MECHANISTIC MODELING OF GAS-LIQUID

TWO - PHASE FLOW IN PIPES

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SCOPE

The objectives of this book are twofold:

- To provide insight and understanding of two-phase flow phenomena and
- To develop analytical tools for either designing two-phase flow systems or conducting research in this area.

The traditional approach for two-phase flow prediction has been the empirical approach. The approach was based on development of empirical correlation from experimental data. This book presents the recent approach in which mathematical mechanistic models are developed, based on the physical phenomena, for the prediction of two-phase flow behavior. The models can be verified and refined with limited experimental data. However, as these models incorporate the physical phenomena and the important flow variables, they can be extended to different operational conditions and enable scale-up with significant confidence.

The first part of the book includes an introduction on two-phase flow phenomena and a review of the early "black box" flow pattern independent models. The main part of the book is a detailed study of models for predicting flow pattern transition boundaries, and separate models for each flow pattern for the prediction of the flow characteristics, such as the liquid holdup and the pressure drop. The analysis is carried out for vertical flow, horizontal and near horizontal flow and inclined flow. Introduction to transient flow and application examples are also presented.
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   This book is one of the earliest attempts to analyze systematically two-phase flow. The first part of the book presents general approaches and models such as the homogeneous model, the separated model, the drift flux model and wave flow. The second part includes separate models of the various flow patterns. In general it is outdated.

   This general book contains only few chapters, which deal with two-phase flow systems. Chapter 7 is an introduction to multiphase flow, which includes the derivation of the conservative equations for two-phase flow, and discussion about the important phenomena of slippage and holdup. Chapters 8 and 10 discuss vertical and horizontal flows, respectively. Each chapter includes general discussion and separate models for the various flow patterns.

   This is Ishii’s Ph.D. Dissertation. The work is highly theoretical and includes the development of the basic equations for two-phase flow. First the local instantaneous equations are developed, after which the averaging procedures are presented.

   This is a general book on multiphase flow. It includes analysis of gas-liquid, liquid-liquid, gas-solid, liquid-solid, boiling condensation and particle flow. Only Chapter 2 deals with gas liquid systems and Chapter 10 is an excellent review of measurement techniques in two-phase flow.

   Volume 3 has three sections on atomized flow, flow regimes, and on reactors and industrial applications. Only the second section includes pertinent material on flow pattern prediction, liquid holdup and pressure drop.

   These two books include collections of invited lectures from experts in particular two-phase flow areas with phase change applied to the nuclear industry. Volume I include modeling and experimental results on fundamental two-phase flow phenomena such as interfacial area analysis, void fraction distribution etc. The second volume includes applications related to the nuclear industry.

Material is included on multiphase flow in chemical, mechanical and nuclear engineering. Chapter 1 covers specific topics on closure relationship, interfacial shear and on non-equilibrium flows. Chapters 2 and 3 contain unique information on two-phase flow: chapter 2 includes data sets on different phenomena, and chapter 3 presents numerical benchmark tests. Excellent review on experiment, theory and numerical methods.


This book focuses on a specific topic in multiphase flow, namely, hydrodynamics of dispersions. The dispersed phase consists of droplets, solid particles and bubbles. The analysis includes particle distributions and the associated constitutive equations of motion, namely, the forces acting on the particles.


This book analyzes two-phase occurring in complex systems of water-cooled nuclear power reactor. Although it is mainly applied to the nuclear industry, general material is presented on environmental applications for global climate prediction. Also, an updated overview of flow patterns and empirical and mechanistic models for pipe flow.


This is an SPE monograph that focuses on empirical correlations and mechanistic modeling of multiphase flow in wellbores. Also included is a comprehensive review of Petroleum fluids physical properties prediction methods.


This book presents a systematic approach to modeling multiphase flow heat transfer for a large variety of wellbore operating conditions. The first six chapters present the modeling approach for multiphase flow (Chapters 2-4) and wellbore heat transfer (Chapters 5-6). The rest of the book (Chapters 7-10) is devoted to specific well operations, such as drilling, gas-lift, etc., where the models find applications. Chapter 7 focuses on modeling transient transport processes and is directly applicable to well test design and test data interpretation.
Chapter 1
Fundamentals of Two-Phase Flow

1.1 Two-Phase Flow Occurrence and Applications

Gas-liquid two-phase flow occurs ubiquitously in various major industrial fields. Examples are the petroleum, chemical, nuclear, and geothermal industries. Two-phase flow is encountered in these industries, as well as others, in a wide range of engineering applications. It has been the need for design methods for these applications that has stimulated an extensive two-phase flow research since the 1950s. Following is a brief review of the occurrence and applications of two-phase flow.

1.1.1 The Petroleum Industry. Two-phase flow occurs in the petroleum industry during the production and transportation of oil and gas. The flow occurs in horizontal, inclined, or vertical pipes, in both the wellbore and the flowline. In offshore production, these lines can be of substantial lengths before reaching separation facilities. Piping components, separators, or slug catchers are commonly used for the flow control and processing. Design methods are needed to determine the pressure drop and the liquid holdup, or liquid volume, in order to size the flowlines and the separation facilities.

1.1.2 The Chemical and Process Industry. The chemical and process industry encounters two-phase flow in most of its applications. Examples are reactors, boilers, condensers, evaporators, and distillation plants. The design of such facilities requires methods for predicting, in addition to the pressure drop and liquid holdup in the pipes, also the heat and mass transfer processes. This includes the wall to fluid heat transfer coefficient and the mass transfer coefficient through the gas-liquid interface. Prediction of the residence time distribution of the two-phases is required too, especially for proper reactor design.

1.1.3 The Nuclear Reactor Industry. The nuclear reactor industry considers two-phase flow mainly for safety purposes. When loss of coolant accident (LOCA) occurs, boiling may take place near the core, and emergency water is injected to cool the core. This counter-current flow situation must be analyzed carefully to predict flooding conditions and possible core melting. The complex hydrodynamics of two-phase flow splitting in branching conduits is of primary importance for this process.

1.1.4 Geothermal Energy Plants. In this industry two-phase flow occurs in the form of steam-water flow in the vertical risers and gathering system pipelines. Pressure, temperature, and phase behavior predictions are essential for a proper design. Elimination of slug flow is a necessity to avoid operational problems.

1.1.5 Space Industry. A special form of two-phase flow occurs in space, namely, zero-gravity flow. Zero-gravity two-phase flow and phase change occur in power generation, energy storage, thermal management, and life support systems of space stations. Under zero-gravity two-phase flow, no stratification occurs, and the flow configurations are always symmetric. The inclination angle concept does not apply in space, and the flow behavior is the same in all directions. Experimental data for two-phase flow under zero-gravity conditions have been collected in parabolic flight research aircrafts and free fall drop-tower facilities to study the phenomena and develop design procedures for such space applications.

A reliable design of the above-mentioned systems requires understanding of two-phase flow behavior. The objective of this book is to provide insight and understanding of the basic principles of two-phase flow. This knowledge can be used by either researchers or designers of two-phase flow systems.
1.2 The Modeling Approach

The modeling concept can be understood by first conducting a review of the fundamental approaches for solving engineering problems, as presented next.

1.2.1 The Experimental (Empirical) Approach. In this approach experiments are conducted and empirical correlations are developed. Scientifically, this approach should be based on dimensional analysis, yielding a universal solution. Correlations that are developed without dimensional analysis can be applied with confidence only in the range of conditions similar to those under which the experimental data are taken. Their general applicability, however, is questionable. An example of a universal correlation is the correlation of friction factor, or frictional pressure loss for turbulent pipe flow.

1.2.2 The Exact Solution (Rigorous) Approach. This approach requires the solution of the conservation equations with the proper boundary conditions. No doubt, this is the most accurate approach, but unfortunately there are very few systems that can be treated rigorously. One example is the solution of laminar flow in pipes.

1.2.3 The Numerical Simulation Approach. In this approach, the conservation equations are solved numerically. The advancement of computer technology has prompted utilization of this approach in recent years. This is most useful for Computational Fluid Dynamics (CFD) and transient flow calculations. This approach usually leads to very large codes, which are costly, complex, and time consuming. Problems with the numerical schemes and closure relationships are still unresolved. An example is the OLGA transient flow simulator.

1.2.4 The Modeling Approach. The modeling approach is an intermediate approach between the experimental and rigorous approaches. A simplified physical model, which attempts to describe the phenomenon closely, is built. This physical model is then expressed mathematically to provide an analytical tool for prediction and design purposes. Experimental measurements can be conducted to check the model and refine it. Various models can be developed for the same system. The closer the physical model is to the real phenomenon, the better is the mathematical model and its predictions.

Two-phase flow is characterized by a large number of flow variables, almost doubled than for single-phase flow. Also, the flow configuration is very complex. This is especially true for gas-liquid systems with a compressible phase and a deformable interface. Such a system makes the empirical, exact solution and numerical simulation approaches either impractical or too complicated. The large numbers of variables lead to large numbers of dimensionless groups when dimensional analysis is applied. For a simple case of horizontal flow, at least six groups can be formed. Clearly, it is impossible to identify and correlate the important groups among a given set of six dimensionless groups. This will require a very large number of experimental runs. Exact solution, on the other hand, is practically impossible because of the complexity of the system. Usually, the shape and the velocity of the boundaries between the two phases, i.e., the interface, are complex and not known a priori. The transport processes of mass momentum and heat transfer across the interface add more complexity to the system. Thus, exact solution for each of the phases is rather impossible because of the complexity of the boundary conditions. Finally, the numerical simulation approach leads to large codes that still suffer from numerical and closure relationship problems.

Earlier predictive means for two-phase flow were through the empirical approach. This is caused by both the complex nature of the system and the need for design methods for the industry. Usually, the dimensionless groups for the data correlations were guessed without any physical basis. This approach was very successful for solving two-phase flow problems for decades, with an update performance of
±30% error. However, the empirical approach has never addressed the “why” and “how” problems for two-phase flow phenomena. Also, it is believed that no further or better accuracy can be achieved through this approach.

The modeling approach has emerged during the recent years. This approach attempts to shed more light on the physical phenomena. The flow mechanisms causing two-phase flow to occur are determined and modeled mathematically. A fundamental postulate in this method is the existence of various flow configurations or flow patterns in two-phase flow. For each flow pattern, the transport processes are similar to some extent. The first objective in this approach is, thus, to predict the existing flow pattern for a given system. Then, for each flow pattern, a separate model is developed, which predicts the hydrodynamics and heat transfer for this flow pattern. The models may be validated against limited experimental data. However, these models are expected to be more general and reliable for other flow conditions as they incorporate the mechanisms and the important parameters of the flow. These are the pipe diameter, inclination angle, gas and liquid flow rates, and their physical properties.

The state-of-the-art in the industry is the use of both the empirical and the modeling approaches. The empirical approach is still the main approach used for design. However, all current research is conducted through the modeling approach. Applications of models in the field are now underway showing the potential of this method. Future development and testing of the modeling approach is needed before it becomes the main tool for design.

1.3 Two-Phase Flow Variables

1.3.1 Mass Flow Rate, \( W \) (kg/s).

\[
W_L = \text{liquid mass flow rate},
\]
\[
W_G = \text{gas mass flow rate},
\]
\[
W = \text{total mass flow rate},
\]
and
\[
W = W_L + W_G. \quad \text{..................................................................................................................(1.1)}
\]

1.3.2 Volumetric Flow Rate, \( q \) (m³/s).

\[
q_L = \text{liquid volumetric flow rate},
\]
\[
q_G = \text{gas volumetric flow rate},
\]
\[
q = \text{total volumetric flow rate},
\]
and
\[
q = q_L + q_G. \quad \text{..................................................................................................................(1.2)}
\]

1.3.3 Liquid Holdup, \( H_L \), and Gas Void Fraction, \( \alpha \) (-). The liquid holdup is the fraction of a volume element in the two-phase flow field occupied by the liquid-phase. Similarly, the gas void fraction is the fraction of the volume element that is occupied by the gas-phase. For two-phase flow,
0 < H_L, α < 1, and \( H_L + \alpha = 1 \). For single-phase flow, \( \alpha \) or \( H_L \) are either 0 or 1. Different definitions of the volume fractions can be used, as discussed next.

The instantaneous liquid holdup, \( H_L(r,t) \), refers to a differential volume element, and represents the holdup at a given time and space point in the flow field. For this condition of a very small volume element, \( H_L(r,t) \) can be either 1 or 0. Integration of the instantaneous liquid holdup over time will yield the local liquid holdup at a given location. For pipe flow, however, more useful definitions are required. The space and time average of the instantaneous liquid holdup is given by

\[
\langle H_L \rangle = \frac{\int \int H_L(r,t)dr\,dt}{\int dr\int dt}. \tag{1.3}
\]

For simplicity, the average liquid holdup is designated by \( H_L \). Two practical averages are used for pipe flow: the cross-sectional average liquid holdup and the volumetric average liquid holdup. These averages refer to a cross-sectional area of the pipe, and a finite volume of the pipe bounded by the pipe wall and two imaginary vertical planes, respectively. One must realize that both the cross-sectional and the volumetric liquid holdup parameters are functions of both time and space.

1.3.4 Superficial Velocities, \( v_{sl} \) and \( v_{sg} \) (m/s). The superficial velocity of a phase is the volumetric flux of the phase, which represents the phase volumetric flow rate per unit area. In other words, the superficial velocity of a phase is the velocity which would occur if only that phase alone flows in the pipe. Thus, the superficial velocities of the liquid and gas phases are, respectively,

\[
v_{sl} = \frac{q_L}{A_p}, \quad \text{and} \quad v_{sg} = \frac{q_G}{A_p}, \tag{1.4}
\]

where \( A_p \) is the cross-sectional area of the pipe.

1.3.5 Mixture Velocity, \( v_M \) (m/s). The mixture velocity is the total volumetric flow rate of both phases per unit area, which is referred to as the center of volume velocity, and is given by

\[
v_M = \frac{q_L + q_G}{A_p} = v_{sl} + v_{sg}. \tag{1.5}
\]

The no-slip liquid holdup is the ratio of the liquid volumetric flow rate to the total volumetric flow rate, namely,

\[
\lambda_L = \frac{q_L}{q_L + q_G} = \frac{v_{sl}}{v_{sl} + v_{sg}}. \tag{1.6}
\]

1.3.6 Mass Flux, \( G \) (kg/m²s).

\[
G_L = \frac{W_L}{A_p} = \text{liquid mass flux},
\]

\[
G_G = \frac{W_G}{A_p} = \text{gas mass flux},
\]

\( G \) = total mass flux,
and
\[ G = \frac{W_L + W_G}{A_p} = G_L + G_G. \]...

(1.7)

### 1.3.7 Actual Velocity, \( v \) (m/s).
The superficial velocities previously defined are not the actual velocities of the phases, as each phase occupies only a fraction of the pipe cross section. Thus, the actual velocities of the liquid and gas phases are, respectively,

\[ v_L = \frac{v_{SL}}{H_L}, \quad \text{and} \quad v_G = \frac{v_{SG}}{1-H_L}. \]...

(1.8)

### 1.3.8 Slip Velocity, \( v_{SLIP} \) (m/s).
The actual velocities of the liquid and gas phases are usually different. The slip velocity represents the relative velocity between the two phases, as given by

\[ v_{SLIP} = v_G - v_L. \]...

(1.9)

### 1.3.9 Drift Velocity, \( v_D \) (m/s).
The drift velocity of a phase is the velocity of the phase relative to a surface moving at the mixture velocity (center of volume).

\[ v_{DL} = v_L - v_M, \quad \text{and} \quad v_{DG} = v_G - v_M. \]...

(1.10)

### 1.3.10 Drift Flux, \( J \) (m/s).
The drift flux represents the flow rate of a phase, per unit area, through a surface moving at the center of volume velocity.

\[ J_L = H_L(v_L - v_M), \quad \text{and} \quad J_G = (1-H_L)(v_G - v_M). \]...

(1.11)

### 1.3.11 Diffusion Velocity, \( v_M \) (m/s).
The diffusion velocity is the velocity of a phase relative to a surface moving at the center of mass velocity, as given by

\[ v_{ML} = v_L - \frac{G}{\rho_M}, \quad \text{and} \quad v_{MG} = v_G - \frac{G}{\rho_M}. \]...

(1.12)

where \( G \) is the total mass flux, and \( \rho_M \) is the average density of the mixture, given by Eq. 1.15.

### 1.3.12 Quality, \( x \) (-).
The quality is the ratio of the gas mass flow rate to the total mass flow rate across a given area.

\[ x = \frac{W_G}{W_G + W_L} = \frac{W_G}{W}. \]...

(1.13)

### 1.3.13 Mass Concentration, \( c \) (-).
The mass concentration is the ratio of the mass of a phase to the total mass in a given volume. The liquid and gas mass concentrations are, respectively,

\[ c_L = \frac{H_L \rho_L}{\rho_M}, \quad \text{and} \quad c_G = \frac{(1-H_L) \rho_G}{\rho_M}. \]...

(1.14)

### 1.3.14 Average Fluid Properties.
The average two-phase density and viscosity are given, respectively, by

\[ \rho_M = \rho_L H_L + \rho_G (1-H_L), \]...

(1.15)
and

\[ \mu_L = \mu_L H_L + \mu_L (1 - H_L). \]  

When the liquid-phase contains oil and water, the liquid-phase density, viscosity, and surface tension are averaged based on the volumetric flow rate fraction of the water-phase, assuming no-slip conditions between the water and the oil, as follows

\[ \rho_L = \rho_W f_{WC} + \rho_O (1 - f_{WC}). \]  

\[ \mu_L = \mu_W f_{WC} + \mu_O (1 - f_{WC}), \]  

and

\[ \sigma_{LG} = \sigma_{WG} f_{WC} + \sigma_{OG} (1 - f_{WC}). \]

where \( f_{WC} \), the water volume fraction (water-cut), is the ratio of the volumetric flow rate of the water phase to the total volumetric flow rate of the liquid-phase, given by

\[ f_{WC} = \frac{q_W}{q_W + q_O}. \]

Note that calculation of the liquid viscosity in Eq. 1.18 is based on a simple averaging of the water and oil viscosities, using their respective volume fractions. In reality, oil-water mixtures exhibit complex physical phenomena, which have prompted the development of special correlations for the prediction of their viscosities.

1.4 Two-Phase Flow Fundamental Phenomena

The hydrodynamics of single-phase flow in pipes is well understood at the present time. Both the pressure drop vs. flow rate behavior and the heat transfer processes for single-phase flow pipelines can be determined in a straightforward manner. The simultaneous flow of two phases in a pipe complicates considerably the transport processes. Consider a pipeline carrying both gas and liquid, as shown schematically in Fig. 1.1. A typical given set of flow conditions include the mass or volume flow rates of the two phases, their physical properties, and the pipe diameter and inclination angle. These data are sufficient for single-phase flow calculations. However, for two-phase flow systems, additional information is required. This is illustrated in the next section.

![Fig. 1.1 — Schematic of gas-liquid two-phase pipe flow.](image)

1.4.1 Complications in Basic Equations. Consider a single-phase flow system where the mass flow rate, pipe diameter and inclination angle, and physical properties of the fluid are given. For this case, as
shown in Fig. 1.2 Part A, at any axial location downstream, it is possible to calculate the velocity of the fluid from the continuity equation, given by

$$ W = \rho v A_p, \quad \text{and} \quad v = \frac{W}{\rho A_p}. \quad \text{.................................................................(1.21)} $$

Once the velocity is determined, one can proceed with the calculations to determine the transport processes, such as the pressure drop or the heat transfer. Similar analysis can be carried out for a two-phase flow system. For this case, as shown in Fig. 1.2 Part B, the input parameters include the gas and liquid mass flow rates, pipe diameter and inclination, and the physical properties of the phases. Two continuity equations can be written for the gas and liquid phases, yielding

$$ W_L = \rho_L v_L A_L, \quad \text{and} \quad W_G = \rho_G v_G A_G. \quad \text{.................................................................(1.22)} $$

Substituting for the areas of the phases in terms of the liquid holdup results in

$$ W_L = \rho_L v_L A_L H_L, \quad \text{and} \quad W_G = \rho_G v_G A_p (1 - H_L). \quad \text{.................................................................(1.23)} $$

The two continuity equations given in Eq. 1.23 have three unknowns, $v_L$, $v_G$, and $H_L$, and cannot be solved in a straightforward manner, as done for the single-phase system. Additional information is required to solve the equations and proceed with the calculations of the pressure drop and heat transfer processes. A simplification of the system can be carried out by assuming that both phases are moving at the same velocity ($v_G = v_L$, no-slip condition), which is not true in general. With this assumption, Eq. 1.23 has two unknowns and can be solved for the liquid holdup and the common velocity of the phases. This will allow proceeding with the calculations of the transport processes. However, in the more general case, when the gas and liquid velocities are not equal, further analysis is required.

![Fig. 1.2—Single-phase and two-phase flow continuity consideration.](image)

1.4.2 Slippage and Holdup. Fig. 1.3 is a schematic description of the relationship between slippage and liquid holdup. (Note: refer to Figs. 1.3 Parts A and B as schematics, whereby the gas and liquid phases are separated, in the form of stratified flow, for illustration purposes only). Fig. 1.3 Part A shows the no-slip condition case, in which the gas and the liquid phases travel at the same velocity, namely $v_G = v_L$. For this condition, it is possible to show from the definition of the slip velocity, that the in-situ liquid holdup is equal to the no-slip liquid holdup, as shown in Eq. 1.24.

$$ v_{SLIP} = 0 = v_G - v_L = \frac{v_{SG}}{1 - H_L} - v_{SL}. \quad \text{.................................................................(1.24)} $$
Solving for the liquid holdup, $H_L$,

$$H_L = \frac{v_{SL}}{v_{SL} + v_{SG}} = \lambda_L.$$ ..........................................................(1.25)

Physically, for no-slip condition, as both phases travel at the same velocity, the liquid holdup is simply equal to the ratio of the liquid volumetric flow rate to the total volumetric flow rate, which is the no-slip liquid holdup. No-slip condition occurs, for example, in homogeneous flow or dispersed-bubble flow, with high liquid and low gas flow rates. Under this flow condition, the gas-phase is dispersed as small bubbles in a continuous liquid-phase. Because of the high liquid flow rate, the gas bubbles are carried out by the liquid-phase at the same velocity, resulting in zero slippage. Thus, for this flow condition, the in-situ liquid holdup is equal to the no-slip liquid holdup, namely, $H_L = \lambda_L$.

![Diagram of no-slip and slip conditions](image)

Fig. 1.3—Schematic of slippage and liquid holdup relationship.

Usually, however, the gas and the liquid do not move at the same velocity, and slippage takes place between the two phases. The gas-phase moves at a higher velocity than the liquid-phase because of buoyancy and lower frictional forces. From continuity consideration, if the gas-phase moves faster than the liquid-phase, as shown in Fig. 1.3 Part B, the cross-sectional area of the gas-phase reduces while the cross-sectional area of the liquid-phase increases. This results in liquid accumulation in the pipe and the in-situ liquid holdup being larger than the no-slip liquid holdup. An example for this case is bubble flow in vertical pipes, at low liquid flow rates. Under this flow condition, because of buoyancy, the gas-phase moves faster than the liquid-phase, or slips through it, in a velocity $v_0$ called the bubble rise velocity. This results in liquid holdup being higher than the no-slip liquid holdup, namely, $H_L > \lambda_L$.

There is an exception for the slippage phenomenon. For downward flow, under very low gas flow rate condition, the liquid-phase may move faster than the gas-phase because of gravity. For this case, the liquid holdup is less than the no-slip liquid holdup, namely, that $H_L < \lambda_L$.

1.5 Flow Pattern Definitions and Classifications

The fundamental difference between single-phase flow and gas-liquid two-phase flow is the existence of flow patterns or flow regimes in two-phase flow. The term flow pattern refers to the geometrical
configuration of the gas and the liquid phases in the pipe. When gas and liquid flow simultaneously in a pipe, the two phases can distribute themselves in a variety of flow configurations. The flow configurations differ from each other in the spatial distribution of the interface, resulting in different flow characteristics, such as velocity and holdup distributions.

The existing flow pattern in a given two-phase flow system depends on the variables listed next:

- Operational parameters, namely, gas and liquid flow rates.
- Geometrical variables, including pipe diameter and inclination angle.
- The physical properties of the two phases, i.e., gas and liquid densities, viscosities, and surface tension.

Determination of flow patterns is a central problem in two-phase flow analysis. Indeed all the design variables of the flow are strongly dependent on the existing flow pattern. The design variables are the pressure drop, liquid holdup, heat and mass transfer coefficients, residence time distribution, and rate of chemical reaction.

In the past, there has been a lack of agreement between two-phase flow investigators on the definition and classification of flow patterns. Some investigators detailed as many flow patterns as possible, while others try to define a set with minimum flow patterns. The disagreement was mainly caused by the complexity of the flow phenomena and to the fact that the flow patterns were usually determined subjectively by visual observations. Also, the flow patterns are dependent on the inclination angle, and usually they were reported for either one inclination or a narrow range of inclination angles. In recent years, there has been a trend to define an acceptable set of flow patterns. On the one hand, the set must be minimal, but on the other hand, it must include acceptable definitions with minor changes. Also, it must apply to all the range of inclination angles.

An attempt to define an acceptable set of flow patterns has been made by Shoham (1982). The definitions are based on experimental data acquired over the entire range of inclination angles, namely horizontal flow, upward and downward inclined flow, and upward and downward vertical flow. Figs. 1.4 and 1.5 show the flow patterns existing in horizontal and near-horizontal pipes and in vertical and sharply inclined pipes, respectively. Following are the definitions and classifications of the flow patterns.

1.5.1 Horizontal and Near-Horizontal Flow. The existing flow patterns in these configurations can be classified as Stratified flow (Stratified-Smooth and Stratified-Wavy), Intermittent flow (Slug flow and Elongated-Bubble flow), Annular flow and Dispersed-Bubble flow. Refer to Fig. 1.4.

Stratified Flow (ST). This flow pattern occurs at relatively low gas and liquid flow rates. The two phases are separated by gravity, where the liquid-phase flows at the bottom of the pipe and the gas-phase on the top. The Stratified flow pattern is subdivided into Stratified-Smooth (SS), where the gas-liquid interface is smooth, and Stratified-Wavy (SW), occurring at relatively higher gas rates, at which stable waves form at the interface.

Intermittent Flow (I). Intermittent flow is characterized by alternate flow of liquid and gas. Plugs or slugs of liquid, which fill the entire pipe cross-sectional area, are separated by gas pockets, which contain a stratified liquid layer flowing along the bottom of the pipe. The mechanism of the flow is that of a fast moving liquid slug overriding the slow moving liquid film ahead of it. The liquid in the slug body may be aerated by small bubbles, which are concentrated towards the front of the slug and at
the top of the pipe. The Intermittent flow pattern is divided into **Slug (SL)** and **Elongated-Bubble (EB)** patterns. The flow behavior of Slug and Elongated-Bubble patterns are the same with respect to the flow mechanism. The Elongated-Bubble pattern is considered the limiting case of Slug flow, when the liquid slug is free of entrained bubbles. This occurs at relatively lower gas rates when the flow is calmer. At higher gas flow rates, where the flow at the front of the slug is in the form of an eddy (caused by picking up of the slow moving film), the flow is designated as Slug flow.

**Annular Flow (A).** Annular flow occurs at very high gas flow rates. The gas-phase flows in a core of high velocity, which may contain entrained liquid droplets. The liquid flows as a thin film around the pipe wall. The interface is highly wavy, resulting in a high interfacial shear stress. The film at the bottom is usually thicker than that at the top, depending upon the relative magnitudes of the gas and liquid flow rates. At the lowest gas flow rates, most of the liquid flows at the bottom of the pipe, while aerated unstable waves are swept around the pipe periphery and wet the upper pipe wall occasionally. This flow occurs on the transition boundary between Stratified-Wavy, Slug and Annular flow. It is not Stratified-Wavy because liquid is swept around and wets the upper pipe wall with a thin film. It is also not Slug flow because no liquid bridging of the pipe cross section is formed. As a result, the frothy waves are not accelerated to the gas velocity but move slower than the gas phase. Also, it is not fully developed Annular flow, which requires a stable film around the pipe periphery. This flow pattern is designated sometimes as a “Proto Slug” flow. Based on the definitions and mechanisms of Slug and Annular flows, this regime is termed as **Wavy-Annular (WA)** and classified as a subgroup of Annular flow. The difference between Slug flow and Wavy-Annular flow is more distinguishable in upward inclined flow. During Slug flow, back flow of the liquid film between slugs is observed, whereas in Wavy-Annular flow, the liquid moves forward uphill with frothy waves superimposed on the film. These waves move much slower than the gas-phase.

**Dispersed-Bubble Flow (DB).** At very high liquid flow rates, the liquid-phase is the continuous-phase, in which the gas-phase is dispersed as discrete bubbles. The transition to this flow pattern is defined either by the condition where bubbles are first suspended in the liquid or when gas pockets, which touch the top of the pipe, are destroyed. When this happens, most of the bubbles are located near the upper pipe wall. At higher liquid rates, the gas bubbles are dispersed more uniformly in the entire cross-sectional area of the pipe. Under Dispersed-Bubble flow condition, as a result of high liquid flow rates, the two phases are moving at the same velocity, and the flow is considered homogeneous no-slip.
1.5.2 Vertical and Sharply Inclined Flow. In this range of inclination angles, the Stratified regime disappears and a new flow pattern is observed, namely, Churn flow. Usually, the flow patterns are more symmetric around the pipe axis and less dominated by gravity. The existing flow patterns are Bubble flow, Slug flow, Churn flow, Annular flow, and Dispersed-Bubble flow. Refer to Fig. 1.5.

**Bubble Flow (B).** In bubble flow the gas-phase is dispersed into small discrete bubbles, moving upwards in a zigzag motion, in a continuous liquid-phase. For vertical flow, the bubble distribution is approximately homogeneous through the pipe cross section. Bubble flow occurs at relatively low liquid rates and is characterized by slippage between the gas and the liquid phases, resulting in large values of liquid holdup.

**Slug Flow (SL).** The Slug flow regime in vertical pipes is symmetric around the pipe axis. Most of the gas-phase is located in a large bullet shape gas pocket termed “Taylor bubble” with a diameter almost equal to the pipe diameter. The flow consists of successive Taylor bubbles and liquid slugs, which bridge the pipe cross section. A thin liquid film flows downward between the Taylor bubble and the pipe wall. The film penetrates into the following liquid slug and creates a mixing zone aerated by small gas bubbles.
1.5.2 Flow patterns in vertical and sharply inclined pipes.

**Churn (CH).** This flow pattern is characterized by an oscillatory motion of the liquid-phase. Churn flow is similar to slug flow but looks much more chaotic with no clear boundaries between the two phases. It occurs at higher gas flow rates, where the liquid slugs bridging the pipe become shorter and frothy. The slugs are blown through by the gas-phase, and then they break, fall backwards, and merge with the following slug. As a result, the bullet-shaped Taylor bubble is distorted and churning occurs.

**Annular Flow (A).** As in the horizontal case, the flow is characterized by a fast moving gas core with entrained liquid droplets and a slow moving liquid film flowing around the pipe wall. The flow is associated with a wavy interfacial structure, which results in a high interfacial shear stress. In vertical flow, the liquid film thickness around the pipe wall is approximately uniform.

**Dispersed-Bubble Flow (DB).** Similar to the horizontal flow case, Dispersed-Bubble flow in vertical and sharply inclined pipes occurs at relatively high liquid flow rates, under which conditions the gas-phase is dispersed as discrete bubbles into the continuous liquid-phase. For this flow pattern, the dominant liquid-phase carries the gas bubbles, and no slippage takes place between the phases. Hence, the flow is considered homogeneous no-slip.

1.5.3 Downward Inclined and Vertical Flow. Fig. 1.6 presents the flow patterns in the entire range of inclination angles in a unified manner. For downward inclined flow, the dominant flow pattern is Stratified-Wavy flow, occurring over a wide range of downward inclination angles, namely, between horizontal flow and up to $-80^\circ$, and covering a wider range of gas and liquid flow rates. As observed in horizontal, upward inclined, and upward vertical flow, Dispersed-Bubble flow and Annular flow occur at high liquid and high gas flow rates, respectively. For vertical downward flow, the Stratified flow pattern disappears and the Annular regime exists also at low gas flow rates, in the form of Falling-Film. The Slug flow pattern in vertical downward flow is similar to that occurring in upward flow, except that usually the Taylor bubble is unstable.
and eccentrically located off the pipe axis. The Taylor bubble may either rise or descend, depending on the relative flow rates of the gas and liquid phases.

Fig. 1.6 — Flow patterns in entire range of inclination angles.

Any attempt to have a general and unique solution for two-phase problems for all flow patterns is quite impossible. However, as for each existing flow pattern the flow behavior is rather similar, two-phase flow becomes somewhat easier, as it is possible to analyze each flow pattern separately. Thus, the general approach is to first predict the existing flow pattern in the pipe. Once the flow pattern is
determined, a separate model for each flow pattern is developed, which can predict the flow characteristics, such as the pressure drop, liquid holdup, and the heat transfer coefficient.

1.5.4 Flow Pattern Prediction. The earlier approach for predicting flow patterns has been the empirical approach. Determination of the flow patterns was carried out mainly by visual observations. Usually, the data were mapped on a two-dimensional plot, and the transition boundaries between the different flow patterns were determined. Such a map is called a Flow Pattern Map. In most cases, the coordinates were chosen arbitrarily, without a physical basis. Thus, such empirical maps are expected to be reliable only in the range of conditions similar to those under which the data were acquired, and extension to other flow conditions is uncertain.

Many different coordinate systems have been proposed for constructing flow pattern maps, most of which are dimensional, such as the mass flow rates, the momentum fluxes, or the superficial velocities, as used by Mandhane et al. (1974). The Mandhane flow pattern map for horizontal flow, shown in Fig. 1.7, is unique, being based on a large data bank (the AGA-API Data Bank). The actual map is based on 1,178 data points for air-water systems in small diameter pipes, i.e., 1.27 to 5.1 cm.

![Flow pattern map for horizontal pipes (after Mandhane et al., 1974).](image)

Fig. 1.7—Flow pattern map for horizontal pipes (after Mandhane et al., 1974).

Various investigators have tried to extend the validity of their flow pattern maps by choosing dimensionless coordinates or correction factors for fluid physical properties. An example of a flow pattern map with correction factors, $X$ and $Y$, for physical properties is the Govier and Aziz (1972) map, shown in Fig. 1.8, developed for vertical flow. Dimensionless coordinates have been proposed, for example, by Griffith and Wallis (1961) and Spedding and Nguyen (1980). Griffith and Wallis showed that the transition from slug to annular flow is governed by the dimensionless groups $v_{SG} / u_M$ and $v_M^2 / gd$ and presented their flow pattern map, as shown in Fig. 1.9, using these groups as coordinates.
During discussion of flow pattern maps, one must mention the work published by Baker (1954) who has certainly been a pioneer in this area. His flow pattern map, given in Fig. 1.10, utilizes mixed dimensionless/dimensional coordinates, where $G_L$ and $G_G$ are the mass fluxes of the liquid and gas phases, respectively, and $\lambda = \left( \frac{\rho_G}{0.075} \right) \left( \frac{\rho_L}{62.3} \right)^{1/2}$ and $\psi = 73 \left( \frac{\mu_L}{1} \right) \left( \frac{\rho_L}{62.3} \right)^{1/3}$ are correction factors for fluid properties in field units. Baker’s flow pattern map is probably the most durable one, as it is still in use in the petroleum industry.

![Flow pattern map for vertical pipes (after Govier and Aziz, 1972).](image1)

![Flow pattern map for vertical pipes (after Griffith and Wallis, 1961).](image2)

More recently developed mechanistic models for flow pattern prediction are presented in following chapters, for horizontal and near horizontal flow, vertical flow, and inclined flow. A unified model applicable for all inclination angles is also presented. These models are based on the physical
mechanisms, which determine the transition between the different flow regimes. Once the transition mechanisms are defined, a mathematical (mechanistic) model and analytical expressions for the transition boundaries can be developed. The models incorporate the effect of the input variables, such as gas and liquid flow rates (operational parameters), pipe diameter and inclination angle (geometrical parameters), and the physical properties of the fluids. Therefore, prediction of flow patterns under different flow conditions can be carried out in a more reliable way. Examples of such mechanistic models are the Taitel and Dukler (1976) and the Taitel et al. (1980) studies. As is presented in Chap. 3, the generalized Taitel and Dukler flow pattern map for horizontal and near horizontal flow is governed by four different dimensionless coordinates. These dimensionless groups were guessed but emerged from the analysis and are based on the physical phenomena and the flow mechanisms of the transition boundaries. Thus, the mechanistic models are expected to be more reliable for a wide range of flow conditions, namely, flow rates, pipe diameters, and fluid properties.

Fig. 1.10—Flow pattern map for horizontal pipes (after Baker, 1954).

1.5.5 Flow Pattern Dependent Pressure Gradient. As previously mentioned, the design parameters, including the pressure gradient, are strongly dependent on the existing flow pattern. Figs. 1.11 and 1.12 present the total pressure gradient for the different flow patterns in horizontal and vertical pipes, respectively. The data were acquired in 0.0254 m ID pipes with air-water at standard conditions (STP), as reported by Govier and Aziz (1972).

Fig. 1.11 presents the flow-pattern dependent pressure gradient for horizontal flow. The $x$-axis is the superficial air velocity; the $y$-axis is a dimensionless total pressure gradient, and the parameter is the superficial water velocity. As can be seen, for low superficial gas and liquid velocities, the flow pattern is Stratified-Smooth flow, exhibiting a very low pressure gradient. Keeping the superficial liquid velocity low, as the superficial gas velocity is increased, the flow pattern changes to Stratified-Wavy flow, increasing the pressure gradient. At higher superficial gas velocity, the flow pattern becomes Annular, which is associated with a very high pressure gradient. On the other hand, keeping the superficial gas velocity constant and increasing the superficial liquid velocity, the flow pattern changes.
from Stratified flow to Intermittent flow (Elongated-Bubble or Slug), for which the pressure gradient increases. Further increase of the superficial liquid velocity (beyond 5 ft/s) results in transition to Dispersed-Bubble flow (not presented in the original figure), which exhibits very high pressure gradients, comparable to Annular flow. Thus, for horizontal flow, increasing either the gas or liquid flow rate results in a higher total pressure gradient.

The pressure gradient results for vertical flow are presented in Fig. 1.12, using the same coordinates as in Fig. 1.11. As can be seen, a different behavior is observed, as compared to horizontal flow, whereby for vertical flow the pressure gradient curves exhibit a minimum. For low gas and liquid superficial velocities, the flow pattern is Bubble flow, which is associated with a high pressure gradient.

![Fig. 1.11](image)

Fig. 1.11—Flow pattern dependent pressure gradient in horizontal flow for 2.54 cm pipe with air-water at STP (after Govier and Aziz, 1972).

This is due to the large liquid holdup that occurs in Bubble flow, which results in a large gravitational pressure gradient. For this case, the frictional pressure gradient is low because of the low velocities occurring in Bubble flow, and the gravitational pressure gradient is the main contributor to the
total pressure gradient. Keeping the superficial liquid velocity relatively low, upon increasing the superficial gas velocity, the flow pattern becomes Intermittent flow (Slug or Churn), which exhibits lower pressure gradient. This is because of the fact that with the increase of the gas flow rate, the liquid holdup reduces, resulting in a lower gravitational pressure gradient. On the other hand, the velocities encountered in Intermittent flow are not high enough to cause high frictional pressure gradient. As a result, the total pressure gradient decreases, exhibiting a minimum for this flow pattern. For very high superficial gas velocities, a transition to Annular flow occurs, whereby the frictional pressure gradient is very high, resulting in a high total pressure gradient. Note that for this case, because of the low liquid holdup, the gravitational pressure gradient is very low. For high superficial liquid velocities, the flow pattern is Dispersed-Bubble flow (beyond 5 ft/s, not indicated in the original figure), for which the total pressure gradient is very high. This is a result of either high liquid holdup (low gas flow rates) that causes high gravitational pressure gradient or high velocities (high gas flow rates) that result in high frictional pressure gradient.

The total pressure gradient behavior for vertical flow is shown schematically in Fig. 1.13. The figure is for a constant superficial liquid velocity. As can be seen, the gravitational pressure gradient decreases as the superficial gas velocity increases, because of lower liquid holdup in the system. On the other hand, the frictional pressure gradient increases as the superficial gas velocity is increased. Thus, the total pressure gradient, which is the sum of the gravitational and frictional pressure gradient components, exhibits a minimum value, usually occurring under Slug flow conditions. This explains why intermittent flow is preferred in the production of oil and gas in vertical wells, especially when artificial-lift gas injection is used.
1.6 Two-Phase Flow Computation Algorithm

1.6.1 Pressure Gradient Calculations. The concept of the pressure gradient is illustrated in Fig. 1.14, showing a pipe segment of diameter \( d \) and length \( dL \) inclined at inclination angle of \( \theta \). The flow conditions are given, including the flow rate, fluid properties, and the pressure at one boundary of the pipe segment, say section 1. It is required to determine the pressure at the other boundary, namely, section 2. The pressure can be calculated as given below

\[
p_2 = p_1 - \left( \frac{dp}{dL} \right) dL = p_1 - (-dp),
\]

where \(-dp / dL\) is the pressure gradient, and \(-dp\) is the pressure drop. The pressure gradient represents the pressure drop per unit length along the pipe. Note that the term \(dp / dL\), based on the definition of a derivative, is negative because the pressure usually drops along the pipe, namely, \( p_2 < p_1 \). Thus, because it is more convenient to work with positive pressure drop values, it is preceded with a minus sign, as shown in Eq. 1.27.

\[
- \frac{dp}{dL} = \frac{p_2 - p_1}{dL}
\]
The total pressure gradient consists of the frictional, gravitational, and accelerational pressure gradient components, as given, respectively, by

\[
\frac{dp}{dL} = \left(\frac{dp}{dL}\right)_F - \left(\frac{dp}{dL}\right)_G - \left(\frac{dp}{dL}\right)_A. \tag{1.28}
\]

For single-phase flow, the three pressure gradient components and the total pressure gradient can be determined rigorously from the conservation laws of mass and momentum, yielding

\[
\frac{dp}{dL} = \frac{2}{d} f_F \rho v^2 + \rho g \sin \theta + \rho v \frac{dv}{dL}, \tag{1.29}
\]

and

\[
\frac{dp}{dL} = \frac{f_M \rho v^2}{2d} + \rho g \sin \theta + \rho v \frac{dv}{dL}. \tag{1.30}
\]

Note that Eq. 1.29 is given in terms of the Fanning friction factor, \( f_F \), while Eq. 1.30 is given in terms of the Moody friction factor, \( f_M \), where \( f_M = 4 f_F \). Determination of the friction factor and frictional pressure drop is based on dimensional analysis, as is discussed in Chap. 2. Determination of the friction factor is based on the Reynolds number defined as

\[
Re = \frac{\rho \left[ \frac{\text{kg}}{\text{m}^3} \right] v \left[ \frac{\text{m}}{\text{s}} \right] d [\text{m}]}{\mu \left[ \frac{\text{kg}}{\text{ms}} \right]}. \tag{1.31}
\]

As can be seen from Eq. 1.31, the Reynolds number should be dimensionless and unitless. However, when common oilfield units are used, one should exercise caution, as the set of units is not consistent. For this case, the Reynolds number is calculated as

\[
Re = \frac{1}{1.488} \frac{\rho \left[ \frac{\text{lbm}}{\text{ft}^3} \right] v \left[ \frac{\text{ft}}{\text{s}} \right] d [\text{ft}]}{\mu \left[ \text{cp} \right]}. \tag{1.32}
\]

Note that the coefficient 1,488 is a units conversion factor, namely, 1 cp = 1,488 lbm/ft s.

Based on dimensional analysis (presented in Chap. 2), the friction factor for rough pipes is a function of the Reynolds number and the pipe relative roughness, namely, \( f = f(Re, \varepsilon / d) \). The functional relationship is given in the Moody chart. Alternatively, the friction factor can be determined using convenient explicit expressions, such as the one given by Hall (1957) presented next for the Moody and Fanning friction factors, respectively,

\[
f_M = 0.0055 \left( 1 + \left[ 2 \times 10^4 \frac{\varepsilon}{d} + \frac{10^6}{Re} \right]^{1/3} \right), \tag{1.33}
\]

and
\[ f_F = 0.001375 \left\{ 1 + \left[ \frac{2 \times 10^4 \varepsilon}{d} + \frac{10^6}{\text{Re}} \right]^{1/3} \right\}. \]                             (1.34)

For smooth pipes, the friction factor is a function of the Reynolds number only, \( f = f(\text{Re}) \), and a Blasius-type equation can be used, as follows:

\[ f = C(\text{Re})^{-n}. \]                              (1.35)

For laminar flow, it can be shown rigorously that the exponent is \( n = 1 \), and that the values of the constant are \( C_F = 16 \) and \( C_M = 64 \) for the Fanning and Moody friction factors, respectively. For turbulent flow, different values are available for different ranges of the Reynolds number. For all practical purposes, the correlation covering the most wide range of the Reynolds number is \( n = 0.2 \), \( C_F = 0.046 \) for the Fanning friction factor, and \( C_M = 0.184 \) for the Moody friction factor.

For multiphase flow, the total pressure gradient cannot be determined in a straightforward manner, as done for single-phase flow, because of the complexity of the flow. Simple extension of the single-phase pressure gradient equation to two-phase flow conditions yields

\[ -\frac{dp}{dL} = \frac{2}{d} f_{TP(F)} \rho_{TP} \nu_{TP}^2 + \rho_{TP} g \sin \theta + \rho_{TP} \nu_{TP} \frac{dv_{TP}}{dz}, \]                             (1.36)

and

\[ -\frac{dp}{dL} = f_{TP(M)} \rho_{TP} \nu_{TP}^2 + \rho_{TP} g \sin \theta + \rho_{TP} \nu_{TP} \frac{dv_{TP}}{dz}. \]                             (1.37)

Eqs. 1.36 and 1.37 demonstrate the complexity of two-phase flow. One must determine the two-phase flow variables, such as the two-phase friction factor, \( f_{TP(F)} \), the two-phase velocity, \( \nu_{TP} \), and the two-phase density, \( \rho_{TP} \). By all means this cannot be done in a straightforward manner.

Predictive means for two-phase flow attempt to provide the answer to the problem just discussed. The available empirical correlations mainly provide methods to determine the two-phase friction factor and the two-phase liquid holdup (which can be used to determine the two-phase density). These can be used to solve for the pressure gradient from the previous equations. The mechanistic models attempt to predict the detailed characteristics of the flow for the different flow patterns, from which both the liquid holdup and the pressure gradient can be determined. One must keep in mind that in both cases, the solution is not rigorous but rather is a practical engineering approximation.

As is shown in the next section, determination of the pressure drop in a long pipeline or wellbore is carried out by dividing the pipe into calculation increments. This book focuses on the understanding of the flow behavior and the calculations in one increment of the flow system. For one calculation increment, it is necessary to determine the fluid properties, such as the solution gas-oil ratio, \( R_S \), the formation volume factors of the oil and gas, \( B_o \) and \( B_g \), respectively, the densities and viscosities of the two phases, and the surface tension (assuming \( f_{WC} = 0 \)). Determination of these variables, which are out of the scope of this book, can be carried out utilizing available empirical correlations or computer programs.
Given the gas and oil flow rates at standard conditions, \( q'_G \) and \( q'_O \), and knowing the fluid properties at the average pressure and temperature of the calculation increment, the superficial velocities of the phases can be determined as

\[
v_{sG} = \frac{q'_G - q'_O R_s}{A_p} B_G, \quad \text{........................................................................................................}(1.38)\]

and

\[
v_{sL} = \frac{q'_L - q'_O B_L}{A_p}, \quad \text{........................................................................................................}(1.39)\]

The fluid properties and the superficial velocities of the phases are the required input variables for all the predictive means that are covered in following chapters, including empirical correlations and mechanistic models. This results in the determination of the flow characteristics in a given calculation increment, including the liquid holdup and the pressure gradient.

1.6.2 The Need for “Marching Algorithm”. A special computation algorithm, called the “marching algorithm,” is used for the determination of the pressure distribution in long two-phase flow pipes, including pipelines and wellbores. This computation algorithm is the basis for all two-phase flow computer codes and software. Following is an illustration for the need for such a computation algorithm.

Consider single-phase liquid flow under isothermal conditions in a pipe segment of length \( L \), as shown in Fig. 1.15 Part A. The flow conditions are given, including the pipe diameter and inclination angle, flow rate, fluid properties, and the pressure at one boundary of the pipe, say section 1. The pressure usually drops along the pipe, namely, \( p_1 > p_2 \). The question is now how to determine the pressure in the other location, say section 2.

One can determine the pressure gradient in section 1, namely, \(-dp/dL)_{L1}\), where all flow conditions are given. Because the flow is incompressible, for isothermal condition, although the pressure drops and is lower in section 2, the flow characteristics do not change significantly, namely, the density is almost the same (\( \rho_{L1} \approx \rho_{L2} \)), resulting in a similar velocity, \( v_{L1} \approx v_{L2} \). Thus, the pressure gradient in section 2, \(-dp/dL)_{L2}\), is approximately the same as in section 1. Moreover, for this case of incompressible
isothermal flow, the pressure gradient in the pipe is approximately constant, independent of the pressure distribution, as given in Eq. 1.40,

\[- \frac{dp}{dL}_{L_1} \simeq - \frac{dp}{dL}_{L_2} \simeq - \frac{dp}{dL} \equiv \text{const.} \]

and the pressure drop is

\[- \Delta p_{2-1} = - \frac{dp}{dL} \]

Consider now a similar case but for gas-liquid two-phase flow, as shown in Fig. 1.15 Part B. As before, the flow conditions are given, including the gas and liquid flow rates and the pressure at one boundary of the pipe, say section 1. Again, the pressure drops along the pipe, namely, \( p_1 > p_2 \), and it is required to determine the pressure in the other location, say section 2.

As opposed to the previous case of single-phase liquid flow, this case is more complicated. It includes a compressible gas-phase, which expands as the pressure drops along the pipe. Also, because of the pressure drop, mass transfer occurs between the phases in the form of gas being liberated from the liquid-phase into the gas-phase. Thus, the flow conditions in section 2 are different than those in section 1, including the velocities \( (v_{L_1} \neq v_{L_2}) \) and \( (v_{G_1} \neq v_{G_2}) \), the liquid holdup \( (H_{L_1} \neq H_{L_2}) \), the mixture densities \( (\rho_{M_1} \neq \rho_{M_2}) \), and the existing flow pattern. As a result, the two-phase pressure gradient in sections 1 and 2 are also not equal but are rather dependent on the pressure and temperature, as shown in Eq. 1.42.

\[- \frac{dp}{dL}_{TP_1} \neq - \frac{dp}{dL}_{TP_2} \neq \text{const.} = \text{function (} p, T \text{)} \]

Thus, the pressure drop between two locations must be determined by integration, as given by

\[- \Delta p_{2-1} = \int_{L_1}^{L_2} - \frac{dp}{dL}(p, T) dL. \]

**1.6.3 Computation Procedure.** As shown in the previous section, the pressure gradient in two-phase flow pipes is not constant but rather varies along the pipe as a function of the pressure and temperature. Thus, the pressure drop must be calculated by integrating the pressure gradient along the pipe. The integration is carried out numerically by dividing the pipe into calculation increments, as shown in Fig. 1.16. As can be seen, the pipe between sections 1 and 2 is divided into \( n \) calculation increment. The pressure gradient changes from one increment to the other, but at each increment, the pressure gradient is assumed to be constant, at the average flow conditions in the increment. Thus, the pressure drop in the pipe is determined as

\[- \Delta p_{2-1} = \sum_{i=1}^{n} - \frac{dp}{dL}(i) dL_i, \]

where \(- \frac{dp}{dL}_i\) and \( dL_i \) are the average pressure gradient and the length of the increment \( i \), respectively.
1.16—Schematic of calculation increments in two-phase flow pipes.

Fig. 1.17 presents a more general schematic of the pressure drop computation procedure, showing a more elaborate piping system. As can be seen, the system is divided into \( m \) segments, in each of which a major change in a flow variable occurs, namely either a change of pipe diameter, pipe inclination angle, or flow rate. Each of the segments, in turn, is divided into \( n \) computation increments, as shown in Fig. 1.16. The pressure drop in each segment is calculated based on Eq. 1.44. The total pressure drop in the piping system is the sum of the pressure drops in the different segments, as shown in Eq. 1.45.

\[
\Delta p_{\text{TOT}} = \sum_{j=1}^{m} \sum_{i=1}^{n} -\frac{dp}{dL}_{ji} \ dL_{ji}. \ 

\textit{Computation Algorithm Flow Chart.} Two cases can be considered when calculating the pressure distribution along a pipe:

1. The temperature distribution along the pipe is known, either isothermal, linear, or given.
2. The temperature distribution along the pipe is unknown and must be determined as part of the calculations.

Only the first case is considered in this book. For this case, the pressure distribution is determined using the pressure gradient equation. For the second case, the energy equation should be used in addition to the pressure gradient (momentum) equation for the prediction of the temperature distribution.
The computation algorithm for case 1 consists of a trial-and-error calculation of the pressure drop in each calculation increment. The computation algorithm flow chart for one segment is given in Fig. 1.18. In general, one boundary pressure is given, either that of the inlet or the outlet of the piping system. Thus, one pressure boundary of each of the calculation increments is known. Referring to Fig. 1.18 that shows increment $i$, the inlet pressure $p_i$ or the outlet pressure $p_{i+1}$ are known. The objective is to determine the other pressure boundary or the pressure drop. Suppose the inlet pressure of the increment, $p_i$, is given; the trial-and-error process is carried out as discussed next.

1. Read all input data. These include the standard condition flow rates, $q'_G$ and $q'_O$, the specific gravities of the gas and the oil, $\gamma_G$ and $\gamma_O$, the segment length ($L$), diameter and inclination angle, the temperature distribution along the system, and one boundary pressure, say $p_{in}$, the segment inlet pressure.

2. Determine the calculation increment length, $\Delta L = L/n$. Set the increment inlet pressure, $p_i = p_{in}$.

3. Guess the increment outlet pressure, $p_{i+1(g)}$. For the first increment, the outlet pressure can be estimated assuming a pressure gradient of 0.0452 kPa/m (or 0.002 psi/ft). The calculations are not sensitive to this initial guess. For subsequent increments, use the determined previous increment pressure gradient to estimate the current increment outlet pressure. Set iteration clock to $Iteration=0$.

4. Calculate the average pressure in the increment, namely, $\bar{p} = (p_i + p_{i+1(g)})/2$. The average temperature increment, $\Delta T$, is known.

5. Determine the fluid properties, in-situ flow rates, and phase velocities (Eqs. 1.38 and 1.39), based on the average pressure and temperature $\bar{p}$ and $\bar{T}$.

6. Determine the average pressure gradient in the increment, $\frac{dp}{dL}$, (Eqs. 1.36 or 1.37).

7. Determine the outlet-calculated pressure of the increment based on the pressure gradient calculated, namely, $p_{i+1(c)} = p_i - \left( -\frac{dp}{dL} \right) \Delta L$.

8. Compare the guessed and calculated outlet pressures, $\left| \left( p_{i+1(g)} - p_{i+1(c)} \right) / p_{i+1(c)} \right| \leq \varepsilon$. If the values agree within less than the tolerance $\varepsilon$, convergence is achieved. If not, increase the iteration clock by 1, $Iteration=Iteration+1$, and the current calculated outlet pressure becomes the next guessed outlet pressure, namely, $p_{i+1(g)} = p_{i+1(c)}$.

9. Repeat steps 1 through 6 until convergence is achieved on the increment. Convergence should be achieved within a few iterations, say 10. So if convergence is not achieved in 10 iterations print an error message and continue the calculations by making an assumption on the pressure drop in the segments.
10. If convergence is achieved, move to the next increment. The outlet pressure of the previous increment becomes the inlet pressure of the current increment.

11. Repeat the calculations for all the increments and determine the pressure distribution and the pressure drop of the segments.

12. Repeat all calculations for all the segments in the piping system.

![Computational algorithm flow chart for one piping segment.](image-url)
1.7 Examples

Example 1.1. Oil and natural gas flow in a 5.1 cm ID horizontal pipe. The in-situ flow rates of the oil and natural gas are 0.025 m$^3$/min and 0.25 m$^3$/min, respectively. The corresponding liquid holdup is 0.35. Determine

1. The gas and liquid superficial velocities, mixture velocity, and no-slip liquid holdup.
2. The actual velocities of the two phases.
3. The slip velocity between the gas-phase and the liquid-phase.
4. The drift velocities and drift fluxes of the two phases. Show that $J_G = -J_L$.

Solution 1.1.

\[ A_p = \frac{\pi}{4} (0.051)^2 = 0.002043 \text{ m}^2 \]

1. \[ v_{SL} = \frac{q_L}{A_p} = \frac{(0.025 \text{ m}^3/\text{min.})}{(60 \text{s/min.})(0.002043 \text{ m}^2)} = 0.20 \text{ m/s,} \]
2. \[ v_{SG} = \frac{q_G}{A_p} = \frac{(0.25 \text{ m}^3/\text{min.})}{(60 \text{s/min.})(0.002043 \text{ m}^2)} = 2.04 \text{ m/s,} \]
3. \[ v_M = v_{SL} + v_{SG} = 2.24 \text{ m/s,} \quad \text{and} \quad \lambda_L = \frac{v_{SL}}{v_M} = \frac{0.20}{2.24} = 0.09. \]
4. \[ v_L = \frac{v_{SL}}{H_L} = \frac{0.20}{0.35} = 0.57 \text{ m/s,} \quad \text{and} \quad v_G = \frac{v_{SG}}{1 - H_L} = \frac{2.04}{0.65} = 3.14 \text{ m/s.} \]
5. \[ v_{SLIP} = v_G - v_L = 3.14 - 0.57 = 2.57 \text{ m/s.} \]
6. \[ v_{DL} = v_L - v_M = 0.57 - 2.24 = -1.67 \text{ m/s, and} \]
7. \[ v_{DG} = v_G - v_M = 3.14 - 2.24 = 0.90 \text{ m/s.} \]
8. \[ J_L = H_L v_{DL} = 0.35(-1.67) = -0.58 \text{ m/s,} \]
9. \[ J_G = (1 - H_L)v_{DG} = 0.65(0.90) = 0.58 \text{ m/s, and} \]
10. \[ J_G + J_L = 0.58 - 0.58 = 0 \]
Example 1.2. Starting from basic two-phase flow definitions, prove that $J_G = -J_L$.

Solution 1.2. Starting from the definitions of the drift fluxes:

$J_G = (1 - H_L) (v_G - v_M)$ and $J_L = H_L (v_L - v_M)$.

Then:

$J_G + J_L = (1 - H_L) (v_G - v_M) + H_L (v_L - v_M)$,

$J_G + J_L = (1 - H_L) v_G - (1 - H_L) v_M + H_L v_L - H_L v_M$

$= (1 - H_L) v_G + H_L v_L - v_M$.

Recall:

$v_M = v_{SL} + v_{SG} = v_L H_L + v_G (1 - H_L)$,

Thus,

$J_G + J_L = v_M - v_M = 0$. 
Chapter 2
Early Black Box Models

Rigorous solutions for two-phase flow systems are complex and not possible at the present time. This is because of the large number of flow variables associated with the two phases and also the complex nature of the flow. It seems natural that the early models developed for two-phase systems were flow pattern independent. These models, referred to as “black box” models, simply ignore the complex two-phase flow configuration and treat the flow with tools developed for single-phase flow. Four of these earlier models are presented in this chapter.

In the Homogeneous No-Slip Model, the two-phase mixture is treated as a pseudo single-phase fluid with average velocity and fluid properties. The mixture fluid properties are determined from the single-phase gas and liquid properties, which are averaged based on the no-slip liquid holdup.

The opposite approach is taken in the Separated Model. In this model, the gas-phase and the liquid-phase are assumed to flow separately from each other. Thus, each of the phases can be analyzed based on single-phase flow methods utilizing the hydraulic diameter concept, such as friction factor or heat transfer coefficient.

Similarity Analysis is a powerful technique to develop generalized solutions. This is achieved by generating the governing dimensionless groups that control a given flow system. The dimensionless groups can be generated either from the system variables (Dimensional Analysis), or from the analysis of the principal forces and the kinematic relationship of the system (Similarity through Dynamic and Kinematic Ratios), or from the basic equations governing the analyzed system (Similarity through Basic Equations). A general solution is obtained that can be applied to all similar systems.

The Drift Flux Analysis treats the two phases as a homogeneous mixture but allows slippage between the gas and the liquid phases. This is a significant improvement of the Homogeneous No-Slip Model, as the actual in-situ liquid holdup can be determined. However, additional information is required about the relative movement of the two phases, which is not always available.

2.1 Homogeneous No-Slip Model

The Homogeneous No-Slip Model is a simple yet versatile model for predicting two-phase flow behavior. In this model, the two phases are combined into one pseudo single-phase fluid with an average velocity and average fluid properties, assuming no-slip condition. The mixture can then be treated using standard single-phase flow methods. The derivation of the model follows Wallis (1969). Refer to Fig. 2.1 for a schematic of the flow.

Following are the assumptions made in the development of the model:

- Steady-state one-dimensional flow.
- The two phases are well mixed and are in equilibrium.
- No slippage occurs between the phases.
- Both phases are compressible, i.e., \( \nu_G = \nu_G(p) \) and \( \nu_L = \nu_L(p) \), where \( \nu \) is the specific volume.
• The pipe cross-sectional area $A_p$ is not constant and can vary along the axial direction, namely, $A_p = A_p(L)$.

• Mass transfer occurs between the phases, and the quality varies along the pipe, $x = x(L)$.

Fig. 2.1—Schematic of Homogeneous No-Slip Model.

The second assumption implies that both phases are present at any point of the flow field, and that the values of the velocity and the temperature of the mixture, the gas-phase and the liquid-phase are identical. The third assumption is the most severe of the Homogeneous No-Slip Model, as it implies that the in-situ liquid holdup is the no-slip liquid holdup. While these assumptions limit the applicability and accuracy of the Homogeneous No-Slip Model, the last three assumptions represent the versatility of the model, allowing compressible flow and variations of the pipe diameter and gas quality.

2.1.1 Conservation Equations. The conservation equations of mass, momentum and energy for the model are given next.

**Continuity.**

$$W = \rho_M v_M A_p = \text{const}, \quad \text{.................................................................(2.1)}$$

where $W$ is the total mass flow rate, and $\rho_M$ and $v_M$ are the mixture average density and velocity, respectively.

**Momentum.**

$$W \frac{dv_M}{dL} = -A_p \frac{dp}{dL} - S_p \tau_w - A_p \rho_M g \sin \theta, \quad \text{.................................................................(2.2)}$$

where $L$ is the axial direction, $p$ is the pressure, $\tau_w$ is the shear stress at the wall, $S_p$ is the pipe perimeter, and $\theta$ is the inclination angle from the horizontal. Dividing by the cross-sectional area $A_p$ and solving for the pressure gradient yields:

$$-\frac{dp}{dL} = S_p \frac{\tau_w}{A_p} + \rho_M g \sin \theta + \frac{W}{A_p} \frac{dv_M}{dL}$$

$$= -\frac{dp}{dL} \bigg|_F - \frac{dp}{dL} \bigg|_G - \frac{dp}{dL} \bigg|_A \quad \text{.................................................................(2.3)}$$
The total pressure gradient is the sum of the frictional, gravitational and accelerational pressure gradient components. Expressions for these components will be developed in a later section.

**Energy.**

\[
\frac{dQ}{dL} - \frac{dw_s}{dL} = W \frac{d}{dL} \left( h_M + \frac{v_m^2}{2} + g L \sin \theta \right), \quad \text{..............................................................................(2.4)}
\]

where \( dQ/dL \) and \( dw_s/dL \) are the heat transfer and the shaft work rate per a unit length of the pipe, respectively, and \( h_M \) is the mixture enthalpy. Usually, in pipe flow, the shaft work is assumed zero. Note that the heat is considered positive when transferred into the system.

**2.1.2 Mixture Average Properties.** The Homogeneous No-Slip Model requires determination of the average velocity and the average fluid properties, such as density, viscosity, and enthalpy. The mixture average velocity is the total volumetric flow rate divided by the pipe cross-sectional area, as given by

\[
v_M = \frac{q_L + q_G}{A_p} = v_{SL} + v_{SG}. \quad \text{..............................................................................(2.5)}
\]

The mixture density can be expressed either in terms of volume fraction, which in this case is the no-slip liquid holdup, namely,

\[
\rho_M = \rho_{NS} = \rho_L \lambda_L + \rho_G (1 - \lambda_L), \quad \text{..............................................................................(2.6)}
\]

where

\[
\lambda_L = \frac{q_L}{q_L + q_G} = \frac{v_{SL}}{v_{SL} + v_{SG}}, \quad \text{..............................................................................(1.6)}
\]

or in terms of mass fraction (quality),

\[
\frac{1}{\rho_M} = \frac{1}{\rho_{NS}} = \frac{x}{\rho_G} + \frac{1-x}{\rho_L} = x \nu_G + (1-x) \nu_L = x \nu_{GL} + \nu_L, \quad \text{..............................................................................(2.7)}
\]

where \( \nu_{GL} = \nu_G - \nu_L \). Eq. 2.7 can be developed from Eq. 2.6, using the relationship between liquid holdup and quality, as

\[
x = \frac{W_G}{W_G + W_L} = \frac{\rho_G v_G A_p (1 - H_L)}{\rho_G v_G A_p (1 - H_L) + \rho_L v_L A_p H_L}, \quad \text{..............................................................................(2.8)}
\]

or no-slip conditions \( \nu_L = \nu_G \) and \( H_L = \lambda_L \), and Eq. 2.8 reduces to

\[
x = \frac{\rho_G (1 - \lambda_L)}{\rho_{NS}}, \quad \text{..............................................................................(2.9)}
\]

Note that Eqs. 2.9 and 2.7 are applicable only for no-slip condition.

Determination of two-phase mixture viscosity is not a simple task. Various methods have been proposed by two-phase flow researchers to estimate this parameter. By all means, these are not rigorous methods, and they must be treated as approximations. Either the expression based on the slip in-situ
liquid holdup ($\mu_{\text{SLIP}}$) or the Dukler et al. (1964) expression, based on the no-slip liquid holdup ($\mu_{\text{NS}}$), as given next, can be used.

$$\mu_m = \mu_{\text{SLIP}} = \mu_L H_c + \mu_G (1 - H_c),$$ .................................................................(2.10)

and

$$\mu_m = \mu_{\text{NS}} = \mu_L \lambda_L + \mu_G (1 - \lambda_L).$$ .................................................................(2.11)

Other expressions for the mixture viscosity can be found in the literature. Note that the no-slip average mixture viscosity, $\mu_{\text{NS}}$, has some physical justification, as the no-slip liquid holdup represents the ratio of the flow rates of the gas and liquid phases, which is important for determining the frictional losses. Clearly, for the Homogeneous No-Slip Model, the no-slip average mixture viscosity, $\mu_{\text{NS}}$, is used.

The mixture enthalpy must be averaged based on mass fraction and is given by

$$h_m = h_G x + h_L (1 - x).$$ .........................................................................................(2.12)

The mixture properties can be utilized to calculate the pressure distribution (Eq. 2.3) and the temperature distribution (Eq. 2.4) of the flow, applying standard single-phase flow methods. In the following section, only the pressure distribution is considered, and the equations for determination of the three pressure gradient components are developed.

**2.1.3 Pressure Drop.** The total pressure gradient given in Eq. 2.3 is composed of three components: frictional, gravitational, and accelerational components. This equation is further developed here to enable the calculation of each of the pressure gradient components and the total pressure gradient.

**Frictional Pressure Gradient Component.** The frictional pressure gradient component, from Eq. 2.3, is given by

$$-\frac{dp}{dL}_{F} = \frac{S_p}{A_p} \tau_w = \frac{4\tau_w}{d}.$$ .................................................................(2.13)

This pressure gradient is developed into a more practical form by expressing the wall shear stress in terms of a friction factor. The average wall shear stress can be expressed in terms of the friction factor (Fanning), as

$$\tau_w = \frac{1}{2} f_F \rho_{\text{NS}} v_M^2,$$ .........................................................................................(2.14)

where $f_F$ is the Fanning friction factor. Substituting Eq. 2.14 into Eq. 2.13 results in the final form for the frictional pressure gradient component, namely,

$$-\frac{dp}{dL}_{F} = \frac{2}{d} f_F \rho_{\text{NS}} v_M^2 = \frac{2}{d} f_F \frac{G^2}{\rho_{\text{NS}}},$$ .................................................................(2.15)

where $G$ is the mixture total mass flux, and $G = \rho_{\text{NS}} v_M$ for no-slip condition. The mixture density can be determined either from Eq. 2.6 or Eq. 2.7. Determination of the friction factor is based on the mixture no-slip Reynolds number defined as
\[ \text{Re}_M = \text{Re}_{NS} = \frac{\rho_{NS} v_M d}{\mu_{NS}}. \] .................................................................(2.16)

For rough pipes, \( f_r = f_F (\text{Re}_M, \varepsilon / d) \), and it can be determined from a Moody chart or from Eq. 1.34. For smooth pipes, \( f_r = f_F (\text{Re}_M) \), and a Blasius'-type equation can be used, as given in Eq. 1.35.

**Gravitational Pressure Gradient Component.** The gravitational pressure gradient component can be determined directly from Eq. 2.3, as

\[-\frac{dp}{dL} = \rho_M g \sin \theta = \rho_{NS} g \sin \theta. \] .................................................................(2.17)

Again, the mixture density can be determined from Eq. 2.6 or Eq. 2.7. In general, the use of the density based on the in-situ liquid holdup in the determination of the gravitational pressure gradient component is rigorous. This is because the gravitational head depends on the weight of the two phases, which are related to the in-situ volume fractions of the two phases. In the Homogeneous No-Slip Model, however, the no-slip liquid holdup is used in the determination of the mixture density, resulting in an underprediction of the gravitational pressure gradient. This represents the most severe limitation of the homogeneous no-slip model.

**Accelerational Pressure Gradient Component.** Determination of the accelerational pressure gradient component is the most difficult one. In many models or correlations, it is simply ignored. The Homogeneous No-Slip Model provides a systematic method, well attached to the physical phenomena, to predict this pressure gradient component. Starting from the expression in Eq. 2.3, substituting for the mixture velocity \( v_M = \frac{W}{A_p \rho_{NS}} \) and expanding the derivative yields

\[-\frac{dp}{dL} = \frac{w}{A_p} \frac{dv_M}{dL} = G \frac{d}{dL} \left( \frac{w}{A_p \rho_{NS}} \right) \]

\[ = G^2 \frac{d}{dL} \left( \frac{1}{\rho_{NS}} \right) - G^2 \frac{1}{\rho_{NS}} \frac{dA_p}{dL}. \] .........................................................(2.18)

The derivative \( \frac{d}{dL} \left( \frac{1}{\rho_{NS}} \right) \) can be obtained by differentiating Eq. 2.7 in terms of the specific volumes of the two phases, yielding

\[ \frac{d}{dL} \left( \frac{1}{\rho_{NS}} \right) = \frac{d}{dL} \left[ x v_G + (1-x)v_L \right] = v_{GL} \frac{dx}{dL} + x \frac{dv_G}{dL} + (1-x) \frac{dv_L}{dL}. \] .........................................................(2.19)

As the two phases are treated as compressible fluids, namely, \( v_G = v_G(p) \) and \( v_L = v_L(p) \), Eq. 2.19 can be expanded in terms of the pressure as

\[ \frac{d}{dL} \left( \frac{1}{\rho_{NS}} \right) = v_{GL} \frac{dx}{dL} + \frac{dp}{dL} \left( x \frac{dv_G}{dp} + (1-x) \frac{dv_L}{dp} \right). \] .........................................................(2.20)
Substituting Eq. 2.20 into Eq. 2.18 yields the final form for the accelerational pressure gradient, given by

\[- \frac{dp}{dL} \bigg|_A = G^2 \left[ \nu_{GL} \frac{dx}{dL} + \frac{dp}{dL} \left( x \frac{d\nu_G}{dp} + (1-x) \frac{d\nu_L}{dp} \right) \right] - \frac{G^2}{\rho_{NS}} \frac{1}{A_p} \frac{dA_p}{dL}. \] .................(2.21)

**Total Pressure Gradient.** Combining the frictional, gravitational, and accelerational pressure gradient components, given respectively in Eqs. 2.15, 2.17, and 2.21, yields the total pressure gradient, which is written as

\[- \frac{dp}{dL} = \frac{2}{d} f_f \rho_{NS} \nu_m^2 + \rho_{NS} g \sin \theta \]

\[+ G^2 \left[ \nu_{GL} \frac{dx}{dL} + \frac{dp}{dL} \left( x \frac{d\nu_G}{dp} + (1-x) \frac{d\nu_L}{dp} \right) \right] \] .................................(2.22)

\[- \frac{G^2}{\rho_{NS}} \frac{1}{A_p} \frac{dA_p}{dL}. \]

Note that the total pressure gradient appears in both sides of the equation because the accelerational pressure gradient is a function of the total pressure gradient. Solving for the total pressure gradient yields

\[- \frac{dp}{dL} = \frac{2}{d} f_f \rho_{NS} \nu_m^2 + \rho_{NS} g \sin \theta + G^2 \nu_{GL} \frac{dx}{dL} - \frac{G^2}{\rho_{NS}} \frac{1}{A_p} \frac{dA_p}{dL} \]

\[1 + G^2 \left( x \frac{d\nu_G}{dp} + (1-x) \frac{d\nu_L}{dp} \right) \] .................................(2.23)

The numerator of the right-hand side of Eq. 2.23 shows, respectively, the effect of friction, gravitation, phase change, and area change on the total pressure gradient. The denominator represents the effect of compressibility on the pressure drop. For special cases, the equation can be reduced as shown next.

1. Constant cross-sectional area: \( A_p = \text{const.} \) \quad \frac{dA_p}{dL} = 0,

2. No phase change: \( x = \text{const.} \) \quad \frac{dx}{dL} = 0,

3. Incompressible liquid: \( \nu_L = \text{const.} \) \quad \frac{d\nu_L}{dL} = 0.

**2.1.4 Mixture Velocity of Sound.** The denominator of the right-hand side of Eq. 2.23 can be related to the velocity of sound of the two-phase mixture, namely,

\[ M^2 = \left( \frac{\nu_m}{c_M} \right)^2 = -G^2 \left( x \frac{d\nu_G}{dp} + (1-x) \frac{d\nu_L}{dp} \right) \] .................................(2.24)

where "M" is mixture Mach number, and \( c_M \) is the mixture velocity of sound. Solving for \( c_M \) yields
\[
\frac{1}{c_m^2} = -\rho_N^2 \left( x \frac{d\rho_G}{dp} + (1-x) \frac{d\rho_L}{dp} \right) \] .................................................................(2.25)

Substituting the mixture density from Eq. 2.6 and the relationship between the quality and holdup from Eq. 2.9 into Eq. 2.25 yields

\[
\frac{1}{c_m^2} = \left[ \rho_l \lambda_L + \rho_G (1-\lambda_L) \right] \left[ (1-\lambda_L) \rho_G \left( -\frac{d\rho_G}{dp} \right) + \lambda_L \rho_L \left( -\frac{d\rho_L}{dp} \right) \right] \] .................................................................(2.26)

The velocity of sound of the gas and the liquid phases are defined as

\[
\frac{1}{c_G^2} = \frac{d\rho_G}{dp} = \rho_G \left( -\frac{d\rho_G}{dp} \right), \quad \text{and} \quad \frac{1}{c_L^2} = \frac{d\rho_L}{dp} = \rho_L \left( -\frac{d\rho_L}{dp} \right) \] .................................................................(2.27)

Substituting into Eq. 2.26 yields

\[
\frac{1}{c_m^2} = \left[ \rho_l \lambda_L + \rho_G (1-\lambda_L) \right] \left[ \frac{1-\lambda_L}{\rho_G c_G^2} + \frac{\lambda_L}{\rho_L c_L^2} \right] \] .................................................................(2.28)

Eq. 2.28 provides some insight into the two-phase flow velocity of sound. For the case where \( \rho_L >> \rho_G \) and \( \rho_L c_L^2 >> \rho_G c_G^2 \), the equation reduces to

\[
c_m^2 = \frac{\rho_G c_G^2}{\rho_L \lambda_L (1-\lambda_L)} \] .................................................................(2.29)

Clearly, the velocity of sound of the mixture is less than the velocity of sound of the gas phase. It has a minimum at \( \lambda_L = 1/2 \). For an air-water mixture at atmospheric pressure, the sonic velocity at this value of liquid holdup is about 22 m/s.

### 2.2 Separated Model

The Separated Model (Lockhart And Martinelli, 1949) is limited to the calculation of the frictional pressure losses in horizontal pipes. In this model, the gas-phase and the liquid-phase are assumed to flow separately from each other. Each phase flows in part of the cross-sectional area of the pipe, as shown in Fig.2.2. Single-phase flow methods based on the hydraulic diameter concept are assumed to apply to each of the phases. Matching conditions are required to couple the two phases in order to obtain a solution.

![Fig. 2.2—Schematic of Separated Model.](image)
2.2.1 Basic Equations. The frictional pressure gradient in the liquid-phase may be written in terms of the single-phase Fanning friction factor, utilizing the hydraulic diameter concept, as

\[- \frac{dp}{dL} = \frac{2}{d_L} f_L \rho_L v_L^2. \] \hfill (2.30)

Similarly for the gas-phase,

\[- \frac{dp}{dL} = \frac{2}{d_G} f_G \rho_G v_G^2. \] \hfill (2.31)

The sum of the cross-sectional areas of the liquid-phase and the gas-phase equals to the total cross-sectional area of the pipe, namely

\[ A_L + A_G = A_p. \] \hfill (2.32)

2.2.2 Auxiliary Relationship. In Eqs. 2.30 and 2.31, \( d_L \) and \( d_G \) are the hydraulic diameters of the liquid-phase and the gas-phase, respectively. The actual cross-sectional area of each phase can be related to its hydraulic diameter as

\[ A_L = a \left( \frac{\pi}{4} d_L^2 \right), \] \hfill (2.33)

and

\[ A_G = b \left( \frac{\pi}{4} d_G^2 \right). \] \hfill (2.34)

The parameters \( a \) and \( b \) are the ratio of the actual cross-sectional area of flow of each phase to the area of a circle based on the hydraulic diameter of the phase. Eqs. 2.33 and 2.34 can be utilized to determine \( v_L \) and \( v_G \), the respective liquid- and gas-phase velocities, namely,

\[ v_L = \frac{W_L}{A_L \rho_L} = \frac{q_L}{A_L} a \left( \frac{\pi}{4} d_L^2 \right), \] \hfill (2.35)

and

\[ v_G = \frac{W_G}{A_G \rho_G} = \frac{q_G}{A_G} b \left( \frac{\pi}{4} d_G^2 \right). \] \hfill (2.36)

The friction factors \( f_L \) and \( f_G \) can be expressed in terms of the Blasius' equation as

\[ f_L = C_L (Re_L)^{-n} = C_L \left[ \frac{q_L}{a \left( \frac{\pi}{4} d_L^2 \right)} \rho_L d_L \frac{1}{\mu_L} \right]^{-n}. \] \hfill (2.37)
2.2.3 Pressure Drop. Substituting Eqs. 2.35 and 2.37 into Eqs. 2.30 results in

\[
\frac{-dp}{dL} = \frac{2}{d} C_L \left[ \left( \frac{q_L}{\frac{\pi}{4} d_L^2} \right) / \mu_L \right]^{n} \frac{\rho_L d_L}{\rho_L \left( \frac{q_L}{\frac{\pi}{4} d_L^2} \right)} \left( \frac{q_L}{\frac{\pi}{4} d_L^2} \right)^2, \quad \text{..................................................................................(2.39)}
\]

and similarly for the gas:

\[
\frac{-dp}{dL} = \frac{2}{d} C_G \left[ \left( \frac{q_G}{\frac{\pi}{4} d_G^2} \right) / \mu_G \right]^{m} \frac{\rho_G d_G}{\rho_G \left( \frac{q_G}{\frac{\pi}{4} d_G^2} \right)} \left( \frac{q_G}{\frac{\pi}{4} d_G^2} \right)^2. \quad \text{..................................................................................(2.40)}
\]

Rearranging Eqs. 2.39 and 2.40 yields

\[
\frac{-dp}{dL} = \left( \frac{2}{d} C_L \right) \left[ \left( \frac{q_L}{\frac{\pi}{4} d_L^2} \right) / \mu_L \right]^{n} \frac{\rho_L d_L}{\rho_L \left( \frac{q_L}{\frac{\pi}{4} d_L^2} \right)} \left( \frac{q_L}{\frac{\pi}{4} d_L^2} \right)^2 \quad \frac{1}{\left( \frac{d}{d_L} \right)^{5-n}}. \quad \text{..................................................................................(2.41)}
\]

and

\[
\frac{-dp}{dL} = \left( \frac{2}{d} C_G \right) \left[ \left( \frac{q_G}{\frac{\pi}{4} d_G^2} \right) / \mu_G \right]^{m} \frac{\rho_G d_G}{\rho_G \left( \frac{q_G}{\frac{\pi}{4} d_G^2} \right)} \left( \frac{q_G}{\frac{\pi}{4} d_G^2} \right)^2 \quad \frac{1}{\left( \frac{d}{d_G} \right)^{5-m}}. \quad \text{..................................................................................(2.42)}
\]

Note that the bracketed term in Eq. 2.41 is the pressure gradient, which would occur if only the liquid-phase flows in the pipe. This is termed the liquid superficial pressure gradient and is designated as \( -\frac{dp}{dL} \) \text{\textsubscript{SL}}. Similarly, \( -\frac{dp}{dL} \) \text{\textsubscript{SG}}; the gas superficial pressure gradient, is the bracketed term in Eq. 2.42. Expressing Eq. 2.41 in terms of the liquid superficial pressure gradient yields

\[
\frac{-dp}{dL} = \frac{2}{d} C_L \left[ \left( \frac{q_L}{\frac{\pi}{4} d_L^2} \right) / \mu_L \right]^{n} \frac{\rho_L d_L}{\rho_L \left( \frac{q_L}{\frac{\pi}{4} d_L^2} \right)} \left( \frac{q_L}{\frac{\pi}{4} d_L^2} \right)^2 \quad \frac{1}{\left( \frac{d}{d_L} \right)^{5-n}}. \quad \text{..................................................................................(2.43)}
\]

Defining the liquid-phase dimensionless pressure group, \( \phi_L \), Eq. 2.43 can be rewritten as

\[
\text{........................................................................(2.44)}
\]
\[
\phi_L = \sqrt{-\frac{dp}{dL}_{SL}}_L - a^{(n-2)/2} \left( \frac{d}{dL} \right)^{(5-n)/2} \ldots \ldots \ldots \ldots \ldots \ldots (2.44)
\]

Similarly, for the gas-phase,
\[
-\frac{dp}{dL}_G = -\frac{dp}{dL}_{SG} b^{(m-2)/2} \left( \frac{d}{dL} \right)^{(5-m)/2} \ldots \ldots \ldots \ldots \ldots \ldots (2.45)
\]

and
\[
\phi_G = \sqrt{-\frac{dp}{dL}_G / -\frac{dp}{dL}_{SG}} = b^{(m-2)/2} \left( \frac{d}{dL} \right)^{(5-m)/2} \ldots \ldots \ldots \ldots \ldots \ldots (2.46)
\]

Dividing \( \phi_G \) by \( \phi_L \) results in
\[
\left(\frac{\phi_G}{\phi_L}\right) = \sqrt{-\frac{dp}{dL}_G / -\frac{dp}{dL}_{SG}} \ldots \ldots \ldots \ldots \ldots \ldots (2.47)
\]

It is assumed that the pressure gradient in the liquid-phase and the pressure gradient in the gas-phase are equal during steady-state flow, namely,
\[
-\frac{dp}{dL}_G = -\frac{dp}{dL}_L \ldots \ldots \ldots \ldots \ldots \ldots (2.48)
\]

Thus, Eq. 2.47 becomes
\[
\left(\frac{\phi_G}{\phi_L}\right) = X \ldots \ldots \ldots \ldots \ldots \ldots (2.49)
\]

Eq. 2.49 is the definition of the Lockhart and Martinelli parameter. This parameter is the square root of the ratio of the liquid superficial pressure gradient to the gas superficial pressure gradient. As it depends only on the phases’ flow rates and fluid properties, it can be determined from inlet flow conditions data. In more explicit form, \( X \) can be calculated as
\[ X^2 = \frac{-dp}{dL_{SL}} = \frac{2}{d} C_L \left[ \frac{v_{SL} \rho_L d^2}{\mu_L} \right]^n \rho_L v_{SL}^2 \]

\[ -\frac{dp}{dL_{SG}} = \frac{2}{d} C_G \left[ \frac{v_{SG} \rho_G d^2}{\mu_G} \right]^m \rho_G v_{SG}^2 \]

Dividing Eq. 2.46 by Eq. 2.44 and using the definition of \( X \) given by Eq. 2.49 yields

\[ X = \frac{b^{(m-2)y2} \left( \frac{d}{d_G} \right)^{(5-m)y2}}{a^{(n-2)y2} \left( \frac{d}{d_L} \right)^{(5-n)y2}} \]

### 2.2.4 Liquid Holdup

Substituting the expressions for the cross-sectional area for flow of the liquid and gas phases, given respectively in Eqs. 2.33 and 2.34, into Eq. 2.32 yields

\[ a \frac{\pi}{4} d^2 + b \frac{\pi}{4} d^2 = \frac{\pi}{4} d^2 \]

and dividing both sides by the right-hand side, Eq. 2.52 becomes

\[ a \left( \frac{d_L}{d} \right)^2 + b \left( \frac{d_G}{d} \right)^2 = 1 \]

The liquid holdup can be determined in a straightforward manner as

\[ H_L = a \left( \frac{d_L}{d} \right)^2 = 1 - b \left( \frac{d_G}{d} \right)^2 \]

### 2.2.5 Summary and Conclusions

So far, the model consists of two equations, namely Eqs. 2.51 and 2.53, for which solution is not possible, as they have four unknowns including \( a, b, \frac{d_L}{d}, \) and \( \frac{d_G}{d} \). At this point, it is postulated that all the four unknowns are unique functions of the Lockhart and Martinelli parameter \( X \). It follows that the dimensionless pressure parameters \( \phi_G \) and \( \phi_L \), given in Eqs. 2.44 and 2.46, and the liquid holdup \( H_L \), given by Eq. 2.54, are also unique functions of the parameter \( X \). The hypothesis must be subject to experimental verification. This has been carried out by plotting \( \phi_G \) and \( \phi_L \), and \( H_L \) vs. \( X \) using several experimental data sets. The data included results from small-diameter pipes with diameters between 0.15 to 2.54 cm. The experiments utilized air as the gas-phase, and several liquids were used, including water, kerosene, diesel fuel, and various oils. The data showed a reasonable correlation with the parameter \( X \). Four cases have been proposed, depending on the phases being in turbulent or laminar flow. This is consistent with Eq. 2.51, in which different combinations of the exponents \( m \) (for the gas-phase) and \( n \) (for the liquid-phase) are possible, as given in Table 2.1.
The results are shown in Fig. 2.3. As can be seen, there is some overlap of the \( \phi \) parameter curves. Moreover, for the liquid holdup, all the different cases collapse into one curve.

### Solution Procedure

For a given set of flow conditions, the solution is obtained by first calculating the Lockhart and Martinelli parameter from Eq. 2.50. This can be done from the given input data, as it does not require any additional information. When calculating the parameter \( X \), one can determine the liquid and gas flow regimes, namely, turbulent or laminar. (Note that this is done as an approximation, as the determination of the phase regimes should be based on the actual velocities and actual Reynolds numbers, and not on the superficial values of these parameters). The liquid holdup can be read directly from Fig. 2.3 with the calculated value of \( X \). To determine the pressure drop, one can read the corresponding value of either \( \phi_G \) or \( \phi_L \). The frictional pressure gradient can then be calculated.
from either Eq. 2.44 (if $\phi_L$ is used) or from Eq. 2.46 (if $\phi_G$ is used), as given in Eq. 2.55. The results from both equations should be the same.

$$-\frac{dp}{dL}_{L(F)} = \phi_L^2 \left(-\frac{dp}{dL}_{SL}\right), \quad \text{and} \quad -\frac{dp}{dL}_{G(F)} = \phi_G^2 \left(-\frac{dp}{dL}_{SG}\right). \quad \text{.................................(2.55)}$$

Chisholm (1967) curve fitted the original plots and provided convenient equations for both $\phi_L$ and $H_L$, as given below:

$$\phi_L^2 = 1 + \frac{C}{X} + \frac{1}{X^2}, \quad \text{.................................................(2.56)}$$

where the constant $C$ can be determined as given in Table 2.2, and the liquid holdup is given by

$$H_L = 1 - (1 + X^{0.8})^{-0.378}. \quad \text{.................................................(2.57)}$$

### Table 2.2—Chisholm (1967) Correlation Coefficients

<table>
<thead>
<tr>
<th>Liquid-Phase</th>
<th>Gas-Phase</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turbulent</td>
<td>Turbulent</td>
<td>20</td>
</tr>
<tr>
<td>Laminar</td>
<td>Turbulent</td>
<td>12</td>
</tr>
<tr>
<td>Turbulent</td>
<td>Laminar</td>
<td>10</td>
</tr>
<tr>
<td>Laminar</td>
<td>Laminar</td>
<td>5</td>
</tr>
</tbody>
</table>

#### 2.3 Similarity Analysis

Engineers are faced many times with the problem of drawing conclusions from experiments conducted on a small-scale model in the laboratory, before designing a full-scale prototype. For example, if a full-scale aircraft prototype must be designed, experimental testing may be run on a small-scale aircraft model in a wind tunnel. However, one must address the question whether it is possible to predict the full-scale aircraft behavior from the laboratory model measurements. This is possible only if similarity exists between the model and the prototype. The conditions that ensure similarity are divided into three categories, namely, geometric similarity, kinematic similarity and dynamic similarity.

Geometric similarity implies that the model and the prototype are geometrically similar. This means that all lengths of the model that are scaled with reference to a characteristic length of the model (such as the model diameter) are equal to all lengths of the prototype scaled with reference to the same characteristic length of the prototype (prototype diameter). Thus, geometric similarity ensures that the model and the prototype have a similar shape. Kinematic similarity implies that all velocities scaled with reference to a characteristic velocity (such as the average velocity) are equal in the model and prototype, ensuring similar velocity profiles. Dynamic similarity implies that the ratios between the different forces in the model are the same as the ratio of the respective forces in the prototype. These forces are the viscous, pressure, and inertia forces. Dynamic similarity is the most restrictive category as it implies both geometric and kinematic similarities. Thus, if the model and the prototype are dynamically similar, measurements obtained with a small-scale model in the laboratory can be related to the full-scale prototype.

For example, as shown in a later section, for single-phase flow in a smooth pipe, similarity exists when the Euler number, Eu, (ratio of pressure to inertia forces) and the Reynolds number, Re, (ratio of
inertia to viscous forces) are the same in the model and prototype. This also implies that there is a unique and universal relationship between the Reynolds number and the Euler number. The functional relationship between the Euler and Reynolds numbers must be determined from experimental data. As similarity is ensured, the results obtained from the small-scale model can be applied to predict the full-scale prototype behavior. In other words, a solution developed based on the model may be extrapolated beyond the conditions under which the data were taken. In fact, if the similarity analysis is carried out successfully, a universal solution results that can be applied to all similar systems, which for this example are all smooth pipes.

There are three basic approaches to similarity analysis:

1. **Dimensional Analysis** is a technique through which the governing dimensionless groups that control a given system are determined. This method does not require knowledge of the mechanisms governing the considered system. The only needed input includes the variables that control the given system. The variables are used to determine the governing dimensionless groups by using the Buckingham $\Pi$ Theorem. The resulting dimensionless groups ensure similarity to exist. Once the dimensionless groups are determined, experimental tests are conducted on a small-scale model, to find the functional relationship between the dimensionless groups, namely the correlation. The experimental work is significantly reduced when the dimensionless groups, rather than the dimensional variables, are considered. The final result is a universal solution, which can be applied to all similar systems.

2. The analysis of Similarity through Dynamic and Kinematic Ratios, as well, does not require a complete knowledge of the mechanisms governing the analyzed system. However, as opposed to dimensional analysis, this method does require some knowledge of the behavior of the system. The analysis is carried out by considering the principal forces and the kinematic relationship of the system, making some simplifying assumptions for the process involved. This results in a much smaller number of governing dimensionless groups, as compared to dimensional analysis. As before, the dimensionless groups are correlated with experimental data to develop the unique and general relationship applicable to all similar systems. One must realize that this method is approximate, as it involves simplifying assumptions of the system.

3. The Similarity through Basic Equations analysis requires knowledge of the basic equations governing the analyzed system. This substantial information is not always available, and also, sometimes simplifying assumptions are made. The approach is to nondimensionalize the governing equations. This is carried out by scaling the different variables with respect to characteristic reference variables. For example, length variables are scaled with respect to the pipe diameter and velocities with respect to the average velocity, etc. In this process, the set of governing dimensionless groups, which must be equal in the model and prototype, emerge from the nondimensional equations. Once again, this set is much smaller than the one obtained from dimensional analysis. Moreover, the functional relationship between the dimensionless groups is also given by the nondimensional equations, which can be verified by limited number of experiments. Again, this method is not rigorous but must be considered as approximate.

For the simple example of flow in smooth pipes, all three methods yield the same result. This is because of the fact that this is a simple problem, which is involved with only a few variables. The result is that for this case, the Euler number is a unique function of the Reynolds number, namely, $Eu = Eu(Re)$. For two-phase flow, the flow behavior is more complex, for which the different similarity methods cannot be applied in a straightforward manner. In the following sections, the Dimensional Analysis and
Similarity through Dynamic and Kinematic Ratios are applied to two-phase flow conditions. Similarity through Basic Equations is demonstrated in the following chapters with the mechanistic models.

2.3.1 Dimensional Analysis. An important aspect of Dimensional Analysis is that it can be used to obtain a solution when all other methods fail. An example is the prediction of single-phase frictional pressure drop or friction factor. For laminar flow, it is possible to solve for the friction factor rigorously from the equations of motion and obtain \( f = \frac{16}{Re} \). For turbulent flow, however, no analytical solution can be obtained because of the complexity of the flow. For this case, there is no other choice but to develop a solution through Dimensional Analysis, as discussed in the next section.

**Review: Dimensional Analysis for Single-Phase Flow.** Dimensional Analysis has been applied successfully to single-phase flow problems. Examples are the prediction of turbulent flow frictional pressure drop and heat transfer. The solution is obtained using the Buckingham \( \Pi \) Theorem. The step-by-step procedure is reviewed and demonstrated in the following example.

**Example.** Determination of the frictional pressure drop for turbulent flow in smooth pipes. Refer to Fig. 2.4.

![Fig. 2.4 — Variables for Buckingham \( \Pi \) theorem.](image)

1. List all parameters involved, including the dependent and the independent variables:

\[ \rho, v, d, \mu, -\frac{dp}{dL}_F \]  

\( m = 5 \) parameters).

2. Select a set of primary (fundamental) dimensions. Choose the \( MLt \) or \( FLt \) system. For heat transfer problems, the \( MLtT \) system is needed. For this example,

\( M, L, t \)  

\( n = 3 \) primary dimensions).

3. Express all the parameters in terms of the primary dimensions:

\[ \rho = \frac{M}{L^3}, \quad v = \frac{L}{t}, \quad d = L, \quad \mu = \frac{M}{Lt}, \quad -\frac{dp}{dL}_F = \frac{M}{L^2 t^2}. \]

4. Select from the list of the parameters a number of repeating parameters equal to the number of primary dimensions. The repeating parameters should include all the primary dimensions. The dependent parameter should be excluded.
\( \rho, \nu, d \) \((n = 3\) repeating parameters). 

5. Form \( m-n \) dimensionless groups, combining the repeating parameters with each of the remaining parameters.

\[
\Pi_1 = -\left(\frac{dp}{dL}\right)_F \rho^a \nu^b d^c \\
= \left(\frac{M}{L^2 \nu^2}\right)^a \left(\frac{L}{t}\right)^b (L)^c = M^0 L^0 t^0.
\]

Summing exponents:

\[
M: \quad 1 + a = 0 \quad a = -1 \\
t: \quad -2 - b = 0 \quad b = -2 \\
L: \quad -2 - 3a + b + c = 0 \quad c = 1
\]

Therefore, the first dimensionless group is

\[
\Pi_1 = \text{Eu} = 2f_F = \frac{-\left(\frac{dp}{dL}\right)}{\rho \nu^2}d_t. \quad \text{...........................................................}(2.58)
\]

Similarly, the second dimensionless group is formed as

\[
\Pi_2 = \mu \rho^d \nu^e d^f \\
= \left(\frac{M}{Lt}\right)^d \left(\frac{L}{t}\right)^e (L)^f = M^0 L^0 t^0.
\]

Summing of the exponents yields

\[
M: \quad 1 + d = 0 \quad d = -1 \\
t: \quad -1 - e = 0 \quad e = -1 \\
L: \quad -1 - 3d + e + f = 0 \quad f = -1
\]

and the second dimensionless group is

\[
\Pi_2 = \text{Re} = \frac{\rho \nu d}{\mu}. \quad \text{...........................................................}(2.59)
\]

Note that the governing two dimensionless groups formed in the process are the Euler number (twice the Fanning friction factor) and the Reynolds number. In conclusion, the solution for the frictional pressure drop in a smooth pipe is

\[
\text{Eu} = \text{Eu(Re).} \quad \text{...........................................................}(2.60)
\]
For rough pipes, an additional parameter must be considered (i.e., the pipe roughness \( \varepsilon \)). Thus, the number of parameters increases to six, yielding three dimensionless groups, namely the Euler and Reynolds numbers and the relative roughness, as shown in Eq. 2.61.

\[
\text{Eu} = \text{Eu}(\text{Re}, \varepsilon / d).
\]

The conclusion from the analysis is that if the Reynolds number and the relative roughness are the same in a model and in a prototype, then the Euler number in the two systems must be the same. In other words, similarity exists between the model and the prototype. Thus, experiments conducted on the model can be extrapolated and applied to the full-scale prototype.

In the second part of the dimensional analysis procedure, the functional relationship between the Euler number, the Reynolds number, and the relative roughness must be determined experimentally. This is, in fact, the well known Moody chart, for which an explicit expression for the Fanning friction factor is given in Eq. 1.34.

**Discussion.** The example previously given reveals the advantages and disadvantages of the Dimensional Analysis method.

**Advantages.**

- Dimensional Analysis is suited for problems that cannot be solved rigorously because of their complexity, such as the pressure loss and heat transfer processes in turbulent flow. No other solution is available for these problems at the present time.

- As the Dimensional Analysis ensures similarity, the solution obtained is universal and can be applied to all similar systems. For the case of the frictional pressure drop, the developed solution is applicable to any pipe diameter and roughness, any fluid velocity, and any density or viscosity.

- Although experimental data are required, dimensional analysis reduces the number of experimental runs substantially. This is achieved through the use of dimensionless groups rather than dimensional variables. For the example of turbulent flow frictional pressure drop, to obtain the necessary data on the pressure gradient, \(-dp/dL\), using the dimensional variables \( \rho, v, d, \mu \) and \( \varepsilon \) is unmanageable. Suppose 10 runs are needed to examine the effect of changing one variable while holding the others constants. Not only will it be needed to use 10 different pipe diameters, 10 fluids with different density etc., but a total of \(10^5\) runs are required. This will take many years of experimental work. Also, there will be a problem presenting this large amount of data graphically. Using the dimensionless groups, on the other hand, the number of parameters is reduced from 6 dimensional variables to 3 dimensionless groups, which is manageable. For this case, of the order of only \(5 \times 10^3\) experiments (and not \(10^5\)) will be needed. Thus, meaningful results are obtained with significantly less effort and significantly less experimental data acquisition.

**Disadvantages.**

- Dimensional analysis does not present a systematic analysis of the problem that is being solved. Some basic knowledge about the problem is required in order to list all the variables involved. However, no insight into the physical phenomena is provided. Even though a solution is obtained, it does not address the “why” and “how” concepts of the problem, and no in-depth understanding of the phenomena is developed.
• The first step of the Buckingham $\Pi$ Theorem requires the listing of all the variables that may affect the problem to be solved. An apriori selection of all the pertinent parameters is difficult. It is possible either to exclude an important parameter or to include a parameter that does not affect the system. However, even if such an error is made, the experimental results will resolve this problem. If an important parameter is not considered, no consistent solution will be obtained. If an extraneous parameter is included, the experimental data will show that this parameter is not important and should be eliminated.

• The process of forming the dimensionless groups is arbitrary. First, the selection of different sets of repeating parameters will result in different sets of dimensionless groups. Also, any set of dimensionless groups formed from a selected set of repeating parameters is not unique. Any combination of these groups is also an acceptable solution. For example, if $\Pi_1$ and $\Pi_2$ are generated in the analysis, then $\Pi_1\Pi_2$ and $\Pi_1/\Pi_2$ are also an accepted set of dimensionless groups. Indeed, there are infinite numbers of possible sets of dimensionless groups for a given problem. Within each set of dimensionless groups, the experimental work will show which groups are important and which are not. A set with the minimum number of dimensionless groups is preferable. Thus, this method is heavily dependent on the experimental work, which can take time and effort.

The previous discussion is illustrated in Fig. 2.5. For this case, three dimensionless groups are considered, namely $\Pi_1$, $\Pi_2$, and $\Pi_3$. Fig. 2.5 Part A demonstrates the case with no consistent solution, as no correlation exists between the dimensionless groups. Fig. 2.5 Part B shows the case where all the three dimensionless groups are important. The case where one dimensionless group ($\Pi_3$) is not important is demonstrated in Fig. 2.5 Part C. Finally, in Fig. 2.5 Part D, both $\Pi_2$ and $\Pi_3$ are not important, and in fact, $\Pi_3$ is constant. Again, the preferred set of dimensionless groups is the set with the minimum number of groups.

Fig. 2.5—Illustration of Dimensional Analysis results.
Dimensional Analysis for Two-phase Flow. Dimensional Analysis has been applied successfully to single-phase flow problems, as shown in the previous section. For two-phase flow systems, the method cannot be applied in a straightforward manner because of the large number of variables involved, which results in a large number of dimensionless groups. Additional assumptions must be made to reduce the number of dimensionless groups. Following is the study presented by Duns and Ros (1963) and Ros (1961).

As in single-phase flow, the first step in applying the Buckingham $\Pi$ Theorem to a two-phase flow system is to list the independent and dependent variables involved in the problem. The dependent variable for a two-phase flow system is the “flow mechanism.” This can be either the pressure gradient, the liquid holdup or the flow pattern. Assuming isothermal flow, Duns and Ros listed 13 relevant variables, as given next.

\[-\frac{dp}{dL}, v_{SL}, v_{SG}, \rho_L, \rho_G, \mu_L, \mu_G, \sigma, \psi, d, \varepsilon, \theta, g\]

The independent variable in this case is the pressure gradient. The dependent variables are, respectively, liquid superficial velocity, gas superficial velocity, liquid density, gas density, liquid viscosity, gas viscosity, surface tension, wall contact angle, pipe diameter, pipe roughness, inclination angle, and acceleration due to gravity.

The list of variables demonstrates clearly the problem faced when attempting to apply Dimensional Analysis to a two-phase flow system. Given the 13 variables, an unmanageable number of 10 dimensionless groups evolves from the process. Even for a smooth horizontal pipe, and neglecting the wall contact angle, the number of variables reduces to 10, which yields 7 dimensionless groups. This minimum number of dimensionless groups is equal to the number of variables involved with the determination of the frictional pressure drop in single-phase turbulent flow. It is impossible to correlate 10 or even 7 dimensionless groups experimentally.

Proceeding with the 13 variables and choosing the liquid density, $\rho_L$, the acceleration due to gravity, $g$, and the surface tension, $\sigma$, as the repeating parameters, the following 10 dimensionless groups can be developed:

\[
G = -\frac{1}{\rho_L g} \frac{dp}{dL}, \quad N_L = \mu_L \left(\frac{g}{\rho_L \sigma^3}\right)^{1/4},
\]

\[
N_{LV} = v_{SL} \left(\frac{\rho_L}{\sigma g}\right)^{1/4}, \quad N_{GV} = v_{SG} \left(\frac{\rho_L}{\sigma g}\right)^{1/4}, \quad N_D = d \left(\frac{\rho_L g}{\sigma}\right)^{1/2}, \quad N_{GL} = \mu_L \left(\frac{g}{\rho_L \sigma^3}\right)^{1/4}.
\]

\[\varepsilon = \frac{\varepsilon}{d}, \quad \theta, \quad \psi\]
where $G$ is the dimensionless pressure gradient; $N_{LV}$ and $N_{GV}$ are the liquid and gas velocity numbers, respectively; $N_D$ is the diameter number; $N_L$ and $N_G$ are the liquid and gas viscosity numbers, respectively; $N_\rho$ is the density ratio number; $\varepsilon/d$ is the relative roughness; and $\theta$ and $\psi$ are the inclination angle and wall contact angle, respectively. Note that this is not a unique set of dimensionless groups. An equivalent and acceptable set can be obtained by adding and dividing the gas and liquid velocity numbers, $N_{LV}$ and $N_{GV}$, as given next,

$$
G, \ (v_{SL} + v_{SG}) \left( \frac{\rho_L}{\sigma g} \right)^{1/4}, \ \frac{v_{SL}}{v_{SG}}, \ N_D, \ N_L, \ N_G, \ N_\rho, \ \frac{\varepsilon}{d}, \ \theta, \ \psi.
$$

The addition of the two velocity numbers yields the mixture velocity number, and their division yields the ratio of the volumetric flow rates. The remaining dimensionless groups are the same.

Being aware that 10 dimensionless groups are not manageable experimentally, Duns and Ros carried out an “elimination of irrelevant groups” using their engineering judgment. The following dimensionless groups were eliminated:

- Inclination angle, $\theta$: This parameter was eliminated assuming most wells to be vertical or nearly vertical.
- Wall contact angle, $\psi$: Visual observation confirmed that the pipe wall is always wet; hence the wall contact angle may be neglected.
- Gas viscosity number, $N_G$: The gas viscosity may have an influence on the flow during very high gas flow rates. However, under high gas flow rates, mist flow occurs, where the flow is highly turbulent, and the effect of the gas viscosity is negligible.
- Pipe wall relative roughness, $\varepsilon/d$: The relative roughness can affect the frictional pressure losses. Vertical flow is usually dominated by gravitational pressure losses, while the frictional losses are negligible. This might not be accurate for high flow rates where both the frictional and the gravitational pressure losses may be of the same order.
- Density ratio number, $N_\rho$: This parameter incorporates the effect of the gas density. The gas density is negligible in comparison to the liquid density, except at very high pressures. However, at high pressures, the gas flow rates are very small, as most of the gas is in solution, and the free gas is highly compressed.

Elimination of the above unimportant dimensionless groups, the two-phase flow system, represented by $G$, the dependent group, is controlled by four dimensionless groups. These are the liquid and gas velocity numbers, $N_{LV}$ and $N_{GV}$, the diameter number, $N_D$ and the liquid viscosity number $N_L$, namely,

$$
G = G(N_{LV}, N_{GV}, N_D, N_L).
$$

Experimental work must be conducted to verify the validity of the four governing dimensionless groups and to find out the functional relationship among them. An extensive experimental program was carried out for these purposes. It included 4,000 test runs with 20,000 data points, covering a wide range of pipe diameters, liquid and gas flow rates, and fluid properties. The experiments included pipe diameters
between 3.2 to 14.2 cm and liquid and gas superficial velocities up to 3.2 m/s and 100 m/s, respectively. Fluids used were gas, several types of oil, and water. For each run the pressure gradient, liquid holdup and flow pattern were obtained.

The extensive experimental work revealed that the pressure gradient, liquid holdup, and flow patterns are controlled, to a large extent, by the proposed four governing dimensionless groups. A flow pattern map was proposed, using the liquid and gas velocity numbers as coordinates. Empirical correlations have been developed for the pressure gradient and liquid holdup, for each of the existing flow patterns, in terms of the four dimensionless groups. These correlations, not given here, are used even today by the petroleum industry.

2.3.2 Similarity through Dynamic and Kinematic Ratios. As shown in the previous section, applying dimensional analysis to two-phase flow systems results in a large and unmanageable number of dimensionless groups. This is the drawback that can be overcome by using the Similarity through Dynamic and Kinematic Ratios approach, as presented by Dukler et al. (1964).

Dukler et al. considered formulating the viscous, pressure, and inertia forces for two-phase flow, and analyzed dynamic similarity by developing the ratios of these forces. The ratio of inertia to viscous forces in the model and prototype is given by

$$\frac{\rho_L H_L v_L \frac{\partial v_L}{\partial L} + \rho_G \alpha v_G \frac{\partial v_G}{\partial L}}{\mu_L H_L \frac{\partial^2 v_L}{\partial n^2} + \mu_G \alpha \frac{\partial^2 v_G}{\partial n^2}} = \frac{\rho_L H_L v_L \frac{\partial v_L}{\partial L} + \rho_G \alpha v_G \frac{\partial v_G}{\partial L}}{\mu_L H_L \frac{\partial^2 v_L}{\partial n^2} + \mu_G \alpha \frac{\partial^2 v_G}{\partial n^2}}, \text{ .........................}(2.64)$$

and the ratio of pressure to inertia force is

$$\frac{-\frac{dp}{dL}}{\rho_L H_L v_L \frac{\partial v_L}{\partial L} + \rho_G \alpha v_G \frac{\partial v_G}{\partial L}} = \frac{-\frac{dp}{dL}}{\rho_L H_L v_L \frac{\partial v_L}{\partial L} + \rho_G \alpha v_G \frac{\partial v_G}{\partial L}} \text{ .........................}(2.65)$$

Note that subscripts \(M\) and \(P\) refer to the model and the prototype, respectively. Next, the flow variables are scaled, as shown in Eqs. 2.66, 2.67, and 2.68.

Local velocity: \(v'_G = \frac{v_G}{v_G} \) and \(v'_L = \frac{v_L}{v_L} \), .................................(2.66)

Local holdup: \(\alpha' = \frac{\alpha}{\alpha} \) and \(H'_L = \frac{H_L}{H_L} \), .................................(2.67)

Lengths: \(L' = \frac{L}{d} \) and \(n' = \frac{n}{d} \), .................................(2.68)
where $\bar{v}_G$ and $\bar{v}_L$ are the average gas and liquid velocities, and $\bar{\alpha}$ and $\bar{H}_L$ are the average void fraction and liquid holdup, respectively. Similarity requires that the scaled variables are the same in the model and prototype. Substituting Eqs. 2.66, 2.67, and 2.68 into Eqs. 2.64 and 2.65 yields

$$\text{Re}_{TP} = v_M \frac{H_L}{\mu_L} \frac{\rho_L \bar{H}_L (\bar{v}_G/v_M)^2 + \rho_G \bar{\alpha} (\bar{v}_G/v_M)^2 C_1}{\mu_L \bar{H}_L (\bar{v}_L/v_M) + \mu_G \bar{\alpha} (\bar{v}_G/v_M) C_2},$$

and

$$\text{Eu}_{TP} \equiv 2 f_{TP(P)} = \left( \frac{-dp}{dL} \right)_F \frac{1}{v_M^2 / d} \frac{\rho_L \bar{H}_L (\bar{v}_L/v_M)^2 + \rho_G \bar{\alpha} (\bar{v}_G/v_M)^2 C_1}{\rho_L \bar{H}_L (\bar{v}_L/v_M) + \rho_G \bar{\alpha} (\bar{v}_G/v_M) C_2},$$

where $v_M$ is the mixture velocity and the coefficients $C_1$ and $C_2$ are given, respectively, by

$$C_1 = \frac{\alpha \bar{v}_G}{H_L \bar{v}_L} \frac{d\bar{v}_G}{dL}, \quad \text{and} \quad C_2 = \frac{\alpha \bar{v}_G}{H_L} \frac{d^2 \bar{v}_G}{d\bar{n}^2}.$$  

By definition $H_L = \frac{v_{SL}}{\bar{v}_L}$ and $\lambda_L = \frac{v_{SL}}{v_M}$. Dividing these two expressions yields

$$\frac{\bar{v}_L}{v_M} = \frac{\lambda_L}{\bar{H}_L}. \quad \text{...................................................(2.72)}$$

Similarly for the gas-phase

$$\frac{\bar{v}_G}{v_M} = \frac{1 - \lambda_L}{\bar{\alpha}}. \quad \text{...................................................(2.73)}$$

Substituting Eqs. 2.72 and 2.73 into Eqs. 2.69 and 2.70 yields

$$\text{Re}_{TP} = v_M d \left\{ \frac{\rho_L \lambda_L^2}{H_L} + \frac{\rho_G (1 - \lambda_L)^2}{\bar{\alpha} C_1} \right\}, \quad \text{...................................................(2.74)}$$

and

$$\text{Eu}_{TP} = \left( \frac{-dp}{dL} \right)_F \frac{1}{v_M^2 / d} \left\{ \frac{\rho_L \lambda_L^2}{H_L} + \frac{\rho_G (1 - \lambda_L)^2}{\bar{\alpha} C_2} \right\}, \quad \text{...................................................(2.75)}$$

Eqs. 2.74 and 2.75 demonstrate that the Reynolds and Euler numbers for two-phase flow can be determined as done for single-phase flow, provided the velocity is taken as the average mixture velocity and the mixture density and viscosity are calculated by (note that $\bar{\alpha} = 1 - \bar{H}_L$)
\[ \rho_{TP} = \rho_L \frac{\lambda_L^2}{H_L} + \rho_G \frac{(1-\lambda_L)^2}{(1-H_L)} C_1, \]  


and

\[ \mu_{TP} = \mu_L \lambda_L + \mu_G (1-\lambda_L) C_2. \]  

As can be seen, using proper definitions of the two-phase Reynolds and Euler numbers, the two-phase problem can be treated similarly to the single-phase problem. Next, it is necessary to correlate the \( \text{Eu}_{TP} \) with the \( \text{Re}_{TP} \), along with the kinematic condition \( \lambda_L \), in order to get a generalized relationship of the form

\[ \text{Eu}_{TP} = \text{Eu}_{TP} (\text{Re}_{TP}, \lambda_L) \quad \text{or} \quad f_{TP} = f_{TP} (\text{Re}_{TP}, \lambda_L). \]  

However, additional assumptions pertaining to the constants, \( C_1 \) and \( C_2 \), must be made where experimental data will be used to verify the validity of these assumptions. Two cases are analyzed next.

**Case I: Homogeneous No-Slip Flow.** For this case, \( v_G = v_L, \quad \overline{v}_G = \overline{v}_L \), and \( v'_G = v'_L \). Also, \( H_L = \overline{H}_L \) and \( \alpha = \overline{\alpha} \) yielding \( \alpha' = 1 \) and \( H'_L = 1. \) Thus, for this case \( C_1 = C_2 = 1 \), and the expressions for the two-phase density and viscosity reduce, respectively, to

\[ \rho_{TP} = \rho_L \frac{\lambda_L^2}{H_L} + \rho_G \frac{(1-\lambda_L)^2}{(1-H_L)} \]  

\[ \mu_{TP} = \mu_{NS} = \mu_L \lambda_L + \mu_G (1-\lambda_L). \]

Note that the two-phase mixture viscosity that emerged from the analysis is the no-slip viscosity. Also, because of no-slip condition, \( H_L = \lambda_L \), and \( \rho_{TP} = \rho_{NS} \) (Eq. 2.6). Thus, the determination of \( \text{Re}_{TP} \) and \( \text{Eu}_{TP} \) numbers is straightforward. For this case, a single-phase Blasius-type equation was suggested for the correlation \( \text{Eu}_{TP} = \text{Eu}_{TP} (\text{Re}_{TP}) \). This means that the two-phase friction factor, \( f_{TP} \), can be determined from a single-phase Blasius equation, using a two-phase Reynolds number, as given by

\[ \text{Re}_{TP} = \frac{\rho_T v_M d}{\mu_{TP}} \]  

where \( \rho_{TP} = \rho_{NS} \) and \( \mu_{TP} = \mu_{NS}. \) Thus, the current analysis reduces to the Homogeneous No-Slip model with proper assumptions (see section 2.1).

**Case II: Slip Flow with \( C_1 = C_2 = 1. \)** For this case, \( v_G \neq v_L \) and \( \overline{v}_G \neq \overline{v}_L \) but it is assumed that \( v'_G = v'_L \), namely, that the ratio of each phase velocity to the average phase velocity is the same for both phases. This implies similar velocity profiles for the gas and liquid phases. As the coefficients \( C_1 \) and \( C_2 \) are assumed equal to 1, the expressions for the density and viscosity of the mixture are the same as the ones given by Eqs. 2.79 and 2.80, whereby for this case \( H_L \neq \lambda_L \). Note that even for this case the mixture viscosity is essentially the no-slip viscosity, \( \mu_{NS} \). There is some physical justification for the use of the no-slip viscosity. It is used to determine the two-phase Reynolds number that is needed to
determine the friction factor, $f_{TP}$. The no-slip liquid holdup, $\lambda_L$, represents the ratio of the volumetric flow rates of the phases, which is important in determining the frictional losses.

Dukler et al. showed that for this case the experimental data were correlated successfully. In other words, for a wide range of experimental conditions, the scatter of the data plotted in the form of $f_{TP} = f_{TP} (Re_{TP}, \lambda_L)$ was very small. Furthermore, normalizing $f_{TP}$ with respect to $f_N$, (single-phase Fanning friction factor calculated with the mixture Reynolds number), yielded a single curve as a function of $\lambda_L$, as shown in Fig. 2.6. An empirical correlation of this result is given by

$$\frac{f_{TP}}{f_N} = 1 + \frac{y}{1.281 - 0.48 y + 0.444 y^2 - 0.094 y^3 + 0.00843 y^4}, \hspace{1cm} \text{equation (2.81)}$$

where $y = -\ln(\lambda_L)$. The normalizing friction factor $f_N$ (Fanning) is calculated from a Blasius-type expression, using the two-phase Reynolds number, as follows:

$$f_N = 0.0014 + 0.125 Re_{TP}^{0.32}. \hspace{1cm} \text{equation (2.82)}$$

The two-phase friction factor can then be determined as

$$f_{TP} = \frac{f_{TP}}{f_N} f_N. \hspace{1cm} \text{equation (2.83)}$$

Finally, the frictional pressure gradient can be calculated as

$$\left(-\frac{dp}{dL}\right)_F = \frac{2}{d} f_{TP} \rho_{TP} v_M^2. \hspace{1cm} \text{equation (2.84)}$$

where $\rho_{TP}$ is calculated from Eq. 2.79.

A value of the liquid holdup or the gas void fraction, $H_L$ or $\alpha$, are needed for the calculation. Thus, Dukler et al. provided a correlation for $H_L$, based on the experimental data. The Dukler et al. correlation for $H_L$ underpredicts the liquid holdup and approaches $\lambda_L$ at high Reynolds numbers. However, any other correlation can be used. The modified Dukler et al. method is presented in Chap. 3.

![Normalized friction factor](image-url)
2.4 Drift Flux Analysis

The Drift Flux Analysis presented in this section is applicable to vertical flow with low frictional losses, such as bubbly flow. The basic assumption in the model is a constant slippage between the gas and the liquid phases. In a more general form, because of the constant slippage, the drift flux is a function of the gas void fraction only. The derivation of the Drift Flux Analysis follows Wallis (1969).

2.4.1 Theory. Starting from the basic definition of the drift flux, namely,

\[ J_G = \alpha (v_G - v_L) \]  

and using the definition of the slip velocity \( v_{SL} \), the following relationship can be developed

\[ J_G = \alpha (1 - \alpha) v_{SL} \]  

From Eq. 2.85, if the slippage is constant and is given, then the drift flux is a function of the gas void fraction only, namely,

\[ J_G = J_G(\alpha) \]  

The relationship between the drift flux and the liquid and gas superficial velocities can be developed from the basic definition of the drift flux as

\[ J_G = (1 - \alpha) v_{SL} - \alpha v_{SG} \]  

2.4.2 Solution. Eqs. 2.85 and 2.87 constitute the drift flux model. Eq. 2.85 is a property of the system, as it depends on the slippage between the phases. Eq. 2.87 gives the operational conditions, namely, the gas and liquid superficial velocities. Simultaneous solution of these equations yields the gas void fraction \( \alpha \). The solution, however, requires specification of the slip velocity.

The Drift Flux Analysis can be used to illustrate graphically the possible solutions that can be obtained in a two-phase flow system. This is given in Fig. 2.7. The x-axis is the gas void fraction, \( \alpha \), while the y-axis is the drift flux, \( J_G \). Eq. 2.85, which is a property of the system, independent of the operational conditions, is represented by the dashed curve. As can be seen, the drift flux is zero at \( \alpha = 0 \) and \( \alpha = 1 \), reaching a maximum value in between these two boundaries. The straight lines, predicted by Eq. 2.87, represent different operational conditions. The solution is obtained at the intersection of the curve and the operational line. The following cases are demonstrated, assuming upward flow is positive:

- Curve (a): Co-current upward flow, \( v_{SL} > 0 \) and \( v_{SG} > 0 \). Only one solution is possible under these conditions.
- Curve (b): Counter-current flow, downward liquid flow, \( v_{SL} < 0 \), and upward gas flow, \( v_{SG} > 0 \). Two solutions might be obtained for this case, as shown in Fig. 2.7. Consider a pipe where the gas flows from the bottom and the liquid is injected at some point along the pipe. The liquid-phase will be partially carried upward by the gas-phase, and partially flow downward. The two solutions correspond to different flow configurations above and below the injection point of the liquid-phase.
- Curve (c): Counter-current, downward liquid flow, \( v_{SL} < 0 \), and upward gas flow, \( v_{SG} > 0 \). For this case, the operational line is tangent to the drift flux curve, and one solution exists. This is the flooding phenomenon, where the gas-phase carries all the liquid-phase upward.
2.5 Examples

Example 2.1. Oil and natural gas ($\gamma_g=0.7$) flow through a 5.1 cm ID pipeline inclined at $\theta=10^\circ$ upward from the horizontal. The in-situ flow rates are $W_L=1.5$ kg/s and $W_G=0.015$ kg/s. At one location of the pipeline, the temperature is 25°C and the physical properties of the fluids are

\[
\rho_L = 850 \text{ kg/m}^3 \quad \text{and} \quad \mu_L = 2.0 \text{ cp} = 2.0 \times 10^{-3} \text{kg/ms} ,
\]

and

\[
\rho_G = 1.5 \text{ kg/m}^3 \quad \text{and} \quad \mu_G = 0.02 \text{ cp} = 0.02 \times 10^{-3} \text{kg/ms} .
\]

1. Using the Homogeneous No-Slip Model, determine the frictional, gravitational, accelerational, and total pressure gradients and the liquid holdup in this location.

2. For the same input data, but assuming a horizontal pipe, $\theta=0^\circ$, determine the liquid holdup and frictional pressure gradient utilizing the Lockhart and Martinelli Separated Model. Estimate the accelerational pressure gradient.

3. For the same input data, but assuming a horizontal pipe, $\theta=0^\circ$, determine the liquid holdup and frictional pressure gradient utilizing the Dukler et al. Similarity Analysis. Assume a constant slip
velocity of $v_{\text{SL}} = 7.18 \text{ m/s}$. State any additional assumptions made. Estimate the accelerational pressure gradient.

**Solution.**

Preliminary calculations:

$$A_p = \frac{\pi}{4} (0.051)^2 = 0.002043 \text{ m}^2,$$

$$v_{\text{SL}} = \frac{W_L}{\rho_L A_p} = \frac{1.5}{850 \times 0.002043} = 0.86 \text{ m/s},$$

$$v_{\text{SG}} = \frac{W_G}{\rho_G A_p} = \frac{0.015}{1.5 \times 0.002043} = 4.89 \text{ m/s},$$

$$v_M = v_{\text{SL}} + v_{\text{SG}} = 5.75 \text{ m/s},$$

$$\lambda = \frac{v_{\text{SL}}}{v_M} = \frac{0.86}{5.75} = 0.15,$$

$$G = \frac{W_L + W_G}{A_p} = \frac{1.5+0.015}{0.002043} = 741.5 \text{ kg/m}^2\text{s},$$

and

$$x = \frac{W_G}{W_L + W_G} = 0.015 = 0.0099.$$

1. Homogeneous No-Slip Model:

$$\rho_M = \rho_{\text{NS}} = \lambda \rho_L + (1 - \lambda) \rho_G = 0.15 \times 850 + 0.85 \times 1.5 \times 128.77 = 128.77 \text{ kg/m}^3,$$

and

$$\mu_M = \mu_{\text{NS}} = \lambda \mu_L + (1 - \lambda) \mu_G = 0.15 \times 0.002 + 0.85 \times 0.00002 = 0.000317 \text{ kg/ms}.$$

Gravitational pressure gradient:

$$\left( - \frac{dp}{dL} \right)_G = \rho_{\text{NS}} g \sin \theta = 128.77 \times 9.81 \times \sin 10 = 219.4 \text{ Pa/m}.$$

Frictional pressure gradient:

$$\text{Re} = \frac{v_M \rho_{\text{NS}} d}{\mu_{\text{NS}}} = \frac{5.75 \times 128.77 \times 0.051}{0.000317} = 119,122,$$

$$f_f = 0.046 (119,122)^{-0.2} = 0.00444,$$

and

$$\left( - \frac{dp}{dL} \right)_f = \frac{2}{d} f_f \rho_{\text{NS}} v_M^2 = \frac{2}{0.051} \times 0.00444 \times 128.77 \times 5.75^2 = 741.3 \text{ (N/m}^2)/\text{m} = 741.3 \text{ Pa/m.}$$
Accelerational pressure gradient:

For \( x=\text{const.}, \ \nu_t=\text{const.}, \ \text{and} \ A_p=\text{const.}, \) Eq. 2.21 reduces to

\[
- \frac{dp}{dL} = G^2 x \frac{d\nu_G dp}{dL}
\]

Assume Ideal Gas Law:

\[
pV = nRT \quad \rightarrow \quad \rho_G = \frac{PM}{RT} \quad \text{or} \quad \nu_G = \frac{RT}{pM}.
\]

where \( R=8314 \text{Nm/kgmole}^6\text{K} \) \quad and \quad \( M = 29 \gamma_G = 29\times0.7 = 20.30 \text{kg/kgmole}, \)

\[
\frac{d\nu_G}{dp} = -\frac{RT}{M} = -\frac{1}{RT} \frac{M}{\rho_G^2} = -\frac{20.30}{8314\times298\times1.5^2} = -3.64\times10^{-6},
\]

and

\[
\frac{dp}{dL} A = 741.5^2 \times 0.0099\times(-3.64\times10^{-6}) \frac{dp}{dL} = -0.0202 \frac{dp}{dL}.
\]

Total pressure gradient:

\[
- \frac{dp}{dL} = \left( - \frac{dp}{dL} \right)_f + \left( - \frac{dp}{dL} \right)_G + \frac{dp}{dL} A = 741.3 + 219.4 - 0.0202 \frac{dp}{dL},
\]

\[
\frac{dp}{dL} = 980.5 \text{Pa/m},
\]

and

\[
\frac{dp}{dL} A = 0.0202 \times 980.5 = 19.8 \text{Pa/m}.
\]

2. Lockhart and Martinelli Separated Model: Calculate Lockhart and Martinelli parameter \( X \):

\[
Re_{SL} = \frac{\nu_S\rho_L d}{\mu_L} = \frac{0.86 \times 850 \times 0.051}{0.002} = 18,640 \text{ (turbulent)},
\]

\[
f_L = 0.046(18,640)^{-0.2} = 0.00644,
\]

\[
- \frac{dp}{dL} \bigg|_{SL} = \frac{2}{d} f_L \rho_L \nu_S^2 = \frac{2}{0.00644} \times 0.00644 \times 850 \times 0.86^2 = 158.8 \text{ Pa/m},
\]

\[
Re_{SG} = \frac{\nu_S\rho_G d}{\mu_G} = \frac{4.89 \times 1.5 \times 0.051}{0.00002} = 18,704 \text{ (turbulent)},
\]

\[
f_G = 0.046(18,704)^{-0.2} = 0.00643,
\]

\[
- \frac{dp}{dL} \bigg|_{SG} = \frac{2}{d} f_G \rho_G \nu_{SG}^2 = \frac{2}{0.00643} \times 0.00643 \times 1.5 \times 4.89^2 = 9.04 \text{ Pa/m},
\]

and
\[
X = \sqrt{\frac{-\frac{dp}{dL}}{SL} - \frac{-\frac{dp}{dL}}{SG}} = \sqrt{\frac{158.8}{9.04}} = 4.19.
\]

Liquid holdup:

\[
H_L = 1 - (1 + X^{0.8})^{-0.378} = 1 - (1 + 4.19^{0.8})^{-0.378} = 0.42.
\]

Frictional pressure gradient:

\[
\phi_L^2 = 1 + \frac{20}{X} + \frac{1}{X^2} = 1 + \frac{20}{4.19} + \frac{1}{4.19^2} = 5.83,
\]

and

\[
-\frac{dp}{dL} = \phi_L^2 \left( -\frac{dp}{dL} \right)_{SL} = 5.83 \times 158.8 = 925.8 \text{ Pa/m}.
\]

Accelerational and total pressure gradients:

From the Homogeneous Model:

\[
-\frac{dp}{dL} = -0.0202 \frac{dp}{dL},
\]

Thus,

\[
-\frac{dp}{dL} = \left( -\frac{dp}{dL} \right)_{F} - \left( -\frac{dp}{dL} \right)_{G} - \left( -\frac{dp}{dL} \right)_{A} = 925.8 + 0 - 0.0202 \frac{dp}{dL},
\]

\[
-\frac{dp}{dL} = 944.9 \text{ Pa/m},
\]

and

\[
-\frac{dp}{dL} = 0.0202 \times 944.9 = 19.1 \text{ Pa/m}.
\]

3. Dukler et al. Similarity Analysis:

Liquid holdup:

\[
v_{SLIP} = \frac{v_{SG} - v_{SG}}{1 - H_L} \quad \text{and} \quad 7.18 = \frac{4.89}{1 - H_L} - \frac{0.86}{H_L},
\]

\[
H_L = 0.46.
\]

Fluid properties:

\[
\rho_{TP} = \rho_L \frac{\lambda_L^2}{H_L} + \rho_g \frac{(1 - \lambda_L)^2}{(1 - H_L)} = 850 \frac{0.15^2}{0.46} + 1.5 \frac{(1 - 0.15)^2}{(1 - 0.46)} = 43.58 \text{ kg/m}^3,
\]

and
\[ \mu_{TP} = \mu_{NS} = \mu_L \lambda_L + \mu_G (1 - \lambda_L) = 0.000317 \text{ kg/ms}. \]

Frictional pressure gradient:
\[ \text{Re}_{TP} = \frac{\rho_{TP} v_M d}{\mu_{TP}} = \frac{43.58 \times 5.75 \times 0.051}{0.000317} = 40,315, \]

and
\[ f_N = 0.046(40,315)^{-0.2} = 0.00552. \]

From Eq. 2.82 (or Fig. 2.6):
\[ \frac{f_{TP}}{f_N} = 2.32, \]
\[ f_{TP} = 2.32 \times 0.00552 = 0.0128, \]

and
\[ - \left( \frac{dp}{dL} \right)_F = \frac{2}{d} f_{TP} \rho_{TP} v_M^2 = \frac{2}{d} \times 0.0128 \times 43.58 \times 5.75^2 = 723.6 \text{ Pa/m}. \]

Accelerational and total pressure gradients:

From the Homogeneous Model:
\[ - \left( \frac{dp}{dL} \right)_A = -0.0202 \left( \frac{dp}{dL} \right)_A, \]

Thus,
\[ - \left( \frac{dp}{dL} \right)_L = - \left( \frac{dp}{dL} \right)_F - \left( \frac{dp}{dL} \right)_G - \left( \frac{dp}{dL} \right)_A = 723.6 + 0 - 0.0202 \left( \frac{dp}{dL} \right)_A, \]
\[ - \left( \frac{dp}{dL} \right)_L = 738.5 \text{ Pa/m}, \]

and
\[ - \left( \frac{dp}{dz} \right)_A = 0.0202 \times 738.5 = 14.9 \text{ Pa/m}. \]
Chapter 3
Flow in Pipelines

This chapter presents analysis for horizontal and slightly inclined pipelines with the accepted range of inclination angles between ±10°. The most commonly used pipeline pressure loss empirical correlations are introduced briefly. Next, mechanistic models for flow pattern prediction (Taitel and Dukler, 1976) and separated models for stratified flow (Taitel and Dukler, 1976b) and slug flow (Dukler and Hubbard, 1975 and Taitel and Barnea, 1990) are presented. Finally, an overall mechanistic model for pipeline design and evaluation of predictive methods (Xiao et al., 1990) is given.

3.1 Pipeline Empirical Correlations

Several empirical correlations have been developed for the determination of pressure drop in two-phase flow pipelines. In this section, the two most commonly used correlations are presented, namely, the Dukler-Eaton-Flanigan and Beggs and Brill correlations.

One should note that the empirical correlations in Sec. 3.1 are given for oilfield system of units and utilize the Moody friction factor. The superficial velocities are expressed in ft/s, the density in lbm/ft³, the surface tension in dynes/cm, the viscosity in cp, the pressure in psi and the diameter in ft.

3.1.1 Dukler-Eaton-Flanigan Correlation. The original correlation developed by Dukler et al. (1964) is limited for horizontal flow conditions and also usually underpredicts the liquid holdup. Both limitations can be removed by utilizing the Eaton et al. (1967) correlation for liquid holdup and the Flanigan (1958) correlation for gravitational pressure losses in inclined flow.

Eaton et al. (1967) Liquid Holdup Correlation. The Eaton et al. (1967) liquid holdup correlation is given in Fig. 3.1. The x-axis coordinate, namely, \( x_{EATON} = [1.84 N_{LF}^{0.575} (p/14.7)^{0.05} N_L^{0.1}]/(N_{GV} N_d^{0.0277}) \), is based on the dimensionless groups developed by Ros (1961) and Duns and Ros (1963) (see Eq. 2.62), which are given next in oilfield units,
\[ N_{LV} = 1.938 v_{SL} \sqrt[4]{\mu L / \sigma}, \]
\[ N_{GV} = 1.938 v_{SG} \sqrt[4]{\mu L / \sigma}, \]
\[ N_d = 120.872 d \sqrt[4]{\mu L / \sigma}, \]

and
\[ N_L = 0.15726 \mu L \sqrt[4]{1/(\rho L \sigma^3)}. \] ................................................................. (3.1)

**Flanigan (1958) Correlation for Gravitational Pressure Losses.** The Flanigan (1958) method is based on a correlation for liquid holdup in upward inclined pipes. The liquid holdup correlation was developed from field data acquired in a 16-in. inside diameter (ID) pipe. The liquid holdup correlation is given by
\[ H_{L(UP)} = \frac{1}{1 + 0.3264 v_{SG}^{1.006}}. \] ................................................................. (3.2)

The main assumption made by Flanigan is that pressure drop occurs in the upward sections of a hilly terrain pipeline, but no pressure recovery is obtained in the downward sections. This is because of possible slugging in the upward sections and stratification of the flow in the downward sections. Thus, the gravitational pressure gradients in the upward and downward sections are, respectively,
\[ -\frac{dp}{dL}_{G(UP)} = \frac{\rho_{SLIP(UP)} g \sin \theta}{g_c}, \quad \text{and} \quad -\frac{dp}{dL}_{G(DOWN)} = 0. \] ......................................................... (3.3)

The slip density for the upward inclined section is approximated by \( \rho_{SLIP(UP)} = \rho_L H_{L(UP)} \). The total gravitational pressure loss in a hilly terrain pipeline can be determined by summing the vertical heights of all the upward sections of the pipeline, \( y_i \), namely, \( -\Delta p_G = (\rho_{SLIP(UP)} g / g_c) \sum_{i=1}^{n} y_i \). Note that for this case, the pipeline average superficial gas velocity is used to calculate the liquid holdup in Eq. 3.2.

**Dukler et al. (1964) Correlation for Frictional Losses.** The Dukler et al. (1964) correlation was developed from two-phase flow similarity analysis, as presented in Sec. 2.3.2. The important dimensionless groups that control the flow behavior emerged from the analysis. A large multiphase flow data bank was used to provide the functional relationship among the developed dimensionless groups. Note that in Sec. 2.3.2, the analysis is presented in SI units with the Fanning friction factor while, here, field units are used with the Moody friction factor. The frictional pressure gradient is given by
\[ -\frac{dp}{dL}_f = \frac{f_{TP} \rho_{TP} v_M^2}{2 d g_c}, \] ................................................................. (3.4)

where the two-phase density is calculated as given in the original analysis by
\[ \rho_{TP} = \rho_L \frac{\lambda_f^2}{H_L} + \rho_G \frac{(1 - \lambda_f)^2}{(1 - H_L)} \] ................................................................. (2.79)

The Reynolds number is calculated as
\[ \text{Re}_{tp} = 1.488 \frac{\rho_{tp} v_M d}{\mu_{ns}}, \]  

where \( v_M \) is the mixture velocity and \( \mu_{ns} \) is the no-slip mixture viscosity (averaged based on the no-slip liquid holdup as given in Eq. 2.11).

The friction factor is determined from a normalized friction factor, namely, \( f_{tp} / f_N \). The normalizing friction factor \( f_N \) (Moody) is calculated from a Blasius-type expression using the two-phase Reynolds number

\[ f_N = 0.0056 + 0.5\text{Re}_{tp}^{-0.32}, \]

and the normalized friction factor can be calculated from the following correlation

\[ \frac{f_{tp}}{f_N} = 1 + \frac{y}{1.281 - 0.48 y + 0.444 y^2 - 0.094 y^3 + 0.00843 y^4}, \]

where \( y = -\ln(\lambda_L) \).

Fig. 2.6 shows the normalized friction factor, as a function of the no-slip liquid holdup. The two-phase friction factor can be determined as

\[ f_{tp} = f_{tp} / f_N. \]

**Summary.** The liquid holdup and frictional pressure drop in a horizontal pipe can be determined from the Eaton et al. and Dukler et al. correlations, respectively. For a hilly terrain pipeline, the additional gravitational pressure losses can be determined from the Flanigan correlation.

### 3.1.2 Beggs and Brill (1973) Correlation

The Beggs and Brill (1973) correlation is applicable to the entire range of inclination angles, namely, from vertical upward through horizontal to vertical downward conditions. However, it is not recommended for vertical upward flow because it underpredicts the pressure loss for this case.

**Horizontal Flow Pattern Map.** The flow pattern map for horizontal flow is given in Fig. 3.2. The coordinates are the mixture Froude number, \( \text{Fr}_M^2 = \frac{v_M^2}{gd} \), and the no-slip liquid holdup, \( \lambda_L = \frac{v_{SL}}{v_M} \). As can be seen, the three flow patterns in horizontal flow are the segregated, intermittent, and distributed patterns. Note that the flow pattern is used as a correlating parameter and does not represent the actual flow pattern unless the pipe is horizontal. The parameters in Eq. 3.7 are used to predict the flow pattern under horizontal conditions.

\[ L_1 = 316 \lambda_L^{0.302}, \quad L_2 = 0.0009252 \lambda_L^{-2.4684}, \]

and

\[ L_3 = 0.10 \lambda_L^{-1.4516}, \quad L_4 = 0.5 \lambda_L^{2.738}. \]

The criteria for the horizontal flow pattern existence are given in Eq. 3.8.

- **Segregated:** \( \lambda_L < 0.01 \) and \( \text{Fr}_M^2 < L_1 \), or \( \lambda_L \geq 0.01 \) and \( \text{Fr}_M^2 < L_2 \).
- **Transition:** \( \lambda_L \geq 0.01 \) and \( L_2 \leq \text{Fr}_M^2 \leq L_3 \).
Intermittent: \[ 0.01 \leq \lambda_L \leq 0.4 \quad \text{and} \quad L_3 \leq Fr_M^2 \leq L_4, \] or \[ \lambda_L \geq 0.4 \quad \text{and} \quad L_3 \leq Fr_M^2 \leq L_4. \]

Distributed: \[ \lambda_L < 0.4 \quad \text{and} \quad Fr_M^2 \geq L_1, \] or \[ \lambda_L \geq 0.4 \quad \text{and} \quad Fr_M^2 > L_4. \] .................................................. (3.8)

\[ H_L = H_{L(0)} \psi, \] .................................................. (3.9)

where \( H_{L(0)} \) is the liquid holdup that would exist in a horizontal pipe with the same flow conditions, and \( \psi \) is a correction factor for the inclination angle. The liquid holdup for horizontal conditions can be determined from

\[ H_{L(0)} = \frac{a \lambda_L^b}{Fr_M^c}, \] .................................................. (3.10)

where the coefficients \( a, b, \) and \( c \) are functions of the flow pattern, as given in Table 3.1. When the flow pattern falls is in the transition region, the liquid holdup must be averaged using the segregated and intermittent liquid holdup values, as follows:

\[ H_{L(\text{TRANSITION})} = A \cdot H_{L(\text{SEGREGATED})} + (1 - A) \cdot H_{L(\text{INTERMITTENT})}, \] .................................................. (3.11)

where \( A = (L_3 - Fr_M^2)/(L_3 - L_2). \)

**TABLE 3.1—COEFFICIENTS FOR LIQUID HOLDUP CORRELATION**

<table>
<thead>
<tr>
<th>Flow Pattern</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>Segregated</td>
<td>0.98</td>
<td>0.4846</td>
<td>0.0868</td>
</tr>
<tr>
<td>Intermittent</td>
<td>0.845</td>
<td>0.5351</td>
<td>0.0173</td>
</tr>
<tr>
<td>Distributed</td>
<td>1.065</td>
<td>0.5824</td>
<td>0.0609</td>
</tr>
</tbody>
</table>

Note that \( H_{L(0)} \geq \lambda_L \) must be satisfied for all flow conditions.

Fig. 3.2—Horizontal flow pattern map (Beggs and Brill, 1973).
The correction factor for the effect of inclination angle is determined by
\[ \psi = 1 + C [\sin(1.8\theta) - 0.333 \sin^3(1.8\theta)] \]
and
\[ C = (1 - \lambda) \ln(d' \lambda_L \xi \frac{N_{LV}^L}{\lambda_L^2} F_{M}^L), \]
where \( \theta \) is the inclination angle of the pipe, and the coefficients \( d' \), \( e \), \( f \), and \( g \) are given in Table 3.2 for the different flow conditions (with the constraint \( C \geq 0 \)).

<table>
<thead>
<tr>
<th>Pattern</th>
<th>( d' )</th>
<th>( e )</th>
<th>( f )</th>
<th>( g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>All flow patterns</td>
<td>4.70</td>
<td>-0.3692</td>
<td>0.1244</td>
<td>-0.5056</td>
</tr>
<tr>
<td>downhill</td>
<td>0.011</td>
<td>-3.768</td>
<td>3.539</td>
<td>-1.614</td>
</tr>
<tr>
<td>Segregated uphill</td>
<td>2.96</td>
<td>0.305</td>
<td>-0.4473</td>
<td>0.0978</td>
</tr>
<tr>
<td>Intermittent uphill</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Distributed uphill</td>
<td>No correction</td>
<td>( C = 0 ) and ( \psi = 1 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Frictional Pressure Gradient.** The frictional pressure drop is determined similarly to Dukler et al. The frictional pressure gradient is given by
\[ -\frac{dp}{dL} = \frac{f_{TP} \rho_{NS} \frac{v_2^2}{M}}{2 g_c d}. \]
The friction factor is determined based on a normalized friction factor, \( f_{TP} / f_N \), which can be calculated from
\[ \frac{f_{TP}}{f_N} = e^s, \]
where
\[ s = \frac{\ln(y)}{-0.0523 + 3.182 \ln(y) - 0.8725 \left[\ln(y)^3 + 0.01853 \left[\ln(y)\right]^{3} \right]}. \]
and
\[ y = \frac{\lambda_L}{H_L^L}. \]
The function \( s \) becomes unbounded in the interval \( 1 < y < 1.2 \). For this interval, \( s \) is calculated from
\[ s = \ln(2.2 y - 1.2). \]
In the original correlation, the no-slip normalizing friction factor, \( f_N \), was determined from a smooth pipe correlation. However, as the correlation tended to underpredict the pressure gradient, it was modified later. In the modified correlation, \( f_N \) is based on a rough pipe friction factor. An example of rough pipe friction factor (Moody) is the convenient explicit form given by Hall (1957), as given by
\[ f_N = 0.0055 \left\{ 1 + \left[ 2 \times 10^4 \frac{\varepsilon}{d} + \frac{10^6}{\text{Re}_{NS}} \right]^{1/3} \right\} \] ..........................................................(1.33)

The no-slip Reynolds number is given by
\[ \text{Re}_{NS} = 1.488 \rho_{NS} \frac{\lambda D}{\mu_{NS}}, \] ..........................................................(3.19)

where the density and the viscosity of the mixture are determined, respectively, using the no-slip liquid holdup as
\[ \rho_{NS} = \rho_L \lambda_L + \rho_G (1 - \lambda_L), \] ..........................................................(3.20)

and
\[ \mu_{NS} = \mu_L \lambda_L + \mu_G (1 - \lambda_L). \] ..........................................................(2.11)

**Gravitational Pressure Gradient.** The gravitational pressure gradient is determined from Eq. 3.21.
\[ - \frac{d \rho}{dL} = \frac{\rho_{SLIP} g \sin \theta}{g_c}, \] ..........................................................(3.21)

where the slip density is calculated based on the in-situ liquid holdup (Eq. 3.9), given by
\[ \rho_{SLIP} = \rho_L H_L + \rho_G (1 - H_L). \] ..........................................................(3.22)

**Accelerational Pressure Gradient.** The accelerational pressure gradient, as given in Eq. 3.23, is usually neglected, except for low-pressure and high-velocity conditions.
\[ - \frac{d \rho}{dL} = \frac{\rho_{SLIP} \rho_M \rho_{SG}}{g_c p} \left( - \frac{dp}{dL} \right). \] ..........................................................(3.23)

**Total Pressure Gradient.** The total pressure gradient is the sum of the frictional, gravitational, and accelerational pressure gradient components.
\[ - \frac{d \rho}{dL} = \left( - \frac{d \rho}{dL} \right)_F - \left( - \frac{d \rho}{dL} \right)_G, \] ..........................................................(3.24)

where
\[ E_k = \frac{\rho_{SLIP} \rho_M \rho_{SG}}{g_c p}. \] ..........................................................(3.25)

Caution should be exercised when calculating pressure gradient including the accelerational component. When \( E_k \) tends to unity, the pressure gradient will approach infinity.
3.2 Flow Pattern Prediction in Pipelines

Extensive studies on two-phase flow pattern transitions have been conducted since the early 1950s. Most of the initial work has been focused on horizontal or vertical flow. Inclined flow studies were initiated in the 1970s, leading to a complete understanding of flow pattern transitions in the entire range of inclination angles, namely from –90° to 90°.

The most common approach for two-phase flow pattern determination has been through visual observation of the flow in a transparent pipe. Usually, the data have been mapped on a two-dimensional (2D) plot and the boundaries between the different flow patterns have been determined. In the initial studies empirical correlations have been developed. However, no physical basis has been suggested for the selection of mapping coordinates. Therefore, these empirical maps are reliable only in the narrow range of conditions under which the data have been acquired, and extension for other flow conditions is uncertain. Also, different flow pattern classifications and definitions have been suggested by the various investigators, resulting in poor agreement between their proposed maps.

Because visual observations are often subjective and difficult, especially at high flow rates, efforts have been devoted to developing flow pattern detection techniques, which are objective and can also be used in opaque pipes. Many such devices have been suggested, including hot wire anemometry, x-ray, pressure transducers, and conductance probes. All efforts done in this approach have resulted in partial success because no single technique is able to distinguish between all the flow patterns under different flow conditions confidently.

Beginning in the mid 1970s, analytical models for flow pattern prediction, based on the physical phenomena, have been reported. The main advantage of these models is that they can be extrapolated with more confidence to conditions for which no data are available. Also, they provide physical insight and understanding of the flow pattern transition phenomena.

A summary of published experimental flow pattern studies for horizontal systems is given in Table 3.3. As can be observed, a large number of coordinate systems have been proposed for plotting the data on a 2D map. Most of the coordinates are dimensional, such as the mass flow rates, \( W_L \) and \( W_G \), as used by Bergelin and Gazley (1949) and Abou Sabe and Johnson (1952); or the superficial velocities, \( v_{SL} \) and \( v_{SG} \), as used by Alves (1954). A flow regime map with such coordinates is not expected to apply under conditions different than the actual experimental conditions used for the data acquisition. Several correction factors for physical properties have been suggested, such as \( \lambda \) and \( \psi \) by Baker (1954), but the general applicability of such a generalization is questionable. Another generalization approach is the choice of dimensionless coordinates, such as \( Re_{TP} \) and \( We_{TP} \), as used by Eaton et al. (1967); or mixed coordinates, such as \( v_{SG}/v_M \) and \( v_M \), as proposed by Kosterin (1949) and Hoogendorn (1959). Though they seem more generalized, these coordinates still lack any theoretical basis.

Mandhane et al. (1974) utilized an American Gas Association/American Petroleum Institute (AGA/API) data bank covering a wide range of pipe diameters (1.3 to 15 cm), flow rates, and fluid properties. Based on 1,178 data points for air-water systems, they constructed a flow regime map using \( v_{SL} \) and \( v_{SG} \) as coordinates. When plotting the map of Mandhane et al. together with previous horizontal maps, one can clearly observe that the Mandhane et al. map represents “an average” of all the other maps. The map, as shown in Fig. 1.5, includes the accepted set of flow patterns, as defined before, and has been used as a reference map for horizontal flow. The weakness of the Mandhane et al. map is that
no physical basis was given for the choice of the superficial velocities of the two phases as the mapping coordinates. Also, most of the data were based on experiments conducted in pipes with diameters of 1.3 to 5 cm and, therefore, extension of the map to larger pipes should be cautiously considered.

Weisman et al. (1979) investigated the effects of fluid properties and pipe diameter on flow pattern transitions. They conducted experiments in air-water, air-low surface tension solutions, and air-Glycerin systems in 1.2, 2.54, and 5.1 cm pipes of 6 m length. Experiments were also performed in a 113 Freon vapor-Freon liquid system in a 3-m long pipe of 2.54 cm diameter. The visual observations were supplemented by an analysis of the pressure drop fluctuations as a means of detection. Based on their data, the authors proposed dimensionless correlations for the transition boundaries between the various flow patterns and constructed an overall flow regime map. Their map is presented for a 2.54 cm pipe with air-water flow at standard conditions (STP), and correction factors to account for different operating conditions are proposed.

<table>
<thead>
<tr>
<th>Author</th>
<th>Pipe Diameter, cm</th>
<th>Fluid System</th>
<th>Mapping Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kosterin (1949)</td>
<td>2.54, 5.1, 7.62, 10.16</td>
<td>air-water</td>
<td>$\frac{v_{SG}}{v_M}, \frac{v_M}{v_L}$</td>
</tr>
<tr>
<td>Bergelin and Gazley (1949)</td>
<td>2.54</td>
<td>air-water</td>
<td>$W_G, W_L$</td>
</tr>
<tr>
<td>Abou Sabe and Johnson (1952)</td>
<td>2.21</td>
<td>air-water</td>
<td>$W_G, W_L$</td>
</tr>
<tr>
<td>Alves (1954)</td>
<td>2.54</td>
<td>air-water/oil</td>
<td>$\frac{v_{SG}}{v_{SL}}$</td>
</tr>
<tr>
<td>Baker (1954)</td>
<td>data from others</td>
<td>oil-gas</td>
<td>$G_G, \left(\frac{\lambda}{\mu G}\right)^{/G_G}$</td>
</tr>
<tr>
<td>White and Huntington (1955)</td>
<td>2.54, 3.8, 5.1</td>
<td>air/natural gas-water/oil</td>
<td>$G_G, G_L$</td>
</tr>
<tr>
<td>Hoogendorn (1959)</td>
<td>2.54, 9.1, 14</td>
<td>air-water/oil</td>
<td>$\frac{v_{SG}}{v_M}, \frac{v_M}{v_L}$</td>
</tr>
<tr>
<td>Govier and Omer (1962)</td>
<td>2.54</td>
<td>air-water</td>
<td>$G_G, G_L$</td>
</tr>
<tr>
<td>Eaton et al. (1967)</td>
<td>5.1, 10.16, 43.2</td>
<td>natural gas-water/crude oil</td>
<td>$Re_{TP}, We_{TP}$</td>
</tr>
<tr>
<td>Al-Sheikh et al. (1970)</td>
<td>data bank</td>
<td>various gas-liquid systems</td>
<td>10 different coordinates</td>
</tr>
<tr>
<td>Govier and Aziz (1972)</td>
<td>data from others</td>
<td>air-water/oil</td>
<td>$X_{v_SL}, Y_{v_sg}$</td>
</tr>
<tr>
<td>Mandhane et al. (1974)</td>
<td>data bank</td>
<td>air-water</td>
<td>$v_{SL}, v_{SG}$</td>
</tr>
<tr>
<td>Simpson et al. (1977)</td>
<td>12.7, 21.6</td>
<td>air-water</td>
<td>$v_{SL}, v_{SG}$</td>
</tr>
<tr>
<td>Weisman et al. (1979)</td>
<td>1.2, 2.54, 5.1</td>
<td>various gas-liquid systems</td>
<td>$v_{SL} / \phi_2, v_{SG} / \phi_1$</td>
</tr>
</tbody>
</table>
3.2.1 Taitel and Dukler Model. The Taitel and Dukler (1976) model is applicable for steady-state, fully developed, Newtonian flow in horizontal and slightly inclined pipelines, namely, between $\pm 10^\circ$. Transient, entrance and exit effects can cause deviation from the prediction. The model was tested successfully mainly against data collected in small diameter pipes under low-pressure conditions. More work is needed to test the model for large diameter, high-pressure flow. Note that in the original paper, the inclination angle is positive downward, while in the following sections, it is positive upwards.

The starting point of the model is equilibrium stratified flow. Assuming stratified flow to occur, the flow variables, including the liquid level in the pipe, are determined. A stability analysis is then performed to determine whether or not the flow configuration is stable. If the flow is stable, stratified flow occurs. If the flow is not stable, a change to nonstratified flow occurs, and the resulting flow pattern is determined.

Equilibrium Stratified Flow. Equilibrium stratified flow configuration is shown schematically in Fig. 3.3. The pipeline is inclined at $\theta$ inclination angle from the horizontal, and the gas and liquid average velocities are $v_G$ and $v_L$, respectively. Also shown is a cross-sectional area of the pipe with the geometrical parameters. The area for flow and the wetted perimeter of the gas and the liquid phases are $A_G$ and $S_G$ and $A_L$ and $S_L$, respectively. The interface length is $S_I$, and the liquid level (under equilibrium conditions) is $h_L$.

![Fig. 3.3—Equilibrium stratified flow.](image)

The objective of this part of the model is to determine the equilibrium liquid level in the pipe, $h_L$, for a given set of flow conditions, namely, the gas and liquid flow rates, pipe diameter and inclination angle, and the fluid physical properties of the phases. This is carried out by applying momentum balances on the gas and the liquid phases in a differential control volume with an axial length of $\Delta L$, as shown in Fig. 3.3.

Fig. 3.4 is an expansion of the control volume, where the two phases are shown separated from each other, and the forces acting on each of the phases are indicated. For steady-state flow, neglecting the rate of change of momentum across the control volume, the momentum balances are reduced to force balances.
The momentum (force) balances for the liquid and gas phases are given, respectively, by

\[-A_L \frac{dp}{dL} - \tau_{WL} S_L + \tau_I S_I - \rho_L A_L g \sin \theta = 0, \]                           \hspace{1cm} (3.26)

and

\[-A_G \frac{dp}{dL} - \tau_{WG} S_G - \tau_I S_I - \rho_G A_G g \sin \theta = 0. \]                           \hspace{1cm} (3.27)

Eliminating the pressure gradient from Eq. 3.26 and Eq. 3.27, the combined momentum equation for the two phases is obtained, as shown in Eq. 3.28.

\[\tau_{WG} S_G S_L A_G \left( \frac{1}{A_L} + \frac{1}{A_G} \right) - (\rho_L - \rho_G) g \sin \theta = 0. \] \hspace{1cm} (3.28)

The combined momentum equation is an implicit equation for \(h_L\), the liquid level in the pipe. It combines all the forces that act on the liquid and gas phases, which in turn determines the location of the liquid level in the pipe. In order to solve the equation for \(h_L\), it is necessary to determine the different geometrical and force variables in the equation. The calculation of the forces in the equation is carried out utilizing the single-phase flow method based on the hydraulic diameter concept. The respective hydraulic diameters of the liquid and gas phases are given by

\[d_L = \frac{4A_L}{S_L},\]

and

\[d_G = \frac{4A_G}{S_G + S_I}. \] \hspace{1cm} (3.29)
The Reynolds numbers and the friction factors (for a smooth pipe) of each of the phases are

\[ f_L = C_L \left( \text{Re}_L \right)^{-n} = C_L \left( \frac{d_L v_L \rho_L}{\mu_L} \right)^{-n} \]

and

\[ f_G = C_G \left( \text{Re}_G \right)^{-m} = C_G \left( \frac{d_G v_G \rho_G}{\mu_G} \right)^{-m} \]

where \( C_L = C_G = 16 \) and \( m = n = 1 \) for laminar flow, and \( C_L = C_G = 0.046 \) and \( m = n = 0.2 \) for turbulent flow. The wall shear stresses corresponding to each phase are

\[ \tau_{WL} = f_L \frac{\rho_L v_L^2}{2} \]

and

\[ \tau_{WG} = f_G \frac{\rho_G v_G^2}{2} \]

The interfacial shear stress is given, by definition, as

\[ \tau_i = f_i \frac{(v_G - v_L)^2}{2} \]

In this model, it is assumed that \( f_i \approx f_G \), namely, a smooth interface exists. In addition, the interface velocity is neglected (i.e., \( v_G \gg v_L \)). With these approximations, the interfacial shear stress is equal to the gas-phase wall shear stress.

Substitution of Eq. 3.29 through Eq. 3.32 into Eq. 3.28 enables the determination of the liquid level in the pipe. However, the final solution for \( h_L \) is presented in a dimensionless form. All the variables can be written in nondimensional form by choosing scale parameters. The diameter, \( d \), is used for the length variables, \( d^2 \) for the area, and \( v_{SL} \) and \( v_{SG} \) for the liquid and gas velocities, respectively. The dimensionless variables are designated by a tilde (~), as given in Eq. 3.33.

\[ \tilde{S}_L = \frac{S_L}{d}, \quad \tilde{h}_L = \frac{h_L}{d}, \quad \tilde{A}_L = \frac{A_L}{d^2}, \quad \tilde{v}_L = \frac{v_L}{v_{SL}}, \quad \text{and} \quad \tilde{v}_G = \frac{v_G}{v_{SG}}. \]

Rearranging Eq. 3.28 yields

\[ \frac{\tau_{WL}}{\tau_{WG}} \frac{S_L}{A_L} - \left( \frac{S_G}{A_G} + \frac{S_L}{A_L} + \frac{S_I}{A_G} \right) + \left( \frac{\rho_L - \rho_G}{\rho_G} g \sin \theta \right) \frac{\tau_{WG}}{\tau_{WG}} = 0. \]

Substituting the dimensionless parameters into Eq. 3.28’ results in the dimensionless form of the combined momentum equation, namely,

\[ X^2 \left( \tilde{v}_L \tilde{d}_L \right)^{-n} \tilde{S}_L \tilde{A}_L^2 \left[ \left( \tilde{v}_G \tilde{d}_G \right)^{-m} \tilde{S}_G \tilde{A}_G \right] - \left( \tilde{v}_G \tilde{d}_G \right)^{-m} \tilde{v}_G \left( \frac{\tilde{S}_G \tilde{S}_L}{\tilde{A}_G \tilde{A}_L} + \frac{\tilde{S}_I}{\tilde{A}_G} \right) + 4Y = 0. \]
Two dimensionless groups emerge from the analysis, namely $X$, the Lockhart and Martinelli parameter and $Y$, an inclination angle parameter, given by

$$
X^2 = \frac{4C_L}{d} \left( \frac{\rho_L v_{SL}}{\mu_L} \right)^{\frac{n}{2}} \frac{\rho_L v_{SL}^2}{2} = -\frac{dp}{dL}_{SL},
$$

and

$$
Y = \frac{(\rho_L - \rho_G) g \sin \theta}{4C_G \left( \frac{\rho_G v_{SG}}{\mu_G} \right)^{\frac{m}{2}} \frac{\rho_G v_{SG}^2}{2}} = -\frac{dp}{dL}_{SG}.
$$

All the dimensionless (tilde) variables in Eq. 3.34 are unique functions of the dimensionless liquid level $\tilde{h}_L = h_L / d$. Thus, it is proved that

$$
\tilde{h}_L = h_L (X,Y).
$$

The functional relationships between the tilde dimensionless variables and $\tilde{h}_L$ are given in Eq. 3.38 (refer to Fig. 3.5).

$$
\tilde{A}_L = 0.25 \left[ \pi - \cos^{-1}(2\tilde{h}_L - 1) + (2\tilde{h}_L - 1)\sqrt{1 - (2\tilde{h}_L - 1)^2} \right],
$$

$$
\tilde{A}_G = 0.25 \left[ \cos^{-1}(2\tilde{h}_L - 1) - (2\tilde{h}_L - 1)\sqrt{1 - (2\tilde{h}_L - 1)^2} \right],
$$

$$
\tilde{S}_L = \pi - \cos^{-1}(2\tilde{h}_L - 1),
$$

$$
\tilde{S}_G = \cos^{-1}(2\tilde{h}_L - 1),
$$

$$
\tilde{S}_t = \sqrt{1 - (2\tilde{h}_L - 1)^2},
$$

$$
\tilde{v}_L = \frac{\tilde{A}_p}{\tilde{A}_L}, \quad \text{and} \quad \tilde{v}_G = \frac{\tilde{A}_p}{\tilde{A}_G},
$$

and

$$
\tilde{d}_L = \frac{4\tilde{A}_L}{\tilde{S}_L}, \quad \text{and} \quad \tilde{d}_G = \frac{4\tilde{A}_G}{\tilde{S}_G + \tilde{S}_t}.
$$
Fig. 3.5—Geometrical parameters for stratified flow.

This is an example of similarity analysis through basic equations. The dimensionless groups controlling the phenomena are obtained based on the proposed model. Thus, Eq. 3.34 should be applicable to all flow conditions (for which stratified flow exists) of pipe diameter and inclination angle, phase flow rates and fluid properties, subject to the model assumptions. Note that there are four possible solutions, depending upon the gas and liquid phases being in laminar or turbulent flow. Also, note that $X$ and $Y$ can be determined from the inlet flow conditions. Thus, Eq. 3.34 can be solved by trial-and-error to obtain the liquid level $\tilde{h}_L$.

A generalized plot of $\tilde{h}_L$ as a function of $X$ and $Y$ (based on the solution of Eq. 3.34) is given in Fig. 3.6. The solid lines represent the case where both phases are in turbulent flow ($C_L = C_g = 0.046$ and $m = n = 0.2$), while the dashed line is for the case of turbulent liquid and laminar gas ($C_L = 0.046, C_g = 16, n = 0.2$, and $m = 1$). The two sets of curves are close for horizontal and upward flow and collapse for downward flow. The other cases of the conditions of the gas and liquid phases can be solved using Eq. 3.34. Note that the determination of the flow conditions of the gas and liquid phases (turbulent or laminar) should be based on the actual Reynolds number of the phase, with actual phase velocity and hydraulic diameter, and not on the superficial Reynolds number (based on the phase superficial velocity and pipe diameter).

Note that multiple solutions, namely, three, are obtained for low values of $X$ ($X < 0.1$) and $Y$ ($0 < Y \leq 5$) for upward inclined flow only. It has been shown that the smallest root of the three possible ones is the stable solution; the middle root is not physical; and, the upper root is unstable. Thus, the smallest root should always be chosen, as long as multiple solutions exist. Thereafter, only one solution exists (the continuation of the upper root), which usually leads to unstable stratified flow, namely, transition to nonstratified flow pattern. No multiple solutions are obtained for horizontal and downward flow conditions.
Fig. 3.6—Equilibrium liquid level in stratified flow.
Stratified to Nonstratified Transition (Transition A). In the first part of the model, the equilibrium liquid level is determined assuming stratified flow conditions exist. The question to ask, for a given set of flow conditions, is whether the flow configuration is stable or not. If the flow is stable, the resulting flow pattern will be stratified flow. For an unstable flow condition, there will be a transition to other flow patterns. This requires stability analysis. In this model, a simplified Kelvin-Helmholtz stability analysis is applied.

In general, the Kelvin-Helmholtz stability theory (Milne-Thomson, 1960) deals with two fluid layers of different densities $\rho_1$ and $\rho_2$, flowing with velocities $v_1$ and $v_2$, respectively, between horizontal parallel flat plates. The theory predicts whether a small (infinitesimal) disturbance on the surface will lead to the interface being stable with a wavy structure, or unstable with wave growth destroying the stratification between the two layers. The Kelvin-Helmholtz analysis can be developed for inviscid flow (frictionless flow neglecting fluid viscosities) or viscous flow (friction flow considering fluid viscosities). The governing mechanisms, according to this analysis, are on the one hand, the gravity and surface tension forces that tend to stabilize the flow. On the other hand, the relative motion of the two layers creates a suction pressure force over the wave, owing to the Bernoulli effect, tending to destroy the stratified structure of the flow. Through this analysis, a stability criterion is developed in terms of the propagation velocity of the waves and the wavelength.

Taitel and Dukler extended the inviscid flow analysis first to the case of flow between parallel plates, with a finite wave on a layer of liquid sheet at the bottom and the gas at the top. Then, the analysis is extended to the case of a finite wave on a stratified flow gas-liquid interface in an inclined pipe. For both cases, the surface tension effect is neglected. Fig. 3.7 shows a schematic of the first case, namely, parallel plates analysis. The equilibrium liquid and gas heights are $h_L$ and $h_G$, respectively, while the respective heights associated with the wave peak are $h'_L$ and $h'_G$.

![Schematic of simplified Kelvin-Helmholtz stability analysis.](image)

The stabilizing gravity force acting on the wave is

$$ (h_G - h'_G) (\rho_L - \rho_G) g \cos \theta .................................................................(3.39) $$

Assuming a stationary wave, the pressure suction force causing wave growth is given by

$$ p - p' = \frac{1}{2} \rho_G (v_G^2 - v'_G^2) .................................................................(3.40) $$

From the continuity relationship, one obtains

$$ v_G A_G = v'_G A'_G .................................................................(3.41) $$
The condition for wave growth, leading to instability of the stratified configuration, is when the suction force is greater than the gravity force. Thus, combining Eq. 3.39 through Eq. 3.41 yields the criterion for instability, as shown in Eq. 3.42.

\[ v_G > c_1 \left[ \frac{g (\rho_L - \rho_G) h_G}{\rho_G} \right]^{0.5} \] .................................(3.42)

where \( c_1 \) depends on the wave size

\[ c_1 = \left[ \frac{2}{(h_G / h_G')(h_G / h_G' + 1)} \right]^{0.5} \] .................................(3.43)

For infinitesimal wave, \( h_G / h_G' \to 1.0 \) and \( c_1 \to 1.0 \), and Eq. 3.42 reduces to the original Kelvin-Helmholtz inviscid wave growth criterion.

The analysis presented for parallel plates can be extended to inclined pipe stratified flow, yielding

\[ v_G > c \left[ \frac{2 (\rho_L - \rho_G) g \cos \theta (h_L' - h_L)}{\rho_G} \frac{A_G^2}{A_G^2 - A_G'^2} \right]^{0.5} \] .................................(3.44)

For a small finite wave, \( A_G \) can be expanded around \( A_G' \) in Taylor series, resulting in

\[ v_G > c \left[ \frac{(\rho_L - \rho_G) g \cos \theta A_G}{\rho_G S_I} \right]^{0.5} \] .................................(3.45)

where

\[ c^2 = 2 \frac{(A_G' / A_G)^2}{(1 + A_G' / A_G)} \] .................................(3.46)

For low liquid level in the pipe \( A_G' \approx A_G \), therefore \( c = 1 \). Similarly, for high liquid level in the pipe, \( A_G' \to 0 \) and, hence, \( c = 0 \). On the basis of these boundary conditions, it is hypothesized that

\[ c = 1 - \frac{h_L}{d} \] .................................(3.47)

Substituting Eq. 3.47 into 3.45 results in the final criterion for this transition boundary, namely,

\[ v_G \geq \left( 1 - \frac{h_L}{d} \right) \left[ \frac{(\rho_L - \rho_G) g \cos \theta A_G}{\rho_G S_I} \right]^{0.5} \] .................................(3.48)

If the gas velocity on the left-hand side (LHS) is greater than the expression on the right-hand side (RHS), then the Bernoulli suction force overcomes the gravity force causing the flow to be unstable and the transition from stratified to nonstratified flow. On the other hand, if the inequality is not satisfied, namely, LHS < RHS, then the flow is stable and stratified flow will exist in the pipe. The loci of all the operational points that satisfy Eq. 3.48 constitute the transition boundary between stratified to
nonstratified flow. Once the liquid level \( \tilde{h}_L \) is determined from the first part of the model, it is possible to determine all the other parameters in the equation and to check if the flow is stable or not.

Note that Eq. 3.48 is the stratified to nonstratified transition criterion represented in a dimensional form. Applying the same dimensionless variables given in Eq. 3.33, the transition criterion can be transformed into a nondimensional form, as shown in Eq. 3.49.

\[
F^2 \left[ \frac{1}{(1 - \tilde{h}_L)^2} \frac{\tilde{v}_G^2 \tilde{S}_l}{A_G} \right] \geq 1. \quad ....................................................(3.49)
\]

All the tilde dimensionless parameters in the square bracket of Eq. 3.49 are unique functions of \( \tilde{h}_L \), as given in Eq. 3.38. Thus, this transition boundary is controlled by two dimensionless groups, namely, \( \tilde{h}_L \) and \( F \), where

\[
F = \frac{\rho_G}{\sqrt{(\rho_L - \rho_G) d \cos \theta}}. \quad ....................................................(3.50)
\]

Note that in the original paper, the term \( d\tilde{A}_L / d\tilde{h}_L \) was used instead of \( \tilde{S}_l \) (\( \tilde{S}_l = d\tilde{A}_L / d\tilde{h}_L \)). The stratified to nonstratified flow transition is designated as transition \( A \) in Fig. 3.8, where \( F \) is plotted against \( \tilde{h}_L \), as given by Eq. 3.49. This figure is a generalized and dimensionless flow pattern map applicable to horizontal and slightly inclined flow conditions.

The equilibrium liquid level \( \tilde{h}_L \) is, in turn, function of \( X \) and \( Y \). For horizontal flow, \( Y = 0 \) and \( \tilde{h}_L \) is a function of only \( X \). Thus, it can be concluded that for horizontal flow, the criterion for the transition between stratified and nonstratified flow for horizontal conditions is a function of \( X \) and \( F \). The generalized flow pattern map for horizontal conditions is given in Fig. 3.9, where this transition boundary is designated again by \( A \).

**Intermittent or Dispersed-Bubble to Annular Transition (Transition B).** The criterion for the transition from intermittent or dispersed-bubble to annular flow is described schematically in Fig. 3.10. When either the gas or the liquid flow rates are increased, the stratified flow structure becomes unstable, and transition from stratified to nonstratified flow occurs. Under unstable flow conditions, the following may occur: at low gas and high liquid flow rates, the liquid level in the pipe is high and the growing waves have sufficient liquid supply from the film, eventually blocking the cross-sectional area of the pipe. This blockage forms a stable liquid slug, and slug flow develops. This is shown schematically in Fig. 3.10 (Part A), whereby, schematically, the crest of the wave reaches the top pipe wall before the trough of the wave reaches the bottom of the pipe.

However, at low liquid and high gas flow rates, the liquid level in the pipe is low. For this case, the wave at the interface do not have sufficient liquid supply from the film, and they are swept up and around the pipe be the high gas velocity. Under these conditions, a liquid film annulus is created rather than a slug, resulting in a transition to annular flow. This is described in Fig. 3.10 (Part B), where the trough of the wave reaches the bottom of the pipe before the crest of the wave touches the top wall of the pipe.
Fig. 3.8—Generalized flow pattern map for horizontal and slightly inclined flow (Taitel and Dukler, 1976).

Fig. 3.9—Generalized flow pattern map for horizontal flow (Taitel and Dukler, 1976).
Thus, it is suggested that this transition depends uniquely on the liquid level in the pipe. Intuitively, a value of \( \tilde{h}_L = 0.5 \) was proposed for this transition. As shown schematically in Fig. 3.10 (Part C), for this case, the crest of the wave reaches the top pipe wall at the same time that the trough of the wave reaches the bottom of the pipe. In a later paper presented by Barnea et al. (1980), the criterion was modified to account for the fact that the slug body does not consist of liquid only. If the average holdup in the slug body is assumed to be 0.7, then a sufficient liquid level to ensure a blockage of the pipe cross-sectional area is \( 0.5 \times 0.7 = 0.35 \), resulting in the following criterion for this transition

\[
\frac{\tilde{h}_L}{d} = 0.35.
\]

Thus, if the stratified flow configuration is not stable, for \( \tilde{h}_L \leq 0.35 \), transition to annular flow occurs, otherwise, for \( \tilde{h}_L > 0.35 \), the flow pattern will be slug or dispersed-bubble flow. This transition boundary is shown as a vertical line \( B \) in both Figs. 3.8 and 3.9. In Fig. 3.8, the constant line represents \( \tilde{h}_L = 0.35 \) while in Fig. 3.9, it represents a corresponding constant value of \( X = 0.65 \).

![Schematic of transition between intermittent or dispersed-bubble flow to annular flow.](image)

Stratified-Smooth to Stratified-Wavy Transition (Transition C). Transition from stratified-smooth to stratified-wavy flow occurs when the gas-phase velocity is, on the one hand, sufficiently high to cause waves to form on the interface, but on the other hand, it is lower than the velocity needed to cause instability and transition to nonstratified flow. In general, waves develop on the interface of a stratified flow when pressure and shear forces exerted by the gas-phase overcome the viscous dissipation forces in the liquid-phase. The present model utilizes Jeffrey’s Theory, given in Eq. 3.52, as the criterion for wave initiation.

\[
(v_G - c_w)^2 c_w > \frac{4 \mu_L (\rho_L - \rho_G) g \cos \theta}{\rho_L \rho_G s},
\]

where \( v_G \) is the gas velocity, \( c_w \) is the velocity of propagation of the waves, and \( s \) is a sheltering coefficient associated with pressure recovery downstream of the wave. Assuming \( v_G \) is much greater than the wave propagation velocity (i.e., \( v_G \gg c_w \)), and approximating the wave propagation velocity by the liquid average velocity, namely, \( c_w \approx v_L \), the final criterion to determine transition between stratified-smooth to stratified-wavy can be represented by

\[
\left(\frac{4 \mu_L (\rho_L - \rho_G) g \cos \theta}{s \rho_L \rho_G v_L}\right)^{0.5}.
\]

As done before, this criterion can be represented in nondimensional form, namely,
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\[ K \geq \frac{2}{\sqrt{\bar{v}_L \bar{v}_G} \sqrt{s}}. \]  

(3.54)

Note that a value of \( s = 0.01 \) is used. A new dimensionless group has emerged, \( K \), given by

\[ K^2 = \frac{\rho_G v_{SG}^2 \rho_L v_{SL}}{(\rho_L - \rho_G) \mu_L g \cos \theta} = \left[ \frac{\rho_G v_{SG}^2}{(\rho_L - \rho_G) g \cos \theta} \right] \left( \frac{\rho_L v_{SL} \mu_L}{\mu_L} \right) = F^2 \text{Re}_{SL}. \]  

(3.55)

Thus, the transition criterion is a function of \( \bar{h}_L \) and \( K \), as shown in Fig. 3.8. For horizontal conditions, the transition is a function of \( X \) and \( K \), as shown in Fig. 3.9. This transition boundary, namely, transition \( C \), applies for conditions under which waves are generated by the interfacial shear stress. However, for downward inclined flow, waves can be generated because of instability of the interface, also for negligible gas flow rates. This mechanism has not been considered by the original model. Thus, transition \( C \) in Fig. 3.8 is applicable to horizontal and upward inclined flow and for downward inclined flow at relatively high gas flow rates, whereby the waves are generated by the interfacial shear. For downward inclined flow with low gas flow rates, where waves are generated because of instability of the interface, Barnea et al. (1982) suggested the following criterion for wave appearance, in terms of a critical Froude number, designated as transition \( K \).

\[ \text{Fr} = \frac{v_L}{\sqrt{g h_L}} > 1.5. \]  

(3.56)

**Intermittent to Dispersed-Bubble Transition (Transition D).** This transition occurs at high liquid flow rates. For these conditions, the equilibrium level in the pipe is high and approaches the upper pipe wall. The gas-phase occurs in the form of a thin gas pocket located at the top of the pipe because of the buoyant forces. For sufficiently high liquid velocities, the gas pocket is shattered into small dispersed bubbles that mix with the liquid-phase. Therefore, the transition to dispersed-bubble flow occurs when the turbulent fluctuations in the liquid-phase are strong enough to overcome the net buoyant forces, which tend to retain the gas as a pocket at the top of the pipe.

The net buoyant forces per unit length and the turbulent forces per unit length, acting on the gas pocket, are given, respectively, by

\[ F_B = A_G g (\rho_L - \rho_G) \cos \theta, \]  

(3.57)

and

\[ F_T = \frac{1}{2} \rho_L v'^2 S_I, \]  

(3.58)

where \( A_G \) is the gas pocket cross-sectional area, \( S_I \) is the interface length, and \( v' \) is the turbulent radial velocity fluctuating component of the liquid-phase. This velocity is determined when the Reynolds stress is first approximated by

\[ \tau_R = \rho_L u' v' \approx \rho_L v'^2. \]  

(3.59)

The wall shear stress can be expressed in terms of the wall friction velocity as

\[ \tau_w = \frac{1}{2} \int_L \rho_L v_L^* \]  

(3.60)
Assuming that $\tau_R \approx \tau_w$, the radial fluctuation velocity component becomes approximately equal to the friction velocity,

$$v' = (\nu'^2)^{0.5} \approx v^* = \nu_L \left( \frac{f_L}{2} \right)^{0.5}.$$ .......................................................... (3.61)

The transition to dispersed-bubble flow will occur when $F_T \geq F_B$. Equating Eq. 3.57 and Eq. 3.58 and substituting Eq. 3.61 for $v'$, the intermittent to dispersed-bubble transition criterion is

$$v_L \geq \frac{4 A_G}{S_I} \frac{g \cos \theta}{f_L} \left( 1 - \frac{\rho_G}{\rho_L} \right)^{0.5}.$$ .......................................................... (3.62)

The same transition represented in a nondimensional form is

$$T^2 \geq \left[ \frac{8 \tilde{A}_G}{S_I \tilde{v}_L^2} \right],$$ .......................................................... (3.63)

where

$$T = \left[ \frac{- \frac{dp}{dL}}{\rho_L - \rho_G} g \cos \theta \right]^{0.5}.$$ .......................................................... (3.64)

Thus, the transition criterion is a function of $\tilde{h}_L$ and $T$ for the general case, and a function of $X$ and $T$ for horizontal flow. These are shown as lines $D$ in Figs. 3.8 and 3.9, respectively.

**Summary.** According to the proposed model, flow pattern transition boundaries for horizontal and near horizontal flow are determined by four dimensionless groups, namely, $\tilde{h}_L$, $F$, $T$, and $K$. For horizontal flow conditions, the four groups are $X$, $F$, $T$, and $K$. These groups emerged from the analysis, and they are based on the physical phenomena and flow mechanisms of the transition boundaries. As the transition criteria incorporate the important flow parameters, such as the gas and liquid flow rates, pipe diameter, inclination angle, and physical properties of the phases, they can be extended to different flow systems with more confidence.

The existence of four different dimensionless groups, whereby different groups control different transition boundaries, explains the difficulties encountered in the past, trying to map flow patterns on a simple 2D plot. Luckily, one dimensionless group (i.e., $\tilde{h}_L$ or $X$) is common to all the transition boundaries. Thus, it is possible to plot the results on a “two-dimensional” plot, as given in Figs. 3.8 and 3.9. However, one must realize that these maps are actually multidimensional, and the prediction of flow patterns must be done in a sequential manner. This should be done when utilizing either the flow pattern maps or the transition criteria equations as follows.

- Determine the equilibrium liquid level and all the dimensionless parameters.
- Check the stratified to nonstratified transition boundary. If the flow is stratified, check the stratified-smooth to stratified-wavy transition boundary.
If the flow is nonstratified, check the transition to annular flow.
If the flow is not annular, check the intermittent to dispersed-bubble transition boundary.

**Inclination Angle Range Validity.** At the time the Taitel and Dukler (1976) model was published, experimental data have been available only for horizontal flow conditions. A comparison between the prediction of the model and experimental data for air-water flow at atmospheric conditions in horizontal and slightly inclined pipes was initially presented by Barnea et al. (1980) and later by Shoham (1982). Figs. 3.11 through 3.31 give the comparison presented by the later reference. Based on this comparison, the inclination angle range, for which the model is valid has been determined.

Fig. 3.11 presents comparison between the Taitel and Dukler (1976) model predictions, the Mandhane et al. (1974) map, and the experimental data for horizontal flow. As can be seen, a symbol is assigned for each flow pattern. This legend is used throughout the text for the identification of the experimental flow pattern results.

Fig. 3.11—Comparison of Taitel-Dukler (1976) model and Mandhane et al. (1974) map with data for air-water flow at standard conditions (STP) in 2.54 cm horizontal pipe.
As Fig. 3.11 shows, the predictions of the Taitel and Dukler model and the Mandhane et al. map agree fairly well with the data regarding the stratified to nonstratified transition boundary. With respect to the intermittent to dispersed-bubble transition boundary, the Taitel and Dukler model predictions agree very well with the experimental data. However, the Mandhane et al. map predicts this transition to occur at higher liquid flow rates. This discrepancy occurs because of differences in flow pattern definitions. Mandhane et al. distinguished the dispersed-bubble flow only when the bubbles are uniformly distributed in the entire cross-sectional area of the pipe. On the other hand, according to the Taitel and Dukler model and the experimental data, the transition to dispersed-bubble occurs as soon as the gas pocket, corresponding to intermittent flow, is broken into small bubbles, even though they concentrate in the upper pipe wall. This results in the transition boundary occurring at lower liquid flow rates, as shown in the figure.

The transition to annular flow predicted by the Mandhane et al. map seems to differ from the model predictions and the experimental data. A closer examination reveals that the all methods agree well if the wavy-annular region is considered slug flow. Wavy-annular flow has been interpreted in the past as slug or pseudoslug flow. Thus, again, this discrepancy is based on different classification of flow patterns.

Fig. 3.12 presents the same data as Fig. 3.11. It demonstrates the different criteria for the transition boundary to annular flow, namely, the original criterion, \( \tilde{h}_L = 0.5 \) and the modified criterion, \( \tilde{h}_L = 0.35 \). As can be seen, the modified criterion agrees more closely with the data.

Comparison between the Taitel and Dukler (1976) model and the experimental data for upward and downward inclined flow, as presented in Figs. 3.11 through Fig. 3.31, shows good agreement. Based on this comparison, it is recommended to use the model for the inclination angle range between ±10°.
Fig. 3.12—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 2.54 cm horizontal pipe for (refer to Fig. 3.11 for legend).

Fig. 3.13—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 5.1 cm horizontal pipe (refer to Fig. 3.11 for legend).
Fig. 3.14—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 2.54 cm 0.25° upward inclined pipe (refer to Fig. 3.11 for legend).

Fig. 3.15—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 5.1 cm 0.25° upward inclined pipe (refer to Fig. 3.11 for legend).
Fig. 3.16—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 2.54 cm 0.5° upward inclined pipe (refer to Fig. 3.11 for legend).

Fig. 3.17—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 5.1 cm 0.5° upward inclined pipe (refer to Fig. 3.11 for legend).
Fig. 3.18—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 2.54 cm 1° upward inclined pipe (refer to Fig. 3.11 for legend).

Fig. 3.19—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 5.1 cm 1° upward inclined pipe (refer to Fig. 3.11 for legend).
Fig. 3.20—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 2.54 cm $2^\circ$ upward inclined pipe (refer to Fig. 3.11 for legend).

Fig. 3.21—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 5.1 cm $2^\circ$ upward inclined pipe (refer to Fig. 3.11 for legend).
Fig. 3.22—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 2.54 cm 5° upward inclined pipe (refer to Fig. 3.11 for legend).

Fig. 3.23—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 5.1 cm 5° upward inclined pipe (refer to Fig. 3.11 for legend).
Fig. 3.24—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 2.54 cm $10^\circ$ upward inclined pipe (refer to Fig. 3.11 for legend).

Fig. 3.25—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 5.1 cm $10^\circ$ upward inclined pipe (refer to Fig. 3.11 for legend).
Fig. 3.26—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 2.54 cm –1° downward inclined pipe (refer to Fig. 3.11 for legend).

Fig. 3.27—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 5.1 cm –1° downward inclined pipe (refer to Fig. 3.11 for legend).
Fig. 3.28—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 2.54 cm −5° downward inclined pipe (refer to Fig. 3.11 for legend).

Fig. 3.29—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 5.1 cm −5° downward inclined pipe (refer to Fig. 3.11 for legend).
Fig. 3.30—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 2.54 cm –10° downward inclined pipe (refer to Fig. 3.11 for legend).

Fig. 3.31—Comparison between Taitel-Dukler (1976) model and data for air-water flow at STP in 5.1 cm –10° downward inclined pipe (refer to Fig. 3.11 for legend).
3.3 Stratified Flow Modeling

The first part of the Taitel and Dukler (1976) model for flow pattern prediction presents an analysis of equilibrium stratified flow. Once the flow is found to be stable, namely, stratified flow exists, then the calculated equilibrium liquid level, \( \tilde{h}_L \), is the actual liquid level in the pipe. Note that the original equilibrium stratified flow analysis has been carried out for smooth interface condition, namely, \( \tau_I = \tau_{WG} \). However, for actual stratified flow, one must calculate the interfacial shear stress appropriately. A stratified flow model was presented in a subsequent study by Taitel and Dukler (1976b). This section first presents the closure relationships for stratified flow, namely, the shear stresses, which will be followed by the model equations and the calculation procedure.

3.3.1 Closure Relationships. The required closure relationships for stratified flow are the shear stresses, including the interfacial shear stress and the gas and liquid wall shear stresses. Refer to Eqs. 3.31 and 3.32 for the definitions of the shear stresses.

**Interfacial Shear Stress.** The interfacial shear stress can be approximated by

\[
\tau_I = f_I \rho_G (v_G - v_I)^2 \approx f_I \rho_G (v_G - v_L)^2. 
\]

**Stratified-Smooth Flow.** For stratified smooth flow, the interfacial shear stress can be calculated as suggested by Taitel and Dukler, equating the interfacial shear stress to the gas wall shear stress, namely, \( \tau_I = \tau_{WG} \), resulting in

\[
f_I = f_G, 
\]

where \( f_G \) is the gas wall friction factor that is discussed in the following section.

**Stratified-Wavy Flow.** For stratified wavy flow with small amplitude waves, a simple expression can be used, as given next. This constant value for \( f_I \) was originally suggested by Cohen and Hanratty (1968) and later used by Shoham and Taitel (1984).

\[
f_I = 0.0142. 
\]

Xiao et al. (1990) proposed another method for the determination of the interfacial friction factor. They suggested the use of a combination of the Andritsos and Hanratty (1987) correlation for small diameter pipes and the Baker et al. (1988) correlation for larger pipe diameters as follows:

**Small diameter pipes** \((d \leq 0.127 \text{ m})\). The superficial gas velocity for the transition to the stratified-wavy flow pattern is defined as

\[
v_{SG,T} = 5 \sqrt{\frac{101,325}{p}}, 
\]

where \( p \) is the absolute pressure in Pa \((N/m^2)\). The interfacial friction factor is determined as presented next:

For \( v_{SG} < v_{SG,T} \), stratified smooth flow exists and \( f_I = f_G \).

For \( v_{SG} > v_{SG,T} \).
Large Diameter Pipes \((d \leq 0.127 \text{ m})\). For this case, the calculation is based on \(\varepsilon_I\), the interface absolute roughness.

For \(N_{we}N_\mu \leq 0.005\),

\[
\varepsilon_I = \frac{34\sigma}{\rho_G v_L^2} \nonumber. 
\]

For \(N_{we}N_\mu > 0.005\),

\[
\varepsilon_I = \frac{170\sigma(N_{we}N_\mu)^{0.3}}{\rho_G v_L^2} \nonumber. 
\]

The interface absolute roughness, \(\varepsilon_I\), should be bound between the pipe wall absolute roughness and 
\[0.25\sqrt{h_L/d}.\] The Weber number, \(N_{we}\), and the liquid viscosity number, \(N_\mu\), are given, respectively, by

\[
N_{we} = \frac{\rho_G v_L^2 \varepsilon_I}{\sigma} \nonumber. 
\]

and

\[
N_\mu = \frac{\mu_L^2}{\rho_L \sigma \varepsilon_I} \nonumber. 
\]

Using the value of \(\varepsilon_I\) and the gas-phase Reynolds Number, \(Re_G = \rho_G v_G d_G / \mu_G\), the interfacial friction factor is determined from a pipe flow friction factor equation, such as Eq. 1.34. (Note that in Eqs. 3.70, 3.71, and 3.72, the liquid velocity is substituted for the interface velocity, namely, \(v_i = v_L\), as suggested by Xiao et al.).

**Wall Shear Stresses.** In the Taitel and Dukler (1976) model, both the liquid and gas wall friction factors, \(f_L\) and \(f_G\), are determined based on the phase hydraulic diameter and Reynolds number using a standard pipe flow friction factor method, as given in Eqs. 3.29 and 3.30, respectively (for smooth pipes). However, Ouyang and Aziz (1996) argued that this procedure is appropriate for the gas-phase only. This is because of the fact that the liquid wall friction factor can be affected significantly by the interfacial shear stress, especially for low liquid holdup condition. Thus, it is recommended to determine \(f_G\) based on the hydraulic diameter concept (Eqs. 3.29 and 3.30), while it is proposed to determine \(f_L\) from the correlation developed by Ouyang and Aziz, as given in Eq. 3.74. Note that this correlation incorporates both the gas and liquid flow rates.

\[
f_L = \frac{1.6291}{Re_{Sl}^{0.5161}} \left( \frac{v_{SG}}{v_{SL}} \right)^{0.0926} 
\]

3.3.2 **Calculation Procedure.** Once the flow pattern has been predicted to be either stratified-smooth or stratified-wavy flow, the liquid holdup and the pressure drop can be determined as outlined next.
1. Determine the dimensionless groups $X$ and $Y$, as defined before, namely,

$$
X^2 = \frac{\frac{4C_L}{d} \left( \frac{\rho_L v_{SL} d}{\mu_L} \right)^n}{\rho_L \frac{v_{SL}^2}{2}} - \frac{d}{dL} \frac{dp}{SL}, \quad \text{.................................(3.35)}
$$

and

$$
Y = \frac{\frac{4C_G}{d} \left( \frac{\rho_G v_{SG} d}{\mu_G} \right)^m}{\rho_G \frac{v_{SG}^2}{2}} - \frac{d}{dL} \frac{dp}{SG}, \quad \text{.................................(3.36)}
$$

2. Determine $\tilde{h}_L$ either from Eq. 3.28 or modified Eq. 3.34 (Eq. 3.75, to account for the fact that $\tau_i \neq \tau_{WG}$). In these equations, one must use the appropriate interfacial friction factor for stratified-smooth or stratified-wavy, respectively. (Also, as an approximation, $\tilde{h}_L$ can be determined from Fig. 3.6, which has been constructed under the conditions $\tau_i = \tau_{WG}$).

$$
\tau_{WG} \frac{S_L}{A_G} - \tau_{WL} \frac{S_L}{A_L} + \tau_i \left( \frac{1}{A_L} + \frac{1}{A_G} \right) - (\rho_L - \rho_G) g \sin \theta = 0. \quad \text{.................................(3.28)}
$$

Modifying Eq. 3.34, utilizing $\tau_i$ rigorously, one obtains

$$
X^2 \left[ \left( \frac{v_i d_i}{A_i} \right)^n \frac{S_i}{A_i} \right] - \left[ \left( \frac{v_G d_G}{A_G} \right)^m \frac{S_G}{A_G} \left( \frac{S_G}{A_G} + \frac{f_i}{f_G A_G} \frac{S_i}{A_G} + \frac{f_i}{f_G} \frac{S_i}{A_G} \right) \right] + 4Y = 0. \quad \text{.................................(3.75)}
$$

3. Determine the cross-sectional area of the liquid-phase from the geometrical relationship given by Eq. 3.38. The holdup is simply the cross-sectional area of the liquid-phase divided by the cross-sectional area of the pipe, namely, $H_L = A_L / A_p$.

4. Determine the pressure gradient from momentum equations of either the liquid-phase or the gas-phase given, respectively, in Eqs. 3.26 and 3.27. Note that either equation should yield the same result.

$$
-A_L \frac{dp}{dL}_L - \tau_{WL} S_L + \tau_i S_i - \rho_L A_L g \sin \theta = 0, \quad \text{.................................(3.26)}
$$

and

$$
-A_G \frac{dp}{dL}_G - \tau_{WG} S_G - \tau_i S_i - \rho_G A_G g \sin \theta = 0. \quad \text{.................................(3.27)}
$$

Comparison between experimental data for horizontal stratified flow and the model predictions is given in Fig. 3.32, as presented by Amaravadi (1993). Note that Amaravadi suggested a value of $f_i = 0.03$ for stratified flow with roll waves. Roll waves are large waves occurring near the transition from stratified
flow and slug flow. Good agreement is observed between the different stratified flow patterns and the data.

Fig. 3.32—Comparison between experimental data and stratified flow model (Amaravadi, 1993).

3.3.3 Horizontal Stratified Flow. For horizontal flow, Eq. 3.75 reduces to

\[ X^2 \left( \frac{\tilde{v}_I \tilde{d}_L}{A_L} \right)^{-m} \tilde{S}_L = \left( \frac{\tilde{v}_G \tilde{d}_G}{A_G} \right)^{-m} \tilde{S}_G + \frac{f_I \tilde{S}_I}{f_G A_L} + \frac{f_I \tilde{S}_I}{f_G A_G} \right] = 0. \]  

Thus, for horizontal flow, the liquid level is a unique function of the Lockhart and Martinelli parameter, \( X \), and the friction factor ratio, \( f_I / f_G \). Assuming that the friction factor ratio is approximately constant for the different flow conditions, one can conclude that \( h_L / d = \tilde{h}_L = \tilde{h}_L(X) \).

Similarly, transforming the gas-phase momentum equation (Eq. 3.27 without the gravitational term) into dimensionless form results in

\[ \phi_G^2 = \frac{1}{4} \tilde{v}_G^2 \left( \frac{\tilde{d}_G}{A_G} \right)^{-m} \left( \tilde{S}_G + \frac{f_I \tilde{S}_I}{f_G} \right), \]

where \( \phi_G^2 \) is the dimensionless pressure gradient, as defined earlier by Lockhart and Martinelli (1949), namely,

\[ \phi_G^2 = \frac{-\frac{dp}{dL}}{-\frac{dp}{dL}}_{SG}. \]
Based on Eq. 3.77, it can be shown that the dimensionless pressure gradient is also a unique function of \( X \), namely, \( \phi^G = \phi^G(X) \).

These results, presented by Taitel and Dukler (1976b), prove the hypothesis of Lockhart and Martinelli (1949), which was done about 27 years before this study, that for horizontal flow, both the pressure drop and the liquid holdup are unique functions of the Lockhart and Martinelli parameter, namely \( \tilde{h}_L(X) \) and \( \phi^G(X) \).

### 3.4 Slug Flow Modeling

Slug flow occurs in horizontal, inclined, and vertical pipes over a wide range of gas and liquid flow rates. It is the dominant flow pattern in upward inclined flow. **Fig. 3.33** shows a schematic description of slug flow in vertical and inclined pipes. Slug flow is characterized by an alternate flow of gas pockets and liquid slugs. Most of the gas-phase is concentrated in large bullet-shaped gas pockets, named Taylor bubbles. The Taylor bubbles are separated by liquid slugs, which contain small gas bubbles. Liquid film flows downward between the Taylor bubble and the pipe wall. The Taylor bubble is symmetric around the pipe axis for vertical flow. For inclined and horizontal flow, the gas pocket flows in the upper part of the pipe and the liquid film below it. Recall, as presented in Chap. 1, that the slug flow and elongated-bubble flow patterns belong to the intermittent pattern. However, slug flow is often used for the general intermittent flow pattern. Also, the models developed for slug flow can be used for the elongated-bubble pattern.

![Slug Flow Diagram](image)

**Fig. 3.33**—Slug flow in vertical and inclined pipes.

Slug flow hydrodynamics is complex with unsteady flow behavior characteristics. It has a peculiar velocity, holdup and pressure distributions. Therefore the prediction of the liquid holdup, pressure drop, heat transfer, and mass transfer are difficult and challenging. The flow is also associated with operational problems, such as separation, erosion-corrosion enhancement, and structural problems occurring in turns and bends.
The complex flow behavior of the slug pattern has hindered development of accurate predictive methods. Recently, several mechanistic models have been proposed that enable reasonable prediction of the flow characteristics, such as the average liquid holdup and pressure gradient. Such models were introduced by Dukler and Hubbard (1975) and Nicholson et al. (1978) for horizontal flow; Fernandes et al. (1983), Sylvester (1987) and Vo and Shoham (1989) for vertical flow; and Bonnecaze et al. (1971) for inclined flow. The more recent trend is the development of unified slug flow models that apply to horizontal, upward inclined, and upward vertical flow. These include the comprehensive study by Taitel and Barnea (1990) and the simplified model of Felizola and Shoham (1995). Kouba (1986) and Kouba et al. (1990) presented a comprehensive data set on slug flow characteristics in a horizontal pipe and developed a model for nonintrusive metering of slug flow. Scott and Kouba (1990) presented advances in slug flow characterization in pipelines. Many other studies have been carried out on slug flow related closure relationships, such as the liquid holdup in the slug, slug length, slug frequency, and Taylor bubble rise velocity.

In this chapter the pioneering study of Dukler and Hubbard (1975) is covered first, followed by the detailed and unified study of Taitel and Barnea (1990). Other models are covered in the following chapters. Finally, a summary of the related closure relationships is presented.

**Fig. 3.34** represents the physical model for slug flow (after Taitel and Barnea, 1990) and defines the flow variables that are referred to throughout the entire chapter. The figure shows a slug unit with a total length of $L_u$. The slug unit consists of two main zones, namely, the liquid slug body, $L_s$, and the stratified region behind the slug, $L_f$, including the liquid film at the bottom and the gas pocket at the top. A turbulent mixing zone exists at the front of the slug, $L_m$.

![Fig. 3.34—Physical model for slug flow.](image-url)
A very peculiar velocity distribution is exhibited in slug flow, where $v_{TB}$ is the translational velocity, which is the interface velocity; $v_{LLS}$ and $v_{GLS}$ are the velocities of the liquid and gas phases within the slug body; and $v_{LTB}$ and $v_{GTB}$ are the liquid film velocity and the gas pocket velocity in the stratified region, respectively. The liquid film velocity is not constant, decreasing from its maximum value at the front of the film (back of the slug body) to the minimum value, $v_{LTBe}$, at the end of the film, ahead of the following slug. In general, the velocity distribution is $v_{TB} > v_{GTB} > v_{GLS} > v_{LLS} > v_{LTB}$.

In the slug body, gas is entrained in the form of small bubbles and the liquid holdup is $H_{LLS}$. The liquid holdup in the stratified region is designated $H_{LTB}$. The liquid holdup varies along the stratified region in a similar manner to the film velocity.

### 3.4.1 Dukler and Hubbard (1975) Horizontal Slug Model

The Hubbard and Dukler model was developed through a phenomenological approach. This study contributed a great deal to the understanding of the flow mechanisms and the hydrodynamic behavior of slug flow. Experimental data were acquired utilizing a 3.8 cm diameter 19.8-m long pipe, using air and water at atmospheric conditions. Based on their observations, Hubbard and Dukler defined an idealized slug unit with a length of $L_u$, as shown in Fig. 3.34, and suggested a mechanism for the flow. This was a basis for a detailed mathematical model, which is capable of predicting the hydrodynamic behavior of slug flow, including the length, velocity, holdup, and pressure distributions.

The flow mechanism suggested by Dukler and Hubbard is as follows. The liquid slugs bridge the entire pipe cross-sectional area. As a result, they move at a relatively high velocity, close to the mixture velocity. The liquid film ahead of the slug moves slower. The fast moving slug overruns the slow moving film ahead of it, picks it up, and accelerates it to the slug velocity, creating a turbulent mixing zone in the front of the slug. At the same time, the gas pocket pushes into the slug, causing the slug to shed liquid from its back, creating the film region. For a steady-state flow, the rate of pickup is equal to the rate of shedding, resulting in a constant slug length. Thus, the slug does not simply slide over the liquid film. Also, the fluids in the slug body are continuously changing, picked up in the front and shed in the back, as the case of wave flow. This behavior is discussed in more details in a following section.

The proposed model is not a complete model. As shown in Fig. 3.35, the model requires two additional variables to be specified as input parameters. These are $v_S$ and $H_{LLS}$, the slug frequency, and the liquid holdup in the slug body, respectively. Note that the model is valid only for horizontal flow conditions. Also, the model assumes homogeneous no-slip flow in the slug body, namely, $v_{GLS} = v_{LLS} = v_S$. Details of the model are presented next.
Pressure Drop. The total pressure drop across a slug unit consists of two components: the accelerational pressure drop in the mixing zone and the frictional pressure drop in the slug body. The pressure drop in the stratified region behind the slug is neglected. The accelerational pressure drop, \( -\Delta p_A \), is due to the velocity difference between the slug and the liquid film. Because the slug moves faster than the film ahead, the slug overruns and picks up liquid from the film. The slug accelerates the picked liquid to the slug velocity. This creates the eddy or the mixing zone in the front of the slug. The frictional pressure drop, \( -\Delta p_F \), is caused by the shear between the slug body and the pipe wall.

Neglecting the pressure drop across the stratified zone, the total pressure drop across a slug unit and the corresponding average pressure gradient are given, respectively, by

\[
-\Delta p_U = -\Delta p_A - \Delta p_F, \quad \text{.................................................................(3.78)}
\]

and

\[
\frac{dp}{dL} = -\frac{\Delta p_U}{L_U}. \quad \text{.................................................................(3.79)}
\]

Accelerational Pressure Component. The rate at which mass is picked up by the slug body from the film zone is designated as \( x \), the pickup rate (mass/time). The mass of film picked up by the slug increases its velocity from \( v_{LTBe} \) to \( v_S \). The force acting on the picked-up mass equals the change of momentum, namely, \( F = x(v_S - v_{LTBe}) \). The pressure drop associated with this force is the force divided by the pipe cross-sectional area, as given by

\[
-\Delta p_A = \frac{x}{A_p}(v_S - v_{LTBe}). \quad \text{.................................................................(3.80)}
\]

Frictional Pressure Component. This is the pressure drop, which is required to overcome the shear forces between the moving slug body and the pipe wall. The assumptions made for the flow in the slug body are homogeneous no-slip flow with a fully developed turbulent velocity profile, as in pipe flow. Thus, \( -\Delta p_F \) can be calculated, as in single-phase flow, on the basis of the average slug properties, as shown in Eq. 3.81.

\[
-\Delta p_F = \frac{2f_S\rho_Sv_S^2L_S}{d}, \quad \text{.................................................................(3.81)}
\]
where \( \rho_S \) is the density in the slug, calculated on the basis of the holdup in the slug body, which is shown in Eq. 3.82.

\[
\rho_S = \rho_L H_{LLS} + \rho_G (1 - H_{LLS}).
\]  

The friction factor \( f_s \) (Fanning) can be determined from single-phase flow correlations. For smooth pipes, a Blasius-type equation can be utilized, as given by Eq. 1.35. For rough pipes, it is possible to utilize the Hall (1957) explicit expression, as presented in Eq. 1.34. The Reynolds number in the slug body, \( \text{Re}_S \), is determined by

\[
\text{Re}_S = \frac{\rho_S d v_s}{\mu_S},
\]  

where the slug viscosity is based on the slug liquid holdup as

\[
\mu_S = \mu_L H_{LLS} + \mu_G (1 - H_{LLS}).
\]  

**Slug Velocity.** The two velocities that are associated with a slug are \( v_s \) and \( v_{TB} \), where the former represents the mean true velocity of the fluid in the slug body, and the latter represents the translational velocity of the slug, which is the propagation or the front velocity of the slug, which is the velocity of the interface between the gas and the liquid phases. One must differentiate between these two velocities. Apparently, it seems that the entire slug unit (i.e., the slug body, liquid film, and the gas-zone) moves at the translational velocity \( v_{TB} \). In reality, this is not true because the fluids in the different zones of the slug moves in different velocities. It is obvious that the slug moves much faster than the liquid film.

The key to understanding the relationship between the slug velocity and the translational velocity lies in the continuous process of scooping and shedding. This process can be better understood by following a liquid particle in the film zone, flowing ahead of a slug at a slower velocity \( v_{LTB} \). Because the slug body moves at a higher velocity, \( v_s \), the particle is soon scooped and engulfed by the slug body. After being accelerated to the slug velocity and traveling with the slug body, finally the particle is shed back into the film, decelerating down to the film velocity. The relationship between the slug and the translational velocities can be further illustrated with a tractor, as shown in Fig. 3.36.

![Slug velocities](image)

**Fig. 3.36**—Illustration of slug velocities.
The tractor moves at a velocity of $v_s$, scooping the sand ahead of it. The sand is accumulated in the front of the scoop. The front of the scooped sand moves faster than $v_s$. The front velocity of the sand is equal to the tractor velocity plus the volumetric scooping rate divided by the cross-sectional area of the scoop. Similar phenomenon occurs in a slug flow. The amount of liquid picked up from the slow moving film is added to the front of the slug, hence the front moves faster than the slug body. Under steady-state conditions, the length of the slug remains constant because liquid is shed at the same rate from the back of the slug.

In order to develop an expression for $v_s$, a mass balance is applied on a pipe section operating under slug flow (see Fig. 3.37). The total mass flow rate, namely, $W_L + W_G$ is not constant at any cross section of the pipe because of the intermittent nature of the flow. However, the total volumetric flow of the mixture is constant through any cross section of the pipe, provided that the flow is incompressible and the pipe boundaries are fixed. The volumetric flow rate is given by

$$q_L + q_G = \text{const}.$$  

Dividing Eq. 3.85 by the cross-sectional area of the pipe, the mixture velocity is obtained as

$$v_s = \frac{q_L + q_G}{A_p} = v_M.$$  

Because the gas and liquid in the slug body move at the same velocity, following Eq. 3.86, it is evident that the fluid velocity in the slug body is actually the mixture velocity, $v_M$. Note that this analysis is different than that proposed in the original study, where Dukler and Hubbard reported that $v_s = v_M$ on the basis of their experimental measurements.

![Fig. 3.37—Schematic of mass balance on slug front.](image-url)
The right-hand side equation gives the relationship between the slug velocity and the translational velocity, as

\[ v_{TB} = v_S + \frac{x}{\rho_L A_p H_{LLS}}. \] .................................(3.89)

Thus, the translational velocity is the sum of the slug velocity and the additional velocity gained by the pickup process. Defining the variable \( c \) as

\[ c = \frac{x}{\rho_L A_p H_{LLS} v_S} \] .................................(3.90)

yields

\[ v_{TB} = (1 + c)v_S = c_0 v_S. \] .................................(3.91)

**The Shedding Process.** In the slug body, it is assumed that the flow is fully developed turbulent flow with a fully developed velocity profile, as shown in Fig. 3.38. The velocity is zero at the wall and maximum at the centerline. There exists a particular radial distance, \( r_p \), where the local velocity is equal to the average velocity, \( v_S \). For \( r < r_p \), the local velocity is greater than \( v_S \) and the fluid advances in the direction of the flow. However, for \( r > r_p \), the fluid moves at a local velocity less than \( v_S \). Thus, the fluid in this region is shed behind.

Flow

\[ v = v_S \]

Fig. 3.38—Schematic of shedding process.

The rate of shedding is given by

\[ x = H_{LLS} \rho_L v_S A_p - \int_0^{r_p} 2\pi r H_{LLS} \rho_L v(r) dr. \] .................................(3.92)

Rearranging,

\[ c = 1 - \frac{2}{R^2} \int_0^{r_p} \frac{v(r)}{v_S} r dr. \] .................................(3.93)

The velocity profile in turbulent flow is given by the “Law of the Wall”, as follows

\[ v^+ = A + \frac{1}{K} \ln v^+, \] .................................(3.94)

where \( A = 5.57 \), and \( K = 0.38 \). The shear (wall) velocity, which is given by
\[ v^* = \frac{\tau_s}{\rho_s} = \sqrt{\frac{1}{2} f_s \rho_s v_s^2} = v_s \sqrt{\frac{f_s}{2}} \] ..........................(3.95)

is used to define the dimensionless “wall coordinate” parameters as

\[ y^+ = \frac{y v^* \rho_s}{\mu_s}, \quad \text{and} \quad v^+ = \frac{v}{v^*} \] ..........................(3.96)

where \( y \) is the distance from the pipe wall, \( y = R - r \). Transforming Eq. 3.93 into nondimensional form, using the “wall coordinates” given in Eq 3.94 through Eq. 3.96, yields

\[ c = 1 - \frac{2}{R^+} \int_{y^+_p}^{R^+} (R^+ - y^+) v^+ dy^+, \] ..........................(3.97)

where

\[ R^+ = \frac{R v^* \rho_s}{\mu_s} - \frac{Re_s}{2} \sqrt{\frac{f_s}{2}} \] ..........................(3.98)

At \( y^+_p \), the local velocity is equal to the average velocity, thus \( v^+|_{yp} = \sqrt{2/f_s} \). Substituting the velocity into the velocity profile given by Eq. 3.94, it is possible to solve for \( y^+_p \) as

\[ y^+_p = \EXP \left[ K \left( \frac{2}{\sqrt{f_s}} - A \right) \right]. \] ..........................(3.99)

Examination of Eq. 3.97 reveals that the parameter \( c \) is a unique function of the Reynolds number \( Re_s \) (Eq. 3.83). All the terms in the equation are functions of \( Re_s \), namely \( f_s \), \( R^+ \) (Eq. 3.98) and \( y^+_p \) (Eq. 3.99). The integration has been carried out numerically for a wide range of \( Re_s \), yielding the following correlation for the parameter \( c \).

\[ c = c(Re_s) \approx 0.021 \ln(Re_s) + 0.022. \] ..........................(3.100)
Values for $c$ were also measured experimentally by Fabre (1994), as shown by Fig. 3.39, where $Re_{LS} = \rho_L v_M d/\mu_L$. For fully developed turbulent flow in the slug body, a value of $c = 0.2$ has been found. This results in a value of $c_0 = 1.2$. Referring to Eq. 3.91, $c_0$ is a flow distribution coefficient related to the contribution of the mixture velocity to the translational velocity. It is greater than unity, as the translational velocity is influenced by the velocity profile in the preceding liquid slug. Thus, the translational velocity is viewed as the maximum possible local velocity in the slug body, equal approximately to the centerline velocity in the slug body. The value of $c_0 = 1.2$ is for turbulent flow while for laminar flow $c_0 = 2$.

**Hydrodynamics of the Film.** It is important to discuss the coordinate system that will be used before formulating the analysis. Two kinds of coordinate systems can be considered for the purpose of this analysis, as shown in Fig. 3.40.
The first coordinate system is an external stationary coordinate system (Lagrangian). Because slug flow is an unsteady flow, all the variables in this system are time and space dependent. Any analysis in this coordinate system is, thus, considerably complex. The second coordinate system is not stationary but rather moves at the translational velocity $v_T$. Considering the fact that the interface in this system is stationary, the slug interface can be visualized as a rigid surface. If liquid is introduced at a constant rate $x$ at the slug front, the liquid will flow backwards in the slug at a velocity of $v_T - v_L$, with respect to the coordinate system (relative velocity). In the liquid film the relative velocity is $v_L$. As the cross-sectional area of the film behind the slug decreases, the relative velocity increases. Thus, it is possible to relate the velocities in the slug and the film to the location in a slug unit. The significant advantage for using this coordinate system is that in this system the problem becomes a steady state, as all the flow variables are only space dependent and independent of time. This coordinate system has already been used in the mass balance presented in Eq. 3.87.

The original film hydrodynamics analysis by Dukler and Hubbard was carried out in the stationary coordinate system. The following analysis is given in the translational velocity coordinate system, and thus is much easier to carry out. It represents a simplified version of the analysis by Taitel and Barnea (1990) (see Sec. 3.4.2). A schematic of the film region analysis, carried out in the translational velocity coordinate system, is shown in Fig. 3.41.

In the translational velocity coordinate system, the interface is stationary, and the liquid film flows backwards at the pickup mass flow rate $x$, under steady-state conditions. The axial coordinate is $z$, pointing from the rear of the liquid slug in the backwards direction, as shown in the figure. It is required to determine the liquid film velocity profile $v_F(z)$ and the liquid holdup distribution $H_L(z)$, or the liquid film height profile $h_F(z)$. The analysis is carried out assuming open channel flow, whereby the pressure drop in the stratified region is neglected.

The velocity of the liquid film in the $v_T$ coordinate system is

$$v_F = v_T - v_L = \frac{x}{\rho_L A_p H_L}. \quad \text{.................................................................(3.101)}$$

A momentum balance applied to a differential control volume of liquid film with a width of $dz$, as shown in Fig. 3.27, yielding

![Fig. 3.41—Schematic of film region analysis in translational coordinate system.](image-url)
\[
\frac{d}{dz}(\rho_L v_F^2 A_p H_{LTB}) = \tau_F S_F - \frac{d}{dz}(\bar{p} A_p H_{LTB}) + A_p H_{LTB} \rho_L g \sin \theta, \tag{3.102}
\]

where \( \bar{p} \) is the average hydrostatic pressure acting on a cross-sectional area of the liquid film, and \( \bar{p} A_p H_{LTB} \) is the respective pressure force. Differentiation of the hydrostatic pressure term and the inertial term yield, respectively (see cross-sectional view in Fig. 3.41),

\[
\frac{d}{dz}(\bar{p} A_p H_{LTB}) = \frac{d}{dz} \left[ \rho_L g \cos \theta(h_F - y)bdy = A_p \rho_L g \cos \theta H_{LTB} \frac{dh_F}{dz}, \tag{3.103}\right.
\]

and

\[
\frac{d}{dz}(\rho_L v_F^2 A_p H_{LTB}) = -\frac{x^2}{\rho_L A_p H_{LTB}^2} \frac{dh_F}{dz}, \tag{3.104}\]

where \( H_{LTB}' = dH_{LTB} / dh_F \). Substituting Eqs. 3.103 and 3.104 into Eq. 3.102 and solving for \( dh_F / dz \) yields the final equation for the determination of the film profile, as follows:

\[
\frac{dh_F}{dz} = -\frac{\tau_F S_F - A_p H_{LTB} \rho_L g \sin \theta}{\frac{x^2}{\rho_L A_p H_{LTB}^2} - A_p H_{LTB} \rho_L g \cos \theta}. \tag{3.105}\]

Solving Eq. 3.105 yields the liquid film profile, \( h_F(z) \), which can be used to determine the required liquid holdup and velocity distributions, \( H_{LTB}(z) \) and \( v_{LTB}(z) \), respectively. This is a first-order ordinary differential equation (ODE) that must be solved numerically. The boundary condition is \( h_F(z = 0) = h_s = H_{LLS} \cdot d \) corresponding to \( v_F(z = 0) = v_{TB} - v_s \). Note that although the relative velocity is used in the analysis, namely, \( v_F = v_{TB} - v_{LTB} \), the shear stress force must be expressed utilizing the actual velocity (force is invariant in non-accelerating coordinate systems) as

\[
\tau_F = \frac{f_F}{2} \rho_L v_F^2 H_{LTB} = \frac{f_F}{2} \rho_L (v_{TB} - v_F)^2, \tag{3.106}\]

where \( f_F \) is calculated based on the hydraulic diameter of the liquid film. There are two limiting cases for Eq. 3.105. First is the equilibrium level, \( h_E \), occurring under the conditions shown in Eq. 3.107.

\[
-\tau_F S_F - A_p H_{LTB} \rho_L g \sin \theta = 0
\]

for \( \frac{dh_F}{dz} \rightarrow 0. \tag{3.107}\]

Similarly, the critical level, \( h_C \), can be calculated from

\[
\frac{x^2}{\rho_L A_p H_{LTB}^2} - A_p H_{LTB} \rho_L g \cos \theta = 0
\]

for \( \frac{dh_F}{dz} \rightarrow \infty. \tag{3.108}\]
The equilibrium liquid level occurs at some distance behind the slug body, where the liquid film reaches an equilibrium condition, and its height does not change any further. It is possible to use the equilibrium liquid level as an approximation for the film region by solving Eq. 3.107 for $H_{LTB}$, and assuming it as a constant in the stratified region. The critical level, when exists, is located at the back of the slug body where the shedding occurs. At this region, the interface is concave, and the critical level separates the upper critical region from the bottom subcritical region of the liquid film. Before integrating Eq. 3.105, one must check if $h_s > h_c$. If this is true, then the critical level should be used as the boundary condition, namely, $h_f(z = 0) = h_c$.

**Slug Length.** The slug unit period, namely, the time it takes for a slug unit to pass a given point in the pipe, is given by the inverse of the slug frequency,

$$T_u = \frac{1}{v_s}.$$  

(3.109)

Similarly, the slug body period is $T_s$. These two-time variables enable determination of the different slug length as

$$L_u = v_{TB}T_u = \frac{v_{TB}}{v_s},$$

$$L_s = v_{TB}T_s,$$

and

$$L_f = L_u - L_s = \frac{v_{TB}}{v_s} - L_s.$$  

(3.110)

The liquid slug length is determined through an overall mass balance on the liquid-phase. There are two different methods to carry out this balance. The first method is through integration with space, in which the liquid mass in a slug unit is determined by “freezing” a slug unit at a given time and checking the liquid inventory within it, as shown in Eq. 3.111.

$$W_L = \left( \rho_L L_s A_p H_{LLS} + \int_0^{L_f} \rho_L A_p H_{LTB} dL \right) \frac{1}{T_u} - x.$$  

(3.111)

The second method is through integration with time, whereby one integrates the amount of liquid passing through a cross-sectional area of the pipe at a given point along the pipe, during the passage time of slug unit, namely,

$$W_L = \left( v_s \rho_L A_p H_{LLS} T_s + \int_0^{T_f} v_{LTB} \rho_L A_p H_{LTBe} dt \right) \frac{1}{T_u}.$$  

(3.112)

In this model, the integration with time approach is used. An expression for $v_{LTB}H_{LTB}$ can be obtained from Eq. 3.87, utilizing the relationship between $v_{TB}$ and $v_s$ through the parameter $c$ from Eq. 3.91, as

$$(v_{TB} - v_{LTBe}) \rho_L A_p H_{LTBe} = (v_{TB} - v_s) \rho_L A_p H_{LLS},$$  

(3.87)

and
Eq. 3.112 can be transformed from the time domain into the space domain utilizing the following change of variables.

\[
T_S = \frac{L_S}{v_{TB}}, \quad T_F = \frac{L_F}{v_{TB}}, \quad T_U = \frac{1}{v_S}, \quad \text{and} \quad dt = \frac{dL}{v_{TB}}. \quad \text{(3.114)}
\]

Substituting Eqs. 3.113 and 3.114 into Eq. 3.112 and rearranging result in

\[
\frac{W_L}{\rho_L A_p v_S} = \frac{v_S}{v_{TB}} \left\{ H_{LLS} L_S + \int_0^{L_S} [H_{LTB} - c(H_{LLS} - H_{LTB})] dL \right\}. \quad \text{(3.115)}
\]

The integral in Eq. 3.115 is simplified by assuming an equilibrium liquid film in the stratified region, namely, \( H_{LTB} = H_{LTBe} = \text{const.} \), resulting in

\[
\frac{W_L}{\rho_L A_p v_S} = \frac{v_S}{v_{TB}} \left\{ H_{LLS} L_S + L_F [H_{LTBe} - c(H_{LLS} - H_{LTBe})] \right\}. \quad \text{(3.116)}
\]

Substituting for \( L_F \) in terms of \( L_S \) from Eq. 3.110 and solving for \( L_S \) yields the final equation for the slug length, as shown in Eq. 3.117.

\[
L_S = \frac{v_S}{v_S(H_{LLS} - H_{LTBe})} \left[ \frac{W_L}{\rho_L A_p v_S} - H_{LTBe} + c(H_{LLS} - H_{LTBe}) \right]. \quad \text{(3.117)}
\]

**Gas-Pocket Velocity.** The gas-phase velocity in the gas pocket can be obtained from a mass balance on the gas-phase carried out in the translational velocity coordinate system in a control volume bounded by a plane in the slug body (Point 1) and a plane in the stratified region (Point 2), as shown in Fig. 3.42.

![Fig. 3.42—Schematic of gas-phase mass balance.](image)

The mass balance between the two planes and the resulted equation for the gas-pocket velocity are given, respectively, by

\[
(v_{TB} - v_S)(1 - H_{LLS}) = (v_{TB} - v_{GTB})(1 - H_{LTB}), \quad \text{(3.118)}
\]

and

\[
v_{GTB} = v_{TB} - c \left( \frac{1 - H_{LLS}}{1 - H_{LTB}} \right) v_S. \quad \text{(3.119)}
\]
**Length of Mixing Zone.** The length of the mixing zone is based on a correlation for the “velocity head” $v_H$, as

$$L_M = 0.3v_H = 0.3 \frac{(v_S - v_{LTBe})^2}{2g}.$$  \hspace{1em} (3.120)

**Closure Relationships.** The Dukler and Hubbard model requires two additional input parameters, namely, the liquid holdup in the slug and the slug frequency. Slug flow closure relationships can be found in Sec. 3.4.3.

<table>
<thead>
<tr>
<th>TABLE 3.4—SUMMARY OF EQUATIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $-\Delta p_U = -\Delta p_A - \Delta p_F$ (Eqs. 3.78 and 3.79)</td>
</tr>
<tr>
<td>2. $-\Delta p_A = \frac{x}{A} (v_S - v_{LTBe})$ (Eq. 3.80)</td>
</tr>
<tr>
<td>3. $-\Delta p_F = \frac{2L_S v_S^2}{d}$ (Eq. 3.81)</td>
</tr>
<tr>
<td>4. $f_S = 0.001375 \left[ 1 + \left( \frac{2 \times 10^4 e + 10^6}{Re_S} \right)^{1/3} \right]$ (Eq. 1.34)</td>
</tr>
<tr>
<td>5. $Re_S = \frac{\rho_S d v_S}{\mu_S}$ (Eq. 3.83)</td>
</tr>
<tr>
<td>6. $v_S = \frac{q_L + q_G}{Ap}$ (Eq. 3.86)</td>
</tr>
<tr>
<td>7. $x = (v_{TB} - v_{LTBe}) \frac{\rho_L Ap_{HLS}}{\rho S}$ (Eq. 3.88)</td>
</tr>
<tr>
<td>8. $v_{TB} = v_S + \frac{x}{\rho_L Ap_{HLS}} = (1 + c)v_S = c_0 v_S$ (Eqs. 3.89 and 3.91)</td>
</tr>
<tr>
<td>9. $c_0 = 1.2$ for turbulent flow, $c_0 = 2$ for laminar flow; or $c = c(Re_S = 0.02 \ln(Re_S) + 0.022)$ (Eq. 3.100)</td>
</tr>
<tr>
<td>10. $\frac{dh_L}{dz} = -\frac{r_p SF - Ap_{LTBe} / L G \sin \theta}{\rho_L Ap_{HLS}^2 - Ap_{LTBe} L G \cos \theta}$ (Eq. 3.105)</td>
</tr>
<tr>
<td>11. $L_S = \frac{v_S}{\nu_S (H_{LLS} - H_{LTBe})} \left[ \frac{W_L}{\rho_L Ap v_S} - H_{LTBo} + c \left( H_{LLS} - H_{LTBo} \right) \right]$ (Eq. 3.117)</td>
</tr>
<tr>
<td>12. $L_F = L_J - L_S = \frac{v_{TB} - L_S}{v_S}$ (Eq. 3.110)</td>
</tr>
</tbody>
</table>

**Calculation Procedure.** A summary of the model equations is given in Table 3.4. The calculation procedure is given next:

1. Specify input parameters: $W_L$, $W_G$, $d$, fluid properties, $H_{LLS}$, and $v_S$.

2. Calculate the slug velocity, $v_S$, from Eq. 3.86.

3. Determine $c$ from Eq. 3.100 (or empirically for laminar flow $c_0 = 2$ and for turbulent flow $c_0 = 2$) and $v_{TB}$ and $x$ from Eqs. 3.91 and 3.89.
4. Assume a value for \( L_S \). Carry out the following iterative trial-and-error calculations:

5. Calculate \( L_F \) from Eq. 3.110.

6. Integrate numerically Eq. 3.105 from \( z = 0 \) \( \left( h_F(z = 0) = H_{LLS} \cdot d \right) \) to \( z = L_F \). Determine \( H_{LTB}(z) \), \( v_{LTB}(z) \), \( H_{LTBe} \) and \( v_{LTBe} \).

7. Calculate \( L_S \) from Eq. 3.117.

8. Compare the assumed and calculated values of \( L_S \). If no convergence is reached, update \( L_S \) and repeat Steps 5 through 8.

9. Once convergence is reached, calculate the following output parameters:

   \[ v_{LTBe} \] from the final results of the integration of Eq. 3.105.

   \(- \Delta p_A \) from Eq. 3.80.

   \( \text{Re}_S \) from Eq. 3.83 and \( f_S \) from 1.34.

   \(- \Delta p_F \) from Eq. 3.81.

   \(- \Delta p_U \) and \(- dp / dL \) from Eqs. 3.78 and 3.79, respectively.

Other output parameters, such as \( L_S, L_F, L_U, v_S, v_{TB}, v_{LTB}(z), H_{LTB}(z) \), and \( H_{LTBe} \), have already been calculated in the final iteration.

3.4.2 Taitel and Barnea (1990) Slug Model. In a more recent study, Taitel and Barnea (1990) presented a comprehensive analysis of slug flow. It was also intended to extend the scope of the slug flow studies into a unified model for horizontal, upward inclined, and upward vertical flow. For the study of the hydrodynamics of the film, several approaches were proposed. Also, two methods to predict the pressure drop were given. Refer to Fig. 3.34 for a schematic description of the physical slug flow model. Note that in this model, as shown in the figure, the velocities of the liquid and gas phases in the slug body are not the same, as given by \( v_{LLS} \) and \( v_{GLS} \), respectively.

**Mass Balances.** Liquid mass balances over a slug unit can be determined in two different ways, as described in Sec. 3.4.1. One method is to integrate the fluid flow rate at a fixed cross section over the time of the passage of a slug unit. Using this method yields Eq. 3.121, which is similar to Eq. 3.112.

\[
W_L = \left( v_{LLS} \rho_L A_p H_{LLS} T_S + \int_0^{T_F} v_{LTB} \rho_L A_p H_{LTB} dt \right) \frac{1}{T_U},
\]

where \( W_L \) is the input liquid mass flow rate, \( v_{LLS} \) is the average liquid velocity in the slug body, and \( v_{LTB} \) is the velocity in the liquid film; and \( T_U, T_S \), and \( T_F \) are the times for the passage of the slug unit, the liquid slug zone, and the film and gas-pocket region, respectively. Substituting \( T_S = L_S / v_{TB}, T_F = L_F / v_{TB} \) and \( dt = dL / v_{TB} \) into Eq. 3.121 yields

\[
W_L = v_{LLS} \rho_L A_p H_{LLS} \frac{L_S}{L_U} + \frac{1}{L_U} \left[ v_{LTB} \rho_L A_p H_{LTB} \right] dL.
\]
The second method takes into account the volume of fluid in a slug unit and is given by

$$W_L = \left( \rho_L A_p H_{LLS} + \int_0^{L_p} \rho_L A_p H_{LUB} dL \right) \frac{1}{T_U} - x. \quad \text{(3.111)}$$

The mass pickup rate of the liquid in the front of the slug, \(x\), is given by (similar to Eq. 3.87)

$$x = (v_{TB} - v_{LSS}) \rho_L A_p H_{LSS} = (v_{TB} - v_{LUB}) \rho_L A_p H_{LUB}. \quad \text{(3.123)}$$

Using the expression for \(x\) in Eq. 3.123, it can be shown that both methods presented for the liquid mass balance, namely, Eqs. 3.111 and 3.121, are equivalent. Combining Eqs. 3.123 and 3.122 yields

$$v_{SL} = v_{LSS} H_{LSS} + v_{TB} (1 - H_{LSS}) \frac{L_F}{L_U} - \frac{v_{TB}}{L_U} \int_0^{L_p} (1 - H_{LUB}) dL. \quad \text{(3.124)}$$

A continuity balance on both liquid and gas phases results in a constant volumetric flow rate through any cross section of the slug unit. Applying this balance on a cross section in the liquid slug body gives

$$v_M = v_{SL} + v_{SG} = v_{LSS} H_{LSS} + v_{GLS} (1 - H_{LSS}), \quad \text{(3.125)}$$

where \(v_M\) is the mixture velocity.

**Average Liquid Holdup.** The average liquid holdup of a slug unit is defined as

$$H_{LSU} = 1 - H_{GSU} = 1 - \left( \frac{(1 - H_{LSS}) L_S + \int_0^{L_p} (1 - H_{LUB}) dL}{L_U} \right). \quad \text{(3.126)}$$

The integral term in Eq. 3.126 can be eliminated using the mass balance of Eq. 3.124. This results in

$$H_{LSU} = \frac{v_{TB} H_{LSS} - v_{LSS} H_{LSS} + v_{SL}}{v_{TB}}. \quad \text{(3.127)}$$

The superficial liquid velocity, \(v_{SL}\), in Eq. 3.127 can be substituted in terms of the superficial gas velocity, \(v_{SG}\), from Eq. 3.125, yielding

$$H_{LSU} = \frac{v_{TB} H_{LSS} + v_{GLS} (1 - H_{LSS}) - v_{SG}}{v_{TB}}. \quad \text{(3.128)}$$

Eqs. 3.127 and 3.128 show an interesting result, namely, that the average liquid holdup in a slug unit, \(H_{LSU}\), depends only on the liquid and gas flow rates, \(v_{SG}\) and \(v_{SL}\), the gas bubbles velocity in the liquid slug, \(v_{GLS}\), the translational velocity, \(v_{TB}\), and the liquid holdup within the slug body, \(H_{LSS}\). However, it does not depend on the slug structure.

**Hydrodynamics of the Liquid Film.** The length of the liquid film, \(L_F\), its profile, \(h_F(L)\), the velocity profile along the liquid film, \(v_{LUB}(L)\), and mainly the film thickness and its velocity just before pickup, \(h_{Fe}\) and \(v_{LUB}\), respectively, are important parameters for calculating the pressure drop in slug flow.

The shape of the liquid film is complex, particularly near the tail of the liquid slug. It is a turbulent three-dimensional (3D) structure with a free surface. In this model, it is suggested to use, as an
approximation, the one-dimensional (1D) approach of “channel flow” theory, as presented by Dukler and Hubbard (1975). The analysis is carried out at the translational velocity coordinate system, in which the interface is stationary and the liquid film flows backwards at the pickup mass flow rate, \( x \), in a steady-state manner. The axial coordinate in this analysis is \( z \), pointing backwards from the rear of the slug body into the liquid film zone, as shown in Fig. 3.34.

Momentum balance equations on the film zone are carried out to find solutions for the film velocity, \( v_{LTB} \), and the film holdup, \( H_{LTB} \), as a function of position from the rear of the slug, \( z \). The momentum equations for the liquid film and the gas pocket in the translational velocity coordinate system are given, respectively, by

\[
\rho_L v_F \frac{\partial v_F}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\tau_L S_F}{A_F} - \frac{\tau_L S_I}{A_F} + \rho_L g \sin \theta - \rho_L g \cos \theta \frac{\partial h_F}{\partial z}, \quad \text{..........................(3.129)}
\]

and

\[
\rho_G v_G \frac{\partial v_G}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\tau_G S_G}{A_G} + \frac{\tau_G S_I}{A_F} + \rho_G g \sin \theta - \rho_G g \cos \theta \frac{\partial h_G}{\partial z}, \quad \text{..........................(3.130)}
\]

where \( v_F = v_{TB} - v_{LTB} \) and \( v_G = v_{TB} - v_{GTB} \) are the film and gas velocities relative to the translational velocity coordinate system, respectively. The shear stresses must be expressed utilizing the actual velocities (force is invariant in non-accelerating coordinate systems) as

\[
\tau_F = f_F \frac{\rho_L |v_{LTB}| v_{LTB}}{2}, \quad \text{..........................(3.131)}
\]

\[
\tau_G = f_G \frac{\rho_G |v_{GTB}| v_{GTB}}{2}, \quad \text{..........................(3.132)}
\]

and

\[
\tau_I = f_I \frac{\rho_G (v_{GTB} - v_{LTB})}{2}, \quad \text{..........................(3.133)}
\]

where \( f_F \) and \( f_G \) are the friction factors between the liquid film and the pipe wall and the gas pocket and the pipe wall, respectively, and \( f_I \) is the gas-liquid interfacial friction factor. The shear stresses and the velocities \( v_{LTB} \) and \( v_{GTB} \) are considered positive in the downstream direction, \( L \).

The Blasius correlation can be used for the determination of the friction factors in smooth pipes, utilizing the hydraulic diameter concept. For the liquid film this yields

\[
f_F = C \left( \frac{\rho_L d_F |v_{LTB}|}{\mu_L} \right)^n, \quad \text{..........................(3.134)}
\]

where the hydraulic diameter is \( d_F = 4A_F / S_F \). The gas friction factor can be determined similarly, where \( d_G = 4A_G / (S_G + S_I) \). For laminar flow, \( C = 16 \) and \( n = 1 \), while for turbulent flow, \( C = 0.046 \) and \( n = 0.2 \). For rough pipes, the expression presented by Hall (1957) can be used, as given in Eq. 1.34.
Determination of the interfacial friction factor, \( f_I \), is more complex. For the case of low liquid and gas velocities, the smooth interface friction factor can be used, namely, \( f_I = f_g \). For wavy interface, a constant value of \( f_I = 0.014 \), which was suggested for stratified-wavy flow, can be utilized. For vertical flow, the Wallis (1969) correlation for co-current annular flow can be used as an approximation for the counter-current flow that occurs in the Taylor bubble/liquid film zone, as given by

\[
f_I = 0.005(1 + 300 \frac{\delta_L}{d}),
\]

where \( \delta_L \) is the film thickness around the Taylor bubble.

Eliminating the pressure gradient from Eqs. 3.129 and 3.130 yields

\[
\rho_L v_F \frac{\partial v_F}{\partial z} - \rho_G v_G \frac{\partial v_G}{\partial z} =
\]

\[
\frac{\tau_F S_F}{A_F} - \frac{\tau_G S_G}{A_G} - \tau_p \left( \frac{1}{A_F} + \frac{1}{A_G} \right) + (\rho_L - \rho_G) g \sin\theta - (\rho_L - \rho_G) g \cos\theta \frac{\partial h_F}{\partial z}. \quad \text{(3.136)}
\]

The relative liquid film velocity, \( v_F \), and can be determined from Eq. 3.123 as

\[
v_F = (v_{TB} - v_{LTB}) = \frac{(v_{TB} - v_{GLS}) H_{LLS}}{H_{LTB}}. \quad \text{(3.137)}
\]

In a similar way, the relative gas velocity, \( v_G \), can be determined as

\[
v_G = (v_{TB} - v_{GTB}) = \frac{(v_{TB} - v_{GLS})(1 - H_{LLS})}{1 - H_{LTB}}. \quad \text{(3.138)}
\]

The derivatives \( \frac{\partial v_F}{\partial z} \) and \( \frac{\partial v_G}{\partial z} \) can be obtained by differentiating Eqs. 3.137 and 3.138, respectively. Substituting into Eq. 3.136 and considering that \( H_{LTB} \) is a function of \( h_F \) (or \( \delta_L \)), an ODE for \( h_F \) (or \( \delta_L \)) as a function of \( z \) can be obtained as

\[
\frac{dh_F}{dz} = \frac{\tau_F S_F}{A_F} - \frac{\tau_G S_G}{A_G} - \tau_p \left( \frac{1}{A_F} + \frac{1}{A_G} \right) + (\rho_L - \rho_G) g \sin\theta
\]

\[
- (\rho_L - \rho_G) g \cos\theta - \rho_L v_F (v_{TB} - v_{GLS}) H_{LLS} \frac{dH_{LTB}}{dh_F} - \rho_G v_G (v_{TB} - v_{GLS})(1 - H_{LLS}) \frac{dH_{LTB}}{dh_F}.
\]

where for the liquid film, similar to stratified flow from geometrical relationships,

\[
\frac{dH_{LTB}}{dh_F} = \frac{4}{\pi d} \sqrt{1 - \left( \frac{2 h_F}{d} - 1 \right)^2}. \quad \text{(3.140)}
\]

Eq. 3.139 is the most detailed form of 1D channel-flow analysis. This differential equation can be integrated numerically solving for \( h_F(z) \). The boundary condition for the ODE is \( h_F(z = 0) = h_S = H_{LLS} \cdot d \) corresponding to \( v_F(z = 0) = v_{TB} - v_{GLS} \). Once the film profile is solved, the actual liquid film velocity profile \( v_{LTB}(L) \) can be determined utilizing the mass balance given by Eq.
The integration is performed until the mass balance of Eq. 3.124 is satisfied, yielding the length of the liquid film, \( L_F \), as well as the holdup \( H_{LTBe} \) and the velocity \( v_{LTBe} \) at the end of the liquid film, just ahead of the following liquid slug front.

For large values of \( z \), the equilibrium liquid film level \( h_E \) is reached. For this case, \( dh_f / dz \to 0 \) and an implicit expression for \( h_E \) can be obtained by equating the numerator of Eq. 3.139 to zero, yielding

\[
\frac{\tau_F S_F}{A_F} - \frac{\tau_G S_G}{A_G} - \tau_L S_L \left( \frac{1}{A_F} + \frac{1}{A_G} \right) + (\rho_L - \rho_G)g \sin \theta = 0 \tag{3.141}
\]

The equilibrium liquid level can be used as an approximation for the liquid film region, assuming that the liquid film height is constant and equal to \( h_E \) along the entire film region. This will simplify the calculations, avoiding the numerical integration of Eq. 3.139.

The critical liquid film level, \( h_c \), occurs when \( dh_f / dz \to \infty \). Thus, an implicit expression for \( h_c \) can be obtained by setting the denominator of Eq. 3.139 equal to zero, yielding

\[
(\rho_L - \rho_G)g \cos \theta - \rho_L v_F \frac{(v_{TB} - v_{LLS})H_{LTB}}{H_{LTB}^2} \frac{dH_{LTB}}{df} - \rho_G v_G \frac{(v_{TB} - v_{GLS})(1 - H_{LLS})}{(1 - H_{LTB})^2} \frac{dH_{LTB}}{df} = 0 \tag{3.142}
\]

The critical liquid level, when exists, is located at the back of the slug body where the shedding occurs. At this region, the interface is concave, and the critical level separates the upper critical region from the bottom subcritical region of the liquid film. Before integrating Eq. 3.139, one must check if \( h_s > h_c \). If this is true, then the critical level should be used as the boundary condition, namely, \( h_f(z = 0) = h_c \).

Note that for vertical flow, the denominator of Eq. 3.139 is never zero, and critical film region does not exist.

Other simplification that can be applied in the analysis of the liquid film/gas pocket region is the assumption that the pressure drop in this region is negligible, as suggested by Dukler and Hubbard (1975). Assuming a negligible pressure drop in the liquid film/gas pocket zone, the liquid can be treated as an uncoupled free surface open channel flow, and Eq. 3.129 reduces to

\[
\frac{\rho_L v_F}{\partial z} = \frac{\tau_F S_F}{A_F} + \rho_L g \sin \theta - \rho_L g \cos \theta \frac{\partial h_f}{\partial z} \tag{3.143}
\]

Eq. 3.143 is a simpler ODE that must be integrated numerically to obtain the profile of the liquid film. Neglecting the pressure drop in the liquid film/gas zone may be justified for short film/gas region, but can cause a significant error for long zones. This open-channel-flow analysis is similar to the one presented in Sec. 3.4.1.

**Summary.** Three approaches have been presented for determination of the liquid film hydrodynamics as follows (see Fig. 3.43):

Case 1: A general formulation for 1D channel flow approximation, as given by Eq. 3.139. For this case \( h_f \neq \text{const.} \) and \(- dp / dz \neq 0 \), resulting in the most accurate solution.

Case 2: The liquid film is treated as a free surface open-channel flow (Eq. 3.143). This is a simplified solution, for which \( h_f \neq \text{const.} \) and \(- dp / dz = 0 \).
Case 3: An equilibrium and constant film thickness is assumed along the entire film zone (Eq. 3.141). For this case $h_f = \text{const.} = h_E$ but $-dp/dz \neq 0$.

Cases 1 and 2 must be solved numerically. Case 1 is more rigorous because in Case 2, the pressure drop along the film zone is neglected. Case 3 assumes equilibrium film thickness but takes into account the pressure drop along the film zone. Case 3 is the most practical approach, but Case 1 should be used whenever a more accurate description of the film hydrodynamics and the pressure drop behavior are required.

**Pressure Drop Determination.** The local axial pressure drop in slug flow is not constant because of the intermittent nature of the flow. Thus, it is practical to determine the average pressure drop and pressure gradient across a slug unit. The pressure drop for a slug unit can be calculated using a global force balance along a slug unit between cross sections A and C (see Fig. 3.34). The momentum fluxes in and out of the control volume are identical, and the global pressure drop is reduced to a force balance as shown in Eq. 3.144.

$$-\Delta p_U = \rho_U g \sin \theta L_u + \frac{\tau_s \pi d}{A_p} L_s + \left[ t_f \frac{\tau_f S_f}{A_p} + \tau_g S_g \right] d$, \hspace{1cm} \text{....................................................(3.144)}$$

where $\rho_U$ is the average density of the slug unit given by

$$\rho_U = H_{LSU} \rho_L + (1 - H_{LSU}) \rho_G. \hspace{1cm} \text{....................................................(3.145)}$$

The first term on the RHS of Eq. 3.144 is the gravitational pressure drop, whereas the second and third terms are the frictional pressure drops in the slug body and in the liquid film/gas pocket regions, respectively. The slug shear stress, $\tau_s$, is calculated as proposed by Dukler and Hubbard (1975), namely, Eqs. 3.81 to 3.84.

![Case 1: Accurate Film Profile](image1)

![Case 2: Simplified Film Profile](image2)

![Case 3: Equilibrium Film](image3)

Fig. 3.43—Liquid film hydrodynamic modeling cases.
Another method used for calculating the pressure drop in a slug unit is to neglect the pressure drop in the liquid film/gas pocket region, namely, \(-dp/dz = 0\). Although the pressure drop is neglected in the liquid film/gas pocket region, the liquid film profile is considered in the analysis. For this case, the pressure drop is calculated only for the liquid slug zone between cross sections A and B (see Fig. 3.34). The resulting pressure drop across a slug unit for this case is given by

\[-\Delta p_U = \rho_S g \sin \theta L_S + \frac{\tau_{L, p} d L_S}{A_p} - \Delta p_{\text{MIX}}, \]

where \(\rho_S\) is the average density in the liquid slug body, as given by

\[\rho_S = H_{LLS} \rho_L + (1 - H_{LLS}) \rho_G. \]

The first and second terms on the RHS of Eq. 3.146 are the gravitational pressure drop and the frictional pressure drop across the slug body, respectively. The third term, \(-\Delta p_{\text{MIX}}\), is the pressure loss in the mixing region, at the front of the liquid slug body. Dukler and Hubbard (1975) identified this term as the acceleration pressure drop (see Sec. 3.4.1), defined by

\[-\Delta p_{\text{MIX}} = -\Delta p_A = \frac{x(v_{\text{LLL}} - v_{\text{LLTe}})}{A_p} = \rho_L H_{LLS} (v_{\text{TB}} - v_{\text{LLL}})(v_{\text{LLL}} - v_{\text{LLTe}}). \]

However, careful analysis of the pickup process occurring at the front of the slug body reveals that the contribution to \(-\Delta p_{\text{MIX}}\) is not only caused by the accelerational pressure drop but is also caused by the change in the liquid elevation between the film zone and the liquid slug zone. In other words, the liquid picked up from the film into the slug body gains both higher kinetic energy (acceleration) and potential energy, which must both be accounted for. Thus, a more rigorous expression for \(-\Delta p_{\text{MIX}}\) is given by

\[-\Delta p_{\text{MIX}} A_p = \rho_L g \cos \theta \int_{0}^{h_S} (h_{Fe} - y) dy - \rho_L g \cos \theta \int_{0}^{h_S} (h_S - y) dy + \rho_L H_{LLS} A_p (v_{\text{TB}} - v_{\text{LLL}})(v_{\text{LLL}} - v_{\text{LLTe}}). \]

Recall that \(h_S\) is the liquid level at the back of the liquid slug, namely, \(h_S = H_{LLS} \cdot d\). Note that when \(h_S > h_C\), one must set \(h_S = h_C\) (given by Eq. 3.142) and replace the term \((v_{\text{LLL}} - v_{\text{LLTe}})\) by \((v_C - v_{\text{LLTe}})\), where \(v_C\) is the film velocity associated with \(h_C\). The integration in Eq. 3.149 results in an explicit expression, as given in Eq. 3.150.

\[
\int_{0}^{h_S} (h_{Fe} - y) dy = \frac{d^3}{12} \left[ 1 - \left( \frac{2 h_{Fe}}{d} - 1 \right)^2 \right] + \frac{d^3}{8} \left( 2 \frac{h_{Fe}}{d} - 1 \right) \pi \left[ \frac{\pi}{2} + \sin^{-1} \left( \frac{2 h_{Fe}}{d} - 1 \right) + \left( 2 \frac{h_{Fe}}{d} - 1 \right) \sqrt{1 - \left( \frac{2 h_{Fe}}{d} - 1 \right)^2} \right]. \]

As can be seen from Eq. 3.149, \(-\Delta p_{\text{MIX}}\) is always less than \(-\Delta p_A\). The use of \(-\Delta p_{\text{MIX}} = -\Delta p_A\) may cause a negligible error in small diameter pipes, but can lead to a more significant error in large diameter pipes because of the large change of elevation from the film to the slug zones.
Summary. Two methods have been presented for the pressure drop calculation:

Method 1: The first method consists of a global force (momentum) balance on the entire slug unit, as given in Eq. 3.144.

Method 2: In the second method, the momentum balance is applied only to the liquid slug zone, neglecting the pressure drop in the film zone. This is given in Eq. 3.146.

It is possible to show that the two methods are identical for the case where the liquid-phase is decoupled from the gas-phase in the liquid film/gas pocket region and is treated as an open-channel flow. Thus, the pressure drop in the liquid film/gas pocket is neglected for Method 1, as done for Method 2 \((-dp/\partial z=0\)). However, for both methods, the liquid film profile is taken into account, namely, \(h_p(h_f(z))\). The starting point is Eq. 3.143, which neglects the pressure drop along the liquid film/gas pocket zone. Multiplying by \(A_p = A_p H_{LTB}\) and integrating along the film zone results in

\[
\int_{v_{TB} - v_{LTS}}^{v_{TB} - v_{LTS}} \rho_L A_p H_{LTB} (v_{TB} - v_{LTS}) \frac{\partial (v_{TB} - v_{LTS})}{\partial z} dz = \int_0^{L_p} \tau_f S_f dz + \rho_L g \sin \theta \int_0^{L_p} A_p dz
\]

\[
- \rho_L g \cos \theta \int_h^{h_f} A_F \frac{dh_f}{\partial z} dz. ...........................................................(3.151)
\]

Carrying out the integration on the LHS of Eq. 3.151 and substituting \(H_{LTB}(v_{TB} - v_{LTS}) = H_{LTS}(v_{TB} - v_{LTS})\) from Eq. 3.123, it can be shown that this term is the force associated with the accelerational pressure drop, namely, \(-Ap\). Utilizing the following relationship

\[
\int_{h_s}^{h_f} A_F dh_f = \int_0^{h_f} (h_f - y) dy - \int_0^{h_s} (h_f - y) dy, ........................................................(3.152)
\]

Eq. 3.151 becomes

\[
\rho_L H_{LTS} A_p (v_{TB} - v_{LTS}, v_{LTS} - v_{LTS}) = \int_0^{L_p} \tau_f S_f dz + \rho_L g \sin \theta \int_0^{L_p} A_p dz
\]

\[
- \rho_L g \cos \theta \int_h^{h_f} (h_f - y) dy + \rho_L g \cos \theta \int_0^{h_s} (h_f - y) dy. .........................(3.153)
\]

Substituting Eq. 3.153 into Eq. 3.149 yields

\[
- \Delta p_{mix} A_p = \int_0^{L_p} \tau_f S_f dz + \rho_L g \sin \theta \int_0^{L_p} A_p dz. ...........................................................(3.154)
\]

Substituting Eq. 3.154 into Eq. 3.146, it can be shown that Eq. 3.146 is identical to Eq. 3.144, for the conditions of free surface liquid film analysis, neglecting the pressure drop in the liquid film/gas pocket zone and taking the liquid film profile into account.

For upward inclined and vertical flows, usually the film reaches its equilibrium level after a relatively short distance from the back of the liquid slug. For this case, the film consists of a short curved zone (around the nose of the Taylor bubble) and an equilibrium zone. In the equilibrium film zone, where the film thickness is constant, the wall shear stress balances the gravity force. For this condition, the upper limit of the integral in Eq. 3.154 can be set to the length of the curved zone. This implies that the pressure drop in the mixing zone \(-\Delta p_{mix}\) equals to the sum of the wall shear stress and gravitational forces in the curved zone of the liquid film. This observation has some consequences when the simplified approach of a uniform thickness along the entire film region is used, as explained next.
When the approximation of a uniform equilibrium thickness in the liquid film is used, the pressure drop calculations that use Eqs. 3.144 and 3.146 are not consistent. For a uniform film thickness, Eq. 3.144 reduces to

\[ -\Delta p_U = \rho_s g \sin\theta L_s + \frac{\tau_s \pi d}{A_p} L_s + \rho_f g \sin\theta L_F + \frac{\tau_F S_F}{A_p} L_F + \frac{\tau_G S_G}{A_p} L_F, \] ..........................(3.155)

where

\[ \rho_f = H_{LTB} \rho_L + (1 - H_{LTB}) \rho_G. \] ..........................(3.156)

As mentioned before, for this case of uniform film thickness, gravity is balanced by the shear forces in the film zone. Assuming that the pressure drop in the gas pocket/liquid film zone is negligible, only the first two terms remain in the RHS of the equation. This is in contradiction to Eq. 3.146, which was developed neglecting the pressure drop in the liquid film/gas pocket zone, as Eq. 3.146 has the additional term, \( -\Delta p_{MI} \). Barnea (1989) showed that for the case of equilibrium liquid film, and neglecting the pressure drop in the gas pocket/liquid film zone, the acceleration term should not be considered. Thus, the result from Eq. 3.155 not Eq. 3.146 must be used for this case. Note that previous studies for vertical slug flow, such as Fernandes et al. (1983) and Sylvester (1987), included the accelerational pressure drop, although considering equilibrium liquid film and neglecting the pressure drop in the liquid film/gas pocket zone in their analysis, which results in over-prediction of the pressure drop.

**Calculation Procedure.** For a given set of flow conditions, including the gas and liquid flow rates, their fluid properties, pipe diameter, and inclination angle, the developed model along with the closure relationships (given in Sec. 3.4.3) can be solved to predict the slug characteristics including the pressure gradient. The calculation procedure presented is for Case 3, namely, for the equilibrium liquid film case.

Calculation of the average slug unit liquid holdup, \( H_{LSU} \), can be performed in a straightforward manner, independent of slug characteristics, as given by either Eq. 3.127 or Eq. 3.128,

\[ H_{LSU} = \frac{v_{TB} H_{LLS} - v_{LSS} H_{LLS} + v_{SL}}{v_{TB}}, \] ..........................(3.127)

and

\[ H_{LSU} = \frac{v_{TB} H_{LLS} + v_{GLS} (1 - H_{LLS}) - v_{SG}}{v_{TB}}. \] ..........................(3.128)

The solution procedure is based on the implicit solution of Eq. 3.141 for \( h_f \) by trial and error, as follows:

1. Determine the mixture velocity, namely, \( v_M = v_{SL} + v_{SG} \).
2. Determine \( v_{TB}, v_{GLS}, H_{LLS}, L_s \) (or \( v_s \)) from closure relationships (Sec. 3.4.3).
3. Determine \( v_{LSS} \) from Eq. 3.125.

\[ v_M = v_{SL} + v_{SG} = v_{LSS} H_{LLS} + v_{GLS} (1 - H_{LLS}). \] ..........................(3.125)
4. Guess \(h_F\) and determine \(H_{LTB}, A_F, A_G, S_F, S_G, S_I, d_F\) from geometrical relationships similar to Eq. 3.38, as

\[
H_{LTB} = \frac{1}{\pi} \left[ \pi - \cos^{-1}\left(\frac{2 h_F}{d} - 1\right) + \left(\frac{2 h_F}{d} - 1\right) \sqrt{1 - \left(\frac{2 h_F}{d} - 1\right)^2} \right],
\]

\[
S_F = d \left[ \pi - \cos^{-1}\left(\frac{2 h_F}{d} - 1\right) \right],
\]

\[
S_G = \pi d - S_F,
\]

and

\[
S_I = d \sqrt{1 - \left(\frac{2 h_F}{d} - 1\right)^2}. \tag{3.157}
\]

For vertical flow with a symmetrical annular film, simpler geometrical relationships can be obtained.

5. Calculate \(v_{LTB}\) from Eq. 3.137.

\[
v_{TB} - v_{LTB} = \frac{(v_{TB} - v_{LSS})H_{LSS}}{H_{LTB}}. \tag{3.137}
\]

6. Determine \(v_{GTB}\) either from Eq. 3.138 or from a continuity balance applied to the liquid film/gas pocket zone, similar to the continuity balance on the slug zone given by Eq. 3.125, namely,

\[
v_M = v_{LTB} H_{LTB} + v_{GTB} (1 - H_{LTB}). \tag{3.158}
\]

7. Determine \(f_F\) (and similarly \(f_G\)) for smooth pipes from Eq. 3.134, and for rough pipes from Eq. 1.34.

\[
f_F = C \left( \frac{\rho_L d_F}{\mu_L} \right)^n \left| \frac{v_{LTB}}{v_{LTB}} \right|^n. \tag{3.134}
\]

For the interfacial friction factor use \(f_I = 0.0142\) for horizontal and inclined pipes, and Eq. 3.135 for vertical flow.

\[
f_I = 0.005(1 + 300 \frac{\delta_i}{d}). \tag{3.135}
\]

8. Calculate \(\tau_F, \tau_G\) and \(\tau_I\) from Eqs. 3.131 through 3.133.

\[
\tau_F = f_F \frac{\rho_L v_{LTB}^2}{2}, \tag{3.131}
\]

\[
\tau_G = f_G \frac{\rho_G v_{GTB}^2}{2}, \tag{3.132}
\]

and
\[ \tau_f = f_f \frac{\rho_G (v_{GTB} - v_{LTB}) (v_{GTB} - v_{LTB})}{2} \] ........................................(3.133)

9. Check Eq. 3.141 for convergence. If convergence is not reached, repeat Steps 4 through 9. Standard methods for ensuring fast convergence can be used. When convergence is reached, proceed to Step 10.

\[ \frac{\tau_f S_F}{A_F} - \frac{\tau_G S_G}{A_G} - \tau_f S_f \left( \frac{1}{A_F} + \frac{1}{A_G} \right) + (\rho_L - \rho_G)g \sin \theta = 0. \] ........................................(3.141)

10. The slug length, \( L_S \), is given. Determine \( L_U \) from Eq. 3.122, which for the case of equilibrium liquid film reduces to

\[ L_U = \frac{L_S (v_{LSS} H_{LLS} - v_{LTB} H_{LTB})}{v_{SL} - v_{LTB} H_{LTB}} \] ........................................(3.159)

and \( L_F = L_U - L_S \).

11. The total pressure drop (for equilibrium liquid film) can be determined from Eq. 3.155

\[ -\Delta p_U = \rho_s g \sin \theta L_S + \frac{\tau_f \pi d}{A_p} L_S + \rho_F g \sin \theta L_F + \frac{\tau_f S_F}{A_p} L_F + \frac{\tau_G S_G}{A_p} L_F, \] ........................................(3.155)

and the pressure gradient is \( -dp/dL = -\Delta p_U / L_U \).

### 3.4.3 Closure Relationships

The Dukler and Hubbard (1975) model requires two closure relationships, namely, the liquid holdup in the slug body, \( H_{LLS} \), and the slug frequency, \( \nu_s \) (or slug length, \( L_s \)). The Taitel and Barnea (1990) model requires, in addition to \( H_{LLS} \) and \( L_S \) (or \( \nu_s \)), the translational velocity, \( V_{TB} \), and the velocity of the small bubbles in the liquid slug, \( V_{GLS} \). Following is a summary of commonly used correlations for the required closure relationships.

**Translational Velocity, \( V_{TB} \)**. The translational velocity of a Taylor bubble is the sum of the Taylor bubble rise velocity, or drift velocity, which is the velocity of a Taylor bubble in a stagnant liquid, plus the contribution of the mixture velocity in the preceding slug, as

\[ V_{TB} = c_v V_M + V_D, \] ........................................(3.160)

where \( c_v \) is a flow distribution coefficient, and \( V_D \) is the drift velocity. As presented in Sec. 3.4.1, \( c_v \) is 1.2 for turbulent flow and 2 for laminar flow, based on the slug body Reynolds number, as defined in Eq. 3.83.

The drift velocity of a Taylor bubble has been treated in the past separately for the vertical and horizontal cases. For vertical flow, based on potential flow analysis and the studies of Davies and Taylor (1949) and Dumitrescu (1943), the drift velocity in vertical pipes is given by

\[ V_{DV} = 0.35 \sqrt{gd}. \] ........................................(3.161)

The drift velocity in horizontal pipes is not as clear as in the vertical case. Several investigators, such as Dukler and Hubbard (1976) (see Eq. 3.91), did not consider drift velocity in horizontal flow, arguing that gravity cannot affect the flow in horizontal pipes. However, other studies have confirmed that drift
velocity does exist in horizontal flow, owing to the difference in hydrostatic head between the nose of the Taylor bubble and the liquid film. Benjamin (1968) analyzed the propagation velocity of a large bubble formed when liquid is emptied from a horizontal pipe, on the basis of potential theory. In principle, this propagation velocity is the same as the drift velocity in slug flow. Based on the analysis of Benjamin, the drift velocity in horizontal pipes is

$$v_{D,H} = 0.54 \sqrt{gd}.$$ ..........................................................(3.162)

For the inclined case, Alves et al. (1993) proposed a model for the prediction of the drift velocity for upward inclination angles, namely, $0^\circ \leq \theta \leq 90^\circ$. Also, Bendiksen (1984) proposed a simple correlation for the drift velocity in this inclination range, as given by

$$v_D = v_{D,H} \cos \theta + v_{D,V} \sin \theta \quad \text{for} \quad 0^\circ \leq \theta \leq 90^\circ.$$ ..........................................................(3.163)

Thus, combining Eqs. 3.160 through 3.163, the final expression for the Taylor bubble translational velocity is

$$v_{TB} = c_0v_d + 0.54 \sqrt{gd} \cos \theta + 0.35 \sqrt{gd} \sin \theta \quad \text{for} \quad 0^\circ \leq \theta \leq 90^\circ.$$ ..........................................................(3.164)

The drift velocity exhibits an interesting behavior as a function of the inclination angle. Fig. 3.44 presents the predictions of the Bendiksen correlation for the variation of the drift velocity with inclination. First, it is observed that the drift velocity in the horizontal case is larger than the drift velocity in the vertical case. Also, as the inclination angle reduces from the vertical, the drift velocity increases, reaching a maximum around $30^\circ$ from the horizontal. Further decrease in the inclination angle towards horizontal flow causes a reduction in of the drift velocity. This trend was confirmed by several experimental studies, such as Zukoski (1966), Bonnecaze et al. (1971) and Bendiksen (1984). A possible explanation for this phenomenon is based on the competition between gravitational potential and liquid drainage in the film around the Taylor bubble. In vertical flow, the gravitational potential is high, but the liquid drainage in the film is low because of the thin film created by the axi-symmetric Taylor bubble that pushes radially in all directions toward the pipe wall. As the inclination angle reduces from the vertical to around $30^\circ$, the Taylor bubble becomes asymmetric, and the liquid film at the bottom gets thicker, increasing the liquid drainage and, as a result, the drift velocity. Further decrease in the inclination angle reduces the gravitational potential and the drift velocity. However, the drainage in the liquid film is high because of its large thickness. This explains why the drift velocity in the horizontal case is larger than the one in the vertical case.
Gas Bubbles Velocity in Slug, \( v_{GLS} \). Similarly to the case of the Taylor bubble translational velocity, the velocity of the dispersed bubbles in the slug body is also assumed to be the sum of the bubbles rise velocity, or drift velocity, and the mixture velocity, namely,

\[
v_{GLS} = c_0v_M + 1.53 \left[ \frac{g \sigma (\rho_L - \rho_G)}{\rho_L^2} \right]^{0.25} H_{LLS}^{0.5} \sin \theta.
\] ......................................................(3.165)

For vertical flow, as the bubbles tend to move in the center of the pipe, the same values are used for \( c_0 \) as used in the Taylor bubble velocity calculation, namely 1.2 for turbulent flow and 2 for laminar flow. However, for horizontal and near-horizontal flow, as the bubbles tend to accumulate at the top of the pipe, it is commonly assumed that the bubbles move at the average mixture velocity, hence the flow distribution coefficient is equal to one.

Note that the drift velocity of the bubbles is based on the Harmathy (1960) expression for a single bubble, corrected by the factor \( H_{LLS}^{0.5} \) to account for the hindered bubble swarm velocity, and multiplied by the \( \sin \theta \) for the inclination angle effect. This will be explained in more detail in the section on Bubble Flow in Chap. 4.

Slug Liquid Holdup, \( H_{LLS} \). The Gregory et al. (1978) correlation, valid only for horizontal flow, is given by

\[
H_{LLS} = \frac{1}{1 + \left( \frac{v_M}{8.66} \right)}^{1.39}, \quad (v_M \text{ in m/s}). \quad ......................................................(3.166)
\]

The Barnea and Brauner (1985) model is based on the transition boundary to dispersed-bubble flow, as given in Eq. 4.23. The assumption is that the liquid holdup in the slug body equals to the liquid holdup predicted by the transition boundary to dispersed-bubble flow, under the same mixture velocity. This is given by
\[ H_{LLS} = 1 - 0.058 \left[ 2 \sqrt{\frac{0.4 \sigma}{(\rho_L - \rho_g)g}} \left( \frac{\rho_L}{\sigma} \right)^{0.6} \left( \frac{2}{d} f_s v_M^3 \right)^{0.4} - 0.725 \right]^2. \]  

Note that the model given in Eq. 3.167 is independent of the inclination angle, as the transition to dispersed-bubble flow is inclination angle independent.

The recently developed Gomez et al. (2000) correlation is the most general one, as it is applicable for the inclination angle range from horizontal to upward vertical flow.

\[ H_{LLS} = 1.0 \exp\left[-(7.85 \times 10^{-3} \theta + 2.48 \times 10^{-6} Re_{LS})\right] \quad \text{for} \quad 0^\circ \leq \theta \leq 90^\circ, \]  

where the liquid slug Reynolds number is calculated as

\[ Re_{LS} = \frac{\rho_L v_M d}{\mu_L}. \]

**Slug Frequency, \( \nu_s \).** An evaluation of slug frequency prediction methods, including empirical correlations and mechanistic models, is given by Zabaras (2000). A mechanistic model for the prediction of slug frequency has been developed by Dukler and Taitel (1977). However, this model is complex and requires numerical solution of the unsteady-state conservation equations of mass and momentum by a finite difference scheme, which requires significant computer time. In this section, two empirical correlations for slug frequency are presented, namely, Hill and Wood (1994) and Zabaras (2000), which apply to small and large diameter pipes.

The Hill and Wood (1994) correlation has been developed from large amount of laboratory and field data, including large-diameter pipes for horizontal and near-horizontal conditions, and is given by

\[ \pi_v = -24.729 + 0.00766 \exp(9.91209 \pi_{\bar{h}_L}) + 24.721 \exp(0.20524 \pi_{\bar{h}_L}), \]

where \( v_{SL} \) and \( v_{SG} \) are in m/s and \( d \) is in m. Note that the frequency \( \nu_s \) is in hr\(^{-1}\). The dimensionless frequency group, \( \pi_v \), and equilibrium liquid level group, \( \pi_{\bar{h}_L} \), are given, respectively, by

\[ \pi_v = \frac{\nu_s d}{v_M} (1 - 0.05 v_{SG}) d^{0.3}, \quad \text{and} \quad \pi_{\bar{h}_L} = \frac{\nu_s}{v_{SL}} \left( 1 - \frac{0.068}{v_{SL}} \right). \]

Note that \( \bar{h}_L \) is the equilibrium stratified flow liquid level, calculated under the given operating slug flow conditions (see Sec. 3.2.1).

The Zabaras (2000) slug frequency correlation is a modification of the correlation originally proposed by Gregory and Scott (1969) for horizontal flow. A 399-points databank was used for the modification, covering pipe diameters between 0.0254 to 0.20 m and inclination angles from 0\(^\circ\) to 11\(^\circ\). Thus, the modified correlation, given in Eq. 3.172, is applicable to horizontal as well as near-horizontal upward inclined pipes.

\[ \nu_s = 0.0226 \left( \frac{v_{SL}}{g d} \right)^{12} \left[ \frac{212.6}{v_M} + v_M \right]^{12} \left[ 0.836 + 2.75 (\sin \theta)^{0.25} \right]. \]
Note that the correlation is given in English units: $v_{SL}$ and $v_{SL}$ in ft/s, $d$ in ft, $\nu_S$ in s$^{-1}$ and $g = 32.2$ ft/s$^2$. The square brackets represent the modification carried out by Zabaras, while the rest of the correlation is exactly the original correlation of Gregory and Scott.

**Slug Length, $L_s$.** The slug length is an input variable to the Taitel and Barnea model. Also, for the Dukler and Hubbard model, it is possible to give the slug length instead of the slug frequency, as an input. For small diameter pipelines ($d \leq 2$ in.), a value of $L_s = 30d$ is recommended. This value is based on a fully developed stable slug length. For large diameter pipelines ($d > 2$ in.), the Scott et al. (1989) correlation, based on data from the Prudhoe Bay field, is recommended and given by

$$\ln(L_s) = -25.4 + 28.5 \left[ \ln(d) \right]^{0.1}, \quad (d \text{ in inches and } L_s \text{ in ft}) \quad \cdots \quad (3.173)$$

Note that slugs in large-diameter pipelines are much longer than $30d$. This is because of slug growth occurring along large-diameter pipelines. A model for slug growth was proposed by Scott (1987) and Scott et al. (1987). The model is based on the phenomenon that in large-diameter pipelines, the operating point is usually near the transition boundary between slug flow and stratified flow. Thus, the pipeline is mainly under stratified flow, whereby slugs are generated periodically. When a slug is formed, it sweeps the liquid film ahead of it, which is much thicker than the film the slug sheds. As a result, the slug grows along the pipeline, reaching lengths significantly greater than $30d$. Note that the length predicted by Eq. 3.173 is an average value for the slug length.

### 3.5 Pipeline Overall Model and Evaluation of Methods

The overall pipeline mechanistic model developed by Xiao et al. (1990) combines the models presented in the previous sections of this chapter. It consists of a flow pattern prediction model and separate models for the different existing flow patterns, such as stratified flow, slug flow, annular flow, and dispersed-bubble flow. For a given set of flow conditions, the overall pipeline model enables the prediction of the flow characteristics, including the existing flow pattern and the liquid holdup and pressure drop. Thus, the overall model can be used to design pipelines, as done previously utilizing empirical correlations.

The objectives of the Xiao et al. study have been to develop an overall pipeline model, to generate a pipeline data bank, and to evaluate the proposed overall model, as well as the most commonly used correlations, against the data bank. The overall model and the evaluation carried out are presented next.

#### 3.5.1 Overall Pipeline Model.

**Flow Pattern Prediction Model.** For flow pattern prediction, Xiao et al. used the Taitel and Dukler (1976) model with some modifications, as presented in Sec. 3.2. Other studies considered were Lin and Hanratty (1986), Andritsos and Hanratty (1987), and Wu et al. (1987). The later studies focused mainly on the prediction of onset to slugging through linear stability theory, resulting in complex mathematical analysis.

**Stratified Flow Model.** Several models, with different degrees of complexity, have been proposed for stratified flow. These models include Taitel and Dukler (1976), which is a one-dimensional two-fluid model based on average gas and liquid velocities, applicable to horizontal and slightluy inclined pipes; the Cheremisinoff and Davis (1979) model that is one-dimensional but considers the liquid-phase velocity profile in the radial direction, which is valid only for horizontal flow; and the Shoham and
Taitel (1984) model, which solves numerically a two-dimensional velocity profile in the liquid-phase, which is also applicable to horizontal and slightly inclined pipes. The later two models that consider the liquid-phase velocity profile are neither easy to use nor guaranteed to give more accurate results. This is the reason that the more practical model of Taitel and Dukler (given in Sec. 3.3) has been chosen for the overall model, with different closure relationships, as given next.

The following closure relationships (given in Sec. 3.3.1) are used:

- For the interfacial friction factor, \( f_I \), the combination of the Andritsos and Hanratty (1987) correlation for small diameter pipes and the Baker et al. (1988) correlation for larger pipe diameters is used, as given in Eqs. 3.68 through 3.73.

- Both the liquid and gas wall friction factors, \( f_L \) and \( f_G \), are determined on the basis of the phase hydraulic diameter and Reynolds number, using a standard pipe flow friction factor method. This was originally suggested by Taitel and Dukler (1976), as given in Eqs. 3.29 and 3.30, respectively (for smooth pipes). As reported by Xiao et al., utilization of the Ouyang and Aziz (1996) correlation for \( f_L \), given in Eq. 3.74, did not result in significant improvement of the model.

**Slug Flow Model.** The models considered in this study include the original Dukler and Hubbard (1975) model and subsequent models by Nicholson et al. (1978) and Kokal and Stanislaw (1989). The more recent model by Taitel and Barnea (1990), as given in Sec. 3.4, was chosen for the overall pipeline model, being the most detailed one. The following options and closure relationships are utilized in the model:

- Equilibrium liquid film method, namely, Case 3 (given by Eq. 3.141), which is the most practical because it does not require numerical integration and is also believed to be sufficiently accurate for actual applications.

- Global pressure drop across a slug unit, with equilibrium liquid level option, as given in Eq. 3.155.

- Gregory et al. (1978) correlation for liquid holdup in the slug body, \( H_{LLS} \), as given by Eq. 3.166. Note that this correlation is valid only for horizontal pipes.

- Scott et al. (1989) correlation for the slug length, as given in Eq. 3.173.

- In the liquid film/gas pocket zone, the gas and liquid wall shear stresses, as well as the interfacial shear stress, are calculated as suggested by Taitel and Barnea (Eqs. 3.131 through 3.134).

- Bendiksen (1984) correlation for the determination of the Taylor bubble translational velocity, \( v_{TB} \), as given in Eq. 3.164.

- The velocity of the small bubbles in the liquid slug is determined by Eq. 3.165. For the bubble swarm correction, Xiao et al. used a value of 0.1 for the exponent, namely, \( H_{LLS}^{0.1} \).

**Annular Flow Model.** Most of the models developed for annular flow are for vertical flow conditions. Usually, a one-dimensional two-fluid approach is used, resulting in a relatively simple model that does not require numerical solution. Examples are the Oliemans et al. (1986) and Alves et al. (1991) models, which assume a constant film thickness. While this assumption is valid for vertical flow, the film thickness distribution in horizontal and inclined pipes is not uniform. Because of gravity, the
film thickness at the bottom is larger than that one at the top (Paz and Shoham, 1999). The film thickness variation also results in variation of the deposition and entrainment rates. Thus, two-dimensional models have been proposed for pipeline conditions to incorporate these phenomena. The resulting models, such as Laurinat et al. (1985) and James et al. (1987), are complex and require numerical methods for the solution.

In the overall model, Xiao et al. chose the Alves et al. model (presented in Sec. 4.5.5), which was developed for vertical and sharply inclined pipe, applying it as an approximation for pipeline conditions. The following closure relationships are used:

- The film wall shear stress is determined as proposed by Alves et al. in Eq. 4.94.
- The entrainment is determined using the Oliemans et al. correlation, as given in Eq. 4.76.
- The interfacial friction factor is determined on the basis of the core velocity, fluid properties, hydraulic diameter, and Reynolds number, as defined in Eqs. 4.64 and 4.65. A standard pipe flow friction factor method is used, based on the calculated core superficial Reynolds number, as given in Eq. 4.93.

**Dispersed-Bubble Flow Model.** The homogeneous no-slip model, as given in Sec. 2.1, is used for the prediction of the flow characteristics for the dispersed-bubble flow pattern. For all practical purposes, only the gravitational and frictional pressure losses are considered here, neglecting the accelerational pressure drop.

### 3.5.2 Pipeline Data Bank.

The applicability of the proposed overall pipeline model, as well as other methods, is evaluated through comparison with actual laboratory and field data. The pipeline data bank established by Xiao et al. for this purpose is given in **Table 3.5.** As shown in the table, the database includes a total of 426 laboratory and field data points from different sources. Field data are from the American Gas Association (AGA) pipeline database (Crowley, 1988) and Mcleod et al. (1971). Laboratory data are provided by Eaton and Brown (1965) and Payne et al. (1979).

<table>
<thead>
<tr>
<th>Data Source</th>
<th>Nominal Pipe Diameter, mm</th>
<th>No. of Data Points</th>
<th>Fluid System</th>
</tr>
</thead>
<tbody>
<tr>
<td>AGA</td>
<td>76.2 – 660.4</td>
<td>79</td>
<td>54 back oil system</td>
</tr>
<tr>
<td>Mcleod et al</td>
<td>152.4</td>
<td>12</td>
<td>25 compositional system</td>
</tr>
<tr>
<td>Eaton et al</td>
<td>50.8 – 101.6</td>
<td>139 – 97</td>
<td>Black oil system</td>
</tr>
<tr>
<td>Payne et al</td>
<td>50.8</td>
<td>99</td>
<td>Natural gas/water, crude or distillate</td>
</tr>
<tr>
<td>Total number of data points:</td>
<td></td>
<td>426</td>
<td></td>
</tr>
</tbody>
</table>

The original 1988 AGA database included 455 data points, covering a wide range of flow conditions in oil and gas pipelines. However, many of the data points are identical or not reliable. For example, unrealistically low-pressure drop was reported for very long pipelines. Also, to avoid the effect of errors from the prediction of the fluid properties, the compositional data containing free water were omitted because of the possibility of occurrence of oil-water emulsions. As a result of this data culling process,
only 79 data points were selected from the AGA database, including 25 points from compositional pipelines and 54 from black oil systems.

The other set of 12 data points field measurements are from Mcleod et al. (1971). This high-quality data set was taken in a 0.15 m diameter offshore pipeline. The fluid properties for this data set are calculated on the basis of the black oil model.

The laboratory data from by Eaton and Brown (1965) and Payne et al. (1979) were obtained in small diameter pipes of 0.051 and 0.102 m. However, different fluids were used, including natural gas, crude oil, distillate, and water. Also, the data were taken at high pressures, similar to field operational conditions.

3.5.3 Evaluation of Overall Model and Other Methods. The data bank presented in Table 3.5 is used to assess the applicability of the proposed overall model, as well as four other most commonly used empirical correlations. These include Beggs and Brill (1973), Mukherjee and Brill (1985), the original Dukler (1964), and the Dukler-Eaton Flanigen correlations (see Sec. 3.1).

Statistical Parameters. Evaluation of methods is carried out only for the pressure drop, as most of the data points do not include the liquid holdup. The evaluation is carried out by comparing the measured pressure drop, $-\Delta p_M$, with the predicted pressure drop, $-\Delta p_C$, for all the points in the data bank. Six statistical parameters have been used in the evaluation process, as defined next. The first three parameters, $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$, which are expressed in $\%$, are based on the relative error that is given by

$$e_R = \frac{-\Delta p_C - (-\Delta p_M)}{-\Delta p_M}.$$

Average percent error, $\varepsilon_1$ ($\%$):

$$\varepsilon_1 = \frac{1}{n} \left( \sum_{i=1}^{n} e_{R,i} \times 100 \right).$$

Absolute average percent error, $\varepsilon_2$ ($\%$):

$$\varepsilon_2 = \frac{1}{n} \left( \sum_{i=1}^{n} |e_{R,i}| \times 100 \right).$$

Percent standard deviation, $\varepsilon_3$ ($\%$):

$$\varepsilon_3 = \frac{\sqrt{\frac{1}{n} \left( \sum_{i=1}^{n} (e_{R,i} \times 100 - \varepsilon_1)^2 \right)} }{n - 1}.$$

The other three parameters, $\varepsilon_4$, $\varepsilon_5$, and $\varepsilon_6$, are based on the actual error, given in pressure units (Pascal), namely,

$$e = -\Delta p_C - (-\Delta p_M).$$

Average error, $\varepsilon_4$ (Pa):

$$\varepsilon_4 = \frac{1}{n} \left( \sum_{i=1}^{n} e_i \right).$$
Absolute average error, $\varepsilon_4$ (Pa):

$$
\varepsilon_4 = \frac{1}{n} \left( \frac{1}{n} \sum_{i=1}^{n} |\varepsilon_i| \right)
$$

Standard deviation, $\varepsilon_6$ (Pa):

$$
\varepsilon_6 = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (\varepsilon_i - \varepsilon_4)^2}
$$

The average percent error, $\varepsilon_1$, and the average error, $\varepsilon_4$, are a measure of the agreement between the predicted values and the measured data. Positive values of these parameters imply overprediction, while negative values imply underprediction. The absolute average percent error, $\varepsilon_2$, and the absolute average error, $\varepsilon_5$, are considered to be more important than $\varepsilon_1$ and $\varepsilon_4$, because in the absolute parameters ($\varepsilon_2$ and $\varepsilon_5$) the positive and negative errors do not cancel out. The percent standard deviation, $\varepsilon_3$, and standard deviation, $\varepsilon_6$, indicate the scatter of the errors with respect to their corresponding average percent error, $\varepsilon_1$, and average error, $\varepsilon_4$. The first three parameters, $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$, which are based on the percent error, are more suitable for the evaluation of small values of pressure drop, while the other three, $\varepsilon_4$, $\varepsilon_5$, and $\varepsilon_6$ that are based on the actual error, are more suitable for the evaluation of large values.

**Overall Evaluation.** The overall evaluation of the model and the four correlations against the entire data bank (all 426 data points) is shown in Table 3.6. As can be seen from the table, the model exhibits the smallest absolute average percent error, namely, $\varepsilon_2 = 30.5\%$, and the smallest absolute average error, namely $\varepsilon_5 = 12.2 \times 10^4$ Pa, as compared to the four empirical correlations evaluated. All the other statistical parameters of the overall model, except for $\varepsilon_1$ (which is very close to the minimum value), are the smallest, too. This demonstrates the superior performance of the proposed overall model as compared to the four evaluated correlations. The overall model has negative values for the average percent error, $\varepsilon_1$, and the average error, $\varepsilon_4$, indicating underprediction against the measured data. A plot of the calculated pressure drop vs. the measured pressure drop for the overall model is presented in Fig. 3.45, showing the overall performance of the model, namely, its accuracy and scatter. Of all the 426 cases of the data bank, the overall model had not converged for one case only, as compared to 6 to 15 cases of convergence problems exhibited by the evaluated correlations. The overall model is the best in this respect.

<table>
<thead>
<tr>
<th>No.</th>
<th>Model or Correlation</th>
<th>No. of Data Points</th>
<th>$\varepsilon_1$ (%)</th>
<th>$\varepsilon_2$</th>
<th>$\varepsilon_3$</th>
<th>$\varepsilon_4$</th>
<th>$\varepsilon_5$ x $10^4$ (Pa)</th>
<th>$\varepsilon_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Xiao et al.</td>
<td>425</td>
<td>-11.7</td>
<td>30.5</td>
<td>50.6</td>
<td>-8.6</td>
<td>12.2</td>
<td>22.0</td>
</tr>
<tr>
<td>2</td>
<td>Beggs and Brill</td>
<td>415</td>
<td>10.9</td>
<td>35.0</td>
<td>94.2</td>
<td>9.6</td>
<td>13.2</td>
<td>31.6</td>
</tr>
<tr>
<td>3</td>
<td>Mukherjee and Brill</td>
<td>411</td>
<td>39.4</td>
<td>60.5</td>
<td>128.0</td>
<td>16.3</td>
<td>21.3</td>
<td>40.5</td>
</tr>
<tr>
<td>4</td>
<td>Original Dukler</td>
<td>415</td>
<td>32.9</td>
<td>43.0</td>
<td>107.0</td>
<td>17.3</td>
<td>18.7</td>
<td>36.7</td>
</tr>
<tr>
<td>5</td>
<td>Dukler-Eaton-Flanigen</td>
<td>419</td>
<td>21.5</td>
<td>35.4</td>
<td>89.3</td>
<td>13.8</td>
<td>16.0</td>
<td>33.9</td>
</tr>
</tbody>
</table>
Fig. 3.45—Performance of Xiao et al. (1990) overall model using entire data bank.

Table 3.6 shows that the most commonly used correlations of Beggs and Brill and Dukler-Eaton-Flanigan, also show a good performance, somewhat close to the overall model. This reflects the substantial use of these correlations over the years, and their modification and improvement during these years. Thus, for a proper prediction and design of pipelines, it is recommended to run the overall model and these two correlations to establish the operational envelop of the given pipeline.

Finally, the overall model performance is of the order of 30% error. While this result is encouraging for a first time model, the model should be improved to decrease its prediction error. This was done in subsequent years, and will be presented in Chap. 5.

**Individual Flow Pattern Models Evaluation.** The overall model consists of separate flow pattern models. In order to evaluate these models separately, the data bank has been separated into groups, in each of which all the cases have the same dominant flow pattern in more than 75% of the total pipeline length. Thus, separate data groups have been created for the stratified flow (89 cases), slug flow (129 cases), and annular flow (123 cases), enabling evaluation of the models for these flow patterns. There were no data for the dispersed-bubble flow pattern.

The evaluations of the stratified flow, slug flow, and annular flow patterns are given in Tables 3.7, 3.8, and 3.9, respectively. Figs. 3.46, 3.47, and 3.48 show the performance of the overall model for these flow patterns, respectively. As can be seen from the results, the performance of all these models, particularly the slug flow model, with $\epsilon_2 = 22.7\%$ is better than any of the four correlations evaluated. This reflects the better suitability of the mechanistic modeling approach, in which separate models are developed for the different flow patterns, as compared to the flow pattern independent correlations.

The degree of uncertainty in the calculation of the liquid wall friction factor for stratified flow has been studied. The sensitivity study was carried out by varying the value of $f_L$ by $\pm 25\%$ around its
predicted value for all the cases presented in Table 3.7. It was found that the pressure drop calculation results were not sensitive to the variation in the value of $f_L$. As mentioned before, Xiao et al. recommended calculating the value of $f_L$, using the standard method based on the hydraulic diameter concept.

<table>
<thead>
<tr>
<th>No.</th>
<th>Model or Correlation</th>
<th>No. of Data Points</th>
<th>Statistical Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\bar{\varepsilon}$</td>
</tr>
<tr>
<td>1</td>
<td>Xiao et al.</td>
<td>89</td>
<td>-18.0</td>
</tr>
<tr>
<td>2</td>
<td>Beggs and Brill</td>
<td>86</td>
<td>-9.0</td>
</tr>
<tr>
<td>3</td>
<td>Mukherjee and Brill</td>
<td>83</td>
<td>16.7</td>
</tr>
<tr>
<td>4</td>
<td>Original Dukler</td>
<td>86</td>
<td>32.1</td>
</tr>
<tr>
<td>5</td>
<td>Dukler-Eaton-Flanigen</td>
<td>86</td>
<td>23.1</td>
</tr>
</tbody>
</table>

Fig. 3.46—Performance of stratified flow model against cases with more than 75% stratified flow.

<table>
<thead>
<tr>
<th>No.</th>
<th>Model or Correlation</th>
<th>No. of Data Points</th>
<th>Statistical Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\bar{\varepsilon}$</td>
</tr>
<tr>
<td>1</td>
<td>Xiao et al.</td>
<td>121</td>
<td>-18.9</td>
</tr>
<tr>
<td>2</td>
<td>Beggs and Brill</td>
<td>129</td>
<td>23.3</td>
</tr>
<tr>
<td>3</td>
<td>Mukherjee and Brill</td>
<td>127</td>
<td>27.1</td>
</tr>
<tr>
<td>4</td>
<td>Original Dukler</td>
<td>128</td>
<td>28.5</td>
</tr>
<tr>
<td>5</td>
<td>Dukler-Eaton-Flanigen</td>
<td>129</td>
<td>22.5</td>
</tr>
</tbody>
</table>
Fig. 3.47—Performance of slug flow model against pipeline cases with more than 75% slug flow.

![Graph showing performance of slug flow model against pipeline cases.](image)

**TABLE 3.9—EVALUATION OF SLUG FLOW MODEL**

<table>
<thead>
<tr>
<th>No.</th>
<th>Model or Correlation</th>
<th>No. of Data Points</th>
<th>$\varepsilon_1$ (%)</th>
<th>$\varepsilon_2$ (%)</th>
<th>$\varepsilon_3$ (%)</th>
<th>$\varepsilon_4$ (%)</th>
<th>$\varepsilon_5$ (%)</th>
<th>$\varepsilon_6$ $\times 10^3$ (Pa)</th>
<th>$\varepsilon_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Xiao et al.</td>
<td>123</td>
<td>-3.6</td>
<td>39.2</td>
<td>76.5</td>
<td>-17.9</td>
<td>24.0</td>
<td>33.7</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Beggs and Brill</td>
<td>114</td>
<td>33.3</td>
<td>41.7</td>
<td>163.0</td>
<td>21.9</td>
<td>24.7</td>
<td>50.2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Mukherjee and Brill</td>
<td>117</td>
<td>83.0</td>
<td>94.1</td>
<td>215.0</td>
<td>35.4</td>
<td>43.3</td>
<td>63.3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Original Dukler</td>
<td>114</td>
<td>49.8</td>
<td>56.5</td>
<td>187.0</td>
<td>27.8</td>
<td>30.4</td>
<td>53.7</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Dukler-Eaton-Flanigen</td>
<td>119</td>
<td>29.8</td>
<td>41.8</td>
<td>149.0</td>
<td>22.1</td>
<td>27.3</td>
<td>54.2</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 3.48—Performance of annular flow model against pipeline cases with more than 75% annular flow.

![Graph showing performance of annular flow model against pipeline cases.](image)
3.6 Examples

**Example 3.1—Flow Pattern Prediction in Pipelines.** A mixture of air-water flows in a 5 cm ID horizontal pipe. The flow rate of the water is $q_L = 0.707$ m$^3$/hr and that of the air is $q_G = 21.2$ m$^3$/hr. The physical properties of the fluids are given as:

$$\rho_L = 993 \text{ kg/m}^3, \quad \rho_G = 1.14 \text{ kg/m}^3,$$

and

$$\mu_L = 0.68 \times 10^{-3} \text{ kg/m \cdot s}, \quad \mu_G = 1.9 \times 10^{-5} \text{ kg/m \cdot s}.$$

Determine the flow pattern, which exists in the pipe under the previous conditions.

**Solution.**

1. Calculate superficial velocities and Reynolds numbers:

   $$A_p = \frac{\pi}{4} \left( \frac{5}{100} \right)^2 = 1.9635 \times 10^{-3} \text{ (m}^2),$$

   $$v_{SL} = \frac{q_L}{A_p} = \frac{(0.707 \text{ m}^3/\text{hr}) \times \left( \frac{1}{3,600 \text{ s}} \right)}{(1.9635 \times 10^{-3} \text{ m}^2)} = 0.1 \text{ (m/s)},$$

   $$v_{SG} = \frac{q_G}{A_p} = \frac{(21.2 \text{ m}^3/\text{hr}) \times \left( \frac{1}{3,600 \text{ s}} \right)}{(1.9635 \times 10^{-3} \text{ m}^2)} = 3.0 \text{ (m/s)},$$

   $$Re_{SL} = \frac{(993 \text{ kg/m}^3) \times (0.1 \text{ m/s}) \times (0.05 \text{ m})}{(0.68 \times 10^{-3} \text{ kg/m} \cdot \text{s})} = 7,301 \text{ (turbulent flow),}$$

   and

   $$Re_{SG} = \frac{(1.14 \text{ kg/m}^3) \times (3.6 \text{ m/s}) \times (0.05 \text{ m})}{(1.9 \times 10^{-5} \text{ kg/m} \cdot \text{s})} = 9,000 \text{ (turbulent flow).}$$

2. Calculate the Lockhart-Martinelli parameter (Eq. 3.35) and the equilibrium liquid level in the pipe. For turbulent/turbulent flow ($C_G = C_L = 0.046$ and $n = m = 0.2$) the Lockhart-Martinelli parameter reduces to

   $$X^2 = \frac{4C_L}{d} \left( \frac{\rho_L v_{SL} \mu_L}{\rho_L} \right)^{-n} \frac{\rho_L v_{SL}^2}{2} = \frac{Re_{SG}^{0.2}}{Re_{SG}} \frac{\rho_L}{\rho_G} \left( \frac{v_{SL}}{v_{SG}} \right)^2.$$
Thus,

\[ X^2 = \left( \frac{9000}{7301} \right)^{0.2} \left( \frac{993}{1.14} \right) \left( \frac{0.1}{3.0} \right)^2 = 1.009 \text{ and } X = 1.005. \]

From Fig. 3.6, for \( X = 1.005 \) and \( Y = 0 \) (horizontal flow),

\[ \tilde{h}_L = \frac{h_I}{d} = 0.42. \]

3. Calculate dimensionless (tilde) variables (Eq. 3.38):

\[
(2\tilde{h}_L - 1) = -0.16, \\
\cos^{-1}(2\tilde{h}_L - 1) = 1.7315 \text{ (rad)}, \\
\tilde{S}_G = \cos^{-1}(2\tilde{h}_L - 1) = 1.7315, \\
\tilde{S}_L = \pi - \tilde{S}_G = 1.41, \\
\tilde{S}_I = \sqrt{1 - (-0.16)^2} = 0.987, \\
\tilde{A}_G = 0.25(1.7315 + 0.16 \times 0.987) = 0.472, \\
\tilde{A}_L = \frac{\pi}{4} - \tilde{A}_G = 0.313, \\
\tilde{v}_L = \frac{\tilde{A}_p}{\tilde{A}_L} = 2.51, \\
\tilde{v}_G = \frac{\tilde{A}_p}{\tilde{A}_G} = 1.66, \\
\tilde{d}_L = \frac{4\tilde{A}_L}{\tilde{S}_L} = 0.888,
\]

and

\[ \tilde{d}_G = \frac{4\tilde{A}_G}{\tilde{S}_G + \tilde{S}_I} = 0.694. \]

4. Check stratified to nonstratified transition criterion (Eq. 3.49):

\[
F^2 \left[ 1 \left( \frac{\tilde{v}_G^2 \tilde{S}_I}{\tilde{A}_G} \right) \right] \geq 1, \text{ where}
\]

\[ F = \frac{\rho_g}{\sqrt{\rho_L - \rho_g}} \frac{v_{SG}}{\sqrt[3]{\tilde{d}_g}} \cos \theta = \sqrt{\frac{1.14}{993 - 1.14}} \left( \frac{3}{\sqrt{0.05 \cdot 9.81}} \right) = 0.145. \]

Substituting the value of \( F \) and the other dimensionless variables in the criterion equation results,

\[ 0.145^2 \left[ 1 \left( \frac{(1.66)^2 \times 0.987}{0.472} \right) \right] = 0.36 < 1. \]
The criterion is not satisfied. This implies that the flow is stable and stratified flow exists.

5. Check for stratified-smooth to stratified-wavy transition criterion (Eq. 3.54):

\[ K \geq \frac{2}{\sqrt{\bar{v}_L \bar{v}_G} \sqrt{s}} \], where \( s = 0.01 \).

From Eq. 3.55

\[ K = F \sqrt{Re_{SL}} = 0.145 \times \sqrt{7301} = 12.39 \]

Substituting the values in the criterion equation results,

\[ \frac{2.0}{\sqrt{2.51 \times 1.66 \times 0.01}} = 7.6 < 12.39. \]

The criterion is satisfied. The flow pattern is stratified-wavy.

**Example 3.2.** Apply “similarity through basic equations” to the “combined momentum equation” (Eq. 3.28) in the Taitel and Dukler (1976) flow pattern transition model, and obtain the dimensionless form of the equation (Eq. 3.34), as given by

\[
X^2 \left( \bar{v}_L \bar{d}_L \right)^n \bar{S}_L = \left( \bar{v}_G \bar{d}_G \right)^n \left( \frac{\bar{S}_G}{A_G} + \frac{\bar{S}_L}{A_L} \right) + 4Y = 0.
\]

**Solution.** Starting from Eq. 3.28, the dimensional combined momentum equation is

\[
\tau_{wg} \frac{S_G}{A_G} - \tau_{wl} \frac{S_L}{A_L} + \tau_s \left( \frac{1}{A_L} + \frac{1}{A_G} \right) - (\rho_L - \rho_G) g \sin \theta = 0.
\]

Assuming smooth interface, namely, \( \tau_s = \tau_{wg} \) and rearranging yields (Eq. 3.28’)

\[
\frac{\tau_{wl}}{\tau_{wg}} \frac{S_L}{A_L} \left( \frac{S_G}{A_G} + \frac{S_L}{A_L} + \frac{S_L}{A_G} \right) + \left( \frac{\rho_L - \rho_G}{\tau_{wg}} g \sin \theta \right) = 0.
\]

The liquid and wall shear stresses are given by (Eq. 3.31)

\[
\tau_{wl} = f_L \frac{\rho_L \bar{v}_L^2}{2}, \quad \tau_{wg} = f_G \frac{\rho_G \bar{v}_G^2}{2}.
\]

The friction factors can be calculated from the Blasius equation, utilizing the hydraulic diameter concept, as (Eq. 3.30)

\[
f_L = C_L (Re_L)^n = C_L \left( \frac{d_L \bar{v}_L}{\mu_L} \right)^n, \quad \text{and} \quad f_G = C_G (Re_G)^m = C_G \left( \frac{d_G \bar{v}_G}{\mu_G} \right)^m,
\]

where \( C_L = C_G = 16 \) and \( m = n = 1 \) for laminar flow, \( C_L = C_G = 0.046 \), and \( m = n = 0.2 \) for turbulent flow, and the hydraulic diameters of the liquid and gas phases are given, respectively, by (Eq. 3.29)

\[
d_L = \frac{4A_L}{S_L}, \quad \text{and} \quad d_G = \frac{4A_G}{S_G + S_I}.
\]
Substituting the shear stresses into the rearranged combined momentum equation gives

\[
\frac{C_L}{C_G} \left( \frac{d_i v_i \rho_i}{\mu_i} \right)^{-n} \frac{\rho_i v_i}{2} S_i = \frac{S_L}{A_L} + \frac{S_L}{A_L} + \frac{S_L}{A_L} + \frac{(\rho_L - \rho_G) g \sin \theta}{2} = 0
\]

Next, the following dimensionless variables, designated by a tilde \( \sim \), are used to nondimensionalize the equation (Eq. 3.33)

\[
\tilde{S}_L = \frac{S_L}{d}, \quad \tilde{h}_L = \frac{h_i}{d}, \quad \tilde{A}_L = \frac{A_L}{d^2}, \quad \tilde{v}_L = \frac{v_L}{v_{SL}}, \quad \text{and} \quad \tilde{v}_G = \frac{v_G}{v_{SG}}.
\]

Substituting the dimensionless variables, the first term becomes

\[
\frac{4}{d} C_L \left( \frac{\rho_i v_i \tilde{v}_L d \tilde{d}_G}{\mu_i} \right)^{-n} \frac{\rho_i v_i^2 \tilde{v}_L^2}{2} \tilde{S}_L d = \frac{4}{d} C_L \left( \frac{\rho_i v_i d}{\mu_i} \right)^{-n} \frac{\rho_i v_i^2}{2} \frac{(\tilde{v}_L \tilde{d}_L)^{-n} \tilde{v}_L^2 \tilde{S}_L}{A_L} \frac{1}{d}.
\]

Similarly, the second term is nondimensionalized to yield

\[
- \left( \frac{\tilde{S}_G}{A_G} + \frac{\tilde{S}_L}{A_L} + \frac{\tilde{S}_L}{A_G} \right) \frac{1}{d},
\]

and the third term in dimensionless form is

\[
\frac{(\rho_L - \rho_G) g \sin \theta}{4} \frac{dP}{dL} = 4(\rho_L - \rho_G) g \sin \theta \frac{1}{d} - \frac{dP}{dL} \frac{1}{d}.
\]

Combining the three dimensionless terms, the final dimensionless combined momentum equation can be obtained, namely,

\[
X^2 \left[ (\tilde{v}_L \tilde{d}_L)^{-n} \tilde{v}_L^2 \tilde{S}_L \right] - \left[ (\tilde{v}_G \tilde{d}_G)^{-n} \tilde{v}_G^2 \left( \frac{\tilde{S}_G}{A_G} + \frac{\tilde{S}_L}{A_L} + \frac{\tilde{S}_L}{A_G} \right) \right] + 4Y = 0,
\]

where \( X \), the Lockhart and Martinelli parameter, and \( Y \), an inclination angle parameter, are given by (Eqs. 3.35 and 3.36)
\[ X^2 = \frac{4C_L}{d} \left( \frac{\rho_L v_{SL} d}{\mu_L} \right)^{-n} \frac{\rho_L v_{SL}^2}{2} = -\frac{dp}{dL}_{SL}, \]

\[ \frac{4C_G}{d} \left( \frac{\rho_G v_{SG} d}{\mu_G} \right)^{-m} \frac{\rho_G v_{SG}^2}{2} = -\frac{dp}{dL}_{SG}, \]

and

\[ Y = \frac{(\rho_L - \rho_G) g \sin \theta}{4C_G} \left( \frac{\rho_G v_{SG} d}{\mu_G} \right)^{-m} \frac{\rho_G v_{SG}^2}{2} = (\rho_L - \rho_G) g \sin \theta. \]
Chap. 4 presents analysis for vertical and sharply inclined pipes for the accepted range of inclination angles between 90 to 60°. The most commonly used pressure loss empirical correlation, namely, the Hagedorn and Brown (1965) correlation, is presented briefly. Next, a mechanistic model for flow pattern prediction (Taitel et al., 1980) is given, followed by separated models for bubble flow, slug flow [Sylvester’s (1987) simplification of Fernandez et al. (1986) model] and annular flow (Alves et al., 1991). Finally, an overall mechanistic model for wellbore design and evaluation of predictive methods (Ansari et al., 1994) is presented.

4.1 Wellbore Empirical Correlations

The most commonly used correlations for wellbores are Hagedorn and Brown (1965), Duns and Ros (1963), Orkiszewski (1967), Aziz et al. (1972) and Hasan and Kabir (1988). As will be shown in the evaluation section of this chapter, the Hagedorn and Brown correlation shows the best performance. Thus, only this correlation will be covered in this section. However, the evaluation study will be carried out for all these correlations.

One should note that the Hagedorn and Brown empirical correlation in Sec. 4.1 is given for oilfield system of units and utilizes the Moody friction factor. The superficial velocities are expressed in ft/s, the density in lbm/ft³, the surface tension in dynes/cm, the viscosity in cp, the pressure in psi, and the diameter in ft.

4.1.1 Hagedorn and Brown (1965) Correlation. The Hagedorn and Brown (1965) correlation was developed from experimental data acquired in a 1,500 ft well. The correlation considers slippage between the gas and liquid phases but is flow pattern independent. The experimental data included measurements of the pressure gradient but not the liquid holdup, which was back calculated from the experimental measurements.

The correlation utilizes the same dimensionless groups developed by Duns and Ros (1963), which are also used in the Eaton et al. (1967) liquid holdup correlation. The dimensionless groups are repeated next for completeness.

\[
\begin{align*}
N_{LV} &= 1.938 \, v_{SL} \, \sqrt[4]{\rho_L / \sigma}, \\
N_{GV} &= 1.938 \, v_{SG} \, \sqrt[4]{\rho_L / \sigma}, \\
N_{d} &= 120.872 \, d \, \sqrt[4]{\rho_L / \sigma},
\end{align*}
\]

and

\[
N_L = 0.15726 \, \mu_L \, \sqrt[4]{1 / (\rho_L \sigma^3)}. \tag{3.1}
\]

**Liquid Holdup.** The liquid holdup correlation is shown in Fig. 4.1 in the form of a holdup-parameter \( H_L / \psi \) on the \( y \)-axis vs. a correlating function on the \( x \)-axis, where \( p \) is the operational pressure, and \( p_a \) is the atmospheric pressure in psia. Values of \( C_{NL} \) and \( \psi \), required for the \( x \)-axis correlating function, can be determined from Figs. 4.2 and 4.3, respectively. Thus, Figs. 4.1, 4.2, and
4.3 enable the determination of the liquid holdup for a given set of flow conditions. The calculated liquid holdup is subject to the constraint that \( H_L \geq \lambda_L \).

Fig. 4.1 — Liquid holdup factor (after Hagedorn and Brown, 1965).

Fig. 4.2 — Viscosity number correlation.
Frictional Pressure Gradient. The frictional pressure gradient is determined from

\[- \frac{dp}{dL} = \frac{f_{TP} \rho_{TP} v_M^2}{2 g_c d}, \] ..................................................... (4.1)

where the two-phase density term is given by

\[ \rho_{TP} = \frac{\rho_{NS}^2}{\rho_{SLIP}}. \] ..................................................... (4.2)

The no-slip mixture density, \( \rho_{NS} \), is defined in Eq. 3.20, and \( \rho_{SLIP} \), the slip density, is calculated based on the calculated in-situ holdup, as given by Eq. 3.22. The friction factor, \( f_{TP} \), is correlated with the following Reynolds number,

\[ \text{Re}_{TP} = 1.488 \frac{\rho_{NS} v_M d}{\mu_S}, \] ..................................................... (4.3)

where the viscosity of the mixture is determined from the expression

\[ \mu_S = \mu_L^{1-H_S} \mu_G^{(1-H_S)}. \] ..................................................... (4.4)

Gravitational and Accelerational Pressure Gradients. The gravitational pressure gradient is determined on the basis of the slip density as

\[- \frac{dp}{dL} = \frac{\rho_{SLIP} g \sin \theta}{g_c}. \] ..................................................... (4.5)

The accelerational pressure gradient is given by
\[-\frac{dp}{dL}_A = \frac{\rho_{\text{SLIP}}}{2 g_e} \frac{d(v_M^2)}{dp} \left( -\frac{dp}{dL} \right) \]  

\[(4.6)\]

**Total Pressure Gradient.** The total pressure gradient is the sum of the frictional, gravitational, and accelerational pressure gradient components as

\[\frac{-dp}{dL} = \frac{-\frac{dp}{dL}_F - \frac{dp}{dL}_G}{1 - E_K}, \]  

\[(4.7)\]

where

\[E_K = \frac{\rho_{\text{SLIP}}}{2 g_e} \frac{d(v_M^2)}{dp} \]  

\[(4.8)\]

Caution should be exercised when calculating pressure gradient including the accelerational component. As \(E_K\) tends to unity, the pressure gradient will approach infinity.

### 4.2 Flow Pattern Prediction In Wellbores

Published vertical upward flow pattern maps in the past differ from each other both in absolute value and trend, even more than in the case of horizontal flow. The reason for the poor agreement between the maps is because of the subjective description and classification of the flow patterns, which were usually observed visually. In upward flow, visual observations are much more difficult, as compared to horizontal flow, owing to the chaotic flow that can occur in vertical pipes. A summary of experimental flow pattern maps for upward vertical flow is given in Table 4.1.

As in the horizontal case, a large number of flow pattern mapping coordinate systems have been used by different investigators, usually chosen arbitrarily. Some of them are dimensional, such as the mass flow rates, \(W_L\) and \(W_G\), as used by Govier et al. (1957), or the superficial momentum fluxes, \(\rho_L v_{SL}^2\) and \(\rho_G v_{SG}^2\), which were used by Hewitt and Roberts (1969). Others proposed coordinates are dimensionless, such as \(v_{SG} / v_M\) and \(v_M^2 / g d\), used by Kozlov (1954). The disadvantage of these coordinate systems has been discussed earlier. Govier and Aziz (1972) suggested correction factors \((X\) and \(Y\)) for physical properties. A partial attempt to determine the mapping coordinates theoretically was made by Griffith and Wallis (1961). They showed that the transition from slug to annular flow is governed by two dimensionless groups, namely, \(v_{SG} / v_M\) and \(v_M^2 / g d\). However, the validity of these groups for determining other transition boundaries has not been established.
<table>
<thead>
<tr>
<th>Author</th>
<th>Pipe ID, cm</th>
<th>System</th>
<th>Mapping Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kosterin (1949)</td>
<td>2.54</td>
<td>air-water</td>
<td>$\frac{v_{SG}}{v_M}$ $\gamma_M$</td>
</tr>
<tr>
<td>Kozlov (1954)</td>
<td>2.54</td>
<td>air-water</td>
<td>$\frac{v_{SG}}{v_M}$ $\frac{v^2}{g \delta}$</td>
</tr>
<tr>
<td>Galegar et al. (1954)</td>
<td>1.2, 5.1</td>
<td>air-water/kerosene</td>
<td>$G_G, G_L$</td>
</tr>
<tr>
<td>Govier et al. (1957, 1958)</td>
<td>2.54</td>
<td>air-water</td>
<td>$W_G, W_L$</td>
</tr>
<tr>
<td>Griffith and Wallis (1961)</td>
<td>1.2–5.75</td>
<td>steam-water</td>
<td>$\frac{v_{SG}}{v_M}$ $\frac{v^2}{g \delta}$</td>
</tr>
<tr>
<td>Duns and Ros (1963)</td>
<td>8</td>
<td>air-oil</td>
<td>$\gamma_{SL}$ $\gamma_{SG}$ $0.25$ $\gamma_S$, $\gamma_{SL}$ $0.25$ $\gamma_S$</td>
</tr>
<tr>
<td>Sterling (1965)</td>
<td>2.54</td>
<td>air-water</td>
<td>$v_{SL}$, $v_{SG}$</td>
</tr>
<tr>
<td>Wallis (1969)</td>
<td>2.54</td>
<td>air-water</td>
<td>$v_{SL}$, $v_{SG}$</td>
</tr>
<tr>
<td>Hewitt and Roberts (1969)</td>
<td>3.18</td>
<td>air-water</td>
<td>$\rho_{G}^2 Y_{SG}$, $\rho_{L}^2 Y_{SL}$</td>
</tr>
<tr>
<td>Govier and Aziz (1972)</td>
<td>2.54</td>
<td>air-water</td>
<td>$X_{SL}$, $Y_{SG}$</td>
</tr>
<tr>
<td>Oshinowo and Charles (1974)</td>
<td>2.54</td>
<td>air-water/glycerin</td>
<td>$\frac{v^2}{g \delta \sqrt{d}}$ $\gamma_{SG}$ $0.5$ $\gamma_{SL}$ $0.25$ $\gamma_S$, $\gamma_{SL}$ $0.25$ $\gamma_S$</td>
</tr>
<tr>
<td>Gould (1974)</td>
<td>data from others</td>
<td>air-water/oil</td>
<td>$\rho_{G}^2 Y_{SG}$, $\rho_{L}^2 Y_{SL}$</td>
</tr>
<tr>
<td>Gould et al. (1974)</td>
<td>data from others</td>
<td>various gas-liquid systems</td>
<td>$\rho_{G}^2 Y_{SG}$, $\rho_{L}^2 Y_{SL}$</td>
</tr>
</tbody>
</table>

### 4.2.1 Taitel et al. (1980) Model

A similar approach to that of Taitel and Dukler (1976) for modeling horizontal and slightly inclined flow pattern transitions was used by Taitel et al. (1980) for the prediction of the transition boundaries between the occurring flow patterns in vertical upward flow. The authors suggested physical mechanisms for the transition boundaries between the various flow patterns and modeled each transition on the basis of the mechanism by which it occurs. In this way, the important flow variables are incorporated, and the model is expected to be more general and applicable to a wide range of flow conditions. The flow patterns considered in the model are the accepted set, namely, bubble flow, slug flow, churn flow, annular flow, and dispersed-bubble flow. The transitions between these flow patterns are predicted as given next.

**Bubble-Slug Transition (Transition E).** This transition occurs at low gas flow rates and relatively low liquid flow rates (to be quantified later). A schematic of the transition mechanism is shown in Fig. 4.4. When low gas and liquid flow rates are introduced into a vertical pipe, the gas-phase is distributed as small bubbles, dispersed into a continuous liquid-phase. Under these low flow rate conditions, no turbulent forces exist, and the discrete bubbles move upward in a linear path, similarly to rigid spheres, without colliding and coalescing. At relatively higher gas flow rates, keeping the liquid rate low, the bubbles become larger, and above a critical size [about 0.3 mm in air-water flow at standard conditions (STP)], they start to deform and move in a zigzag path. Under these conditions, the bubbles collide randomly and coalesce, forming clusters of bubbles. Occasionally, a cluster can completely coalesce and form a cap bubble. These bubbles are similar to the tip of a Taylor bubble, but
they are smaller, less than 1 diameter in length, and do not occupy the entire cross-sectional area of the pipe. Thus, the cap bubbles do not cause slug flow to occur. However, at relatively higher gas flow rates (but still low), the bubble density increases, promoting more collisions and coalescence, which finally results in the formation of Taylor bubbles. The formed Taylor bubbles occupy the entire cross-sectional area of the pipe and promote slugging. This constitutes the transition from bubble to slug flow.

Published experimental data show that the maximum measured void fraction in bubble flow (without foaming) is about 0.3. Below a void fraction of 0.2, the bubbles behave as rigid spheres and do not coalesce. Bubbles movement and coalescence modeling was presented by Radovicich and Moissis (1962). They assumed that the bubbles were structured in a cubic lattice geometry, whereby they could fluctuate and move randomly. The results of the model are shown in Fig. 4.5, where the predicted dimensionless bubble collision frequency is plotted against the bubble void fraction. As can be seen, for void fractions below 0.2, the collision frequency is low. However, for void fraction values higher than 0.2, the collision frequency increases exponentially. A maximum void fraction of 0.3 is reached when the collision frequency is very high.

A simple exercise in geometry can shed more light on bubble flow phenomenon. Consider the bubbles to have the same diameter and to be arranged in a cubic lattice geometry touching each other, as shown in Fig. 4.6 Part A. This configuration is assumed to be the maximum allowable packing of bubbles arrangement. A simple geometry calculation results in a void fraction of $\alpha_{\text{MAX}} = 0.52$. However, bubble coalescence occurs for void fractions well below this maximum value, where the bubbles do not need to touch each other. In fact, bubbles are attracted to each other, promoting sharply increased coalescence, when the distance between them is decreased. It is assumed that bubble coalescence is increased sharply when the distance between them is equal to half of the radius of a bubble in a similar cubic lattice arrangement’ as shown in Fig. 4.6 Part B. The resulted calculated gas void fraction for this case is 0.25

Fig. 4.4—Schematic of transition from bubble to slug flow.
A simple exercise in geometry can shed more light on bubble flow phenomenon. Consider the bubbles to have the same diameter and to be arranged in a cubic lattice geometry touching each other, as shown in Fig. 4.6 Part A. This configuration is assumed to be the maximum allowable packing of bubbles arrangement. A simple geometry calculation results in a void fraction of $\alpha_{\text{MAX}} = 0.52$. However, bubble coalescence occurs for void fractions well below this maximum value, where the bubbles do not need to touch each other. In fact, bubbles are attracted to each other, promoting sharply increased coalescence, when the distance between them is decreased. It is assumed that bubble coalescence is increased sharply when the distance between them is equal to half of the radius of a bubble in a similar cubic lattice arrangement’ as shown in Fig. 4.6 Part B. The resulted calculated gas void fraction for this case is 0.25.
Based on the experimental and theoretical results previously discussed, Taitel et al. suggested that the criterion for the transition between bubble and slug flow, at low liquid flow rates in the absence of turbulent forces, is a void fraction of 0.25. When the void fraction reaches 0.25, bubble coalescence increases sharply promoting the formation of Taylor bubbles and slugging. The mathematical formulation of this transition, based on the proposed mechanism and criterion, is given next.

The gas and liquid velocities are given, respectively, by

\[ v_G = \frac{v_{SG}}{\alpha}, \quad \text{and} \quad v_L = \frac{v_{SL}}{1-\alpha}. \]  

The bubble rise velocity is the terminal velocity of a single bubble in an infinite stagnant liquid medium. For low velocity liquid flow, the bubble rise velocity, designated as \( v_{0*} \), is also the slip velocity. Thus,

\[ v_{0*} = v_G - v_L. \]  

Substituting the phase velocities from Eq. 4.9 into Eq. 4.10 yields

\[ v_{SL} = v_{SG} \frac{1-\alpha}{\alpha} - (1-\alpha)v_{0*}. \]  

An expression for the bubble rise velocity was presented by Harmathy (1960) and is given in Eq. 4.12.

\[ v_{0*} = 1.53 \left[ \frac{g(\rho_L - \rho_g)\sigma}{\rho_L^2} \right]^{0.25}. \]  

Eq. 4.12 is valid for bubbles with a diameter larger than the critical diameter (0.3 cm for air-water flow at STP). Bubbles in this size range are deformable and because of balance between buoyancy and drag have a constant velocity that is insensitive to the bubble size, as shown in Fig. 4.7.

![Graph of bubble rise velocity vs. bubble diameter](image)

**Fig. 4.7**—Schematic of bubble rise velocity as function of bubble diameter.

Combining Eqs. 4.12 and 4.11 and substituting \( \alpha = 0.25 \), which is the void fraction causing transition to slug flow, results in the final form of the transition equation, namely,
Eq. 4.13 relates the gas and liquid superficial velocities, their physical properties, and the void fraction (\( \alpha = 0.25 \)) on the bubble-slug transition boundary. The equation can be used to construct this transition boundary on a \( v_{SL} \) vs. \( v_{SG} \) flow pattern map, which will be valid only for a given system. This transition, predicted by the proposed model for air-water flow in 5.1 cm diameter pipe at STP, is shown in Fig. 4.8. As can be seen, for low liquid flow rates (\( v_{SL} < 0.1 \) m/s) where the liquid velocity is negligible, the transition is controlled by the bubble rise velocity and is independent of the liquid flow rate. At higher liquid flow rates (\( 0.1 < v_{SL} < 1 \) m/s), on the other hand, the bubble rise velocity is negligible in comparison to the liquid velocity, and the transition boundary is linear on log-log coordinates. This transition occurs at relatively low liquid flow rates, namely, at superficial liquid velocities below 1 m/s.

**Bubble Region Existence.** Transition “E”, the transition between bubble to slug flow, may or may not occur depending on the pipe diameter. This is shown schematically in Fig. 4.9. Note that the other transition boundaries are designated by “F” and “G” (transition to dispersed-bubble flow), “H” (transition between slug and churn flow) and “J” (transition to annular flow).
The criterion for the existence of the bubble flow region is based on the relative velocities of small bubbles and large Taylor bubbles, as demonstrated in Fig. 4.10. The rise velocity (at stagnant liquid conditions) of small and deformable bubbles, \( v_{0b} \), is given by Eq. 4.12. Similarly, the rise velocity of a Taylor bubble at stagnant liquid vertical pipes, \( v_{TB} \), which is actually its drift velocity, is given by

\[
v_{TB} = v_D = 0.35 \sqrt{gd} .
\]  

Fig. 4.10—Bubble flow existence physical phenomena.

There is a significant difference between the two expressions for the rise velocity. The small bubble rise velocity is independent of pipe diameter (depends only on physical properties), while the Taylor bubble
rise velocity is diameter dependent (and independent of fluid properties). Thus, for large diameter pipes \( v_{0e} < v_{TB} \), while for small diameter pipes \( v_{0e} > v_{TB} \).

Now consider bubble flow occurring in a large diameter pipe to the left of transition “E” and below the transition to dispersed-bubble flow, as shown in Fig. 4.8. For this condition, mainly small and discrete bubbles move upwards in the pipe. However, occasionally a small cap Taylor bubble occurs, which is shorter than the pipe diameter. As the cap bubble rise velocity is higher than the small bubbles rise velocity, namely, \( v_{TB} > v_{0e} \), the cap bubble moves through the array of small bubbles. At the nose of the cap bubble, the small bubbles are swept around the bubble and no coalescence occurs, as shown schematically in Fig. 4.10 Part A, and bubble flow prevails in the pipe. On the other hand, for similar flow conditions in a small diameter pipe, the small bubbles rise velocity is larger than the cap bubble rise velocity, namely, \( v_{0e} > v_{TB} \). For this case, as shown in Fig. 4.10 Part B, the small bubbles approach the tail of the cap bubble and coalesce with it, increasing its size, which eventually results in a Taylor bubble. Thus, for small diameter pipes, bubble flow does not occur and transition “E” does not exist, whereby slug flow occurs in the entire region below the transition to dispersed-bubble flow, “F-G”, as shown in Fig. 4.9.

Equating the rise velocities of a small bubble and a Taylor bubble given, respectively, by Eqs. 4.12 and 4.14, a criterion for the existence of the bubble region can be developed, as shown in Eq. 4.15:

\[
\left[ \frac{\rho_i^2 g d^2}{(\rho_l - \rho_i) \sigma} \right]^{0.25} \geq 4.36. \quad \text{..................................................................................................................................(4.15)}
\]

For pipe diameters satisfying the criterion in Eq. 4.15, the bubble flow region exists to the left of transition “E”. For air-water at STP, the pipe diameter that satisfies the equality of the criterion in Eq. 4.15 is \( d = 5 \) cm. For this diameter \( v_{0e} = v_{TB} \). Thus, for air-water flow at STP in large diameter pipes, \( d \geq 5 \) cm, transition “E” exists and bubble flow occurs for flow rate conditions to the left of the curve. This is shown in Fig. 4.8, which is valid for 5.1 cm diameter pipe. For small diameter pipes, \( d < 5 \) cm, transition “E” does not exist, and slug flow occurs in the entire region below transition “F-G”. This is demonstrated in Fig. 4.11, which presents the flow pattern map predicted by the model upward vertical air-water flow at STP in a 2.54-cm-diameter pipe.

The analysis for the existence of transition “E” demonstrates the complexity of two-phase flow phenomena. As shown, with respect to this transition boundary, experimental results obtained for small diameter pipes cannot be directly scaled up to larger diameter pipes. This emphasizes the importance of understanding the physical phenomena to develop prediction tools. Also, it is interesting to note that previous investigators have conducted experiments utilizing air-water at STP with pipe diameters between 2 to 6 cm. This analysis explains the discrepancies reported by them on the bubble to slug transition boundary, as will be shown in a later section.
Transition to Dispersed-Bubble Flow (Transition F-G). The transition to dispersed-bubble flow occurs at high liquid flow rates associated with high turbulent forces, which act to breakup the gas-phase into small bubbles and disperse them through the continuous liquid-phase. This can occur for gas void fractions higher than 0.25. Dispersion of an immiscible-phase into another immiscible-phase, as a result of breakup by turbulence forces, was studied by Hinze (1955) and later confirmed by Sevik and Park (1973). Hinze discovered that the characteristic diameter of the dispersed-phase results from a balance between surface tension forces and turbulent forces. He proposed the expression shown in Eq. 4.16 for calculating the maximum stable diameter of the dispersed-phase.

\[
d_{\text{MAX}} = k \frac{(\sigma / \rho)}{(\varepsilon)^{0.4}},
\]

where \( \varepsilon \) is the rate of energy dissipation per unit mass, which for turbulent flow in a pipe can be calculated from the pressure gradient as

\[
\varepsilon = \left| \frac{dp}{dL} \right| \frac{v_M}{\rho_M}.
\]

The pressure gradient is determined on the basis of the homogeneous no-slip model, namely,

\[
- \frac{dp}{dL} = \frac{2f_M \rho_{NS} v_M^2}{d}.
\]

The mixture friction factor, \( f_M \), can be determined from the Blasius equation, utilizing a homogeneous no-slip Reynolds number, as given by
\[ f_M = C \left( \frac{\rho_{NS} v_M d}{\mu_{NS}} \right)^{-n}, \]  

where \( C = 0.046 \) and \( n = 0.2 \). As for the parameter \( k \) in Eq. 4.16, Hinze suggested a constant value of 0.725. However, with this value the predicted transition boundary to dispersed-bubble flow reported by the original study of Taitel et al. (1980) did not follow accurately the trend of the data. In a subsequent study, Barnea et al. (1985) modified the value of the constant \( k \), resulting in a more accurate prediction of this transition. The modified value of \( k \) is based on the fact that \( k = 0.725 \) is valid only for low concentration of the dispersed-phase. However, Calderbank (1958), investigating dispersion phenomena in gas-liquid flow, found that the bubble diameter increases proportionally to the gas void fraction in the system, \( \alpha \). Combining the Hinze and the Calderbank results for the value of \( k \), Eq. 4.14 can be rewritten as

\[ d_{\text{MAX}} = (0.725 + 4.15 \sqrt[3]{\alpha}) (\sigma/\rho_t)^{0.6} (\varepsilon)^{-0.4}. \]  

The bubbles resulting from the breakup process can exhibit different types of behavior depending on their size. Referring to Fig. 4.7, if the resulted bubble size is smaller than the critical bubble diameter, \( d_{\text{CRIT}} \), the bubbles are spherical, behaving as solid spheres, with low rates of collision and coalescence. However, when the resulted bubble size is larger than \( d_{\text{CRIT}} \), the bubbles start to deform, and the rate of collision and coalescence is high. Thus, if large bubbles are formed, subject to deformation and coalescence, the bubbles can agglomerate and form a Taylor bubble, resulting in slug flow. However, if the turbulent forces are sufficiently high to break the gas into small bubbles, the bubbles will not agglomerate and dispersed-bubble flow will prevail. An expression for the critical bubble diameter separating these two types of bubble flow behavior is given by Brodkey (1967). As suggested by Barnea et al. (1985), based on the data of Miyagi (1925), the critical bubble diameter is estimated to be double the value given by the original Brodkey expression, namely,

\[ d_{\text{CD}} = 2 \left[ \frac{0.4 \sigma}{(\rho_L - \rho_G) g} \right]^{0.5}. \]  

Thus, if the turbulent forces are sufficiently high to break the bubbles into small bubbles with diameter less than the critical diameter, the coalescence rate is hindered, ensuring dispersed-bubble flow to occur. This criterion is presented in Eq. 4.22.

\[ d_{\text{MAX}} \leq d_{\text{CD}}. \]  

Substituting Eqs. 4.20 and 4.21 into Eq. 4.22 yields the final form of the transition boundary, as given by

\[ 2 \left[ \frac{0.4 \sigma}{(\rho_L - \rho_G) g} \right]^{0.5} \left( \frac{\rho_L}{\sigma} \right)^{0.6} \left( \frac{2 \times 0.046 \left( \frac{\rho_M d}{\mu_M} \right)^{-0.2} \varepsilon}{n} \right)^{0.4} \left( v_M \right)^{(3-0.2)0.4} = 0.725 + 4.15 \left( \frac{v_{SG}}{v_M} \right)^{0.5}. \]  

Eq. 4.23 gives the relationship between the superficial gas and liquid velocities and the physical properties of the phases on the transition boundary to dispersed-bubble flow. For a given system, slug flow cannot exist for higher \( v\) and \( v_S \) (or \( v_L \)) as defined by this equation. This transition is shown in
Figs. 4.8 and 4.11 as transition “F” for air-water flow at STP in 5.1-cm and 2.54-cm ID pipes, respectively.

Transition “F” terminates at transition “G”. Transition “G” is based on the maximum possible gas void fraction allowable in a cubic lattice bubble packing, namely, \( \alpha_{\text{MAX}} = 0.52 \), as demonstrated by Fig. 4.6. Independent of turbulence or surface tension forces, dispersed-bubble flow cannot exist for void fractions larger than 0.52. Under this void fraction, coalescence of the bubble occurs to form Taylor bubbles, characteristic of slug flow. Thus, the simple criterion for transition “G” is based on the combination of superficial gas and liquid velocities that yield a no-slip void fraction of 0.52, as shown in Eq. 4.24.

\[
\frac{v_{SG}}{v_{SG} + v_{SL}} = 0.52. \quad \text{........................................................................................................(4.24)}
\]

In summary, referring to Fig. 4.8, for large diameter pipes to the left of transition “E” and below transition “F-G”, bubble flow occurs. Above transition “F-G” is the region where fine dispersed-bubble occurs. Below transition “F-G” and to the right of transition “E” slug flow may occur. For small diameter pipes, as shown in Fig. 4.11, transition “E” does not exist, and transition “F-G” separates the slug flow and dispersed-bubble flow regions.

**Slug-Churn Transition (Transition H).** Starting from bubble flow conditions, namely, the region to the left of transition “E” (see Figs. 4.8 and 4.9), upon increasing the gas flow rate, the bubble concentration increases resulting in a higher rate of coalescence between the bubbles. When the transition boundary is crossed, Taylor bubbles are formed resulting in a transition to slug flow. Further increase in the gas flow rate causes the liquid slugs to become shorter and frothy and the Taylor bubble to increase in length. Eventually, the liquid slugs are blown through by the gas-phase and transition to churn flow occurs.

Slug flow is characterized by orderly alternate flow of Taylor bubbles and liquid slugs, whereby the liquid slugs are stable with a well defined front and tail, moving at a constant velocity. In churn flow, on the other hand, the interface between the gas and liquid phases is not well defined because of the blow through and breakup of the slugs. An upward and downward oscillatory motion of the liquid-phase is observed. The blown through slugs fall backwards, causing local liquid accumulation, which is then pushed upwards by the gas-phase.

The transition from slug to churn flow is complex and not fully understood at the present time. Several mechanisms have been proposed in the past for this transition, such as flooding conditions in the liquid film region around the Taylor bubble caused by higher gas velocities or Helmholtz instability of the liquid film. The transition mechanism proposed in this model is based on an entry region concept, namely, that churn flow is an entry region to slug flow that occurs downstream along the pipe. The authors claim, based on experimental observations, that in vertical and sharply inclined pipes, whenever slug flow occurs downstream in the pipe, the flow is churn at the entry region. The entry region length is the distance along the pipe required to develop a stable slug. This region is not constant but rather is dependent on the flow conditions, namely, the flow rates and pipe diameter. Thus, for a given set of flow conditions, if the entry region is shorter than the pipe length, the observed flow pattern beyond the entry region is slug flow. On the other hand, if the entry region is longer than the given pipe length, churning will be observed in the entire pipe, and the existing flow pattern is churn flow.
The proposed analysis for the prediction of the slug to churn transition boundary is based on a method for determining the entry region length, which is the length required to develop a stable slug. The length of a stable slug in this analysis is taken as \( L_s/d = 16 \). This stable slug length can be obtained assuming that the falling liquid film from the preceding Taylor bubble region acts as a jet, plunging into the slug body. A stable slug has a sufficient length for the jet velocity to decay and attain the slug velocity. According to this approach, the length required for the “jet” to be decayed and absorbed by the surrounding liquid in the slug is approximately 16 pipe diameters.

Figure 4.12 illustrates the phenomenon involved with respect to a stable slug and an unstable slug. The figure shows a stable slug of length of \( L_s/d = 16 \) at the top, and an unstable slug at the bottom with length \( L_s < 16d \). The Taylor bubbles following the stable slug and the unstable slug are designated numbers 1 and 2, respectively. As can be seen, the falling liquid film into the slug causes a distorted velocity profile at the front of the slug, whereby the flow is downwards near the wall and upwards at the pipe center. This flow reversal near the wall is decayed along the slug. In a stable slug, at a distance of 16 pipe diameters from the slug front, the falling film “jet” is completely decayed resulting in a normal turbulent velocity profile in the slug. This does not occur in an unstable slug. As the figure shows, the unstable slug is not sufficiently long to absorb and decay the falling film “jet”, and a normal velocity profile is not established. As a result, a distorted velocity profile exists along the unstable slug. This phenomenon has a significant effect on the velocities of the trailing Taylor bubbles, as explained next.

![Fig. 4.12—Velocity profiles in stable and unstable slugs.](image)

The velocity of Taylor bubble number 1, which trails a stable slug, is given by

\[
v_{TB} = 1.2v_M + 0.35\sqrt{gd},
\]

(4.25)

where the first term on the right-hand side is the center-line velocity ahead of the Taylor bubble, namely, at the tail of the preceding slug. For a stable slug, as a normal velocity profile is established, for
turbulent flow, the centerline velocity is given by \( v_{C(1)} = 1.2v_M \). The second term is the terminal rise velocity of a Taylor bubble, as given before in Eq. 4.14.

The velocity of Taylor bubble number 2, which follows the unstable slug, is quite different. Because of the short slug length, the velocity distribution ahead of the Taylor bubble is not established to a normal velocity profile. As a result, the velocity profile at the back of the slug (ahead of the nose of the Taylor bubble) exhibits flow reversal near the wall. To maintain conservation of mass (continuity), the centerline velocity in the slug must increase. This results in a higher centerline velocity than that of the developed velocity profile at the back of a stable slug, namely, \( v_{C(2)} > v_{C(1)} \). Because the Taylor bubble rise velocity is the sum of the centerline velocity in the slug ahead of it and the terminal rise velocity, it is obvious that \( v_{TB(2)} > v_{TB(1)} \). This results in Taylor bubble number 2 overtaking Taylor bubble number 1, whereby the two bubbles coalesce and the unstable slug is broken, falls back, and merges with the succeeding slug. This flow phenomenon is typical of churn flow, and determines the entry region length leading to the establishment of a stable slug, as explained next.

At the inlet of the pipe, short slugs and Taylor bubbles are formed. A short slug is unstable, and as described previously, it is blown through by the trailing Taylor bubble, falls back and merges with the following slug, approximately doubling the slug length. At the same time, the Taylor bubble coalesces with the preceding Taylor bubble, as the slug between them falls back, doubling its length as well. This process continues along the entry region until a stable slug is formed. In the entire entry region, unstable slugs rise and fall back, typical of churn flow. Thus, churn flow exists in the entry region to slug flow.

**Churn Flow Entry Region Determination.** A schematic of the entry region analysis is shown in Fig. 4.13. Consider a coordinate system \( z \) attached to the tail of the leading Taylor bubble (number 1) and pointing downwards. At \( z = 0 \), the centerline velocity in the slug is \( v_C = v_{TB} \). At the tail of the slug, namely, \( z = L_S \) (where \( L_S = 16d \) is the length of a stable slug), the velocity distribution corresponds to normal turbulent flow, and the centerline velocity is \( v_C = 1.2v_M \). Based on these two boundary values, an

![Fig. 4.13—Schematic of entry region analysis.](image-url)
exponential profile is assumed for the centerline velocity along the axial direction \( z \), as given in Eq. 4.26.

\[
v_c = v_{TB} \exp(-\beta z / L_s) + 1.2 v_M \left[ 1 - \exp(-\beta z / L_s) \right]. \tag{4.26}
\]

The coefficient \( \beta \) determines the decay of the centerline velocity, taken as \( \beta = \ln(100) = 4.6 \), to ensure that at \( z = L_s \) the centerline velocity decay is 99\%. The results are not sensitive to the value of \( \beta \) and to the particular profile assumed, provided it satisfies the two boundary velocities at \( z = 0 \), and \( z = L_s \).

With Eq. 4.26, it is possible to determine the centerline velocity (ahead of a Taylor bubble) at the tail of an unstable slug of any given length. Using Eqs. 4.25 and 4.26 to calculate the velocities of two consecutive Taylor bubbles, the approach velocity between the two bubbles, \( -\dot{z} \), can be determined as

\[
-\dot{z} = \frac{dz}{dt} = v_{TB(2)} - v_{TB(1)} = (v_{TB} - 1.2 v_M) \exp(-\beta z / L_s) = 0.35 \sqrt{g d} \exp(-\beta z / L_s). \tag{4.27}
\]

Integrating Eq. 4.27 yields the time required for each coalescence to take place between two consecutive Taylor bubbles, as a function of the distance between them, \( L_{Li} \) (the slug length between Taylor bubbles).

\[
t_i = \int_0^{L_{Li}} \frac{dz}{0.35 \sqrt{g d} \exp(-\beta z / L_s)} = \frac{L_s}{0.35 \sqrt{g d}} \left[ \exp(\beta L_{Li}/L_s) - 1 \right]. \tag{4.28}
\]

At the inlet of the pipe, short slugs and short Taylor bubbles occur. As the flow proceeds upwards, coalescence between pairs of Taylor bubbles occurs, doubling the size of the Taylor bubble. At the same time, larger slugs develop as a result of the fall back of unstable slugs overtaken by the Taylor bubbles, which merge with the trailing slugs. This process continues along the entry region until a stable slug is formed. It is further assumed that the entry region exists up to the point where a slug of a length of \( L_L = 8d = L_s/2 \) is formed, as the last merger of two slugs of this size to form a 16\( d \)-long stable slug is quite slow. Thus, assigning \( L_{Li} \) in Eq. 4.28 values from 0 to \( L_s/4 \), namely, \( L_{Li} = L_s/4, L_s/8, L_s/16, \ldots, 0 \), results in an infinite series for \( t_i \), whose sum yields the time required for forming a stable slug, as given by

\[
t_E = \sum_i t_i = \frac{L_s}{0.35 \sqrt{g d}} \left[ \exp(\beta / 4) - 1 + \exp(\beta / 8) - 1 + \exp(\beta / 16) - 1 + \ldots \right] = \\
= \frac{L_s}{0.35 \sqrt{g d}} \sum_{n=2}^\infty \left[ \exp(\beta / 2n) - 1 \right]. \tag{4.29}
\]

Multiplying \( t_E \) from Eq. 4.29 by the \( v_{TB} \) yields the length of the entry region, as shown in Eq. 4.30.

\[
L_E = t_E v_{TB} = \frac{L_s v_{TB}}{0.35 \sqrt{g d}} \sum_{n=2}^\infty \left[ \exp(\beta / 2n) - 1 \right]. \tag{4.30}
\]

Substituting \( \beta = 4.6 \), \( L_s = 16d \), and \( v_{TB} \) from Eq. 4.25 into Eq. 4.30 yields the final expression for the entry region length, given by
\[ \frac{L_E}{d} = 40.6 \left( \frac{v_M}{\sqrt{gd}} + 0.22 \right). \]  

Eq. 4.31 provides a dimensionless criterion for the transition boundary between slug and churn flow. The transition depends on one dimensionless group, namely, \( v_M / \sqrt{gd} \). Thus, for a given set of flow conditions, including the length of the pipe, it is possible to construct this transition boundary on a \( v_{SL} \) vs. \( v_{SG} \) flow pattern map. The predicted results of this transition boundary for air-water flow at STP, in 10-m long vertical pipes with 5.1-cm and 2.54-cm diameter, are shown in Figs. 4.8 and 4.11, respectively, designated as curve “H”. The dimensionless lengths of the two pipes, which are \( L/d = 196 \) and \( L/d = 394 \), respectively, are set to the entry region length \( L_E/d \) in Eq. 4.31. As \( v_M = v_{SL} + v_{SG} \), this provides the relationship between the gas and liquid superficial velocities that satisfy Eq. 4.31, providing the transition curve “H”. Slug flow occurs to the left of the transition boundary, while churn flow occurs to the right.

**Transition to Annular Flow (Transition J).** The transition to annular flow occurs at high gas flow rates. Under this condition, the gas flows at the center of the pipe as a core, while the liquid-phase is pushed to the wall and flows upwards in the form of a thin film, with a wavy interface. The liquid film moves upwards because of the high interfacial shear and form drag exerted by the fast-moving gas core on the wavy interface. Also, liquid droplets are torn away from the wavy film, entrained in the core, and are pushed upward by the gas-phase.

The proposed mechanism for this transition is based on the “droplet model” suggested by Turner et al. (1969), originally proposed for unloading gas well. Adopting the Turner et al. model, it is proposed that the transition to annular flow occurs when the gas velocity in the core is sufficiently high to lift the liquid droplets upward. If the gas velocity is not sufficient to lift the droplets, they will fall back and accumulate, forming a local blockage of liquid in the core, which may result in either churn or slug flow.

The transition mechanism to annular flow is shown schematically in **Fig. 4.14**. Shown in the figure are the forces acting on a droplet, namely the drag force and the gravity force, given, respectively, by

\[ F_D = C_D \frac{\pi d_D^2 \rho_G v_G^2}{4}, \]  

and

\[ F_G = \frac{\pi d_D^3}{6} g (\rho_L - \rho_G). \]

Note that this analysis is applied to a droplet in the gas core. However, the same analysis can be applied instead to the crest of the waves occurring on the liquid film. If for a given set of conditions the drag force is higher than the gravity force, namely, \( F_D > F_G \), the gas velocity is sufficiently high to lift the droplets upward and annular flow occurs. On the other hand, if \( F_D < F_G \), the droplets fall back, and churn or slug flow occur. Thus, the criterion for the transition boundary is \( F_D = F_G \), namely,

\[ C_D \frac{\pi d_D^2 \rho_G v_G^2}{4} = \frac{\pi d_D^3}{6} g (\rho_L - \rho_G). \]
Solving for $v_G$ from Eq. 4.34 yields the gas velocity on the transition boundary, which corresponds to the minimum gas velocity required to lift up the droplets in the core and maintain annular flow, that is,

$$v_G = \frac{2}{\sqrt{3}} \left[ \frac{g (\rho_L - \rho_G) d_D}{\rho_G C_D} \right]^{0.5}.$$  \hspace{1cm} (4.35)

The droplet size is determined from a balance between surface tension forces that promote larger droplets and the impact forces of the gas-phase that tend to break the droplets into smaller ones. This phenomenon may be quantified by the Weber number, which is defined as

$$\text{We} = \frac{d_D \rho_G v_G^2}{\sigma}.$$  \hspace{1cm} (4.36)

Very small particles (droplets or bubbles) correspond to conditions where $\text{We} = 8$. At the transition to annular flow, larger droplets occur, where it is assumed that the We number is between 20 to 30. Thus, once the We number is given, the droplet diameter can be determined from Eq. 4.36 as

$$d_D = \frac{\sigma \text{We}}{\rho_G v_G^2}.$$  \hspace{1cm} (4.37)

Substituting Eq. 4.37 into Eq. 4.35 yields

$$v_G = \left( \frac{4\text{We}}{3 C_D} \right)^{0.25} \left[ \frac{\sigma g (\rho_L - \rho_G)}{\rho_G^{0.5}} \right]^{0.25}.$$  \hspace{1cm} (4.38)

As suggested by Turner et al., values of $C_D = 0.44$ (for fully developed turbulent flow) and $\text{We} = 30$ (for large droplets) are used in Eq. 4.38. Also, as the liquid holdup associated with annular flow is very small, the gas velocity in Eq. 4.38 is approximated by the superficial gas velocity, namely, $v_G \equiv v_{SG}$. These substitutions lead to the final form of the transition boundary as

$$v_{SG} \rho_G^{0.5} = 3.1 \left[ \frac{\sigma g (\rho_L - \rho_G)}{\rho_G^{0.5}} \right]^{0.25}, \hspace{1cm} \text{or} \hspace{1cm} v_{SG} = \frac{3.1}{\rho_G^{0.5}} \left[ \frac{\sigma g (\rho_L - \rho_G)}{\rho_G^{0.5}} \right]^{0.25}.$$  \hspace{1cm} (4.39)

where the dimensionless group in the left-hand side (LHS) of Eq. 4.39 is the Kutateladze number. Eq. 4.39 demonstrates that the transition to annular flow occurs at a constant superficial velocity and is
independent of the liquid flow rate. This transition is shown in Figs. 4.8 and 4.11 (designated as curve “J”) for air-water flow at STP, in 5.1-cm and 2.54-cm diameter pipes, respectively.

An interesting phenomenon is predicted by Eq. 4.39. At high pressures, as the gas density increases, the transition boundary shifts to lower superficial gas velocities. This phenomenon must be verified with high-pressure experimental data.

**Comparison with Previous Studies.** Figs. 4.15 and 4.16 present comparisons between the prediction of the model and previously published flow pattern maps for air-water flow in vertical pipes at STP. The model predictions are compared with studies carried out by well-known two-phase flow investigators.

Fig. 4.15 shows the comparison for the bubble-slug and the dispersed-bubble transition boundaries. This figure demonstrates the disagreement that has existed between the different investigators about these transition boundaries. However, as explained in an earlier section, for air-water flow at STP, transition “E” exists in a 5.1-cm pipe while it does not occur in a 2.54-cm pipe. Indeed, some of the investigators used a 5.1 cm pipe, observing transition “E”, while others who used smaller diameter pipes did not observe it but rather reported transition “F-G” to dispersed-bubble flow. The predictions of the model agree with both groups. A similar comparison is presented in Fig. 4.16 for the transition to annular flow. As can be seen, with an exception of one of the studies, the prediction of the proposed model agrees well with the other studies, cutting through the other reported transitions boundaries. Finally, a comparison between the predictions of the model and air-water upward flow at STP data in 5.1-cm and 2.54-cm diameter vertical pipes are shown, respectively, in Figs. 4.8 and 4.11, exhibiting good agreement.

![Fig. 4.15—Comparison between model predictions and previously published maps for the bubble to slug and dispersed-bubble transition boundaries.](image)
4.3 Bubble Flow Modeling

The formulation of Transition “E” in the Taitel et al. (1980) model, namely, the transition from bubble to slug flow, as given in Eq. 4.11, constitutes a rudimentary model for bubble flow. Indeed, Eq. 4.11, given again, can be used to predict the void fraction in bubble flow (see Example 4.1). Once the liquid holdup is determined, it is possible to predict the pressure gradient in a straightforward manner.

$$v_{SG} = \frac{1 - \alpha}{\alpha} - (1 - \alpha)v_{0\infty}.$$ .................................................................(4.11)

Nevertheless, this model can be improved, as shown by Hasan and Kabir (1988). A schematic of the physical model for bubble flow is given in Fig. 4.17. Details of the bubble flow model follow.

Fig. 4.17—Schematic of physical model for bubble Flow.
4.3.1 Void Fraction. The starting point of the model is the expression for the gas and liquid velocities, as presented before by

\[ v_G = \frac{v_{SG}}{\alpha}, \quad \text{and} \quad v_L = \frac{v_{SL}}{1-\alpha}. \]  \hspace{1cm} (4.9)

However, the slippage between the gas and liquid-phase is not taken simply as the difference between the gas and liquid velocities, as done previously in Eq. 4.10. In bubble flow, most of the bubbles are moving at the pipe centerline. Thus, the slippage (bubble-rise velocity) is taken as the relative velocity between the gas velocity and the centerline velocity, as given next (for turbulent flow condition).

\[ v_0 = v_G - 1.2v_M. \]  \hspace{1cm} (4.40)

The bubble-rise velocity for a single bubble in a stagnant infinite medium is designated as \( v_{0\infty} \), and is given by Harmathy (1960) as

\[ v_{0\infty} = 1.53 \left[ \frac{g(\rho_L - \rho_G)\sigma}{\rho_L^2} \right]^{0.25}. \]  \hspace{1cm} (4.12)

Under bubble flow conditions, bubbles move as a “bubble-swarm” with a hindered velocity. Zuber and Hench (1962) modified the Harmathy equation by using a correction factor based on the liquid holdup, namely, \((1-\alpha)^n\), to provide an expression for the bubble-swarm rise velocity, \( v_0 \). Various values have been used for the exponent \( n \). In this study, a value of \( n = 0.5 \) is used, resulting in Eq. 4.41 for the bubble-swarm rise velocity

\[ v_0 = v_{0\infty}(1-\alpha)^{0.5} = 1.53 \left[ \frac{g(\rho_L - \rho_G)\sigma}{\rho_L^2} \right]^{0.25}(1-\alpha)^{0.5}. \]  \hspace{1cm} (4.41)

For a pipe that is inclined at inclination angle \( \theta \), the axial component of the bubble rise velocity should be considered, namely, \( v_0 \sin \theta \). Substituting Eqs. 4.9 (for the gas velocity) and 4.41 into Eq. 4.40 yields

\[ 1.53 \left[ \frac{g(\rho_L - \rho_G)\sigma}{\rho_L^2} \right]^{0.25}(1-\alpha)^{0.5} \sin \theta = \frac{v_{SG}}{\alpha} - 1.2v_M. \]  \hspace{1cm} (4.42)

Eq. 4.42 is an implicit equation for the in-situ gas void fraction for bubble flow. The equation is solved by trial and error for the value of \( \alpha \). A suitable initial value for \( \alpha \) can be obtained from the solution of Eq. 4.11, which is a quadratic equation. As mentioned before, once the liquid holdup is determined, it is possible to predict the pressure gradient in a straightforward manner, as given next.

4.3.2 Pressure Gradient. The mixture density and viscosity can be determined based on the actual liquid holdup, as determined from Eq. 4.42. These properties are given, respectively, by

\[ \rho_M = \rho_L H_L + \rho_G(1-H_L), \]  \hspace{1cm} (4.43)

and

\[ \mu_M = \mu_L H_L + \mu_G(1-H_L), \]  \hspace{1cm} (4.44)
and the Reynolds number can be determined as
\[
Re_M = \frac{\rho_M v_M d}{\mu_M}.
\]

The gravitational pressure gradient is given by
\[
-\frac{dp}{dL} = \rho_M \sin \theta.
\]

The frictional pressure gradient is
\[
-\frac{dp}{dL} = \frac{2 f_M \rho_M v_M^2}{d},
\]

where the friction factor can be calculated from the standard pipe flow method, namely, \( f_M = f_M(Re_M, \varepsilon/d) \). Neglecting the accelerational pressure gradient component, the total pressure gradient can be determined from
\[
-\frac{dp}{dL} = -\frac{dp}{dL} - \frac{dp}{dL}.
\]

### 4.4 Slug Flow Modeling

The Taitel and Barnea (1990) slug flow model, presented in Sec. 3.4, is a unified model that can be applied for the range of inclination angles from horizontal to vertical. Also, a detailed slug flow model for vertical pipes was presented by Fernandez et al. (1986). In this section, a simplified slug flow model applicable for wellbore conditions is presented. This is the Sylvester (1987) simplified version of the Fernandez et al. model. The simplification is carried out by utilizing a correlation for the liquid holdup in the slug body, \( H_{LLS} \). This simplifies the original model, reducing considerably the number of equations involved. The simplified Sylvester model and the solution procedure developed by Vo and Shoham (1989) are presented next.

#### 4.4.1 Sylvester (1987) Simplified Slug Flow Model

A schematic of the physical model for slug flow is given in Fig. 4.18, with the definition of a slug unit. Note that for fully developed slug flow, the length of the Taylor bubble cap is negligible in comparison to the length of the Taylor bubble itself. Thus, in this model the film is treated as having a constant equilibrium film thickness. Following are the model equations.

The first two equations are overall mass balances on the gas and liquid phases carried out on a slug unit (as given by Eq. 3.121). For an equilibrium liquid film, an overall gas mass balance on a slug unit yields
\[
v_{SG} = \beta v_{GTB} (1 - H_{L TB}) + (1 - \beta) v_{GLS} (1 - H_{LLS}),
\]

where \( \beta \) is the ratio of the Taylor bubble/liquid film zone length to the slug unit length, namely,
\[
\beta = \frac{L_F}{L_U}.
\]
Similarly, an overall mass balance for the liquid-phase over the slug unit gives (similar to Eq. 3.159)

\[ v_{SL} = (1 - \beta)v_{LLS} H_{LLS} - \beta v_{LTB} H_{LTB}. \] ...........................................................(4.51)

A mass balance for the liquid-phase between two cross sections, one in the slug body and the other in the liquid film/Taylor bubble region, carried out in a coordinate system moving with the translational Taylor bubble velocity, \( v_{TB} \), yields

\[ (v_{TB} - v_{LLS})H_{LLS} = [v_{TB} - (-v_{LTB})]H_{LTB}. \] ...........................................................(4.52)
Note that the negative sign for \( v_{LTB} \) indicates downward flow of liquid in film. A similar mass balance for the gas-phase gives

\[ (v_{TB} - v_{GLS})(1 - H_{LLS}) = (v_{TB} - v_{GTB})(1 - H_{LTB}). \] ...........................................................(4.53)

The Taylor bubble rise velocity is equal to the centerline mixture velocity plus the Taylor bubble drift velocity (same as Eq. 3.160), namely,

\[ v_{TB} = c_0 v_M + v_D. \] ........................................................................................................ (4.54)

where \( c_0 \) is a flow coefficient, equal to 1.2 for turbulent flow and 2 for laminar flow. Utilizing the Bendiksen (1984) correlation for the drift velocity, \( v_D \), the Taylor bubble rise velocity is given by

\[ v_{TB} = c_0 v_M + 0.54 \sqrt{gd} \cos \theta + 0.35 \sqrt{gd} \sin \theta \quad \text{for} \quad 0^\circ \leq \theta \leq 90^\circ. \] ..................................................................(3.164)

Similarly, the rise velocity of the small bubbles in the slug body is (same as Eq. 3.165)
Eqs. 4.49 to 4.55 represent six independent equations with eight unknowns. Two more equations are obtained from empirical correlations for \( v_{LTB} \) and \( H_{LLS} \), as shown next.

The liquid film velocity, \( v_{LTB} \), can be considered as a falling film velocity. For turbulent flow conditions, the falling film velocity can be correlated with film thickness, \( \delta_L \), as given by Brotz (1954), namely,

\[
v_{LTB} = \sqrt{196.7g\delta_L} . \quad .................................................(4.56)
\]

Assuming a constant film thickness and using geometrical relationships, the film thickness can be expressed in terms of the liquid holdup in the liquid film/Taylor bubble region, \( H_{LTF} \), as

\[
\delta_L = \frac{d}{2}(1-\sqrt{1-H_{LTB}}) . \quad ...................................................(4.57)
\]

Substituting Eq. 4.57 into Eq. 4.56 results in the final expression for the liquid film velocity, given by

\[
v_{LTB} = 9.916\left[gd(1-\sqrt{1-H_{LTB}})\right]^{0.5} . \quad ...................................................(4.58)
\]

Sylvester developed a correlation for the liquid holdup in the slug body based on the data presented by Fernandes et al. and Schmidt (1976). The more recently developed general correlation by Gomez et al. (2000), given in Eq. 3.168, is recommended instead, namely

\[
H_{LLS} = 1.0\exp[-(7.85\times10^{-3}\theta + 2.48\times10^{-6}\text{Re}_{LS})] \quad \text{for} \quad 0^\circ \leq \theta \leq 90^\circ , \quad ....................(3.168)
\]

where the liquid slug Reynolds number is given by

\[
\text{Re}_{LS} = \frac{\rho_L v_M d}{\mu_L} . \quad ...................................................(3.169)
\]

The model so far (without the determination of the pressure drop that will be given later) consists of eight equations with eight unknowns. The eight equations are Eqs. 4.49, 4.51 to 4.55, 4.58, and 3.168. The eight unknowns to be solved are \( \beta, H_{LTB}, H_{LLS}, v_{TB}, v_{LLS}, v_{GLS}, v_{LTF}, \) and \( v_{GTB} \). The eight equations can be solved iteratively. However, a convenient solution procedure was proposed by Vo and Shoham (1989), as presented next.

### 4.4.2 Vo and Shoham (1989) Solution Procedure

Vo and Shoham (1989) showed that the eight equations constituting the simplified Sylvester model can be reduced to one equation with one unknown, \( H_{LTB} \), as given by

\[
F(H_{LTB}) = (9.916\sqrt{gd})(1-\sqrt{1-H_{LTB}})^{0.5}H_{LTB} - v_{TB}(1-H_{LTB}) + \tilde{A} = 0 , \quad .............(4.59)
\]

where

\[
\tilde{A} = (1-H_{LLS})v_{TB} + H_{LLS}\left(v_M - (1-H_{LLS}) \left[1.53\left[\frac{\sigma g(\rho_L-\rho_G)}{\rho_L^2}\right]^{0.25}H_{LLS}^{0.5}\sin\theta\right]\right) , \quad .......(4.60)
\]
All the terms in Eq. 4.60 are known, including $v_{TB}$, $v_M$, and $H_{LLS}$ (or $H_{GLS}$). Thus, calculating $\tilde{A}$ from Eq. 4.60 and substituting into Eq. 4.59, it is possible to determine $H_{LTB}$ in an iterative calculation process. The derivative of the function $F$ in Eq. 4.59, with respect to $H_{LTB}$, is given by

$$F'(H_{LTB}) = v_{TB} + (9.916 \sqrt{gd}) \left[ (1 - \sqrt{1 - H_{LHB}})^{0.5} + \frac{H_{LHB}}{4 \sqrt{(1 - H_{LHB}) (1 - \sqrt{1 - H_{LHB}})}} \right]. \quad (4.61)$$

Thus, a Newton-Raphson method can be used to accelerate the convergence and find the root of Eq. 4.59, namely, $H_{LHB}$, as

$$H_{LHB_{j+1}} = H_{LHB_j} - \frac{F(H_{LHB_j})}{F'(H_{LHB_j})}. \quad ..................................................................................... (4.62)$$

The step-by-step solution procedure is given next.

1. Calculate $v_{TB}$ and $H_{LLS}$ from Eqs. 4.54 and 3.168, respectively.
2. Using Eqs. 4.59 to 4.62, determine $H_{LHB}$. A good initial guess is $H_{LHB} = 0.15$.
3. Solve Eq. 4.58 for $v_{LHB}$.
4. Solve Eq. 4.52 for $v_{LLS}$.
5. Solve Eq. 4.55 for $v_{GLS}$.
6. Solve Eq. 4.53 for $v_{GTB}$.
7. Solve either Eq. 4.49 or Eq. 4.51 for $\beta$.
8. Assuming that the slug length is $L_S = 20d$, calculate $L_U$ and $L_F$ from the definition of $\beta$. Note that $L_F = L_U - L_S$.

**Pressure Drop.** Once the slug variables are calculated, it is possible to determine the pressure drop across a slug unit. Note that in the original Selvester simplified model, the pressure drop calculation is carried out only on the slug body, neglecting the pressure drop in the Taylor bubble/liquid film zone. However, although an equilibrium liquid film is used, the pressure drop equation also includes, in addition to the gravitational and frictional components, the accelerational pressure drop. This results in an over-prediction of the pressure drop. As mentioned at the end of Sec. 3.4.2, Barnea (1989) showed that for the case of equilibrium liquid film, when the pressure drop in the gas pocket/liquid film zone is neglected, the acceleration pressure drop term should not be considered. Thus, it is recommended to calculate pressure drop using the Taitel and Barnea (1990) model, Case 3 for equilibrium film thickness (see Sec. 3.4.2), as given by

$$-\Delta p_U = \rho_S g \sin \theta L_S + \frac{\tau_s \pi d}{A_p} L_S + \rho_F g \sin \theta L_F + \frac{\tau_F S_F}{A_p} L_F + \frac{\tau_G S_G}{A_p} L_F. \quad ................. \quad (3.155)$$

If the pressure drop in the Taylor bubble/liquid film zone is neglected, only the first two terms on the RHS should be used. The pressure gradient is given by

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\[
- \frac{dp}{dL} = - \frac{\Delta p_U}{L_U}.
\]

(3.79)

4.5 Annular Flow Modeling

A schematic of annular flow is given in Fig. 4.19. Annular flow is dominated by high gas flow rates. The gas-phase moves fast at the center of the pipe (gas core), while the liquid-phase flows as an annular film around the pipe wall. Two phenomena are associated with annular flow:

- High interfacial shear stress, \( \tau_f \), caused by high shear velocities introduced by the high gas flow rates.

- Continuous processes of droplet entrainment (out of the liquid film) and droplet deposition (into the liquid film). At steady-state conditions, the rates of deposition and entrainment are equal, resulting in an equilibrium entrainment fraction of droplets in the gas core, \( f_E \).

![Fig. 4.19—Schematic of annular flow.](image)

The following parameters are important for understanding annular flow and developing models to predict the flow behavior:

- Interfacial shear stress.
- Fraction of liquid entrained as droplets in gas core.
- Liquid film thickness.
- Heat transfer coefficient to the pipe wall.
- Slippage between droplets and the gas core.
4.5.1 Annular Flow Triangle Hydrodynamic Relationship. A fundamental relationship for annular flow is the “triangle hydrodynamic relationship.” This is the relationship between the following three variables:

- \( q_L \) = liquid flow rate.
- \( \delta_L \) = liquid film thickness.
- \(-dp/\,dL\) = pressure gradient.

The triangle relationship can be developed by applying a momentum (force) balance in the axial flow direction on a differential control volume in the gas core, resulting in

\[
-\tau_f\pi d_c^2\Delta L - \Delta p\frac{\pi d_c^2}{4} - \rho_L g \frac{\pi d_c^2}{4} \Delta L = 0. 
\]

In general, it is possible to develop continuity and momentum equations for the gas core and the liquid film. However, no analytical solution is possible because of the complexity of the flow, including the interfacial structure and the droplet entrainment. Thus, empirical closure relationships for the interfacial shear stress and the entrainment are needed. These are reviewed in Secs. 4.5.3 and 4.5.4.


Extensive studies have been carried out on the physical phenomena associated with annular flow. Turner et al. (1969) and Ilobi and Ikoku (1981) studied the minimum gas velocity required for liquid removal in vertical pipes. Taitel et al. (1980) adapted this concept to predict the transition boundary to annular flow. Several investigators developed correlations for the interfacial friction factor, including Wallis (1969), Henstock and Hanratty (1976), Whally and Hewitt (1978), and Asali et al. (1985). Other studies, namely, Wallis (1969), Hanratty and Asali (1883), and Ishii and Mishima (1989) have studied the entrainment process. Oliemans et al. (1986) presented a comprehensive review of both interfacial shear and entrainment and developed their own correlations for these parameters. Some of these correlations will be covered in subsequent Secs. 4.5.3 and 4.5.4. In this section, the model developed by Alves et al. (1991), using a similar approach to the one used by Oliemans et al. (1986), will be covered.

4.5.3 Interfacial Shear in Annular Flow. The interfacial shear stress for annular flow is defined by

\[
\tau_f = f_i \frac{\rho_L (v_c - v_f)^2}{2}, 
\]

\((4.64)\)
where $v_c$ and $v_F$ are the core velocity and liquid film velocity, as given by Eqs. 4.88 and 4.85, respectively, and $f_i$ is the interfacial friction factor. Several interfacial friction factor correlations have been proposed. The most commonly used practical ones are given next. Usually, the interfacial friction factor is expressed in terms of a dimensionless parameter $I$ as

$$I = \frac{f_i}{f_{SG}},$$

where $f_{SG}$ is the superficial gas-phase friction factor in a smooth pipe flow, which can be calculated using the Blasius equation (Eq. 1.35). As the flow of the gas-phase is always turbulent in annular flow, $f_{SG}$ is given by

$$f_{SG} = 0.046 \text{Re}_{SG}^{-0.2} = 0.046 \left( \frac{\rho_G v_{SG} d}{\mu_G} \right)^{-0.2}.$$

**Wallis (1969).** The Wallis (1969) correlation for the dimensionless interfacial friction factor parameter $I$ is a function of the dimensionless liquid film thickness, $\delta_l/d$, taking into account an equivalent roughness caused by the wavy structure of the liquid film. For large film thickness, large waves may be generated, increasing the interfacial shear stress. For small film thickness, smaller waves may be generated, causing lower interfacial shear stress. The Wallis correlation is given by

$$I = 1 + 300 \frac{\delta_l}{d}.$$

Note that sometimes, for simplicity, a constant value for the superficial gas friction factor $f_{SG} = 0.005$ is used in Eq. 4.65 for the determination of $f_i$ for the Wallis correlation.

**Whalley and Hewitt (1978).** The Whalley and Hewitt correlation for $I$ is a function of the dimensionless film thickness and the PVT properties, as shown in Eq. 4.68.

$$I = 1 + 24 \left( \frac{\rho_L}{\rho_G} \right)^{1/3} \left( \frac{\delta_l}{d} \right).$$

**Henstock and Hanratty (1976).** The Henstock and Hanratty (1976) correlation is the most versatile one. It applies to different flow conditions, including upward concurrent flow, downward concurrent flow, and horizontal flow. The following dimensionless groups are defined for the correlations.

$$F = \frac{(0.42 \text{Re}_{F,SG}^{1.25} + 2.8 \times 10^{-4} \text{Re}_{F,SG}^{2.25})^{0.4}}{\text{Re}_{SG}^{0.9}} \frac{\mu_L}{\mu_G} \left( \frac{\rho_G}{\rho_L} \right)^{0.5}, \quad \text{and} \quad G = \frac{\rho_L d g}{\rho_G v_{SG}^2 f_{SG}}, \quad (4.69)$$

where the superficial gas Reynolds number, $\text{Re}_{SG}$, and the superficial gas friction factor, $f_{SG}$, are given in Eq. 4.66. The liquid film Reynolds number, $\text{Re}_F$, is defined in Eq. 4.92. The dimensionless interfacial friction factor parameter $I$ for the different flow conditions can be determined from Eqs. 4.70 through 4.73.

Upward concurrent Flow:
\[ I = 1 + 1.400 F \]  
\[ \text{Downward concurrent Flow:} \]
\[ I = 1 + 1.400 F \left\{ 1 - \text{EXP} \left[ -\frac{(1+1.400 F)^{1.5}}{13.2 F G} \right] \right\}. \]
\[ \text{Horizontal flow:} \]
\[ I = 1 + 850 F. \]

\textit{Oliemans et al. (1986).}
\[ I = 1 + 2.250 \frac{\delta_l/d}{\rho_c (v_c - v_F)^2 \delta_L / \sigma}, \]
where \( v_F \) and \( v_c \) are the liquid film and core velocities, respectively.

\textbf{4.5.4 Entrainment Fraction.} The entrainment fraction, \( f_E \), is defined as the fraction of the liquid flow rate, which is entrained in the gas core as droplets. Given next are three of the most commonly used correlations for the entrainment fraction.

\textit{Wallis (1969).} The Wallis (1969) correlation for the entrainment fraction is given by
\[ f_E = 1 - \text{EXP}[-0.125(\phi - 1.5)], \]
where
\[ \phi = 10^4 \frac{\nu_{SG} \mu_G (\rho_G / \rho_L)^{0.5}}{\sigma}. \]
A graphical representation of the correlation is given in \textbf{Fig. 4.20}.

![Fig. 4.20—Wallis (1969) entrainment fraction correlation.](image-url)
**Oliemans et al. (1986).** The Oliemans et al. (1986) correlation has been developed from a regression analysis of a large database in the form

\[
\frac{f_E}{1-f_E} = 10^{\beta_0} \rho_L^{\beta_1} \rho_G^{\beta_2} \mu_L^{\beta_3} \mu_G^{\beta_4} \sigma^{\beta_5} d^{\beta_6} v_{SL}^{\beta_7} v_{SG}^{\beta_8} g^{\beta_9},
\]

where the exponent \( \beta \) parameters are given as

\[
\begin{array}{cccccccccc}
\beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 & \beta_9 \\
-2.52 & 1.08 & 0.18 & 0.27 & 0.28 & -1.80 & 1.72 & 0.70 & 1.44 & 0.46 \\
\end{array}
\]

**Ishii and Mishima (1989).** The Ishii and Mishima (1989) correlation is based on a modified Weber number defined as

\[
\text{We'} = \text{We} \left( \frac{\rho_L - \rho_G}{\rho_G} \right)^{1/3} = \rho_G \gamma_G d \left( \frac{\rho_L - \rho_G}{\rho_G} \right)^{1/3}. \]

The entrainment fraction can be calculated from

\[
f_E = \tanh \left[ 7.26 \times 10^7 (\text{We'})^{1.25} \text{Re}_{SL}^{0.25} \right],
\]

where \( \text{Re}_{SL} \) is the liquid superficial Reynolds number, as given in Eq. 3.35.

**4.5.5 Alves et al. (1991) Annular Flow Model.** The Alves et al. (1991) mechanistic model for annular flow in vertical and off-vertical pipes is a one-dimensional (1D) two-fluid model. The two fluids are the core fluid (gas and entrained droplets) and the liquid film. In this respect, the approach is similar to the approach used in the development of the stratified flow model. However, the annular model is based on different geometry and different physical phenomena. The model enables detailed prediction of the annular pattern flow characteristics, including the velocity distribution, liquid film thickness, gas void fraction, and pressure gradient.

A schematic of the physical model for annular flow is given in Fig. 4.21 Part A. As can be seen, the analysis is carried out utilizing a control volume with a differential axial length of \( dL \). The pipe diameter is \( d \) and the inclination angle is \( \theta \). The liquid flows as a film around the pipe perimeter, while the gas core, including entrained liquid droplets, flows at the center. The following assumptions are used in the development of the model:

- Fully developed flow.
- Uniform film thickness.
- Homogeneous no-slip flow of the gas and the entrained droplets in the core.
- Average-velocity concept (no velocity profiles considered).
- Isothermal and incompressible flow.
Fig. 4.21—Schematic of annular flow physical model.

Fig. 4.21 Part B presents the general approach of separating the flow in the control volume into two-fluids, namely, the core fluid and the liquid film fluid. The figure also shows the forces acting on the core and liquid film regions and the geometrical parameters involved. The forces are the gravity, pressure, and shear forces. The liquid wall shear stress and the interfacial shear stress are designated by $\tau_w$ and $\tau_f$, respectively. The area for flow of the core and the liquid film regions are $A_c$ and $A_f$, respectively. The wetted perimeter of the liquid-phase is $S_w$, which is the pipe perimeter; $S_I$ is the interface length and the wetted perimeter of the core, and the film thickness is $\delta_L$.

Momentum Equations. The model is derived by applying the momentum balances to the liquid film and the core, respectively. Note that because the flow is considered incompressible at a given location, the rate of change of momentum is neglected, resulting in the reduction of the momentum balances to force balances. The momentum (force) balances for the liquid film and core are given, respectively, by

$$\frac{-\tau_w}{A_F} S_L + \frac{\tau_f}{A_F} S_I - \frac{dp}{dL} - \rho_L g \sin \theta = 0,$$

and
\[-\tau_l \left( \frac{S_L}{A_c} - \frac{dp}{dL} \right) - \rho_c g \sin \theta = 0. \]  \hspace{1cm} \text{(4.80)}

For fully developed flow, the pressure gradient in the film and core are equal. Thus, Eqs. 4.79 and 4.80 can be combined by eliminating the pressure gradient, resulting in the combined momentum equation for annular flow, given by

\[-\tau_{wl} \frac{S_L}{A_F} + \tau_l \frac{S_I}{A_F} \left( \frac{1}{A_F} + \frac{1}{A_c} \right) - (\rho_L - \rho_c) g \sin \theta = 0. \]  \hspace{1cm} \text{(4.81)}

**Geometrical Relationship.** The geometrical parameters are derived based on a uniform film thickness, \(\delta_L\), as shown in Eq. 4.82.

\[A_c = \pi (d - 2\delta_L)^2 / 4, \]
\[A_F = \pi \delta_L (d - \delta_L), \]
\[S_I = \pi (d - 2\delta_L), \]

and

\[S_L = \pi d. \]  \hspace{1cm} \text{(4.82)}

The hydraulic diameters of the liquid film and core are given, respectively, by

\[d_F = 4\delta_L (d - 2\delta_L) / \pi, \quad \text{and} \quad d_C = (d - 2\delta_L). \]  \hspace{1cm} \text{(4.83)}

**Mass Balances.** Mass balances are carried out to determine the film and core velocities, core average physical properties, and the gas void fraction. The liquid flow rate in the film can be determined from the total liquid flow rate based on the entrainment fraction as

\[q_F = q_L (1 - f_E) = A_F v_{SL} (1 - f_E) = v_F A_F. \]  \hspace{1cm} \text{(4.84)}

The different correlations for determining the entrainment fraction, \(f_E\), are given in Sec. 4.5.4. Solving for the film velocity from Eq. 4.84 and substituting for \(A_F\) from Eq. 4.82, the film velocity is determined as

\[v_F = v_{SL} \frac{(1 - f_E)}{4\delta_L (d - \delta_L)} \]  \hspace{1cm} \text{(4.85)}

The core velocity is determined in a similar manner. The flow rate in the core is

\[q_C = q_G + q_L f_E = A_F (v_{SG} + v_{SL} f_E) = A_C v_C, \]  \hspace{1cm} \text{(4.86)}

and the core superficial velocity and actual velocity are, respectively,

\[v_{SC} = v_{SG} + v_{SL} f_E, \]  \hspace{1cm} \text{(4.87)}

and
\[ v_C = \frac{(v_{SG} + v_{SL} f_E) d^2}{(d - 2\delta_L)^2}. \]  .................................................................(4.88)

Assuming a homogeneous no-slip mixture of the gas-phase and the entrained droplets in the core, the core void fraction is determined as

\[ \alpha_c = \frac{v_{SG}}{v_{SG} + v_{SL} f_E}, \]  .................................................................(4.89)

and the core density and viscosity are given, respectively, by

\[ \rho_c = \rho_g \alpha_c + \rho_I (1 - \alpha_c), \quad \text{and} \quad \mu_c = \mu_g \alpha_c + \mu_I (1 - \alpha_c). \]  ............................................(4.90)

As the flow in the core is homogeneous no-slip flow, the gas velocity is equal to the core velocity (given in Eq. 4.88), namely, \( v_g = v_C \). Thus, the total void fraction of the flow can be determined from the relationship \( \alpha_T = v_{SG} / v_g \), or also from geometrical consideration, namely, \( \alpha_T = \alpha_c A_c / A_p \), either of which yield the same result, as shown in Eq. 4.91.

\[ \alpha_T = \alpha_c \left(1 - 2 \frac{\delta_L}{d}\right)^2. \]  .................................................................(4.91)

**Reynolds Numbers.** It is possible now to determine the core and film Reynolds numbers given, respectively, by

\[ \text{Re}_C = \frac{\rho_c v_C d_C}{\mu_c}, \quad \text{and} \quad \text{Re}_F = \frac{\rho_I v_F d_F}{\mu_L}. \]  .................................................................(4.92)

Note that the liquid film wall shear stress, \( \tau_w \), is determined on the basis of \( \text{Re}_F \). The core superficial Reynolds number is defined as

\[ \text{Re}_{SC} = \frac{\rho_c v_{SC} d}{\mu_c}. \]  .................................................................(4.93)

**Shear Stresses.** The liquid film wall shear stress is determined as

\[ \tau_{WL} = f_F \frac{\rho_f v_F^2}{2}, \]  .................................................................(4.94)

where \( f_F \) is the film friction factor, which can be determined from the standard pipe flow method using the film Reynolds number based on the hydraulic diameter (given in Eq. 4.92), namely, \( f_F = f_F (\text{Re}_F, \varepsilon / d) \). For a smooth pipe, the Blasius equation can be used, as given by

\[ f_F = C_F \text{Re}_F^n, \]  .................................................................(4.95)

where \( C_F \) is 16 for laminar flow and 0.046 for turbulent flow, and the exponent \( n \) is 1 for laminar flow and 0.2 for turbulent flow. The interfacial shear stress is determined from

\[ \tau_I = f_I \frac{\rho_c (v_C - v_F)^2}{2}, \]  .................................................................(4.64)
where the interfacial friction factor is expressed in terms of a dimensionless parameter $I$, namely,

$$ I = \frac{f_f}{f_{SG}}. \hspace{1cm} (4.65) $$

The different correlations for the determination of $I$ are given in Sec. 4.5.3.

**Final Dimensional Form of Model.** Substituting the geometrical relationship, velocities, and shear stresses into Eq. 4.81 results in the final dimensional form of the combined momentum equation, as follows

$$ -\frac{dp}{dL}_{SL} \left(1 - f_K \right)^{2-n} \frac{dL^6}{64} - \frac{dL^7 I}{4d(d - \delta_L)(d - 2\delta_L)^{5-m}} + g \sin \theta (\rho_L - \rho_C) \left( -\frac{dp}{dL} \right)_{SC} = 0. \hspace{1cm} (4.96) $$

Similar treatment of Eqs. 4.79 and 4.80 results in the final dimensional form of the momentum equations for the core and film that are given, respectively, by

$$ -\frac{dp}{dL} = -\frac{dp}{dL}_{SC} \left( d - 2\delta_L \right)^{5-m} + \rho_C g \sin \theta, \hspace{1cm} (4.97) $$

and

$$ -\frac{dp}{dL} = -\frac{dp}{dL}_{SL} \left(1 - f_K \right)^{2-n} \frac{dL^6}{64} - \frac{dL^7 I}{4d(d - \delta_L)(d - 2\delta_L)^{5-m}} + \rho_L g \sin \theta. \hspace{1cm} (4.98) $$

In Eqs. 4.96 through 4.98, $-\frac{dp}{dL}_{SL}$ is the liquid superficial pressure gradient (as defined in Eq. 3.35), and $-\frac{dp}{dL}_{SC}$ is the core superficial pressure gradient given by

$$ -\frac{dp}{dL}_{SC} = \frac{4}{d} f_{SC} \rho_C v_{SC}^2. \hspace{1cm} (4.99) $$

The superficial core friction factor, $f_{SC}$, can be determined from the standard pipe flow method using the core superficial Reynolds number (given in Eq. 4.93), namely, $f_{SC} = f_{SC} (\text{Re}_{SC}, \nu/d)$. The Blasius equation can be used for smooth pipes, namely

$$ f_{SC} = C_{SC} \text{Re}_{SC}^{m}. \hspace{1cm} (4.100) $$

As the core in annular flow is always in turbulent flow, $C_{SC} = 0.046$, and $m = 0.2$.

Eqs. 4.96 through 4.98 constitute the final dimensional form of the model. Eq. 4.96 is an implicit equation for the liquid film thickness, $\delta_L$, which can be solved iteratively utilizing the equations for the geometrical relationship, velocities and shear stresses. Once $\delta_L$ is determined, the gas void fraction can be calculated with Eq. 4.91, and the pressure gradient can be determined from either Eq. 4.97 or Eq. 4.98.
**Dimensionless Model Form.** The model developed in the previous sections in a dimensional form can be generalized by transforming it into a dimensionless form. In this process, the governing dimensionless groups emerge. Also, this allows plotting the results in a generalized form, which is practically useful and also sheds light on the physical phenomena. For example, the reference scale variable for the length is the diameter. Thus, the dimensionless film thickness is \( \delta_L = \delta_l/d \). The transformation into a dimensionless form is carried out by substituting the dimensionless film thickness, \( \delta_L \), for the dimensional film thickness, \( \delta_l \), in the model equations. Transforming Eq. 4.96 into dimensionless form yields

\[
X_M^2 \frac{(1-f_E)^{2-n}}{[1-(1-2\delta_L)^2]^3} = \frac{I}{[1-(1-2\delta_L)^2][1-2\delta_L]^{5-m}} + Y_M = 0. \tag{4.101}
\]

Two dimensionless parameters emerge from the analysis. The first is \( X_M \), a modified Lockhart and Martinelli parameter given by

\[
X_M^2 = \frac{-\frac{dp}{dL}}{\frac{dp}{dL}}_{SL}, \quad \text{.................................................................(4.102)}
\]

and the second dimensionless group, \( Y_M \), is a modified Taitel and Dukler (1976) inclination angle parameter, namely

\[
Y_M = \frac{g \sin \theta (\rho_L - \rho_C)}{-\frac{dp}{dL}}_{SC}. \quad \text{.................................................................(4.103)}
\]

Examining Eq. 4.101 reveals that the dimensionless film thickness is controlled by six dimensionless groups, namely, \( \tilde{\delta}_L = \tilde{\delta}_l(X_M,Y_M,f_E,I,m,n) \). However, the number of dimensionless groups can be reduced by combining 3 of them, as shown in Eq. 4.104.

\[
X_{M_0}^2 = X_M^2 (1-f_E)^{2-n}. \quad \text{.................................................................(4.104)}
\]

Note that the parameter \( X_{M_0} \) is equal to \( X_M \) for zero entrainment. The number of dimensionless groups can be further reduced by noting that the gas-phase is always in turbulent flow, namely, \( m = 0.2 \). Thus, the film thickness depends on three dimensionless groups, namely, \( \tilde{\delta}_L = \tilde{\delta}_l(X_{M_0},Y_M,I) \). Furthermore, if a Wallis-type correlation is used for the interfacial shear stress, namely, \( I = I(\tilde{\delta}_L) \), the number of dimensionless groups is reduced further, and the dimensionless film thickness is a function of only two dimensionless groups, namely, \( \tilde{\delta}_L = \tilde{\delta}_l(X_{M_0},Y_M) \).

Similarly, Eqs. 4.97 and 4.98 can be transformed into dimensionless form, yielding, respectively, the dimensionless pressure gradient equations for the core and the film, namely,

\[
\phi_C^2 = \frac{I}{(1-2\tilde{\delta}_L)^{5-m}}, \quad \text{.................................................................(4.105)}
\]
and

$$\phi_C^2 = \frac{(1 - f_E)^{2-n}}{[1 - (1 - 2\tilde{\delta}_L)^2]^2} \left\{ \frac{I}{(1 - 2\tilde{\delta}_L)^{2\cdot m} - Y_M} \right\}, \hspace{1cm} \text{..................................(4.106)}$$

where $\phi_C^2$ and $\phi_F^2$ are modified Lockhart and Martinelli dimensionless pressure parameters for the core and the film that are given, respectively, by

$$\phi_C = \frac{-\frac{dp}{dL}_C}{\frac{dp}{dL}_{SC}} - \rho_C g \sin \theta,$$

and

$$\phi_F = \frac{-\frac{dp}{dL}_F}{\frac{dp}{dL}_{SL}} - \rho_L g \sin \theta.$$

Eqs. 4.105 and 4.106 show that the dimensionless pressure gradient parameters $\phi_C^2$ and $\phi_F^2$ are functions of six dimensionless groups, including $Y_M, f_E, I, \tilde{\delta}_L, m, n$. As done before for Eq. 4.101, the number of dimensionless groups in these equations can be reduced by defining the dimensionless group

$$\phi_F^2 = \phi_F^2(1 - f_E)^{2-n}.$$

Also, assuming that the core is in turbulent flow, $m = 0.2$, and that the interfacial friction factor is $I = I(\tilde{\delta}_L)$, the dimensionless pressure parameter for the core is a function of one dimensionless group only, namely, $\phi_C^2 = \phi_C^2(\tilde{\delta}_L)$, and the dimensionless pressure parameter for the film is function of two dimensionless groups, namely, $\phi_F^2 = \phi_F^2(\tilde{\delta}_L)$. Finally, a relationship can be developed among the dimensionless groups as

$$X_M = \frac{\phi_C^2 - Y_M}{\phi_F^2}.$$

**Generalized Graphical Solution.** The final form of the dimensionless model is given by Eq. 4.101 for the film thickness and Eqs. 4.105 and 4.106 for the pressure gradient. In this section, the model is presented in a form of dimensionless plots, as given in Figs. 4.22 and 4.23. Fig. 4.22 presents the generalized solution for the dimensionless film thickness given by Eq. 4.101 in the form of $\tilde{\delta}_L = \tilde{\delta}_L(X_{M0}, Y_M)$. For the development of this figure, the Wallis correlations for both $I$ and $f_E$ have been used. The figure can be used conveniently to determine the film thickness, as follows: $X_M$ and $Y_M$ can be determined from inlet conditions. If $f_E$ can also be determined from inlet conditions (as in
the case of the Wallis correlation), $X_{M0}$ can be determined directly from Eq. 4.104. The calculated values of $X_{M0}$ and $Y_M$ can then be entered into Fig. 4.22 to obtain the value of $\tilde{\delta}_L$. If $f_E$ cannot be determined from inlet conditions, a trail-and-error procedure must be carried out for determining $\tilde{\delta}_L$. Note that the value of $n$ depends on whether the flow in the film is laminar ($n = 1$) or turbulent ($n = 0.2$). This should be determined utilizing the actual Reynolds number of the film, as given in Eq. 4.92 (and Eq. 4.95), based on the hydraulic diameter of the film.

![Fig. 4.22—Dimensionless liquid film thickness.](image)

Fig. 4.23 presents the generalized solution for the pressure gradient given by Eqs. 4.105 and 4.106 (and Eq. 4.109) in the form of $\phi^2_C = \phi^2_C(\tilde{\delta}_L)$ and $\phi^2_{F0} = \phi^2_{F0}(\tilde{\delta}_L,Y_M)$, respectively. Again, the Wallis correlation for $I$ has been used in the development of the figure. Once the values of $Y_M$ and $\tilde{\delta}_L$ are known, it is possible to obtain either $\phi^2_C$ or $\phi^2_{F0}$ from the figure and proceed to determine the pressure gradient. Note that the same results should be obtained using either $\phi^2_C$ or $\phi^2_{F0}$.

As can be seen, $\phi^2_C$ and $\phi^2_{F0}$ exhibit different behaviors. While $\phi^2_C$ increases monotonically with $\tilde{\delta}_L$, the behavior of $\phi^2_{F0}$ is more complex and depends on the parameter $Y_M$. For small values of $Y_M$, $\phi^2_{F0}$ decreases monotonically as $\tilde{\delta}_L$ increases. For large values of $Y_M$ and higher values of $\tilde{\delta}_L$, the behavior is the same until $\phi^2_{F0}$ reaches a maximum and then drops sharply. This behavior is better understood by rearranging Eq. 4.108 in terms of Eq. 4.79, yielding
As shown in Eq. 4.111, the behavior of $\phi_F^2$ depends on the relative magnitude of the wall shear stress, $\tau_w$, and the interfacial shear stress, $\tau_I$. For large values of $\tilde{\delta}_L$, implying a large liquid flow rate, $\tau_w$ is dominant; $\phi_F^2$ is positive, and the curves for all values of $Y_M$ converge. The curves finally collapse into a single line at the limit, which represents single-phase liquid flow, where $\phi_F^2 = 1$. The opposite behavior is observed for very small values of $\tilde{\delta}_L$, as the flow approaches single-phase gas flow. Under these conditions, $\tau_I$ is dominant and $\phi_F^2$ becomes negative. As the liquid phase is introduced and increased, $\tilde{\delta}_L$ increases, $\tau_w$ increases, and $\phi_F^2$ becomes positive. Between these two types of behavior, $\phi_F^2$ reaches a maximum, as shown in the figure.

**Comparison with Experimental Data.** The developed annular flow model has been tested against the Tulsa University Fluid Flow Projects (TUFFP) wellbore data bank and also against two field measurements conducted on a wet gas well in Brazil. The TUFFP wellbore data bank is presented in more detail in Sec. 4.6. It consists of 1,775 field well cases, representing a wide range of flow conditions. Using the Taitel *et al.* (1980) flow pattern prediction model, 75 cases were identified to be
under annular flow throughout the entire wellbore length. Each data case included both the wellhead and bottomhole pressures.

For comparison purpose, the developed model was incorporated into a two-phase flow pressure traverse computation algorithm code, as explained in Sec. 1.6.3. Comparison between the model predictions and the data has been carried out by using the wellhead pressure as an input. The code with the annular flow model has been used to calculate the bottomhole pressure. The calculated bottomhole pressure, \( p_{BH(C)} \), and measured bottomhole pressure, \( p_{BH(M)} \), have then been compared. This has been done for all the 75 data points. The statistical parameters used in the evaluation are based on the actual error, namely, \( p_{BH(M)} - p_{BH(C)} \), given in pressure units (Pascals). Thus, the parameters used are \( \varepsilon_4 \) (average error), \( \varepsilon_5 \) (absolute average error), and \( \varepsilon_6 \) (standard deviation). The statistical parameters are discussed in Sec. 3.5.3. The evaluation has been carried out for the developed model and also for six commonly used empirical correlations, as presented in Table 4.2. As can be seen, the model predictions are superior to all other correlations evaluated, exhibiting the lowest values for all the three statistical parameters. The very good performance of the model is also demonstrated in Fig. 4.24, where the calculated and measured pressure drops are compared.

<table>
<thead>
<tr>
<th>TABLE 4.2—PERFORMANCE OF DIFFERENT METHODS AGAINST TUFFP DATA BASE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>Alves et al. (1991)</td>
</tr>
<tr>
<td>Aziz et al. (1972)</td>
</tr>
<tr>
<td>Hagedorn and Brown (1965)</td>
</tr>
<tr>
<td>Duns and Ros (1963)</td>
</tr>
<tr>
<td>Mukherjee and Brill (1985)</td>
</tr>
<tr>
<td>Beggs and Brill (1973)</td>
</tr>
<tr>
<td>Orkiszewski (1967)</td>
</tr>
</tbody>
</table>

Field data from a gas well in the Alagoas basin in Brazil have also been used to evaluate the proposed model. Natural gas, condensate, and free water have been produced from this 1510-m-long vertical well having a 0.062-m tubing ID. Both the pressure and temperature profiles have been measured along the well in 100-m intervals. Two runs have been conducted, for which the operational conditions are given in Table 4.3. The evaluation has been carried out by utilizing the computer code with the annular flow model to predict the pressure traverse for both runs. The pressure profiles have been calculated starting from the known measured bottomhole pressure. The measured temperature profile has been used as an input to the code.
Comparison between the model predictions and the measured data for the pressure traverse along the well for Run No. 1 and Run No. 2 are shown in Figs. 4.25 and 4.26, respectively. Excellent agreement is observed for both runs between the model predictions and the measured data, with less than 3% error for pressure and less than 10% error for the total pressure drop. Examining the predicted profiles, it can be observed that the profiles are almost linear. This can be explained based on the fluid system. For both runs, the gas velocity was almost constant despite the reduction in pressure. The condensation of heavy ends occurring, caused by pressure and temperature reduction, compensates the gas expansion that is caused by pressure reduction. This results in only a small variation in the pressure gradient along the wellbore.
Fig. 4.25—Comparison between model prediction and field data for Run No. 1.

Fig. 4.26—Comparison between model prediction and field data for Run No. 2.
4.6 Wellbore Overall Model and Evaluation of Methods

Several overall wellbore mechanistic models have been developed. Examples are the Ozon et al. (1987), Hasan and Kabir (1988), Ansari et al. (1994) and Chokshi et al. (1996) models. In this section, the Ansari et al. (1994) overall model is presented, including its evaluation, as well as the evaluation of six most commonly used empirical correlations and one mechanistic model, against a wellbore data bank.

The overall mechanistic model for the prediction of two-phase flow in vertical and sharply inclined wellbores has been formulated by Ansari et al. (1994) by combining the models presented in the previous sections of this chapter. This is carried out in a similar manner as done in the development of the Xiao et al. (1990) overall pipeline model, presented in Sec. 3.5. Similarly, the overall wellbore model first predicts the existing flow pattern for a given set of flow conditions. This is followed up by the prediction of the flow characteristics, including the liquid holdup and pressure gradient to name but two, using separate models for the different existing flow patterns such as bubble flow, slug flow, annular flow, and dispersed-bubble flow. Thus, the overall model can be used to design wellbores, as done previously utilizing empirical correlations.

4.6.1 Overall Wellbore Model. A schematic of the flow behavior along a wellbore is given in Fig. 4.27. As can be seen, the flow pattern varies along the wellbore changing the flow characteristics, including the pressure gradient. In the schematic figure, just as an illustration, the flow pattern at the bottomhole may be single-phase flow because the relatively high pressure causes all the gas to be in solution in the liquid phase. As the flow proceeds upwards, because of the pressure reduction, gas evolves out of the liquid phase in the form of free gas. The flow pattern for these conditions can be bubble flow or dispersed bubble flow, depending on the liquid flow rate. As the flow moves further upwards, the pressure reduces further, resulting in more gas evolving from the liquid phase and expansion of the free gas. For these conditions, the flow pattern may change to slug flow. Further decrease in the pressure along the wellbore, as the flow proceeds further upwards, results in more free
gas and further expansion of the gas-phase, causing the flow to change to churn flow and finally to annular flow. One must realize that this is just a schematic description of the flow behavior in wellbores. In reality, there are gas wells that exhibit the annular flow pattern all along the wellbore (as demonstrated in Sec. 4.5). Similarly, the flow pattern in oil wells may be slug flow over its entire length. Fig. 4.26 also demonstrates the need for two-phase flow pressure traverse computation algorithm code, as explained in Sec. 1.6.3, when analyzing flow in long two-phase flow pipes (pipelines and wellbores). The code should also incorporate a physical properties prediction package to account for the varying fluid properties along the pipe. The computer code based on the Ansari et al. overall model incorporates all the phenomena and analysis presented in this paragraph.

**Flow Pattern Prediction Model.** Taitel et al. (1980) presented the fundamental mechanistic model for flow pattern prediction in vertical upward two-phase flow, as given in Sec. 4.2.1. The model enables the prediction of the existing flow pattern, namely, bubble flow, slug flow, churn flow, annular flow, and dispersed-bubble flow. In a subsequent study, Barnea et al. (1985) extended the original model to enable flow pattern prediction in inclined pipes (as given in Chap. 5). Finally, Barnea (1987) combined these previous models, applicable to different inclination angle ranges, into a unified model applicable to the entire range of inclination angles, namely, between ±90° (as given in Chap. 5). Ansari et al. have utilized the different models reviewed here for the prediction of the different transition boundaries, as follows:

- **Bubble to slug transition:** Taitel et al. model, as given in Eq. 4.13, with the existence criterion given in Eq. 4.15. Note that no transition to churn flow is considered, and the churn flow pattern is analyzed as part of the slug pattern.

- **Transition to dispersed-bubble flow:** Barnea et al. (1985) modified transition boundary, as given in Eq. 4.23.

- **Transition to annular flow:** Barnea (1987) unified model, as given in Sec. 5.2.1.

**Bubble Flow and Dispersed-Bubble Flow Models.** Ansari et al. have utilized the rudimentary bubble flow model for the calculation of the void fraction, as given in Eq. 4.11. The pressure gradient has been determined, as given in Sec. 4.3.2. The homogeneous no-slip model, as given in Sec. 2.1, has been used for the prediction of the flow behavior for dispersed-bubble flow, considering only the gravitational and frictional pressure losses.

**Slug Flow Model.** The Simplified Sylvester (1987) slug flow model has been used, as given in Sec. 4.4.1, with the Vo and Shoham (1989) solution procedure given in Sec. 4.4.2. The following closure relationships have been used in the model:

- The original Sylvester correlation for the liquid holdup in the slug body, $H_{LLS}$.

- A liquid slug length of $L_s = 30d$.

Note that Ansari et al. have used the original Sylvester pressure drop equation, including the accelerational pressure drop. This resulted in an overprediction of the pressure drop, especially for cases where the Taylor bubble is small, and the length of the bubble cap cannot be neglected. For this case, Ansari et al. considered a second model, namely, the developing slug flow model. However, as presented at the end of section 3.4.2, Barnea (1989) showed that for the case of equilibrium liquid film, when the pressure drop in the gas pocket/liquid film zone is neglected, the acceleration pressure drop
term should not be considered. Thus, if the pressure drop is calculated in a consistent manner (Taitel and Barnea, 1990), as given in Eq. 3.155, there is no need for the developing slug model.

**Annular Flow Model.** The Alves et al. model, presented in Sec. 4.5.5, has been used, with the following closure relationships:

- Wallis correlation (1969) for entrainment fraction, $f_E$, as given in Eq. 4.74.
- A combination of two correlations for the dimensionless interfacial friction factor parameter, $I$, depending on the entrainment fraction, as follows: the Wallis (1969) correlation (given in Eq. 4.67) for $f_E > 0.9$; and the Whalley and Hewitt (1978) correlation (given in Eq. 4.68) for $f_E \leq 0.9$.

4.6.2 **Wellbore Data Bank.** The evaluation of the model, as well as the evaluation of six most commonly used empirical correlations and one mechanistic model, has been carried out against the Tulsa University Fluid Flow Projects (TUFFP) updated data bank, which consists of 1,712 field and laboratory cases including a wide range of operational conditions, as given in Table 4.4. The previous TUFFP data bank includes data from Poettmann and Carpenter, Fancher and Brown, Hagedorn and Brown, Baxendell and Thomas, Orkiszewski, Espanol, Messulam, Camacho, and field data from several oil companies. The additional data for the updated data bank are from Govier and Fogarasi, Asheim, Chierici et al., and Prudhoe Bay field. Refer to Ansari et al. (1994) for the references of these data sources.

<table>
<thead>
<tr>
<th>Source</th>
<th>Nominal Diameter, in.</th>
<th>Oil Rate, STBD</th>
<th>Gas Rate, $10^3$ scf/D</th>
<th>Oil Gravity, °API</th>
</tr>
</thead>
<tbody>
<tr>
<td>Old TUFFP* data bank</td>
<td>1–8</td>
<td>0–10,150</td>
<td>1.5–10,567</td>
<td>9.5–70.5</td>
</tr>
<tr>
<td>Govier and Fogarasi</td>
<td>2–4</td>
<td>8–1,600</td>
<td>114–27,400</td>
<td>17–112</td>
</tr>
<tr>
<td>Asheim</td>
<td>2–6</td>
<td>720–27,000</td>
<td>740–55,700</td>
<td>35–86</td>
</tr>
<tr>
<td>Chierici et al.</td>
<td>2–5</td>
<td>0.3–69</td>
<td>6–27,914</td>
<td>8.3–46</td>
</tr>
<tr>
<td>Prudhoe Bay</td>
<td>5–7</td>
<td>600–23,000</td>
<td>200–110,000</td>
<td>24–86</td>
</tr>
</tbody>
</table>

*Includes data from Poettmann and Carpenter; Fancher and Brown; Hagedorn, Baxendell and Thomas; Orkiszewski et al.; and field data from several oil companies. [Refer to Ansari et al. (1994) for references of data sources.]

4.6.3 **Statistical Parameters.** The evaluation is carried out by comparing the measured pressure drop, $\Delta p_M$, with the predicted pressure drop, $\Delta p_C$, for all the points in the data bank. Similarly to Xiao et al. (1990), as discussed in Sec. 3.5.3 and as given in Eqs. 3.174 through 3.181, six statistical parameters have been used in the evaluation process. The first three parameters, $\epsilon_1$, $\epsilon_2$, and $\epsilon_3$, are based on the relative error, $\epsilon_{r,j}$, and are expressed in %. The other three parameters, $\epsilon_4$, $\epsilon_5$, and $\epsilon_6$, are based on the actual error, $\epsilon_i$, given in pressure units (Pascals).
4.6.4 Evaluation of Overall Model and Other Methods. The proposed overall wellbore model has been tested against the previously presented data bank. One other mechanistic model, namely, Hasan and Kabir (1988), and six commonly used correlations have been evaluated as well. These include: Hagedorn and Brown (1965), Duns and Ros (1963), Orkiszewski (1967), Beggs and Brill (1973), Mukherjee and Brill (1985), and Aziz et al. (1972). The evaluation is given in Table 4.5. Comparison between the performance of the overall model and the other seven methods is carried out by defining and using a “relative performance factor,” \( F_{RP} \), which is based on the maxima and minima values of the six statistical parameters, as shown in Eq.4.112.

\[
F_{RP} = \frac{|e_1 - e_{1\text{MIN}}|}{e_{1\text{MAX}} - e_{1\text{MIN}}} + \frac{|e_2 - e_{2\text{MIN}}|}{e_{2\text{MAX}} - e_{2\text{MIN}}} + \frac{|e_3 - e_{3\text{MIN}}|}{e_{3\text{MAX}} - e_{3\text{MIN}}} + \\
\frac{|e_4 - e_{4\text{MIN}}|}{e_{4\text{MAX}} - e_{4\text{MIN}}} + \frac{|e_5 - e_{5\text{MIN}}|}{e_{5\text{MAX}} - e_{5\text{MIN}}} + \frac{|e_6 - e_{6\text{MIN}}|}{e_{6\text{MAX}} - e_{6\text{MIN}}}. \quad \text{(4.112)}
\]

<table>
<thead>
<tr>
<th></th>
<th>1</th>
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<th>3</th>
<th>4</th>
<th>5</th>
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<th>8</th>
<th>9</th>
<th>10</th>
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</tr>
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<tbody>
<tr>
<td>n</td>
<td>1.712</td>
<td>1.086</td>
<td>626</td>
<td>755</td>
<td>1.381</td>
<td>29</td>
<td>1.052</td>
<td>654</td>
<td>745</td>
<td>387</td>
<td>70</td>
</tr>
<tr>
<td>MODEL</td>
<td>0.700</td>
<td>1.121</td>
<td>1.378</td>
<td>0.081</td>
<td>0.000</td>
<td>0.143</td>
<td>1.295</td>
<td>1.461</td>
<td>0.112</td>
<td>0.142</td>
<td>0.000</td>
</tr>
<tr>
<td>HAGBR</td>
<td>0.585</td>
<td>0.600</td>
<td>0.919</td>
<td>0.876</td>
<td>0.774</td>
<td>2.029</td>
<td>0.386</td>
<td>0.485</td>
<td>0.457</td>
<td>0.939</td>
<td>0.546</td>
</tr>
<tr>
<td>AZIZ</td>
<td>1.312</td>
<td>1.108</td>
<td>2.085</td>
<td>0.803</td>
<td>1.062</td>
<td>0.262</td>
<td>1.798</td>
<td>1.764</td>
<td>1.314</td>
<td>1.486</td>
<td>0.214</td>
</tr>
<tr>
<td>BUNRS</td>
<td>1.719</td>
<td>1.678</td>
<td>1.678</td>
<td>1.711</td>
<td>1.792</td>
<td>1.128</td>
<td>2.056</td>
<td>2.280</td>
<td>1.852</td>
<td>2.296</td>
<td>1.213</td>
</tr>
<tr>
<td>HASKA</td>
<td>1.940</td>
<td>2.005</td>
<td>2.201</td>
<td>1.836</td>
<td>1.780</td>
<td>0.009</td>
<td>2.575</td>
<td>2.590</td>
<td>2.044</td>
<td>1.998</td>
<td>1.043</td>
</tr>
</tbody>
</table>

**Data:** EBD=entire databank; 2. VW=vertical well cases; 3. DW=deviated well cases; 4. VNH=vertical well cases without Hagedorn and Brown data; 5. ANH=all well cases without Hagedorn and Brown data; 6. AB=all well cases with 75% bubble flow; 7. AS=all well cases with 100% slug flow; 8. VS=vertical well cases with 100% slug flow; 9. SNH=all well cases with 100% slug flow without Hagedorn and Brown data; 10. VSNH=vertical well cases with 100% slug flow without Hagedorn and Brown data; 11. AA=all well cases with 100% annular flow.

**Correlations:** HAGBR=Hagedorn and Brown correlation; AZIZ=Aziz et al. correlation; DUNRS=Duns and Ros correlation; HASKA=Hasan and Kabir mechanistic model; BEGBR=Beggs and Brill correlation; ORKIS=Orkiszewski correlation.

**Overall Evaluation.** The overall evaluation of the model and the seven other methods against the entire data bank (all 1,712 data points) is shown in Column 1 of Table 4.5. The performance has also been checked for vertical well cases only (1,086 cases), as given in Column 2 of the table, and for deviated well cases only (626 cases), resulting in Column 3 of the table. Also, because the data bank includes 333 data points from Hagedorn and Brown, and to make the comparison unbiased, a second, reduced data bank has been created that excluded these 331 cases, resulting in 1,381 data points. For
this reduced data bank, the results of all vertical wells cases (755 cases) are shown in Column 4, and the results for all 1,381 cases, namely, vertical and deviated wells are given in Column 5.

The following conclusions can be made from the overall evaluation, as given in Columns 1 through 5. When the Hagedorn and Brown data points are excluded, as can be seen from Columns 4 and 5, the performance of the overall model is superior to all other methods considered. However, the performance of the Hagedorn and Brown correlation is comparable to that of the overall model. The excellent performance of the Hagedorn and Brown correlation can be attributed to the extensive data used in its development, and to the significant modifications that have been carried out to the correlation. Also, although the Hagedorn and Brown correlation performed better than all other methods for deviated wells (column 3), none of the methods gave satisfactory results for this evaluation.

**Individual Flow Pattern Models Evaluation.** The overall model consists of separate flow pattern models. To evaluate these models separately, the data bank has been separated into sets of data that have the same dominant flow pattern. Thus, for bubble flow, a separate data set has been created for which bubble flow exists more than 75% of the length of the well (29 cases). The results for bubble flow are presented in Column 6 of Table 4.5. Columns 7 through 10 give the results for the data sets for which slug flow exists through 100% of the well length. Column 7 results are for slug flow cases selected from the entire data bank (1,052 cases), whereby Column 8 is for vertical wells under slug flow, selected from the entire data bank (654 cases). Columns 9 and 10 present similar results as Columns 7 and 8, but for slug flow cases selected from the reduced data bank (excluding Hagedorn and Brown data cases). Column 9 presents the results for all wells in the reduced data bank under slug flow (745 cases), while Column 10 results are for the vertical wells only (387 cases). Finally, Column 11 shows the results selected from the entire data bank for which annular flow exists through the entire well (70 cases). Note that Table 4.5 presents the results in terms of the overall relative performance factor, $F_{RP}$. Complete performance results for the overall model and the other methods evaluated, in terms of all individual statistical parameters, namely, $\varepsilon_i$ through $\varepsilon_6$, can be found in Ansari (1988).

The following conclusions can be reached from the examination of Columns 6 to 11 in Table 4.5, with respect to the evaluation of individual flow pattern models:

1. For the bubble flow pattern, the Hasan and Kabir model gave the best results, while the overall model was second best, as presented in Column 6. The reason for these results is that Hasan and Kabir used the modified bubble flow model, as given in Eq. 4.42, while the Ansari et al. model is based on the rudimentary model given in Eq. 4.11.

2. For the entire data bank, the performance of the slug flow model is exceeded by the Hagedorn and Brown correlation, as shown in Columns 7 and 8. However, for the reduced data bank, when the Hagedorn and Brown data are excluded, the overall model performed the best for all wells (Column 9) and for vertical wells (Column 10).

3. The performance of the annular flow model is significantly superior to all the other methods evaluated, as shown in Column 11.
4.7 Examples

Example 4.1. Water and air flow upwards through a 5.1-cm ID vertical pipe at STP conditions. The in-situ flow rates of the water and air are 2.12 m³/hr and 0.425 m³/hr, respectively. Using the Taitel et al. (1980) model for flow pattern prediction, and assuming the air is introduced into the pipe in a form of a bubbles swarm with a bubble rise velocity of \( v_0 = 0.3 \) m/s, determine:

1. The air void fraction. Show that the flow pattern in the pipe is bubble flow.
2. The air flow rate is increased slowly. At what air flow rate will transition to slug flow occur?

Solution 4.1.

1. Calculation of void fraction:

\[
A = \frac{\pi}{4} (0.051)^2 = 0.0019635 \text{ m}^2,
\]

\[
v_{SL} = \frac{q_L}{A_p} = \frac{2.12}{3600 \times 0.0019635} = 0.3\text{ m/s},
\]

\[
v_{SG} = \frac{q_G}{A_p} = \frac{0.425}{3600 \times 0.0019635} = 0.06\text{ m/s},
\]

\[
v_0 = v_G - v_L = \frac{v_{SG}}{\alpha} - \frac{v_{SL}}{1 - \alpha},
\]

\[
0.3 = \frac{0.06}{\alpha} - \frac{0.3}{1 - \alpha},
\]

\[
10\alpha^2 - 22\alpha + 2 = 0,
\]

and solving for \( \alpha \),

\[
\alpha = \frac{11 \pm \sqrt{121 - 20}}{10} = \frac{11 \pm 10.05}{10} = 0.095.
\]

Thus, as the void fraction is small, namely, \( \alpha < 0.25 \), the flow pattern may be bubble flow. It is necessary to check the existence of bubble flow using the criterion for the existence of transition “E”, which is \( v_{TB} \geq v_0 \). The critical pipe diameter, above which bubble flow occurs, can be calculated from the equality \( v_{TB} = v_0 \), namely, \( 0.35 \sqrt{gd} = 1.53 \left[ g \left( \rho_L - \rho_G \right) \sigma \right]^{0.25} / \rho_L^{1.25} \). This leads to the existence criterion of transition “E” as (Eq. 4.15)

\[
\left[ \frac{g d^2 \rho_L^2}{\left( \rho_L - \rho_G \right) \sigma} \right]^{-0.25} = 4.36.
\]

Solving for the diameter from Eq. 4.15 yields

\[
d = 4.36 \left[ \frac{\rho_L - \rho_G}{g \rho_L^2} \sigma \right]^{0.5}.
\]
Substituting the fluid properties for air-water system at STP gives

\[ d = 19 \left( \frac{(1,000 - 1.2) \times 0.072}{9.8 \times 1,000^2} \right)^{0.5}, \text{ or } d = 0.051 \text{ m}. \]

Thus, for air-water system at STP, bubble flow exists for pipe diameters \( d \geq 0.051 \text{ m} \) if \( \alpha < 0.25 \).

Since the diameter in this problem satisfies the above critical diameter criterion, namely, \( d = 0.051 \text{ m} \), and also the void fraction is \( \alpha = 0.095 \), bubble flow exists.

2. Transition to slug flow will occur when due to the increase in the air flow rate, the gas void fraction increases and reaches the value corresponding to transition “E”, namely, \( \alpha = 0.25 \), as

\[
0.3 = \frac{v_{SG}}{0.25} - \frac{0.3}{0.75},
\]

\[ 4v_{SG} = 0.7, \]

and

\[ v_{SG} = \frac{0.7}{4} = 0.175 \text{ m/s}. \]

Thus, transition to slug flow will occur when the air flow rate is increased to \( v_{SG} = 0.175 \text{ m/s} \).
Chapter 5
Flow in Inclined Pipes

Analysis of two-phase flow in horizontal and slightly inclined pipelines was presented in Chap. 3, with the accepted range of inclination angles between $\pm 10^\circ$. Similarly, analysis for vertical and sharply inclined wellbores, for inclination angle range between $90^\circ$ to $60^\circ$, was presented in Chap. 4. This chapter provides analysis of two-phase flow in the entire range of inclination angles, namely, $-90^\circ \leq \theta \leq 90^\circ$, including downward vertical flow, downward inclined flow, horizontal flow, upward inclined flow and upward vertical flow. This is carried out in two ways, namely, by either extension of the pipeline and wellbore models to the entire range of inclination angles, or by the development of unified models applicable to the entire range of inclination angles or part of it.

Flow pattern maps are presented for upward inclined flow, downward inclined flow and downward vertical flow. These maps, together with Figs. 3.11 to 3.31 for horizontal and slightly inclined flow (inclination angles between $\pm 10^\circ$) and Figures 4.8 and 4.11 for vertical flow, form a complete set of maps for the entire range of inclination angles. The associated physical phenomena of two-phase flow in the entire range of inclination angles are discussed. The flow pattern prediction models for pipelines (Taitel and Dukler, 1976) and wellbores (Taitel et al., 1980) are extended to cover the entire range of inclination angles. Also, a unified model for flow pattern prediction (Barnea, 1986, 1987) applicable to the entire range of inclination angles is presented. Comparisons between both the extended model and the unified model for flow pattern prediction and experimental data are presented. Next, unified models for slug flow (Felizola and Shoham, 1995) and stratified flow (Shoham and Taitel, 1984) are given. Finally, a unified overall model for pipelines and wellbores is presented (Gomez et al., 2000).

5.1 Flow Pattern Transitions in Inclined Pipes

Experimental data and mechanistic modeling of upward and downward inclined flow are scarce, as compared to the horizontal and vertical flow cases. Several investigators reported experimental maps for a limited range of inclination angles, while others presented studies on related topics, such as pressure drop and liquid holdup, but have not considered flow patterns, as reviewed briefly below.

Inclined Flow Studies: Experiments in an air-water system at shallow upward inclination angles were carried out by Singh and Griffith (1970) in 5 pipes, 1.7 to 3.8 cm in diameter, enabling flow pattern observations. Gould (1974) and Gould et al. (1974) published flow regime maps for horizontal and vertical flows and for $45^\circ$ upward inclined flow, using the coordinate system proposed by Duns and Ros (1963). Four flow patterns were observed, including continuous liquid-phase, continuous gas-phase, continuous liquid and gas phases and alternating phases. These flow patterns are equivalent to dispersed-bubble, annular, churn and slug flow, respectively. They concluded that the dispersed-bubble and annular flow regimes do not vary significantly with inclination.

A limited study on flow pattern transition in upward inclined and vertical pipes was conducted by Weisman and Kang (1981). Air-water, air-glycerol and Freon-Freon vapor fluid systems were used. Pipe diameters varied between 1.2 to 2.5 cm, with inclination angles of 0.5, 2, 7, 30, 45 and 90 degrees. The restricted range of data collected cannot provide an overall and systematic evaluation of the phenomena in upward inclined flow. The authors proposed empirical correlations for the transition boundaries to annular flow, to dispersed-bubble flow and between bubble and intermittent flow. Their
overall flow pattern map for vertical and sharply inclined pipes is given in $v_{SL}$ and $v_{SG}$ coordinates, using correction factors for physical properties and geometry ($\phi_1$ and $\phi_2$).

Very few studies have been published on flow pattern transitions in downward inclined flow. Example is the study of Singh and Griffith (1976) conducted for downward inclination between $0^\circ$ to $-9^\circ$ in 1.27, 2.54 and 3.8 cm diameter pipes. They developed correlations for the holdup in each observed flow pattern, and suggested that the transition boundaries between each pair of flow patterns may be found by matching their respective holdup correlations. Their flow regime map utilizes as coordinates the dimensionless volumetric fluxes of the two phases.

**Downward Vertical Studies:** One of the first downward vertical flow regime maps was proposed by Golan and Stenning (1969-1970). The experiments were conducted in an inverted u-tube consisting of a vertical riser followed by a vertical down comer, both 3.8 cm and 3 m long. Their flow regime map is given in the gas and liquid superficial velocity coordinates, and consists of 3 flow regimes: slug and bubble flow, oscillatory flow and annular flow. The observed oscillatory flow was probably caused by exit effects.

An experimental study on flow patterns, liquid holdup and pressure drop in downward vertical air-water flow was carried out by Yamazaki and Yamaguchi (1979) in a 2.54 cm diameter 3.3 m long pipe. Their flow regime map is given in both $G_L$ vs. $G_G$ and $v_M$ vs. $v_{SG} / v_M$ coordinates systems. Yamazaki and Yamaguchi (1979) did not observe the dispersed-bubble flow pattern. Also, the occurrence of slug flow at very low liquid and gas flow rates reported in their map is questionable (in this range annular falling film is the natural flow regime).

Studies of slug flow in air-water systems in downward vertical pipes have been carried out by Martin (1973, 1976). He investigated the single transition boundary from bubble to slug flow in a 14 cm diameter 8 m long pipe. The transition was observed visually and detected by pressure and holdup measurements. Kulov et al. (1979) studied downward vertical annular flow. Based on experimental data taken in a 2 m long pipe of 2.54 cm, they proposed correlations for film thickness, pressure drop and entrainment.

**Studies for Entire Range of Inclination Angles:** Experimental flow pattern maps for air-water flow in the entire range of inclination angles, from vertically upward to vertically downward, were reported by Spedding and Nguyen (1980). The maps are based on data collected in a 4.55 cm diameter 6 m long transparent pipe. Although the maps are presented for the full range of inclination angles, the maps do not form a sequential set, in which the development and changes of the transition boundaries with the inclination angle can be observed. The results are presented in flow pattern maps using two dimensionless coordinates, $(v_M / \sqrt{gd})^{0.5}$ and $v_{SL} / v_{SG}$. A partial explanation for the choice of these particular coordinates is given by an analysis of the transition mechanism from stratified to slug flow.

Beggs and Brill (1973) developed a correlation for the prediction of liquid holdup and pressure drop for the entire range of inclinations, namely, between $-90^\circ$ to $90^\circ$ (see Section 3.1.2). They presented a flow pattern map for horizontal flow using $v_{SL} / v_M$ and $v_M^2 / gd$ as coordinates. The map differentiates among only 3 flow patterns, including segregated flow, intermittent flow and distributed flow. No other maps have been presented.

A systematic experimental and theoretical study on the effect of pipe inclination on flow pattern transition was conducted by Shoham (1982). A 10 m long facility was used, supporting 2.5, 3.8 and 5.1
cm pipes, capable of rotating through the entire range of inclination angles. A comprehensive experimental investigation of flow pattern transitions occurring through the entire range of inclinations, namely, from -90° to 0° to +90°, has been carried out. Emphasis has been placed on identification of the physical mechanistic phenomena, which cause transition among the different flow patterns. The experimental data are presented in the form of flow pattern maps using $v_{SL}$ and $v_{SG}$ as coordinates, which form a complete sequential set of maps from upward vertical to downward vertical flow. The mathematical models of Taitel and Dukler (1976) and Taitel et al. (1980) for horizontal and near horizontal and upward vertical flow, respectively, have been extended to inclined flow, providing a predictive means for determining flow patterns through the entire range of inclinations.

A unified model for the prediction of flow pattern transitions for the entire range of inclination angles was presented by Barnea (1986, 1987). This model provides continuous and smooth variation of the transition boundaries for the entire range of inclination angles.

5.1.1 Flow Pattern Maps for Upward Inclined Flow. The effect of inclination angle on flow pattern transition boundaries for upward inclined flow has been examined by consistently varying the inclination angle in the range 0° to 90° (Shoham, 1982). Flow pattern maps were reported for air-water flow at STP, in both 2.54 and 5.1 cm pipes, for horizontal flow, vertical flow and upward inclinations of 0.25°, 0.5°, 1°, 2°, 5°, 10°, 15°, 20°, 30°, 50°, 70°, 80° and 85°. The maps for inclination angles from horizontal up to 10° are shown in Section 3.2.1, while the maps for vertical flow are given in Section 4.2 by Figs. 4.8 and 4.11. In this section, only the flow pattern maps for upward inclined flow in the inclination angle range $15° \leq \theta \leq 85°$ are presented in Figs. 5.1 to 5.12. A brief discussion of the physical phenomena observed follows.

The major effect of small inclinations from the horizontal on the flow pattern is observed in the transition between stratified to non-stratified flow. Upward inclinations cause the intermittent flow regime to expand, taking place over a wider range of flow conditions. The stratified-intermittent transition is very sensitive to angle of inclination, whereby even for upward inclinations less than 1° the regime of stratified flow shrinks into a small, bell-shaped region of stratified wavy flow. Stratified-smooth flow exists only in very shallow inclinations, less than 0.25°. For upward inclinations higher than about 20°, stratified flow is not observed at all.

In shallow, upward inclinations, the intermittent-annular transition line passes to the left of the bell-shaped stratified region. For increasing upward inclination this transition line shifts slightly towards higher gas flow rates, as the stratified region disappears. For inclinations greater than 20°, the intermittent-annular transition is almost unaffected by the pipe’s inclination angle.

The transition to dispersed-bubble flow is relatively insensitive to the inclination angle. For the 2.54 cm pipe this transition is almost unaffected by the inclination angle throughout the whole range, from 0° to 90°. For the 5.1 cm pipe, however, for inclination angles higher than 60°, the bubble flow regime occurs at low gas and liquid flow rates, similarly to vertical flow. As discussed earlier, the bubble flow region occurs only in large diameter pipes.

Churn flow is typical to vertical upward flow. It persists in off vertical inclination angles down to approximately 70°. As the angle of inclination decreases, the churn region decreases until it shrinks into a very small region at a 70° inclination (for air-water system), and disappears completely for inclination angles less than 70°.
Fig. 5.1—Flow pattern map, 15° upward inclination, 2.54 cm pipe, air-water at STP
—— Experiment       ----- Model (See Fig. 3.11 for legend).

Fig. 5.2—Flow pattern map, 15° upward inclination, 5.1 cm pipe air-water at STP
—— Experiment       ----- Model (See Fig. 3.11 for legend).
Fig. 5.3—Flow pattern map, 20° upward inclination, 2.54 cm pipe, air-water at STP
— Experiment ------Model (See Fig. 3.11 for legend).

Fig. 5.4—Flow pattern map, 20° upward inclination, 5.1 cm pipe, air-water at STP
— Experiment ------Model (See Fig. 3.11 for legend).
Fig. 5.5—Flow pattern map, 30° upward inclination, 2.54 cm pipe, air-water at STP
— Experiment —— Model (See Fig. 3.11 for legend).

Fig. 5.6—Flow pattern map, 30° upward inclination, 5.1 cm pipe, air-water at STP
— Experiment —— Model (See Fig. 3.11 for legend).
Fig. 5.7—Flow pattern map, 50° upward inclination, 2.54 cm pipe, air-water at STP
—— Experiment  ------Model (See Fig. 3.11 for legend).

Fig. 5.8—Flow pattern map, 50° upward inclination, 5.1 cm pipe, air-water at STP
—— Experiment  ------Model (See Fig. 3.11 for legend).
Fig. 5.9—Flow pattern map, $70^\circ$ upward inclination, 2.54 cm pipe, air-water at STP
— Experiment ——— Model (See Fig. 3.11 for legend).

Fig. 5.10—Flow pattern map, $70^\circ$ upward inclination, 5.1 cm pipe, air-water at STP
— Experiment ——— Model (See Fig. 3.11 for legend).
Fig. 5.11—Flow pattern map, 80° upward inclination, 5.1 cm pipe, air-water at STP
— Experiment —— Model (See Fig. 3.11 for legend).

Fig. 5.12—Flow pattern map, 85° upward inclination, 5.1 cm pipe, air-water at STP
— Experiment —— Model (See Fig. 3.11 for legend).
5.1.2 Flow Pattern Prediction for Upward Inclined Flow. Flow pattern transition in slightly inclined pipes (up to 10° in air-water systems) can be predicted by Taitel and Dukler (1976). Above this inclination angle, flow patterns can be predicted by a modification of the upward vertical flow model, presented by Taitel et al. (1980), to include the effect of inclination. This extension has been performed by Barnea et al. (1985), as presented below. The predictions of the modified model are compared to the experimental flow pattern maps, which are presented in the previous section.

**Bubble - Slug Transition (Transition E).** The bubble-slug transition occurs at low liquid rates. Under these conditions the turbulent forces are negligible, and the transition to slug flow is caused by coalescence of bubbles, when the gas void fraction reaches \( \alpha = 0.25 \). The transition boundary given by Taitel et al. (1980) (Eq. 4.13) is modified by taking the component of the bubble rise velocity along the pipe axis, as follows

\[
v_{SL} = 3.0v_{SG} - 1.15 \left[ \frac{g(p_l - p_g)\sigma}{\rho_l^2} \right]^{0.25} \sin \theta. \quad \text{(5.1)}
\]

Similar to the case of vertical flow (Eq. 4.15), bubble flow exists in pipe diameters larger than \( \frac{d}{\left(1547250\frac{\rho_r}{\rho_m}\right)^{0.5}} \) (d > 5 cm for air-water system at STP) and only in the inclination angle range of 60° to 90° (for air-water systems at STP). For inclinations less than 60°, the small bubbles tend to concentrate at the upper pipe wall, due to the buoyancy forces, resulting in coalescence and formation of large bubbles and transition to slug flow.

**Transition to Dispersed-Bubble Flow (Transition F-G).** The transition to dispersed-bubble flow occurs at high liquid flow rates, in which the turbulent forces act against surface tension forces to break the Taylor bubbles into small dispersed-bubbles. This mechanism is independent of pipe inclination and, thus, the transition boundary expression is the same as presented in Chap. 4 for vertical flow, namely,

\[
2 \left[ \frac{0.4\sigma}{(\rho_l - \rho_g)g} \right]^{0.5} \left( \frac{\rho_l}{\sigma} \right)^{0.6} \left( \frac{2 \times 0.046 (\rho_m d)}{d} \left( \frac{\rho_m d}{\mu_m} \right)^{-0.2} \right)^{0.4} \left( \frac{v_m}{v_m} \right)^{3(0.2)0.4} = 0.725 + 4.15 \left( \frac{v_{SG}}{v_m} \right)^{0.5}. \quad \text{(4.23)}
\]

The transition boundary predicted by Eq. (4.23) is valid for void fractions up to \( \alpha_{\text{MAX}} = 0.52 \), where a maximum bubble “packing” is reached. For void fractions greater than 0.52, bubbles cannot exist because of their high concentration, promoting coalescence to form Taylor bubbles. Thus, for higher gas void fractions, \( \alpha > 0.52 \), regardless of the turbulent forces, slug flow occurs, and Eq. (4.23) is not valid. For this case, a simple criterion for transition to dispersed-bubbles is proposed, based on the combination of superficial gas and liquid velocities that yield a no-slip void fraction of 0.52, as follows

\[
\frac{v_{SG}}{v_{SG} + v_{SL}} = 0.52. \quad \text{(4.24)}
\]

**Intermittent - Annular Transition (Transition J).** The transition to annular flow in vertical upward flow is based on the minimum gas velocity required to suspend a droplet of entrained liquid in the gas core (Eq. 4.39). The same concept can be easily modified for inclined pipes by taking the \( g \) component along the pipe axis. Thus, the criterion for the existence of annular flow is
\[
\frac{v_{SG} \rho_G^{0.5}}{[\sigma g \sin \theta (\rho_L - \rho_G)]^{0.25}} = 3.1, \quad \text{or} \quad v_{SG} = \frac{3.1 [\sigma g \sin \theta (\rho_L - \rho_G)]^{0.25}}{\rho_G^{0.5}}.
\] 

**Slug - Churn Transition (Transition H).** Inclination angle has a substantial effect on this transition boundary, suppressing the chaotic nature of churn flow by enhancing the separation between the gas and liquid phases. As a result, the region of churn flow shrinks considerably in off vertical inclinations, and disappears at about 70°. Thus, as a first approximation, it is suggested that the narrow region of churn flow is considered as part of the intermittent pattern.

The transition boundaries of the proposed modified model are given in **Figs. 5.1 to 5.12** by the dashed lines. Comparison between the experimental results and the predicted boundaries shows quite a good agreement.
5.1.3 Flow Pattern Maps for Downward Inclined Flow. Flow pattern maps were reported for 2.54 and 5.1 cm pipes for downward inclination angles of $-1^\circ$, $-5^\circ$, $-10^\circ$, $-30^\circ$, $-50^\circ$, $-70^\circ$ and $-80^\circ$ (Shoham, 1982). The maps for inclination angles from horizontal up to $-10^\circ$ are shown in Section 3.2.1, while the maps for downward vertical flow are given in Section 5.1.5. In this section the flow pattern maps for downward inclination angles of $-30^\circ$, $-50^\circ$, $-70^\circ$ and $-80^\circ$, for air-water flow at STP in 2.54 and 5.1 cm pipes, are presented in Figs. 5.13 to 5.20. Discussion of the physical phenomena for downward inclined flow, as exhibited in these figures follows.

Downward inclinations affect the stratified flow regime significantly. In downward inclined stratified flow the liquid moves more rapidly than in the horizontal case, owing to downward gravity forces. Thus, for downward inclined stratified flow, the liquid level in the pipe is smaller, as compared to horizontal flow. As a result, higher liquid flow rates are required to cause transition from stratified flow, and the stratified flow region is considerably expanded as the angle of inclination increases. This change takes place primarily at the lower inclination angles range, from $0^\circ$ to $-10^\circ$, whereas in the range from $-10^\circ$ to $-70^\circ$ the region of stratified flow is almost unchanged. From about $-70^\circ$ downward inclination, increasing the inclination angle results in a gradual change from stratified to annular falling-film flow. The annular region is expanded and as a result the stratified region shrinks until it disappears completely at vertical downward flow.

Stratified-smooth flow is observed up to $-1^\circ$ for the 2.5 cm pipe and up to $-5^\circ$ for the 5.1 cm pipe. As compared to the case of horizontal or slightly inclined upward flow, for downward flow, waves can be generated not only by the gas action on the interface (at high gas flow rates), but also due to natural instability of stratified flow (at low gas flow rates). For the later case, the transition for low gas flow rates from a smooth to wavy interface depends only on the liquid rate, as demonstrated by the “horizontal” transition line (for $-1^\circ$ inclination, see Chap. 3), independent of the gas flow rates.

As a result of the expansion of the stratified region with downward inclinations, the intermittent region shrinks considerably for inclinations up to about $-10^\circ$. For downward inclination angles larger than $-10^\circ$, the region of intermittent flow is observed only at very high liquid and gas flow rates, while at lower gas flow rates a direct transition from stratified to dispersed bubble flow occurs, as shown in Figs. 5.13 and 5.14.

The transition to annular flow in the inclination angle range between $0^\circ$ and $-70^\circ$ takes place at high gas flow rates. At lower inclinations, between $-70^\circ$ to $-90^\circ$, annular flow appears also at low gas flow rates in the form of falling-film. This phenomenon was observed by Yamazaki and Yamaguchi (1979), who defined it as a separate, new flow pattern, termed “wet wall” regime. According to the flow pattern definitions given in Chap. 1, this flow configuration belongs to the annular regime, since it satisfies the continuity of the gas core. As shown in Figs. 5.17 to 5.20, the transition from stratified to annular flow, at low gas flow rates, depends mainly on the liquid flow rate, as exhibited by the “horizontal” transition line. Also, it is observed from these figures that as the inclination angle approaches the downward vertical case, the annular region expands considerably, while the stratified region shrinks until it disappears completely at a $-90^\circ$ inclination.

The transition to dispersed-bubble flow in downward inclinations occurs at high liquid flow rates due to the turbulent forces that shatter the gas-phase into small bubbles and disperse them into the continuous liquid-phase. As shown, the dispersed-bubble transition is almost unaffected by the inclination angle.
Fig. 5.13—Flow pattern map, –30° downward inclination, 2.54 cm pipe, air-water at STP
— Experiment       -----Model (See Fig. 3.11 for legend).

Fig. 5.14—Flow pattern map, –30° downward inclination, 5.1 cm pipe, air-water at STP
— Experiment       -----Model (See Fig. 3.11 for legend).
Fig. 5.15—Flow pattern map, −50° downward inclination, 2.54 cm pipe, air-water at STP

- Experiment

-----Model (See Fig. 3.11 for legend).

Fig. 5.16—Flow pattern map, −50° downward inclination, 5.1 cm pipe, air-water at STP

- Experiment

-----Model (See Fig. 3.11 for legend).
Fig. 5.17—Flow pattern map, −70° downward inclination, 2.54 cm pipe, air-water at STP
—— Experiment ——— Model (See Fig. 3.11 for legend).

Fig. 5.18—Flow pattern map, −70° downward inclination, 5.1 cm pipe, air-water at STP
—— Experiment ——— Model (See Fig. 3.11 for legend).
Fig. 5.19—Flow pattern map, –80° downward inclination, 2.54 cm pipe, air-water at STP
—— Experiment       ------Model (See Fig. 3.11 for legend).

Fig. 5.20—Flow pattern map, –80° downward inclination, 5.1 cm pipe, air-water at STP
—— Experiment       ------Model (See Fig. 3.11 for legend).
5.1.4 Flow Pattern Prediction for Downward Inclined Flow. The model of Taitel and Dukler (1976) for flow pattern transitions in horizontal and slightly inclined flow can be used for downward inclination angles up to \(-10^\circ\), with the exception of the transition between stratified-smooth to stratified-wavy at low gas flow rates, as discussed in the previous section. The modification of the transition boundary between stratified smooth to stratified wavy regimes, as well as the extension of the Taitel and Dukler (1976) model to downward inclination below \(-10^\circ\), has been carried out by Barnea et al. (1982a), as given below. The modified model predictions are compared to the experimental flow pattern maps presented in the previous section.

**Stratified - Non Stratified Transition (Transition A).** This transition boundary can be predicted accurately by the Taitel and Dukler (1976) model for the whole range of downward inclination angles. However, at inclinations below \(-10^\circ\) the transition boundary is terminated by the transition boundary to dispersed bubble flow, while at inclinations below \(-70^\circ\) it is terminated by the stratified to annular (falling film) transition boundary. The comparison between the experimental boundaries with the predicted boundaries for this transition shows very good agreement for the entire range of inclination angles.

**Intermittent - Annular Transition (Transition B).** This transition occurs when stratified flow is unstable, for low liquid and high gas flow rates, whereby no sufficient liquid is available to form slugs. The modified Barnea et al. (1980) criterion is used for the prediction of this transition boundary, namely,

\[
\frac{h_L}{d} = 0.35. \quad \text{................................................................. (3.51)}
\]

As this transition is applicable only when stratified flow is unstable, it is terminated by the stratified-non-stratified boundary at moderate inclinations, as shown in Figs. 5.13 and 5.14. However, at steeper inclinations, from \(-70^\circ\) to \(-90^\circ\), it continues down to very low gas flow rates, as shown in Figs. 5.17 to 5.20. Again good agreement is observed between theory and experiment.

**Transition to Dispersed-Bubble Flow (Transition F-G).** As presented for upward inclined flow, the transition boundary developed for vertical flow (Eqs. 4.23 and 4.24) is independent of pipe inclination, and, therefore, can be used for downward inclined flow, as well.

**Stratified-Smooth to Stratified-Wavy Transition (Transitions C and K).** Stratified-smooth flow occurs in downward inclinations only in slightly inclined pipes (up to \(-5^\circ\) for air-water system at STP). Transition “C”, given by Eq. 3.53 or 3.54, is applicable to horizontal and upward inclined flow, and for downward inclined flow at relatively high gas flow rates, whereby the waves are generated by the interfacial shear. However, for downward inclination angles, waves can also be generated due to instability of the interface, even at low gas flow rates. For these conditions, Barnea et al. (1982) suggested using a criterion for wave generation in terms of a critical Froude number, as follows:

\[
Fr = \frac{v_L}{\sqrt{gh_L}} > 1.5. \quad \text{................................................................. (3.56)}
\]

The values of \(h_L\) and \(v_L\) can be calculated as in the Taitel and Dukler (1976) model. Thus, the transition boundary between stratified-smooth to stratified-wavy for low gas flow rate conditions is predicted by Eq. 3.56 (transition “K”) and by Eq. 3.53 or 3.54 for high gas flow rates (transition “C”). The prediction of transition boundary K-C compares well with the experimental results, as can be seen.
in Figs. 3.26 and 3.27. For larger downward inclinations, less than $-5^\circ$ inclination, both theory and experiments show that the stratified smooth flow pattern does not exist.

**Stratified - Annular (Falling-Film) Transition (Transition L).** For steep inclination angles between $-70^\circ$ and $-90^\circ$, annular flow appears also at negligible gas flow rates, in the form of falling-film, depending mainly on the liquid flow rate and inclination angle. This transition is shown in Figs. 5.17 to 5.20 by the “horizontal” boundary.

The observed mechanism by which stratified flow undergoes transition into annular flow at low gas flow rates is through liquid particles, which are torn away from the stratified-wavy flow turbulent interface, and are thrown upwards. Once the flow energy is sufficient to throw the liquid particles all the way to the upper pipe wall, the impinging liquid droplets will form a continuous liquid film on the upper pipe wall, causing transition to annular flow. Clearly, for these conditions the liquid film flowing on the upper pipe wall is much thinner than the film at the bottom of the pipe. When the flow energy is not sufficient, at lower liquid flow rates or at shallow inclination, this phenomenon causes the stratified interface to become concave. The concaved wedges of the liquid climb the tube periphery as the liquid flow rate or inclination angle increase. Finally, the entire tube periphery becomes continuously wetted and annular flow occurs.

![Fig. 5. 21—Trajectory of a torn away liquid particle in downward stratified flow.](image)

Referring to Fig. 5.21, a liquid particle at the stratified flow interface is considered to be thrown radially with a velocity $v'$, while having an axial velocity $v_L$ (the liquid film velocity). The radial distance which the particle passes depends only on its initial velocity $v'$ and the radial component of gravity. It is assumed that $v'$ is of the order of the average radial turbulent velocity fluctuation, which can be estimated by (Taitel and Dukler, 1976),

$$\overline{v'} = \left(\overline{v'^2}\right)^{0.5} = v_L \left(\frac{f_L}{2}\right)^{0.5}. \quad \text{(3.61)}$$

The maximum radial velocity of the particle is assumed to be twice the average velocity, namely,
\[ v' = 2\sqrt{v} = 2v_L \left( \frac{f_L}{2} \right)^{0.5}. \]  

(5.3)

With this initial velocity, the radial distance that the particle passes is

\[ Sr = \frac{v'^2}{2g \cos \theta}. \]  

(5.4)

The criterion for annular flow to exist is \( Sr > d - h_L \), namely, that the particle reaches the upper pipe wall and wets it. Rearranging yields the final form of the transition boundary, as follows

\[ v_L^2 > \frac{g d \cos \theta (1 - h_L/d)}{f_L}, \]  

(5.5)

where \( v_L \), \( f_L \), \( h_L / d \) are calculated as in the Taitel and Dukler model. This transition is designated as transition “L”. Comparison between the prediction of this transition and the experimental data in Figs. 5.16 to 5.20 shows good agreement.

Thus, for downward inclined flow, the stratified-annular transition is composed of two boundaries. The first, given by Eq. 5.5, occurs at steep inclination angles, at high liquid flow rates and relatively low gas flow rates. The second boundary, given by Eq. 5.2, is the “normal” transition to annular flow occurring at high gas flow rates. The first transition boundary is represented by an almost “horizontal” line, while the second one is presented by an almost “vertical” line.

5.1.5 Flow Pattern Maps for Downward Vertical Flow. Downward vertical flow regime maps for 2.54 and 5.1 cm pipes, for air-water at STP, are presented in Figs. 5.22 and 5.23, respectively. As can be seen, only three flow regimes are observed: annular flow, slug flow and dispersed-bubble flow.

The natural and dominant flow pattern in downward vertical flow is annular flow, which takes the form of falling-film at low gas flow rates, and typical “normal” annular flow at high gas flow rates.

For low gas flow rates, the transition from annular to slug flow occurs at a relatively constant liquid flow of \( v_{SL} \approx 0.6 \text{ m/s}. \) At high gas rates the transition boundary shifts slightly towards higher liquid flow rates.

At even higher liquid flow rates, transition to dispersed-bubble flow occurs. For the 2.54 cm pipe this transition is almost at the same position, as in the case of horizontal or vertical upward flow. For the 5.1 cm pipe the transition occurs at relatively lower liquid flow rates, due to the occurrence of bubble flow, and the slug flow region shrinks somewhat as a result.

5.1.6 Flow Pattern Prediction for Downward Vertical Flow. A model for the prediction of flow pattern transition in downward vertical flow was proposed by Barnea et al. (1982 b), as given below.

When liquid at a low flow rate is introduced into a vertical downward pipe, without gas, it flows as a symmetrical annular falling-film. When gas is introduced cocurrently with the liquid, the gas flows along the pipe core. Therefore, the process of analyzing the transition boundaries between flow patterns in downward flow starts from the condition of annular falling-film flow. Criteria are then developed for the transition from annular to slug and from slug to dispersed-bubble flow. The first step is to develop a relationship between the film thickness and the other flow parameters for annular flow.
Fig. 5.22—Flow pattern map, –90° downward inclination, 2.54 cm pipe, air-water at STP
— Experiment —— Model (See Fig. 3.11 for legend).

Fig. 5.23—Flow pattern map, –90° downward inclination, 5.1 cm pipe, air-water at STP
— Experiment —— Model (See Fig. 3.11 for legend).
Equilibrium Annular Falling-Film. The analysis presented in this section is similar to the one presented by Alves et al. (1991) for normal annular flow (see Section 4.5.5). However, for this case neither entrainment nor annular flow interfacial friction factor are considered, as the falling-film annular flow occurs at low gas flow rates. Refer to Fig. 5.24 for equilibrium annular falling-film flow.

![Schematic of downward vertical annular falling-film flow.](image)

Momentum balances on the liquid and gas phases yield, respectively,

\[-A_L \left(\frac{dP}{dL}\right)_L = -\tau_{W_L} S_L + \tau_I S_I + \rho_L A_L g = 0, \quad \text{...............................}(5.6)\]

and

\[-A_G \left(\frac{dP}{dL}\right)_G = -\tau_I S_I + \rho_G A_G g = 0. \quad \text{...............................}(5.7)\]

Equating the pressure gradients of the two phases results in

\[\tau_I S_I \left(\frac{1}{A_L} + \frac{1}{A_G}\right) + g(\rho_L - \rho_G) - \tau_{W_L} \frac{S_L}{A_L} = 0. \quad \text{...............................}(5.8)\]

Substituting for the geometrical relationships
\[ S_L = \pi d, \]
\[ S_I = \pi(d - 2\delta_L), \]
\[ A_L = \pi(d\delta_L - \delta_L^2), \] and
\[ A_G = \pi(d/2 - \delta_L)^2, \] .................................................................(5.9)

into Eq. (5.8) gives
\[ \tau_l \frac{1}{d(\delta_L - \delta_L^2)(1 - 2\delta_L^2)} + g(\rho_L - \rho_G) - \tau_{wl} \frac{1}{d(\delta_L - \delta_L^2)} = 0 \] ...........................................(5.10)

where \( \delta_L = \delta_l/d \) is the dimensionless film thickness. The shear stresses are evaluated by
\[ \tau_{wl} = f_{L} \rho_L v_L^2 \]

\[ \tau_{l} = f_{I} \rho_G (v_G - v_l)^2, \] .....................................................(5.11)

with the liquid and the interfacial friction factors calculated based on the hydraulic diameter concept, utilizing the Blasius equation. Note that for this case, the interfacial friction factor is approximated by a gas-phase friction factor, as follows
\[ f_{L} = C_L \left( \frac{d_L v_L \rho_L}{\mu_L} \right)^n \]

\[ f_{I} \approx f_{G} = C_G \left( \frac{d_G v_G \rho_G}{\mu_G} \right)^m, \] ...............................................(5.12)

where the hydraulic diameters of the liquid and gas phases are given, respectively, by
\[ d_L = \frac{4A_L}{S_L} = 4d(\delta_L - \delta_L^2), \] and \[ d_G = \frac{4A_G}{S_G} = (1 - 2\delta_L)d, \] ..................................(5.13)

and the respective phase velocities are
\[ v_L = \frac{v_{SL}}{4(\delta_L - \delta_L^2)}, \]

\[ v_G = \frac{v_{SG}}{1 - 4\delta_L + 4\delta_L^2}. \] ..................................................(5.14)

The commonly used values of \( C_G = C_L = 0.046 \) and \( m = n = 0.2 \) are utilized for turbulent flow, and \( C_G = C_L = 16 \) and \( m = n = 1 \) for laminar flow.

A solution of Eq. (5.10), with the auxiliary relationships in Eqs. 5.11 to 5.14, yields the film thickness as a function of the superficial liquid and gas velocities, the fluids physical properties and pipe diameter.

**Annular - Slug Transition (Transition B):** As the liquid flow rate increases, the liquid film thickness increases too, and larger waves appear on the interface. These waves can cause a temporary bridging of the pipe, which can lead to the formation of stable slug and transition to slug flow. The criterion for this transition is based on the same concept as in Taitel and Dukler (1976). A stable slug is formed when the supply of liquid from the film to the wave is large enough to provide the liquid needed to bridge the pipe and maintain a stable slug. It is assumed that when the liquid holdup in the slug is twice the liquid holdup in annular flow, the resulted liquid slugs are stable, and the flow pattern is that of slug flow. Assuming further that an average liquid holdup in a slug is 0.7, the transition to slug flow occurs at
The transition boundary based on the above criterion is plotted in Figs. 5.22 and 5.23. Good agreement with the experimental data is shown for \( d = 2.54 \) cm, while for \( d = 5.1 \) cm slightly higher values are predicted.

**Transition to Dispersed-Bubble Flow (Transition F-G):** As presented for upward and downward inclined flow, the transition boundary developed for vertical flow (Eqs. 4.23 and 4.24) is independent of pipe inclination and, therefore, can be used for vertical downward flow, as well. The predicted transition boundary to dispersed-bubble flow is given in Figs. 5.22 and 5.23.

For the 5.1 cm pipe the transition line is composed of four different sections, which depends on the different mechanisms involved. Proceeding from right to left in Figure 5.23, the first section is the transition line based on the maximum possible bubble-packing concept (Eq. 4.24, transition “G”), below which slug flow occurs. The second section is the transition line caused by the turbulent breakup of bubbles, as given by Eq. 4.23 (transition “F”). The third section is the transition to bubble flow at lower liquid rates, which is given by Eq. 5.1 (transition “E”). Finally, the fourth section is the extension of the transition to annular falling-film flow, for low gas rates.

For the 2.54 cm pipe, the transition to dispersed-bubble, as shown in Fig. 5.22, consists of only two sections. The maximum bubble packing (Eq. 4.24, transition “G”) line and the turbulent breakup of bubbles section (Eq. 4.23, transition “F”). The predicted transition boundaries agree well with the experimental results, both by trend and value.

5.2 Unified Modeling for Flow Pattern Prediction

Models for flow pattern prediction have been presented in previous chapters, which are applicable only to some range of inclination angles, including:

- Horizontal and slightly inclined flow (Taitel and Dukler, 1976, see Section 3.2).
- Upward vertical and sharply inclined flow (Taitel et al, 1980, see Section 4.2).
- Upward and downward inclined flow and vertical downward flow (extension of horizontal and upward vertical models, Barnea et al., 1985, 1982a and 1982b, see Section 5.1).

While providing flow pattern prediction for the entire range of inclination angles, the above models have one disadvantage. As the inclination angle varies, one must switch from one model to another, whereby discontinuity in the mechanism and transition boundary occurs, especially for annular flow and dispersed-bubble flow. It is desirable and practical to develop a unified model for the prediction of the transition boundaries between the different flow patterns, applicable to the entire range of inclination angles, namely, from \(-90^\circ\) to \(0^\circ\) to \(90^\circ\). The same transition mechanisms are applicable for the entire range of inclination angles. With this approach, the transition boundaries will show a smooth behavior as the inclination angle varies, eliminating the discontinuity problem occurring when switching between different models and mechanisms.

The main transition boundaries occurring in the entire range of inclination angles are the stratified to non-stratified transition, the transition to annular flow and the transition to dispersed-bubble flow. The transition from stratified to non-stratified flow occurs in horizontal, upward inclined flow (up to \(10^\circ\)) and in all the range of downward inclined flow (excluding \(-90^\circ\)). As shown in Section 5.1, the Taitel...
and Dukler (1976) transition equation agrees very well with the data for the entire range of inclinations where stratified flow occurs. Thus, this transition mechanism is unified and requires no further development. In this section the unified model presented by Barnea (1986), applicable for the entire range of inclination angles, for the transitions to annular flow and to dispersed-bubble flow, is presented.

5.2.1 Transition to Annular Flow (Transition J). Two different mechanisms have been proposed for the transition to annular flow. For pipeline conditions, Taitel and Dukler (1976) proposed that annular flow occurs when stratified flow is unstable and the liquid level in the pipe is low so that no complete bridging of the pipe by the liquid-phase can occur (leading to slug flow). Under these conditions, the liquid is swept around the pipe wall in the form of a film, resulting in annular flow. For wellbore conditions, Taitel et al. (1980) suggested a different mechanism, namely, that annular flow exists when the gas-phase velocity is sufficient to lift the largest stable droplet in order to maintain annular flow. Following is the description of the unified transition boundary to annular flow, as presented by (Barnea 1986).

The physical model is given in Fig. 5.25. The gas flows in the core (no entrainment is assumed), while the liquid-phase flows in the form of annular film with a uniform thickness around the pipe periphery. Transition to slug flow occurs when liquid waves/lumps block the gas core, destroying the annular structure and creating a stable slug.

Fig. 5.25—Schematic of Annular Flow.

Momentum balances are carried out for the liquid film and the gas core, similarly to Alves et al. (1991). However, no entrainment is assumed in this study. A momentum (force) balance on the liquid film for steady-state annular flow yields

\[ -A_L \frac{dP}{dL} - \tau_{wL} S_L + \tau_L S_f - \rho_L A_L g \sin \theta = 0. \]

A similar balance on the gas-phase in the core is given by
\[ -A_G \frac{dp}{dL} - \tau_i S_i - \rho_g A_G g \sin \theta = 0. \]  \hspace{1cm} (5.17)

Equating and eliminating the pressure gradient from both liquid and gas momentum equations yields the combined momentum equation for annular flow, as follows:

\[ \tau_i S_i \left( \frac{1}{A_L} + \frac{1}{A_G} \right) - g(\rho_L - \rho_G) \sin \theta - \tau_{WL} \frac{S_L}{A_L} = 0 \]  \hspace{1cm} (5.18)

The geometrical relationships are as given before, namely,

\[ S_L = \pi d, \]
\[ S_i = \pi (d - 2\delta_L), \]
\[ A_L = \pi (d \delta_L - \delta_L^2), \]
\[ A_G = \pi (d/2 - \delta_L^2). \]  \hspace{1cm} (5.9)

The liquid shear stress is

\[ \tau_{WL} = f_L \frac{\rho_L v_L^2}{2}, \]  \hspace{1cm} (5.11)

where the friction factor \( f_L \) is calculated as given in Eqs. 5.12 and 5.13. Substituting the geometrical relationships and the liquid shear stress expression into Eq. 5.18 and solving for the interfacial shear stress yields

\[ \tau_i = g(\rho_L - \rho_G) d \sin \theta (\delta_L^2 - \delta_L^2) (1 - 2\delta_L) + \frac{1}{32} C_i \rho_L \left( \frac{\rho_L}{\mu_L} \right)^{\frac{2}{n}} \left( \frac{1 - 2\delta_L}{(\delta_L - \delta_L^2)^2} \right) \]  \hspace{1cm} (5.19)

Eq. 5.19 relates the required interfacial shear stress to maintain the annular flow structure, \( \tau_i \), to the dimensionless film thickness, \( \delta_L \), for a given superficial liquid velocity, \( v_{SL} \). This relationship for upward vertical flow in a 2.54 cm pipe with air-water at STP, is shown by the solid line in Fig. 5.26, for superficial liquid velocities between 0.01 to 1.5 m/s. Note that \( \tilde{\tau}_i \) is the dimensionless interfacial shear stress given by \( \tilde{\tau}_i = \tau_i / (\rho_L - \rho_G) g d \).

The supplied interfacial shear stress is provided by the gas-phase. The interfacial shear stress for annular flow is given by

\[ \tau_i = \frac{1}{2} f_i \rho_G \frac{v_{SG}^2}{(1 - 2\delta_L)^4}, \]  \hspace{1cm} (5.20)

where the gas velocity in the core is given by

\[ v_G = \frac{v_{SG}}{(1 - 2\delta_L)^3}. \]  \hspace{1cm} (5.21)

The interfacial friction factor used is the one developed by Wallis (1969), as given by
\[ f_I = f_{SG}(1 + 300\tilde{\delta_L}), \]  

where the superficial gas friction factor is determined by

\[ f_{SG} = C_G \left( \frac{\rho_G v_{SG} d}{\mu_G} \right)^m \]  

Eq. 5.20 relates the provided interfacial shear stress to the dimensionless film thickness for a given superficial gas velocity, \( v_{SG} \). This relationship is given by the dashed line in Fig. 5.26 for superficial gas velocities between 6 to 25 m/s. Any intersection between the solid line (constant \( v_{SL} \)) and the dashed line (constant \( v_{SG} \)) is a possible solution for steady-state annular flow (provided that annular flow exists), for the given set of operational conditions, including the particular pair of superficial velocities. The solution for the film thickness is designated as \( \tilde{\delta}_{L,OPR} \).

Transition from annular to slug flow occurs when the liquid blocks the gas core promoting slugging. Blockage of the gas core can occur due to two different mechanisms, namely, (1) instability of annular flow configuration; and, (2) spontaneous blockage of the core by wave growth on the liquid film. These two mechanisms are discussed next.

Fig. 5.26—Steady-state solutions for vertical annular flow in 2.54 cm pipe with air-water at STP.
The Instability Criterion. Referring to Fig. 5.26, the solid lines representing the required interfacial shear stress (solution of Eq. 5.19), display the following behavior: For large values of \( v_{SL} \), the required \( \tau_l \) increases as the film thickness decreases, which is the expected behavior. However, for small values of \( v_{SL} \), the curves display a minimum, as presented by the \( v_{SL} = 0.1 \text{ m/s} \) line. The branch to the left of the minimum, which is represented by the point A, is the stable steady-state solution, corresponding to upward velocity of the liquid film. The branch to the right of the minimum (point B), is the unstable solution, corresponding to downward velocity of the liquid film. For this case, the annular flow configuration is unstable, with backwards flow of the film, resulting in liquid accumulation and blockage of the core and transition to slug flow. The locus of all the minima points represents the transition boundary between annular flow and slug flow, due to the instability mechanism.

The condition at the minima points is obtained by differentiating Eq. 5.19 with respect to \( \delta_L \) and equating it to zero \((\frac{\partial \tau_l}{\partial \delta_L} = 0)\), yielding

\[
g(\rho_L - \rho_G) d \sin \theta \left[ (1-2\delta_L)^2 - 2(\delta_L - \delta_L^2) \right] - \frac{1}{16} C_L \rho_L \left( \frac{\rho_L d}{\mu_L} \right)^{-n} (v_{SL})^{2-n} \left[ \frac{(\delta_L - \delta_L^2)^2 + (1-2\delta_L)^2}{(\delta_L - \delta_L^2)^2} \right] = 0. (5.24)
\]

Equation 5.24 gives the value of the film thickness at the minima point, namely, \( \delta_{L,MIN} \) for a given value of \( v_{SL} \). Simultaneous solution of Eqs. 5.19 and 5.20, with \( \delta_{L,MIN} \) that satisfies Eq. 5.24, yields the corresponding value of \( v_{SG} \) on the transition boundary. The locus of the superficial liquid and gas velocities at the minima points represent the neutral stability condition corresponding to the transition between annular flow and slug flow, as given by the dotted line in Fig. 5.26.

Spontaneous Blockage due to Wave Growth Criterion. Spontaneous blockage of the gas core may also occur due to wave growth on the liquid film. This may happen at relatively higher liquid flow rates, when the liquid film is thick enough to provide sufficient liquid supply needed for wave growth. The large wave can cause the formation of a liquid bridge across the pipe cross-sectional area that blocks the gas passage, resulting in a transition to slug flow. The criterion for this transition is based on the minimum liquid holdup in the slug body, namely, \( H_{LLS,MIN} = 0.48 \). The value of 0.48 corresponds to the maximum possible packing of bubbles. Smaller values of liquid holdup, less than 0.48, correspond to large void fractions in the slug body, under which conditions it is impossible to form a competent bridging of the pipe, which is necessary for slug flow. Similar to the transition to slug flow in pipelines presented by Taitel and Dukler (1976), the transition to slug flow caused by blockage occurs when

\[
\frac{A_L}{A_p H_{LLS,MIN}} = \frac{H_L}{H_{LLS,MIN}} \geq 0.5 \quad \text{................................................................. (5.25)}
\]

Note that the value of \( H_{LLS,MIN} = 0.48 \) corresponds to the actual liquid holdup in the slug only on the transition boundary to annular flow, and is not valid for fully developed slug flow, away from the transition boundary. The final transition boundary is given by

\[
H_L \geq 0.24, \quad \text{or} \quad \delta_L \geq 0.065, \quad \text{................................................................. (5.26)}
\]

and is shown as the vertical broken line in Fig. 5.26. The liquid holdup for annular flow is given by
The Combined Criteria. The combined criteria are illustrated in Fig. 5.26. The figure presents the line connecting all the minima points corresponding to the instability criterion predicted by Eq. 5.24 (dotted line), as well as the line predicted by Eq. 5.26 (broken line), corresponding to the criterion of wave growth and pipe blockage. These two lines represent the combined criteria, namely, annular flow exists on the left-hand side, region of these two lines. On the right-hand side region of these two lines slug flow occurs, due to either instability of annular flow (occurring at relatively lower liquid flow rates) or blockage of the pipe cross-sectional area (occurring at relatively higher liquid flow rates).

As an illustration, point “A” in Fig. 5.26, corresponding to \( v_{SL} = 0.1 \text{ m/s} \) and \( v_{SG} = 25 \text{ m/s} \), for which conditions annular flow exists. Keeping the superficial liquid velocity constant, while decreasing the superficial gas velocity, the operational conditions move along the stable branch toward higher film thickness values. The minimum point for this case occurs at \( v_{SG} = 15 \text{ m/s} \), which represents the neutral stability for this condition. Stable annular flow will occur for all superficial gas velocity corresponding to the stable branch, namely, \( v_{SG} > 15 \text{ m/s} \) for this case. Decreasing the gas flow rates below 15 m/s results in an unstable annular flow (instability criterion), namely, transition to slug flow, as illustrated by point “B”. However, as can be seen from the figure, the location of the minima points shift to the right for increasing superficial liquid velocities. Thus, for higher superficial liquid velocities, such as \( v_{SL} = 0.5 \text{ m/s} \), upon decreasing the superficial gas velocity while keeping the superficial liquid velocity constant, (moving along the stable branch) it is possible to have a transition to annular flow due to wave growth before reaching the minimum neutral stability point. This is illustrated by point “C” for which the superficial gas velocity is about 15 m/s. Note that for \( v_{SL} = 0.5 \text{ m/s} \), the minimum point occurs at superficial gas velocity of about 10 m/s.

Effect of Pipe Inclination. Figure 5.27 presents results for \( \tau_{f} \) vs. \( \delta_{L} \) along lines of constant \( v_{SL} \), as predicted by Eq. 5.19, for the entire range of inclination angles, for air-water flow in a 2.54 cm pipe at STP. As can be seen from the figures, a minimum in the \( \tau_{f} \) curves, which separates the stable and the unstable branches, occurs only in upward flow, and does not occur in horizontal and downward flow. For upward flow, the minima points occur at relatively low superficial liquid velocities. As the inclination angle deviates from the vertical towards the horizontal, the minima points shift towards lower values of \( \tau_{f} \) and higher values of \( \delta_{L,MIN} \), until they disappear in horizontal flow. Also, no minima points in \( \tau_{f} \) are exhibited at higher superficial liquid velocities. Thus, for vertical upward flow, both the instability and wave growth mechanisms occur. However, as the pipe is declined from the vertical, the conditions for the occurrence of the transition due to instability reduce and are limited to very low liquid flow rates, until it disappears at horizontal flow. At horizontal and downward inclinations, as the flow is always stable, the transition from annular to slug flow occurs only due to the wave growth mechanism only.
Comparison with Experimental Data. A comparison between the prediction of the unified model for the annular to slug flow transition boundary and experimental data, for air-water flow in 2.54 cm pipe at STP, is presented in Fig. 5.28. As can be seen, for upward flow, both the instability and wave growth and blockage mechanisms occur, the former at low liquid superficial velocities (vertical line) and the later at relatively higher superficial velocities (inclined line). For horizontal and downward inclinations, the transition occurs only due to the wave growth and blockage mechanism. Note that for these conditions the transition lines might be terminated by the stratified to non-stratified transition boundary, as the proposed unified model is valid only outside the region of stable stratified flow. Good agreement is observed between the model predictions and the experimental data for the entire range of inclination angles.
The transition equations can be transformed into a dimensionless form in order to determine the dimensionless groups that control this transition, and also to generalize the results. Transforming the steady-state solution for the liquid holdup, which can be obtained from the simultaneous solution of Eqs. 5.19 and 5.20 (the combined momentum equation), into dimensionless form yields

\[ Y = \frac{1 + 75 H_L}{(1 - H_L)^{2.5} H_L} - \frac{1}{H_L^2} X^2. \]  \hspace{1cm} (5.28)

Similarly, Eq. 5.24, which predicts the minima value of \( \tau_f \) for a given value of \( v_{SL} \), in dimensionless form is

\[ Y = \frac{2 - 1.5 H_L}{H_L^2 (1 - 1.5 H_L)} X^2, \]  \hspace{1cm} (5.29)

where the dimensionless parameters, as defined before, are the Lockhart and Martinelli parameter \( X \), namely, and an inclination angle parameter \( Y \), given, respectively,
\[
X^2 = \frac{4C_L}{d} \left( \frac{\rho_L v_{SL} d}{\mu_L} \right)^{-n} \frac{\rho_L v_{SL}^2}{2} = -\frac{dp}{dL}_{SL}, \quad \text{(3.35)}
\]

and

\[
Y = \frac{4C_G}{d} \left( \frac{\rho_G v_{SG} d}{\mu_G} \right)^{-m} \frac{\rho_G v_{SG}^2}{2} = -\frac{dp}{dL}_{SG}, \quad \text{...............................................................(3.36)}
\]

The additional transition equation, presenting the criterion for the wave growth and blockage mechanism, is already given in dimensionless form by Eq. 5.26.

It is possible now to construct a generalized map for combined transition mechanisms, utilizing \(X\) and \(Y\) as the dimensionless coordinates, as shown in Fig. 5.29. Note that the upper part of the figure is for upward flow, while the lower part is for downward flow. Curve (a) represents the transition mechanism due to instability. It is obtained by the simultaneous solution of Eqs. 5.28 and 5.29, yielding the loci of the \(X\) and \(Y\) pairs along the neutral stability curve. Thus, for conditions below curve (a) stable annular flow occurs, while the conditions above the curve is for unstable flow and transition to slug flow. Curve (b) is obtained by the simultaneous solution of Eqs. 5.28 and 5.26, which represents the transition mechanism due to wave growth and blockage. In practice, Eq. 5.28 is plotted utilizing a constant value of \(H_L = 0.24\). Thus, annular flow occurs in the region to the left of curve (b), while slug flow occurs in the region to the right of the curve.

**Calculation Procedure.** Prediction of the transition to annular flow can be carried out utilizing Fig. 5.29. For this case \(X\) and \(Y\) are calculated from Eqs. 3.35 and 3.36, respectively. For determination of the transition boundary via calculations, the following procedure should be followed:

1. Solve Eqs. 5.19 and 5.20 to obtain the operational film thickness, \(\tilde{\delta}_{LOPR}\).
2. If \(\tilde{\delta}_{LOPR} > 0.065\), then slug flow occurs.
3. If \(\tilde{\delta}_{LOPR} < 0.065\) solve for \(\tilde{\delta}_{LMIN}\) from Eq. 5.24 and check:
   - If \(\tilde{\delta}_{LOPR} < \tilde{\delta}_{LMIN}\), the flow is stable and annular flow occurs.
   - If \(\tilde{\delta}_{LOPR} \geq \tilde{\delta}_{LMIN}\), the flow is not stable and slug flow occurs.
5.2.2 Transition to Dispersed-Bubble Flow (Transition F-G). Different mechanisms have been proposed in the past for the transition boundary to dispersed-bubble flow. For horizontal and slightly inclined flow, Taitel and Dukler (1976) proposed a mechanism based on turbulence versus buoyancy force balance (see Section 3.2.1). For vertical and off-vertical flow, Taitel et al. (1980) and Barnea et al. (1985) proposed that the transition mechanism is based on turbulence versus surface tension force balance (see Section 4.2.1). The model of Barnea (1986) attempts to combine both mechanisms into a unified mechanism that applies for the entire range of inclination angles. As presented next, the unified model suggests that transition to dispersed-bubble occurs as a result of either mechanism:

- Agglomeration of bubbles
- Migration of bubbles, due to buoyancy, to the upper part of the pipe (creaming)

**Transition due to Bubble Agglomeration.** This is the mechanism based on the concept of turbulent forces overcoming surface tension forces, dispersing the gas-phase into small bubbles in the continuous liquid-phase. This transition boundary is previously given in Section 4.2.1, and is summarized below. Based on Hinze’s (1955) study and the modifications based on Calderbank’s (1958) results, Barnea et al. (1985) suggested the following expression for the maximum stable diameter of the dispersed bubbles:

\[
d_{\text{MAX}} = (0.725 + 4.15\sqrt{\alpha})(\sigma/\rho_L)^{0.6}(\varepsilon)^{-0.4}. \]

Fig. 5.29—Generalized map for unified annular to slug flow transition boundary.
The critical bubble size diameter, above which bubbles start to deform and coalesce, was originally given by Brodkey (1967) and later modified by Barnea et. al (1982), as given by

$$d_{CD} = 2 \left[ \frac{0.4 \sigma}{(\rho_L - \rho_G)g} \right]^{0.5}. \quad \text{...........................................................................................................(4.21)}$$

Thus, if the turbulent forces are sufficiently high to break the bubbles into small bubbles with diameter less than the critical diameter, the coalescence rate is hindered, ensuring dispersed-bubble flow to occur. This criterion is presented below

$$d_{\text{MAX}} \leq d_{CD}. \quad \text{...........................................................................................................(4.22)}$$

**Transition due to Bubble Creaming.** This mechanism, based on the concept of turbulent forces overcoming buoyancy forces, was presented originally by Taitel and Dukler (1976), as given in section 3.2.1. In the original study the analysis was carried out on the stratified flow gas-liquid interface. However, in the unified model presented here, the analysis is carried out on a single bubble, with a diameter $d_B$. As shown in Fig. 5.30, the buoyancy force tends to lift the bubble towards the upper pipe wall. Thus, bubbles can concentrate near the upper pipe wall (creaming), promoting coalescence and transition to slug flow. The turbulent forces, on the other hand, tend to disperse the bubbles, promoting dispersed-bubble flow. The balance between these forces yields the critical bubble diameter for “creaming”.

![Fig. 5.30—Forces acting on single bubble in dispersed-bubble flow.](image)

The buoyancy force and the turbulent force acting on a bubble in the radial direction, are given, respectively, by

$$F_B = (\rho_L - \rho_G)g \cos \theta \frac{\pi d_B^3}{6}. \quad \text{...........................................................................................................(5.30)}$$

and

$$F_T = \frac{1}{2} \rho_L v'^2 \frac{\pi d_B^2}{4}. \quad \text{...........................................................................................................(5.31)}$$

where $v'$ is the average radial velocity fluctuations, estimated to equal to the friction velocity, as follows:

$$\left( \bar{v'^2} \right)^{0.5} \approx v_* = v_M \left( \frac{f_M}{2} \right)^{0.5}. \quad \text{...........................................................................................................(5.32)}$$
Transition to slug flow occurs when migration of bubbles towards the upper pipe wall occurs, leading to agglomeration of bubbles, namely,

\[ F_B > F_T. \]  \hspace{1cm} \text{................................................................. (5.33)}

Substituting for \( F_B \) and \( F_T \) into Eq. 5.33, enables the solution of the critical bubble diameter for this transition, given by

\[ d_{CB} = \frac{3}{8} \frac{\rho_l}{\rho_l - \rho_G} f_M v_M^2 g \cos \theta. \] \hspace{1cm} \text{................................. (5.34)}

The final criterion for the transition boundary based on bubble creaming is

\[ d_{\text{MAX}} \leq d_{CB}. \] \hspace{1cm} \text{................................................................. (5.35)}

**The combined Criterion.** For dispersed-bubble flow to occur, the maximum bubble diameter, \( d_{\text{MAX}} \), should be less than both \( d_{CD} \) and \( d_{CB} \), namely,

\[ d_{\text{MAX}} \leq d_{CD} \quad \text{and} \quad d_{\text{MAX}} \leq d_{CB}. \] \hspace{1cm} \text{................................................................. (5.36)}

This will ensure that neither agglomeration of bubbles, nor “creaming” occurs, and that the dispersed-bubble flow pattern exists.

**Comparison with Experimental Data.** Figure 5.31 presents a comparison between the unified model for the transition to dispersed-bubble flow and experimental data, for air-water flow in a 2.5 cm pipe at STP. The transition boundary predicted by \( d_{\text{MAX}} = d_{CD} \) (Eqs. 4.20 and 4.21) is shown by the solid line, while the dashed line is the transition boundary predicted by \( d_{\text{MAX}} = d_{CB} \) (Eqs. 4.20 and 5.34).

As can be seen both transition boundaries can be predicted for the entire range of inclination angles. The transition boundary that actually occurs (the combined criterion) is given by the outer (higher superficial liquid velocity) boundary of the two, which is marked with a thicker line. For horizontal and slightly inclined flow \( d_{\text{CB}} < d_{\text{CD}} \) and the transition boundary occurs when \( d_{\text{MAX}} = d_{CB} \) (at higher mixture velocities than those predicted by \( d_{\text{CD}} \)). For vertical and off-vertical flow \( d_{\text{CD}} < d_{\text{CB}} \) and the transition occurs when \( d_{\text{MAX}} = d_{\text{CD}} \), whereby the effect of buoyancy can be ignored. As can be seen, the two mechanisms do not differ substantially from each other over the entire range of inclination angles. Note that the unified model presented is valid for gas void fractions of \( \alpha < 0.52 \). The value of \( \alpha = 0.52 \) represents the maximum possible packing of bubbles, above which bubble flow is not possible, yielding a transition to slug flow (see Section 4.2.1 and Eq. 4.24).
5.2.3 Overall Unified Model. In a follow up paper, Barnea (1987) combined all the transition boundaries in an overall unified model for flow pattern prediction. This model includes the unified transition boundaries presented in this section, namely, the transition boundaries to annular flow and to dispersed-bubble flow, and other transition boundaries, such as the transition from stratified to non-stratified. Figure 5.32 presents comparison between the predictions of the overall unified model for flow pattern prediction with experimental data for air-water flow in a 5.1 cm pipe at STP, for the entire range of inclination angles. As can be seen, good agreement is observed between the model predictions and the data for the entire range of inclination angles.
Fig. 5.32—Overall comparison between unified model predictions and experimental Data for air-water flow in a 5.1 cm. pipe at STP for entire range of inclination angles.
5.3 Unified Modeling For Upward Slug Flow

Most of the studies on two-phase slug flow have been carried out for either horizontal or vertical conditions. Changes of the physical phenomena occur as the pipe inclination angle varies from vertical through inclined to horizontal conditions. Due to gravity, the gas-phase tends to accumulate in the upper cross section of the pipe. As a result, both the small bubbles in the liquid slugs and the Taylor bubbles tend to flow along the upper pipe wall, causing significant variation of the Taylor bubble rise velocity with inclination angle. Since the liquid holdup and the pressure gradient of the flow are strongly dependent on the Taylor bubble rise velocity, they are also affected. Models are needed to be developed in order to account for this phenomenon.

Very few studies have considered two-phase slug flow in inclined pipes. Pertinent publications are by Hasan and Kabir (1986), Crowley and Rothe (1986) and Stanislav et al. (1986). The study by Taitel and Barnea (1990) presents a comprehensive and detailed analysis of slug flow, applicable for the range of inclination angles from horizontal to vertical (see section 3.4.2). The two main contributions of this elaborated study are the analysis of the hydrodynamics of the liquid film and a rigorous prediction of the pressure drop for a slug unit.

In this section, the simplified slug flow model by Felizola and Shoham (1995) is presented. The objectives of this study are two-fold: first, to carry out an experimental investigation on the effect of the inclination angle on slug flow phenomena. This is achieved by acquiring data on slug flow characteristics for the range of inclination angles from horizontal to vertical (upward flow). The data include the liquid holdup in the slug, translational velocity, slug length, slug frequency and pressure drop. The second objective is to develop a simplified and unified model for the prediction of slug flow behavior in the inclination angle range from horizontal to vertical.

5.3.1 Experimental Program. The test facility, shown schematically in Fig. 5.33, consists of two legs. The first one, shown on the left-hand side, is the test section, constructed of 0.051 m diameter 15 m long PVC R-4000 transparent pipe. The second leg, shown in the right-hand side, is the return line and a bypass to the separator. The return line is a 0.076 m diameter, standard PVC pipe. The test facility is capable of rotating to cover the range of inclination angles from horizontal to vertical conditions. Air and kerosene were used in the tests, whereby the properties of the kerosene, at 25° C, were: density 800 kg/m³, viscosity 0.0016 kg/m·s and surface tension of 0.028 N/m.

The following instruments were installed in the test section to acquire the slug flow characteristics data: Three capacitance sensors to measure the instantaneous liquid holdup, velocity and length distributions, and two sets of pressure transducers to measure both absolute and differential pressures. All instrumentation was connected to a computerized data acquisition system.

Three actuated ball valves were used to control the flow in the test section. These valves allow two different modes of operation: One mode of operation allowed the acquisition of the experimental data. Under this mode the actuated ball valves, designated as “a” and “b” in Fig. 5.33, were open and valve “c” was closed. The two-phase mixture passed through the test section, where the data were acquired, and then were flown down the return line to the separator. The second mode of operation allowed the dynamic calibration of the capacitance sensors. This was accomplished for each inclination angle prior to the acquisition of the data. After establishing a steady-state flow in the test section, valves “a” and “b” were closed and valve “c” opened simultaneously. The two-phase mixture was deviated from the test section through the bypass line to the separator. At the same time, the mixture was trapped in the test section for measuring the average liquid holdup. A more detailed description of the experimental
facility and the procedure for the dynamic calibration of the capacitance sensors is given by Felizola (1992).

5.3.2 Modeling. The model developed in this study is unified and simplified. Its formulation follows the methodology of the Fernandes et al. (1983) model as simplified by Sylvester (1987), utilizing the Vo and Shoham (1989) approach for the solution. The main difference between the present model and the Sylvester (1987) model is that the Sylvester model is valid only for upward vertical flow, while the present model is a unified one, applicable for the range of inclination angles from horizontal to vertical. This requires a different analysis for the liquid film region and the Taylor bubble rise velocity, as will be presented next. Refer to Fig. 3.34 for the physical model and nomenclature.

From an overall mass balance for the gas-phase in a slug unit, the superficial gas velocity, $v_{SG}$, can be expressed as:

$$v_{SG} = \beta v_{GB} (1 - H_{LTB}) + (1 - \beta) v_{GLS} (1 - H_{LLS}),$$ ................................. (4.49)

where $\beta$ is the ratio of the Taylor bubble/liquid film zone length to the slug unit length, namely,
\[ \beta = \frac{L_F}{L_U}. \]  

(4.50)

Similarly, an overall mass balance for the liquid-phase over the slug unit the superficial liquid velocity, \( v_{SL} \), is given by:

\[ v_{SL} = (1 - \beta)H_{LLS} - \beta v_{LTB} H_{LTB}. \]  

(4.51)

The following two equations are cross-sectional area mass balances for the gas and liquid phases, respectively. The two cross-sectional areas are located in the slug body and in the film/gas region. Both balances are carried out in a reference frame moving at the Taylor bubble translational velocity, \( v_{TB} \). In this reference frame, the problem becomes a steady-state one. The cross-sectional area mass balance for the liquid-phase yields:

\[ (v_{TB} - v_{LLS})H_{LLS} = [v_{TB} - (v_{LTB})]H_{LTB}. \]  

(4.52)

Similarly, a mass balance for the gas-phase carried out relative to the Taylor bubble translational velocity between the two cross-sectional areas gives:

\[ (v_{TB} - v_{GLS})(1 - H_{LLS}) = (v_{TB} - v_{GTB})(1 - H_{LTB}). \]  

(4.53)

The Taylor bubble translational velocity, \( v_{TB} \), is given by

\[ v_{TB} = c_0v_M + v_D, \]  

(4.54)

where \( c_0 \) is a flow coefficient, equal to 1.2 for turbulent flow and 2 for laminar flow. The drift velocity \( v_D \) can be determined either from the Bendiksen (1984) correlation or from the model developed by Alves et al. (1993). Utilizing the Bendiksen correlation, the Taylor bubble translational velocity becomes

\[ v_{TB} = c_0v_M + 0.54 \sqrt{gd} \cos \theta + 0.35 \sqrt{gd} \sin \theta \quad 0^\circ \leq \theta \leq 90^\circ. \]  

(3.164)

The gas-phase velocity in the liquid slug is the sum of the medium velocity in the slug and the entrained bubbles rise velocity, and can be written as:

\[ v_{GLS} = c_0v_M + 1.53 \left[ \frac{g \sigma (\rho_L - \rho_G)}{\rho_L^2} \right]^{0.25} H_{LLS}^{0.5} \sin \theta. \]  

(4.55)

A force balance over the film zone yields (assuming that the film flows backwards):

\[ \rho_L H_{LTB} g \sin \theta = \frac{f_c \rho_G}{2} (v_{GTB} + v_{LTB})^2 \frac{S_L}{A_p} + \frac{f_l \rho_L}{2} v_{LTB}^2 \frac{S_L}{A_p}. \]  

(5.37)

The correlation for the liquid holdup in the slug body, \( H_{LLS} \), which has been developed based on the original study experimental data is complex. Instead, it is recommended to use the simple correlation developed more recently by Gomez et al. (2000), given by

\[ H_{LLS} = 1.0 \cdot \text{EXP}[-(7.85 \times 10^3 \theta + 2.48 \times 10^6 \text{Re}_{LS})], \quad 0^\circ \leq \theta \leq 90^\circ, \]  

(3.168)
\[ Re_{LS} = \frac{\rho_L v_M d}{\mu_L} \]. .................................................................(3.169)

Once the set of 8 equations is solved, the characteristic lengths corresponding to a slug unit and the pressure drop can be determined. The film length, \( L_F \), is determined from

\[ L_F = L_S \frac{\beta}{1-\beta} \]. .................................................................(5.38)

where the slug length, \( L_S \), is an input parameter for the model. The length of the slug unit, \( L_U \), can be then obtained as

\[ L_U = L_S + L_F \]. .................................................................(5.39)

The pressure drop in a slug unit can now be determined. As suggested by Taitel and Barnea (1990), consistent with the physical model, the pressure drop is calculated from Eq. (3.155), which is valid for the case of uniform film thickness analysis

\[-\Delta p_U = \rho_S g \sin \theta L_S + \frac{\tau_S \pi d}{A_p} L_S + \rho_F g \sin \theta L_F + \frac{\tau_F S_F}{A_p} L_F + \frac{\tau_G S_G}{A_p} L_F, \] ...........(3.155)

and the pressure gradient is \(- \frac{dp}{dL} = \frac{-\Delta p_U}{L_U}\). If the pressure drop in the liquid film/gas pocket is neglected, the last 3 terms on the right-hand side of Eq. 3.155 are dropped out.

5.3.3 Numerical Solution. The 8 equations with 8 unknowns system presented above can be reduced to a single algebraic equation with only one unknown, namely, \( h_F \), as given by

\[ f_G \rho_G \left[ \frac{K_1}{H_{LTB}} + \frac{K_2}{1-H_{LTB}} \right]^2 S_I + f_L \rho_L \left[ \frac{K_1}{H_{LTB}} - v_{TB} \right]^2 S_L - 2 \rho_L g A_p H_{LTB} \sin \theta = 0, \] ...........(5.40)

where \( K_1 \) and \( K_2 \) are given by

\[ K_1 = v_{TB} H_{LSS} + v_{GLS} (1 - H_{LSS}) - v_M \] .................................................................(5.41)

and

\[ K_2 = K_1 + v_M - v_{TB} \]. .................................................................(5.42)

Note that Eq. (5.40) is a unique function of \( h_F \), the liquid level in the stratified region behind the slug. The variables \( K_1, K_2 \) and \( v_{TB} \) can be determined from input conditions, and the geometrical parameters \( S_I, S_L \) and \( H_{LTB} \) are functions of \( h_F \), as given, respectively, by

\[ S_I = d \sqrt{1-(2h_F/d-1)^2}, \]

\[ S_L = d[(\pi-\cos^2(2(2h_F/d-1)], \] and

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\[ H_{LTB} = \frac{1}{\pi} \left[ \pi - \cos^{-1}(2h_F / d - 1) + (2h_F / d - 1)\sqrt{1-(2h_F / d - 1)^2} \right]. \] 

Solution of Eq. (5.40) yields the value of \( h_F \), which enables the solution of the entire set of 8 equations sequentially. A second order Newton-Raphson convergence scheme can be used to find the solution, as given by:

\[
(h_F)_{j+1} = (h_F)_j - \frac{F}{F'}, \]

where \( F \) is given in Eq. 5.40. The derivative, \( F' \), is obtained by differentiating Eq. (5.40) with respect to \( h_F \), yielding

\[
\frac{dF}{dh_F} = 2 \left[ f_g \rho_g S_I \left( \frac{K_1}{H_{LTB}} + \frac{K_2}{1-H_{LTB}} \right) \left( \frac{K_2}{(1-H_{LTB})^2} - \frac{K_1}{H_{LTB}^2} \right) 
- \frac{f_L \rho_L S_L}{H_{LTB} - v_{TB}} \left( \frac{K_1}{H_{LTB}} - \rho_L g A_p \sin \theta \right) \frac{dH_{LTB}}{dh_F} 
+ \frac{f_g \rho_g}{H_{LTB} + \frac{K_1}{1-H_{LTB}}} \left( \frac{K_1}{H_{LTB}} + \frac{K_2}{1-H_{LTB}} \right)^2 \frac{dS_I}{dh_F} 
+ \frac{f_L \rho_L}{H_{LTB} - v_{TB}} \left( \frac{K_1}{H_{LTB}} - \rho_L g A_p \sin \theta \right) \frac{dS_L}{dh_F} \right].
\] 

The derivatives of the geometrical variables with respect to \( h_F \), are given by

\[
\frac{dS_I}{dh_F} = \frac{-2(2h_F / d - 1)}{\sqrt{1-(2h_F / d - 1)^2}}, \\
\frac{dS_L}{dh_F} = \frac{2}{\sqrt{1-(2h_F / d - 1)^2}}, \text{ and} \\
\frac{dH_{LTB}}{dh_F} = \frac{4}{\pi d} \sqrt{1-(2h_F / d - 1)^2}.
\] 

Solution of Eq. 5.40 yields two roots. The smaller of the two roots is the correct root. The second root, the larger of the two, of the order 0.9d to 0.99d, is not the physical one and should be ignored. If a value of \( h_F = 0.1d \) is used as the initial guess, the solution procedure will always yield the correct root.

The step-by-step procedure for determining all slug flow variables is:

1. Calculate \( v_{TB}, v_{GLS} \) and \( H_{LSS} \) from Eqs. (3.164), (4.55) and (3.168), respectively.
2. Using Eqs. (5.40) to (5.43), determine \( h_F \) and \( H_{LTB} \).
3. Determine \( K_1 \) and \( K_2 \) from Eqs. (5.41) and (5.42).
4. Calculate \( v_{LTB} \) and \( v_{GTB} \) from the following equations, respectively,
\[ v_{LTB} = \frac{K_1}{H_{LTB}} v_{TB}, \quad \text{.................................................................(5.47)} \]

and

\[ v_{GTB} = \frac{K_2}{1-H_{LTB}} + v_{TB}, \quad \text{.................................................................(5.48)} \]

5. Solve Eq. (4.52) for \( v_{LLS} \).

6. Solve for \( \beta \) from

\[
\beta = \frac{v_{LLS} H_{LLS} - v_{SL}}{v_{LLS} H_{LLS} + v_{LTB} H_{LTB}}. \quad (5.49)
\]

Note that equations (5.47), (5.48) and (5.49) can be developed from Eqs. (4.49) to (4.53).

7. Determine the liquid film region, \( L_F \), and the slug unit, \( L_U \), lengths from Eqs. (5.38) and (5.39), respectively.

8. Determine the pressure drop and the pressure gradient from Eq. (3.155).

5.3.4 Closure Relationships. The model requires, as an input, the liquid holdup in the slug body and the drift velocity, which are given by Eqs. (3.168) and (3.164), respectively. Another input variable is the liquid slug length. Based on the experimental measurements, it is proposed to use a value of \( L_s = 3d \) for horizontal flow and a value of \( L_s = 20d \) for vertical flow. A linear distribution between these two values is assumed for inclined flow with respect to the inclination angle. For the flow coefficient \( c_0 \) the recommended values are the ones proposed by Alves et al. (1993), as follows:

\[
c_0 = \begin{cases} 
1.0 & \text{for } 0^\circ < \theta \leq 50^\circ \\
1.15 & \text{for } 50^\circ < \theta \leq 60^\circ \\
1.25 & \text{for } 60^\circ < \theta \leq 90^\circ 
\end{cases} \quad (5.50)
\]

5.3.5 Results. The proposed model was used to predict the pressure gradient for the experimental data flow conditions, excluding the horizontal cases. The horizontal runs were excluded because of their low accuracy. As mentioned before, the pressure drop in the test section was measured by two absolute pressure transducers, located at both ends of the test section. For inclined and vertical flow the pressure drop is significant due to the gravitational component, and accurate results can be obtained from the difference of the results of the two transducers. However, for horizontal flow the pressure drop is small, as it is due to frictional losses only. For these conditions the measured pressure drop is not accurate and is omitted. Figure 5.34 shows a comparison between the predicted and measured pressure gradients for all the experimental runs between 10° and 90°. As can be seen, the majority of the predicted values agree within ±30% with the experimental data. Also shown in the figure are the pressure gradients predicted by the Taitel and Barnea (1990) elaborated model. A larger spread of the predicted values is observed for their model. The main reason for this larger discrepancy is the use of the Barnea and Braunen (1985) model for liquid holdup in the slug body, which is independent of the inclination angle. Note that the liquid holdup in the slug body used by the proposed model was based on the
correlation developed in original study. It is believed that better predictions of the model could be obtained if the recommended Gomez et al. (2000) correlation (Eq. 1.168) was used, since it is based on a larger and more accurate data set.

Fig. 5.34—Comparison between experimental data and predictions of Taitel and Barnea (1990) and Felizola and Shoham (1995) models.

5.4 Unified Two-Dimensional Modeling For Stratified Flow

Stratified flow exists in horizontal flow as well as in downward and upward inclined flow. In downward inclinations stratified flow is the dominant flow regime for the entire range of inclinations, excluding $-90^\circ$. In upward flow, on the other hand, it occurs only in slightly inclined pipes and over a small range of flow rates. As the inclination angle increases, the region of stratified flow decreases and finally completely disappears at $10^\circ$ (for air-water at STP). It does, however, usually exist at the entrance region in slug flow, which is the dominant flow regime in the upward configuration. Following is a brief review of literature on stratified flow for turbulent and laminar flow conditions.

Turbulent Flow: A solution for turbulent stratified flow is quite complex due to the two dimensionality of the flow field and also due the uncertainty of the interfacial shear determination. As a result, the approaches taken in the past were considerably simplified using generally either empirical correlations or a one-dimensional approach.

One of the first and most durable empirical correlation for pressure drop and holdup in horizontal flow was proposed by Lockhart and Martinelli (1949). Although the analysis is that of a separated flow configuration, the developed correlation was based on experimental data of other flow patterns. This is the main reason for large errors reported on the application of the correlation for stratified flow. An
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extension of this model was done by Johannessen (1972) who evolved a theoretical model, which agrees better than Lockhart and Martinelli’s correlation with the experimental data.

Mechanistic approaches have been proposed for turbulent stratified flow by Etchells (1970), Govier and Aziz (1972), Agrawal et al. (1973) and Russell et al. (1974). In these studies, the frictional pressure drop was calculated as in single-phase flow, using a geometrical model for the flow cross-section.

Taitel and Dukler (1976) model for stratified flow is a unified one-dimensional model applicable to horizontal and inclined flow. In this model, both phases are treated as bulk flow. This approach neglects the detailed velocity profile structure, and the shear stresses are calculated via empirical correlations based on the average velocity. Such an approach can result in error when calculating the shear stresses at the bottom pipe for upward inclined flows. In this case the flow near the bottom of the pipe may be negative resulting in a negative value for the shear stress compared to its value calculated on the basis of the “one-dimensional” theory.

An extension of the model of Russell et al. (1974) to turbulent-turbulent gas-liquid stratified flow in horizontal pipes was carried out by Cheremisinoff and Davis (1979). Their analysis of the turbulent liquid-phase is based on the approach of Dukler (1960), who used successfully available turbulent mixing length theory for downward annular two-phase flow. The expressions used for the eddy viscosity were Deissler’s equation for the region near the wall and Von Karman’s equation for the turbulent core. In order to simplify their model, Cheremisinoff and Davis assumed a constant shear stress in the whole liquid region, namely, \( \tau = \tau_c = \text{const.} \), and a velocity profile, which depends only on the radial position \( \nu = \nu(r) \). Solving for the velocity profile under these assumptions, a universal dimensionless turbulent profile was developed. Though the Cheremisinoff and Davis results should apply fairly well for shallow liquid layers, \( h_L / d < 1 \), it may introduce significant errors for thick layers and inclined pipes where both assumptions of constant shear stress and the dependence of the velocity profile on the radial position only do not apply.

Kadambi (1981) suggested an analytical procedure to determine the pressure drop between two parallel plates. He showed that the velocity profile of the gas and the liquid could be represented by two hypothetical fully-developed flow profiles using matching conditions at the interface. Pai polynomial velocity profile was used, which is applicable over the whole range of Reynolds number, so that the solution is valid for both laminar and turbulent flow. The use of the equivalent pipe diameter concept has enabled the results to be of general use for pipes, as well as for flat plates and rectangular channels.

**Laminar Flow:** Exact mathematical solutions for laminar stratified-smooth flow in horizontal pipes have been proposed by various researchers, aimed at formulating analytical solutions for the velocity distribution in two-phase flow. Bentwich (1964) developed a solution, which has an exact integral representation in terms of bipolar coordinates. The integrals are of Fourier transform type, infinitely extended, which makes velocity calculations somewhat difficult. Ranger and Davis (1979) extended the above solution in order to obtain expressions for the volumetric fluxes of the two phases. Their contribution is that these expressions are in the form of one variable integral depending on \( \gamma \), the angle that defines the interface location, and on physical properties. Numerical results are also difficult to obtain for this model.

A simpler case of laminar liquid flow in open circular channels was presented by Buffham (1968). His work is divided into two parts: 1) A numerical solution for deep channels using bipolar coordinates; 2) An approximate analytical solution for shallow channels. The analytical solution is in the form of a
Fourier transform integral, which is solved numerically. Comparison between the two solutions reveals that the approximate solution is applicable for $h_L / d \leq 0.1$.

Masliyah and Shook (1978) proposed a solution for steady, counter-current, zero-net flow of two immiscible fluids in inclined pipes, under laminar conditions. Such zero-net flow occurs at the shutdown of an inclined slurry pipeline. Since the time scale of the particles settling is fairly long, inertial effects are negligible, resulting in a steady-state equation of motion. The equation is transformed into bipolar coordinates and solved numerically with the proper boundary conditions, as done by Buffham, to give the velocity distribution of the two fluids. Once the velocity profiles are determined, the local shear stress along the upper and lower pipe wall, as well as at the interface, can be evaluated, and the respective average shear stresses can be found by integration.

In this section, a two-dimensional model for stratified turbulent-turbulent gas-liquid flow in inclined pipes is presented (Shoham and Taitel, 1984). The gas-phase is treated as bulk flow, but an exact solution is carried out for the liquid-phase, applying the eddy viscosity theory to model the turbulent viscosity. Appropriate correlations are used for the interfacial shear stress. The model is capable of predicting the liquid velocity field, holdup and pressure drop for a given set of flow conditions. Finally, the predicted pressure drop and holdup are tested against published experimental data.

5.4.1 Modeling. Figure 5.35 presents the physical model for stratified flow. The liquid flows at the bottom, with a liquid level of $h_L$, while the gas flows at the top. A cross-sectional area of the pipe is shown on the left-hand side. with the flow geometrical parameters. The inclination angle is $\theta$.

For gas-liquid stratified flow under steady state-conditions, the inertial terms in the Navier-Stokes equations are zero, and the equations of motion for each of the phases reduce to

\[
\nabla^2 v_L = \frac{1}{\mu_{EL}} \left( \frac{dp}{dL} + \rho_L g \sin\theta \right), \quad \text{.................................................................}(5.51)
\]

and

\[
\nabla^2 v_G = \frac{1}{\mu_{EG}} \left( \frac{dp}{dL} + \rho_G g \sin\theta \right), \quad \text{.................................................................}(5.51')
\]

where $\mu_E$ denotes the effective viscosity, which equals the sum of the molecular viscosity and the turbulent eddy viscosity, namely, $\mu_E = \mu + \mu_T$. The boundary conditions are:

1) $v_G = 0$ at the upper wall (no-slip condition).

2) $v_L = 0$ at the lower wall (no-slip condition).

3) $v_L = v_G$ and $\mu_{EL} \frac{\partial v_L}{\partial y} = \mu_{EG} \frac{\partial v_G}{\partial y}$ at the interface (continuity of velocity and shear stress).

4) $\frac{\partial v_L}{\partial x} = \frac{\partial v_G}{\partial x} = 0$ at the normal mid-plane (symmetry).
Introducing dimensionless variables

\[
\begin{align*}
    v'_L &= \frac{v_L}{v_{SL}}; \\
    v'_G &= \frac{v_G}{v_{SG}}; \\
    y' &= \frac{y}{d}; \\
    x' &= \frac{x}{d}; \\
    A' &= \frac{A_p}{d^2}; \\
    \rho'_L &= \frac{\rho_L}{\rho_{SL}}; \\
    \rho'_G &= \frac{\rho_G}{\rho_{SG}}; \\
    \mu'_L &= \frac{\mu_L}{\rho_L v_{SL} d}; \\
    \mu'_G &= \frac{\mu_G}{\rho_G v_{SG} d},
\end{align*}
\]

the governing equations become

\[
\nabla^2 v'_L = \frac{1}{\mu_{EL}} \left( \frac{dp}{dL} + \rho_L g \sin \theta \right) = \frac{K'_L}{\mu_{EL}}, \tag{5.53}
\]

and

\[
\nabla^2 v'_G = \frac{1}{\mu_{EG}} \left( \frac{dp}{dL} + \rho_G g \sin \theta \right) = \frac{K'_G}{\mu_{EG}}, \tag{5.53'}
\]

The dimensionless effective viscosities are, respectively,

\[
\mu'_{EL} = \frac{1}{Re_{SL} + \mu'_{TL}} \quad \text{and} \quad \mu'_{EG} = \frac{1}{Re_{SG} + \mu'_{TG}}, \tag{5.54}
\]

where

\[
Re_{SL} = \frac{v_{SL} d \rho_L}{\mu_L} \quad \text{and} \quad Re_{SG} = \frac{v_{SG} d \rho_G}{\mu_G}. \tag{5.55}
\]

**Bipolar Coordinates.** One of the difficulties in solving the problem is caused by the inhomogeneity of the boundaries, which consist of a circular surface (the pipe wall) and a plane (the interface). This difficulty may be alleviated by using bipolar coordinates, as shown in Fig. 5.36. The bipolar coordinates are the natural coordinate system to be used in dealing with field problems, whose boundaries consists of intersecting circular cylinder with a plane.
If the intersection points between the plane and the circle are \( S_1(c, o) \) and \( S_2(-c, o) \), as shown in Fig. 5.36-Part-A, the conformal transformation, which define the bipolar coordinates, is

\[
\omega = ic \cot \left( \frac{\zeta}{2} \right),
\]

(5.56)

where \( \omega = x + iy \), \( \zeta = \xi + i\eta \), and \((x, y)\) and \((\xi, \eta)\) are the coordinates in the \( \omega \)-plane and the \( \zeta \)-plane, respectively. The relationships between \((x, y)\) and \((\xi, \eta)\) are given by (see Fig. 5.36-Part-B):

\[
\begin{align*}
\xi &= \angle S_2 DS_1 \\
\eta &= \ln \frac{r_2}{r_1} = \ln \left[ \frac{(x-c)^2 + y^2}{(x-c)^2 - y^2} \right] \\
x &= \frac{c}{\cosh \eta - \cos \xi} \sinh \eta \\
y &= \frac{c}{\cosh \eta - \cos \xi} \sin \xi
\end{align*}
\]

(5.57)

This conformal mapping allows the transformation of the gas and liquid cross-sectional areas into two infinite strips, as shown in Fig. 5.36-Part-C. For the gas-phase \( \gamma < \xi < \pi \) and \(-\infty < \eta < \infty\), while for the liquid-phase \( \pi < \xi < \pi + \gamma \) and \(-\infty < \eta < \infty\).

Useful mathematical expressions for the bipolar coordinate system are:
• The Jacobian:

\[
\text{Ja} = \frac{\partial (x,y)}{\partial (\xi,\eta)} = \frac{-c^2}{(\cosh \eta - \cos \xi)^2}, \quad \text{.................................................. (5.58)}
\]

• The Laplacian:

\[
\nabla^2 = \frac{(\cosh \eta - \cos \xi)}{c^2} \left[ \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right], \quad \text{.................................................. (5.59)}
\]

• An arc length element on constant \( \xi \) and \( \eta \) lines, respectively:

\[
d\ell_\xi = \frac{c}{\cosh \eta - \cos \xi} d\xi, \quad \text{and} \quad d\ell_\eta = \frac{c}{\cosh \eta - \cos \xi} d\eta. \quad \text{.................................................. (5.60)}
\]

• An area element

\[
dA = d\ell_\eta d\ell_\xi = \frac{c^2}{(\cosh \eta - \cos \xi)^2} d\eta d\xi = -\text{Ja} \cdot d\eta d\xi. \quad \text{.................................................. (5.61)}
\]

Transforming the dimensionless governing equations into bipolar coordinates yields

\[
\frac{\partial^2 v_L}{\partial \xi^2} + \frac{\partial^2 v_L}{\partial \eta^2} = \frac{K_L}{\mu_{EL}} \frac{c^2}{(\cosh \eta - \cos \xi)^2}, \quad \text{.................................................. (5.62)}
\]

and

\[
\frac{\partial^2 v_G}{\partial \xi^2} + \frac{\partial^2 v_G}{\partial \eta^2} = \frac{K_G}{\mu_{EG}} \frac{c^2}{(\cosh \eta - \cos \xi)^2}. \quad \text{.................................................. (5.62')} \]

A solution for both the liquid and the gas regions can be obtained by solving Eqs. (5.62) and (5.62'), satisfying matching conditions at the interface. Such a solution will be quite complex and also applicable only to smooth surfaces. In principle, there is a major difference in the hydrodynamic behavior of the two phases: while the solution for the liquid velocity profile needs to be exact, to take into account velocity differences near the interface and near the wall (in upward inclination the liquid velocity near the pipe wall can be negative), the gas flow can be approximated realistically by duct flow. Therefore, the gas-phase can be treated as bulk flow, and the shear stresses can be calculated using empirical correlations based on the average velocity. Such an approach simplifies considerably the solution and is capable of taking the effective interfacial shear generated by a wavy interface. Note also that this approach is restricted to low pressure, where the density of the gas is small as compared to the liquid density.
**Liquid-Phase Solution.** The governing equation for the liquid region is given by Eq. (5.62). Transforming the liquid boundary conditions into the \( \xi, \eta \) plane yields:

1) At \( \xi = \pi + \gamma \): \( v_L' = 0 \) (no-slip condition at pipe wall).

2) At \( \xi = \pi \):
   \[
   \frac{\partial v_L'}{\partial \xi} = \frac{\tau'_L}{\mu_{EL}} \frac{c'}{1 + \cosh \eta} \] (interfacial shear stress).

3) At \( \eta = 0 \): \( \frac{\partial v_L'}{\partial \eta} = 0 \) (symmetry).

4) As \( \eta \to \infty \): \( v_L' = 0 \) (no-slip condition at wall-interface junction).

The liquid flow rate is calculated by integrating the velocity over the liquid cross-sectional area. Transforming into bipolar coordinates using Eq. (5.61) yields

\[
W_L' = \frac{W_L}{\rho_L v_{SL} d^2} = 2 \int_0^\pi \int_0^{\pi/2} v_L' d\xi d\eta = 2c^2 \int_0^\pi \int_0^{\pi/2} v_L' \sin \theta d\xi d\eta \].................................(5.63)

Note that the dimensionless input flow rate is \( W_L' = \pi / 4 \).

**Gas-Phase Solution.** In the gas-phase region a uniform flow in a closed duct is assumed, bounded by the pipe wall and the interface. Wall shear stress and pressure drop are calculated using correlations based on the hydraulic diameter. The flow geometrical variables, shown in Fig. 5.35, are given by:

\[
\cos \gamma = 1 - 2h'_L, \quad \gamma = \cos^{-1}(1 - 2h'_L),
\]

\[
A_L = 0.25 d^2 (\gamma - \sin \gamma \cos \gamma), \quad A_G = 0.25 d^2 (\pi - \gamma + \sin \gamma \cos \gamma),
\]

\[
H_L = A_L / A_p, \quad \alpha = A_G / A_p,
\]

\[
S_L = d \cdot \gamma, \quad S_G = (\pi - \gamma) d, \quad S_I = d \sin \gamma, \quad d_L = \frac{4A_L}{S_L}, \quad d_G = \frac{4A_G}{S_G + S_L}. \].................................(5.64)

Single-phase correlation, based on the hydraulic diameter, is used to calculate the shear stress, namely,

\[
\tau_{WG} = f_G \frac{\rho_G v_G^2}{2}, \quad \text{and} \quad f_G = C_G (Re_G)^m. \].................................(5.65)

The values used in this study for \( C_G \) and \( m \) are: \( C_G = 0.046 \) and \( m = 0.2 \) for turbulent flow, and \( C_G = 16 \) and \( m = 1 \) for laminar flow. An overall force balance on the gas-phase gives:

\[
- A_G \frac{dp}{dL} - \tau_{WG} S_G - \tau_I S_I - \rho_G A_G g \sin \theta = 0 \].................................(5.66)
Measurements of interfacial shear stress for air-water stratified flow were reported by Cohen and Hanratty (1968) for three-dimensional small amplitude waves. Based on the data, an interfacial friction factor correlation was developed, namely,

\[ f_I = 0.0142. \]  \hspace{4cm} (5.67)

whereby the interfacial shear stress is given by

\[ \tau_I = 0.0142 \frac{\rho_G v_G^2}{2}. \]  \hspace{4cm} (5.68)

The pressure drop, as calculated from the gas-phase (Eq. 5.66), in dimensionless form is given by

\[ \frac{dp'}{dL} = \left( -\tau_{WG} \frac{S_G}{A_G} - \tau_I \frac{S_I}{A_G} - \rho_G g \sin \theta \right) \frac{d}{\rho_I v_{SL}^2}. \]  \hspace{4cm} (5.69)

This expression for \( \frac{dp'}{dL} \) is used in Eq. (5.62) to calculate the velocity profile in the liquid-phase.

**Turbulent Viscosity.** The hydrodynamics of turbulent cannot be solved rigorously at the present time. One of the most acceptable engineering approaches for a solution of turbulent flow is to use the concept of the eddy viscosity, which views the Reynolds stresses as an effective turbulent viscosity. The Prandtl mixing length theory has been commonly used for this purpose, given by:

\[ \mu_T = \rho \ell_M^2 \left| \frac{dv}{dr} \right|. \]  \hspace{4cm} (5.70)

Thus, once the mixing length \( \ell_M \) is determined, the turbulent eddy viscosity, \( \mu_T \), can be calculated and the solution of turbulent flow can proceed. Unfortunately, the determination of the mixing length is not straightforward. Various correlations have been suggested for \( \ell_M \), usually limited to a certain region of the flow, based on the distance from the wall.

The most common correlations for the mixing length and eddy viscosity calculations for pipe flow are given in “wall coordinates”, based on the frictional velocity \( v^* = \sqrt{\tau_w / \rho} \), which defines the dimensionless distance \( y^+ = \frac{y v^* \rho}{\mu} \). Launder and Spalding (1972) summarized many such correlations. Some of the most widely used are:

1) Deissler’s equation for the region near the wall \((0 \leq y^+ \leq 20)\):

\[ \epsilon_T = \frac{\mu_T}{\rho} = 0.01 v y [1 - \text{EXP}(-0.01 v y \rho / \mu)]. \]  \hspace{4cm} (5.71)

2) Von Karman equation for the turbulent core \((y^+ > 20)\):

\[ \epsilon_T = \frac{\mu_T}{\rho} = 0.36 \left( \frac{dv/dr}{dr} \right)^3 \frac{(dv/dr)^3}{(d^2 v/dr^2)^2}. \]  \hspace{4cm} (5.72)

3) Van Driest model for the entire flow cross-section:
The aforementioned correlations are strictly for full pipe flow. Extension to stratified flow conditions needs further elaboration. The correlations are given in terms of the radial coordinate \( r \) or the radial distance from the wall \( y \). For stratified flow, where two different phases occupy (each) part of the pipe cross sectional area, the use of the distance along the radial direction is not appropriate, because the flow field is two-dimensional. Furthermore, for \( h_L / d > 0.5 \), this distance is meaningless for the region above the pipe centerline. Thus, it is suggested to measure the distance from the wall along the lines \( \eta = \text{const.} \). Note that near the wall the \( \eta \) lines coincide with the \( y \) (or \( r \)) lines.

Another difficulty arises in applying the eddy viscosity correlation for pipe flow (Eqs. 5.71 to 5.73) to the turbulent “core” of stratified flow, namely, to the region near the interface. Strict application of Eq. (5.73) in this case will result in very high values of the eddy viscosity, due to the large velocity gradients obtained at the interface. This is in contradiction to the values of the eddy viscosity obtained previously in similar geometries. Shlichting (1960) shows that the eddy viscosity decreases close to the pipe centerline. For a similar geometry of closed channel flow, Quarmby and Quirk (1972) and Hussain and Reynolds (1975) showed that the eddy viscosity approaches approximately a constant value near the channel center. Experiments in open channel flow (Ueda et al., 1977) showed that the eddy viscosity decreases near the interface. From the above discussion, it is clear that applying the full pipe turbulent eddy viscosity method to stratified flow might result in erroneously high values for the eddy viscosity away from the wall (near the interface region), owing to the large velocity gradients at the interface. The eddy viscosity should probably be either constant or decreases close to the interface.

In this work a tentative suggestion is made for stratified flow, namely, to use the Van Driest model near the wall and to apply a constant value for the eddy viscosity in the turbulent core. The Van Driest model is assumed to be valid for \( y^+ \leq 30 \), while for \( y^+ > 30 \) a constant value, which is the same value as that calculated at \( y^+ = 30 \), is used. Admittedly, the choice of \( y^+ = 30 \) is somewhat arbitrary. For full pipe flow this method would give reasonable results at low Reynolds number and it is quite possible that the results of this study are limited to Reynolds number below 10,000. Note that for stratified flow to exist the flow rates should be low and the Reynolds number is usually below 10,000. Also, it should be noted that the actual variation of the eddy viscosity in this case is unknown, and is a subject for a separate study. The procedure of solution can be modified latter to accommodate more recent information. Nevertheless, the assumption will be tested by comparing the model predictions against experimental data for pressure drop, whereby agreement with the data will be an indication that this procedure is sufficiently accurate, at least for the prediction of the pressure drop. Note also that for evaluating \( y^+ \), the wall shear is based on the average absolute value of \( \tau_{WL} \). Thus, the modified Van Driest model, for stratified flow, is given by

\[
\ell'_M = k\ell'_\xi \left[ 1 - \exp(-y^+ / 26) \right], \quad k = 0.4. \tag{5.73}
\]

And,

\[
\mu'_{WL} = \ell'_M \frac{\partial \bar{v}'_L}{\partial \ell'_\xi} = \ell'_M \frac{\partial \bar{v}'_L}{\partial \xi} \left[ \cosh \eta - \cos \xi \right], \tag{5.74}
\]

\[
\mu'_{WL} = \ell'^2_M \frac{\partial \bar{v}'_L}{\partial \ell'_\xi} = \ell'_M \frac{\partial \bar{v}'_L}{\partial \xi} \left[ \cosh \eta - \cos \xi \right], \tag{5.75}
\]
where $\ell_\xi$ is the distance from the pipe wall, as measured along the $\eta = \text{const.}$ lines, and 

$$\ell_\xi^* = (\ell_\xi d \sqrt{\tau_{WL}/\rho_L}) \frac{\rho_L}{\mu_L}.$$  

The average shear stress at the pipe wall, $\tau_{WL}$, can be calculated from an overall momentum balance on the liquid-phase, as follows:

$$\tau_{WL} = \frac{S_L}{S_L} \frac{g \sin \theta}{A_L} \frac{dL}{dS_L} - \frac{dp}{dL} \frac{A_L}{S_L}.$$  

(5.76)

An arc length element along the line $\eta = \text{const.}$ is given by

$$d\ell_\xi' = \frac{c'}{\cosh \eta - \cos \xi'} d\xi'.$$  

(5.77)

Integrating $d\ell_\xi'$ from $\xi$ to $\xi = \pi + \gamma$ results in

$$\ell_\xi' = \int_{\xi}^{\pi + \gamma} \frac{c'}{\cosh \eta - \cos \xi'} d\xi.$$  

(5.78)

Along $\eta = \text{const.}$ lines, $\cosh \eta = \text{const.} = B \geq 1$, thus

$$\ell_\xi' = \frac{2c'}{\sqrt{B^2 - 1}} \left[ \tan^{-1} \left( \frac{B+1}{\sqrt{B-1}} \frac{\tan \frac{\pi + \gamma}{2}}{2} \right) - \tan^{-1} \left( \frac{B+1}{\sqrt{B-1}} \frac{\tan \frac{\xi}{2}}{2} \right) \right].$$  

(5.79)

Equation (5.79) permits the calculation of the distance of any point $(\xi, \eta)$ from the pipe wall along the $\eta$ lines. Note, however, that for the points along the interface where $\xi = \pi$

$$\lim_{\xi \to \pi + \gamma} \tan^{-1} \left( \frac{B+1}{\sqrt{B-1}} \frac{\tan \frac{\pi}{2}}{2} \right) = \frac{\pi}{2}.$$  

(5.80)

### 5.4.2 Numerical Solution.

The momentum equation for the liquid-phase, as given by Eq. (5.62), with the boundary conditions, is solved numerically using a finite difference technique. Due to symmetry with respect to the vertical diameter, the solution is carried out on half of the pipe cross-section. Figure 5.37 shows the numerical solution scheme and the boundary conditions in the $\xi, \eta$ plane. As can be seen, the region of solution is unbounded in the $\eta$ direction. In practice the upper limit of $\eta$ is taken as a finite large number. Taking $\eta = 10$, for example, replaces the pipe wall-interface junction point by a circular arc with a radius of $10^{-5} R$. At this region the velocity approaches zero and the error introduced by truncating $\eta$ is minimal. In order to avoid a large number of grid points in the $\eta$ direction, a variable increasing increment step is used. The increment step in the $\xi$ direction is also variable. A finer grid is used near the pipe wall and the interface, where the velocity gradients in the $\xi$ direction change substantially.
Expanding $v_{L(i,j)}$ in Taylor’s series and solving for the first and second partial derivatives, Eq. (5.62) can be converted into a discrete form. (For brevity $v_{i,j}$ is used instead of $v'_{L(i,j)}$):

$$
\frac{(v_{i,j+1} - v_{i,j})}{\xi_d/2} \Delta \xi_j - \frac{(v_{i,j} - v_{i,j-1})}{\xi_d/2} \Delta \xi_{j+1} + \frac{(v_{i+1,j} - v_{i,j})}{\eta_d/2} \Delta \eta_i - \frac{(v_{i,j} - v_{i-1,j})}{\eta_d/2} \Delta \eta_{i+1} = \frac{K'_L}{\mu_{EL(i,j)}} \left( \frac{c^2}{cosh \eta_i - cosh \xi_j} \right), \quad \text{..............................................................}(5.81)
$$

where

$$
\eta_d = \Delta \eta_i \Delta \eta_{i+1} (\Delta \eta_i + \Delta \eta_{i+1}),
$$

$$
\xi_d = \Delta \xi_j \Delta \xi_{j+1} (\Delta \xi_j + \Delta \xi_{j+1}),
$$

and

$$
\mu_{EL(i,j)} = \frac{1}{Re_{SL} + \epsilon_{M(i,j)}^\text{L}} \left\{ \frac{(v_{i,j+1} - v_{i,j}) \Delta \xi_j - (v_{i,j+1} - v_{i,j}) \Delta \xi_{j+1}}{\xi_d} \cosh \eta_i - \cosh \xi_j \right\}.
$$

Solving for $v_{i,j}$ from Eq. (5.81) yields

$$
v_{i,j} = P_1 v_{i,j-1} + P_2 v_{i+1,j} + P_3 v_{i,j+1} + P_4 v_{i-1,j} - P_5, \quad \text{..............................................................}(5.82)
$$

where
\[ P_d = \eta_d (\Delta \xi_j + \Delta \xi_{j+1}) + \xi_d (\Delta \eta_i + \Delta \eta_{i+1}), \]
\[ P_1 = \frac{\Delta \xi_{j+1} \eta_d}{P_d}, \]
\[ P_2 = \frac{\Delta \eta_i \xi_d}{P_d}, \]
\[ P_3 = \frac{\Delta \xi_j \eta_d}{P_d}, \]
\[ P_4 = \frac{\Delta \eta_{i+1} \xi_d}{P_d}, \]
\[ P_5 = \frac{K'_L \xi_d \eta_d}{2P_d} \left( \cosh \eta_i - \cos \xi_j \right)^2 \left( \frac{c^2}{\mu_{i+1,L}} \right). \]

Equation (5.82) is solved iteratively in order to calculate interior grid point velocities. Along the boundary \( \eta = 0 \), the first derivative with respect to \( \eta \) is set to zero, using the mirror image approach, i.e., applying Eq. (5.82) with \( v_{i-1,j} = v_{i+1,j} \). At the interface, the first derivative with respect to \( \xi \) is prescribed by the interfacial shear stress (boundary condition 2 in Section 5.4.1). Thus, points along the interface are relaxed by the equation

\[ v_{i,j} = \Delta \xi_i \frac{\tau' c'}{\mu_{i+1,L} (1 + \cosh \eta_i)} + v_{i+2}. \]

**Solution Procedure.** The solution is carried out iteratively along the following steps:

1) Inlet flow conditions are set, including \( v_{SL}, v_{SG}, d, \) and the physical properties of the phases

2) A value for \( h_L / d \) is assumed and all the geometrical variables, which depend on \( h_L / d \), are calculated, as given in Eq. (5.64). Note that the first guess for \( h_L / d \) is taken based on Taitel and Dukler (1976).

3) The pressure gradient is calculated Eq. (5.69).

4) The average wall shear stress \( \tau_{WL} \) is calculated using Eq. (5.76), and the mixing length \( \ell_M \) at each grid point is determined.

5) The velocity profile of the liquid-phase is calculated using the Gauss-Seidel iterative technique (Eqs. 5.82 and 5.84).

6) The dimensionless liquid flow rate is calculated by Eq. (5.63) and is compared with the input flow rate, namely, \( W'_L = \pi / 4 \).

7) Steps 2 to 6 are repeated until convergence of the flow rate is achieved.

**5.4.3 Results and Discussion.** The results of the model predictions reported in this section are for an air-water flow in a 2.54 cm pipe at STP, with flow rates of \( v_{SL} = 0.1 \) m/s and \( v_{SG} = 1 \) m/s. Note that
turbulent stratified flow for air-water flow in horizontal pipes at STP takes place over quite a narrow range of liquid flow rates, namely $0.08 < v_{sl} < 0.12$ m/s.

**Velocity Profiles.** Typical results of the velocity field over half the liquid phase cross-section are shown in Figures 5.38, 5.39 and 5.40 for horizontal, upward inclined and downward inclined flow, respectively. The abscissa is the distance from the normal mid-plane and the ordinate the distance from the bottom pipe wall. The velocity is normalized with respect to the maximum velocity, namely, $v'_l / v'_{l(\text{MAX})}$, where $0 \leq \left| v'_l / v'_{l(\text{MAX})} \right| \leq 1$.

**Figure 5.38** shows the velocity field for the horizontal configuration. As can be observed, the velocity depends mostly on the radial coordinate, varying from zero on the pipe wall to the maximum velocity near the pipe center. In such a case the velocity can be approximated by $v = v(r)$, as assumed by Cheremisinoff and Davis (1979).

![Fig. 5.38—Velocity field for horizontal flow ($v_{sl} = 0.1$ m/s and $v_{sg} = 1$ m/s).](image)

**Figure 5.39** represents the velocity field for a $10^\circ$ upward inclined flow. The velocity field in this configuration is completely different than that of the horizontal case. The upper liquid layer moves at a higher velocity, in the flow direction, due to the shear exerted by the gas-phase at the interface. However, near the lower pipe wall the liquid flows backwards, resulting in a negative shear on the pipe surface. In this region curves of constant velocity are closed contours. The zero velocity line intersects the pipe wall, separating the flow cross-section into forward and backward flows. Clearly the one-dimensional approach, which is based on the calculation of the shear stress using the average velocity, does not apply here (as it yields a positive shear).
Fig. 5.39— velocity field for upward inclined flow \((\theta = 10^\circ, v_{L} = 0.1 \text{ m/s and } v_{G} = 1 \text{ m/s})\).

In downward inclined flow, as shown in Fig. 5.40, all the liquid moves forward in the flow direction, with a considerably lower equilibrium liquid level, as compared to the horizontal flow. The velocity field is, however, similar to that of the horizontal case, as it is always in the positive direction.

Fig. 5.40— Velocity field for downward inclined flow \((\theta = -10^\circ, v_{L} = 0.1 \text{ m/s and } v_{G} = 1 \text{ m/s})\).
**Liquid Holdup and Pressure Drop.** Experimental data for liquid holdup and pressure drop in stratified flow are available in the literature mainly for horizontal flow only. Such data are found in the studies of Berglin and Gazley (1949), Hoogendoorn (1959), Govier and Omer (1962), Agrawal et al. (1973) and Cheremisinoff and Davis (1979). The data are usually presented in the form of \( H_L \) and \( \phi_G \) vs. \( X \), where \( \phi_G = \frac{dp_g/\partial g}{dp/\partial L} \) is the ratio of the two-phase to single gas-phase pressure drop and \( X \) is the Lockhart and Martinelli parameter (see Section 2.2.3).

Figures 5.41 and 5.42 show, respectively comparisons between the holdup and pressure drop predicted by the present model and the experimental measurements obtained by the aforementioned investigators. Also shown in the figures are the predictions of the Lockhart and Martinelli (1949) correlation and the Taitel and Dukler (1976) one-dimensional model. The results of the present model are plotted for two cases of interfacial shear stress: the first case assumes smooth interface conditions, while the second case is for small amplitude waves flow.

![Comparison between theory and experimental data for liquid holdup.](image)

As shown in Fig. 5.41, the results of the analysis for the liquid holdup under smooth interface \( (f_I = f_G) \) conditions and the results of the Taitel and Dukler model are very close. For wavy interfacial structure, however, the present analysis predicts lower values for \( H_L \) than the Taitel and Dukler model for \( X < 0.2 \), and shows better agreement with the experimental measurements. This is due to the fact that the present model takes into account the interfacial shear stress for wavy flow \( (f_I = 0.0142, \text{ Eq. 5.68}) \). However, for \( X < 0.1 \) the data fall below the predictions of the model with Eq. 5.68. For these conditions roll wave develop on the interface, and a different correlation for \( f_I \) should be used. Amaravadi (1993) proposed a value of \( f_I = 0.03 \) for the interfacial friction factor for stratified flow with
roll waves, (see Sections 3.3.2 and 3.3.3). Utilization of \( f_I = 0.03 \) in the proposed model will yield lower values for \( H_L \), which would agree better with the experimental data for \( X < 0.1 \) (similarly to the results in Fig. 3.32).

A similar comparison of the dimensionless pressure drop \( \phi_G \) vs. is given in Fig. 5.42. As can be seen, the Lockhart and Martinelli correlation predicts considerably higher values of \( \phi_G \), as compared to the experimental data. The Taitel and Dukler model, on the other hand, forms a lower bound for the data points. As expected, the predictions of this analysis for a wavy-interface is somewhat higher than those for a smooth-interface condition. The data points above the curve for the wavy interface correspond to the development of roll waves on the interface. These data points can be predicted more accurately by the present model provided that a suitable correlation for the interfacial friction factor applicable for roll waves condition is used, such as \( f_I = 0.03 \) (Amaravadi, 1993).

![Fig. 5.42—Comparison between theory and experimental data for pressure drop.](image)

The effect of pipe inclination on the liquid equilibrium level and pressure drop is demonstrated in Figs. 5.43, 5.44 and 5.45. In these figures the solution for a smooth interface is used and a comparison with the Taitel and Dukler model is made. The liquid flow rate, \( v_{SL} \), varies between 0.05 to 1 m/s. In the lower range the flow is usually not yet turbulent, while in the upper range the flow is no longer stratified in horizontal and upward flows. Though the range of liquid flow rates covered exceeds the range of the actual turbulent stratified flow conditions, it does however clearly demonstrate the effect of the liquid flow rate on the holdup and pressure gradient. The gas flow rate, \( v_{SG} \), varies between 0.01 to 100 m/s, covering laminar and turbulent flows.
Fig. 5.43—Comparison of holdup and pressure drop for horizontal air-water flow at STP.
As expected, it can be observed from Figures 5.43, 5.44 and 5.45, which under the same flow conditions the liquid level, \( h_L / d \), is considerably higher in upward inclined flow, as compared to horizontal flow, whereas in downward inclined flow the liquid level is considerably lower than that for horizontal flow. As an example, for the flow rates of \( v_{SL} = 0.1 \) and \( v_{SG} = 1 \) m/s, the liquid level \( h_L / d \) is 0.66, 0.89 and 0.18 for horizontal, upward inclined and downward inclined flows, respectively. In upward inclined flow gravity forces act opposite to the flow direction, resulting in a lower liquid velocity and higher liquid level in the pipe. In downward inclined flow, on the other hand, the velocity is much higher and the liquid level in the pipe is lower, as the gravity acts in the direction of the flow for this case.

The same trends are observed the figures describing the pressure drop. Comparing \( \phi_g \) for the same flow conditions, it is again observed, as expected, that the pressure drop in upward inclined flow is much higher than in horizontal flow and considerably lower in downward inclined flow.
5.4.4 Generalization of Results. The solution of Eq. 5.62 with the proper boundary conditions and the turbulent viscosity calculated by Eq. 5.75 shows that the liquid level in stratified turbulent flow depends on the following dimensionless groups:

\[
\frac{h_L}{d} = F(K_L', \text{Re}_{SL}, \tau'_L, \text{Re}^*),
\]

where

\[
K_L' = \left( \frac{dp}{dL} + \rho_L g \sin \theta \right) \left( \frac{d}{\rho_L v_{SL}^2} \right),
\]

\[
\text{Re}_{SL} = \frac{v_{SL} d \rho_L}{\mu_L},
\]

\[
\tau'_L = \frac{\tau_L}{\rho_L v_{SL}^2},
\]

and

\[
\text{Re}^* = \frac{v_{SL}}{\rho_L v_{SL}^2}.
\]
\[ \text{Re}^* = \frac{d \mu_l}{\tau_{WL}} \left( \frac{\rho_{WL}}{\rho_l} \right). \qquad \text{(5.86)} \]

The average liquid wall shear stress \( \tau_{WL} \) is calculated from a momentum balance on the liquid phase, as given in Eq. 5.76. Converting this equation into dimensionless form yields

\[- A_L K'_L - \left( \frac{\text{Re}^*}{\text{Re}_L} \right)^2 S'_L + \tau'_L S'_L = 0. \qquad \text{(5.87)}\]

Note the \( \text{Re}^* \) can be expressed by the other groups and eliminated from Eq. 5.86. Thus, the solution depends on

\[ \frac{h_L}{d} = F(K'_L, \text{Re}_L, \tau'_L). \qquad \text{(5.88)} \]

It is suggested, however, to present the results in the form

\[ \frac{h_L}{d} = F(K'_L, \text{Re}_L, G'_E), \qquad \text{(5.89)} \]

where

\[ G'_E = \frac{4 \tau'_L}{K'_L} = \frac{4 \tau'_L/d}{dp dL + \rho_l g \sin \theta} \quad \text{(5.90)} \]

since \( G'_E \) is not sensitive to the liquid and gas flow rates (as contrary to \( \tau'_L \)).

For horizontal flow

\[ G'_E = \frac{4 f_I \rho_l v^0_l}{2 d} = \frac{d_g}{d} \frac{f_I}{f_G} \quad \text{(5.91)} \]

and for smooth interface conditions where \( f_I \rightarrow 1 \), the solution depends only on

\[ \frac{h_L}{d} = F(K'_L, \text{Re}_L). \qquad \text{(5.92)} \]

The result of Eq. (5.92) is also obtained for \( f_I / f_G \approx \text{const.} \). Equation (5.92) shows that \( h_L / d \) depends on two parameters, \( K'_L \) and \( \text{Re}_L \). This is equivalent also to the relationship

\[ \frac{h_L}{d} = F(X, \text{Re}_L), \quad \text{(5.93)} \]
namely, that the equilibrium liquid level, or holdup, depends on the Lockhart and Martinelli parameter and the superficial Reynolds number. Note, however, that Eq. (5.93) is for given values of $f_l / f_G$.

Figure 5.41 shows the experimental results in the form of $h_L / d = F(X, \text{Re}_{SL}, f_l / f_G)$. As explained above, this is indeed a valid single representation of the liquid holdup. Likewise, it can be shown that $\phi_G$, the dimensionless pressure drop, is only a function of $h_L / d$ and $f_l / f_G$, and, thus, can be represented as a function of the said variables. This is shown in Figure 5.42.

The generalized solution of Eq. 5.89, which covers a range of practical importance, is given in Figs. 5.46, 5.47 and 5.48 for $G'_E = 0.1$, 1 and 10, respectively. Following is the solution procedure:

1) Assume value for $h_L / d$. As an initial guess, one can utilize the value predicted by Taitel and Dukler (1976) model.

2) Calculate $K'_L$, $\text{Re}_{SL}$ and $G'_E$ from Eqs. 5.86 and 5.90 using Eqs. 5.69 and 5.68 to determine the pressure drop and the interfacial shear stress.

3) Find a new value for $h_L / d$ using Figs. 5.46 to 5.48.

4) Repeat steps 2 and 3 with the new value of $h_L / d$ until convergence is reached.

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**Fig. 5.46—Generalized liquid level chart for $G'_E = 0.1$.**
Fig. 5.47—Generalized liquid level chart for $G'_E = 1$.

Fig. 5.48—Generalized liquid level chart for $G'_E = 10$. 
5.5 A Unified Mechanistic Model for Pipelines and Wellbores

This section presents the recent study by Gomez et al. (2000), which is a unified model that is applicable to pipelines and wellbores, namely, for the range of inclinations of \(-10^\circ \leq \theta \leq 90^\circ\). The model can be further extended all the way to downward vertical flow, to cover the entire range of inclination angles. This study is actually a summary of most of the studies presented in Chapters 1 to 5.

5.5.1 Introduction. The mechanistic models developed over the past three decades have been formulated separately for pipelines and wellbores. Following is a brief review of the literature for these two cases.

**Pipeline Models.** These models are applicable for horizontal and near horizontal flow conditions, namely, between \(\pm 10^\circ\). The pioneering and most durable model for flow pattern prediction in pipelines was presented by Taitel and Dukler (1976). Other studies have been carried out for the prediction of specific transitions, such as the onset of slug flow (Lin and Hanratty, 1986) or different flow conditions, such as high pressure (Wu et al., 1987). Separate models have been developed for stratified flow (Taitel and Dukler, 1976, Cheremisinoff and Davis, 1979, Shoham and Taitel, 1984 and Issa, 1988); slug flow (Dukler and Hubbard, 1975, Nicholson et al., 1978 and Kokal and Stanislav, 1989); annular flow (Laurinat et al., 1985 and James et al., 1987); and dispersed-bubble flow (homogeneous no-slip model, Wallis, 1969). A comprehensive mechanistic model, incorporating a flow pattern prediction model and separate models for the different flow patterns, was presented by Xiao et al. (1990) for pipeline design.

**Wellbore Models.** These models are applicable mainly for vertical flow, but can also be applied as an approximation for off vertical sharply inclined flow \((60^\circ \leq \theta \leq 90^\circ)\). A flow pattern prediction model was proposed by Taitel et al. (1980) for upward vertical flow, which was later extended to inclined flow by Barnea et al. (1985). Specific models for the prediction of the flow behavior have been developed for bubble flow, such as Hasan and Kabir (1988) and Caetano et al. (1992); slug flow (Fernandes et al., 1983, Sylvester, 1987 and Vo and Shoham, 1989); and annular flow, including Oliemans et al. (1986) and Alves et al. (1991). Comprehensive mechanistic models for vertical flow have been presented by Ozon et al. (1987), Hasan and Kabir (1988), Ansari et al. (1994) and Chokshi et al. (1996).

**Unified Models.** Attempts have been made in recent years to develop unified models that are applicable for the range of inclination angles between horizontal \((0^\circ)\) and upward vertical \((90^\circ)\) flow. These models are practical as they incorporate the inclination angle. Thus, there is no need to apply different models for the different inclination angles encountered in horizontal, inclined and vertical pipes. A unified flow pattern prediction model was presented by Barnea (1987), valid for the entire range of inclination angles \(-90^\circ \leq \theta \leq 90^\circ\). Felizola and Shoham (1995) presented a unified slug flow model, applicable to the inclination angle range from horizontal to upward vertical flow. A unified mechanistic model applicable to horizontal, upward and downward flow conditions has been presented by Petalas and Aziz (1996), which was tested against a large number of laboratory and field data. Recently, Gomez et al. (2000) presented a unified correlation for the prediction of the liquid holdup in the slug body.

The above literature review reveals that separate comprehensive mechanistic models are available for pipeline flow and wellbore flow. Also, only very few studies have been published on unified modeling. The objective of this study is to present a systematic, comprehensive and unified model applicable for both pipelines and wellbores \(-10^\circ \leq \theta \leq 90^\circ\). This will provide more efficient computing
algorithms, as the model can be applied conveniently for both pipelines and wellbores, without the need to switch among different models. The proposed model will be evaluated against new field data, along with other published models and correlations.

5.5.2 Unified Model Formulation. The unified model consists of a unified flow pattern prediction model and separate unified models for the different existing flow patterns. These are briefly described below.

Unified Flow Pattern Prediction Model. The Barnea (1987) model is applicable for the entire range of inclination angles, namely, from upward vertical flow to downward vertical flow \((-90^\circ \leq \theta \leq 90^\circ)\). Below is a summary of the pertinent different transition boundaries of the model, including the stratified to non-stratified, slug to dispersed-bubble, annular to slug and bubble to slug flow.

Stratified to Non-Stratified Transition (Transition A). The criterion for this transition is the same as the original one proposed by Taitel and Dukler (1976), based on a simplified Kelvin-Helmholtz stability analysis given by

\[
F^2 \left[ \frac{1}{(1 - \tilde{h}_L)^2} \frac{\tilde{v}_S^2 \tilde{S}_L}{A_G} \right] \geq 1, \quad \text{..........................................................(3.49)}
\]

where the superscript tilde symbol (\(^\sim\)) represents a dimensionless parameter (length and area are normalized with \(d\) and \(d^2\), respectively, and the phase velocity is normalized with the corresponding superficial velocity). \(F\) is a dimensionless group given by

\[
F = \left( \frac{\rho_G}{\rho_L - \rho_G} \right) \frac{v_{SG}}{\sqrt{d g \cos \theta}}. \quad \text{................................................(3.50)}
\]

This transition boundary requires the knowledge of the equilibrium liquid level, \(\tilde{h}_L = h_L / d\), which is discussed in section 3.2.1.

Slug to Dispersed-Bubble Transition (Transition F-G). This transition occurs at high liquid flow rates, where the turbulent forces overcome the interfacial tension forces, dispersing the gas-phase into small bubbles in the continuous liquid-phase. The resulting maximum bubble size can be obtained from (similar to Eq. 4.20):

\[
d_{\text{MAX}} = \left[ 0.725 \sqrt[5]{0.415 \frac{v_{SG}}{v_M}} \left( \frac{\sigma}{\rho_L} \right)^{0.6} \left( \frac{2 f_M v_M^3}{d} \right)^{-0.4} \right]. \quad \text{..................................................(5.94)}
\]

Two critical bubble diameters are considered. First is the critical diameter, below which bubbles do not deform, avoiding agglomeration or coalescence, given by

\[
d_{\text{CD}} = 2 \left[ \frac{0.4 \sigma}{(\rho_L - \rho_G) g} \right]^{0.5}. \quad \text{............................................................(4.21)}
\]

The other critical diameter is applicable to shallow inclinations (between \(\pm 10^\circ\)), where due to buoyancy bubbles larger than this diameter migrate to the upper part of the pipe causing “creaming” and transition to slug flow, as follows
\[ d_{CB} = \frac{3}{8 (\rho_L - \rho_g)} \frac{f_M v_M^2}{g \cos \theta} \] .................................(5.34)

Transition to dispersed-bubble flow occurs when the maximum possible bubble diameter, given by Eq. 5.94, is less than both critical diameters given by Eqs. 4.21 or 5.34, namely

\[ d_{MAX} \leq d_{CD} \text{ and } d_{MAX} \leq d_{CB} \] .................................(5.36)

The transition boundary given by Eq. 5.36 is valid for \( \alpha < 0.52 \), which represents the maximum possible packing of bubbles for a cubic lattice configuration. For larger values of void fraction, agglomeration of bubbles occurs, independent of the turbulence forces, resulting in a transition to slug flow. This criterion is given by

\[ \frac{v_{SG}}{v_{SG} + v_{SL}} = 0.52. \] .....................................................(4.24)

**Slug to Annular Transition (Transition J).** Two mechanisms are responsible for the transition to annular flow, causing blockage of the gas core by the liquid-phase. The two mechanisms are based on the characteristic film structure in annular flow, as follows:

1) Instability of the liquid film due to downward flow near the pipe wall. The criterion for the instability of the film is obtained from the simultaneous solution of the following two dimensionless equations:

\[ Y = \frac{1 + 75 H_L}{(1 - H_L)^{2.5} H_L} - \frac{1}{H_L^2} X^2, \] .................................(5.28)

and

\[ Y = \frac{2 - 1.5 H_L}{H_L^2 (1 - 1.5 H_L)} X^2 \] .................................(5.29)

where \( X \) is the Lockhart and Martinelli parameter and \( Y \) is a dimensionless gravity group defined, respectively, by

\[ X^2 = \frac{4C_L}{d} \left( \frac{\rho_L v_{SL} d}{\mu_L} \right)^{\frac{1}{2}} \frac{\rho_L v_{SL}^2}{2} - \frac{dp}{dL}_{SL}, \] .................................(3.35)

and

\[ \frac{4C_G}{d} \left( \frac{\rho_G v_{SG} d}{\mu_G} \right)^{\frac{1}{2}} \frac{\rho_G v_{SG}^2}{2} - \frac{dp}{dL}_{SG} \]

\[ Y = \frac{(\rho_L - \rho_G) g \sin \theta}{\frac{4C_G}{d} \left( \frac{\rho_G v_{SG} d}{\mu_G} \right)^{\frac{1}{2}} \frac{\rho_G v_{SG}^2}{2} \frac{dp}{dL}_{SG}} \] .................................(3.36)

Note that Eq. 5.28 yields the steady-state solution for the liquid holdup \( H_L \), while Eq. 5.29 yields the value of the liquid holdup that satisfies the condition of film instability.
2) Wave growth on the interface due to large liquid supply from the film. If sufficient liquid is provided, the wave will grow and bridge the pipe, resulting in slug flow. The condition for wave growth is given by

\[ H_L \geq 0.24 \text{ or } \tilde{\delta}_L \geq 0.065. \] .................................................................(5.26)

Thus, simultaneous solution of Eqs. 5.28 and 5.26 provides the transition boundary criterion due to wave growth.

Transition from annular to slug flow will occur whenever one of the two criteria is satisfied. A smooth change between the two mechanisms is obtained when the inclination angle varies over the entire range of inclinations, or when a change occurs in the operational conditions.

**Bubble to Slug Transition (Transition E).** The transition from bubble to slug flow occurs at relatively lower liquid flow rates, as compared to the transition from slug to dispersed-bubble flow. Under these conditions the turbulent forces are negligible, and the transition is caused by coalescence of bubbles at a critical gas void fraction of \( \alpha = 0.25 \), as follows

\[ v_{SL} = 3.0v_{SG} - 1.15 \left[ \frac{g(\rho_L - \rho_G)\sigma}{\rho_L^2} \right]^{0.25} \sin \theta. \] .................................................................(5.1)

Bubble flow exists in vertical and sharply inclined pipes, with inclination angles between 90° to 60°, at low liquid flow rates, provided that the pipe diameter is larger than

\[ d \geq 19 \left[ \frac{(\rho_L - \rho_G)\sigma}{\rho_L^2 g} \right]^{0.5}. \] .................................................................(4.15)

**Elimination of Transition Discontinuities.** Mechanistic models for the prediction of pressure traverses in multiphase flow are notorious for creating discontinuities. This is the result of switching from one flow pattern model to another as the transition boundary is crossed. Different models are used for different flow patterns to predict the liquid holdup and pressure drop, which might result in a discontinuity. In order to avoid this problem in the proposed model, the following criteria have been implemented to smoothen the transitions between the different flow patterns.

**Bubble to Slug and Slug to Dispersed-Bubble Transitions.** Near the transition boundaries from slug to bubble or dispersed-bubble flow, the liquid film/gas pocket region behind the slug body, namely, \( L_F \), becomes small. The short film/gas length can prevent the slug flow model from converging. Thus, to solve this problem, when slug flow is predicted near these transition boundaries, the following constraints have been developed:

If \( L_F \leq 1.2 \) and \( v_{SL} \leq 0.6 \text{ m/s} \) then Bubble Flow exists,

If \( L_F \leq 1.2 \) and \( v_{SL} > 0.6 \text{ m/s} \) then Dispersed-Bubble Flow exists. ..............................(5.95)

The value \( L_F / d = 1.2 \) is based on the mechanism that once the Taylor bubble length approaches the pipe diameter, it becomes unstable and might break into small bubbles. Under these conditions, for high superficial liquid velocities, due to turbulence intensity and bubble break-up and dispersion, the resulting flow pattern will be dispersed-bubble flow. However, for low superficial liquid velocities, due
to low turbulence intensity and coalescence of the small bubble to larger ones, the resulting flow pattern will be bubble flow.

**Slug to Annular Transition.** A two-fold problem is associated with this transition boundary. First, a discontinuity in the pressure gradient between slug flow and annular flow occurs. Also, if slug flow is predicted near this transition boundary, due to the high gas rates, the film/gas zone becomes long, resulting in a very thin film thickness, approaching zero. This can cause the slug flow model convergence problems. To alleviate the two problems, a transition zone is created between slug flow and annular flow, based on the superficial gas velocity. The transition zone is predicted by the critical velocity corresponding to the droplet model used by Taitel et al. (1980), as follows

\[
V_{SG, (CRIT)} = \frac{3.1 \sigma g \sin \theta (\rho_l - \rho_g)^{0.25}}{\rho_g^{0.5}}.
\]

Thus, for a given superficial liquid velocity, the transition region is defined when the superficial gas velocity is greater than the critical superficial gas velocity, as given in Eq. 5.2, and less than the superficial gas velocity on the transition boundary to annular flow, as predicted by the Barnea model (1987) (Eqs. 5.26, 5.28 and 5.29). Hence, when slug flow is predicted in the transition zone, the pressure gradient is averaged between the pressure gradient under slug flow and annular flow conditions. The corresponding slug flow pressure gradient is calculated at the given superficial liquid velocity and the critical superficial gas velocity, given by Eq. 5.2. Similarly, the corresponding pressure gradient under annular flow is calculated at the given superficial liquid velocity and the superficial gas velocity on the transition boundary to annular flow, as predicted by the Barnea model. This averaging eliminates numerical problems and ensures a smooth pressure gradient across the slug to annular transition boundary.

Flow pattern prediction is the first phase of the analysis. It is also required to develop separate models for each of the existing flow patterns, which are capable of predicting the flow characteristics of each flow pattern, including the liquid holdup and the pressure gradient. These models are covered next.

**Unified Stratified Flow Model.** The physical model for stratified flow is given in Fig. 5.49. A modified form of the Taitel and Dukler (1976) model is used here. Two modifications are introduced: The liquid wall friction factor is determined by Ouyang and Aziz (1996) and the interfacial friction factor as suggested by Xiao et al. (1990), as will be presented next.

![Fig. 5.49—Physical model for stratified flow.](image-url)
**Momentum Balances.** The momentum (force) balances for the liquid and gas phases are given, respectively, by

\[
-A_L \frac{dp}{dL} = -\tau_{wl} S_L + \tau_i S_I - \rho_L g Sin \theta = 0, \quad \text{(3.26)}
\]

and

\[
-A_G \frac{dp}{dL} = -\tau_{wg} S_G - \tau_i S_I - \rho_G A_G g Sin \theta = 0. \quad \text{(3.27)}
\]

Eliminating the pressure gradient from Eqs. (3.26) and (3.27), the combined momentum equation for the two phases is obtained, as follows

\[
\frac{\tau_{wg}}{A_G} S_G - \frac{\tau_{wl}}{A_L} S_L + \tau_i \left( \frac{1}{A_L} + \frac{1}{A_G} \right) - (\rho_L - \rho_G) g Sin \theta = 0. \quad \text{(3.28)}
\]

The combined momentum equation is an implicit equation for \( h_L \) (or \( \bar{h}_L = h_L / d \)), the liquid level in the pipe. Solution of the equation, carried out by a trial and error procedure, requires the determination of the different geometrical, velocity and shear stress variables (see section 3.2.1.1). Note that under high gas and low liquid flow rates, multiple solutions can occur. It can be shown that the smallest of the three solutions is the physical and stable solution.

Once the liquid level \( h_L / d \) is determined, the liquid holdup, \( H_L \), can be calculated in a straightforward manner from geometrical relationships, as follows

\[
H_L = \frac{1}{\pi} \left[ \pi \cos^{-1} \left( \frac{2h_L}{d - 1} \right) + (2h_L / d - 1) \sqrt{1 - (2h_L / d - 1)^2} \right]. \quad \text{(5.96)}
\]

**Pressure Drop.** Once the pressure drop can be calculated from either Eq. 3.26 or 3.27. Either equation provides the frictional and the gravitational pressure losses, and neglects the accelerational pressure losses.

**Closure Relationship.** The wall shear stresses corresponding to each phase are determined based on single-phase analysis utilizing the hydraulic diameter concept, as follows:

\[
\tau_{wl} = f_L \frac{\rho_L v_L^2}{2}, \quad \text{and} \quad \tau_{wg} = f_G \frac{\rho_G v_G^2}{2}. \quad \text{(3.31)}
\]

The respective hydraulic diameters of the liquid and gas phases are given by

\[
d_L = \frac{4A_L}{S_L}, \quad \text{and} \quad d_G = \frac{4A_G}{S_G + S_I}. \quad \text{(3.29)}
\]

The Reynolds numbers of each of the phases are

\[
Re_L = \frac{d_L v_L \rho_L}{\mu_L}, \quad \text{and} \quad Re_G = \frac{d_G v_G \rho_G}{\mu_G}. \quad \text{(3.30)}
\]

Taitel and Dukler (1976) proposed that both the liquid and gas wall friction factors, \( f_L \) and \( f_G \), can be calculated using a standard friction factor chart. However, Ouyang and Aziz (1996) found this procedure to be appropriate for the gas-phase only. This is due to the fact that the liquid wall friction factor can be
affected significantly by the interfacial shear stress, especially for low liquid holdup conditions. Thus, \( f_G \) is determined from a standard friction factor chart, while \( f_L \) is determined by the correlation developed by Ouyang and Aziz, incorporating the gas and liquid flow rates, as given below

\[
f_G = \frac{16}{Re_G} \quad \text{for} \quad Re_G \leq 2300, \quad \text{and}
\]

\[
f_G = 0.001375 \left[ 1 + \left( \frac{2 \times 10^4 \cdot \gamma}{d} + \frac{10^6}{Re_G} \right)^{1/3} \right] \quad \text{for} \quad Re_G \leq 2300. \quad \text{(5.97)}
\]

and

\[
f_L = \frac{1.6291}{Re_{SL}} \left( \frac{v_{SG}}{v_{SL}} \right)^{0.0926}. \quad \text{(3.74)}
\]

The interfacial shear stress is given, by definition, as

\[
\tau_I = f_I \rho_G (v_G - v_L)^2 \quad \text{for} \quad f_I = f_G. \quad \text{(3.65)}
\]

The interfacial friction factor for stratified smooth flow is taken as the friction factor between the gas phase and the wall \((f_I = f_G)\). However, for stratified wavy flow, the interfacial friction factor is determined as suggested by Xiao et al. (1990) (see Section 3.3.1).

**Unified Slug Flow Model.** The unified and comprehensive analysis of slug flow, presented by Taitel and Barnea (1990), is utilized in the present study with the following features:

1. A uniform film along the liquid film/gas pocket zone.
2. A global momentum balance on a slug unit for pressure drop calculations.

The original Taitel and Barnea model was extended to vertical flow by assuming a symmetric film around the Taylor bubble for inclination angles between 86° and 90°.

With the above characteristics, the original model is simplified considerably, as given below, avoiding the need for a numerical integration along the liquid film region. The proposed simplified model is considered sufficiently accurate for practical applications. Refer to Fig. 5.50 for the physical model for slug flow.

**Mass Balances.** An overall liquid mass balance over a slug unit results in (similar to Eq. 3.159)

\[
v_{SL} = v_{L_{LS}} H_{L_{LS}} \frac{L_S}{L_U} + v_{L_{LTB}} H_{L_{LTB}} \frac{L_F}{L_U}. \quad \text{(5.98)}
\]

A mass balance can also be applied between two cross sectional areas, namely, in the slug body and in the gas pocket/liquid film region, in a coordinate system moving with the translational velocity, \( v_{TB} \), yielding (similar to Eq. 3.123)
A continuity balance on both liquid and gas phases results in a constant volumetric flow rate through any cross section of the slug unit. Applying this balance on cross sections in the liquid slug body and in the liquid film region gives, respectively,

\[ (v_{TB} - v_{LLS})H_{LLS} = (v_{TB} - v_{LTB})H_{LTB}. \] ..................(5.99)

Equation 3.125 can be used to determine \( v_{LLS} \), the liquid velocity in the slug body, as the other variables are given later in the form of closure relationships. The liquid film velocity, \( v_{LTB} \), can be determined from Eq. 5.98 for a given liquid holdup in this region, \( H_{LTB} \). Once \( v_{LTB} \) is determined, it is possible to determine \( v_{GTB} \), the gas velocity in the gas pocket, from Eq. 3.158.

The average liquid holdup in a slug unit is defined as

\[ H_{LSU} = \frac{H_{LLS}L_S + H_{LTB}L_F}{L_U}. \] ..................(5.100)

Using Eqs. 5.98, 5.99 and 3.125, it is possible to show that the average liquid holdup is

\[ H_{LSU} = \frac{v_{TB}H_{LLS} + v_{GLS}(1 - H_{LLS}) - v_{SG}}{v_{TB}}. \] ..............(3.128)

Equation 3.128 shows an interesting result, namely, that the average liquid holdup in a slug unit is independent of the lengths of the different slug zones.
Hydrodynamics of the Liquid Film. Considering a uniform liquid film thickness, a combined momentum equation, similar to the case of stratified flow, can be obtained for the liquid film/gas pocket zone, as follows:

\[
\frac{\tau_f}{A_f} S_f - \frac{\tau_g}{A_g} S_g - \tau_f S_f \left( \frac{1}{A_f} + \frac{1}{A_g} \right) + (\rho_L - \rho_G) g \sin \theta = 0. \quad (3.141)
\]

Solution of Eq. 3.141 yields the uniform (equilibrium) film thickness and the liquid holdup in this region, \( H_{LTB} \). This value can be used to determine the gas and liquid velocities in the slug and liquid film/gas pocket regions, as discussed below Eq. 3.158. The liquid film length can be determined from

\[
L_F = L_U - L_S. \quad (3.110)
\]

The slug length, \( L_S \), is given as a closure relationship while the slug unit length, \( L_U \), can be determined from Eq. 5.98, as follows

\[
L_U = \frac{L_S (v_{LLS} H_{LLS} - v_{LTB} H_{LTB})}{v_{SL} - v_{LTB} H_{LTB}}. \quad (3.159)
\]

Pressure Drop. The pressure drop for a slug unit can be calculated using a global force balance along a slug unit. Since the momentum fluxes in and out of the slug unit control volume are identical, the pressure gradient across this control volume for a uniform liquid film is (similar to Eq. 3.155):

\[
\frac{dp}{dL} = \rho_U g \sin \theta + \frac{\tau_s \pi d}{A_p} \frac{L_S}{L_U} + \frac{\tau_f S_f + \tau_g S_g L_F}{A_p} \frac{L_F}{L_U}, \quad (5.101)
\]

where \( \rho_U \) is the slug unit average density, as given by

\[
\rho_U = H_{LSU} \rho_L + (1 - H_{LSU}) \rho_G, \quad (5.102)
\]

and \( H_{LSU} \), the slug unit average liquid holdup, can be determined from Eq. 3.128.

Closure Relationships. The proposed model requires four closure relationships, namely, the liquid slug length, \( L_S \), the liquid holdup in the slug body, \( H_{LLS} \), the slug translational velocity, \( v_{TB} \), and the gas velocity of the small bubbles entrained in the liquid slug, \( v_{GLS} \). The closure relationships are given below:

1. A constant length of \( L_S = 30d \) and \( L_S = 20d \) is used for fully developed and stable slugs in horizontal and vertical pipes, respectively. For inclined flow, an average slug length is used based on inclination angle. However, for horizontal and slight inclinations (\( \theta \) between \( \pm 1^\circ \)) large diameter pipes (\( d > 2 \) inch), the Scott et al. (1989) correlation is used, as given below

\[
\ln(L_S) = -25.4 + 28.5 [\ln(d)]^{0.1}, \quad (3.173)
\]

where \( d \) is expressed in inch and \( L_S \) is in ft.

2. The liquid holdup in the slug body, \( H_{LLS} \), is predicted using the Gomez et al. (2000) unified correlation, given by
\[ H_{LLS} = 1.0 \exp\left[-(7.85 \times 10^3 \theta + 2.48 \times 10^6 \text{Re}_{LS})\right] \text{ for } 0^\circ \leq \theta \leq 90^\circ. \] ..................................(3.168)

where the liquid slug Reynolds number is calculated as
\[ \text{Re}_{LS} = \frac{\rho_L v_M d}{\mu_L}. \] .................................................................(3.169)

3. The slug translational velocity is determined from the Bendiksen (1984) correlation, namely,
\[ v_{TB} = c_0 v_M + 0.54 \sqrt{gd} \cos \theta + 0.35 \sqrt{gd} \sin \theta \text{ for } 0^\circ \leq \theta \leq 90^\circ. \] .............................................(3.164)

4. The gas velocity of the small bubbles entrained in the liquid slug, \( v_{GLS} \), is given by
\[ v_{GLS} = c_0 v_M + 1.53 \left[ \frac{g(\rho_L - \rho_G)}{\rho_G^2} \sigma \right]^{0.25} H_{LLS}^{0.5} \sin \theta. \] .............................................(3.165)

**Unified Annular Flow Model.** The model of Alves *et al.* (1991) developed originally for vertical and sharply inclined flow has been extended in the present study to the range of inclination angles covering pipelines and wellbores \((-10^\circ \leq \theta \leq 90^\circ\), as given below. The physical model for annular flow is given in Fig. 5.51.

![Physical model for annular flow](image)

**Fig. 5.51—Physical model for annular flow**

The annular flow model equations are similar to the stratified flow model ones, as both patterns are separated flow. The differences between the two models are the different geometrical and closure relationships, and the fact that the gas core in annular flow includes liquid entrainment.

**Momentum Balances.** The linear momentum (force) balances for the liquid and gas core phases are given, respectively, by
\[ -\tau_{WL} \frac{S_L}{A_F} + \tau_L \frac{S_L}{A_F} \left( \frac{dp}{dL} \right)_{F} - \rho_L g \sin \theta = 0, \] .............................................(4.79)

and
\[-\tau \frac{S}{A_c} \frac{dp}{dL} - \rho_c g \sin \theta = 0. \] .................................................................(4.80)

Eliminating the pressure gradients from the equations results in the combined momentum equation for annular flow, namely

\[-\tau_{\text{int}} \frac{S}{A_F} + \tau \frac{S}{A_c} \left[ \frac{1}{A_F} + \frac{1}{A_c} \right] (\rho - \rho_c) g \sin \theta = 0. \] ...................................................(4.81)

Equation 4.81 is an implicit equation for the film thickness \( \delta \) (or \( \delta \) \( \delta \) / 2) that can be solved by trial and error, provided that the proper geometrical, velocity and closure relationships are given. These are described below.

**Mass Balances.** The velocities of the liquid film and the gas core can be determined from simple mass balance calculations yielding, respectively,

\[ v_F = v_{SL} \frac{(1 - f_E) d^2}{4 \delta (d - \delta)} , \] .................................................................(4.85)

and

\[ v_C = \frac{(v_{SG} + v_{SL} f_E) d^2}{(d - 2 \delta)^2} . \] .................................................................(4.88)

Assuming a homogeneous no-slip flow of the gas and the entrained liquid droplets in the core, the gas void fraction in the core and the core average density and viscosity are given, respectively, by

\[ \alpha_C = \frac{v_{SG}}{v_{SG} + v_{SL} f_E} , \] .................................................................(4.89)

and

\[ \rho_C = \rho_g \alpha_C + \rho_l (1 - \alpha_C) , \quad \text{and} \quad \mu_C = \mu_g \alpha_C + \mu_l (1 - \alpha_C) . \] ..............................................(4.90)

The total void fraction is given by

\[ \alpha_T = \alpha_C \left(1 - 2 \frac{\delta}{d} \right)^2 . \] .................................................................(4.91)

**Pressure Drop.** Once the combined momentum equation, Eq. 4.79, is solved (with the proper geometrical, velocity and closure relationships) for film thickness \( \delta \) , the pressure gradient can be determined from either of the momentum equations, namely, Eq. 4.79 or 4.80.

**Closure Relationships.** The liquid wall shear stress is determined from single-phase flow calculations, based on the hydraulic diameter concept. The most difficult task in modeling annular flow is the determination of the interfacial shear stress, \( \tau_f \), and the entrainment fraction, \( f_e \). By all means this is an unresolved problem even for vertical or horizontal flow conditions. The definition of the interfacial shear stress for annular flow is
\[ \tau_I = f_I \frac{\rho_C (v_C - v_F)^2}{2}. \] \hspace{1cm} (4.64)

The interfacial friction factor can be expressed by

\[ I = \frac{f_I}{f_{SG}}. \] \hspace{1cm} (4.65)

where \( f_{SG} \) is the superficial gas-phase friction factor in a smooth pipe flow, which is always in turbulent flow, and can be calculated from

\[ f_{SG} = 0.046 Re_{SG}^{-0.2} = 0.046 \left( \frac{\rho_G v_{SG} d}{\mu_G} \right)^{-0.2}. \] \hspace{1cm} (4.66)

The interfacial correction parameter \( I \) is used to take into account the roughness of the interface. Different expressions for \( I \) are given by Alves et al. (1991). In the present study, the parameter \( I \) is an average between a horizontal factor and a vertical factor, based on the inclination angle, \( \theta \), as follows

\[ I_\theta = I_H \cos^2 \theta + I_V \sin^2 \theta. \] \hspace{1cm} (5.103)

The horizontal interfacial correction parameter is given by Henstock and Hanratty (1976)

\[ I_H = 1 + 850 F, \] \hspace{1cm} (4.72)

where

\[ F = \left[ 0.42 Re_F^{1.25} + 2.8 \times 10^{-4} Re_{SG}^{2.25} \right]^{0.4} \frac{\mu_L}{\mu_G} \left( \frac{\rho_G}{\rho_L} \right)^{0.5}. \] \hspace{1cm} (4.69)

The superficial gas Reynolds number, \( Re_{SG} \), is given in Eq. 4.66, and the liquid film Reynolds number, \( Re_F \), is defined as

\[ Re_F = \frac{\rho_L v_F d_F}{\mu_L}. \] \hspace{1cm} (4.92)

The vertical interfacial correction parameter is given by Wallis (1969)

\[ I_V = 1 + 300 \frac{\delta}{d}. \] \hspace{1cm} (4.67)

The entrainment fraction, \( f_E \), is calculated by the Wallis (1969) correlation, given by

\[ f_E = 1 - \text{EXP}[-0.125(\phi - 1.5)], \] \hspace{1cm} (4.74)

where

\[ \phi = 10^4 \frac{v_{SG} \mu_G}{\sigma} \left( \frac{\rho_G}{\rho_L} \right)^{0.5}. \] \hspace{1cm} (4.75)
**Unified Bubble Flow Model.** The extension of the Hasan and Kabir (1988) bubble flow model for the entire range of wellbore inclination angles was carried out by taking the component of the bubble rise velocity in the direction of the flow, as given below (see Fig. 5.52 for the bubble flow physical model). The liquid holdup can be determined from

\[
1.53 \left[ \frac{g(\rho_L - \rho_G)\sigma}{\rho_L^2} \right]^{0.25} (1 - \alpha)^{0.5} \sin\theta = \frac{v_{SG}}{\alpha} - 1.2v_M. \tag{4.42}
\]

Equation 4.42 must be solved numerically to determine the gas void fraction, \(\alpha\). Once the liquid holdup is computed, the gravitational and frictional pressure gradients are determined in a straightforward manner, as given in Section 4.3.2. For dispersed-bubble flow, the homogeneous no-slip model is used, as presented in Section 2.1.

---

**5.5.3 Results and Discussion.** This section includes the validation of the developed unified model with published laboratory and field data, and the performance of the model with new field data.

**Unified Model Validation.** Initially, the individual flow pattern models for slug flow, stratified flow, bubble flow and annular flow have been validated against several sets of available laboratory and limited field data. **Tables 5.1 and 5.2** present the range of data and the validation results, respectively.

**Unified Slug Model.** The validation of the proposed slug flow model has been carried out utilizing the following sets of data:

a) The Felizola and Shoham (1995) data provide detailed slug characteristics, liquid holdup and pressure drop, for the entire range of upward inclination angles between 10° and 90° at 10° increments.

b) The Nuland et al. (1997) data for 10°, 20°, 45°, 60° and 80° including liquid holdup and pressure drop.

c) The Schmidt (1977) data for vertical flow with liquid holdup only.
**TABLE 5.1. DATABASE FOR INDIVIDUAL FLOW PATTERN MODELS VALIDATION**

<table>
<thead>
<tr>
<th>Data Source</th>
<th>Flow Pattern</th>
<th>Inclination</th>
<th>Pipe Diameter Inch</th>
<th>Fluids</th>
<th>Liquid Density lbm/ft³</th>
<th>Pressure psi</th>
<th>Data Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minami (1982)</td>
<td>Stratified</td>
<td>θ = 0°</td>
<td>3</td>
<td>Air - Kerosene/Water</td>
<td>50/62.4</td>
<td>50</td>
<td>100</td>
</tr>
<tr>
<td>Nuland et al. (1997)</td>
<td>Slug</td>
<td>10° &lt; θ &lt; 60°</td>
<td>4</td>
<td>Dense Gas (SF₆) - Oil</td>
<td>51</td>
<td>145</td>
<td>52</td>
</tr>
<tr>
<td>Felizola &amp; Shoham (1995)</td>
<td>Slug</td>
<td>0° &lt; θ &lt; 90°</td>
<td>2</td>
<td>Air-Kerosene</td>
<td>50</td>
<td>250</td>
<td>72</td>
</tr>
<tr>
<td>Schmidt (1977)</td>
<td>Slug</td>
<td>θ = 90°</td>
<td>2</td>
<td>Air-Kerosene</td>
<td>50</td>
<td>225</td>
<td>15</td>
</tr>
<tr>
<td>Caetano et al. (1992)</td>
<td>Bubble</td>
<td>θ = 90°</td>
<td>Annulus</td>
<td>Air - Kerosene/Water</td>
<td>50/62.4</td>
<td>45</td>
<td>19</td>
</tr>
<tr>
<td>Alves et al. (1991)</td>
<td>Annular</td>
<td>θ = 90°</td>
<td>2.5</td>
<td>Natural Gas - Crude</td>
<td>27</td>
<td>1750</td>
<td>2 (75)</td>
</tr>
</tbody>
</table>

**TABLE 5.2. PERFORMANCE OF INDIVIDUAL FLOW PATTERN MODELS**

<table>
<thead>
<tr>
<th></th>
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<tr>
<td>Minami (1982)</td>
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<td>θ = 0°</td>
<td>Avg. Error</td>
<td>-20.8</td>
<td>33.5</td>
<td>-</td>
<td>-</td>
<td>100</td>
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<td>Nuland et al. (1997)</td>
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<td>10° &lt; θ &lt; 60°</td>
<td>Avg. Error</td>
<td>-6.7</td>
<td>9.6</td>
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<td>16.2</td>
<td>52</td>
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<tr>
<td>Felizola &amp; Shoham (1995)</td>
<td>Slug</td>
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<td>Avg. Error</td>
<td>0.6</td>
<td>13.2</td>
<td>20.6</td>
<td>25.5</td>
<td>72</td>
</tr>
<tr>
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<td>Slug</td>
<td>θ = 90°</td>
<td>Avg. Error</td>
<td>-0.3</td>
<td>15</td>
<td>-</td>
<td>-</td>
<td>15</td>
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<tr>
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<td>Bubble</td>
<td>θ = 90°</td>
<td>Avg. Error</td>
<td>-2.3</td>
<td>2.7</td>
<td>-</td>
<td>-</td>
<td>19</td>
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<tr>
<td>Alves et al. (1991)</td>
<td>Annular</td>
<td>θ = 90°</td>
<td>Avg. Error</td>
<td>-</td>
<td>-</td>
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<tr>
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<td>Avg. Error</td>
<td>-</td>
<td>-</td>
<td>-0.9</td>
<td>9.8</td>
<td>75</td>
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</table>

**Figure 5.53** presents a comparison of the predictions of the Gomez et al. (2000) slug body liquid holdup correlation with published experimental data (including additional data other than the above mentioned three sets). As can be seen, the correlation follows the trend of decreasing slug liquid holdup as the inclination angle increases.

![Fig. 5.53—Comparison between slug liquid holdup predictions by Gomez et al. (2000) and experimental data](image-url)
Comparisons between the predictions of the unified slug model and the experimental data were carried out for both the liquid holdup, $H_{LSU}$, and the pressure gradient, averaged over a slug unit, $-dp/dL\rangle$. The results for the different data sources are given in Table 5.2. As can be seen, the average error for liquid holdup is between $-6.7\%$ to $0.6\%$, while the absolute average error is between $9.6\%$ to $15\%$. For the pressure gradient the average error is between $7.5\%$ to $20.6\%$ and the absolute average error is between $10.2\%$ to $25\%$.

**Unified Bubble Model.** The data of Caetano et al. (1992) were utilized to test the model for bubble flow. Note that the Caetano et al. data were acquired in an annulus configuration with a 3-in. casing ID and 1.66-in. tubing OD. For this reason the comparison was carried out only for the liquid holdup. An equivalent diameter was used that provides the same cross sectional area and superficial velocities occurring in the annulus. The results in Table 5.2 show an excellent agreement with an average error and an absolute average error of $-2.3\%$ and $2.7\%$, respectively.

**Unified Stratified Model.** The stratified flow model was tested against the liquid holdup data of Minami (1982). The data were collected for air-water and air-kerosene. The model systematically under predicted the data, with an average error and average absolute error of $-20.8\%$ and $33.5\%$, respectively, as shown in Table 5.2. Note that, as reported by Minami, the original Taitel and Dukler (1976) model performed poorly against his data. The modification of both the liquid wall friction factor and the interfacial friction factor, implemented in the present study model, improve the predictions of the stratified model considerably.

**Unified Annular Model.** As shown in Table 5.1, Alves et al. (1991) provided 2 new field data points, in addition to the 75 data points taken from the Tulsa University Fluid Flow Projects (TUFFP) database, for which the wells are under annular flow (see Section 4.5.5). The model of Alves et al. shows an excellent agreement with the pressure drop data. As can be seen in Table 5.2, for the 2 data points the average error and average absolute errors are both $1.5\%$. For the 75-database points the average error is $-0.9\%$ and the average absolute error is $9.8\%$.

**Entire Unified Model Validation.** Following the validation of the individual flow pattern models, the entire unified model was evaluated against the TUFFP wellbore databank, as reported by Ansari et al. (1994). The databank includes a total of 1,723 laboratory and field data, for both vertical and deviated wells. The data cover a wide range of flow conditions: pipe diameter 1 to 8 in.; oil rate up to 27,000 bbl/d; gas rate up to 110,000 scf/d and oil gravity 8.3 to 112 °API. Additionally, 6 commonly used correlations and models have been evaluated against the databank. These are Ansari et al. (1994), Chokshi et al. (1996), modified Hagedorn and Brown (1965), Duns and Ros (1963), Beggs and Brill (1973) and Hasan and Kabir (1988). The modifications of the Hagedorn and Brown are the Griffith and Wallis (1961) correlation for bubble flow and the use of no-slip liquid holdup if greater than calculated liquid holdup. Note that except for the Beggs and Brill (1973) correlation, the other 5 methods were developed for vertical upward flow only. These methods are adopted in this study for deviated well conditions by incorporating the inclination angle in the gravitational pressure gradient calculations. The proposed unified model is the only mechanistic model applicable to all the inclination angle range, namely, $-10\° \leq \theta \leq 90\°$.

The overall performance of the unified model showed an average error of $-3.8\%$ and an absolute average error of $12.6\%$. The Hagedorn and Brown (1965) correlation showed the minimum average error and absolute average error of $1.2\%$ and $9.3\%$, respectively. However, the databank includes about
400 data points collected by Hagedorn and Brown (1965) to develop their correlation. An objective comparison should exclude these data points from the databank.

**Unified Model Performance and Results.** The ultimate goal of any model is to predict the flow behavior under field conditions. The performance of the proposed Gomez et al. (2000) unified model under field conditions was evaluated by comparison between its predictions and directional well field data provided by British Petroleum and Statoil. Two sets of data were provided. The first data set includes 21 data points (see Table 5.3) while the second data set includes 65 cases (see Table 5.4). The data include wells with different flow conditions: pipe diameter 2-7/8 to 7 in.; inclination angles 0° to 90°; oil rate 79 to 2,658 bbl/d; gas rate 42 to $23,045 \times 10^3$ scf/d, and water-cut 0 to 80%. Of the total cases, 59 wells were producing naturally and the remaining 27 were on artificial lift. Each data point included, in addition to the geometrical and operational variables, the well-head pressure, the well-head and bottom-hole temperatures and the total pressure drop.

**Fluid Physical Properties.** The fluid properties correlations used are summarized by Brill and Mukherjee (1999). The Glaso correlation was used for the prediction of the solution gas/oil ratio, oil formation volume factor and oil viscosity. The Standing z-factor was used in the calculations of the gas-phase properties. The Lee et al. correlation was used for the gas viscosity. The gas/oil surface tension was predicted by the Baker and Swerdloff correlation. The liquid-phase (oil and water) properties, namely, density, viscosity and surface tension, are calculated based on the volume fraction of the oil and water in the liquid-phase. The volume fractions were calculated based on the in-situ flow rates, assuming no-slip between the oil and water.

For the gas lift wells, the gas properties are calculated as follows. Up to the point of gas injection, the calculations are performed utilizing the flow rate and specific gravity of the formation gas. At the point of gas injection, the formation gas flow rate is combined with the injection gas rate to give the total gas flow rate. Also, a weighted average total specific gravity is used, based on the formation gas and injected gas corresponding specific gravities and their respective flow rates at STP. From the point of injection to the surface, the fluid properties, including the solution gas oil ratio (and hence free gas quantity), are determined based on the combined total gas specific gravity. No tuning of the fluid properties data has been done.

**Results and Discussion.** Table 5.3 reports the pressure drop prediction performance of the unified model, along with that of Chokshi et al. (1996), Hagedorn and Brown (1965) and Ansari et al. (1994), against the first data set (21 data points). Note that the table includes, in addition to the pressure drop, the gas/liquid ratio and the water-cut. The comparison shows a good agreement, with an average error of -5.2% (and a corresponding standard deviation, s.d., of 14.7) and an average absolute error of 13.1% (with a s.d. of 8.1) for the unified model. Corresponding errors for the other methods are as follows: –10.5% (s.d. 12.2) and 12.3% (s.d. 10.3) for Chokshi et al. (1966), –11.7% (s.d. 12.1) and 14.5% (s.d. 8.3) for Hagedorn and Brown (1965), and –16.1% (s.d. 14.0) and 17.5% (s.d. 12) for the Ansari et al. (1994) model.

Figure 5.54 shows a comparison between the predicted results of the unified model and measured pressure drops for the 65 cases of the second data set. The predictions of the proposed unified model show an excellent agreement against this data set, with an average error of 0% (s.d. 3.9), as compared to 4.5% (s.d. 4.5) for the Chokshi et al. model. The average absolute errors for the unified model and the Chokshi et al. model are 3.0% (s.d. 2.5) and 5.5% (s.d. 3.2), respectively.
Table 5.3. Performance of Gomez et al. (2000) Model and Other Methods for Dataset No. 1

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<td>Error (%)</td>
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<td>660</td>
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<td>632</td>
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Average Error [%]  
Std. Dev. Avg. Error  
Average Absolute Error [%]  
Std. Dev. Abs. Avg. Error

Fig. 5.54—Comparison between unified model predictions and data set no. 2.

The overall performance of the model was evaluated against the combined two data sets, including all 86 well cases. The results were compared with the predictions of only the Chokshi et al. (1996) model. In addition, a sensitivity analysis was carried out based on maximum deviation angle of the well, production method (natural or artificial lift) and tubing diameter. All the results are summarized in Table 5.4. Figure 5.55 presents the overall performance of the unified model against the combined two data sets in a graphical form.
As can be seen from Table 5.4, for the combined data sets the unified model shows an excellent performance, with an average error of -1.3% (s.d. 8.2) and absolute error of 5.5% (s.d. 6.2). These results are also shown graphically in Fig. 5.55. The Chokshi et al. (1996) model shows average error and absolute error of 0.9% (s.d. 9.6) and 7.1% (s.d. 6.4), respectively. As can be seen from the table, except for the 3 small diameter well cases, the unified model shows a better performance than the Chokshi et al. (1996) model, especially for large diameter tubing and deviated wells. It is believed that
the unified slug flow model is the main reason for this behavior, since it is more suitable for directional flow. Both models perform equally well for the entire range of water-cuts.

5.5.4 Conclusions. A unified steady-state two-phase flow mechanistic model for the prediction of flow pattern, liquid holdup and pressure drop is presented, applicable to the range of inclination angles for pipelines and wellbores, namely, $-10^\circ \leq \theta \leq 90^\circ$. The model consists of a unified flow pattern prediction model and 5 individual unified models for the stratified, slug, bubble, annular and dispersed-bubble flow patterns.

The proposed unified model was validated and compared to other 6 most commonly used models or correlations. This was carried out by running the unified model and the other methods against the TUFFP wellbore databank. The databank includes a total of 1723 laboratory and field data for both vertical and deviated wells. The overall performance of the unified model showed an average error of $-3.8\%$ and an absolute average error of $12.6\%$.

The performance of the unified model and other models and correlations were evaluated against 86 new directional well field data cases provided by British Petroleum and Statoil. The predictions of the unified model show an excellent agreement with data, with an average error of $-1.3\%$ and an absolute average error of $5.5\%$, with respective standard deviations, s.d., of 8.2 and 6.2. The predictions of the unified model were carried out without any tuning of either the model or the fluid properties data. It provides an accurate two-phase flow mechanistic model for research and design for the industry.

5.6 Examples

Example 5.1. Gas (air) lift is used to pump water to elevation of $L-H$, as shown in Fig. 5.56. The following data are given:

Pipe diameter: $d = 0.05$ m,

Submergence level: $H = 10$ m,

Bubble rise velocity: $v_{0x} = 0.25$ m/s,

Fluid properties: $\rho_L = 1000$ kg/m$^3$ and $\rho_G = 1.2$ kg/m$^3$,

$\mu_L = 7 \times 10^{-4}$ kg/m·s and $\mu_G = 1.9 \times 10^{-5}$ kg/m·s

Determine the liquid flow rate $v_{SL}$ as a function of the gas flow rate $v_{SG}$ and the gas lift pump efficiency for $L/H = 1.2$ and 1.4, as follows:

1. Neglecting fricitonal losses.
2. 2. Including fricitonal losses.
Solution 5.1.

For bubble flow:

\[ v_{SL} = v_{SG} \frac{1-\alpha}{\alpha} - v_0 (1-\alpha). \] .................................................................(1)

Bubble swarm rise velocity, \( v_0 \):

\[ v_0 = v_{0e} (1-\alpha)^{0.5} = 0.25 (1-\alpha)^{0.5}. \] .................................................................(2)

Substituting (2) into (1) yields the liquid flow rate as function of the gas flow rate, namely,

\[ v_{SL} = v_{SG} \frac{1-\alpha}{\alpha} - 0.25 (1-\alpha)^{1.5}. \] .................................................................(3)

Determination of pump efficiency:

\[ \eta = \frac{v_{SL} \rho_l g (L-H)}{v_{SG} \rho_l g H} = \frac{v_{SL}}{v_{SG}} \left( \frac{L}{H} - 1 \right). \] .................................................................(4)

Need: The void fraction, \( \alpha \), in order to proceed with calculations. Next, \( \alpha \) is determined, neglecting frictional losses (case 1) and including frictional losses (case 2).

Case 1. Neglecting Frictional Losses.

Pressure balance:

\[ \rho_l g H = \rho_m g L \quad \text{or} \quad \frac{\rho_m}{\rho_l} = \frac{H}{L}. \] .................................................................(5)

Substituting the mixture density into Eq. (5) results in:

\[ \frac{\alpha \rho_G + (1-\alpha) \rho_l}{\rho_l} = \frac{H}{L}. \]
Assuming \( \rho_g \ll \rho_L \) and solving for the void fraction \( \alpha \):

\[
\alpha = 1 - \frac{H}{L}.
\] ................................................................. (6)

**Case 1a - \( L/H = 1.2 \):**

Determination of liquid production rate:

Substituting \( L/H = 1.2 \) into Eq. (6) yields \( \alpha = 1 - 1/1.2 = 0.167 \). Substituting the value of \( \alpha \) into Eq. 3 yields the liquid production rate as a function of the gas rate, as follows:

\[
v_{SL} = v_{SG} \frac{0.833}{0.167} - 0.25(0.833)^{1.5} \quad \text{or} \quad v_{SL} = 5v_{SG} - 0.19.
\] ................................................................. (7)

Determination of pump efficiency:

Substituting the flow rate expressions, Eq. (7), into the efficiency equation, Eq. (4), yields the pump efficiency as function of the superficial gas velocity, as follows:

\[
\eta = 1 - 0.038/v_{SG}.
\] ................................................................. (8)

**Case 1b - \( L/H = 1.4 \):**

Similarly to Case 1a, for this case the void fraction is \( \alpha = 1 - 1/1.4 = 0.29 \), and the liquid flow rate and pump efficiency are given, respectively, by

\[
v_{SL} = 2.5v_{SG} - 0.15,
\] ................................................................. (9)

and

\[
\eta = 1 - 0.06/v_{SG}.
\] ................................................................. (10)

**Case 2. Including Frictional Losses.**

The basic equation for this case, including frictional losses, are:

\[
v_{SL} = v_{SG} \frac{1 - \frac{\alpha}{\alpha}}{0.25(1 - \alpha)^{1.5}}.
\] ................................................................. (1')

\[
\rho_L g H = \left[ -\frac{dp}{dL}_G - \frac{dp}{dL}_F \right] L = \left[ \rho_M g - \frac{dp}{dL}_F \right] L.
\] ................................................................. (2')

\[
-\frac{dp}{dL}_F = \frac{4}{d} f_M \frac{\rho_M v_M^2}{2},
\] ................................................................. (3')

\[
f_M = 0.046 \left( \frac{\rho_M v_M d}{\mu_M} \right)^{-0.2},
\] ................................................................. (4')

and

\[
5-85
\]
\[ \eta = \frac{v_{SL} \rho_L g (L-H)}{v_{SG} \rho_G g H} = \frac{v_{SL}}{v_{SG}} \left( \frac{L}{H} - 1 \right) \] ................................................................. (5')

Solution Procedure: The solution is obtained by a trial and error process, as follows:

1) Assume a value for the void fraction \( \alpha \). Use the frictionless solution void fraction as a first guess.

2) Calculate \( \rho_M \) and \( \mu_M \).

3) Assume a value for the mixture velocity, \( v_M \).

4) Calculate \( \text{Re}_M, f_M \) and \( -\left( \frac{dp}{dL} \right)_F \) from Eqs. 3' and 4'.

5) Calculate \( \rho_M \) from Eq. 2', namely, \( \rho_M = \rho_L \frac{H}{L} + \frac{1}{g} \left( \frac{dp}{dL} \right)_F \).

6) Calculate \( \alpha \) : \( \alpha = \frac{\rho_L - \rho_M}{\rho_L - \rho_G} \).

7) Repeat steps 2 to 6 until convergence is achieved with respect to \( \alpha \).

8) Calculate \( v_{SL} \) and \( v_{SG} \) from Eq. 1' and \( v_M = v_{SL} + v_{SG} \) using the values of \( \alpha \) and \( v_M \).

9) Calculate \( \eta \) from Eq. 5'.

Note that the frictional losses for bubble flow are much smaller than the gravitational losses. As a result, only small changes in \( \alpha \) are expected, between the frictional and frictionless solutions. Thus, starting from the frictionless solution of \( \alpha \), only one iteration will be carried out.

**Case 2a - \( \frac{L}{H} = 1.2 \):**

Following the steps presented in the solution procedure:

1) \( \alpha = 0.167 \).

2) \( \rho_M = 1000 \times 0.833 + 1.2 \times 0.167 = 833.2 \) and \( \mu_M = 7 \times 10^{-4} \times 0.833 + 1.9 \times 10^{-5} \times 0.167 = 5.86 \times 10^{-4} \).

3) \( v_M = \) assume a value.

4) \( \text{Re}_M = \frac{\rho_M dv}{\mu_M} = \frac{833.2 \times 0.05}{5.86 \times 10^{-4}} v_M = 71092 \times v_M, \quad f_M = 0.046 (\text{Re}_M)^{-0.2} \) and \( -\left( \frac{dp}{dL} \right)_F = \frac{4}{d} f_M \frac{\rho_M v_M^2}{2} = 2\rho_M \frac{v_M^2}{d} \times f_M \times v_M = 33328 \times f_M \times v_M^2 \).

5) \( \rho_M = \frac{1000}{1.2} + \frac{1}{g} \left( \frac{dp}{dL} \right)_F = 833.3 + \frac{1}{g} \left( \frac{dp}{dL} \right)_F \).

6) \( \alpha = \frac{1000 - \rho_M}{\rho_L - \rho_G} = \frac{1000 - \rho_M}{998.8} \).
7) Only 1 iteration is carried out.

8) \( v_{SG} = v_M \alpha + 0.25 \alpha (1 - \alpha)^{1.5} \) and \( v_{SL} = v_M - v_{SG} \).

9) \( \eta = \frac{v_{SL}}{v_{SG}} \left( \frac{L}{H} - 1 \right) = \frac{v_{SL}}{v_{SG}} \times 0.2 \).

The results for this case are summarized in Table 5.5.

### Table 5.5
RESULTS FOR CASE 2A \( L/H = 1.2 \) WITH FRICTIONAL LOSSES

<table>
<thead>
<tr>
<th>( v_{SL} + v_{SG} )</th>
<th>Re_M</th>
<th>( f_M )</th>
<th>(-dp/dL)</th>
<th>( \rho_M )</th>
<th>( \alpha )</th>
<th>v_SL</th>
<th>v_SL</th>
<th>( \eta ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>3550</td>
<td>0.0089</td>
<td>0.75</td>
<td>833.22</td>
<td>0.1669</td>
<td>0.01</td>
<td>0.04</td>
<td>5</td>
</tr>
<tr>
<td>0.1</td>
<td>71092</td>
<td>0.0078</td>
<td>2.6</td>
<td>833.03</td>
<td>0.1672</td>
<td>0.052</td>
<td>0.048</td>
<td>21.7</td>
</tr>
<tr>
<td>0.2</td>
<td>14218.4</td>
<td>0.0068</td>
<td>9.0</td>
<td>832.4</td>
<td>0.1678</td>
<td>0.135</td>
<td>0.065</td>
<td>41.5</td>
</tr>
<tr>
<td>0.5</td>
<td>35546.0</td>
<td>0.0056</td>
<td>47.1</td>
<td>828.5</td>
<td>0.1717</td>
<td>0.38</td>
<td>0.12</td>
<td>63</td>
</tr>
</tbody>
</table>

Case 2b - \( L/H = 1.4 \):

Similarly to case 2a:

1) \( \alpha = 0.29 \).

2) \( \rho_M = 714.64 \) and \( \mu_M = 5.05 \times 10^{-4} \).

3) \( v_M \) = assume a value.

4) \( \text{Re}_M = 70756.4 \times v_M, \ f_M = 0.046 (\text{Re}_M)^{-0.2} \) and \( \frac{dp}{dL} = 28585.6 \times f_M \times v_M^2 \).

5) \( \rho_M = 714.8 + \left( \frac{1}{g} \frac{dp}{dL} \right)_F \).

6) \( \alpha = \frac{1000 - \rho_M}{998.8} \).

7) Only 1 iteration is carried out.

8) \( v_{SG} = v_M \alpha + 0.25 \alpha (1 - \alpha)^{1.5} \) and \( v_{SL} = v_M - v_{SG} \).

9) \( \eta = \frac{v_{SL}}{v_{SG}} \times 0.4 \).

The results for this case are given in Table 5.6.
TABLE 5.6
RESULTS FOR CASE 2B \( L/H = 1.4 \) WITH FRICTIONAL LOSSES

<table>
<thead>
<tr>
<th>( \nu_{SL} + \nu_{SG} )</th>
<th>( Re_M )</th>
<th>( f_M )</th>
<th>( -\frac{dp}{dL} \frac{1}{Re} )</th>
<th>( \rho_M )</th>
<th>( \alpha )</th>
<th>( \nu_{SL} )</th>
<th>( \nu_{SG} )</th>
<th>( \eta (%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>3538</td>
<td>0.0089</td>
<td>0.64</td>
<td>714.21</td>
<td>0.2860</td>
<td>-0.0074</td>
<td>0.0574</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>7076</td>
<td>0.0078</td>
<td>2.23</td>
<td>714.06</td>
<td>0.2863</td>
<td>0.028</td>
<td>0.072</td>
<td>15</td>
</tr>
<tr>
<td>0.2</td>
<td>14151</td>
<td>0.0068</td>
<td>7.77</td>
<td>713.5</td>
<td>0.2808</td>
<td>0.1</td>
<td>0.1</td>
<td>40</td>
</tr>
<tr>
<td>0.5</td>
<td>35378</td>
<td>0.0056</td>
<td>40.46</td>
<td>710.16</td>
<td>0.2902</td>
<td>0.31</td>
<td>0.19</td>
<td>65</td>
</tr>
<tr>
<td>1.0</td>
<td>70756</td>
<td>0.0046</td>
<td>140.9</td>
<td>700.0</td>
<td>0.300</td>
<td>0.66</td>
<td>0.34</td>
<td>77</td>
</tr>
</tbody>
</table>

Note that as presented in Tables 5.5 and 5.6, the assumption of turbulent flow is validated by the results. Finally, Figs. 5.57 and 5.58 presents the results for both Cases 1 and 2, for \( L/H = 1.2 \) and \( L/H = 1.2 \), respectively.

![Flow rates and pump efficiency results for example 5.1](image)

**Fig. 5.57**—Flow rates and pump efficiency results for example 5.1 for \( L/H = 1.2 \).
Fig. 5.58— Flow rates and pump efficiency results for example 5.1 for $L/H = 1.4$. 
CHAPTER 8
NOMENCLATURE AND REFERENCES

8.1 NOMENCLATURE

\( a \) = constant; ratio of area of phase to total area (-)
\( A \) = area, \( L^2 \), m\(^2\); thermal relaxation distance, \( L \), m; coefficient
\( B \) = formation volume factor (-)
\( c \) = concentration (-); velocity of sound, \( L/t \), m/s; coefficient, \( 1 + c = c_0 \)
\( c_0 \) = half interface length, \( L \), m
\( c_f \) = flow distribution coefficient,
\( C \) = coefficient; constant
\( C_p \) = heat capacity at constant pressure, \( L^2/t^2T \), J/kg °C
\( C_{LN} \) = liquid correction factor (-)
\( d \) = diameter, \( L \), m
\( e \) = internal energy per unit mass, \( L^2/t^2 \), m\(^2\)/s\(^2\);
pressure error \( M/Lt^2 \), kg/ms\(^2\) (Pa)
\( e_r \) = relative error (-)
\( E \) = energy, enthalpy, \( ML^2/t^2 \), kgm\(^2\)/s\(^2\) (J); efficiency (-)
\( E_{Eu} \) = Euler number (-)
\( E_{k} \) = kinetic energy parameter (-)
\( f(t) \) = dimensionless transient heat conduction time function for earth (-)
\( f_{E} \) = entrainment fraction (-)
\( F \) = dimensionless group (-); function; force, \( ML/t^2 \), kgm/s\(^2\) (N)
\( Fr \) = Froude number (-)
\( f \) = friction factor (-); volumetric fraction (-)
\( g \) = acceleration due to gravity, \( L/t^2 \), m/s\(^2\)
\( g_c \) = conversion of units parameter, \( g_c = 32.2 \frac{lbm \cdot ft}{lb \cdot ft \cdot s^2} \)
\( g_E \) = environment thermal gradient, \( T/L \), °C/m
\( G \) = mass flux, \( \frac{M}{L^2 t} \), kg/m²s; dimensionless group (-)

\( h \) = height, \( L \), m; enthalpy per unit mass, \( \frac{L^2}{t^2} \), m²/s² ;
convective heat transfer coefficient, \( \frac{M}{t^3 T} \), W/m²°C ,

\( H_L \) = liquid holdup (-)

\( I \) = interfacial annular parameter

\( i, j, k \) = indices

\( J \) = drift flux, \( L/t \), m/s

\( J_a \) = Jacobian

\( K \) = dimensionless group (-); parameter; constant

\( k \) = thermal conductivity, \( \frac{M}{L t^3 T} \), W/m°C; constant (-); kilo \((10^3)\)

\( L \) = length, \( L \), m, axial coordinate, line equation

\( \ell \) = length, \( L \), m

\( \ell_m \) = mixing length, \( L \), m

\( M \) = mass \( M \), kg; Mach number (-); Mega \((10^6)\)

\( n \) = number of increment; normal;

\( N \) = dimensionless number (-); index

\( \text{Nu} \) = Nusselt number (-), \( \text{Nu}_1 \) = bottom, \( \text{Nu}_2 \) = top

\( p \) = pressure, \( \frac{M}{L t^2} \), kg/ms² (Pa)

\( P \) = parameter for numerical scheme

\( \text{Pe} \) = Peclet number (-)

\( \text{Pr} \) = Prandtl number (-)

\( q \) = volumetric flow rate, \( \frac{L^3}{t} \), m³/s; heat flux, \( \frac{M}{t^3} \), W/m²

\( Q \) = heat transfer, \( \frac{ML^2}{t^3} \), W

\( r \) = radial position, \( L \), m

\( R \) = radius, \( L \), m

\( \text{Re} \) = Reynolds number (-)

\( R_p \) = producing gas-oil ratio, \( \frac{L^3}{L^3} \), Sm³/m³

\( R_s \) = Solution gas-oil ratio, \( \frac{L^3}{L^3} \), Sm³/m³

\( s \) = shelter coefficient (-); exponent
\( S = \) perimeter, \( L, \) m; standard conditions
\( Sr = \) radial distance, \( L, \) m
\( t = \) time, \( t, \) s
\( T = \) dimensionless group (-); time period, \( t, \) s; temperature, \( T, \) °C
\( U = \) overall heat transfer coefficient, \( M/t^3T, \) W/m² °C
\( U_o = \) overall heat transfer coefficient times pipe radius, \( ML/t^3T, \) W/m °C
\( v = \) velocity, \( L/t, \) m/s
\( V = \) Volume, \( L^3, \) m³
\( v_{0,e} = \) single bubble rise velocity, \( L/t, \) m/s
\( w = \) width, \( L, \) m
\( w_s = \) shaft work, \( ML^2/t^2, \) kgm²/s²
\( W = \) mass flow rate, \( M/t, \) kg/s
\( We = \) Weber number (-)
\( X = \) Lockhart and Martinelli parameter (-); correction factor for fluid properties (-)
\( x = \) quality (-); x-coordinate; pickup rate, \( M/t, \) kg/s
\( Y = \) inclination angle dimensionless group (-); correction factor for fluid properties (-)
\( y = \) y-coordinate
\( z = \) coordinate opposite to flow direction; gas compressibility factor (-); elevation, \( L, \) m.

**Greek Letters**
\( \alpha = \) void fraction (-); thermal diffusivity, \( L^2/t, \) m²/s
\( \beta = \) ratio of Taylor bubble/film length to total slug unit length (-); constant; parameter
\( \gamma = \) specific gravity (-); half of angle confirmed by liquid wetted periphery in stratified flow, radian
\( \delta = \) film thickness, \( L, \) m
\( \Delta = \) difference
\( \varepsilon = \) rate of energy dissipation per unit mass, \( L^2/t^3 \), m^2/s^3; pipe roughness, \( L \), m; convergence tolerance; statistical parameter

\( \varepsilon_I = \) kinematic eddy viscosity, \( L^2/t \), m^2/s

\( \varepsilon_i = \) wavy interface equivalent roughness, \( L \), m

\( \mu = \) viscosity, \( M/Lt \), kg/ms

\( \nu = \) specific volume, \( L^3/M \), m^3/kg; frequency, \( t^{-1} \), s^{-1}

\( \Pi = \) dimensionless group (-)

\( \pi = 3.1415926 \)

\( \phi = \) dimensionless pressure group (-); parameter;

normalized temperature, \( (T - T_i)/(T_T - T_i) \) (-)

\( \Phi = \) dimensionless correction factor (-)

\( \Lambda = \) correction factor for fluid properties (-)

\( \theta = \) inclination angle from horizontal, positive upward, degree;

angular position around pipe periphery, degree

included angle, radian

\( \rho = \) density, \( M/L^3 \), kg/m^3

\( \tau = \) shear stress, \( M/Lt^2 \), kg/ms^2

\( \sigma = \) surface tension, \( M/t^2 \), kg/s^2

\( \lambda = \) no-slip holdup (-); correction factor for fluid properties (-)

\( \Psi = \) holdup correction factor (-); correction factor for fluid properties (-);

local temperature difference ratio, \( (T_B - T)/(T_T - T) \), (-)

wall contact angle, degree

\( \omega, \zeta = \) bipolar coordinate system

\( \eta = \) Joule-Thomson coefficient, \( T/(M/Lt^3) \), °C/Pascal

\( \eta, \xi = \) bipolar coordinates

**Subscripts**

\( A = \) accelerational; air; absolute (pressure)

\( ACCUM = \) accumulation
AVG = average
B = bubble; buoyancy; bulk; bottom
C = core; calculated; critical; convection
CAT = slug catcher
CRIT = critical
CB = critical buoyancy
CD = critical diameter
CS = control volume surface
D = drift, drag; droplet; downstream
DIS = discharge
e = end position (of film or slug regions)
E = effective (viscosity); entrance region; environment (external); earth
F = film; frictional; Fanning; factor; front
G = gas; gravitational; guessed
H = hydraulic; head; horizontal; hole
I = interface; inlet
K = kinetic
L = liquid
LS = liquid slug
M = mixture; mass; Moody; modified; mixing; grid point
MAX = maximum
MIX = mixing
MIN = minimum
N = normalized
NS = no-slip
O = oil; outlet
0 = bubble swarm velocity; isothermal plate walls
OPR = operation
OUT = outlet
P = pipe; prototype; particular; pig, performance
\( R \) = relative; Reynolds (stress)
\( RP \) = relative performance
\( S \) = slug
\( SC \) = superficial core
\( SG \) = superficial gas
\( SL \) = superficial liquid; slug
\( SLIP \) = slippage
\( SP \) = slug production
\( c \) = turbulent; transition; total; top
\( TB \) = Taylor bubble, translational
\( TP \) = two-phase
\( TRN \) = transition
\( U \) = total slug unit; upstream
\( V \) = vertical
\( W \) = wall; water; wave
\( WC \) = water cut
1 = bottom parallel plate
2 = top parallel plate

**Superscripts**
\(~\) = dimensionless quantity
\('\) = dimensionless quantity; fluctuating velocity component; derivative;
standard conditions
\(+\) = dimensionless quantity
\(-\) = average
\(m, n\) = Blasius equation exponents
\(*\) = friction velocity

**Glossary**
\( D \) = day
k = kilo \((10^3)\)

M = mega \((10^6)\)

s = standard

scf = standard cubic feet

stbo = stock tank barrel of oil
8.2 REFERENCES


Brodkey, R.S.: “The Phenomena of Fluid Motion”, Addison-Wesley Press (1967)


Cunliffe, R.S.: “Prediction of Condensate Flow Rate in Large Diameter High-Pressure Wet-Gas Pipelines”, *APEA J.* 18, pp. 171-177 (1978)


Dittus, F. W. and Boelter, L. M. K.: “Heat Transfer in Automobile Radiators of the Tubular Type”, University of California/Berkeley Publications in Engineering, 2, no. 13, pp. 443 (1930)


Sarica, C., Shoham, O. and Brill, J.P.: “A New Approach for Finger Storage Slug Catcher Design”, OTC 6414, presented at the 22nd Annual OTC, Houston, TX, May 7-10, 1990


Taitel, Y.: “Transient Two-Phase Flow”, class notes (1987)


