-\mathbf{n} A_{n}$ and $\nabla_{t} \equiv \nabla-\mathbf{n} \frac{\partial}{\partial n}$

$$
\begin{gather*}
\mathbf{A}=\mathbf{A}_{t}+\mathbf{n} A_{n}  \tag{C.138}\\
\nabla=\nabla_{t}+\mathbf{n} \frac{\partial}{\partial n}  \tag{C.139}\\
\mathbf{n} \cdot \mathbf{A}_{t}=0 \tag{C.140}
\end{gather*}
$$

$$
\left.\begin{array}{c}
(\mathbf{n} \cdot \nabla) \phi=0 \\
\nabla_{t} \phi=\nabla \phi-\mathbf{n} \frac{\partial \phi}{\partial n} \\
\mathbf{n} \cdot(\nabla \phi)=(\mathbf{n} \cdot \nabla) \phi=\frac{\partial \phi}{\partial n} \\
\mathbf{n} \cdot\left(\nabla_{t} \phi\right)=0 \\
\nabla_{t} \cdot(\mathbf{n} \phi)=0 \\
\nabla_{t} \times(\mathbf{n} \phi)=-\mathbf{n} \times\left(\nabla_{t} \phi\right) \\
\nabla_{t} \times(\mathbf{n} \times \mathbf{A})=\mathbf{n} \nabla_{t} \cdot \mathbf{A}_{t} \\
\mathbf{n} \times\left(\nabla_{t} \times \mathbf{A}\right)=\nabla_{t} A_{u} \\
\mathbf{n} \times\left(\nabla_{t} \times \mathbf{A}_{t}\right)=0 \\
\mathbf{n} \cdot(\mathbf{n} \times \mathbf{A})=0 \\
\mathbf{n} \times(\mathbf{n} \times \mathbf{A})=-\mathbf{A}_{t} \\
\nabla \phi=\nabla_{t} \phi+\mathbf{n} \frac{\partial \phi}{\partial n} \\
\nabla \times \mathbf{A}=\nabla_{t} \cdot \mathbf{A}_{t}+\frac{\partial A_{n}}{\partial n} \\
\nabla_{t} \times \nabla_{t}^{2} \phi+\frac{\partial^{2} \phi}{\partial n^{2}} \\
\nabla_{t}+\mathbf{n} \times\left[\frac{\mathbf{A}_{t}}{\partial u}-\nabla_{t} A_{u}\right. \tag{C.155}
\end{array}\right]
$$

$$
\begin{align*}
\nabla \times \nabla \times \mathbf{A} & =\nabla_{t} \times \nabla_{t} \times \mathbf{A}_{t}-\frac{\partial^{2} \mathbf{A}_{t}}{\partial n^{2}}+\nabla_{t} \frac{\partial A_{n}}{\partial n}+\mathbf{n}\left[\frac{\partial}{\partial n}\left(\nabla_{t} \cdot \mathbf{A}_{t}\right)-\nabla_{t}^{2} A_{n}\right]  \tag{C.156}\\
\nabla^{2} \mathbf{A} & =\nabla_{t}\left(\nabla_{t} \cdot \mathbf{A}_{t}\right)+\frac{\partial^{2} \mathbf{A}_{t}}{\partial n^{2}}-\nabla_{t} \times \nabla_{t} \times \mathbf{A}_{t}+\mathbf{n} \nabla^{2} A_{n} \tag{C.157}
\end{align*}
$$

## Appendix D

## Analytic Geometry

## D. 1 Elementary Geometry

## D.1.1 Plane Triangle

In the following formulas, $\mathbf{S}$ stands for the area of triangle; $\alpha, \beta$ and $\gamma$ are the interior angles; $\mathrm{a}, \mathrm{b}$ and c ; are the sides opposite the angle $\alpha, \beta$ and $\gamma$ respectively; $h_{a}, h_{b}$ and $h_{c}$ are the attitudes corresponding to sides $\mathrm{a}, \mathrm{b}$ and $\mathrm{c} ; r_{a}, r_{b}$ and $r_{c}$ are the radii of escribed circles tangent to sides $\mathrm{a}, \mathrm{b}$ and c , respectively; p is the semiperimeter, $p=\frac{1}{2}(a+b+c)$; R is the radius of a circumscribe circle; $r$ is the radius of an inscribed circle,

- Area of triangle:

$$
\begin{align*}
S & =\frac{1}{2} a h_{a}=\frac{1}{2} b h_{b}=\frac{1}{2} c h_{c} \\
& =\frac{1}{2} a b \sin \gamma=\frac{a b c}{4 R}=p r \\
& =r_{a}(p-a)=r_{b}(p-b)=r_{c}(p-c)=\sqrt{r_{a} r_{b} r_{c} r} \\
& =p(p-a) \tan \frac{\alpha}{2}=p(p-b) \tan \frac{\beta}{2}=p(p-c) \tan \frac{\gamma}{2} \\
& =p^{2} \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2} \\
& =\sqrt{p(p-a)(p-b)(p-c)} \tag{D.1}
\end{align*}
$$

- Law of Cosines:

$$
\begin{align*}
a^{2} & =b^{2}+c^{2}-2 b c \cos \alpha \\
b^{2} & =a^{2}+c^{2}-2 a c \cos \beta \\
c^{2} & =a^{2}+b^{2}-2 a b \cos \gamma \tag{D.2}
\end{align*}
$$

- Law of Sines:

$$
\begin{equation*}
\frac{a}{\sin \alpha}=\frac{b}{\sin \beta}=\frac{c}{\sin \gamma}=2 R \tag{D.3}
\end{equation*}
$$

- Law of Tangents

$$
\begin{align*}
& \frac{a+b}{a-b}=\frac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}}=\frac{\cot \frac{\gamma}{2}}{\tan \frac{\alpha-\beta}{2}} \\
& \frac{b+c}{b-c}=\frac{\tan \frac{\beta+\gamma}{2}}{\tan \frac{\beta-\gamma}{2}}=\frac{\cot \frac{\alpha}{2}}{\tan \frac{\beta-\gamma}{2}} \\
& \frac{c+a}{c-a}=\frac{\tan \frac{\gamma+\alpha}{2}}{\tan \frac{\gamma-\alpha}{2}}=\frac{\cot \frac{\beta}{2}}{\tan \frac{\gamma-\alpha}{2}} \tag{D.4}
\end{align*}
$$

- Remarkable lines in a triangle: The median $m_{a}$ onto side a:

$$
\begin{equation*}
m_{a}=\frac{1}{2} \sqrt{2 b^{2}+2 c^{2}-a^{2}} \tag{D.5}
\end{equation*}
$$

The bisector $l_{a}$ to side a

$$
\begin{equation*}
l_{a}=\frac{2 a b \cos \frac{\gamma}{2}}{a+b}=\frac{2 a c \cos \frac{\beta}{2}}{a+c} \tag{D.6}
\end{equation*}
$$

- Equilateral triangle (with side a)

$$
\begin{equation*}
S=\frac{a^{2} \sqrt{3}}{4} \tag{D.7}
\end{equation*}
$$

The radius of a circumscribed circle and inscribed circle

$$
\begin{equation*}
R=\frac{a \sqrt{3}}{3} \quad r=\frac{a \sqrt{3}}{6} \tag{D.8}
\end{equation*}
$$

## D. 2 The Straight Line in a Plane

The general equation of a straight line is

$$
\begin{equation*}
A x+B y+C=0 \tag{D.9}
\end{equation*}
$$

The distance between the straight line $A x+B y+C=0$ and point $x_{0}, y_{0}$

$$
\begin{equation*}
d=\frac{\left|A x_{0}+B y_{0}+C\right|}{\sqrt{A^{2}+B^{2}}} \tag{D.10}
\end{equation*}
$$

Three points $\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right)$ and $\left(x_{3}, y_{3}\right)$ be in collinear:

$$
\left|\begin{array}{lll}
x_{1} & y_{1} & 1  \tag{D.11}\\
x_{2} & y_{2} & 1 \\
x_{3} & y_{3} & 1
\end{array}\right|=0
$$

Equation of a line connecting two points $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ :

$$
\begin{equation*}
y-y_{1}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}\left(x-x_{1}\right) \tag{D.12}
\end{equation*}
$$

For the straight lines $y=k_{1} x+b_{1}$ and $y=k_{2} x+b_{2}$ :
a) $k_{1}=k_{2}$ parallel condition;
b) $k_{1} k_{2}=-1$ perpendicular condition;
c) The angle between two lines $\tan \phi=\left|\frac{k_{2}-k_{1}}{1+k_{1} k_{2}}\right|$

The condition that the three lines $A_{1} x+B_{1} y+C_{1}=0, A_{2} x+B_{2} y+C_{2}=0$ and $A_{3} x+B_{3} y+C_{3}=0$ intersect at one point is:

$$
\left|\begin{array}{ccc}
A_{1} & B_{1} & C_{1}  \tag{D.13}\\
A_{2} & B_{2} & C_{2} \\
A_{3} & B_{3} & C_{3}
\end{array}\right|=0
$$

## D. 3 Equation of a Plane

$$
\begin{equation*}
A x+B y+C z+D=0 \tag{D.14}
\end{equation*}
$$

Equation of a plane passing through three points $\left(x_{1}, y_{1}, z_{1}\right),\left(x_{2}, y_{2}, z_{2}\right)$ and $\left(x_{3}, y_{3}, z_{3}\right)$

$$
\begin{gather*}
\left|\begin{array}{ccc}
x-x_{1} & y-y_{1} & z-z_{1} \\
x-x_{2} & y-y_{2} & z-z_{2} \\
x-x_{3} & y-y_{3} & z-z_{3}
\end{array}\right|=0  \tag{D.15}\\
\left|\begin{array}{ll}
y_{2}-y_{1} & z_{2}-z_{1} \\
y_{3}-y_{2} & z_{3}-z_{2}
\end{array}\right|\left(x-x_{1}\right)+\left|\begin{array}{ll}
z_{2}-z_{1} & x_{2}-x_{1} \\
z_{3}-z_{2} & x_{3}-x_{2}
\end{array}\right|\left(y-y_{1}\right)+ \\
\left|\begin{array}{ll}
x_{2}-x_{1} & y_{2}-y_{1} \\
x_{3}-x_{2} & y_{3}-y_{2}
\end{array}\right|\left(z-z_{1}\right)=0 \tag{D.16}
\end{gather*}
$$

Distance of point $\left(x_{0}, y_{0}, z_{0}\right)$ from plane $A x+B y+C z+D=0$

$$
\begin{equation*}
d=\frac{\left|A x_{0}+B y_{0}+C z_{0}+D\right|}{\sqrt{A^{2}+B^{2}+C^{2}}} \tag{D.17}
\end{equation*}
$$

Equation of line perpendicular to the plane $A x+B y+C z+D=0$ and through point $\left(x_{0}, y_{0}, z_{0}\right)$.

$$
\begin{equation*}
\frac{x-x_{0}}{A}=\frac{y-y_{0}}{B}=\frac{z-z_{0}}{C} \tag{D.18}
\end{equation*}
$$

The angle between two planes $A_{1} x+B_{1} y+C_{1} z+D_{1}=0$, and $A_{2} x+B_{2} y+$ $C_{2} z+D_{2}=0$

$$
\begin{equation*}
\cos \phi=\frac{\left|A_{1} A_{2}+B_{1} B_{2}+C_{1} C_{2}\right|}{\sqrt{A_{1}^{2}+B_{1}^{2}+C_{1}^{2}} \sqrt{A_{2}^{2}+B_{2}^{2}+C_{2}^{2}}} \tag{D.19}
\end{equation*}
$$

## Appendix E

## Series

$$
\begin{array}{cc}
e^{x}=1+\frac{x}{1!}+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\frac{x^{4}}{4!}+\cdots & |x|<\infty \\
\cosh (x)=1+\frac{x^{2}}{2!}+\frac{x^{4}}{4!}+\frac{x^{6}}{6!}+\frac{x^{8}}{8!}+\cdots & \\
\sinh (x)=\frac{x}{1!}+\frac{x^{3}}{3!}+\frac{x^{5}}{5!}+\frac{x^{7}}{7!}+\frac{x^{9}}{9!}+\cdots & |x|<\infty \\
\tanh (x)=x-\frac{x^{3}}{3}+\frac{2 x^{5}}{3 \cdot 5}-\frac{17 x^{7}}{3^{2} \cdot 5 \cdot 7}+\frac{62 x^{9}}{3^{2} \cdot 5 \cdot 7 \cdot 9}+\cdots & |x|<\frac{\pi}{2} \\
\operatorname{coth}(x)=\frac{1}{x}+\frac{x}{3}-\frac{x^{3}}{3 \cdot 5}+\frac{2 x^{5}}{3^{2} \cdot 5 \cdot 7}+\cdots & |x|<\pi \\
\cos x=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}-\frac{x^{6}}{6!}+\frac{x^{8}}{8!} \cdots & |x|<\infty \\
\sin x=\frac{x}{1!}-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\cdots & |x|<\infty \\
\tan x=x+\frac{x^{3}}{3}+\frac{2 * x^{5}}{3 \cdot 5}+\frac{17 x^{7}}{3^{2} \cdot 5 \cdot 7}+\frac{62 x^{9}}{3^{2} \cdot 5 \cdot 7 \cdot 9}+\cdots & \quad|x|<\frac{\pi}{2} \\
\cot x=\frac{1}{x}-\frac{x}{3}-\frac{x^{3}}{3^{2} \cdot 5}-\frac{2 * x^{5}}{3^{2} \cdot 5 \cdot}-\frac{x^{7}}{3^{2} \cdot 5^{2} \cdot 7}-\cdots & 0<|x|<\pi \\
\cos { }^{-1} x=\frac{\pi}{2}-x-\frac{1}{2} \cdot \frac{x^{3}}{3}-\frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^{5}}{5}-\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot \frac{x^{7}}{7}}-\cdots & -1<|x| \leq 1(\mathrm{E} \tag{E.10}
\end{array}
$$

$$
\begin{array}{rlrl}
\sin ^{-1} x=x+\frac{1}{2} \cdot \frac{x^{3}}{3}+\frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^{5}}{5}+\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^{7}}{7}+\cdots & -1<|x| \leq 1 \\
\tan ^{-1} x & =x-\frac{x^{3}}{3}+\frac{x^{5}}{5}-\frac{x^{7}}{7}+\cdots & -1<|x| \leq 1 \\
\cot ^{-1} x & =\frac{1}{x}-\frac{1}{3 x^{3}}+\frac{1}{5 x^{5}}-\frac{1}{7 x^{7}}+\cdots & 1<|x|  \tag{E.13}\\
& =\pi+\frac{1}{x}-\frac{1}{3 x^{3}}+\frac{1}{5 x^{5}}-\frac{1}{7 x^{7}}+\cdots & |x|<-1
\end{array}
$$

$$
\sinh ^{-1} x=\left\{\begin{array}{cl}
x-\frac{x^{3}}{2 \cdot 3}+\frac{1 \cdot 3 x^{5}}{2 \cdot 4 \cdot 5}-\frac{1 \cdot 3 \cdot 5 x^{7}}{2 \cdot 4 \cdot 6 \cdot 7}+\cdots & |x|<1 \\
\pm\left(\ln |2 x|+\frac{1}{2 \cdot 2 x^{2}}-\frac{1 \cdot 3}{2 \cdot 4 \cdot 4 x^{4}}+\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 6 x^{6}}-\cdots\right) & {\left[\begin{array}{l}
+ \text { if } x \geq 1 \\
- \text { if } x \leq-1
\end{array}\right]}
\end{array}\right.
$$

$$
\cosh ^{-1} x= \pm\left\{\ln (2 x)-\left(\frac{1}{2 \cdot 2 x^{2}}+\frac{1 \cdot 3}{2 \cdot 4 \cdot 4 x^{4}}+\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 6 x^{6}}+\cdots\right)\right\}
$$

$$
\left[\begin{array}{l}
+ \text { if } \cosh ^{-1} x>0, x \geq 1  \tag{E.14}\\
- \text { if } \cosh ^{-1} x<0, x \geq 1
\end{array}\right]
$$

$$
\begin{equation*}
\tanh ^{-1} x=x+\frac{x^{3}}{3}+\frac{x^{5}}{5}+\frac{x^{7}}{7}+\cdots \quad|x|<1 \tag{E.15}
\end{equation*}
$$

$$
\begin{equation*}
\operatorname{coth}^{-1} x=\frac{1}{x}+\frac{1}{3 x^{3}}+\frac{1}{5 x^{5}}+\frac{1}{7 x^{7}}+\cdots \quad|x|>1 \tag{E.16}
\end{equation*}
$$

## E. 1 Taylor Series For Functions of Two Variables

$$
\begin{align*}
& f(x, y)=f(a, b)+(x-a) f_{x}(a, b)+(y-b) f_{y}(a, b) \\
& +\frac{1}{2!}\left\{(x-a)^{2} f_{x x}(a, b)+2(x-a)(y-b) f_{x y}(a, b)+(y-b)^{2} f_{y y}(a, b)\right\}+\cdots \tag{E.17}
\end{align*}
$$

## Appendix F

## Complex Inverse Trigonometric Function

The inverse trigonometric and hyperbolic functions are the multivalued function that are the inverse functions of the trigonometric and hyperbolic functions.

## F. 1 Range of usual principal value

The trigonometric functions are periodic, so we must restrict their domains before we are able to define a unique inverse.

Table F.1: Range of usual principal value

| $y=f(z)$ | Alternate notations | Range of usual principal value |
| :---: | :---: | :---: |
| $y=\sin ^{-1}(z)$ | $y=\arcsin (z)$ | $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ |
| $y=\cos ^{-1}(z)$ | $y=\arccos (z)$ | $0 \leq y \leq \pi$ |
| $y=\tan ^{-1}(z)$ | $y=\arctan (z)$ | $-\frac{\pi}{2}<y<\frac{\pi}{2}$ |
| $y=\cot ^{-1}(z)$ | $y=\operatorname{arccot}(z)$ | $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \quad y \neq 0$ |
| $y=\sec ^{-1}(z)$ | $y=\operatorname{arcsec}(z)$ | $0 \leq y \leq \pi, \quad y \neq \frac{\pi}{2}$ |
| $y=\csc ^{-1}(z)$ | $y=\operatorname{arccsc}(z)$ | $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \quad y \neq 0$ |

## F. 2 Definitions as infinite series

The inverse trigonometric functions can be defined in terms of infinite series.

$$
\begin{gathered}
\arcsin (z)=\sum_{n=0}^{\infty}\left(\frac{(2 n)!}{2^{2 n}(n!)^{2}} \frac{z^{2 n+1}}{2 n+1}\right), \quad|z|<1 \\
\arccos (z)=\frac{\pi}{2}-\arcsin (z)=\frac{\pi}{2}-\sum_{n=0}^{\infty}\left(\frac{(2 n)!}{2^{2 n}(n!)^{2}} \frac{z^{2 n+1}}{2 n+1}\right), \quad|z|<1 \\
\arctan (z)=\sum_{n=0}^{\infty} \frac{(-1)^{n} z^{2 n+1}}{2 n+1}, \quad|z|<1 \\
\operatorname{arccot}(z)=\frac{\pi}{2}-\arctan (z)=\sum_{n=0}^{\infty} \frac{(-1)^{n} z^{2 n+1}}{2 n+1}, \quad|z|<1 \\
\operatorname{arccsc}(z)=\arcsin \left(z^{-1}\right) \\
\operatorname{arcsec}(z)=\arccos \left(z^{-1}\right)
\end{gathered}
$$

## F. 3 Natural logarithm's expressions

The inverse trigonometric functions may be expressed using natural logarithms.

$$
\begin{gather*}
\arcsin (z)=-j \ln \left(j z+\sqrt{1-z^{2}}\right)  \tag{F.1}\\
\arccos (z)=-j \ln \left(z+\sqrt{z^{2}-1}\right)  \tag{F.2}\\
\arctan (z)=\frac{j}{2}(\ln (1-j z)-\ln (1+j z))  \tag{F.3}\\
\operatorname{arccot}(z)=\frac{j}{2}\left(\ln \left(1-\frac{j}{z}\right)-\ln \left(1+\frac{j}{z}\right)\right)  \tag{F.4}\\
\operatorname{arccsc}(z)=-j \ln \left(j z^{-1}+\sqrt{1-z^{-2}}\right)  \tag{F.5}\\
\operatorname{arcsec}(z)=-j \ln \left(z^{-1}+\sqrt{z^{-2}-1}\right) \tag{F.6}
\end{gather*}
$$

F.4. DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS 459

## F. 4 Derivatives of inverse trigonometric functions

$$
\begin{align*}
\frac{d}{d z} \arcsin (z) & =\frac{1}{\sqrt{1-z^{2}}}  \tag{F.7}\\
\frac{d}{d z} \arccos (z) & =-\frac{1}{\sqrt{1-z^{2}}}  \tag{F.8}\\
\frac{d}{d z} \arctan (z) & =\frac{1}{1+z^{2}}  \tag{F.9}\\
\frac{d}{d z} \operatorname{arccot}(z) & =-\frac{1}{1+z^{2}}  \tag{F.10}\\
\frac{d}{d z} \operatorname{arccsc}(z) & =-\frac{1}{z \sqrt{z^{2}-1}}  \tag{F.11}\\
\frac{d}{d z} \operatorname{arcsec}(z) & =\frac{1}{z \sqrt{z^{2}-1}} \tag{F.12}
\end{align*}
$$

## F. 5 Indefinite integrals of inverse trigonometric functions

$$
\begin{array}{r}
\int \arcsin (z) d z=z \arcsin (z)+\sqrt{1-z^{2}}+C \\
\int \arccos (z) d z=z \arccos (z)-\sqrt{1-z^{2}}+C \\
\int \arctan (z) d z=z \arctan (z)-\frac{1}{2} \ln \left(1+z^{2}\right)+C \\
\int \operatorname{arccot}(z) d z=z \operatorname{arccot}(z)+\frac{1}{2} \ln \left(1+z^{2}\right)+C \\
\int \operatorname{arccsc}(z) d z=z \operatorname{arccsc}(z)+\ln \left(z+\sqrt{z^{2}-1}\right)+C \\
\int \operatorname{arcsec}(z) d z=z \operatorname{arcsec}(z)-\ln \left(z+\sqrt{z^{2}-1}\right)+C \tag{F.18}
\end{array}
$$

## Appendix G

## SPECIAL FUNCTIONS

Bessel, Hankel and Legendre functions have been used in various part of electromagnetics such as: circular waveguides and cavity resonators, scattering of EM waves from cylindrical and spherical objects. Series expansions of these functions, and knowledge of the differentiation and recurrence formula often required. In this appendix, we assembled some useful formulas of these functions. For more detailed treatment of these functions, see one of the standard text on those subjects [12]

## G. 1 Bessel Functions

Bessel equation is:

$$
\begin{equation*}
z^{2} \frac{\partial^{2} \psi}{\partial z^{2}}+z \frac{\partial \psi}{\partial z}+\left(z^{2}-\nu^{2}\right) \psi=0 \tag{G.1}
\end{equation*}
$$

Solutions are:
a) Bessel function of first kind $J_{\nu}(z)$

$$
\begin{align*}
J_{\nu}(z) & =\sum_{m=0}^{\infty} \frac{(-1)^{m} z^{\nu+2 m}}{m!\Gamma(m+\nu+1) 2^{\nu+2 m}} \\
J_{-\nu}(z) & =\sum_{m=0}^{\infty} \frac{(-1)^{m} z^{\nu+2 m}}{m!\Gamma(m-\nu+1) 2^{\nu+2 m}} \tag{G.2}
\end{align*}
$$

The function $\Gamma(m-\nu+1)=\Gamma(p)$ is the generalized factorial function known as the Gamma function. It is defined by

$$
\begin{equation*}
\Gamma(p)=\int_{0}^{\infty} x^{p-1} e^{-x} d x \tag{G.3}
\end{equation*}
$$

where for $p=n$ (an integer), the Gamma function becomes the factorial $\Gamma(m+n+1)=(m+n)$ ! The two solutions given by Eq.(G.1) are then no longer independent, but instead are related by

$$
\begin{equation*}
J_{-n}(z)=(-1)^{n} J_{n}(z) \tag{G.4}
\end{equation*}
$$

$$
\begin{equation*}
J_{n}(-z)=(-1)^{n} J_{n}(z) \tag{G.5}
\end{equation*}
$$

The order of equation is given by value of $\nu$. In general $\nu$ will be non-integer. For integer vale of $\nu$ the symbol n is usually used.
b) Bessel function of second kind $Y_{\nu}(z)$ (also called Neumann function)

A second independent solution of Eq.(G.1) is defined by

$$
\begin{equation*}
Y_{\nu}(z)=\frac{J_{\nu}(z) \cos \nu \pi-J_{-\nu}(z)}{\sin \nu \pi} \tag{G.6}
\end{equation*}
$$

and for integer value of $\nu=n$

$$
\begin{equation*}
Y_{-n}(z)=(-1)^{n} Y_{n}(z) \tag{G.7}
\end{equation*}
$$

c) Bessel function of third kind $H_{\nu}^{(1)}(z)$ (also called Hankel function of first kind)
d) Bessel function of forth kind $H_{\nu}^{(2)}(z)$ (also called Hankel function of second kind)

Solution to Eq.(G.1) may be written in term of Bessel function of first and second kind and is called Hankel function of first and second kind.

$$
\begin{gather*}
H_{\nu}^{(1)}(z)=J_{\nu}(z)+j Y_{\nu}(z) \\
H_{\nu}^{(2)}(z)=J_{\nu}(z)-j Y_{\nu}(z)  \tag{G.8}\\
H_{-\nu}^{(1)}(z)=e^{j \nu \pi} H_{\nu}^{(1)}(z) \quad H_{-\nu}^{(2)}(z)=e^{j \nu \pi} H_{\nu}^{(2)}(z) \tag{G.9}
\end{gather*}
$$

## G.1.1 Bessel Functions for Small and Large Arguments

 for $z \ll 1$ :$$
\begin{gather*}
J_{\nu}(z) \approx \frac{z^{\nu}}{z^{\nu} \nu!} \quad Y_{\nu}(z) \approx-\frac{2^{\nu}(\nu-1)!}{\pi z^{\nu}}  \tag{G.10}\\
J_{0}(z) \approx 1, \quad Y_{0} \approx-\frac{2}{\pi} \ln \frac{2}{\gamma_{0} z} \\
J_{n}(z) \approx \frac{1}{n!}\left(\frac{z}{2}\right)^{n}, \quad Y_{n}(z) \approx-\frac{(n-1)!}{\pi}\left(\frac{2}{z}\right)^{n}, \quad n=1,2, \cdots \tag{G.11}
\end{gather*}
$$

$0!=1$ and $\gamma_{0}=e^{\gamma}=1.781072 \cdots$
$\gamma=$ Euler's constant $=0.57721566 \cdots$
for large value arguments:

$$
\begin{align*}
J_{\nu}(z) & \approx \sqrt{\frac{2}{\pi z}} \cos \left(z-\frac{\nu \pi}{2}-\frac{\pi}{4}\right) \\
Y_{\nu}(z) & \approx \sqrt{\frac{2}{\pi z}} \sin \left(z-\frac{\nu \pi}{2}-\frac{\pi}{4}\right) \\
H_{\nu}^{(1)}(z) & \approx \sqrt{\frac{2}{\pi z}} e^{j\left(z-\frac{\nu \pi}{2}-\frac{\pi}{4}\right)} \\
H_{\nu}^{(2)}(z) & \approx \sqrt{\frac{2}{\pi z}} e^{-j\left(z-\frac{\nu \pi}{2}-\frac{\pi}{4}\right)} \tag{G.12}
\end{align*}
$$

## G.1.2 Differentiation and Integration of Bessel Functions

In the following formulas $Z_{\nu}(z)$ may denote any kind of the function: $J_{\nu}(z)$, $Y_{\nu}(z), H_{\nu}^{(1)}(z), H_{\nu}^{(2)}(z)$ and $Z_{\nu}^{\prime}(z)$ means $\frac{d}{d z}\left[Z_{\nu}(z)\right]$

$$
\begin{align*}
Z_{0}^{\prime}(z)= & Z_{1}(z) \\
Z_{1}^{\prime}(z)= & Z_{0}(z)-\frac{1}{z} Z_{1}(z) \\
z Z_{\nu}^{\prime}(z)= & \nu Z_{\nu}(z)-z Z_{\nu+1}(z)=z Z_{\nu-1}(z)-\nu Z_{\nu}(z)  \tag{G.13}\\
& \frac{d}{d z}\left[z^{\nu} Z_{\nu}(z)\right]=z^{\nu} Z_{\nu-1}(z) \\
& \frac{d}{d z}\left[z^{-\nu} Z_{\nu}(z)\right]=-z^{-\nu} Z_{\nu+1}(z) \tag{G.14}
\end{align*}
$$

useful recurrence formulas

$$
\begin{align*}
& 2 \nu Z_{\nu}(z)=z\left[Z_{\nu-1}+Z_{\nu+1}(z)\right]  \tag{G.15}\\
\int z^{\nu} Z_{\nu-1}(z) d z & =z^{\nu} Z_{\nu}(z)  \tag{G.16}\\
\int z^{-\nu} Z_{\nu+1}(z) d z & =-z^{-\nu} Z_{\nu}(z)  \tag{G.17}\\
\int z Z_{\nu}^{2}(\alpha z) d z & =\frac{z^{2}}{2}\left[Z_{\nu}^{2}(\alpha z)-Z_{\nu-1}(\alpha z) Z_{\nu+1}(\alpha z)\right] \\
& =\frac{z^{2}}{2}\left[Z_{\nu}^{\prime 2}(\alpha z)+\left(1-\frac{\nu^{2}}{\alpha^{2} z^{2}}\right) Z_{\nu}^{2}(\alpha z)\right]  \tag{G.18}\\
\int z Z_{\nu}(\alpha z) Z_{\nu}(\beta z) d z & =\frac{z}{\alpha^{2}-\beta^{2}} \\
& \times\left[\beta Z_{\nu}(\alpha z) Z_{\nu-1}(\beta z)-\alpha Z_{\nu-1}(\alpha z) Z_{\nu}(\beta z)\right], \alpha \neq \beta \tag{G.19}
\end{align*}
$$

## G. 2 Bessel Functions of Integer Order

$$
\begin{equation*}
J_{n}(z)=\sum_{m=0}^{\infty} \frac{(-1)^{m} z^{n+2 m}}{m!(m+n)!2^{n+2 m}} \tag{G.20}
\end{equation*}
$$

For small value of the argument:

$$
\begin{gather*}
J_{0}(z)=1 ; \quad J_{n}(z)=\frac{z^{n}}{n!2^{n}}  \tag{G.21}\\
Y_{0}(z)=-\frac{2}{\pi} \ln \frac{2}{z}=\frac{2}{\pi}(\ln z-0.11593) ; \quad Y n(z)=-\frac{2^{n}(n-1)}{\pi(z)^{n}} ; \quad n>0  \tag{G.22}\\
H_{-n}^{(1)}(z)=e^{j n \pi} H_{n}^{(1)}(z) \quad H_{-n}^{(2)}(z)=e^{j n \pi} H_{n}^{(2)}(z) \tag{G.23}
\end{gather*}
$$

useful recurrence formulas

$$
\begin{equation*}
2 n Z_{n}(z)=z\left[Z_{n-1}+Z_{n+1}(z)\right] \tag{G.24}
\end{equation*}
$$

$$
\begin{gather*}
\int z\left[Z_{n}(z)\right]^{2} d z=\frac{z^{2}}{2}\left[Z_{n}^{2}(z)-Z_{n-1}(z) Z_{n+1}(z)\right]  \tag{G.25}\\
Y_{n-1}(z) J_{n}(z)-Y_{n}(z) J_{n-1}(z)=\frac{2}{n z}  \tag{G.26}\\
H_{n+1}^{(1)}(z) H_{n}^{(1)}(z)-H_{n+1}^{(2)}(z) H_{n}^{(2)}(z)=\frac{4}{j \pi n z}  \tag{G.27}\\
J_{n}(z) J_{1-n}(z)+J_{n-1}(z) J_{-n}(z)=\frac{2 \sin n \pi}{n z}  \tag{G.28}\\
J_{0}^{\prime}(z)=-J_{1}(z)  \tag{G.29}\\
J_{2}(z)=J_{0}(z)+2 J_{0}^{\prime \prime}(z)=J_{0}(z)-\frac{1}{z} J_{0}^{\prime}(z)  \tag{G.30}\\
\cos (z)=J_{0}(z)-2 J_{2}(z)+2 J_{4}(z) \cdots  \tag{G.31}\\
\sin (z)=2 J_{1}(z)-2 J_{3}(z)+2 J_{5}(z) \cdots  \tag{G.32}\\
\cos (z \sin \theta)=J_{0}(z)+2 J_{2}(z) \cos 2 \theta+2 J_{4}(z) \cos 4 \theta \cdots  \tag{G.33}\\
\sin (z \sin \theta)=2 J_{1}(z) \sin \theta+2 J_{3}(z) \sin 3 \theta+2 J_{5}(z) \sin 5 \theta \cdots  \tag{G.34}\\
{\left[J_{0}(z)\right]^{2}+2\left[J_{1}(z)\right]^{2}+2\left[J_{2}(z)\right]^{2}+2\left[J_{3}(z)\right]^{2}+\cdots=1} \tag{G.35}
\end{gather*}
$$

Generating function:

$$
\begin{gather*}
e^{z(t / 2-1 / 2 t)}=\sum_{n=-\infty}^{+\infty} t^{n} J_{n}(z)  \tag{G.36}\\
e^{j z \sin \theta}=\sum_{n=-\infty}^{+\infty} e^{j n \theta} J_{n}(z) ; \quad e^{j z \cos \theta}=\sum_{n=-\infty}^{+\infty} e^{j n(\theta+\pi / 2)} J_{n}(z) \tag{G.37}
\end{gather*}
$$

If n is a positive integer and $(\mathrm{m}+2 \mathrm{n}+1)$ is positive

$$
\begin{gather*}
(m-1) \int_{0}^{x} t^{m} J_{n+1}(t) J_{n-1}(t) d t= \\
\quad x^{m+1}\left[J_{n+1}(x) J_{n-1}(x)-J_{n}^{2}(t)\right]  \tag{G.38}\\
J_{0}(z)=\frac{2}{\pi} \int_{0}^{\infty} \sin (x \cosh t) d t ; \int_{0}^{x} t^{m} J_{n+1}^{2}(t) d t  \tag{G.39}\\
Y_{0}(z)=-\frac{2}{\pi} \int_{0}^{\infty} \cos (x \cosh t) d t  \tag{G.40}\\
\int_{0}^{\infty} \cos (x \cos t) d t=\frac{1}{\pi} \int_{0}^{\pi} \cos (x \sin t) d t  \tag{G.41}\\
J_{n}(z)=\frac{1}{\pi} \int_{0}^{\pi} \cos (n t-x \sin t) d t  \tag{G.42}\\
\int_{0}^{x} t J_{0}(t) d t=x \rho J_{1}(x) d k=\left(\rho^{2}-x^{2}\right)^{-1 / 2}  \tag{G.43}\\
\rho \text { and } x \text { are real }  \tag{G.44}\\
\int_{0}^{\infty} \frac{J_{n}(x) d x}{}=1  \tag{G.45}\\
\int_{0}^{\infty} e^{-a t} J_{m}(b t) \frac{d t}{t}=\frac{1}{m b^{m}}\left[\left(a^{2}+b^{2}\right)^{\frac{1}{2}}-a\right]^{m} \tag{G.46}
\end{gather*}
$$

( a and b are real and positive)

$$
\begin{gather*}
\int_{0}^{\infty} e^{-a t} J_{m}(b t) t^{m} d t=\frac{(2 b)^{m} \Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\left(a^{2}+b^{2}\right)^{\left(m+\frac{1}{2}\right)}}  \tag{G.47}\\
J_{n}(z)=\frac{1}{2 \pi j^{m}} \int_{0}^{\pi} e^{j z \cos (n t)} d t  \tag{G.48}\\
J_{n}(z)=\frac{2(2 / z)^{n}}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}-n\right)} \int_{1}^{\infty} \sin z t\left(t^{2}-1\right)^{\left(-n-\frac{1}{2}\right)} d t \tag{G.49}
\end{gather*}
$$

Table G.1: Roots of Bessel Functions $J_{n}\left(z_{m}\right)=0$

| n | $\mathrm{m}=1$ | $\mathrm{~m}=2$ | $\mathrm{~m}=3$ | $\mathrm{~m}=4$ | $\mathrm{~m}=5$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 2.405 | 5.520 | 8.654 | 11.792 | 14.931 |
| 1 | 3.832 | 7.016 | 10.173 | 13.324 | 16.471 |
| 2 | 5.136 | 8.417 | 11.620 | 14.796 | 17.960 |
| 3 | 6.380 | 9.761 | 13.015 | 16.223 | 19.409 |

$$
\begin{array}{r}
Y_{n}(z)=-\frac{2(2 / z)^{n}}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}-n\right)} \int_{1}^{\infty} \cos z t\left(t^{2}-1\right)^{\left(-n-\frac{1}{2}\right)} d t \\
\int_{0}^{\pi / 2}\left[J_{1}(x \sin \theta)\right]^{2} \frac{d \theta}{\sin \theta}=\int_{0}^{1} \frac{\left[J_{1}(x t)\right]^{2}}{t\left(1-t^{2}\right)^{1 / 2}} d t=\frac{1}{2}-\frac{J 1(2 x)}{2 x} \quad \text { (G.50) } \\
\int_{0}^{\pi / 2} J_{m}(z \sin \theta)(\sin \theta)^{m+1}(\cos \theta)^{2 n+1} d \theta=\frac{2^{n}}{z^{n+1}} \Gamma(n+1) J_{m+n+1}(z) \\
\int_{0}^{\infty} J_{m}(t z) d t \int_{0}^{\infty} J_{m}(t u) F(u) u d u=F(z) \quad(\text { Fourier }- \text { Bessel } \quad \text { Integral) } \\
\int_{0}^{\infty} J_{m}(a \sin \theta) J_{n}(b \cos \theta)(\sin \theta)^{m+1}(\cos \theta)^{n+1} d \theta=a^{m} / b^{n} \frac{J_{m+n+1}\left[\left(a^{2}+b^{2}\right)^{1 / 2}\right]}{\left(a^{2}+b^{2}\right)^{(m+n+1) / 2}} \\
\quad(\mathrm{G} .54)  \tag{G.55}\\
\int_{0}^{\pi / 2} \cos (a \cos \theta) \cos (b \sin \theta) d \theta=\frac{\pi}{2} J_{0}\left[\left(a^{2}+b^{2}\right)^{1 / 2}\right] \quad \text { for } \quad b>a \geq 0 \quad(\mathrm{G} .55)
\end{array}
$$

## G. 3 Roots of Bessel Functions

Four roots of $J_{n}\left(z_{m}\right)=0$ are given in Table(G.1), and four roots of $J_{n}^{\prime}\left(z_{m}\right)=0$ are given in Table(G.2), respectively.

Table G.2: Roots of the First Derivative of Bessel Functions $J_{n}^{\prime}\left(z_{m}\right)=0$

| n | $\mathrm{m}=1$ | $\mathrm{~m}=2$ | $\mathrm{~m}=3$ | $\mathrm{~m}=4$ | $\mathrm{~m}=5$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 3.832 | 7.016 | 10.173 | 13.324 |
| 1 | 1.841 | 5.331 | 8.536 | 11.706 | 14.864 |
| 2 | 3.054 | 6.706 | 9.969 | 13.170 | 16.348 |
| 3 | 4.201 | 8.015 | 11.346 | 14.586 | 17.789 |

## G. 4 Modified Bessel Functions

Modified Bessel equation of order $\nu$ is

$$
\begin{equation*}
z^{2} \frac{\partial^{2} \psi}{\partial z^{2}}+z \frac{\partial \psi}{\partial z}-\left(z^{2}+\nu^{2}\right) \psi=0 \tag{G.56}
\end{equation*}
$$

There are two solution for this equation:
a) Modified Bessel Function of first kind $I_{\nu}(z)$, order $\nu$
b) Modified Bessel Function of second kind $K_{\nu}(z)$, order $\nu$ For noninteger values of $\nu$ two independent solutions are:
$I_{\nu}(z)$ and $I_{-\nu}(z)$ where

$$
\begin{equation*}
I_{\nu}(z)=\sum_{m=0}^{\infty} \frac{z^{\nu+2 m}}{m!\Gamma(m+\nu+1) 2^{\nu+2 m}} \tag{G.57}
\end{equation*}
$$

When $\nu$ is an integer, two solution are related by $I_{n}(z)=I_{-n}(z)$.

$$
\begin{equation*}
I_{n}(-z)=(-1)^{n} I_{n}(z) \tag{G.58}
\end{equation*}
$$

Another solution is given by

$$
\begin{equation*}
K_{n}(z)=\frac{\pi}{2 \sin \nu \pi}\left[I_{-\nu}(z)-I_{\nu}(z)\right] \tag{G.59}
\end{equation*}
$$

for integer values of $\nu=n$, Eq.(G.59) reduces to

$$
\begin{equation*}
K_{\nu}(z)=\frac{2}{\cos n \pi}\left[\frac{\partial I_{-n}(z)}{\partial n}-\frac{\partial I_{n}(z)}{\partial n}\right] \tag{G.60}
\end{equation*}
$$

and Eq.(G.60) is second independent solution. Generating function for $I_{\nu}(z)$ is:

$$
\begin{equation*}
e^{z\left(t+\frac{1}{t}\right) / 2}=\sum_{n=-\infty}^{\infty} I_{\nu}(z) t^{\nu} \tag{G.61}
\end{equation*}
$$

## G.4.1 Small and Large Arguments

For $z \ll 1$ the $I_{\nu}(z)$ and $K_{\nu}(z)$ functions are given by

$$
\begin{equation*}
I_{\nu}(z) \approx \frac{z^{\nu}}{2^{\nu} \nu!} \quad K_{\nu} \approx \frac{2^{\nu-1}(\nu-1)}{z^{\nu}} \tag{G.62}
\end{equation*}
$$

and for large $z$ :

$$
\begin{align*}
I_{\nu}(z) & \approx \frac{e^{z}}{\sqrt{2 \pi z}}  \tag{G.63}\\
K_{\nu}(z) & \approx \sqrt{\frac{\pi}{2 z}} e^{-z} \tag{G.64}
\end{align*}
$$

## G.4.2 Recurrence Formulas for Modified Bessel Functions

$$
\begin{gather*}
z I_{\nu}^{\prime}(z)=\nu I_{\nu}(z)+z I_{\nu+1}(z)  \tag{G.65}\\
z K_{\nu}^{\prime}(z)=\nu K_{\nu}(z)-z K_{\nu+1}(z)  \tag{G.66}\\
z I_{\nu}^{\prime}(z)=z I_{\nu-1}(z)-\nu I_{\nu}(z)  \tag{G.67}\\
z K_{\nu}^{\prime}(z)=-z K_{\nu-1}(z)-\nu K_{\nu}(z)  \tag{G.68}\\
\frac{d}{d z}\left[z^{\nu} I_{\nu}(z)\right]=z^{\nu} I_{\nu-1}(z)  \tag{G.69}\\
\frac{d}{d z}\left[z^{\nu} K_{\nu}(z)\right]=-z^{\nu} K_{\nu-1}(z)  \tag{G.70}\\
\frac{d}{d z}\left[z^{-\nu} I_{\nu}(z)\right]=z^{-\nu} I_{\nu+1}(z)  \tag{G.71}\\
\frac{d}{d z}\left[z^{-\nu} K_{\nu}(z)\right]=-z^{-\nu} K_{\nu+1}(z)  \tag{G.72}\\
\left.2 \nu I_{\nu}(z)\right]=z\left[I_{\nu-1}(z)-I_{\nu+1}(z)\right]  \tag{G.73}\\
\left.2 \nu K_{\nu}(z)\right]=-z\left[K_{\nu-1}(z)-I_{\nu+1}(z)\right] \tag{G.74}
\end{gather*}
$$

## G.4.3 Ber and Bei Functions

The real and imaginary parts of $J_{n}\left(x e^{\frac{j 3 \pi}{4}}\right)$ are denoted by $\operatorname{Ber}_{n}(x)$ and $B e i_{n}(x)$, respectively:

$$
\begin{align*}
& \operatorname{Ber}_{n}(x)=\sum_{k=0}^{\infty} \frac{(x / 2)^{2 k+n}}{k!\Gamma(n+k+1)} \cos \frac{(3 n+2 k) \pi}{4}  \tag{G.75}\\
& \operatorname{Bei}_{n}(x)=\sum_{k=0}^{\infty} \frac{(x / 2)^{2 k+n}}{k!\Gamma(n+k+1)} \sin \frac{(3 n+2 k) \pi}{4} \tag{G.76}
\end{align*}
$$

## G.4.4 Ker and Kei Functions

The real and imaginary parts of $K_{n}\left(x e^{\frac{j \pi}{4}}\right)$ are denoted by $\operatorname{Ker}_{n}(x)$ and $K e i_{n}(x)$, respectively:

$$
\begin{array}{r}
\operatorname{Ker}_{n}(x)=-\{\ln (x / 2)+\gamma\} \operatorname{Ber}_{n}(x)+\frac{\pi}{4} B e i_{n}(x) \\
+\frac{1}{2} \sum_{k=0}^{n-1} \frac{(n-k-1)!(x / 2)^{2 k-n}}{k!} \cos \frac{(3 n+2 k) \pi}{4}+ \\
\frac{1}{2} \sum_{k=0}^{\infty} \frac{(x / 2)^{2 k+n}}{k!(n+k)!}\{\Phi(k)+\Phi(n+k)\} \cos \frac{(3 n+2 k) \pi}{4} \\
K e i_{n}(x)=-\{\ln (x / 2)+\gamma\} B e i_{n}(x)-\frac{\pi}{4} B e r_{n}(x) \\
\quad-\frac{1}{2} \sum_{k=0}^{n-1} \frac{(n-k-1)!(x / 2)^{2 k-n}}{k!} \sin \frac{(3 n+2 k) \pi}{4} \\
+\frac{1}{2} \sum_{k=0}^{\infty} \frac{(x / 2)^{2 k+n}}{k!(n+k)!}\{\Phi(k)+\Phi(n+k)\} \sin \frac{(3 n+2 k) \pi}{4} \tag{G.78}
\end{array}
$$

where $\gamma=.5772156 \cdots$ is Euler's constant and

$$
\begin{equation*}
\Phi(n)=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \quad \Phi(0)=0 \tag{G.79}
\end{equation*}
$$

## G. 5 Spherical Bessel Functions

When the coordinate system is spherical, we usually see special type of Bessel function. Spherical Bessel equation is:

$$
\begin{equation*}
z^{2} \psi^{\prime \prime}+2 z \psi^{\prime}+\left[z^{2}-n(n+1)\right] \psi=0 \tag{G.80}
\end{equation*}
$$

Solutions are:
a) Spherical Bessel function of first kind $j_{n}(z)$
b) Spherical Bessel function of second kind $y_{n}(z)$
c) Spherical Bessel function of third kind $h_{n}^{(1)}(z)$
d) Spherical Bessel function of forth kind $h_{n}^{(2)}(z)$

Spherical Bessel function is related to common Bessel function by:

$$
\begin{align*}
j_{n}(z) & =\sqrt{\frac{\pi}{2 z}} J_{n+\frac{1}{2}}(z) \\
j_{n}(-z) & =(-1)^{n} j_{n}(z) \\
j_{-n}(z) & =(-1)^{n} y_{n-1}(z), \quad n>0 \\
y_{n}(z) & =\sqrt{\frac{\pi}{2 z}} Y_{n+\frac{1}{2}}(z) \\
y_{n}(-z) & =(-1)^{n+1} y_{n}(z) \\
h_{n}^{(1)}(z) & =j_{n}(z)+j y_{n}(z) \\
h_{n}^{(2)}(z) & =j_{n}(z)-j y_{n}(z) \tag{G.81}
\end{align*}
$$

The order of these equations is given by value of $n$. The function $j_{n}(z), y_{n}(z)$ for $\mathrm{n}=0,1,2$ are:

$$
\begin{align*}
j_{0}(z)=\frac{\sin z}{z} ; \quad y_{0}(z) & =-\frac{\cos z}{z} \\
h_{0}^{(1)}(z)=\frac{e^{j z}}{j z} ; \quad h_{0}^{(2)}(z) & =\frac{e^{-j z}}{-j z}  \tag{G.82}\\
j_{1}(z)=\frac{\sin z}{z^{2}}-\frac{\cos z}{z} ; \quad y_{1}(z) & =-\frac{\cos z}{z^{2}}-\frac{\sin z}{z} \\
h_{1}^{(1)}(z)=-\frac{e^{j z}}{z}\left(1+\frac{j}{z}\right) ; \quad h_{1}^{(2)}(z) & =-\frac{e^{-j z}}{z}\left(1-\frac{j}{z}\right) \tag{G.83}
\end{align*}
$$

$$
\begin{align*}
& j_{2}(z)=\left(\frac{3}{z^{3}}-\frac{1}{z}\right) \sin z-\frac{3 \cos z}{z^{2}} ; \quad y_{2}(z)=\left(\frac{3}{z^{3}}-\frac{1}{z}\right) \cos z-\frac{3 \sin z}{z^{2}} \\
& h_{2}^{(1)}(z)=-\frac{j e^{j z}}{z}\left(1+\frac{3 j}{z}-\frac{3}{z^{2}}\right) ; \quad h_{2}^{(2)}(z)=\frac{j e^{-j z}}{z}\left(1-\frac{3 j}{z}-\frac{3}{z^{2}}\right)(\mathrm{G} .8 \tag{G.84}
\end{align*}
$$

## G.5.1 Small and Large Arguments Approximation

For $z \ll 1$ the $j_{n}(z)$ and $y_{n}(z)$ functions are given by

$$
\begin{gather*}
j_{n}(z) \approx \frac{z^{n}}{1 \cdot 3 \cdot 5 \cdots(2 n+1)}=\frac{2^{n} n!z^{n}}{(2 n+1)!}  \tag{G.85}\\
y_{n}(z) \approx \frac{1 \cdot 3 \cdot 5 \cdots(2 n-1)}{z^{n+1}} \tag{G.86}
\end{gather*}
$$

And for large arguments $z \gg 0$

$$
\begin{align*}
& j_{n}(z) \approx \frac{1}{z} \cos \left[z-\frac{\pi}{2}(n+1)\right]  \tag{G.87}\\
& y_{n}(z) \approx \frac{1}{z} \sin \left[z-\frac{\pi}{2}(n+1)\right] \tag{G.88}
\end{align*}
$$

## G.5.2 Recurrence Relations

If we denote $j_{n}(z), y(z), h_{n}^{(1)}(z), h_{n}^{(2)}(z)$ for $(n=0, \pm 1, \pm 2 \cdots)$ by $f_{n}(z)$, we can write:

$$
\begin{gather*}
f_{n-1}(z)+f_{n+1}(z)=(2 n+1) z^{-1} f_{n}(z)  \tag{G.89}\\
n f_{n-1}(z)-(n+1) f_{n+1}(z)=(2 n+1) \frac{d}{d z} f_{n}(z)  \tag{G.90}\\
\frac{n+1}{z} f_{n}(z)+\frac{d}{d z} f_{n}(z)=f_{n-1}(z)  \tag{G.91}\\
\frac{n}{z} f_{n}(z)-\frac{d}{d z} f_{n}(z)=f_{n+1}(z) \tag{G.92}
\end{gather*}
$$

## G.5.3 Cross Products

$$
\begin{gather*}
j_{n}(z) y_{n-1}(z)-j_{n-1}(z) y_{n}(z)=z^{-2}  \tag{G.93}\\
j_{n+1}(z) y_{n-1}(z)-j_{n-1}(z) y_{n+1}(z)=(2 n+1) z^{-3} \tag{G.94}
\end{gather*}
$$

Table G.3: Roots of Spherical Bessel Functions

| n | $\mathrm{m}=1$ | $\mathrm{~m}=2$ | $\mathrm{~m}=3$ | $\mathrm{~m}=4$ | $\mathrm{~m}=5$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | $\pi$ | $2 \pi$ | $3 \pi$ | $4 \pi$ | $5 \pi$ |
| 1 | 4.493 | 7.725 | 10.904 | 14.066 | 17.221 |
| 2 | 5.763 | 9.095 | 12.323 | 15.515 | 18.689 |
| 3 | 6.988 | 10.417 | 13.698 | 16.924 | 20.122 |

Table G.4: Roots of First Derivative of Spherical Bessel Functions

| n | $\mathrm{m}=1$ | $\mathrm{~m}=2$ | $\mathrm{~m}=3$ | $\mathrm{~m}=4$ | $\mathrm{~m}=5$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 4.493 | 7.725 | 10.904 | 14.066 |
| 1 | 2.081 | 5.940 | 9.205 | 12.404 | 15.579 |
| 2 | 3.342 | 7.290 | 10.613 | 13.846 | 17.043 |
| 3 | 4.514 | 8.583 | 11.972 | 15.244 | 18.468 |

## G.5.4 Roots of Spherical Bessel Function

Four roots of $j_{n}\left(z_{m}\right)=0$ and $j_{n}^{\prime}\left(z_{m}\right)=0$ are given in Table(G.3), and Table(G.4), respectively.

## G.5.5 Riccati Bessel Functions

Another set of spherical Bessel and Hankel functions which appears in solutions of EM problems are those denoted by $\hat{B_{n}}$ where $\hat{B}$ stands to represent $\hat{J}_{n}, \hat{Y}_{n}, \hat{H}_{n}^{(1)}$ and $\hat{H}_{n}^{(2)}$. These are related to the spherical Bessel and Hankel functions [denoted $b_{n}$ to represent $j_{n}, y_{n}, h_{n}^{(1)}$ and $h_{n}^{(2)}$ ] and regular Bessel and Hankel functions [denoted by $B_{\left(n+\frac{1}{2}\right)}$ to represent $J_{n+\frac{1}{2}}, Y_{n+\frac{1}{2}}, H_{n+\frac{1}{2}}^{(1)}$ and $H_{n+\frac{1}{2}}^{(2)}$ by

$$
\begin{equation*}
\hat{B}_{n}(z)=z b_{n}(z)=\sqrt{\frac{\pi z}{2}} B_{n+\frac{1}{2}}(z) \tag{G.95}
\end{equation*}
$$

The term $\frac{d}{d z}\left[z b_{n}(z)\right]$ is usually used, so by Riccati-Bessel

$$
\begin{equation*}
\hat{B_{n}^{\prime}}(z)=\frac{d}{d z}\left[z b_{n}(z)\right]=b_{n}(z)+z b_{n}^{\prime}(z) \tag{G.96}
\end{equation*}
$$

Table G.5: Roots of First Derivative of Riccati-Bessel Functions

| n | $\mathrm{m}=1$ | $\mathrm{~m}=2$ | $\mathrm{~m}=3$ | $\mathrm{~m}=4$ | $\mathrm{~m}=5$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 2.743 | 6.116 | 9.3166 | 12.485 | 15.643 |
| 2 | 3.870 | 7.443 | 10.713 | 13.920 | 17.102 |
| 3 | 4.973 | 8.721 | 12.063 | 15.313 | 18.524 |
| 4 | 6.061 | 9.967 | 13.380 | 16.674 | 19.915 |

## G. 6 Legendre Functions

The differential equation $(0 \leq x \leq 1)$ :

$$
\begin{equation*}
\left(1-x^{2}\right) \frac{d^{2} T}{d x^{2}}-2 x \frac{d T}{d x}+\left[n(n+1)-\frac{m^{2}}{1-x^{2}}\right] T(x)=0 \tag{G.97}
\end{equation*}
$$

[ m and n are integers]. When $\mathrm{m}=0$, there are two linearly independent solutions:
a) $P_{n}(x)$ Legendre function of first kind.
b) $Q_{n}(x)$ Legendre function of second kind.

If $n=0,1,2, \cdots$ solutions of Eq.(G.97) are Legendre Polynomials $P_{n}(x)$ given by Rodrigue's formula

$$
\begin{equation*}
P_{n}(x)=\frac{1}{2^{n} n!} \frac{d^{n}}{d x^{n}}\left(x^{2}-1\right)^{n} \tag{G.98}
\end{equation*}
$$

The first few of them have explicit form

$$
\begin{equation*}
P_{0}(x)=1 ; \quad P_{1}(x)=x ; \quad P_{2}(x)=\frac{3 x^{2}-1}{2} ; \quad P_{3}(x)=\frac{5 x^{3}-3 x}{2} \tag{G.99}
\end{equation*}
$$

and for second kind:

$$
\begin{align*}
Q_{0}(x) & =\frac{1}{2} \log \frac{1+x}{1-x} \\
Q_{1}(x) & =\frac{x}{2} \log \frac{1+x}{1-x}-1 \\
Q_{2}(x) & =\frac{3 x^{2}-1}{4} \log \frac{1+x}{1-x}-\frac{3 x}{2} \\
Q_{3}(x) & =\frac{5 x^{2}-3 x}{4} \log \frac{1+x}{1-x}-\frac{5}{2} x^{2}+\frac{2}{3} \tag{G.100}
\end{align*}
$$

The singularity occur at points $x \pm 1$

$$
\left.\left.\begin{array}{c}
P_{n}(1)=1 ; \quad P_{n}(-1)=(-1)^{n} \\
P_{n}(0)=\left\{\begin{array}{l}
0 \\
(-1)^{n / 2} \frac{1 \cdot 3 \cdot 5 \cdots(n-1)}{2 \cdot 4 \cdot 6 \cdots n} \quad n=\text { Odd }
\end{array}\right. \\
Q_{n}(0)=\left\{\begin{array}{l}
n=\text { Even } \\
(-1)^{(n+1) / 2} \frac{2 \cdot 4 \cdot 6 \cdots(n-1)}{1 \cdot 3 \cdot 5 \cdots n} n=\text { Oven }
\end{array}\right. \\
Q_{n}(1)=\infty ; \quad Q_{n}(\infty)=0
\end{array}\right\} \begin{array}{l}
P_{n}^{\prime}(0)=-(n+1) P_{n+1}(0) ; \quad P_{n}^{\prime}(1)=\frac{1}{2} n(n+1)
\end{array} \int_{0}^{\pi} P_{2 n}(\cos \theta) d \theta=\pi\left[\frac{(2 n)!}{\left(2^{n} n!\right)^{2}}\right]^{2}\right\}
$$

## G.6.1 Orthogonality and Expansion Series

$$
\begin{gather*}
\int_{-1}^{+1}\left[P_{n}(x)\right]^{2} d x=\frac{2}{2 n+1}  \tag{G.108}\\
\int_{-1}^{+1} P_{n}(x) P_{m}(x) d x=0 \quad \text { if } m \neq n  \tag{G.109}\\
f(x)=A_{0} P_{0}(x)+A_{1} P_{1}(x)+A_{2} P_{2}(x)+\cdots  \tag{G.110}\\
A k=\frac{2 k=1}{2} \int_{-1}^{+1} f(x) P_{k}(x) d x \tag{G.111}
\end{gather*}
$$

## G. 7 Recurrence Relations

$$
\begin{array}{r}
x P_{n}^{\prime}(x)-P_{n-1}^{\prime}(x)=n P_{n-1}(x) \\
\left(x^{2}-1\right) P_{n}^{\prime}(x)=n x P_{n-1}(x)-n P_{n-1}(x) \\
P_{n}^{\prime}(x)-x P_{n}^{\prime}(x)=(n+1) P_{n}(x) \\
(n+1) P_{n+1}(x)-(2 n+1) x P_{n}(x)+n P_{n-1}(x)=0 \\
P_{n+1}^{\prime}(x)-P_{n-1}^{\prime}(x)=(2 n+1) P_{n}(x) \tag{G.116}
\end{array}
$$

Formula Eq.(G.112) to Eq.(G.116) are also valid for $Q_{n}(x)$ and any linear combination of $P_{n}(x)$ and $Q_{n}(x)$.

## G. 8 Associated Legendre Functions

When m is different from zero in Eq.(G.97), the linearly independent solution are the Associated Legendre Functions of the first and the second kind.

$$
\begin{equation*}
\left(1-x^{2}\right) \frac{d^{2} T}{d x^{2}}-2 x \frac{d T}{d x}+\left[n(n+1)-\frac{m^{2}}{1-x^{2}}\right] T(x)=0 \tag{G.117}
\end{equation*}
$$

where m and n are integer, and $m \leq n$.

$$
\begin{gather*}
P_{n}^{m}(x)=0 \quad \text { if } \quad m>n  \tag{G.118}\\
P_{n}^{m}(x)=\left(1-x^{2}\right)^{m / 2} \frac{d^{m}}{d x^{m}} P_{n}(x) \quad m \leq n \tag{G.119}
\end{gather*}
$$

For example:
$P_{1}^{1}(x)=\left(1-x^{2}\right)^{\left(\frac{1}{2}\right)} ; \quad P_{2}^{1}(x)=3 x\left(1-x^{2}\right)^{\left(\frac{1}{2}\right)} ;$
$P_{2}^{2}(x)=3\left(1-x^{2}\right) ; \quad P_{3}^{1}(x)=\frac{3}{2}\left(5 x^{2}-1\right)\left(1-x^{2}\right)^{\left(\frac{1}{2}\right)} ;$
$P_{3}^{2}(x)=15 x\left(1-x^{2}\right) ; \quad P_{3}^{3}(x)=15\left(1-x^{2}\right)^{\left(\frac{3}{2}\right)}$
Associate Legendre Functions of second kind:

$$
\begin{equation*}
Q_{n}^{m}(x)=\left(1-x^{2}\right)^{m / 2} \frac{d^{m}}{d x^{m}} Q_{n}(x) \tag{G.120}
\end{equation*}
$$

For example:
$Q_{1}^{1}(x)=\left(x^{2}-1\right)^{\frac{1}{2}}\left(\frac{x}{x^{2}-1}-\frac{1}{2} \log \frac{x+1}{x-1}\right) ;$
$Q_{2}^{1}(x)=\left(x^{2}-1\right)^{\frac{1}{2}}\left(\frac{3 x^{2}-2}{x^{2}-1}-\frac{3}{2} x \log \frac{x+1}{x-1}\right) ;$
$Q_{2}^{2}(x)=\frac{3}{2}\left(x^{2}-1\right) \log \frac{x+1}{x-1}-\frac{3 x^{2}-5 x}{x^{2}-1}$

## G.8.1 Recurrence Relations

$$
\begin{gather*}
P_{n-1}^{m+1}(x)=x P_{n}^{m+1}(x)-(n-m)\left(1-x^{2}\right)^{\frac{1}{2}} P_{n}^{m}(x)  \tag{G.121}\\
P_{n+1}^{m+1}(x)=x P_{n}^{m+1}(x)+(n+m+1)\left(1-x^{2}\right)^{\frac{1}{2}} P_{n}^{m}(x)  \tag{G.122}\\
P_{n}^{m+2}(x)-\frac{2(m+n) x}{\left(1-x^{2}\right)^{\frac{1}{2}}} P_{n}^{m+1}(x)-(n-m)(n+m+1) P_{n}^{m}(x)=0  \tag{G.123}\\
(n+1-m) P_{n+1}^{m}(x)-(2 n+1) P_{n}^{m}(x)+(n+m) P_{n-1}^{m}(x)=0 \tag{G.124}
\end{gather*}
$$

Relations Eq.(G.121) to Eq.(G.124) are also valid for $Q_{n}^{m}(x)$ and any linear combination of $P_{n}^{m}(x)$ and $Q_{n}^{m}(x)$.

## G.8.2 Derivative of Associated Legendre Functions

$$
\begin{equation*}
\frac{d}{d x} P_{n}^{m}(x)=\frac{(n+m) P_{n-1}^{m}(x)-n x P_{n}^{m}(x)}{1-x^{2}} \tag{G.125}
\end{equation*}
$$

let $x=\cos \theta$, therefore

$$
\begin{equation*}
\frac{d}{d \theta} P_{n}^{m}(x)=\frac{1}{2}\left[(n+m)(n-m+1) P_{n}^{m-1}(x)-P_{n}^{m+1}(x)\right] \tag{G.126}
\end{equation*}
$$

## G.8.3 Special Values

$$
\begin{gather*}
\frac{d}{d \theta} P_{n}^{m}(x)=-\left(1-x^{2}\right)^{1 / 2} \frac{d}{d x} P_{n}^{m}(x)  \tag{G.127}\\
\frac{m P_{n}^{m}(\cos \theta)}{\sin \theta}= \begin{cases}0 & m=0 \\
\frac{1}{2} \cos \theta\left\{(n-m+1)(n+m) P_{n}^{m-1}(\cos \theta)+\right. \\
\left.P_{n}^{m-1}(\cos \theta)\right\}+m \sin \theta P_{n}^{m}(\cos \theta) & m>0\end{cases} \tag{G.128}
\end{gather*}
$$

$$
\begin{gather*}
\frac{d P_{n}^{m}(\cos \theta)}{d \theta}= \begin{cases}-P_{n}^{1}(\cos \theta) & m=0 \\
\frac{1}{2}\left\{(n-m+1)(n+m) P_{n}^{m-1}(\cos \theta)-\right. & m>0 \\
\left.P_{n}^{m+1}(\cos \theta)\right\} & m \neq 1\end{cases}  \tag{G.129}\\
\lim _{\theta \rightarrow 0} \frac{m P_{n}^{m}(\cos \theta)}{\sin \theta}= \begin{cases}n(n+1) / 2 & m=1 \\
0 & m \neq 1\end{cases}  \tag{G.130}\\
\lim _{\theta \rightarrow \pi} \frac{m P_{n}^{m}(\cos \theta)}{\sin \theta}= \begin{cases}(-1)^{n+1} n(n+1) / 2 & m=1 \\
0 & m \neq 1\end{cases}  \tag{G.131}\\
\lim _{\theta \rightarrow 0} \frac{d}{d \theta} P_{n}^{m}(\cos \theta)= \begin{cases}n(n+1) / 2 & m=1 \\
0 & m \neq 1\end{cases}  \tag{G.132}\\
\lim _{\theta \rightarrow \pi} \frac{d}{d \theta} P_{n}^{m}(\cos \theta)= \begin{cases}(-1)^{n+1} n(n+1) / 2 & m=1 \\
0 & m \neq 1\end{cases} \tag{G.133}
\end{gather*}
$$

## G.8.4 Generating Function for $P_{n}^{m}(x)$

$$
\begin{equation*}
\frac{(2 m)!\left(1-x^{2}\right)^{m / 2} t^{m}}{\left.2^{m} m!\left(1-2 t x+t^{2}\right)^{m+\frac{1}{2}}\right)}=\sum_{n=m}^{\infty} P_{n}^{m}(x) t^{n} \tag{G.134}
\end{equation*}
$$

## G.8.5 Orthogonality of $P_{n}^{m}(x)$ and Expansion Series

$$
\begin{array}{r}
\int_{-1}^{+1} P_{n}^{m}(x) P_{l}^{m}(x) d x=0 \quad \text { if } n \neq l \\
\int_{-1}^{+1}\left[P_{n}^{m}(x)\right]^{2} d x=\frac{2}{2 n+1} \frac{(n+m)!}{(n-m)!} \\
f(x)=A_{m} P_{m}^{m}(x)+A_{m+1} P_{m+1}^{m}+A_{m+2} P_{m+2}^{m}+\cdots \tag{G.137}
\end{array}
$$

where

$$
\begin{equation*}
A_{k}=\frac{2 k+1}{2} \frac{(k-m)!}{(k+m)!} \int_{-1}^{+1} f(x) P_{k}^{m}(x) d x \tag{G.138}
\end{equation*}
$$

## G. 9 Parabolic Cylinder Functions

In mathematics, the parabolic cylinder functions are special functions defined as solutions to the differential equation

$$
\begin{equation*}
\frac{\partial^{2} \psi}{\partial z^{2}}+\left(a z^{2}+b z+c\right) \psi=0 \tag{G.139}
\end{equation*}
$$

This equation is found, for example, when the technique of separation of variables is used on differential equations which are expressed in parabolic cylindrical coordinates.
The above equation may be brought into two distinct forms Eq.(G.140) and Eq.(G.141) by complete the square and rescaling $z$, called H. F. Weber's equations (Weber 1869):

$$
\begin{equation*}
\frac{\partial^{2} \psi}{\partial z^{2}}-\left(\frac{z^{2}}{4}+a\right) \psi=0 \tag{G.140}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial^{2} \psi}{\partial z^{2}}+\left(\frac{z^{2}}{4}-a\right) \psi=0 \tag{G.141}
\end{equation*}
$$

If $\psi(a, z)$ is a solution, then so are $\psi(a,-z), \psi(-a, j z)$ and $\psi(-a,-j z)$. If $\psi(a, z)$ is the solution of Eq.(G.140), then $\psi\left(-j a, z e^{j \frac{\pi}{4}}\right)$ is the solution of Eq.(G.141) and by symmetry $\psi\left(-j a,-z e^{j \frac{\pi}{4}}\right), \psi\left(j a,-z e^{-j \frac{\pi}{4}}\right), \psi\left(j a, z e^{-j \frac{\pi}{4}}\right)$ are the solution of Eq.(G.141). There are independent even and odd solutions of the form Eq.(G.140). These are given by (following the notation of Abramowitz and Stegun):

$$
\begin{align*}
& y_{1}(a, z)=e^{-z^{2} / 4} M\left(\frac{1}{2} a+\frac{1}{4}, \frac{1}{2}, \frac{1}{2} z^{2}\right)  \tag{G.142}\\
& y_{2}(a, z)=z e^{-z^{2} / 4} M\left(\frac{1}{2} a+\frac{3}{4}, \frac{3}{2}, \frac{1}{2} z^{2}\right)
\end{align*}
$$

where $M(a, b, z)$ is the confluent hypergeometric function.

## G.9.1 Parabolic Cylinder Functions $D_{\nu}(z)$ or $U(a, z)$

The first solution of Eq.(G.140) is defined by

$$
\begin{equation*}
D_{\nu}(z)=U(a, z)=\cos \frac{\pi \nu}{2} w_{1}(z)+\sin \frac{\pi \nu}{2} w_{2}(z) \tag{G.143}
\end{equation*}
$$

where $\nu=-\left(a+\frac{1}{2}\right)$ and

$$
\begin{align*}
& w_{1}(z)=\frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2} \nu\right)}{2^{-\nu / 2}} y_{1}(z)  \tag{G.144}\\
& w_{2}(z)=\frac{1}{\sqrt{\pi}} \frac{\Gamma\left(1+\frac{1}{2} \nu\right)}{2^{-\nu / 2-1 / 2}} y_{2}(z)
\end{align*}
$$

where $y_{1}(z)$ and $y_{2}(z)$ are given by Eq.(G.142).

$$
\begin{equation*}
D_{-n-1}(z)=j^{n} \sqrt{2^{n-1} \pi} e^{\frac{z^{2}}{4}} \operatorname{erfc}\left(\frac{z}{\sqrt{2}}\right) \tag{G.145}
\end{equation*}
$$

## G.9.2 Parabolic Cylinder Functions $V_{\nu}(z)$ or $V(a, z)$

The second solution of Eq.(G.140) is defined by

$$
\begin{equation*}
V_{\nu}(z)=V(a, z)=\frac{1}{\Gamma(1+\nu)}\left[\cos \frac{\pi \nu}{2} w_{2}(z)-\sin \frac{\pi \nu}{2} w_{1}(z)\right] \tag{G.146}
\end{equation*}
$$

where $w_{1}(z)$ and $w_{2}(z)$ are given by Eq.(G.144).

## G.9.3 Recurrence Relations

$$
\begin{align*}
U^{\prime}(a, x)+\frac{1}{2} x U(a, x)+\left(a+\frac{1}{2}\right) U(a+1, x) & =0  \tag{G.147}\\
U^{\prime}(a, x)-\frac{1}{2} x U(a, x)+U(a-1, x) & =0  \tag{G.148}\\
2 U^{\prime}(a, x)+U(a-1, x)+\left(a+\frac{1}{2}\right) U(a+1, x) & =0  \tag{G.149}\\
V^{\prime}(a, x)-\frac{1}{2} x V(a, x)-\left(a-\frac{1}{2}\right) V(a-1, x) & =0  \tag{G.150}\\
V^{\prime}(a, x)+\frac{1}{2} x V(a, x)-V(a+1, x) & =0  \tag{G.151}\\
2 V^{\prime}(a, x)-V(a+1, x)+\left(a-\frac{1}{2}\right) V(a-1, x) & =0 \tag{G.152}
\end{align*}
$$

## G. 10 Hermite Polynomial

In the Sturm-Liouville Boundary Value Problem, there is a special case called Hermite's Differential Equation which arises in the treatment of the harmonic oscillator in quantum mechanics. Hermite's Differential Equation is defined as:

$$
\begin{equation*}
y^{\prime \prime}-2 x y^{\prime}+2 n y=0 \tag{G.153}
\end{equation*}
$$

The Hermite Polynomials can be expressed by Rodrigues' formula

$$
\begin{equation*}
H_{n}(x)=(-1)^{n} e^{x^{2}} \frac{d^{n}}{d x^{n}}\left(e^{-x^{2}}\right) \quad n=0,1,2,3, \ldots \tag{G.154}
\end{equation*}
$$

$H_{0}(x)=1$, $H_{1}(x)=2 x$, $H_{2}(x)=4 x^{2}-2$,
$H_{3}(x)=8 x^{3}-12 x$, $H_{4}(x)=16 x^{4}-48 x^{2}+12$, $H_{5}(x)=32 x^{5}-160 x^{3}+120 x$,
The Hermite polynomials $H_{n}(x)$ are set of orthogonal polynomials over the domain with weighting function $e^{-x^{2}}$ It can be shown that:

$$
\begin{equation*}
\int_{-\infty}^{\infty} e^{-x^{2}} H_{m}(x) H_{n}(x) d x=2^{n} n!\sqrt{\pi} \delta_{m n} \tag{G.155}
\end{equation*}
$$

By using this orthogonality, a piecewise continuous function $f(x)$ can be expressed in terms of Hermite Polynomials:

$$
\begin{equation*}
f(x)=\sum_{n=0}^{\infty} C_{n} H_{n}(x) \tag{G.156}
\end{equation*}
$$

where

$$
\begin{equation*}
C_{n}=\frac{1}{2^{n} n!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^{2}} f(x) H_{n}(x) d x \tag{G.157}
\end{equation*}
$$

Whether a Hermite Polynomial is an even or odd function depends on its degree $n$.

$$
\begin{equation*}
H_{n}(-x)=(-1)^{n} H_{n}(x) \tag{G.158}
\end{equation*}
$$

$H_{n}(x)$ is an even function, when $n$ is even $H_{n}(x)$ is an odd function, when $n$ is odd

## G.10.1 Recurrence Relations

$$
\begin{equation*}
H_{n+1}(x)=2 x H_{n}(x)-2 n H_{n-1}(x) \tag{G.159}
\end{equation*}
$$

$$
\begin{equation*}
H_{n}^{\prime}(x)=2 n H_{n-1}(x) \tag{G.160}
\end{equation*}
$$

## G.10.2 Special Integrals

let us define:

$$
\begin{equation*}
I_{m n}(a, b)=\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^{2}} H_{m}(a x) H_{n}(b x) d x \tag{G.161}
\end{equation*}
$$

If $m+n$ is odd the value of this integral is $I_{m n}(a, b)=0$. If $m+n$ is even the value of this integral is function of $a$ and $b$.

$$
\begin{align*}
& I_{0,0}(a, b)=1 \\
& I_{0,2}(a, b)=2\left(b^{2}-1\right) \\
& I_{0,4}(a, b)=12\left(b^{2}-1\right)^{2} \\
& I_{0,6}(a, b)=120\left(b^{2}-1\right)^{3} \\
& I_{0,8}(a, b)=1680\left(b^{2}-1\right)^{4} \\
& I_{0,10}(a, b)=30240\left(b^{2}-1\right)^{5} \\
& I_{1,1}(a, b)=2 a b \\
& I_{1,3}(a, b)=12 a b\left(b^{2}-1\right) \\
& I_{1,5}(a, b)=120 a b\left(b^{2}-1\right)^{2} \\
& I_{1,7}(a, b)=1680 a b\left(b^{2}-1\right)^{3} \\
& I_{1,9}(a, b)=30240 a b\left(b^{2}-1\right)^{4} \\
& I_{2,2}(a, b)=4\left(1-a^{2}-b^{2}+3 a^{2} b^{2}\right) \\
& I_{2,4}(a, b)=24\left(b^{2}-1\right)\left(1-a^{2}-b^{2}+5 a^{2} b^{2}\right) \\
& I_{2,6}(a, b)=240\left(b^{2}-1\right)^{2}\left(1-a^{2}-b^{2}+7 a^{2} b^{2}\right) \\
& I_{2,8}(a, b)=3360\left(b^{2}-1\right)^{3}\left(1-a^{2}-b^{2}+9 a^{2} b^{2}\right) \\
& I_{2,10}(a, b)=60480\left(b^{2}-1\right)^{4}\left(1-a^{2}-b^{2}+11 a^{2} b^{2}\right) \\
& I_{3,3}(a, b)=24 a b\left(3-3 a^{2}-3 b^{2}+5 a^{2} b^{2}\right) \\
& I_{3,5}(a, b)=240 a b\left(b^{2}-1\right)\left(3-3 a^{2}-3 b^{2}+7 a^{2} b^{2}\right) \\
& I_{3,7}(a, b)=10080 a b\left(b^{2}-1\right)^{2}\left(1-a^{2}-b^{2}+3 a^{2} b^{2}\right) \\
& I_{3,9}(a, b)=60480 a b\left(b^{2}-1\right)^{3}\left(3-3 a^{2}-3 b^{2}+11 a^{2} b^{2}\right) \\
& I_{4,4}(a, b)=48\left(3-6 a^{2}+3 a^{4}-6 b^{2}+36 a^{2} b^{2}-30 a^{4} b^{2}+3 b^{4}-30 a^{2} b^{4}+35 a^{4} b^{4}\right) \\
& I_{4,6}(a, b)=1440\left(b^{2}-1\right)\left(1-2 a^{2}+a^{4}-2 b^{2}+16 a^{2} b^{2}-14 a^{4} b^{2}+b^{4}-\right. \\
& \left.14 a^{2} b^{4}+21 a^{4} b^{4}\right) \\
& I_{4,8}(a, b)=20160\left(b^{2}-1\right)^{2}\left(1-2 a^{2}+a^{4}-2 b^{2}+20 a^{2} b^{2}-18 a^{4} b^{2}+b^{4}-\right. \\
& \left.18 a^{2} b^{4}+33 a^{4} b^{4}\right) \\
& I_{4,10}(a, b)=120960\left(b^{2}-1\right)^{3}\left(3-6 a^{2}+3 a^{4}-6 b^{2}+72 a^{2} b^{2}-66 a^{4} b^{2}+3 b^{4}-\right. \\
& \left.66 a^{2} b^{4}+143 a^{4} b^{4}\right) \tag{G.162}
\end{align*}
$$

$$
\begin{align*}
& I_{5,5}(a, b)=480 a b\left(15-30 a^{2}+15 a^{4}-30 b^{2}+100 a^{2} b^{2}-\right. \\
& \left.70 a^{4} b^{2}+15 b^{4}-70 a^{2} b^{4}+63 a^{4} b^{4}\right) \\
& I_{5,7}(a, b)=20160 a b\left(b^{2}-1\right)\left(5-10 a^{2}+5 a^{4}-10 b^{2}+\right. \\
& \left.40 a^{2} b^{2}-30 a^{4} b^{2}+5 b^{4}-30 a^{2} b^{4}+33 a^{4} b^{4}\right) \\
& I_{5,9}(a, b)=120960 a b\left(b^{2}-1\right)^{2}\left(15-30 a^{2}+15 a^{4}-30 b^{2}+\right. \\
& \left.140 a^{2} b^{2}-110 a^{4} b^{2}+15 b^{4}-110 a^{2} b^{4}+143 a^{4} b^{4}\right) \\
& I_{6,6}(a, b)=2880\left(5-15 a^{2}+15 a^{4}-5 a^{6}-15 b^{2}+135 a^{2} b^{2}-\right. \\
& 225 a^{4} b^{2}+105 a^{6} b^{2}+15 b^{4}-225 a^{2} b^{4}+525 a^{4} b^{4}-  \tag{G.163}\\
& \left.315 a^{6} b^{4}-5 b^{6}+105 a^{2} b^{6}-315 a^{4} b^{6}+231 a^{6} b^{6}\right) \\
& I_{6,8}(a, b)=40320\left(b^{2}-1\right)\left(5-15 a^{2}+15 a^{4}-5 a^{6}-15 b^{2}+\right. \\
& 165 a^{2} b^{2}-285 a^{4} b^{2}+135 a^{6} b^{2}+15 b^{4}-285 a^{2} b^{4}+ \\
& \left.765 a^{4} b^{4}-495 a^{6} b^{4}-5 b^{6}+135 a^{2} b^{6}-495 a^{4} b^{6}+429 a^{6} b^{6}\right) \\
& I_{6,10}(a, b)=3628800\left(b^{2}-1\right)^{2}\left(1-3 a^{2}+3 a^{4}-a^{6}-3 b^{2}+\right. \\
& 39 a^{2} b^{2}-69 a^{4} b^{2}+33 a^{6} b^{2}+3 b^{4}-69 a^{2} b^{4}+ \\
& \left.209 a^{4} b^{4}-143 a^{6} b^{4}-b^{6}+33 a^{2} b^{6}-143 a^{4} b^{6}+143 a^{6} b^{6}\right)
\end{align*}
$$

$$
\begin{aligned}
& I_{7,7}(a, b)=40320 a b\left(35-105 a^{2}+105 a^{4}-35 a^{6}-105 b^{2}+525 a^{2} b^{2}-\right. \\
& 735 a^{4} b^{2}+315 a^{6} b^{2}+105 b^{4}-735 a^{2} b^{4}+1323 a^{4} b^{4}- \\
& \left.693 a^{6} b^{4}-35 b^{6}+315 a^{2} b^{6}-693 a^{4} b^{6}+429 a^{6} b^{6}\right) \\
& I_{7,9}(a, b)=725760 a b\left(b^{2}-1\right)\left(35-105 a^{2}+105 a^{4}-35 a^{6}-\right. \\
& 105 b^{2}+595 a^{2} b^{2}-875 a^{4} b^{2}+385 a^{6} b^{2}+105 b^{4}-875 a^{2} b^{4}+1771 a^{4} b^{4} \\
& \left.-1001 a^{6} b^{4}-35 b^{6}+385 a^{2} b^{6}-1001 a^{4} b^{6}+715 a^{6} b^{6}\right) \\
& I_{8,8}(a, b)=80640\left(35-140 a^{2}+210 a^{4}-140 a^{6}+35 a^{8}-140 b^{2}+\right. \\
& 1680 a^{2} b^{2}-4200 a^{4} b^{2}+3920 a^{6} b^{2}-1260 a^{8} b^{2}+210 b^{4}-4200 a^{2} b^{4}+ \\
& 14700 a^{4} b^{4}-17640 a^{6} b^{4}+6930 a^{8} b^{4}-140 b^{6}+3920 a^{2} b^{6}-17640 a^{4} b^{6}+ \\
& \left.25872 a^{6} b^{6}-12012 a^{8} b^{6}+35 b^{8}-1260 a^{2} b^{8}+6930 a^{4} b^{8}-12012 a^{6} b^{8}+6435 a^{8} b^{8}\right) \\
& I_{8,10}(a, b)=7257600\left(b^{2}-1\right)\left(7-28 a^{2}+42 a^{4}-28 a^{6}+7 a^{8}-\right. \\
& 28 b^{2}+392 a^{2} b^{2}-1008 a^{4} b^{2}+952 a^{6} b^{2}-308 a^{8} b^{2}+ \\
& 42 b^{4}-1008 a^{2} b^{4}+3892 a^{4} b^{4}-4928 a^{6} b^{4}+ \\
& 2002 a^{8} b^{4}-28 b^{6}+952 a^{2} b^{6}-4928 a^{4} b^{6}+ \\
& 8008 a^{6} b^{6}-4004 a^{8} b^{6}+7 b^{8}-308 a^{2} b^{8}+2002 a^{4} b^{8}- \\
& \left.4004 a^{6} b^{8}+2431 a^{8} b^{8}\right)
\end{aligned}
$$

$$
I_{9,9}(a, b)=1451520 a b\left(315-1260 a^{2}+1890 a^{4}-1260 a^{6}+315 a^{8}-\right.
$$

$$
1260 b^{2}+8400 a^{2} b^{2}-17640 a^{4} b^{2}+15120 a^{6} b^{2}-
$$

$$
4620 a^{8} b^{2}+1890 b^{4}-17640 a^{2} b^{4}+47628 a^{4} b^{4}-
$$

$$
49896 a^{6} b^{4}+18018 a^{8} b^{4}-1260 b^{6}+15120 a^{2} b^{6}-
$$

$$
49896 a^{4} b^{6}+61776 a^{6} b^{6}-25740 a^{8} b^{6}+315 b^{8}-
$$

$$
\left.4620 a^{2} b^{8}+18018 a^{4} b^{8}-25740 a^{6} b^{8}+12155 a^{8} b^{8}\right)
$$

$$
I_{10,10}(a, b)=14515200\left(63-315 a^{2}+630 a^{4}-630 a^{6}+315 a^{8}-63 a^{10}-315 b^{2}+\right.
$$

$$
4725 a^{2} b^{2}-15750 a^{4} b^{2}+22050 a^{6} b^{2}-14175 a^{8} b^{2}+3465 a^{10} b^{2}+630 b^{4}-15750 a^{2} b^{4}+
$$

$$
73500 a^{4} b^{4}-132300 a^{6} b^{4}+103950 a^{8} b^{4}-30030 a^{10} b^{4}-630 b^{6}+22050 a^{2} b^{6}-
$$

$$
132300 a^{4} b^{6}+291060 a^{6} b^{6}-270270 a^{8} b^{6}+90090 a^{10} b^{6}+315 b^{8}-14175 a^{2} b^{8}+
$$

$$
103950 a^{4} b^{8}-270270 a^{6} b^{8}+289575 a^{8} b^{8}-109395 a^{10} b^{8}-63 b^{10}+3465 a^{2} b^{10}-
$$

$$
\begin{equation*}
\left.30030 a^{4} b^{10}+90090 a^{6} b^{10}-109395 a^{8} b^{10}+46189 a^{10} b^{10}\right) \tag{G.164}
\end{equation*}
$$

## G. 11 Struve Functions

In mathematics, Struve functions $H_{\nu}(z)$, are solutions $y(x)$ of the non-homogenous Bessel's differential equation:

$$
\begin{equation*}
z^{2} \frac{d^{2} y}{d x^{2}}+z \frac{d y}{d z}+\left(z^{2}-\nu^{2}\right) y=\frac{4(z / 2)^{\nu+1}}{\sqrt{\pi} \Gamma\left(\nu+\frac{1}{2}\right)} \tag{G.165}
\end{equation*}
$$

introduced by Hermann Struve (1882). The complex number $\nu$ is the order of the Struve function, and is often an integer. The modified Struve functions $L_{\nu}(z)$ are equal to $-j e^{-j \nu \pi / 2} H_{\nu}(j z)$.
Since this is a non-homogenous equation, solutions can be constructed from a single particular solution by adding the solutions of the homogeneous problem. In this case, the homogenous solutions are the Bessel functions, and the particular solution may be chosen as the corresponding Struve function.

## G.11.1 Power series expansion

Struve functions, denoted as $H_{\nu}(z)$ have the following power series form:

$$
\begin{equation*}
H_{\nu}(z)=\sum_{k=0}^{\infty} \frac{(-1)^{k}}{\Gamma\left(k+\frac{3}{2}\right) \Gamma\left(k+\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{2 k+\nu+1} \tag{G.166}
\end{equation*}
$$

where $\Gamma(z)$ is the gamma function.

## G.11.2 Recurrence Relations

$$
\begin{gather*}
H_{\nu-1}(z)+H_{\nu+1}(z)=\frac{2 \nu}{z} H_{\nu}(z)+\frac{(z / 2)^{\nu}}{\sqrt{\pi} \Gamma\left(\nu+\frac{3}{2}\right)}  \tag{G.167}\\
H_{\nu-1}(z)-H_{\nu+1}(z)=2 H_{\nu}^{\prime}(z)-\frac{(z / 2)^{\nu}}{\sqrt{\pi} \Gamma\left(\nu+\frac{3}{2}\right)}  \tag{G.168}\\
H_{0}^{\prime}(z)=2 / \pi-H_{1}(z)  \tag{G.169}\\
\frac{d}{d z}\left(z^{\nu} H_{\nu}\right)=z^{\nu} H_{\nu-1}  \tag{G.170}\\
\frac{d}{d z}\left(z^{-\nu} H_{\nu}\right)=\frac{1}{\sqrt{\pi} 2^{\nu} \Gamma\left(\nu+\frac{3}{2}\right)}-z^{-\nu} H_{\nu+1} \tag{G.171}
\end{gather*}
$$

## G.11.3 Special Properties

$$
\begin{gather*}
H_{\nu}(x) \geq 0 \quad\left(x>0 \text { and } \nu \geq \frac{1}{2}\right)  \tag{G.172}\\
H_{-\left(n+\frac{1}{2}\right)}(z)=(-1)^{n} J_{n+\frac{1}{2}}(z) \quad(n \text { and integer } \geq 0)  \tag{G.173}\\
H_{\frac{1}{2}}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}(1-\cos z)  \tag{G.174}\\
H_{\frac{3}{2}}(z)=\left(\frac{z}{2 \pi}\right)^{\frac{1}{2}}\left(1+\frac{2}{z^{2}}\right)-\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left(\sin z+\frac{\cos z}{z}\right)  \tag{G.175}\\
H_{0}(z)=\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{J_{2 k+1}(z)}{2 k+1}  \tag{G.176}\\
H_{1}(z)=\frac{2}{\pi}-\frac{2}{\pi} J_{0}(z)+\frac{4}{\pi} \sum_{k=1}^{\infty} \frac{J_{2 k}(z)}{4 k^{2}-1} \tag{G.177}
\end{gather*}
$$

## Appendix H

## Numerical Computations

## H. 1 Numerical Integrals Formula

Transformation

$$
\begin{equation*}
\int_{a}^{b} f(x) d x=\frac{1}{2}(b-a) \int_{-1}^{1} f\left(\frac{b-a}{2} t+\frac{b+a}{2}\right) d t \tag{H.1}
\end{equation*}
$$

Trapezoidal Rule

$$
\begin{gather*}
h=(b-a) / n \text { with truncation error: } R_{n}=-\frac{n h^{3} f^{\prime \prime}(\xi)}{12} \quad a \leq \xi \leq b \\
\int_{a}^{b} f(x) d x=h\left(\frac{1}{2} f_{0}+f_{1}+f_{2}+\cdots+f_{n-1}+\frac{1}{2} f_{n}\right) \tag{H.2}
\end{gather*}
$$

## Simpson's Rule

with truncation error: $R_{n}=-\frac{n h^{5} f^{\prime \prime}(\xi)}{180} \quad a \leq \xi \leq b$

$$
\begin{equation*}
\int_{a}^{b} f(x) d x=\frac{h}{3}\left(f_{0}+4 f_{1}+2 f_{2}+4 f_{3}+2 f_{4}+\cdots+2 f_{n-2}+4 f_{n-1}+f_{n}\right) \tag{H.3}
\end{equation*}
$$

Gauss-Legendre Formula The Gauss-Legendre integration formula is the most commonly used form of Gaussian quadratures. It is based on the Legendre polynomials of the first kind $P_{n}(x)$.

$$
\begin{gather*}
\int_{a}^{b} f(x) d x=\frac{1}{2}(b-a)\left[\alpha_{1} f\left(x_{1}\right)+\alpha_{2} f\left(x_{2}\right)+\cdots+\alpha_{r} f\left(x_{r}\right)\right]  \tag{H.4}\\
\alpha_{k}=\frac{2\left(1-u_{k}^{2}\right)}{(r+1)^{2} P_{r+1}^{2}\left(u_{k}\right)}
\end{gather*}
$$

where $k=1,2, \cdots, r$, and $P_{r}(u)=$ Legendre Polynomial of order $r$ and $u_{1}, u_{2}, \cdots, u_{r}$ are zeros of $P_{r}(u)$
Gauss-Chebyshev Formula The Gauss-Chebyshev formula based on the Chebyshev polynomials of the first kind $T_{n}(x)$ has more handy integration points and weights. But more variation of parameters may be needed.

$$
\begin{equation*}
\int_{a}^{b} f(x) d x=\int_{-1}^{1} \frac{g(\zeta)}{\sqrt{1-\zeta^{2}}} d \zeta=\frac{b-a}{2} \sum_{k=1}^{n} w\left(\zeta_{k}\right) \sqrt{1-\zeta_{k}^{2}} f\left(\frac{b-a}{2} \zeta_{k}+\frac{b+a}{2}\right)+R_{n}(\zeta) \tag{H.5}
\end{equation*}
$$

where

$$
\begin{aligned}
& \zeta=\frac{2 x-b-a}{b-a}, \Rightarrow x=\frac{b-a}{2} \zeta+\frac{b+a}{2},-1<\zeta<1, \\
& \zeta_{k}=\cos \frac{(2 n-1) \pi}{2 n}, \\
& w\left(\zeta_{k}\right)=\frac{\pi}{n}, \\
& g(\zeta)=\frac{b-a}{2} \sqrt{1-\zeta^{2}} f\left(\frac{b-a}{2} \zeta+\frac{b+a}{2}\right), \\
& R_{n}(\zeta)=\frac{\pi}{2^{n-1}(2 n)!} g^{2 n}(\zeta)
\end{aligned}
$$

## Appendix I

## Table of Some Important Constants

Table I.1: Relative Permittivity $\epsilon_{r}$ of Some Dielectrics

| Material | $\epsilon_{r}$ | Material | $\epsilon_{r}$ |
| :---: | :---: | :---: | :---: |
| Vacuum | 1.0000 | Nylon | 3.5 |
| Beewax | 2.35 | Barium tetratitanate | 37 |
| Transformer oil | 2.24 | Glass | $3.8-6.8$ |
| Mica | 6.0 | Rubber | $2.3-4$ |
| Lucite | 3.0 | Parafin | 2.24 |
| Silicon | 11.9 | Porcelains | $5-10$ |
| Germanium | 16. | Sea Water | 72 |
| Gallium Arsenide | 13.1 | Distilled water at $20^{\circ} \mathrm{C}$ | 80.0 |
| Etanol | 24.3 | Benzene | 2.28 |
| Metanol | 32.6 | Polyetylene | 2.24 |
| Neoprene | 6.7 | Teflon | 2.08 |
| Plexiglass | 2.60 | Titania | 96 |
| Paper | $2-4$ | Dry Soil | $3-4$ |
| SiO2 | 3.9 | Bakelite | 5.0 |

Table I.2: Physical Constants

| Permittivity of free space | $\epsilon_{0}=8.8541878 \times 10^{-12} \approx \frac{10^{-9}}{36 \pi}$ farad $/ \mathrm{meter}$ |
| :---: | :---: |
| Permeability of free space | $\mu_{0}=4 \pi \times 10^{-7} \mathrm{henry} / \mathrm{meter}$ |
| Velocity of light in vacuum | $c_{0}=2.99792458 \times 10^{8} \approx 3 \times 10^{8} \mathrm{~m} / \mathrm{s}$ |
| Charge of electron | $e=1.6021892 \times 10^{-19} \mathrm{C}$ |
| Mass of electron | $m_{e}=9.109534 \times 10^{-31} \mathrm{~kg}$ |
| Mass of proton | $m_{p}=1.672648 \times 10^{-27} \mathrm{~kg}$ |
| Boltzmann's constant | $k=1.380662 \times 10^{-23} \mathrm{~J} /{ }^{o} \mathrm{~K}$ |
| Plank's constant | $\hbar=1.054 \times 10^{-34} \mathrm{~J}-s$ |
| Gyromagnetic ratio | $\gamma=1.759 \times 10^{11} \mathrm{C} / \mathrm{Kg} \quad$ for $\left.\mathrm{g}=2\right)$ |

Table I.3: Conversion

| $1 \mathrm{eV}=1.6021892 \times 10^{-19}$ Julios |
| :---: |
| 1 Tesla $=10^{4}$ Gauss |
| Magnetic flux $1 \mathrm{~Wb}=10^{8}$ Maxwell |
| Mag. Field $1 \mathrm{~A} / \mathrm{m}=4 \pi \times 10^{-3}$ Oersted |
| Energy $1 \mathrm{~J}=10^{7} \mathrm{ergs}$ |
| Force $1 \mathrm{~N}=10^{5}$ dynes |

Table I.4: Dielectric Strength

| Air | $30 \times 10^{6}$ | Porcelain | $20 \times 10^{6}$ |
| :---: | :---: | :---: | :---: |
| Transformer oil | $15 \times 10^{6}$ | Glass | $10 \times 10^{6}$ |
| Mica | $50 \times 10^{6}$ | Rubber | $40 \times 10^{6}$ |

Table I.5: Conductivity of Some Materials

| Conductors | $\sigma(S / m)$ | Conductors | $\sigma(S / m)$ |
| :---: | :---: | :---: | :---: |
| Silver | $6.173 \times 10^{7}$ | Iron | $1.03 \times 10^{7}$ |
| Copper | $5.813 \times 10^{7}$ | Distilled Water | $2 \times 10^{-4}$ |
| Aluminum | $3.816 \times 10^{7}$ | Lead | $0.48 \times 10^{7}$ |
| Zinc | $1.67 \times 10^{7}$ | Mercury | $1.04 \times 10^{6}$ |
| Tungsten | $1.825 \times 10^{7}$ | Steel(silicon) | $2 \times 10^{6}$ |
| Platinum | $9.52 \times 10^{6}$ | Steel(stainless) | $1.1 \times 10^{6}$ |
| Sea Water | $\approx 4$ | Earth | $10^{-4}-10^{-7}$ |
| Insulators | $\sigma(S / m)$ | Insulators | $\sigma(S / m)$ |
| Fused quartz | $10^{-17}$ | Mica | $10^{-15}$ |
| Glass | $10^{-12}$ | Transformer oil | $10^{-11}$ |
| Porcelain | $10^{-13}$ | Rubber | $10^{-15}$ |

Table I.6: Relative Permeability $\mu_{r}$ of Some Materials

| Material | $\mu_{r}$ | Material | $\mu_{r}$ |
| :---: | :---: | :---: | :---: |
| Bismuth | 0.99983 | Air | 1.00000036 |
| Silver | 0.99998 | Copper | 0.99999 |
| Water | 0.99999 | vacuum | 1.00000 |
| Cobalt | 250 | Nickel | 600 |
| Cold rolled steel | 2,000 | Mu metal | 100,000 |
| Purified iron | 180,000 | 78 Permalloy | 100,000 |
| Soft iron | 5,000 | Supermalloy | 800,000 |

## Appendix J

## Useful Electrical Relations

Let total charge $Q$ be on a conducting ellipsoid with semi-axes $a>b>c$ along the $\mathrm{x}, \mathrm{y}$ and z axes, respectively [13]. The capacitance of it will be found by

$$
\begin{equation*}
C=\frac{Q}{V_{0}}=\frac{8 \pi \epsilon}{\int_{0}^{\infty}\left[\left(a^{2}+\eta\right)\left(b^{2}+\eta\right)\left(c^{2}+\eta\right)\right]^{-\frac{1}{2}} d \eta} \tag{J.1}
\end{equation*}
$$

and the density charge distribution is

$$
\begin{equation*}
\sigma=\frac{Q}{4 \pi a b c}\left[\frac{x^{2}}{a^{4}}+\frac{y^{2}}{b^{4}}+\frac{z^{2}}{c^{4}}\right]^{-\frac{1}{2}} \tag{J.2}
\end{equation*}
$$

When $c=0$ we will have an elliptic disk, so the Eq.(J.2) reduces to

$$
\begin{equation*}
\sigma=\frac{Q}{4 \pi a b}\left[1-\left(\frac{x}{a}\right)^{2}-\left(\frac{y}{b}\right)^{2}\right]^{-1 / 2} \tag{J.3}
\end{equation*}
$$

If $a=b$ the shape reduces to oblate spheroid. The capacitance of which is readily obtained

$$
\begin{equation*}
C=\frac{4 \pi \epsilon \sqrt{a^{2}-c^{2}}}{\tan ^{-1} \sqrt{(a / c)^{2}-1}} \tag{J.4}
\end{equation*}
$$

and the charge density, becomes

$$
\begin{equation*}
\sigma=\frac{Q}{4 \pi a^{2} c}\left[\frac{\rho^{2}}{a^{4}}+\frac{z^{2}}{c^{4}}\right]^{-1 / 2} \tag{J.5}
\end{equation*}
$$

If where $c \rightarrow 0, x^{2}+y^{2}=\rho^{2}$, we have the infinitely thin circular disk with

$$
\begin{equation*}
C=8 \epsilon a \tag{J.6}
\end{equation*}
$$

The charge density derives directly from Eq.(J.3) with $a=b$ and $x^{2}+y^{2}=\rho^{2}$

$$
\begin{equation*}
\sigma=\frac{Q}{4 \pi a \sqrt{a^{2}-\rho^{2}}} \tag{J.7}
\end{equation*}
$$

This value holds, for each side of the disk. For axial symmetry about the x -axis, $b=c$, we have a prolate spheroid, and the capacitance is

$$
\begin{equation*}
C=\frac{4 \pi \epsilon \sqrt{a^{2}-b^{2}}}{\tanh ^{-1} \sqrt{1-(b / a)^{2}}} \tag{J.8}
\end{equation*}
$$


[^0]:    ${ }^{1}$ For more information, see: "Handbook of Atmospheric Electrodynamics, vol. I", by Hans Volland, 1995 published by the CRC Press. Chapter 11 is entirely on Schumann Resonances and is written by Davis Campbell at the Geophysical Institute, University of Alaska, Fairbanks AK, 99775. There is also a history of this research and an extensive bibliography.

