بسم ا... الرحمن الرحیم

سیستم های کنترل دیجیتال

Digital Control Systems

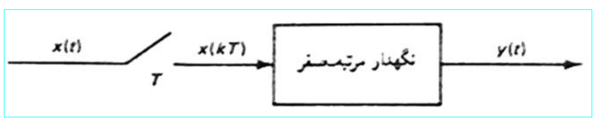
دکتر محمدرضا رمضانی

مراجع

1. K. Ogata, Discrete-time control systems, Prentice Hall, 1995
2- اسلایدهای درس کنترل دیجیتال دانشگاه علم و صنعت دکتر بلندی و دکتر اسمعیل زاده

- سرفصل مطالب
- نگهدارنده مرتبه صفر و یک (ادامه)
- مروری بر مفهوم کانولوشن ، کانولوشن زمان گسسته
- تابع انتقال پالسی Pulse Transfer Function
- کنترل کننده های دیجیتال (تناسبی مشتقی انتگرالی)

از دیدگاه روابط ورودی - خروجی



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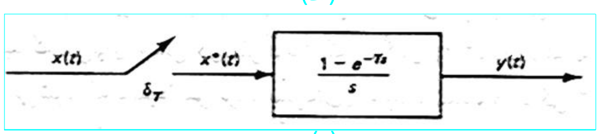

(ب)

نکته ! یک نمونه بردار واقعی و نگهدار مرتبه – صفر را می توان با یک سیستم زمان – پیوسته معادل ریاضی جایگزین کرد که شامل یک نمونه بردار ضربه ای و تابع تبدیل زیر است.

$$\frac{1-e^{-Ts}}{s}$$

$$X^*(s) = L[x^*(t)]$$
$$= x(0)L[\delta(t)] + x(T)L[\delta(t-T)]$$

$$= \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

تعریف می کنیم :

$$e^{Ts} = z$$

$$s = \frac{1}{T}\ln z$$

نتیجه

$$X^*(s)|_{s=\frac{1}{T}\ln z} = \sum_{k=0}^{\infty} x(kT)z^{-k}$$

$$X^*(s)|_{s=\frac{1}{T}\ln z} = X(z)$$

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نتیجه

$$X^*(s)|_{s=\frac{1}{T}\ln z} = X^*(\frac{1}{T}\ln z) = X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k}$$

نکته !

تعاریف فوق ما را قادر می سازند که سیستم های کنترل زمان – گسسته را که شامل نمونه بردارها و مدارهای نگه دار است با روش تبدیل z تحلیل کنیم. این بدان معنا است که با بکار بردن متغیر مختلط z ، تکنیک های توسعه یافته ای برای روش های تبدیل لاپلاس (نظیر پاسخ فرکانسی و روش های مکان ریشه) را می توان براحتی در تحلیل سیستم های کنترل زمان – گسسته که شامل عملیات نمونه برداری است، اعمال نمود.

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مثال: تبدیل Z تابع دلتای کرونکر را به دست آورید

$$\delta_o(kT) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

حل: تعریف می کنیم :

$$x(t) = \delta_o(t)$$

$$x^*(t) = x(t)\delta_T(t) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT)$$

$$x^*(t) = x(0)\delta(t) \longrightarrow X^*(s) = x(0) = \delta_o(0) = 1$$

$$X(z) = Z[\delta_o(kT)] = 1$$

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مثال: تبدیل Z تابع پله واحد را به دست آورید

حل: تعریف می کنیم :

$$x(t) = 1(t)$$

$$x^*(t) = x(t)\delta_T(t) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT) = \sum_{k=0}^{\infty} \delta(t-kT)$$

$$X^*(s) = \sum_{k=0}^{\infty} e^{-kTs} = \frac{1}{1-e^{-Ts}}$$

$$X(z) = Z[1(t)] = \frac{1}{1-z^{-1}}$$

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خلاصه

❑ نشان داده شده که تبدیل لاپلاس سیگنال نمونه برداری شده $x^*(t)$ با تبدیل Z سیگنال $x(t)$ یکسان است اگر تبدیل زیر استفاده شود.

$$z = e^{Ts}$$

❑ در روش تبدیل z ، ما فقط مقادیر سیگنال را در لحظه های نمونه برداری در نظر می گیریم. از این جهت، تبدیل Z $x(t)$ و تبدیل z $x^*(t)$ نتایج یکسانی بدست می دهند :

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❑ به طوری که پیشتر اشاره شد، وقتی که نمونه بردار واقعی و نگهدار مرتبه – صفر از لحاظ ریاضی با نمونه بردار ضربه ای و تابع تبدیل جایگزین شوند، سیستم به یک سیستم زمان – پیوسته تبدیل می گردد .

$$(1-e^{-Ts})/s$$

این امر تحلیل سیستم کنترل زمان – گسسته را ساده تر می کند، زیرا می توان روشهای موجود در سیستم های کنترل زمان – پیوسته را اعمال کرد.

❑ در یک نمونه بردار واقعی، کلیدی در هر دوره نمونه برداری T بسیار سریع باز و بسته می شود و دنباله ای از اعداد $x(kT)$ را به وجود می آورد. در یک نمونه بردار ضربه ای ریاضی و برای ورودی داده شده $x(t)$ کلید خروجی زیر را ایجاد می کند :

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT)$$

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محاسبه تابع تبدیل مدار نگهدار مرتبه – اول (n=1)

$k = 0,1,2,..$ $\quad$ $0 \le \tau < T$

$$h((k-1)T) = x((k-1)T)$$

$$a_1 = \frac{x(kT) - x((k-1)T)}{T}$$

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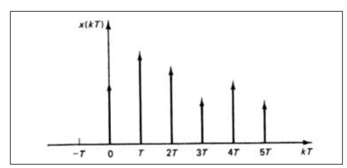
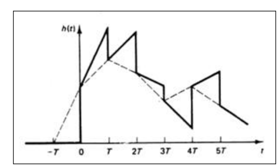
مدار نگهدار مرتبه – اول (n=1)

$$h(kT+\tau) = x(kT) + \frac{x(kT) - x((k-1)T)}{T}\tau$$

سیگنال زمان پیوسته $h(t)$ به دست آمده از نگهدار مرتبه اول یک سیگنال خطی تکه ای است که در شکل زیر نشان داده شده است.



ورودی



خروجی

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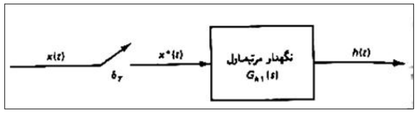
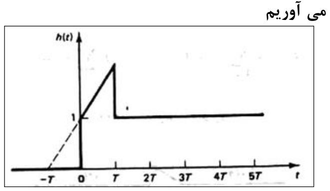


فرض کنید در سیستم فوق، تابع پله واحدی را به عنوان $x(t)$ انتخاب کنیم، در این صورت :

با توجه به اینکه خروجی $h(t)$ نگهدار مرتبه اول خط مستقیمی است که برون یابی دو مقدار نمونه برداری شده پیشین است، خروجی $h(t)$ را به صورت نشان داده شده در شکل زیر به دست می آوریم



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از شکل خروجی $h(t)$ داریم :

$$h(t) = (1+\frac{t}{T})1(t) - \frac{t-T}{T}1(t-T) - 1(t-T)$$

$$H(s) = (\frac{1}{s}+\frac{1}{Ts^2}) - \frac{1}{Ts^2}e^{-Ts} - \frac{1}{s}e^{-Ts}$$

$$= \frac{1-e^{-Ts}}{s} + \frac{1-e^{-Ts}}{Ts^2}$$

$$= (1-e^{-Ts})\frac{Ts+1}{Ts^2}$$

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تبدیل لاپلاس دنباله پله واحد چنین است :

$$X^*(s) = L[1^*(t)] = \frac{1}{1-e^{-Ts}}$$

از این رو، تابع تبدیل نگهدار مرتبه اول به صورت زیر داده می شود :

$$G_{h1}(s) = \frac{H(s)}{X^*(s)} = (1-e^{-Ts})^2\frac{Ts+1}{Ts^2}$$

$$G_{h1}(s) = (\frac{1-e^{-Ts}}{s})^2\frac{Ts+1}{T}$$

توجه کنید که یک نمونه بردار واقعی ترکیب شده با یک نگهدار مرتبه اول، معادل یک نمونه بردار ضربه ای ترکیب شده با تابع تبدیل زیر است :

$$G_{h1}(s) = (1-e^{-Ts})^2\frac{Ts+1}{Ts^2}$$

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Why are we doing this?

- How can we write mathematical description of relation between the sampled output signal and the sampled input signal.
- How D/A attached to the system gives different behavior.
- What is the response from the system during the sampling interval.

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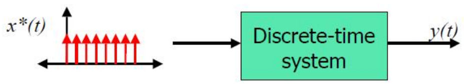
Impulse input

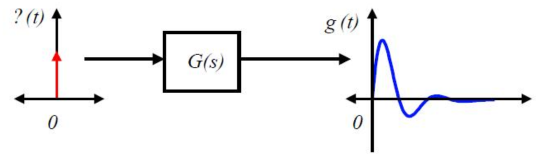
- Consider the normal operations of the sampled-data system.



- The input is series of impulses.

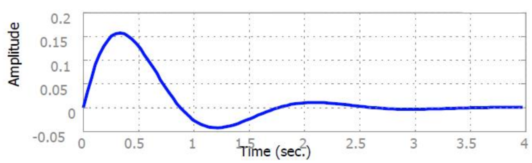
Convolution

- The output of a system when input signal is the impulse is called the impulse response function.



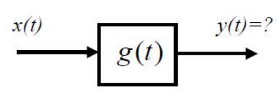
Convolution

- We can see that the impulse response function takes some time before decaying to zero.



Problem

- If we know the system impulse response function $g(t)$, how can we compute the system response for any input $x(t)$?



Convolution

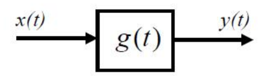
- From system theory, we can find the system response by the convolution integral.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)g(t-\tau)d\tau$$

Convolution

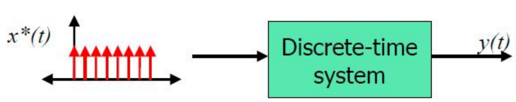
- It is used to calculate the response of the system with known impulse response function.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)g(t-\tau)d\tau \quad \text{or} \quad y(t) = x(t) * g(t)$$



Discrete-time convolution

- Coming back to our problem of having series of impulses as input to a system.

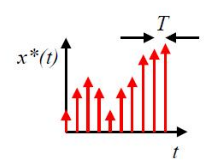


Discrete-time convolution

- In discrete-time, the input signal is series of impulses.

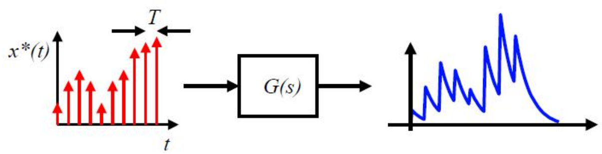
$$x^*(t) = \sum_{k=-\infty}^{\infty} x(t)\delta(t-kT)$$

$$= \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT)$$



Discrete-time convolution

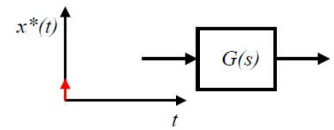
- The output will be the summation of the impulse response function at different time lag (superposition).



Discrete-time convolution

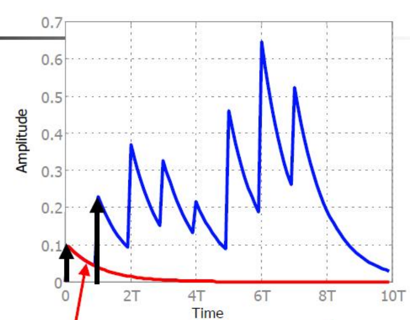
- When the first impulse input sample arrives, the output is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)g(t-\tau)d\tau$$



$$= \int_{-\infty}^{\infty} x(\tau)\delta(\tau)g(t-\tau)d\tau$$

$$= x(0)g(t) = g(t)x(0)$$

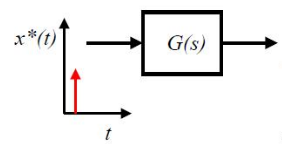


Impulse response from the first impulse

Discrete-time convolution

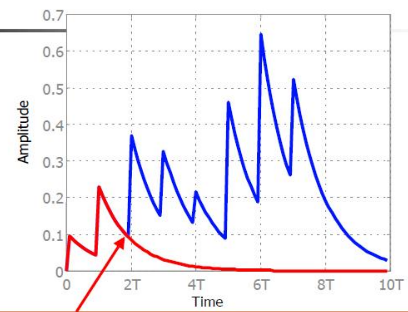
- Consider the output from the second impulse input only

$$y(t) = \int_{-\infty}^{\infty} x(\tau)g(t-\tau)d\tau$$



$$= \int_{-\infty}^{\infty} x(\tau-T)\delta(\tau-T)g(t-\tau)d\tau$$

$$= x(T)g(t-T) = g(t-T)x(T)$$



Impulse response from the second impulse, including The effect from the first impulse.

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Discrete-time convolution

- We can say that during the first sample, the response is

$$y(t) = g(t)x(0) \qquad 0 \le t < T$$

Impulse with intensity $x(0)$

Impulse response function

Discrete-time convolution

- And during the second sample, the response is

$$y(t) = g(t)x(0) + g(t-T)x(T) \qquad T \le t < 2T$$

Impulse with intensity $x(T)$

Impulse response function at time $t-T$

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Discrete-time convolution

- In general,

$$y(t) = g(t)x(0) + g(t-T)x(T) + \cdots + g(t-kT)x(kT) \quad kT \le t < (k+1)T$$

- or

$$y(t) = \sum_{h=0}^{k} g(t-hT)x(hT) \qquad 0 \le t \le kT$$

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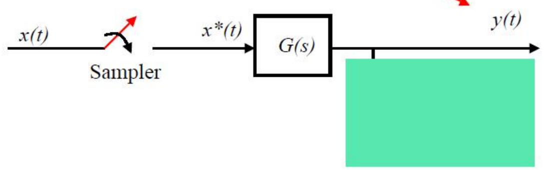
Discrete-time convolution

- The last equation gives the output from G(s) in continuous-time domain.



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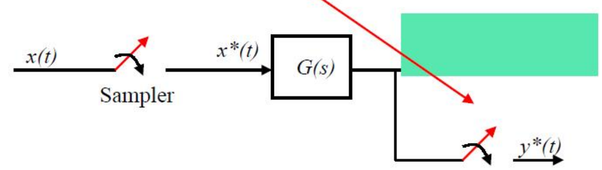
Discrete-time convolution

- If we interest in only the output at the sampling instant t=kT,



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Discrete-time convolution

- From

$$y(t) = \sum_{h=0}^{k} g(t-hT)x(hT) \qquad 0 \le t \le kT$$

- Consider at time t=kT

$$y(kT) = \sum_{h=0}^{k} g(kT-hT)x(hT)$$

System's *weighting sequence*

$$= \sum_{h=0}^{k} x(kT-hT)g(hT)$$

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Discrete-time convolution

- The summation

$$y(kT) = \sum_{h=0}^{k} g(kT-hT)x(hT)$$

$$= \sum_{h=0}^{k} x(kT-hT)g(hT)$$

- is called discrete-time convolution summation

$$y(kT) = x(kT) * g(kT)$$

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Discrete-time convolution

- Note that, since

$$g(kT-hT)=0 \quad \text{if } h>k$$
$$x(kT-hT)=0 \quad \text{if } h>k$$

- We can run the summation to infinity

$$y(kT)=\sum_{h=0}^{\infty} g(kT-hT)x(hT)$$
$$=\sum_{h=0}^{\infty} x(kT-hT)g(hT)$$

Pulse transfer function

Discrete-Time Control System, Ogata Ch 3

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Pulse transfer function

- We now move on to the topic of pulse transfer function.

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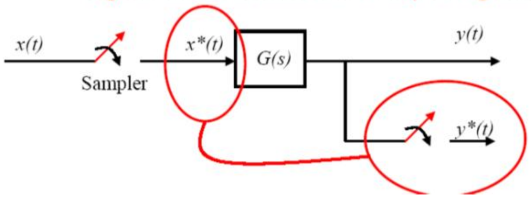
Pulse transfer function

- The pulse transfer function is ratio between the z-transform of output signal and z-transform of input signal.



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Pulse transfer function

- From the sampled output signal

$$y(kT)=\sum_{h=0}^{\infty}g(kT-hT)x(hT) \qquad k=0,1,2,\ldots$$

- Z-transform of $y(kT)$ is

$$Y(z)=\sum_{k=0}^{\infty}y(kT)z^{-k}$$

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Pulse transfer function

$$Y(z)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}g(kT-hT)x(hT)z^{-k}$$

Let $k\text{-}h=m$

- The index k runs from 0.
- When $m<0$; $g(mT)=0$ so m need to run from zero up.

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Pulse transfer function

$$Y(z)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}g(kT-hT)x(hT)z^{-k}$$
$$=\sum_{m=0}^{\infty}\sum_{h=0}^{\infty}g(mT)x(hT)z^{-(m+h)}$$
$$=\sum_{m=0}^{\infty}g(mT)z^{-m}\sum_{h=0}^{\infty}x(hT)z^{-h}$$
$$=G(z)X(z)$$

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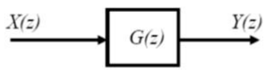
Pulse transfer function

- The pulse transfer function is

$$G(z)=\frac{Y(z)}{X(z)}$$



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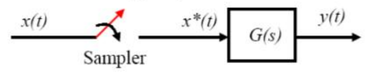
Pulse transfer function

- Note that for the pulse transfer function, both the output signal and the input signal must be sampled.

Pulse transfer function

- For the case of the input is sampled signal, but the output is continuous-signal,



- The output $Y(s)$ is

$$Y(s) = G(s)X^*(s)$$

Pulse transfer function

- We take the starred Laplace's transform of the output signal $Y(s)$.

$$Y^*(s) = \left[G(s)X^*(s)\right]^*$$

$$G^*(s)\Big|_{s=\frac{1}{T}\ln z} = Z\left[G(s)\right] = G(z)$$

Pulse transfer function

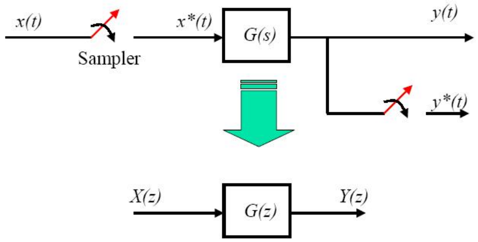
- The starred Laplace's transform of the output then become

$$Y^*(s) = \left[G(s)X^*(s)\right]^* = \left[G(s)\right]^* X^*(s) = G^*(s)X^*(s)$$

This is true only if we have sampled input signal

$$Y(z) = G(z)X(z)$$

Pulse transfer function



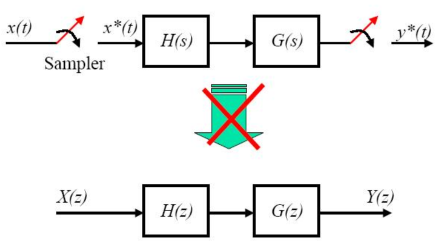
Pulse transfer function

- This is not true if we don't have the sampled input signal.

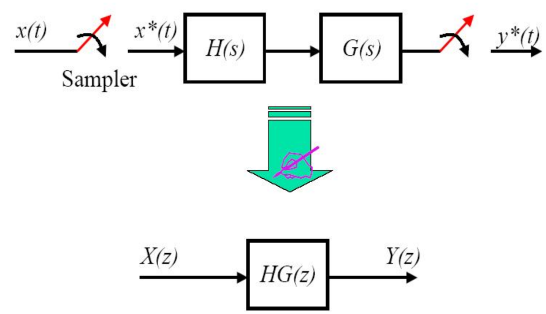
$$Y^*(s) = [G(s)X(s)]^* = [G(s)]^* X^*(s) = G^*(s)X^*(s)$$

$$Y^*(s) = [G(s)X(s)]^* = [GX(s)]^*$$
$$= Z[GX(s)]\big|_{s=\frac{1}{T}\ln z}$$
$$= GX(z)$$

Example



Example



Pulse transfer function

- We also know that

$$G(z) = Z\left[G(s)\right]\Big|_{s=\frac{1}{T}\ln z}$$

- and

$$HG(z) = Z\left[H(s)G(s)\right]\Big|_{s=\frac{1}{T}\ln z} \neq H(z)G(z)$$

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Pulse transfer function

- Quick Review
- Find G(z) of a DC motor

$$G(s) = \frac{1}{s\left(s+1\right)}$$

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Pulse transfer function

- The inverse Laplace's transform (impulse response function) of G(s) is

$$g(t) = 1 - e^{-t}, \qquad 0 \le t$$

$$G(z) = Z\left[1 - e^{-t}\right] = \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-T}z^{-1}}$$

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Pulse transfer function

$$G(z) = Z\left[1 - e^{-t}\right] = \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-T}z^{-1}}$$

$$= \frac{\left(1 - e^{-T}\right)z^{-1}}{\left(1 - z^{-1}\right)\left(1 - e^{-T}z^{-1}\right)}$$

$$= \frac{\left(1 - e^{-T}\right)z}{\left(z - 1\right)\left(z - e^{-T}\right)}$$

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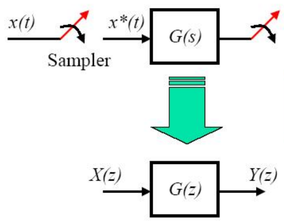
Pulse transfer function

- Let's try something differently

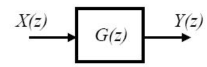


$$G(s)=\frac{1}{(s+a)}$$

Pulse transfer function

- From z-transform table

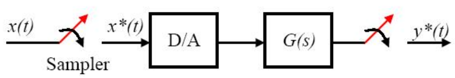
$$G(s)=\frac{1}{(s+a)}$$



$$G(z)=Z\left[\frac{1}{(s+a)}\right]=\frac{1}{1-e^{-aT}z^{-1}}$$

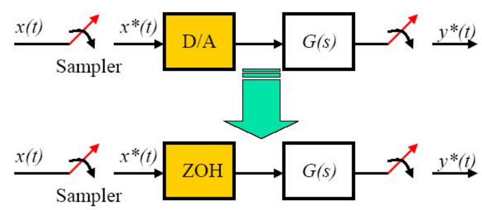
Pulse transfer function

- Note that in the normal situations, the discrete-time signal sent to the plant must passed a D/A.



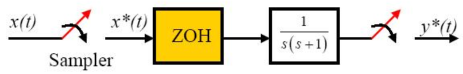
Pulse transfer function

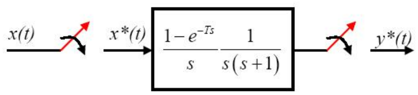
- We can model a A/D as a ZOH



Pulse transfer function

- The total plant transfer function is





Pulse transfer function

$$G(z) = Z\left[(1-e^{-Ts})\frac{1}{s^2(s+1)}\right]$$

$$= (1-z^{-1})Z\left[\frac{1}{s^2(s+1)}\right]$$

$$= (1-z^{-1})Z\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right]$$

Pulse transfer function

$$G(z) = (1-z^{-1})Z\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right]$$

$$= (1-z^{-1})\left[\frac{Tz^{-1}}{(1-z^{-1})^2} - \frac{1}{1-z^{-1}} + \frac{1}{1-e^{-T}z^{-1}}\right]$$

$$= \frac{(T-1+e^{-T})z^{-1} + (1-e^{-T}-Te^{-T})z^{-2}}{(1-z^{-1})(1-e^{-T}z^{-1})}$$

Closed-loop discrete-time Transfer Function

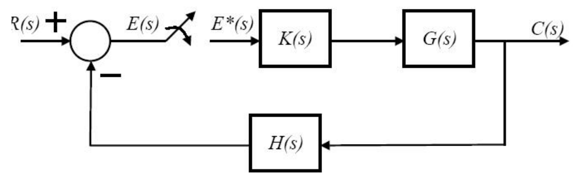
Discrete-Time Control System, Ogata Ch 3

Why are we doing this?

- We will see how the closed-loop discrete-time structure looks like.

Pulse transfer function

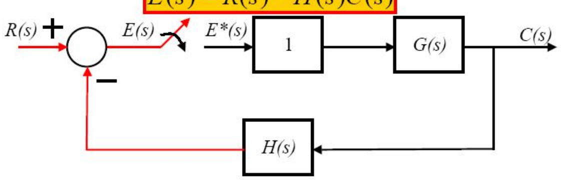
- For the closed-loop system



Pulse transfer function

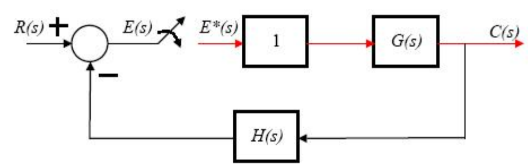
- For K(s)=1

$$E(s) = R(s) - H(s)C(s)$$



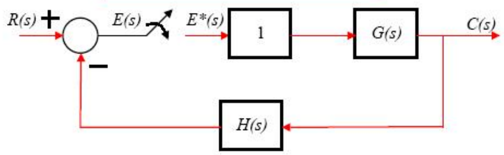
Pulse transfer function

$$C(s) = G(s)E^{*}(s)$$



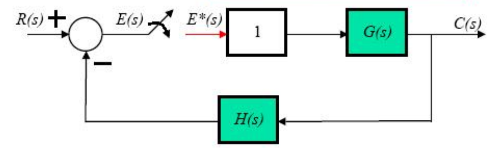
Pulse transfer function

$$E(s) = R(s) - H(s)G(s)E^*(s)$$



Pulse transfer function

$$E^*(s) = R^*(s) - GH^*(s)E^*(s)$$



Pulse transfer function

$$E^*(s) = R^*(s) - GH^*(s)E^*(s)$$

$$= \frac{R^*(s)}{1 + GH^*(s)}$$

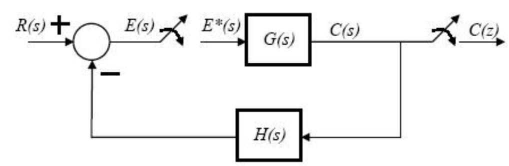
$$C^*(s) = G^*(s)E^*(s)$$

$$= \frac{G^*(s)R^*(s)}{1 + GH^*(s)}$$

$$C(z) = \frac{G(z)R(z)}{1 + GH(z)} \qquad \frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

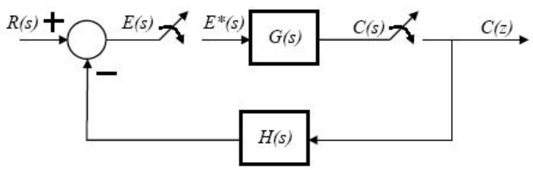
Pulse transfer function

$$C(z) = \frac{G(z)R(z)}{1 + GH(z)}$$



Pulse transfer function

$$C(z)=\frac{G(z)R(z)}{1+G(z)H(z)}$$



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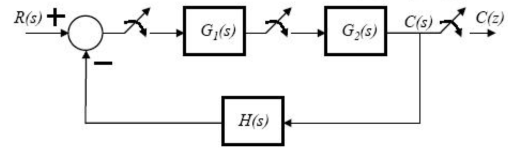
Pulse transfer function

$$C(z)=\frac{G_1(z)G_2(z)R(z)}{1+G_1(z)G_2H(z)}$$



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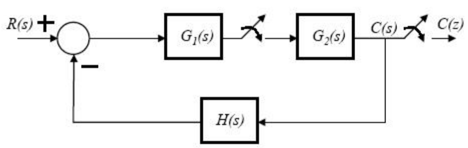
Pulse transfer function

$$C(z)=\frac{G_2(z)G_1R(z)}{1+G_1G_2H(z)}$$



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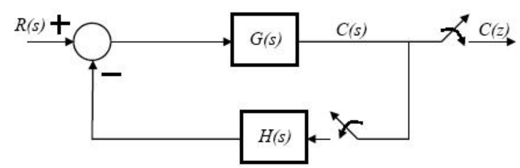
Pulse transfer function

$$C(z)=\frac{GR(z)}{1+GH(z)}$$



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Pulse transfer function

- Note that for the last case,

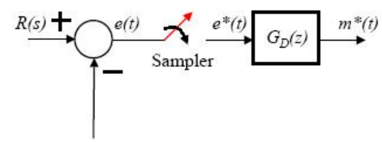
$$C(z) = \frac{GR(z)}{1 + GH(z)}$$

- We cannot explicitly write

$$\frac{C(z)}{R(z)} \neq \frac{G}{1 + GH(z)}$$

Digital Controller

- The general Pulse Transfer Function of a digital controller is written as



Digital Controller

- The difference equation for the digital controller is

$$m(k) + a_1 m(k-1) + a_2 m(k-2) + \cdots + a_n m(k-n)$$
$$= b_0 e(k) + b_1 e(k-1) + \cdots + b_n e(k-n)$$

Digital Controller

- The z-transform is

$$M(z) + a_1 z^{-1} M(z) + a_2 z^{-2} M(z) + \cdots + a_n z^{-n} M(z)$$
$$= b_0 E(z) + b_1 z^{-1} E(z) + \cdots + b_n z^{-n} E(z)$$

$$\left(1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}\right) M(z)$$
$$= \left(b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}\right) E(z)$$

Digital Controller

- The pulse transfer function of the digital controller is

$$G_D(z) = \frac{M(z)}{E(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_n z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_n z^{-n}}$$

- Our task is to design the controller by adjusting the controller parameters bs and as.

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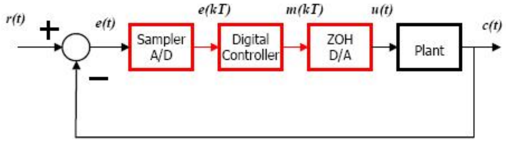
Closed-loop Digital Controller

- The closed-loop structure of the digital controller is



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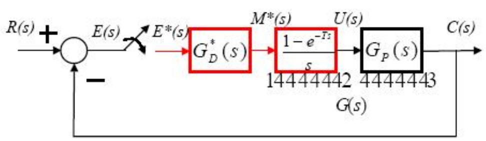
Closed-loop Digital Controller

- We put this into our model as



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Closed-loop Digital Controller

- The closed-loop transfer function is derived by letting

$$G(s) = \frac{1 - e^{-Ts}}{s} G_P(s)$$

- The closed-loop output is

$$C(s) = G(s) G_D^*(s) E^*(s)$$

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Closed-loop Digital Controller

- The sampled output signal is then

$$C^*(s) = G^*(s)G_D^*(s)E^*(s)$$
$$C(z) = G(z)G_D(z)E(z)$$

- And

$$E(z) = R(z) - C(z)$$

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Closed-loop Digital Controller

- Then,

$$C(z) = G(z)G_D(z)\left[R(z) - C(z)\right]$$

- Therefore,

$$\frac{C(z)}{R(z)} = \frac{G(z)G_D(z)}{1 + G(z)G_D(z)}$$

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Digital PID

- The PID (Proportional + Integral + Derivative) control action in analog form is

$$m(t) = K\left[e(t) + \frac{1}{T_i}\int_0^t e(t)dt + T_d\frac{de(t)}{dt}\right]$$

87

Digital PID

- The Proportional part = Gain of *e(t)*, we can use multiplication in digital computer.
- The Integral part = Area under *e(t)* curve, we need an estimation method.
- The Derivative part = Slope of *e(t)* curve, we need an estimation method.

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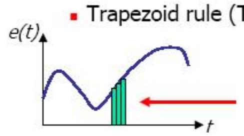
Approximation of Integral part

- The area under the $e(t)$ curve can be estimated using the
  - Forward Rectangular rule (Euler's rule)
  - Backward Rectangular rule
  - Trapezoid rule (Tustin's method)



$\sum$ = estimated area

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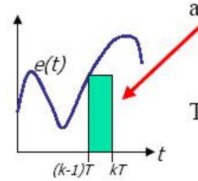
Integral part

- Forward rectangular rule



$$\text{area} = \text{height} \times \text{width}$$
$$= e\left(\left(k-1\right)T\right) \times T$$

Total area from $t = 0$ to $t = kT$

$$= \sum_{n=0}^{k} e\left(nT\right) \times T$$

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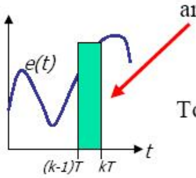
Integral part

- Backward rectangular rule



$$\text{area} = \text{height} \times \text{width}$$
$$= e\left(kT\right) \times T$$

Total area from $t = 0$ to $t = kT$

$$= \sum_{n=1}^{k} e\left(nT\right) \times T$$

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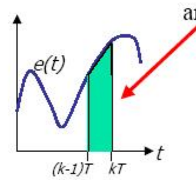
Integral part

- Trapezoid rule



$$\text{area} = \frac{1}{2} \times \text{total prll length} \times \text{height}$$
$$= \frac{1}{2} \times \left[e\left(\left(k-1\right)T\right) + e(kT) \right] \times T$$

Total area from $t = 0$ to $t = kT$

$$= T \sum_{h=1}^{k} \frac{e\left(\left(h-1\right)T\right) + e(hT)}{2}$$

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Integral part

- The digital integral term from time *0* to time *t* using the Trapezoid approximation is

$$m(t) = K\left[e(t) + \frac{1}{T_i}\int_0^t e(t)dt + T_d\frac{de(t)}{dt}\right]$$

$$\frac{T}{T_i}\left[\frac{e(0)+e(T)}{2} + \frac{e(T)+e(2T)}{2} + \cdots + \frac{e\left((k-1)T\right)+e(kT)}{2}\right]$$

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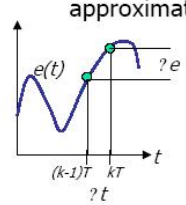
Approximate of Derivative part

- The derivative (slope) of $e(t)$ is approximated as



$$\frac{de(t)}{dt} \approx \frac{\Delta e}{\Delta t} = \frac{e(kT) - e\left((k-1)T\right)}{T}$$

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Derivative part

- The digital derivative term is

$$m(t) = K\left[e(t) + \frac{1}{T_i}\int_0^t e(t)dt + T_d\frac{de(t)}{dt}\right]$$

$$\frac{T_d}{T}\left[e(kT) - e\left((k-1)T\right)\right]$$

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Digital PID

- The digital PID is then,

$$m(t) = K\left[e(t) + \frac{1}{T_i}\int_0^t e(t)dt + T_d\frac{de(t)}{dt}\right]$$

$$m(kT) = K\left[e(kT) + \frac{T}{T_i}\sum_{h=1}^{k}\frac{e\left((h-1)T\right)+e(hT)}{2} + \frac{T_d}{T}\left[e(kT) - e\left((k-1)T\right)\right]\right]$$

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Digital PID

- Taking the z-transform

$$Z\left[\sum_{h=1}^{k}\frac{e\big((h-1)T\big)+e(hT)}{2}\right]=\frac{1+z^{-1}}{2(1-z^{-1})}E(z)$$

- HW : Prove this!

97

Digital PID

- The transfer function is

$$M(z)=K\left[1+\frac{T}{2T_i}\frac{1+z^{-1}}{(1-z^{-1})}+\frac{T_d}{T}(1-z^{-1})\right]E(z)$$

- Convert this term into the digital integrator. $\frac{1}{s}\Leftrightarrow\frac{1}{1-z^{-1}}$

98

Digital PID

- We get

$$M(z)=K\left[1+\frac{T}{2T_i}\frac{1+z^{-1}}{(1-z^{-1})}+\frac{T_d}{T}(1-z^{-1})\right]E(z)$$

$$=K\left[1-\frac{T}{2T_i}+\frac{T}{T_i}\frac{1}{1-z^{-1}}+\frac{T_d}{T}(1-z^{-1})\right]E(z)$$

P term　I term　D term

99

Digital PID

$$M(z)=\left[K_P+K_I\frac{1}{1-z^{-1}}+K_D(1-z^{-1})\right]E(z)$$

$$K_P=K-\frac{KT}{2T_i}=\text{proportional gain}$$

$$K_I=\frac{KT}{T_i}=\text{integral gain}$$

$$K_D=\frac{KT_D}{T}=\text{derivative gain}$$

100

PID

- Let's come back to the analog PID transfer function

$$m(t) = K_P\left[e(t) + \frac{1}{T_i}\int_0^t e(t)dt + T_d\frac{de(t)}{dt}\right]$$

$$M(s) = K_P\left[1 + \frac{1}{T_i s} + T_d s\right]E(s)$$

101

PID

- The PID tuning methods can be studied from many Classical Control textbooks.
- The tuning methods reviewed here will be the Ziegler-Nichols tuning rules.

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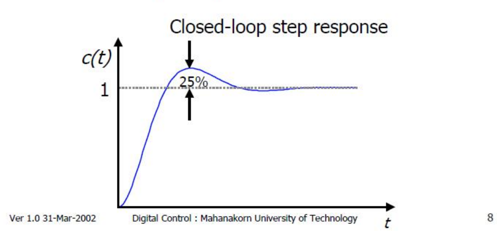
PID Tuning

- Ziegler-Nichols rule will yield target closed-loop response of 25% overshoot.



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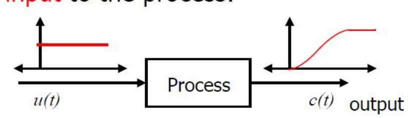
PID Tuning

- There are two methods in tuning the PID according to Ziegler-Nichols.
- The first method is by applying the step input to the process.



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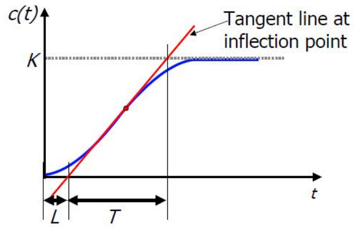
PID Tuning first method

- Measure the output s-curve



PID Tuning first method

- Assumptions for the first method :
  - Plant has no integrator
  - Plant has no dominant complex-conjugate poles
- If the above assumptions do not hold, this first method cannot apply.

PID Tuning first method

- The parameters for PID are

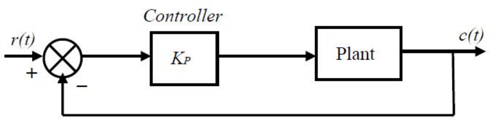
Controller	Kp	Ti	Td
P	$\frac{T}{L}$	∞	0
PI	$0.9\frac{T}{L}$	$\frac{L}{0.3}$	0
PID	$1.2\frac{T}{L}$	$2L$	$0.5L$

PID Tuning second method

- If the first method cannot applied, we can use the second method (closed-loop test).
- By letting the Integral term and the Derivative term OFF.

PID Tuning second method

- Leaving only the Proportional term.



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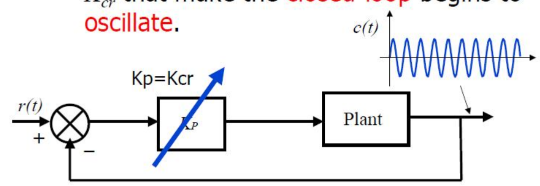
PID Tuning second method

- Increase the gain K_p from 0 to a value K_{cr} that make the closed-loop begins to oscillate.



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PID Tuning second method

- Assumption for the second method
  - The closed-loop response must oscillate for some $Kp=Kcr$ (critical gain)
  - If the closed-loop response does not oscillate, this method cannot be applied.

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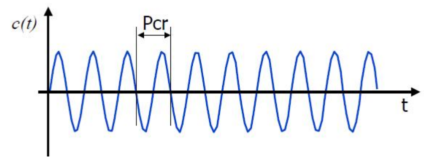
PID Tuning second method

- When the closed-loop response $c(t)$ oscillates with $Kp=Kcr$, we measure the period of oscillation Pcr.



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PID Tuning second method

- The parameters for PID are

Controller	Kp	Ti	Td
P	$0.5K_{cr}$	∞	0
PI	$0.45K_{cr}$	$\frac{1}{1.2}P_{cr}$	0
PID	$0.6K_{cr}$	$0.5P_{cr}$	$0.125P_{cr}$

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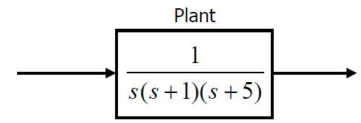
PID Example

- Let's design a PID controller for the plant:



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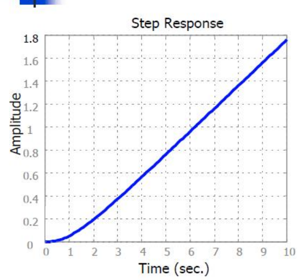
PID Example



- We can see from the step response that the plant has integrator, the first method cannot be used.

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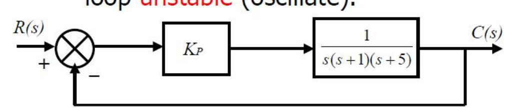
PID Example

- We apply the Ziegler-Nichols' second method, the closed-loop response.
- Find the value of Kp which makes the loop unstable (oscillate).



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PID Example

- The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K_P}{s(s+1)(s+5)+K_P}$$

- Find the value of Kp which makes the loop unstable using Routh's stability criterion.

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PID Example

- We get the Kp=30 (the critical gain Kcr) will make the loop begins to oscillate at rad/sec. $\omega = \sqrt{5}$

- HW : Prove that Kp=30 is the critical gain

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PID Example

- Test this condition with MATLAB step response of the closed-loop transfer function.

$$\frac{C(s)}{R(s)} = \frac{30}{s(s+1)(s+5)+30}$$

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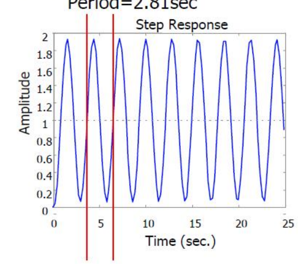
PID Example

Period=2.81sec



- Critical gain
  - Kcr=30
- Critical period
  - Pcr=2.81

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PID Example

- Use the table

Controller	Kp	Ti	Td
P	$0.5K_{cr}$	∞	0
PI	$0.45K_{cr}$	$\frac{1}{1.2}P_{cr}$	0
PID	$0.6K_{cr}$	$0.5P_{cr}$	$0.125P_{cr}$

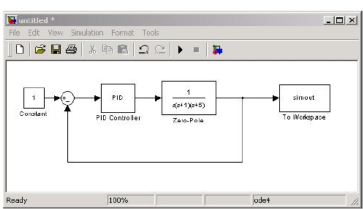
PID Example

- We get the controller parameters as

$$K_P = 0.6K_{cr} = 18$$
$$T_i = 0.5P_{cr} = 1.405$$
$$T_d = 0.125P_{cr} = 0.35124$$

PID Example

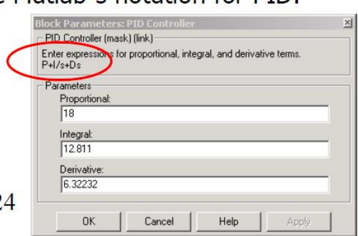
- Simulate the closed-loop response with Simulink.



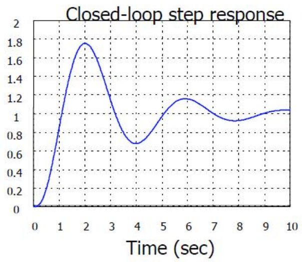
PID Example

- Note the Matlab's notation for PID.

$$K_i = \frac{18}{1.405}$$
$$K_d = 18 \times 0.35124$$



PID Example



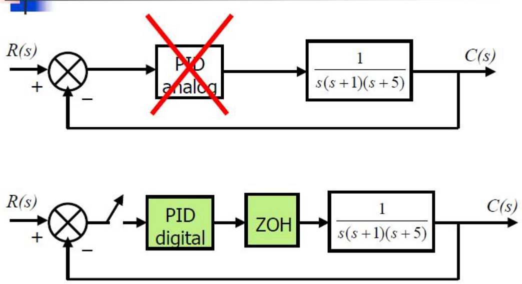
- The response requires some fine tuning [Ogata: *Modern Control Eng*.], but let's say it is ok for us.

Digital PID Example

- We will now convert this analog PID into digital PID follows the method discussed previously.

$$M(z) = \left[K_{PDigital} + K_I \frac{1}{1-z^{-1}} + K_D (1-z^{-1}) \right] E(z)$$

Digital PID Example



Digital PID Example

- Let's the sampling period T = 0.1 sec

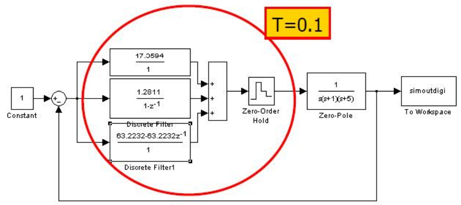
$$K_{PDigital} = K_P - \frac{K_P T}{2T_i} = 17.3594$$

$$K_I = \frac{K_P T}{T_i} = 1.2811$$

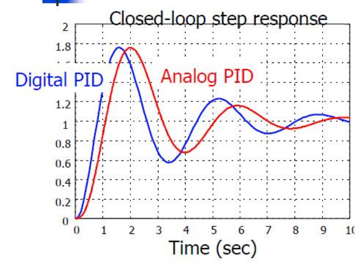
$$K_D = \frac{K_P T_D}{T} = 63.2232$$

Digital PID Example

- Simulate the digital PID using Simulink.



Digital PID Example



- The result is compared against the analog PID control loop.

Summary

- We have looked at the analog PID design using Zeigler-Nichols method.
- We have applied the PID parameters to the approximated digital PID.