4

LINEAR PROGRAMMING II: ADDITIONAL TOPICS AND EXTENSIONS

4.1 INTRODUCTION

If a LP problem involving several variables and constraints is to be solved by using the simplex method described in Chapter 3, it requires a large amount of computer storage and time. Some techniques, which require less computational time and storage space compared to the original simplex method, have been developed. Among these techniques, the revised simplex method is very popular. The principal difference between the original simplex method and the revised one is that in the former we transform all the elements of the simplex tableau, while in the latter we need to transform only the elements of an inverse matrix. Associated with every LP problem, another LP problem, called the *dual*, can be formulated. The solution of a given LP problem, in many cases, can be obtained by solving its dual in a much simpler manner.

As stated above, one of the difficulties in certain practical LP problems is that the number of variables and/or the number of constraints is so large that it exceeds the storage capacity of the available computer. If the LP problem has a special structure, a principle known as the *decomposition principle* can be used to solve the problem more efficiently. In many practical problems, one will be interested not only in finding the optimum solution to a LP problem, but also in finding how the optimum solution changes when some parameters of the problem, such as cost coefficients change. Hence the sensitivity or postoptimality analysis becomes very important.

An important special class of LP problems, known as *transportation problems*, occurs often in practice. These problems can be solved by algorithms that are more efficient (for this class of problems) than the simplex method. Karmarkar's method is an interior method and has been shown to be superior to the simplex method of Dantzig for large problems. The quadratic programming problem is the best-behaved nonlinear programming problem. It has a quadratic objective function and linear constraints and is convex (for minimization problems). Hence the quadratic programming problem can be solved by suitably modifying the linear programming techniques. All these topics are discussed in this chapter.

4.2 REVISED SIMPLEX METHOD

We notice that the simplex method requires the computing and recording of an entirely new tableau at each iteration. But much of the information contained in the tableau is not used; only the following items are needed.

1. The relative cost coefficients $\bar{c}_j$ to compute†

$$\bar{c}_s = \min(\bar{c}_j) \tag{4.1}$$

   $\bar{c}_s$ determines the variable x_s that has to be brought into the basis in the next iteration.
2. By assuming that $\bar{c}_s < 0$, the elements of the updated column

$$\overline{\mathbf{A}}_s = \begin{Bmatrix} \bar{a}_{1s} \\ \bar{a}_{2s} \\ \vdots \\ \bar{a}_{ms} \end{Bmatrix}$$

   and the values of the basic variables

$$\overline{\mathbf{X}}_B = \begin{Bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_m \end{Bmatrix}$$

   have to be calculated. With this information, the visiable x_r that has to be removed from the basis is found by computing the quantity

$$\frac{\bar{b}_r}{\bar{a}_{rs}} = \min_{\bar{a}_{is} > 0} \left\{ \frac{\bar{b}_i}{\bar{a}_{is}} \right\} \tag{4.2}$$

†The modified values of b_i, a_{ij}, and c_j are denoted by overbars in this chapter (they were denoted by primes in Chapter 3).

and a pivot operation is performed on $\bar{a}_{rs}$. Thus only one nonbasic column $\overline{\mathbf{A}}_s$ of the current tableau is useful in finding x_r. Since most of the linear programming problems involve many more variables (columns) than constraints (rows), considerable effort and storage is wasted in dealing with the $\overline{\mathbf{A}}_j$ for $j \neq s$. Hence it would be more efficient if we can generate the modified cost coefficients $\bar{c}_j$ and the column $\overline{\mathbf{A}}_s$, from the original problem data itself. The revised simplex method is used for this purpose; it makes use of the inverse of the current basis matrix in generating the required quantities.

Theoretical Development. Although the revised simplex method is applicable for both phase I and phase II computations, the method is initially developed by considering linear programming in phase II for simplicity. Later, a step-by-step procedure is given to solve the general linear programming problem involving both phases I and II.

Let the given linear programming problem (phase II) be written in column form as

Minimize

$$f(\mathbf{X}) = c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{4.3}$$

subject to

$$\mathbf{AX} = \mathbf{A}_1x_1 + \mathbf{A}_2x_2 + \cdots + \mathbf{A}_nx_n = \mathbf{b} \tag{4.4}$$

$$\underset{n\times 1}{\mathbf{X}} \geq \underset{n\times 1}{\mathbf{0}} \tag{4.5}$$

where the jth column of the coefficient matrix $\mathbf{A}$ is given by

$$\underset{m\times 1}{\mathbf{A}_j} = \begin{Bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{Bmatrix}$$

Assuming that the linear programming problem has a solution, let

$$\mathbf{B} = [\mathbf{A}_{j_1} \ \mathbf{A}_{j_2} \ \cdots \ \mathbf{A}_{j_m}]$$

be a basis matrix with

$$\underset{m\times 1}{\mathbf{X}_B} = \begin{Bmatrix} x_{j_1} \\ x_{j_2} \\ \vdots \\ x_{j_m} \end{Bmatrix} \quad \text{and} \quad \underset{m\times 1}{\mathbf{c}_B} = \begin{Bmatrix} c_{j_1} \\ c_{j_2} \\ \vdots \\ c_{j_m} \end{Bmatrix}$$

representing the corresponding vectors of basic variables and cost coefficients, respectively. If $\mathbf{X}_B$ is feasible, we have

$$\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b} = \overline{\mathbf{b}} \geq \mathbf{0}$$

As in the regular simplex method, the objective function is included as the $(m + 1)$th equation and $-f$ is treated as a permanent basic variable. The augmented system can be written as

$$\sum_{j=1}^{n} \mathbf{P}_j x_j + \mathbf{P}_{n+1}(-f) = \mathbf{q} \tag{4.6}$$

where

$$\mathbf{P}_j = \begin{Bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \\ c_j \end{Bmatrix}, \quad j = 1 \text{ to } n, \quad \mathbf{P}_{n+1} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{Bmatrix} \quad \text{and} \quad \mathbf{q} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \\ 0 \end{Bmatrix}$$

Since $\mathbf{B}$ is a feasible basis for the system of Eqs. (4.4), the matrix $\mathbf{D}$ defined by

$$\underset{m+1 \times m+1}{\mathbf{D}} = [\mathbf{P}_{j_1}\ \mathbf{P}_{j_2} \cdots \mathbf{P}_{j_m}\ \mathbf{P}_{n+1}] = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{c}_B^T & 1 \end{bmatrix}$$

will be a feasible basis for the augmented system of Eqs. (4.6). The inverse of $\mathbf{D}$ can be found to be

$$\mathbf{D}^{-1} = \begin{bmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ -\mathbf{c}_B^T\mathbf{B}^{-1} & 1 \end{bmatrix}$$

Definition The row vector

$$\mathbf{c}_B^T\mathbf{B}^{-1} = \boldsymbol{\pi}^T = \begin{Bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_m \end{Bmatrix}^T \tag{4.7}$$

is called the vector of simplex multipliers relative to the f equation. If the computations correspond to phase I, two vectors of simplex multipliers, one relative to the f equation, and the other relative to the w equation are to be defined as

$$\boldsymbol{\pi}^T = \mathbf{c}_B^T\mathbf{B}^{-1} = \begin{Bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_m \end{Bmatrix}^T$$

$$\boldsymbol{\sigma}^T = \mathbf{d}_B^T\mathbf{B}^{-1} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_m \end{Bmatrix}^T$$

By premultiplying each column of Eq. (4.6) by $\mathbf{D}^{-1}$, we obtain the following canonical system of equations†:

$$\begin{array}{llll} x_{j_1} & & & \bar{b}_1 \\ & x_{j_2} & & \bar{b}_2 \\ & & \ddots \quad + \displaystyle\sum_{j\,\text{nonbasic}} \overline{\mathbf{A}}_j x_j = & \vdots \\ & & x_{j_m} & \bar{b}_m \\ & & -f + \displaystyle\sum_{j\,\text{nonbasic}} \bar{c}_j x_j = & -f_0 \end{array}$$

where

$$\begin{Bmatrix} \overline{\mathbf{A}}_j \\ \bar{c}_j \end{Bmatrix} = \mathbf{D}^{-1}\mathbf{P}_j = \begin{bmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ -\boldsymbol{\pi}^T & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{A}_j \\ c_j \end{Bmatrix} \tag{4.8}$$

From Eq. (4.8), the updated column $\overline{\mathbf{A}}_j$ can be identified as

$$\overline{\mathbf{A}}_j = \mathbf{B}^{-1}\mathbf{A}_j \tag{4.9}$$

†Premultiplication of $\mathbf{P}_j x_j$ by $\mathbf{D}^{-1}$ gives

$$\mathbf{D}^{-1}\mathbf{P}_j x_j = \begin{bmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ -\boldsymbol{\pi}^T & 1 \end{bmatrix} \begin{Bmatrix} \mathbf{A}_j \\ c_j \end{Bmatrix} x_j$$

$$= \begin{Bmatrix} \mathbf{B}^{-1}\mathbf{A}_j \\ -\boldsymbol{\pi}^T\mathbf{A}_j + \mathbf{c}_j \end{Bmatrix} x_j = \begin{cases} x_j & \text{if } x_j \text{ is a basic variable} \\ \mathbf{D}^{-1}\mathbf{P}_j x_j & \text{if } x_j \text{ is not a basic variable} \end{cases}$$

and the modified cost coefficient $\bar{c}_j$ as

$$\bar{c}_j = c_j - \boldsymbol{\pi}^T \mathbf{A}_j \tag{4.10}$$

Equations (4.9) and (4.10) can be used to perform a simplex iteration by generating $\overline{\mathbf{A}}_j$ and $\bar{c}_j$ from the original problem data, $\mathbf{A}_j$ and c_j.

Once $\overline{\mathbf{A}}_j$ and $\bar{c}_j$ are computed, the pivot element $\bar{a}_{rs}$ can be identified by using Eqs. (4.1) and (4.2). In the next step, $\mathbf{P}_s$ is introduced into the basis and $\mathbf{P}_{jr}$ is removed. This amounts to generating the inverse of the new basis matrix. The computational procedure can be seen by considering the matrix:

$$\left[\begin{array}{ccc} \underbrace{\mathbf{P}_{j1}\ \mathbf{P}_{j2} \cdots \mathbf{P}_{jm}\ \mathbf{P}_{n+1}}_{\mathbf{D}} & \underbrace{\mathbf{e}_1\ \mathbf{e}_2 \cdots \mathbf{e}_{m+1}}_{\mathbf{I}} & \overbrace{\begin{matrix} a_{1s} \\ a_{2s} \\ \vdots \\ a_{ms} \\ c_s \end{matrix}}^{\mathbf{P}_s} \\ m+1 \times m+1 & m+1 \times m+1 & \end{array}\right] \tag{4.11}$$

where $\mathbf{e}_i$ is a $(m+1)$-dimensional unit vector with a one in the ith row. Premultiplication of the above matrix by $\mathbf{D}^{-1}$ yields

$$\left[\begin{array}{ccc} \underbrace{\mathbf{e}_1\ \mathbf{e}_2 \cdots \mathbf{e}_r \cdots \mathbf{e}_{m+1}}_{\mathbf{I}} & \mathbf{D}^{-1} & \begin{matrix} \bar{a}_{1s} \\ \bar{a}_{2s} \\ \vdots \\ \bar{a}_{rs} \\ \text{Pivot} \\ \text{element} \\ \vdots \\ \bar{a}_{ms} \\ \bar{c}_s \end{matrix} \\ m+1 \times m+1 & m+1 \times m+1 & m+1 \times 1 \end{array}\right] \tag{4.12}$$

By carrying out a pivot operation on $\bar{a}_{rs}$, this matrix transforms to

$$[[\mathbf{e}_1\ \mathbf{e}_2 \cdots \mathbf{e}_{r-1}\ \boldsymbol{\beta}\ \mathbf{e}_{r+1} \cdots \mathbf{e}_{m+1}]\ \ \mathbf{D}_{\text{new}}^{-1}\ \ \mathbf{e}_r] \tag{4.13}$$

where all the elements of the vector $\boldsymbol{\beta}$ are, in general, nonzero and the second partition gives the desired matrix $\mathbf{D}_{\text{new}}^{-1}$.† It can be seen that the first partition (matrix $\mathbf{I}$) is included only to illustrate the transformation, and it can be dropped in actual computations. Thus, in practice, we write the $m + 1 \times m + 2$ matrix

$$\begin{bmatrix} & \bar{a}_{1s} \\ & \bar{a}_{2s} \\ & \vdots \\ \mathbf{D}^{-1} & \boxed{\bar{a}_{rs}} \\ & \vdots \\ & \bar{a}_{ms} \\ & \bar{c}_s \end{bmatrix}$$

and carry out a pivot operation on $\bar{a}_{rs}$. The first $m + 1$ columns of the resulting matrix will give us the desired matrix $\mathbf{D}_{\text{new}}^{-1}$.

Procedure. The detailed iterative procedure of the revised simplex method to solve a general linear programming problem is given by the following steps.

1. Write the given system of equations in canonical form, by adding the artificial variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$, and the infeasibility form for phase I as shown below:

$$\begin{array}{lllllll}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + x_{n+1} & & & & & = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & + x_{n+2} & & & & = b_2 \\
\qquad\qquad\vdots & & & & & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & & + x_{n+m} & & & = b_m \\
c_1x_1 + c_2x_2 + \cdots + c_nx_n & & & -f & & = 0 \\
d_1x_1 + d_2x_2 + \cdots + d_nx_n & & & & -w & = -w_0
\end{array} \tag{4.14}$$

†This can be verified by comparing the matrix of Eq. (4.13) with the one given in Eq. (4.11). The columns corresponding to the new basis matrix are given by

$$\mathbf{D}_{\text{new}} = [\mathbf{P}_{j_1}\ \mathbf{P}_{j_2} \cdots \mathbf{P}_{j_{r-1}}\ \underbrace{\mathbf{P}_s}_{\substack{\text{brought in} \\ \text{place of } \mathbf{P}_r}}\ \mathbf{P}_{j_{r+1}} \cdots \mathbf{P}_{j_m}\ \mathbf{P}_{n+1}]$$

These columns are modified and can be seen to form a unit matrix in Eq. (4.13). The sequence of pivot operations that did this must be equivalent to multiplying the original matrix, Eq. (4.11), by $\mathbf{D}_{\text{new}}^{-1}$. Thus the second partition of the matrix in Eq. (4.13) gives the desired $\mathbf{D}_{\text{new}}^{-1}$.

Here the constants b_i, $i = 1$ to m, are made nonnegative by changing, if necessary, the signs of all terms in the original equations before the addition of the artificial variables x_{n+i}, $i = 1$ to m. Since the original infeasibility form is given by

$$w = x_{n+1} + x_{n+2} + \cdots + x_{n+m} \tag{4.15}$$

the artificial variables can be eliminated from Eq. (4.15) by adding the first m equations of Eqs. (4.14) and subtracting the result from Eq. (4.15). The resulting equation is shown as the last equation in Eqs. (4.14) with

$$d_j = -\sum_{i=1}^{m} a_{ij} \quad \text{and} \quad w_0 = \sum_{i=1}^{m} b_i \tag{4.16}$$

Equations (4.14) are written in tableau form as shown in Table 4.1.

2. The iterative procedure (cycle 0) is started with $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$, $-f$, and $-w$ as the basic variables. A tableau is opened by entering the coefficients of the basic variables and the constant terms as shown in Table 4.2. The starting basis matrix is, from Table 4.1, $\mathbf{B} = \mathbf{I}$, and its inverse $\mathbf{B}^{-1} = [\beta_{ij}]$ can also be seen to be an identity matrix in Table 4.2. The rows corresponding to $-f$ and $-w$ in Table 4.2 give the negative of simplex multipliers π_i and σ_i ($i = 1$ to m), respectively. These are also zero since $\mathbf{c}_B = \mathbf{d}_B = \mathbf{0}$ and hence

$$\boldsymbol{\pi}^T = \mathbf{c}_B^T \mathbf{B}^{-1} = \mathbf{0}$$

$$\boldsymbol{\sigma}^T = \mathbf{d}_B^T \mathbf{B}^{-1} = \mathbf{0}$$

In general, at the start of some cycle k ($k = 0$ to start with) we open a tableau similar to Table 4.2, as shown in Table 4.4. This can also be interpreted as composed of the inverse of the current basis, $\mathbf{B}^{-1} = [\beta_{ij}]$, two rows for the simplex multipliers π_i and σ_i, a column for the values of the basic variables in the basic solution, and a column for the variable x_s. At the start of any cycle, all entries in the tableau, except the last column, are known.

3. The values of the relative cost factors $\bar{d}_j$ (for phase I) or $\bar{c}_j$ (for phase II) are computed as

$$\bar{d}_j = d_j - \boldsymbol{\sigma}^T \mathbf{A}_j$$

$$\bar{c}_j = c_j - \boldsymbol{\pi}^T \mathbf{A}_j$$

TABLE 4.1 Original System of Equations

Admissible (Original) Variable						Artificial Variable				Objective Variable		
x_1	x_2	$\cdots$	x_j	$\cdots$	x_n	x_{n+1}	x_{n+2}	$\cdots$	x_{n+m}	$-f$	$-w$	Constant
						$\longleftarrow$ Initial basis			$\longrightarrow$			
a_{11}	a_{12}		a_{1j}		a_{1n}	1						b_1
a_{21} $\mathbf{A}_1$	a_{22} $\mathbf{A}_2$		a_{2j} $\mathbf{A}_j$		a_{2n} $\mathbf{A}_n$		1					b_2
$\vdots$	$\vdots$		$\vdots$		$\vdots$							
a_{m1}	a_{m2}		a_{mj}		a_{mn}				1			b_m
c_1	c_2		c_i		c_n	0	0		0	1	0	0
d_1	d_2		d_i		d_n	0	0		0	0	1	$-w_0$

TABLE 4.2 Tableau at the Beginning of Cycle 0

Basic Variables	Columns of the Canonical Form x_{n+1}	x_{n+2}	$\cdots$	x_{n+r}	$\cdots$	x_{n+m}	$-f$	$-w$	Value of the Basic Variable	x_s[a]
x_{n+1}	1								b_1	
x_{n+2}		1							b_2	
$\vdots$									$\vdots$	
x_{n+r}				1					b_r	
$\vdots$									$\vdots$	
x_{n+m}						1			b_m	
	$\longleftarrow$ Inverse of the basis $\longrightarrow$									
$-f$	0	0	$\cdots$	0	$\cdots$	0	1		0	
$-w$	0	0	$\cdots$	0	$\cdots$	0		1	$-w_0 = -\sum_{i=1}^{m} b_i$	

[a]This column is blank at the beginning of cycle 0 and filled up only at the end of cycle 0.

and entered in a tableau form as shown in Table 4.3. For cycle 0, $\boldsymbol{\sigma}^T = \mathbf{0}$ and hence $\overline{d}_j \equiv d_j$.

4. If the current cycle corresponds to phase I, find whether all $\overline{d}_j \geq 0$. If all $\overline{d}_j \geq 0$ and $\overline{w}_0 > 0$, there is no feasible solution to the linear programming problem, so the process is terminated. If all $\overline{d}_j \geq 0$ and $\overline{w}_0 = 0$, the current basic solution is a basic feasible solution to the linear programming problem and hence phase II is started by (a) dropping all variables x_j with $\overline{d}_j > 0$, (b) dropping the w row of the tableau, and (c) restarting the cycle (step 3) using phase II rules.

   If some $\overline{d}_j < 0$, choose x_s as the variable to enter the basis in the next cycle in place of the present rth basic variable (r will be determined

TABLE 4.3 Relative Cost Factor $\overline{d}_j$ or $\overline{c}_j$

Cycle Number		Variable x_j: x_1	x_2	$\cdots$	x_n	x_{n+1}	x_{n+2}	$\cdots$	x_{n+m}
Phase I	0	d_1	d_2	$\cdots$	d_n	0	0	$\cdots$	0
	1		Use the values of σ_i (if phase I) or π_i (if phase II) of the current cycle and compute						
	$\vdots$								
	l		$\overline{d}_j = d_j - (\sigma_1 a_{1j} + \sigma_2 a_{2j} + \cdots + \sigma_m a_{mj})$						
			or						
Phase II	$l+1$		$\overline{c}_j = c_j - (\pi_1 a_{1j} + \pi_2 a_{2j} + \cdots + \pi_m a_{mj})$						
	$l+2$		Enter $\overline{d}_j$ or $\overline{c}_j$ in the row corresponding to the current cycle and choose the pivot column s such that $\overline{d}_s = \min \overline{d}_j$ (if phase I) or $\overline{c}_s = \min \overline{c}_j$ (if phase II)						
	$\vdots$								

later) such that

$$\overline{d}_s = \min(\overline{d}_j < 0)$$

On the other hand, if the current cycle corresponds to phase II, find whether all $\overline{c}_j \geq 0$. If all $\overline{c}_j \geq 0$, the current basic feasible solution is also an optimal solution and hence terminate the process. If some $\overline{c}_j < 0$, choose x_s to enter the basic set in the next cycle in place of the rth basic variable (r to be found later), such that

$$\overline{c}_s = \min(\overline{c}_j < 0)$$

5. Compute the elements of the x_s column from Eq. (4.9) as

$$\overline{\mathbf{A}}_s = \mathbf{B}^{-1}\mathbf{A}_s = \overline{\beta}_{ij}\mathbf{A}_s$$

that is,

$$\begin{aligned}
\overline{a}_{1s} &= \beta_{11}a_{1s} + \beta_{12}a_{2s} + \cdots + \beta_{1m}a_{ms} \\
\overline{a}_{2s} &= \beta_{21}a_{1s} + \beta_{22}a_{2s} + \cdots + \beta_{2m}a_{ms} \\
&\vdots \\
\overline{a}_{ms} &= \beta_{m1}a_{1s} + \beta_{m2}a_{2s} + \cdots + \beta_{mm}a_{ms}
\end{aligned}$$

and enter in the last column of Table 4.2 (if cycle 0) or Table 4.4 (if cycle k).

6. Inspect the signs of all entries $\overline{a}_{is}$, $i = 1$ to m. If all $\overline{a}_{is} \leq 0$, the class of solutions

$$x_s \geq 0 \text{ arbitrary}$$

$x_{j_i} = \overline{b}_i - \overline{a}_{is} \cdot x_s$ if x_{j_i} is a basic variable, and $x_j = 0$ if x_j is a nonbasic variable ($j \neq s$), satisfies the original system and has the property

$$f = \overline{f}_0 + \overline{c}_s x_s \to -\infty \quad \text{as} \quad x_s \to +\infty$$

Hence terminate the process. On the other hand, if some $\overline{a}_{is} > 0$, select the variable x_r that can be dropped in the next cycle as

$$\frac{\overline{b}_r}{\overline{a}_{rs}} = \min_{\overline{a}_{is} > 0} (\overline{b}_i/\overline{a}_{is})$$

In the case of a tie, choose r at random.

TABLE 4.4 Tableau at the Beginning of Cycle k

Basic Variable	Columns of the Original Canonical Form				Value of the Basic Variable	x_s [a]
	$x_{n+1} \cdots x_{n+m}$		$-f$	$-w$		
	$[\beta_{ij}] = [\bar{a}_{i,n+j}]$ ←Inverse of the basis→					
x_{j1}	$\beta_{11} \cdots \beta_{1m}$				$\bar{b}_1$	$\bar{a}_{1s} = \sum_{i=1}^{m} \beta_{1i} a_{is}$
$\vdots$	$\vdots \quad \vdots$				$\vdots$	
x_{j_r}	$\beta_{r1} \cdots \beta_{rm}$				$\bar{b}_r$	$\bar{a}_{rs} = \sum_{i=1}^{m} \beta_{ri} a_{i_s}$
$\vdots$	$\vdots \quad \vdots$				$\vdots$	
x_{j_m}	$\beta_{m1} \cdots \beta_{mm}$				$\bar{b}_m$	$\bar{a}_{ms} = \sum_{i=1}^{m} \beta_{mi} a_{i_s}$
$-f$	$-\pi \cdots -\pi_m$ $(-\pi_j = +\bar{c}_{n+j})$		1		$-\bar{f}_0$	$\bar{c}_s = c_s - \sum_{i=1}^{m} \pi_i a_{i_s}$
$-w$	$-\sigma_1 \cdots -\sigma_m$ $(-\sigma_j = +\bar{d}_{n+j})$			1	$-\bar{w}_0$	$\bar{d}_s = d_s - \sum_{i=1}^{m} \sigma_i a_{is}$

[a]This column is blank at the start of cycle k and is filled up only at the end of cycle k.

TABLE 4.5 Tableau at the Beginning of Cycle $k + 1$

Basic Variables	Columns of the Canonical Form					Value of the Basic Variable	x_s [a]
	x_{n+1}	$\cdots$	x_{n+m}	$-f$	$-w$		
x_{j1}	$\beta_{11} - a_{1s}\beta_{r1}^*$	$\cdots$	$\beta_{1m} - \bar{a}_{1s}\beta_{rm}^*$			$\bar{b}_1 - \bar{a}_{1n}\bar{b}_r^*$	
$\vdots$							
x_s	β_{r1}^*	$\cdots$	β_{rm}^*			$\bar{b}_r^*$	
$\vdots$							
x_{jm}	$\beta_{m1} - \bar{a}_{ms}\beta_{r1}^*$	$\cdots$	$\beta_{mm} - \bar{a}_{ms}\beta_{rm}^*$			$\bar{b}_m - \bar{a}_{ms}\bar{b}_r^*$	
$-f$	$-\pi_1 - \bar{c}_s\beta_{r1}^*$	$\cdots$	$-\pi_m - \bar{c}_s\beta_{rm}^*$	1		$-\bar{f}_0 - \bar{c}_s\bar{b}_r^*$	
$-w$	$-\sigma_1 - \bar{d}_s\beta_{r1}^*$	$\cdots$	$-\sigma_m - \bar{d}_s\beta_{rm}^*$		1	$-\bar{w}_0 - \bar{d}_s\bar{b}_r^*$	

$$\beta_{ri}^* = \frac{\beta_{ri}}{\bar{a}_{rs}} \; (i = 1 \text{ to } m) \quad \text{and} \quad \bar{b}_r^* = \frac{\bar{b}_r}{\bar{a}_{rs}}$$

[a]This column is blank at the start of the cycle.

7. To bring x_s into the basis in place of x_r, carry out a pivot operation on the element $\bar{a}_{rs}$ in Table 4.4 and enter the result as shown in Table 4.5. As usual, the last column of Table 4.5 will be left blank at the beginning of the current cycle $k + 1$. Also, retain the list of basic variables in the first column of Table 4.5 the same as in Table 4.4, except that j_r is changed to the value of s determined in step 4.
8. Go to step 3 to initiate the next cycle, $k + 1$.

Example 4.1

$$\text{Maximize } F = x_1 + 2x_2 + x_3$$

subject to

$$\begin{aligned} 2x_1 + x_2 - x_3 &\leq 2 \\ -2x_1 + x_2 - 5x_3 &\geq -6 \\ 4x_1 + x_2 + x_3 &\leq 6 \\ x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 &\geq 0 \end{aligned}$$

SOLUTION This problem can be stated in standard form as (making all the constants b_i positive and then adding the slack variables):

Minimize

$$f = -x_1 - 2x_2 - x_3 \tag{E$_1$}$$

subject to

$$\begin{aligned} 2x_1 + x_2 - x_3 + x_4 \qquad\qquad &= 2 \\ 2x_1 - x_2 + 5x_3 \qquad + x_5 \qquad &= 6 \\ 4x_1 + x_2 + x_3 \qquad\qquad + x_6 &= 6 \\ x_i \geq 0, \quad i = 1 \text{ to } 6 & \end{aligned} \tag{E$_2$}$$

where x_4, x_5, and x_6 are slack variables. Since the set of equations (E_2) are in canonical form with respect to x_4, x_5, and x_6, $x_i = 0$ $(i = 1,2,3)$ and $x_4 = 2$, $x_5 = 6$, and $x_6 = 6$ can be taken as an initial basic feasible solution and hence there is no need for phase I.

TABLE 4.6 Detached Coefficients of the Original System

Admissible Variables							
x_1	x_2	x_3	x_4	x_5	x_6	$-f$	Constants
2	1	−1	1	0	0		2
2	−1	5	0	1	0		6
4	1	1	0	0	1		6
−1	−2	−1	0	0	0	1	0

Step 1: All the equations (including the objective function) can be written in canonical form as

$$\begin{aligned}
2x_1 + x_2 - x_3 + x_4 \qquad\qquad\qquad &= 2 \\
2x_1 - x_2 + 5x_3 \qquad + x_5 \qquad\qquad &= 6 \\
4x_1 + x_2 + x_3 \qquad\qquad + x_6 \qquad &= 6 \\
-x_1 - 2x_2 - x_3 \qquad\qquad\qquad - f &= 0
\end{aligned} \tag{E$_3$}$$

These equations are written in tableau form in Table 4.6.

Step 2: The iterative procedure (cycle 0) starts with x_4, x_5, x_6, and $-f$ as basic variables. A tableau is opened by entering the coefficients of the basic variables and the constant terms as shown in Table 4.7. Since the basis matrix is $\mathbf{B} = \mathbf{I}$, its inverse $\mathbf{B}^{-1} = [\beta_{ij}] = \mathbf{I}$. The row corresponding to $-f$ in Table 4.7 gives the negative of simplex multipliers π_i, $i = 1,2,3$. These are all zero in cycle 0. The entries of the last column of the table are, of course, not yet known.

TABLE 4.7 Tableau at the Beginning of Cycle 0

Basic Variables	Columns of the Canonical Form				Value of the Basic Variable (Constant)	x_2[a]
	x_4	x_5	x_6	$-f$		
x_4	1	0	0	0	2	$\boxed{\bar{a}_{42} = 1}$ Pivot element
x_5	0	1	0	0	6	$\bar{a}_{52} = -1$
x_6	0	0	1	0	6	$\bar{a}_{62} = 1$
	Inverse of the basis $= [\beta_{ij}]$					
$-f$	0	0	0	1	0	$c_2 = -2$

[a]This column is entered at the end of step 5.

Step 3: The relative cost factors $\bar{c}_j$ are computed as

$$\bar{c}_j = c_j - \boldsymbol{\pi}^T \mathbf{A}_j = c_j, \qquad j = 1 \text{ to } 6$$

since all π_i are zero. Thus

$$\begin{aligned}
\bar{c}_1 &= c_1 = -1 \\
\bar{c}_2 &= c_2 = -2 \\
\bar{c}_3 &= c_3 = -1 \\
\bar{c}_4 &= c_4 = 0 \\
\bar{c}_5 &= c_5 = 0 \\
\bar{c}_6 &= c_6 = 0
\end{aligned}$$

These cost coefficients are entered as the first row of a tableau (Table 4.8).

Step 4: Find whether all $\bar{c}_j \geq 0$ for optimality. The present basic feasible solution is not optimal since some $\bar{c}_j$ are negative. Hence select a variable x_s to enter the basic set in the next cycle such that $\bar{c}_s = \min(\bar{c}_j < 0) = \bar{c}_2$ in this case. Therefore, x_2 enters the basic set.

Step 5: Compute the elements of the x_s column as

$$\bar{\mathbf{A}}_s = [\beta_{ij}]\, \mathbf{A}_s$$

where $[\beta_{ij}]$ is available in Table 4.7 and $\mathbf{A}_s$ in Table 4.6.

$$\bar{\mathbf{A}}_2 = \mathbf{I}\mathbf{A}_2 = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

These elements, along with the value of $\bar{c}_2$, are entered in the last column of Table 4.7.

TABLE 4.8 Relative Cost Factors $\bar{c}_j$

Cycle Number	Variable x_j					
	x_1	x_2	x_3	x_4	x_5	x_6
Phase II						
Cycle 0	-1	$\boxed{-2}$	-1	0	0	0
Cycle 1	3	0	$\boxed{-3}$	2	0	0
Cycle 2	6	0	0	$\frac{11}{4}$	$\frac{3}{4}$	0

Step 6: Select a variable (x_r) to be dropped from the current basic set as

$$\frac{\bar{b}_r}{\bar{a}_{rs}} = \min_{\bar{a}_{is} > 0} \left(\frac{\bar{b}_i}{\bar{a}_{is}} \right)$$

In this case,

$$\frac{\bar{b}_4}{\bar{a}_{42}} = \frac{2}{1} = 2$$

$$\frac{\bar{b}_6}{\bar{a}_{62}} = \frac{6}{1} = 6$$

Therefore, $x_r = x_4$.

Step 7: To bring x_2 into the basic set in place of x_4, pivot on $a_{rs} = a_{42}$ in Table 4.7. Enter the result as shown in Table 4.9, keeping its last column blank. Since a new cycle has to be started, we go to step 3.

Step 3: The relative cost factors are calculated as

$$\bar{c}_j = c_j - (\pi_1 a_{1j} + \pi_2 a_{2j} + \pi_3 a_{3j})$$

where the negative values of π_1, π_2, and π_3 are given by the row of $-f$ in Table 4.9, and a_{ij} and c_i are given in Table 4.6. Here $\pi_1 = -2$, $\pi_2 = 0$, and $\pi_3 = 0$.

$$\bar{c}_1 = c_1 - \pi_1 a_{11} = -1 - (-2)(2) = 3$$

$$\bar{c}_2 = c_2 - \pi_1 a_{12} = -2 - (-2)(1) = 0$$

TABLE 4.9 Tableau at the Beginning of Cycle 1

Basic Variables	Columns of the Original Canonical Form				Value of the Basic Variable	
	x_4	x_5	x_6	$-f$		x_3[a]
x_2	1	0	0	0	2	$\bar{a}_{23} = -1$
x_5	1	1	0	0	8	$\boxed{\bar{a}_{53} = 4}$ Pivot element
x_6	-1	0	1	1	4	$\bar{a}_{63} = 2$
	←Inverse of the basis = $[\beta_{ij}]$→					
$-f$	$2 = -\pi_1$	$0 = -\pi_2$	$0 = -\pi_3$	1	4	$\bar{c}_3 = -3$

[a]This column is entered at the end of step 5.

$$\bar{c}_3 = c_3 - \pi_1 a_{13} = -1 - (-2)(-1) = -3$$

$$\bar{c}_4 = c_4 - \pi_1 a_{14} = 0 - (-2)(1) = 2$$

$$\bar{c}_5 = c_5 - \pi_1 a_{15} = 0 - (-2)(0) = 0$$

$$\bar{c}_6 = c_6 - \pi_1 a_{16} = 0 - (-2)(0) = 0$$

Enter these values in the second row of Table 4.8.

Step 4: Since all $\bar{c}_j$ are not ≥ 0, the current solution is not optimum. Hence select a variable (x_s) to enter the basic set in the next cycle such that $\bar{c}_s = \min(\bar{c}_j < 0) = \bar{c}_3$ in this case. Therefore, $x_s = x_3$.

Step 5: Compute the elements of the x_s column as

$$\overline{\mathbf{A}}_s = [\beta_{ij}]\mathbf{A}_s$$

where $[\beta_{ij}]$ is available in Table 4.9 and $\mathbf{A}_s$ in Table 4.6.

$$\overline{\mathbf{A}}_3 = \begin{Bmatrix} \bar{a}_{23} \\ \bar{a}_{53} \\ \bar{a}_{63} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} -1 \\ 5 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 4 \\ 2 \end{Bmatrix}$$

Enter these elements and the value of $\bar{c}_s = \bar{c}_3 = -3$ in the last column of Table 4.9.

Step 6: Find the variable (x_r) to be dropped from the basic set in the next cycle as

$$\frac{\bar{b}_r}{\bar{a}_{rs}} = \min_{\bar{a}_{is} > 0} \left(\frac{\bar{b}_i}{\bar{a}_{is}} \right)$$

Here

$$\frac{\bar{b}_5}{\bar{a}_{53}} = \frac{8}{4} = 2$$

$$\frac{\bar{b}_6}{\bar{a}_{63}} = \frac{4}{2} = 2$$

Since there is a tie between x_5 and x_6, we select $x_r = x_5$ arbitrarily.

Step 7: To bring x_3 into the basic set in place of x_5, pivot on $\bar{a}_{rs} = \bar{a}_{53}$ in Table 4.9. Enter the result as shown in Table 4.10, keeping its last column blank. Since a new cycle has to be started, we go to step 3.

Step 3: The simplex multipliers are given by the negative values of the numbers appearing in the row of $-f$ in Table 4.10. Therefore, $\pi_1 = -\frac{11}{4}$, $\pi_2 =$

TABLE 4.10 Tableau at the Beginning of Cycle 2

Basic Variables	Columns of the Original Canonical Form				Value of the Basic Variable	x_s [a]
	x_4	x_5	x_6	$-f$		
x_2	$\frac{5}{4}$	$\frac{1}{4}$	0	0	4	
x_3	$\frac{1}{4}$	$\frac{1}{4}$	0	0	2	
x_6	$-\frac{6}{4}$	$-\frac{2}{4}$	1	1	0	
$-f$	$\frac{11}{4}$	$\frac{3}{4}$	0	1	10	

[a]This column is blank at the beginning of cycle 2.

$-\frac{3}{4}$, and $\pi_3 = 0$. The relative cost factors are given by

$$\bar{c}_j = c_j - \boldsymbol{\pi}^T \mathbf{A}_j$$

Then

$$\bar{c}_1 = c_1 - \pi_1 a_{11} - \pi_2 a_{21} = -1 - (-\tfrac{11}{4})\,(2) - (-\tfrac{3}{4})\,(2) = 6$$

$$\bar{c}_2 = c_2 - \pi_1 a_{12} - \pi_2 a_{22} = -2 - (-\tfrac{11}{4})\,(1) - (-\tfrac{3}{4})(-1) = 0$$

$$\bar{c}_3 = c_3 - \pi_1 a_{13} - \pi_2 a_{23} = -1 - (-\tfrac{11}{4})(-1) - (-\tfrac{3}{4})\,(5) = 0$$

$$\bar{c}_4 = c_4 - \pi_1 a_{14} - \pi_2 a_{24} = 0 - (-\tfrac{11}{4})\,(1) - (-\tfrac{3}{4})\,(0) = \tfrac{11}{4}$$

$$\bar{c}_5 = c_5 - \pi_1 a_{15} - \pi_2 a_{25} = 0 - (-\tfrac{11}{4})\,(0) - (-\tfrac{3}{4})\,(1) = \tfrac{3}{4}$$

$$\bar{c}_6 = c_6 - \pi_1 a_{16} - \pi_2 a_{26} = 0 - (-\tfrac{11}{4})\,(0) - (-\tfrac{3}{4})\,(0) = 0$$

These values are entered as third row in Table 4.8.

Step 4: Since all $\bar{c}_j$ are ≥ 0, the present solution will be optimum. Hence the optimum solution is given by

$$x_2 = 4, \quad x_3 = 2, \quad x_6 = 0 \quad \text{(basic variables)}$$

$$x_1 = x_4 = x_5 = 0 \quad \text{(nonbasic variables)}$$

$$f_{\min} = -10$$

4.3 DUALITY IN LINEAR PROGRAMMING

Associated with every linear programming problem, called the *primal*, there is another linear programming problem called its *dual*. These two problems possess very interesting and closely related properties. If the optimal solution to any one is known, the optimal solution to the other can readily be obtained. In fact, it is immaterial which problem is designated the primal since the dual of a dual is the primal. Because of these properties, the solution of a linear programming problem can be obtained by solving either the primal or the dual, whichever is easier. This section deals with the primal–dual relations and their application in solving a given linear programming problem.

4.3.1 Symmetric Primal–Dual Relations

A nearly symmetric relation between a primal problem and its dual problem can be seen by considering the following system of linear inequalities (rather than equations).

Primal Problem

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \\
c_1x_1 + c_2x_2 + \cdots + c_nx_n &= f
\end{aligned}
\tag{4.17}
$$

$(x_i \geq 0,\ i = 1 \text{ to } n, \text{ and } f \text{ is to be minimized})$

Dual Problem. As a definition, the dual problem can be formulated by transposing the rows and columns of Eq. (4.17) including the right-hand side and the objective function, reversing the inequalities and maximizing instead of minimizing. Thus, by denoting the dual variables as $y_1, y_2, \ldots, y_m$, the dual problem becomes

$$
\begin{aligned}
a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m &\leq c_1 \\
a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m &\leq c_2 \\
&\vdots \\
a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m &\leq c_n \\
b_1y_1 + b_2y_2 + \cdots + b_my_m &= \nu
\end{aligned}
\tag{4.18}
$$

$(y_i \geq 0,\ i = 1 \text{ to } m, \text{ and } \nu \text{ is to be maximized})$

Equations (4.17) and (4.18) are called *symmetric primal–dual pairs* and it is easy to see from these relations that the dual of the dual is the primal.

4.3.2 General Primal–Dual Relations

Although the primal–dual relations of Section 4.3.1 are derived by considering a system of inequalities in nonnegative variables, it is always possible to obtain

TABLE 4.11 Correspondence Rules for Primal–Dual Relations

Primal Quantity	Corresponding Dual Quantity
Objective function: Minimize $\mathbf{c}^T\mathbf{X}$	Maximize $\mathbf{Y}^T\mathbf{b}$
Variable $x_i \ge 0$	ith constraint $\mathbf{Y}^T\mathbf{A}_i \le c_i$ (inequality)
Variable x_i unrestricted in sign	ith constraint $\mathbf{Y}^T\mathbf{A}_i = c_i$ (equality)
jth constraint, $\mathbf{A}_j\mathbf{X} = b_j$ (equality)	jth variable y_j unrestricted in sign
jth constraint, $\mathbf{A}_j\mathbf{X} \ge b_j$ (inequality)	jth variable $y_j \ge 0$
Coefficient matrix $\mathbf{A} \equiv \begin{bmatrix} \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_m \end{bmatrix}$	Coefficient matrix $\mathbf{A}^T \equiv [\mathbf{A}_1 \cdots \mathbf{A}_n]$
Right-hand-side vector $\mathbf{b}$	Right-hand-side vector $\mathbf{c}$
Cost coefficients $\mathbf{c}$	Cost coefficients $\mathbf{b}$

the primal–dual relations for a general system consisting of a mixture of equations, less than or greater than type of inequalities, nonnegative variables or variables unrestricted in sign by reducing the system to an equivalent inequality system of Eqs. (4.17). The correspondence rules that are to be applied in deriving the general primal–dual relations are given in Table 4.11 and the primal–dual relations are shown in Table 4.12.

4.3.3 Primal–Dual Relations When the Primal Is in Standard Form

If $m^* = m$ and $n^* = n$, primal problem shown in Table 4.12 reduces to the standard form and the general primal–dual relations take the special form shown in Table 4.13.

It is to be noted that the symmetric primal–dual relations, discussed in Sec-

TABLE 4.12 Primal–Dual Relations

Primal Problem	Corresponding Dual Problem
Minimize $f = \sum_{i=1}^{n} c_i x_i$ subject to	Maximize $v = \sum_{i=1}^{m} y_i b_i$ subject to
$\sum_{j=1}^{n} a_{ij}x_j = b_i,\ i = 1,2,\ldots,m^*$	$\sum_{i=1}^{m} y_i a_{ij} = c_j,\ j = n^* + 1,\ n^* + 2,\ \ldots, n$
$\sum_{j=1}^{n} a_{ij}x_j \ge b_i,\ i = m^* + 1,\ m^* + 2,\ \ldots, m$	$\sum_{i=1}^{m} y_i a_{ij} \le c_j,\ j = 1,2,\ldots,n^*$
where	where
$x_i \ge 0,\ i = 1,2,\ldots,n^*;$	$y_i \ge 0,\ i = m^* + 1,\ m^* + 2,\ \ldots, m;$
and	and
x_i unrestricted in sign, $i = n^* + 1,\ n^* + 2,\ \ldots, n$	y_i unrestricted in sign, $i = 1,2,\ldots,m^*$

TABLE 4.13 Primal-Dual Relations Where $m^* = m$ and $n^* = n$

Primal Problem	Corresponding Dual Problem
Minimize $f = \sum_{i=1}^{n} c_i x_i$	Maximize $\nu = \sum_{i=1}^{m} b_i y_i$
subject to	subject to
$\sum_{j=1}^{n} a_{ij} x_j = b_i,\ i = 1,2,\ldots,m$	$\sum_{i=1}^{m} y_i a_{ij} \le c_j,\ j = 1,2,\ldots,n$
where	where
$x_i \ge 0,\ i = 1,2,\ldots,n$	y_i is unrestricted in sign, $i = 1,2,\ldots,m$
In matrix form	*In matrix form*
Minimize $f = \mathbf{c}^T\mathbf{X}$	Maximize $\nu = \mathbf{Y}^T\mathbf{b}$
subject to	subject to
$\mathbf{AX} = \mathbf{b}$	$\mathbf{A}^T\mathbf{Y} \le \mathbf{c}$
where	where
$\mathbf{X} \ge \mathbf{0}$	$\mathbf{Y}$ is unrestricted in sign

tion 4.3.1, can also be obtained as a special case of the general relations by setting $m^* = 0$ and $n^* = n$ in the relations of Table 4.12.

Example 4.2 Write the dual of the following linear programming problem:

$$\text{Maximize } f = 50x_1 + 100x_2$$

subject to

$$\left.\begin{aligned} 2x_1 + x_2 &\le 1250 \\ 2x_1 + 5x_2 &\le 1000 \\ 2x_1 + 3x_2 &\le 900 \\ x_2 &\le 150 \end{aligned}\right\} n = 2,\ m = 4$$

where

$$x_1 \ge 0 \quad \text{and} \quad x_2 \ge 0$$

SOLUTION Let y_1, y_2, y_3, and y_4 be the dual variables. Then the dual problem can be stated as:

$$\text{Minimize } \nu = 1250y_1 + 1000y_2 + 900y_3 + 150y_4$$

subject to

$$2y_1 + 2y_2 + 2y_3 \ge 50$$

$$y_1 + 5y_2 + 3y_3 + y_4 \ge 100$$

$$\text{where } y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0, \text{ and } y_4 \ge 0.$$

Notice that the dual problem has a lesser number of constraints compared to the primal problem in this case. Since, in general, an additional constraint requires more computational effort than an additional variable in a linear programming problem, it is evident that it is computationally more efficient to solve the dual problem in the present case. This is one of the advantages of the dual problem.

4.3.4 Duality Theorems

The following theorems are useful in developing a method for solving LP problems using dual relationships. The proofs of these theorems can be found in Ref. [4.10].

Theorem 4.1 The dual of the dual is the primal.

Theorem 4.2 Any feasible solution of the primal gives an f value greater than or at least equal to the ν value obtained by any feasible solution of the dual.

Theorem 4.3 If both primal and dual problems have feasible solutions, both have optimal solutions and minimum f = maximum ν.

Theorem 4.4 If either the primal or the dual problem has an unbounded solution, the other problem is infeasible.

4.3.5 Dual Simplex Method

There exist a number of situations in which it is required to find the solution of a linear programming problem for a number of different right-hand-side vectors $\mathbf{b}^{(i)}$. Similarly, in some cases, we may be interested in adding some more constraints to a linear programming problem for which the optimal solution is already known. When the problem has to be solved for different vectors $\mathbf{b}^{(i)}$, one can always find the desired solution by applying the two phases of the simplex method separately for each vector $\mathbf{b}^{(i)}$. However, this procedure will be inefficient since the vectors $\mathbf{b}^{(i)}$ often do not differ greatly from one another. Hence the solution for one vector, say, $\mathbf{b}^{(1)}$ may be close to the solution for some other vector, say, $\mathbf{b}^{(2)}$. Thus a better strategy is to solve the linear programming problem for $\mathbf{b}^{(1)}$ and obtain an optimal basis matrix $\mathbf{B}$. If this basis happens to be feasible for all the right-hand-side vectors, that is, if

$$\mathbf{B}^{-1}\mathbf{b}^{(i)} \geq \mathbf{0} \qquad \text{for all } i \tag{4.19}$$

then it will be optimal for all cases. On the other hand, if the basis $\mathbf{B}$ is not feasible for some of the right-hand-side vectors, that is, if

$$\mathbf{B}^{-1}\mathbf{b}^{(r)} < 0 \qquad \text{for some } r \tag{4.20}$$

then the vector of simplex multipliers

$$\pi^T = \mathbf{c}_B^T \mathbf{B}^{-1} \tag{4.21}$$

will form a dual feasible solution since the quantities

$$\bar{c}_j = c_j - \pi^T \mathbf{A}_j \geq 0$$

are independent of the right-hand-side vector $\mathbf{b}^{(r)}$. A similar situation exists when the problem has to be solved with additional constraints.

In both the situations discussed above, we have an infeasible basic (primal) solution whose associated dual solution is feasible. Several methods have been proposed, as variants of the regular simplex method, to solve a linear programming problem by starting from an infeasible solution to the primal. All these methods work in an iterative manner such that they force the solution to become feasible as well as optimal simultaneously at some stage. Among all the methods, the dual simplex method developed by Lemke [4.2] and the primal–dual method developed by Dantzig, Ford, and Fulkerson [4.3] have been most widely used. Both these methods have the following important characteristics:

1. They do not require the phase I computations of the simplex method. This is a desirable feature since the starting point found by phase I may be nowhere near optimal, since the objective of phase I ignores the optimality of the problem completely.
2. Since they work toward feasibility and optimality simultaneously, we can expect to obtain the solution in a smaller total number of iterations.

We shall consider only the dual simplex algorithm in this section.

Algorithm. As stated earlier, the dual simplex method requires the availability of a dual feasible solution which is not primal feasible to start with. It is the same as the simplex method applied to the dual problem but is developed such that it can make use of the same tableau as the primal method. Computationally, the dual simplex algorithm also involves a sequence of pivot operations, but with different rules (compared to the regular simplex method) for choosing the pivot element.

Let the problem to be solved be initially in canonical form with some of the $\bar{b}_i < 0$, the relative cost coefficients corresponding to the basic variables $\bar{c}_j = 0$, and all other $\bar{c}_j \geq 0$. Since some of the $\bar{b}_i$ are negative, the primal solution will be infeasible, and since all $\bar{c}_j \geq 0$, the corresponding dual solution will be feasible. Then the simplex method works according to the following iterative steps.

1. Select row r as the pivot row such that

$$\bar{b}_r = \min \bar{b}_i < 0 \tag{4.22}$$

2. Select column s as the pivot column such that

$$\frac{\overline{c}_s}{-\overline{a}_{rs}} = \min_{\overline{a}_{rj} < 0} \left(\frac{\overline{c}_j}{-\overline{a}_{rj}} \right) \tag{4.23}$$

   If all $\overline{a}_{rj} \geq 0$, the primal will not have any feasible (optimal) solution.
3. Carry out a pivot operation on $\overline{a}_{rs}$
4. Test for optimality: If all $\overline{b}_i \geq 0$, the current solution is optimal and hence stop the iterative procedure. Otherwise, go to step 1.

Remarks:

1. Since we are applying the simplex method to the dual, the dual solution will always be maintained feasible, and hence all the relative cost factors of the primal ($\overline{c}_j$) will be nonnegative. Thus the optimality test in step 4 is valid because it guarantees that all $\overline{b}_i$ are also nonnegative, thereby ensuring a feasible solution to the primal.
2. We can see that the primal will not have a feasible solution when all $\overline{a}_{rj}$ are nonnegative from the following reasoning. Let $(x_1, x_2, \ldots, x_m)$ be the set of basic variables. Then the rth basic variable, x_r, can be expressed as

$$x_r = \overline{b}_r - \sum_{j=m+1}^{n} \overline{a}_{rj} x_j$$

   It can be seen that if $\overline{b}_r < 0$ and $\overline{a}_{rj} \geq 0$ for all j, x_r can not be made nonnegative for any nonnegative value of x_j. Thus the primal problem contains an equation (the rth one) that cannot be satisfied by any set of nonnegative variables and hence will not have any feasible solution.

The following example is considered to illustrate the dual simplex method.

Example 4.3

$$\text{Minimize } f = 20x_1 + 16x_2$$

subject to

$$x_1 \geq 2.5$$

$$x_2 \geq 6$$

$$2x_1 + x_2 \geq 17$$

$$x_1 + x_2 \geq 12$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

SOLUTION By introducing the surplus variables x_3, x_4, x_5, and x_6, the problem can be stated in canonical form as:

Minimize f

with

$$\begin{aligned}
-x_1 \qquad\quad + x_3 \qquad\qquad\qquad\qquad\qquad &= -2.5 \\
- x_2 \qquad\quad + x_4 \qquad\qquad\qquad &= -6 \\
-2x_1 - x_2 \qquad\qquad\qquad + x_5 \qquad\quad &= -17 \\
-x_1 - x_2 \qquad\qquad\qquad\qquad\qquad + x_6 &= -12 \\
20x_1 + 16x_2 \qquad\qquad\qquad\qquad\qquad - f &= 0
\end{aligned} \tag{E$_1$}$$

$$x_i \geq 0, \quad i = 1 \text{ to } 6$$

The basic solution corresponding to (E_1) is infeasible since $x_3 = -2.5$, $x_4 = -6$, $x_5 = -17$, and $x_6 = -12$. However the objective equation shows optimality since the cost coefficients corresponding to the nonbasic variables are nonnegative ($\bar{c}_1 = 20$, $\bar{c}_2 = 16$). This shows that the solution is infeasible to the primal but feasible to the dual. Hence the dual simplex method can be applied to solve this problem as follows.

Step 1: Write the system of equations (E_1) in tableau form:

Basic Variables	Variables								
	x_1	x_2	x_3	x_4	x_5	x_6	$-f$	$\bar{b}_i$	
x_3	-1	0	1	0	0	0	0	-2.5	
x_4	0	-1	0	1	0	0	0	-6	
x_5	$\boxed{-2}$ Pivot element	-1	0	0	1	0	0	-17	← Minimum, pivot row
x_6	-1	-1	0	0	0	1	0	-12	
$-f$	20	16	0	0	0	0	1	0	

Select the pivotal row r such that

$$\bar{b}_r = \min(\bar{b}_i < 0) = \bar{b}_3 = -17$$

in this case. Hence $r = 3$.

Step 2: Select the pivotal column s as

$$\frac{\bar{c}_s}{-\bar{a}_{rs}} = \min_{\bar{a}_{rj} < 0} \left(\frac{\bar{c}_j}{-\bar{a}_{rj}} \right)$$

Since

$$\frac{\bar{c}_1}{-\bar{a}_{31}} = \frac{20}{2} = 10, \quad \frac{\bar{c}_2}{-\bar{a}_{32}} = \frac{16}{1} = 16, \quad \text{and} \quad s = 1$$

Step 3. The pivot operation is carried on $\bar{a}_{31}$ in the preceding table, and the result is as follows:

Basic Variables	Variables						$-f$	$\bar{b}_i$	
	x_1	x_2	x_3	x_4	x_5	x_6			
x_3	0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	0	6	
x_4	0	$\boxed{-1}$ Pivot element	0	1	0	0	0	-6	← Minimum, pivot row
x_1	1	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0	$\frac{17}{2}$	
x_6	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	1	0	$-\frac{7}{2}$	
$-f$	0	6	0	0	10	0	1	-170	

Step 4: Since some of the $\bar{b}_i$ are <0, the present solution is not optimum. Hence we proceed to the next iteration.

Step 1: The pivot row corresponding to minimum ($\bar{b}_i < 0$) can be seen to be 2 in the preceding table.

Step 2: Since $\bar{a}_{22}$ is the only negative coefficient, it is taken as the pivot element.

Step 3: The result of pivot operation on $\bar{a}_{22}$ in the preceding table is as follows:

Basic Variables	Variables						$-f$	$\bar{b}_i$	
	x_1	x_2	x_3	x_4	x_5	x_6			
x_3	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	3	
x_2	0	1	0	-1	0	0	0	6	
x_1	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{11}{2}$	
x_6	0	0	0	$\boxed{-\frac{1}{2}}$ Pivot element	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	← Minimum, pivot row
$-f$	0	0	0	6	10	0	1	-206	

Step 4: Since all $\bar{b}_i$ are not ≥ 0, the present solution is not optimum. Hence we go to the next iteration.

Step 1: The pivot row (corresponding to minimum $\bar{b}_i \leq 0$) can be seen to be the fourth row.

Step 2: Since

$$\frac{\bar{c}_4}{-\bar{a}_{44}} = 12 \quad \text{and} \quad \frac{\bar{c}_5}{-\bar{a}_{45}} = 20$$

the pivot column is selected as $s = 4$.

Step 3: The pivot operation is carried on $\bar{a}_{44}$ in the preceding table, and the result is as follows:

Basic Variables	Variables							
	x_1	x_2	x_3	x_4	x_5	x_6	$-f$	$\bar{b}_i$
x_3	0	0	1	0	-1	1	0	$\frac{5}{2}$
x_2	0	1	0	0	1	-2	0	7
x_1	1	0	0	0	-1	1	0	5
x_4	0	0	0	1	1	-2	0	1
$-f$	0	0	0	0	4	12	1	-212

Step 4: Since all $\bar{b}_i$ are ≥ 0, the present solution is dual optimal and primal feasible. The solution is

$$x_1 = 5, \quad x_2 = 7, \quad x_3 = \tfrac{5}{2}, \quad x_4 = 1 \quad \text{(dual basic variables)}$$

$$x_5 = x_6 = 0 \quad \text{(dual nonbasic variables)}$$

$$f_{\min} = 212$$

4.4 DECOMPOSITION PRINCIPLE

Some of the linear programming problems encountered in practice may be very large in terms of the number of variables and/or contraints. If the problem has some special structure, it is possible to obtain the solution by applying the decomposition principle developed by Dantzing and Wolfe [4.4]. In the decomposition method, the original problem is decomposed into small subproblems and then these subproblems are solved almost independently. The procedure, when applicable, has the advantage of making it possible to solve large-scale problems that may otherwise be computationally very difficult or infeasible. As an example of a problem for which the decomposition principle can be applied, consider a company having two factories, producing three and two products, respectively. Each factory has its own internal resources for production, namely, workers and machines. The two factories are coupled by the fact that there is a shared resource which both use, for example, a raw material whose availability is limited. Let b_2 and b_3 be the maximum available internal resources for factory 1, and let b_4 and b_5 be the similar availabilities for factory

2. If the limitation on the common resource is b_1, the problem can be stated as follows:

$$\text{Minimize } f(x_1, x_2, x_3, y_1, y_2) = c_1x_1 + c_2x_2 + c_3x_3 + c_4y_1 + c_5y_2$$

subject to

$$\begin{array}{lr}
\boxed{a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}y_1 + a_{15}y_2} & \leq b_1 \\[2ex]
\boxed{\begin{array}{l} a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{array}} & \begin{array}{l} \leq b_2 \\ \leq b_3 \end{array} \\[2ex]
\qquad\qquad\qquad \boxed{\begin{array}{l} a_{41}y_1 + a_{42}y_2 \\ a_{51}y_1 + a_{52}y_2 \end{array}} & \begin{array}{l} \leq b_4 \\ \leq b_5 \end{array}
\end{array} \tag{4.24}$$

where x_i and y_j are the quantities of the various products produced by the two factories (design variables) and the a_{ij} are the quantities of resource i required to produce 1 unit of product j.

$$\underset{(i=1,2,3)}{x_i \geq 0}, \qquad \underset{(j=1,2)}{y_j \geq 0}$$

An important characteristic of the problem stated in Eqs. (4.24) is that its constraints consist of two independent sets of inequalities. The first set consists of a coupling constraint involving all the design variables, and the second set consists of two groups of constraints, each group containing the design variables of that group only. This problem can be generalized as follows:

$$\text{Minimize } f(\mathbf{X}) = \mathbf{c}_1^T\mathbf{X}_1 + \mathbf{c}_2^T\mathbf{X}_2 + \cdots + \mathbf{c}_p^T\mathbf{X}_p \tag{4.25a}$$

subject to

$$\mathbf{A}_1\mathbf{X}_1 + \mathbf{A}_2\mathbf{X}_2 + \cdots + \mathbf{A}_p\mathbf{X}_p = \mathbf{b}_0 \tag{4.25b}$$

$$\left.\begin{array}{lr}
\mathbf{B}_1\mathbf{X}_1 & = \mathbf{b}_1 \\
\qquad \mathbf{B}_2\mathbf{X}_2 & = \mathbf{b}_2 \\
\qquad\qquad\qquad \mathbf{B}_p\mathbf{X}_p & = \mathbf{b}_p
\end{array}\right\} \tag{4.25c}$$

$$\mathbf{X}_1 \geq \mathbf{0}, \quad \mathbf{X}_2 \geq \mathbf{0}, \ldots, \mathbf{X}_p \geq \mathbf{0}$$

where

$$\mathbf{X}_1 = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m_1} \end{Bmatrix}, \quad \mathbf{X}_2 = \begin{Bmatrix} x_{m_1+1} \\ x_{m_1+2} \\ \vdots \\ x_{m_1+m_2} \end{Bmatrix}, \quad \ldots,$$

$$\mathbf{X}_p = \begin{Bmatrix} x_{m_1+m_2+\cdots+m_{p-1}+1} \\ x_{m_1+m_2+\cdots+m_{p-1}+2} \\ x_{m_1+m_2+\cdots+m_{p-1}+m_p} \end{Bmatrix}$$

$$\mathbf{X} = \begin{Bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_p \end{Bmatrix}$$

It can be noted that if the size of the matrix $\mathbf{A}_k$ is $(r_0 \times m_k)$ and that of $\mathbf{B}_k$ is $(r_k \times m_k)$, the problem has $\Sigma_{k=0}^{p} r_k$ constraints and $\Sigma_{k=1}^{p} m_k$ variables.

Since there are a large number of constraints in the problem stated in Eqs. (4.25), it may not be computationally efficient to solve it by using the regular simplex method. However, the decomposition principle can be used to solve it in an efficient manner. The basic solution procedure using the decomposition principle is given by the following steps.

1. Define p subsidiary constraint sets using Eqs. (4.25) as

$$\begin{aligned} \mathbf{B}_1\mathbf{X}_1 &= \mathbf{b}_1 \\ \mathbf{B}_2\mathbf{X}_2 &= \mathbf{b}_2 \\ &\vdots \\ \mathbf{B}_k\mathbf{X}_k &= \mathbf{b}_k \\ &\vdots \\ \mathbf{B}_p\mathbf{X}_p &= \mathbf{b}_p \end{aligned} \tag{4.26}$$

The subsidiary constraint set

$$\mathbf{B}_k\mathbf{X}_k = \mathbf{b}_k, \qquad k = 1,2,\ldots,p \tag{4.27}$$

represents r_k equality constraints. These constraints along with the requirement $\mathbf{X}_k \geq \mathbf{0}$ define the set of feasible solutions of Eqs. (4.27). Assuming that this set of feasible solutions is a bounded convex set, let s_k be the number of vertices of this set. By using the definition of convex combination of a set of points,† any point $\mathbf{X}_k$ satisfying Eqs. (4.27) can be represented as

$$\mathbf{X}_k = \mu_{k,1}\mathbf{X}_1^{(k)} + \mu_{k,2}\mathbf{X}_2^{(k)} + \cdots + \mu_{k,s_k}\mathbf{X}_{s_k}^{(k)} \tag{4.28}$$

$$\mu_{k,1} + \mu_{k,2} + \cdots + \mu_{k,s_k} = 1 \tag{4.29}$$

$$0 \leq \mu_{k,i} \leq 1, \qquad i = 1,2,\ldots,s_k,\ k = 1,2,\ldots,p \tag{4.30}$$

where $\mathbf{X}_1^{(k)}, \mathbf{X}_2^{(k)}, \ldots, \mathbf{X}_{s_k}^{(k)}$ are the extreme points of the feasible set defined by Eqs. (4.27). These extreme points $\mathbf{X}_1^{(k)}, \mathbf{X}_2^{(k)}, \ldots, \mathbf{X}_{s_k}^{(k)}$; $k = 1,2,\ldots,p$, can be found by solving the Eqs. (4.27).

2. These new Eqs. (4.28) imply the complete solution space enclosed by the constraints

$$\left.\begin{aligned} \mathbf{B}_k\mathbf{X}_k &= \mathbf{b}_k \\ \mathbf{X}_k &\geq \mathbf{0}, \qquad k = 1,2,\ldots,p \end{aligned}\right. \tag{4.31}$$

By substituting Eqs. (4.28) into Eqs. (4.25), it is possible to eliminate the subsidiary constraint sets from the original problem and obtain the following equivalent form:

$$\text{Minimize } f(\mathbf{X}) = \mathbf{c}_1^T \left(\sum_{i=1}^{s_1} \mu_{1,i}\mathbf{X}_i^{(1)} \right) + \mathbf{c}_2^T \left(\sum_{i=1}^{s_2} \mu_{2,i}\mathbf{X}_i^{(2)} \right) + \cdots + \mathbf{c}_p^T \left(\sum_{i=1}^{s_p} \mu_{p,i}\mathbf{X}_i^{(p)} \right)$$

†If $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are any two points in an n-dimensional space, any point lying on the line segment joining $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ is given by a convex combination of $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ as

$$\mathbf{X}(\mu) = \mu\, \mathbf{X}^{(1)} + (1 - \mu)\, \mathbf{X}^{(2)}, \qquad 0 \leq \mu \leq 1$$

This idea can be generalized to define the convex combination of r points $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots, \mathbf{X}^{(r)}$ as

$$\mathbf{X}(\mu_1,\mu_2,\ldots,\mu_r) = \mu_1\, \mathbf{X}^{(1)} + \mu_2\, \mathbf{X}^{(2)} + \cdots + \mu_r\, \mathbf{X}^{(r)}$$

where $\mu_1 + \mu_2 + \cdots + \mu_r = 1$ and $0 \leq \mu_i \leq 1$, $i = 1,2,\ldots,r$.

subject to

$$
\begin{aligned}
&\mathbf{A}_1\left(\sum_{i=1}^{s_1}\mu_{1,i}\mathbf{X}_i^{(1)}\right)+\mathbf{A}_2\left(\sum_{i=1}^{s_2}\mu_{2,i}\mathbf{X}_i^{(2)}\right)+\cdots+\mathbf{A}_p\left(\sum_{i=1}^{s_p}\mu_{p,i}\mathbf{X}_i^{(p)}\right)=\mathbf{b}_0\\
&\sum_{i=1}^{s_1}\mu_{1,i}=1\\
&\sum_{i=1}^{s_2}\mu_{2,i}=1\\
&\qquad\vdots\\
&\sum_{i=1}^{s_p}\mu_{p,i}=1\\
&\mu_{j,i}\geq 0,\qquad i=1,2,\ldots,s_j,\quad j=1,2,\ldots,p
\end{aligned}
\tag{4.32}
$$

Since the extreme points $\mathbf{X}_1^{(k)}, \mathbf{X}_2^{(k)}, \ldots, \mathbf{X}_{s_k}^{(k)}$ are known from the solution of the set $\mathbf{B}_k\mathbf{X}_k = \mathbf{b}_k$, $\mathbf{X}_k \geq \mathbf{0}$, $k = 1,2,\ldots,p$, and since $\mathbf{c}_k$ and $\mathbf{A}_k$, $k = 1,2,\ldots,p$, are known as problem data, the unknowns in Eqs. (4.32) are $\mu_{j,i}$, $i = 1,2,\ldots,s_j$; $j = 1,2,\ldots,p$. Hence $\mu_{j,i}$ will be the new decision variables of the modified problem stated in Eqs. (4.32).

3. Solve the linear programming problem stated in Eqs. (4.32) by any of the known techniques and find the optimal values of $\mu_{j,i}$. Once the optimal values $\mu_{j,i}^*$ are determined, the optimal solution of the original problem can be obtained as

$$\mathbf{X}^* = \begin{Bmatrix} \mathbf{X}_1^* \\ \mathbf{X}_2^* \\ \vdots \\ \mathbf{X}_p^* \end{Bmatrix}$$

where

$$\mathbf{X}_k^* = \sum_{i=1}^{s_k} \mu_{k,i}^* \mathbf{X}_i^{(k)}, \qquad k = 1,2,\ldots,p$$

Remarks:

1. It is to be noted that the new problem in Eqs. (4.32) has $(r_0 + p)$ equality constraints only as against $r_0 + \Sigma_{k=1}^{p} r_k$ in the original problem of Eq. (4.25). Thus there is a substantial reduction in the number of constraints due to the application of the decomposition principle. At the same time, the number of variables might increase from $\Sigma_{k=1}^{p} m_k$ to $\Sigma_{k=1}^{p} s_k$, de-

pending on the number of extreme points of the different subsidiary problems defined by Eqs. (4.31). The modified problem, however, is computationally more attractive since the computational effort required for solving any linear programming problem depends primarily on the number of constraints rather than on the number of variables.

2. The procedure outlined above requires the determination of all the extreme points of every subsidiary constraint set defined by Eqs. (4.31) before the optimal values $\mu_{j,i}^*$ are found. However, this is not necessary when the revised simplex method is used to implement the decomposition algorithm [4.5].
3. If the size of the problem is small, it will be convenient to enumerate all the extreme points of the subproblems and use the simplex method to solve the problem. This procedure is illustrated in the following example.

Example 4.4 A fertilizer mixing plant produces two fertilizers, A and B, by mixing two chemicals, C_1 and C_2, in different proportions. The contents and costs of the chemicals C_1 and C_2 are as follows:

	Contents		
Chemical	Ammonia	Phosphates	Cost ($/lb)
C_1	0.70	0.30	5
C_2	0.40	0.60	4

Fertilizer A should not contain more than 60% of ammonia and B should contain at least 50% of ammonia. On the average, the plant can sell up to 1000 lb/hr and due to limitations on the production facilities, not more than 600 lb of fertilizer A can be produced per hour. The availability of chemical C_1 is restricted to 500 lb/hr. Assuming that the production costs are same for both A and B, determine the quantities of A and B to be produced per hour for maximum return if the plant sells A and B at the rates of \$6 and \$7 per pound, respectively.

SOLUTION Let x_1 and x_2 indicate the amounts of chemicals C_1 and C_2 used in fertilizer A, and y_1 and y_2 in fertilizer B per hour. Thus the total amounts of A and B produced per hour are given by $x_1 + x_2$ and $y_1 + y_2$, respectively. The objective function to be maximized is given by

$$\begin{aligned} f &= \text{selling price} - \text{cost of chemicals } C_1 \text{ and } C_2 \\ &= 6(x_1 + x_2) + 7(y_1 + y_2) - 5(x_1 + y_1) - 4(x_2 + y_2) \end{aligned}$$

The constraints are given by

$$(x_1 + x_2) + (y_1 + y_2) \le 1000 \qquad \text{(amount that can be sold)}$$

$$x_1 + y_1 \le 500 \qquad \text{(availability of } C_1\text{)}$$

$$x_1 + x_2 \le 600 \qquad \text{(production limitations on } A\text{)}$$

$$\tfrac{7}{10} x_1 + \tfrac{4}{10} x_2 \le \tfrac{6}{10} (x_1 + x_2) \qquad (A \text{ should not contain more than 60\% of ammonia})$$

$$\tfrac{7}{10} y_1 + \tfrac{4}{10} y_2 \ge \tfrac{5}{10} (y_1 + y_2) \qquad (B \text{ should contain at least 50\% of ammonia})$$

Thus the problem can be restated as:

$$\text{Maximize } f = x_1 + 2x_2 + 2y_1 + 3y_2 \tag{E$_1$}$$

subject to

$$\boxed{\begin{array}{l} x_1 + x_2 + y_1 + y_2 \\ x_1 \qquad\quad + y_1 \end{array}} \begin{array}{l} \le 1000 \\ \le 500 \end{array} \tag{E$_2$}$$

$$\boxed{\begin{array}{l} x_1 + x_2 \\ x_1 - 2x_2 \end{array}} \begin{array}{l} \le 600 \\ \le 0 \end{array} \tag{E$_3$}$$

$$\boxed{-2y_1 + y_2} \le 0 \tag{E$_4$}$$

$$x_i \ge 0, \quad y_i \ge 0, \qquad i = 1,2$$

This problem can also be stated in matrix notation as follows:

$$\text{Maximize } f(\mathbf{X}) = \mathbf{c}_1^T\mathbf{X}_1 + \mathbf{c}_2^T\mathbf{X}_2$$

subject to

$$\begin{aligned} \mathbf{A}_1\mathbf{X}_1 + \mathbf{A}_2\mathbf{X}_2 &\le \mathbf{b}_0 \\ \mathbf{B}_1\mathbf{X}_1 \qquad\qquad &\le \mathbf{b}_1 \\ \mathbf{B}_2\mathbf{X}_2 &\le \mathbf{b}_2 \\ \mathbf{X}_1 \ge \mathbf{0}, \quad \mathbf{X}_2 &\ge \mathbf{0} \end{aligned} \tag{E$_5$}$$

where

$$\mathbf{X}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \mathbf{X}_2 = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}, \quad \mathbf{c}_1 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \quad \mathbf{c}_2 = \begin{Bmatrix} 2 \\ 3 \end{Bmatrix},$$

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad [\mathbf{A}_2] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b}_0 = \begin{Bmatrix} 1000 \\ 500 \end{Bmatrix},$$

$$\mathbf{B}_1 = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{Bmatrix} 600 \\ 0 \end{Bmatrix}, \quad \mathbf{B}_2 = \{-2 \quad 1\}, \mathbf{b}_2 = \{0\},$$

$$\mathbf{X} = \begin{Bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{Bmatrix}$$

Step 1: We first consider the subsidiary constraint sets

$$\mathbf{B}_1\mathbf{X}_1 \le \mathbf{b}_1, \qquad \mathbf{X}_1 \ge \mathbf{0} \tag{E$_6$}$$

$$\mathbf{B}_2\mathbf{X}_2 \le \mathbf{b}_2, \qquad \mathbf{X}_2 \ge \mathbf{0} \tag{E$_7$}$$

The convex feasible regions represented by (E_6) and (E_7) are shown in Fig. 4.1*a* and *b*, respectively. The vertices of the two feasible regions are given by

$$\mathbf{X}_1^{(1)} = \text{point } P = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{X}_2^{(1)} = \text{point } Q = \begin{Bmatrix} 0 \\ 600 \end{Bmatrix}$$

$$\mathbf{X}_3^{(1)} = \text{point } R = \begin{Bmatrix} 400 \\ 200 \end{Bmatrix}$$

$$\mathbf{X}_1^{(2)} = \text{point } S = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{X}_2^{(2)} = \text{point } T = \begin{Bmatrix} 1000 \\ 2000 \end{Bmatrix}$$

$$\mathbf{X}_3^{(2)} = \text{point } U = \begin{Bmatrix} 1000 \\ 0 \end{Bmatrix}$$

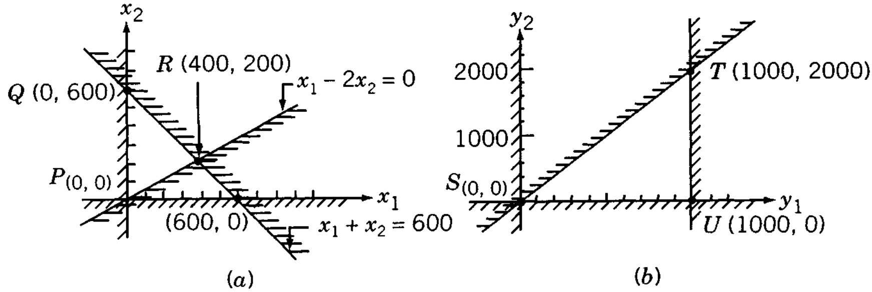

Figure 4.1 Vertices of feasible regions. To make the feasible region bounded, the constraint $y_1 \leq 1000$ is added in view of Eq. (E_2).

Thus any point in the convex feasible sets defined by Eqs. (E_6) and (E_7) can be represented, respectively, as

$$\left.\begin{array}{l} \mathbf{X}_1 = \mu_{11}\begin{Bmatrix}0\\0\end{Bmatrix} + \mu_{12}\begin{Bmatrix}0\\600\end{Bmatrix} + \mu_{13}\begin{Bmatrix}400\\200\end{Bmatrix} = \begin{Bmatrix}400\mu_{13}\\600\mu_{12} + 200\mu_{13}\end{Bmatrix} \\ \text{with} \\ \qquad \mu_{11} + \mu_{12} + \mu_{13} = 1, \quad 0 \leq \mu_{1i} \leq 1, \quad i = 1,2,3 \end{array}\right\} \quad (E_8)$$

and

$$\left.\begin{array}{l} \mathbf{X}_2 = \mu_{21}\begin{Bmatrix}0\\0\end{Bmatrix} + \mu_{22}\begin{Bmatrix}1000\\2000\end{Bmatrix} + \mu_{23}\begin{Bmatrix}1000\\0\end{Bmatrix} \\ \quad = \begin{Bmatrix}1000\mu_{22} + 1000\mu_{23}\\2000\mu_{22}\end{Bmatrix} \\ \text{with} \\ \qquad \mu_{21} + \mu_{22} + \mu_{23} = 1; \quad 0 \leq \mu_{2i} \leq 1, \quad i = 1,2,3 \end{array}\right\} \quad (E_9)$$

Step 2: By substituting the relations of (E_8) and (E_9), the problem stated in Eqs. (E_5) can be rewritten as:

$$\text{Maximize} f(\mu_{11}, \mu_{12}, \ldots, \mu_{23}) = (1 \quad 2)\begin{Bmatrix}400\mu_{13}\\600\mu_{12} + 200\mu_{13}\end{Bmatrix}$$

$$+ (2 \quad 3)\begin{Bmatrix}1000\mu_{22} + 1000\mu_{23}\\2000\mu_{22}\end{Bmatrix}$$

$$= 800\mu_{13} + 1200\mu_{12} + 8000\mu_{22} + 2000\mu_{23}$$

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subject to

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} 400\mu_{13} \\ 600\mu_{12} + 200\mu_{13} \end{Bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} 1000\mu_{22} + 1000\mu_{23} \\ 2000\mu_{22} \end{Bmatrix} \leq \begin{Bmatrix} 1000 \\ 500 \end{Bmatrix}$$

that is,

$$\begin{aligned} 600\mu_{12} + 600\mu_{13} + 3000\mu_{22} + 1000\mu_{23} &\leq 1000 \\ 400\mu_{13} + 1000\mu_{22} + 1000\mu_{23} &\leq 500 \\ \mu_{11} + \mu_{12} + \mu_{13} &= 1 \\ \mu_{21} + \mu_{22} + \mu_{23} &= 1 \end{aligned}$$

with

$$\mu_{11} \geq 0, \quad \mu_{12} \geq 0, \quad \mu_{13} \geq 0, \quad \mu_{21} \geq 0, \quad \mu_{22} \geq 0, \quad \mu_{23} \geq 0$$

The optimization problem can be stated in standard form (after adding the slack variables α and β) as:

$$\text{Minimize } f = -1200\mu_{12} - 800\mu_{13} - 8000\mu_{22} - 2000\mu_{23}$$

subject to

$$\begin{aligned} 600\mu_{12} + 600\mu_{13} + 3000\mu_{22} + 1000\mu_{23} + \alpha &= 1000 \\ 400\mu_{13} + 1000\mu_{22} + 1000\mu_{23} + \beta &= 500 \\ \mu_{11} + \mu_{12} + \mu_{13} &= 1 \\ \mu_{21} + \mu_{22} + \mu_{23} &= 1 \\ \mu_{ij} \geq 0 \ (i = 1,2; j = 1,2,3), \quad \alpha \geq 0, &\quad \beta \geq 0 \end{aligned} \tag{E$_{10}$}$$

Step 3: The problem (E_{10}) can now be solved by using the simplex method.

4.5 SENSITIVITY OR POSTOPTIMALITY ANALYSIS

In most practical problems, we are interested not only in optimal solution of the LP problem, but also in how the solution changes when the parameters of the problem change. The change in the parameters may be discrete or contin-