Abstract

In this paper I’ve gathered almost all of Functional Equation problems from recent years.

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1 Problems

1. (Canada 2015) Find all functions \( f : \mathbb{N} \rightarrow \mathbb{N} \) such that for all \( n \in \mathbb{N} \)
\[ (n - 1)^2 < f(n)f(f(n)) < n^2 + n \]

2. (APMO 2015) Let \( S = \{2, 3, 4, \ldots\} \) denote the set of integers that are greater than or equal to 2. Does there exist a function \( f : S \rightarrow S \) such that for all \( a, b \in S \) with \( a \neq b \)
\[ f(a)f(b) = f(a^2b^2) \]

3. (India 2015) Find all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that for all \( x, y \in \mathbb{R} \)
\[ f(x^2 + yf(x)) = xf(x + y) \]

4. (Zhautykov Olympiad 2015) Find all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that for all \( x, y \in \mathbb{R} \)
\[ f(x^3 + y^3 + xy) = x^2f(x) + y^2f(y) + f(xy) \]

5. (ISI Entrance 2015) Find all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that for all \( x, y \in \mathbb{R} \)
\[ |f(x) - f(y)| = 2|x - y| \]

6. (Moldava TST 2015) Find all functions \( f : \mathbb{N} \rightarrow \mathbb{N} \) that for all \( m, n \in \mathbb{N} \)
\[ f(mf(n)) = n + f(2015m) \]

7. (Turkey TST 2015) Find all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that for all \( x, y \in \mathbb{R} \)
\[ f(x^2) + 4y^2f(y) = (f(x - y) + y^2)(f(x + y) + f(y)) \]
8. (USAJMO 2015) Find all functions \( f : \mathbb{Q} \to \mathbb{Q} \) such that for all \( x < y < z < t \in \mathbb{Q} \)
\[ f(x) + f(t) = f(y) + f(z) \]

9. (Baltic Way 2014) Find all functions \( f : \mathbb{R} \to \mathbb{R} \) such that for all \( x, y \in \mathbb{R} \)
\[ f(f(y)) + f(x - y) = f(xf(y) - x) \]

10. (Albania TST 2014) Find all functions \( f : \mathbb{R} \to \mathbb{R} \) such that for all \( x, y \in \mathbb{R} \)
\[ f(xf(y) - x) = f(x) + f(y) \]

11. (Bulgaria 2014) Find all functions \( f : \mathbb{Q}^+ \to \mathbb{R}^+ \) such that for all \( x, y \in \mathbb{Q}^+ \)
\[ f(xy) = f(x + y)(f(x) + f(y)) \]

12. (European Girls’ Mathematical Olympiad 2014) Find all functions \( f : \mathbb{R} \to \mathbb{R} \) such that for all \( x, y \in \mathbb{R} \)
\[ f(y^2 + 2xf(y) + f(x)^2) = (y + f(x))(x + f(y)) \]

13. (Romanian District Olympiad 2014) Find all functions \( f : \mathbb{N} \to \mathbb{N} \) such that for all \( m, n \in \mathbb{N} \)
\[ f(m + n) - 1|f(m) + f(n) \]
and \( n^2 - f(n) \) is a square.

14. (Romanian District Olympiad 2014) Find all functions \( f : \mathbb{Q} \to \mathbb{Q} \) such that for all \( x, y \in \mathbb{Q} \)
\[ f(x + 3f(y)) = f(x) + f(y) + 2y \]

15. (ELMO 2014) Find all triples \((f, g, h)\) of injective functions from the set of real numbers to itself satisfying
\[ f(x + f(y)) = g(x) + h(y) \]
\[ g(x + g(y)) = h(x) + f(y) \]
\[ h(x + h(y)) = f(x) + g(y) \]
for all real numbers \( x \) and \( y \). (We say a function \( F \) is injective if \( F(a) \neq F(b) \) for any distinct real numbers \( a \) and \( b \).)

16. (ELMO 2014 Shortlist) Let \( \mathbb{R}^* \) denote the set of nonzero reals. Find all functions \( f : \mathbb{R}^* \to \mathbb{R}^* \) such that for all \( x, y \in \mathbb{R}^* \) with \( x^2 + y \neq 0 \)
\[ f(x^2 + y) + 1 = f(x^2 + 1) + \frac{f(xy)}{f(x)} \]

17. (Britain 2014) Find all functions \( f : \mathbb{N} \to \mathbb{N} \) such that for all \( n \in \mathbb{N} \)
\[ f(n) + f(n + 1) = f(n + 2)f(n + 3) - 2010 \]
18. (IMO Shortlist 2013) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$

$$m^2 + f(n) \mid mf(m) + n$$

19. (Zhautykov Olympiad 2014) Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions: for each real $y$ there is a real $x$ such that $f(x) = y$, and

$$f(f(x)) = (x - 1)f(x) + 2$$

for all real $x$?

20. (Iran 2014) Find all continuous function $f : \mathbb{R}^\geq 0 \to \mathbb{R}^\geq 0$ such that for all $x, y \in \mathbb{R}^\geq 0$

$$f(xf(y)) + f(f(y)) = f(x)f(y) + 2$$

21. (Iran TST 2014) Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ satisfying the following conditions: (i)

$$\exists n \in \mathbb{N} : f(n) \neq n$$

(ii) the number of divisors of $m$ is $f(n)$ if and only if the number of divisors of $f(m)$ is $n$

22. (Iran TST 2014) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y \in \mathbb{R}^+$,

$$f \left( \frac{y}{f(x+1)} \right) + f \left( \frac{x + 1}{xf(y)} \right) = f(y)$$

23. (Kazakhstan 2014) Find all functions $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ such that for all $x, y, z \in \mathbb{Q}$

$$f(x, y) + f(y, z) + f(z, x) = f(0, x + y + z)$$

24. (Korea 2014) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$f(xf(x) + f(x)y + y - 1) = f(xf(x) + xy) + y - 1$$

25. (Middle European Mathematical Olympiad 2014) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$xf(y) + f(xf(y)) - xf(f(y)) - f(xy) = 2x + f(y) - f(x + y)$$

26. (Middle European Mathematical Olympiad 2014) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$xf(xy) + xyf(x) \geq f(x^2)f(y) + x^2y$$

27. (Moldava TST 2014) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$f(xf(y) + y) + f(xy + x) = f(x + y) + 2xy$$

28. (Turkey TST 2014) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$f(f(y) + x^2 + 1) + 2x = y + (f(x + 1))^2$$
29. (USAMO 2014) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all $x, y \in \mathbb{Z}$

$$xf(2f(y) - x) + y^2f(2x - f(y)) = \frac{f(x)^2}{x} + f(yf(y))$$

30. (Uzbekistan 2014) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$f(x^3) + f(y^3) = (x + y)(f(x^2) + f(y^2) - f(xy))$$

31. (Balkan 2013) Let $S$ be the set of positive real numbers. Find all functions $f : S^3 \to S$ such that, for all positive real numbers $x, y, z$ and $k$, the following three conditions are satisfied:

(a) $xf(x, y, z) = zf(z, y, x)$,
(b) $f(x, ky, k^2z) = kf(x, y, z)$,
(c) $f(1, k, k + 1) = k + 1$.

32. (Benelux MO 2013) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$f(x + y) + y \leq f(f(f(x)))$$

33. (Baltic Way 2013) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$f(xf(y) + y) + f(-f(x)) = f(yf(x) - y) + y$$

34. (Brazil 2013) Find all injective functions $f : \mathbb{R}^* \to \mathbb{R}^*$ such that for all $x, y \in \mathbb{R}^*$

$$f(x + y)(f(x) + f(y)) = f(xy)$$

35. (ELMO Shortlist 2013) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$f(x) + f(y) = f(x + y)$$

and

$$f(x^{2013}) = f(x)^{2013}$$

36. (Austrian Federal Competition 2013) Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying following conditions:

(a) $f(x) \geq 0$ for all $x \in \mathbb{R}$.
(b) For $a, b, c, d \in \mathbb{R}$ with $ab + bc + cd = 0$, equality $f(a - b) + f(c - d) = f(a) + f(b + c) + f(d)$ holds.

37. (Austrian Federal Competition 2013) Let $k$ be an integer. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and

$$f(x^k y^k) = xyf(x)f(y)$$

for $x, y \neq 0$.

38. (IMO 2013) Let $\mathbb{Q}_{>0}$ be the set of all positive rational numbers. Let $f : \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:

(i) for all $x, y \in \mathbb{Q}_{>0}$, $f(x)f(y) \geq f(xy)$;
(ii) for all $x, y \in \mathbb{Q}_{>0}$, $f(x + y) \geq f(x) + f(y)$;
(iii) there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$
39. (IMO Shortlist 2013) Let $\mathbb{Z}_{\geq 0}$ be the set of all non-negative integers. Find all the functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ satisfying the relation
\[ f(f(f(n))) = f(n + 1) + 1 \]
for all $n \in \mathbb{Z}_{\geq 0}$

40. (IMO Shortlist 2013) Determine all functions $f : \mathbb{Q} \to \mathbb{Z}$ satisfying
\[ f \left( \frac{f(x) + a}{b} \right) = f \left( \frac{x + a}{b} \right) \]
for all $x \in \mathbb{Q}$, $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{>0}$. (Here, $\mathbb{Z}_{>0}$ denotes the set of positive integers.)

41. (India TST 2013) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$
\[ f(x(1 + y)) = f(x)(1 + f(y)) \]

42. (Iran 2013) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
\[ f(0) \in \mathbb{Q} \]
and
\[ f(x + f(y)) = f(x + y)^2 \]

43. (Iran TST 2013) find all functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f$ is increasing and also:

(i) $f(f(x) + 2g(x) + 3f(y)) = g(x) + 2f(x) + 3g(y)$
(ii) $g(f(x) + y + g(y)) = 2x - g(x) + f(y) + y$

44. (Japan 2013) Find all functions $f : \mathbb{Z} \to \mathbb{R}$ such that the equality
\[ f(m) + f(n) = f(mn) + f(m + n + mn) \]
holds for all $m, n \in \mathbb{Z}$

45. (Korea 2013) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$
\[ f(mn) = \text{lcm}(m, n) \cdot \text{gcd}(f(m), f(n)) \]

46. (Middle European Mathematical Olympiad 2013) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$
\[ f(xf(x) + 2y) = f(x^2) + f(y) + x + y - 1 \]

47. (Romania 2013) Given $f : \mathbb{R} \to \mathbb{R}$ an arbitrary function and $g : \mathbb{R} \to \mathbb{R}$ a function of the second degree, with the property: for any real numbers $m$ and $n$ equation $f(x) = mx + n$ has solutions if and only if the equation $g(x) = mx + n$ has solutions. Show that the functions $f$ and $g$ are equal.

48. (Romania 2013) Find all injective functions $f : \mathbb{Z} \to \mathbb{Z}$ that satisfy:
\[ |f(x) - f(y)| \leq |x - y| \]
for any $x, y \in \mathbb{Z}$. 

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49. (Romania 2013) Determine continuous functions \( f : \mathbb{R} \to \mathbb{R} \) such that

\[
(a^2 + ab + b^2) \int_a^b f(x) \, dx = 3 \int_a^b x^2 f(x) \, dx,
\]

for every \( a, b \in \mathbb{R} \).

50. (Romania TST 2013) Determine all injective functions defined on the set of positive integers into itself satisfying the following condition: If \( S \) is a finite set of positive integers such that \( \sum_{s \in S} \frac{1}{s} \) is an integer, then \( \sum_{s \in S} \frac{1}{f(s)} \) is also an integer.

51. (Pan African 2013) Find all functions \( f : \mathbb{R} \to \mathbb{R} \) such that for all \( x, y \in \mathbb{R} \)

\[
f(x)f(y) + f(x + y) = xy
\]

52. (Romanian Masters in Mathematics 2013) Does there exist a pair \((g, h)\) of functions \( g, h : \mathbb{R} \to \mathbb{R} \) such that the only function \( f : \mathbb{R} \to \mathbb{R} \) satisfying \( f(g(x)) = g(f(x)) \) and \( f(h(x)) = h(f(x)) \) for all \( x \in \mathbb{R} \) is identity function \( f(x) \equiv x \)?

53. (Stars of Mathematics 2013) Given a (fixed) positive integer \( N \), solve the functional equation

\[
f : \mathbb{Z} \to \mathbb{R}, \quad f(2^k) = 2f(k) \text{ and } f(N - k) = f(k), \text{ for all } k \in \mathbb{Z}.
\]

54. (USA TSTST 2013) Find all functions \( f : \mathbb{N} \to \mathbb{N} \) that satisfy the equation

\[
f^{abc-a}(abc) + f^{abc-b}(abc) + f^{abc-c}(abc) = a + b + c
\]

for all \( a, b, c \geq 2 \). (Here \( f^1(n) = f(n) \) and \( f^k(n) = f(f^{k-1}(n)) \) for every integer \( k \) greater than 1.)

55. (Turkey TST 2013) Determine all functions \( f : \mathbb{R} \to \mathbb{R}^+ \) such that for all real numbers \( x, y \) the following conditions hold:

i. \( f(x^2) = f(x)^2 - 2xf(x) \)
ii. \( f(-x) = f(x - 1) \)
iii. \( 1 < x < y \implies f(x) < f(y) \).

56. (Vietnam 2013) Find all functions \( f : \mathbb{R} \to \mathbb{R} \) that satisfies \( f(0) = 0, f(1) = 2013 \) and

\[
(x - y)(f(f^2(x)) - f(f^2(y))) = (f(x) - f(y))(f^2(x) - f^2(y))
\]

Note: \( f^2(x) = (f(x))^2 \)

57. (Uzbekistan 2013) Find all functions \( f : \mathbb{Q} \to \mathbb{Q} \) such that

\[
f(x+y)+f(y+z)+f(t+x)+f(t+z)+f(y+t)+f(z+t) \geq 6f(x-3y+5z+7t)
\]

for all \( x, y, z, t \in \mathbb{Q} \).
58. (Albania 2012) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^3) + f(y^3) = (x+y)f(x^2) + f(y^2) - f(xy)$$

for all $x \in \mathbb{R}$.

59. (Albania TST 2012) Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that:

$$x, y > 0 \quad f(x+f(y)) = yf(xy+1).$$

a) Show that $(y-1)(f(y)-1) \leq 0$ for $y > 0$.

b) Find all such functions that require the given condition.

60. (Balkan 2012) Let $\mathbb{Z}^+$ be the set of positive integers. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that the following conditions both hold:

(i) $f(n!) = f(n)!$ for every positive integer $n$,

(ii) $m - n$ divides $f(m) - f(n)$ whenever $m$ and $n$ are different positive integers.

61. (Baltic Way 2012) Find all functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$f(x+y) = f(x-y) + f(f(1-xy))$$

holds for all real numbers $x$ and $y$.

62. (Brazil 2012) Find all surjective functions $f : (0, +\infty) \to (0, +\infty)$ such that

$$2xf(f(x)) = f(x)(x + f(f(x)))$$

for all $x > 0$.

63. (China TST 2012) $n$ being a given integer, find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all integers $x, y$ we have

$$f(x + y + f(y)) = f(x) + ny$$

64. (Czech-Polish-Slovak 2012) Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+f(y)) - f(x) = (x+f(y))^4 - x^4$$

for all $x, y \in \mathbb{R}$.

65. (European Girls’ Mathematical Olympiad 2012) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(yf(x+y) + f(x)) = 4x + 2yf(x+y)$$

for all $x, y \in \mathbb{R}$.

66. (ELMO Shortlist 2012) Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that

$$f(x)f(y)f(x+y) = f(xy)(f(x) + f(y))$$

for all $x, y \in \mathbb{Q}$. 

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67. (Austrian Federal Competition for Advanced Students 2012) Determine all functions \( f : \mathbb{Z} \to \mathbb{Z} \) satisfying the following property:

For each pair of integers \( m \) and \( n \) (not necessarily distinct), \( \gcd(m, n) \) divides \( f(m) + f(n) \).

Note: If \( n \in \mathbb{Z} \), \( \gcd(m, n) = \gcd(|m|, |n|) \) and \( \gcd(n, 0) = n \).

68. (IMO 2012) Find all functions \( f : \mathbb{Z} \to \mathbb{Z} \) such that, for all integers \( a, b, c \) that satisfy \( a + b + c = 0 \), the following equality holds:

\[
f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).
\]

(Here \( \mathbb{Z} \) denotes the set of integers.)

69. (IMO Shortlist 2012) Find all functions \( f : \mathbb{R} \to \mathbb{R} \) that satisfy the conditions

\[
f(1 + xy) - f(x + y) = f(x)f(y) \quad \text{for all} \ x, y \in \mathbb{R},
\]

and \( f(-1) \neq 0 \).

70. (India 2012) Let \( f : \mathbb{Z} \to \mathbb{Z} \) be a function satisfying \( f(0) \neq 0 \), \( f(1) = 0 \) and

(i) \( f(xy) + f(x)f(y) = f(x) + f(y) \)

(ii) \( (f(x - y) - f(0)) f(x)f(y) = 0 \)

for all \( x, y \in \mathbb{Z} \), simultaneously.

(a) Find the set of all possible values of the function \( f \).

(b) If \( f(10) \neq 0 \) and \( f(2) = 0 \), find the set of all integers \( n \) such that \( f(n) \neq 0 \).

71. (Indonesia 2012) Let \( \mathbb{R}^+ \) be the set of all positive real numbers. Show that there is no function \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying

\[
f(x + y) = f(x) + f(y) + \frac{1}{2012}
\]

for all positive real numbers \( x \) and \( y \).

72. (IMO Training Camp 2012) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function such that

\[
f(x + y + xy) = f(x) + f(y) + f(xy)
\]

for all \( x, y \in \mathbb{R} \). Prove that \( f \) satisfies

\[
f(x + y) = f(x) + f(y)
\]

for all \( x, y \in \mathbb{R} \).

73. (IMO Training Camp 2012) Let \( \mathbb{R}^+ \) denote the set of all positive real numbers. Find all functions \( f : \mathbb{R}^+ \to \mathbb{R} \) satisfying

\[
f(x) + f(y) \leq \frac{f(x + y)}{2}, \quad f(x) + \frac{f(y)}{x + y} \geq \frac{f(x + y)}{x + y},
\]

for all \( x, y \in \mathbb{R}^+ \).

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74. (Iran TST 2012) The function \( f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) satisfies the following properties for all \( a, b \in \mathbb{R}_{\geq 0} \):

a) \( f(a) = 0 \iff a = 0 \)

b) \( f(ab) = f(a)f(b) \)

c) \( f(a + b) \leq 2 \max\{f(a), f(b)\} \).

Prove that for all \( a, b \in \mathbb{R}_{\geq 0} \) we have \( f(a + b) \leq f(a) + f(b) \).

75. (Iran TST 2012) Let \( g(x) \) be a polynomial of degree at least 2 with all of its coefficients positive. Find all functions \( f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) such that

\[
f(f(x) + g(x) + 2y) = f(x) + g(x) + 2f(y) \quad \forall x, y \in \mathbb{R}_{\geq 0}.
\]

76. (Japan 2012) Find all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that

\[
f(f(x) + f(x - y)) = x^2 - yf(y)
\]

for all \( x, y \in \mathbb{R} \).

77. (Kazakhstan 2012) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a function such that

\[
f(xf(y)) = yf(x)
\]

for any \( x, y \) are real numbers. Prove that

\[
f(-x) = -f(x)
\]

for all real numbers \( x \).

78. (Kyrgyzstan 2012) Find all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that \( f(f(x)^2 + f(y)) = xf(x) + y, \forall x, y \in \mathbb{R} \).

79. (Macedonia 2012) Find all functions \( f : \mathbb{R} \rightarrow \mathbb{Z} \) which satisfy the conditions:

\[
f(x + y) < f(x) + f(y)
\]

\[
f(f(x)) = \lfloor x \rfloor + 2
\]

80. (Middle European Mathematical Olympiad 2012) Let \( \mathbb{R}^+ \) denote the set of all positive real numbers. Find all functions \( \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that

\[
f(x + f(y)) = yf(xy + 1)
\]

holds for all \( x, y \in \mathbb{R}^+ \).

81. (Pan African 2012) Find all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that

\[
f(x^2 - y^2) = (x + y)(f(x) - f(y))
\]

for all real numbers \( x \) and \( y \).
82. (Puerto Rico TST 2012) Let \( f \) be a function with the following properties:
1) \( f(n) \) is defined for every positive integer \( n \);
2) \( f(n) \) is an integer;
3) \( f(2) = 2 \);
4) \( f(mn) = f(m)f(n) \) for all \( m, n \);
5) \( f(m) > f(n) \) whenever \( m > n \).
Prove that \( f(n) = n \).

83. (Romania 2012) Find all functions \( f : \mathbb{R} \to \mathbb{R} \) with the following property: for any open bounded interval \( I \), the set \( f(I) \) is an open interval having the same length with \( I \).

84. (Romania 2012) Find all differentiable functions \( f : [0, \infty) \to [0, \infty) \) for which \( f(0) = 0 \) and \( f'(x^2) = f(x) \) for any \( x \in [0, \infty) \).

85. (Poland 2012) Find all functions \( f, g : \mathbb{R} \to \mathbb{R} \) satisfying \( \forall x, y \in \mathbb{R} : g(f(x) - y) = f(g(y)) + x \).

86. (Singapore 2012) Find all functions \( f : \mathbb{R} \to \mathbb{R} \) such that \( (x+y)(f(x) - f(y)) = (x-y)f(x+y) \) for all \( x, y \) that belongs to \( \mathbb{R} \).

87. (South Africa 2012) Find all functions \( f : \mathbb{N} \to \mathbb{R} \) such that \( f(km) + f(kn) - f(k)f(mn) \geq 1 \) for all \( k, m, n \in \mathbb{N} \).

88. (Spain 2012) Find all functions \( f : \mathbb{R} \to \mathbb{R} \) such that \( (x-2)f(y) + f(y + 2f(x)) = f(x + yf(x)) \) for all \( x, y \in \mathbb{R} \).

89. (Turkey 2012) Find all non-decreasing functions from real numbers to itself such that for all real numbers \( x, y \)
\[
f(f(x^2) + y + f(y)) = x^2 + 2f(y)
\]
holds.

90. (USA TST 2012) Determine all functions \( f : \mathbb{R} \to \mathbb{R} \) such that for every pair of real numbers \( x \) and \( y \),
\[
f(x + y^2) = f(x) + |yf(y)|.
\]

91. (USAMO 2012) Find all functions \( f : \mathbb{Z}^+ \to \mathbb{Z}^+ \) (where \( \mathbb{Z}^+ \) is the set of positive integers) such that \( f(n!) = f(n)! \) for all positive integers \( n \) and such that \( m - n \) divides \( f(m) - f(n) \) for all distinct positive integers \( m, n \).

92. (Vietnam 2012) Find all functions \( f : \mathbb{R} \to \mathbb{R} \) such that:
   (a) For every real number \( a \) there exist real number \( b ; f(b) = a \)
   (b) If \( x > y \) then \( f(x) > f(y) \)
   (c) \( f(f(x)) = f(x) + 12x \).
2 Solutions

17. http://www.artofproblemsolving.com/community/c6h589266p3489256