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Optimal portfolios with maximum Value-at-Risk constraint under a hidden Markovian regime-switching model[☆]

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ABSTRACT

This paper studies an optimal portfolio selection problem in the presence of the Maximum Value-at-Risk (MVaR) constraint in a hidden Markovian regime-switching environment. The price dynamics of n risky assets are governed by a hidden Markovian regime-switching model with a hidden Markov chain whose states represent the states of an economy. We formulate the problem as a constrained utility maximization problem over a finite time horizon and then reduce it to solving a Hamilton–Jacobi–Bellman (HJB) equation using the separation principle. The MVaR constraint for n risky assets plus one riskless asset is derived and the method of Lagrange multiplier is used to deal with the constraint. A numerical algorithm is then adopted to solve the HJB equation. Numerical results are provided to demonstrate the implementation of the algorithm.

1. Introduction

In modern finance, optimal portfolio allocation of different assets is a prominent issue. Markowitz (1952) pioneered the use of a mean–variance approach in formulating optimal allocation problems. His approach reduces the problem to the situation that one only needs to maximize the expected return under an acceptable level of risk in a single period. The risk level is measured by the variance of the return. Then, Merton (1969, 1971) extended this single-period model to a continuous-time framework which reflects the market environment better. Closed form solutions were then derived using stochastic optimal control techniques with the premise that the coefficients in the price process of the risky assets are constant. However, this assumption may not be realistic. Therefore, some researchers started investigating asset allocation models with non-constant parameters. For example, Boyle and Yang (1997) employed a multi-factor stochastic interest rate model from Duffie and Kan (1996) for the bond price dynamics in their asset allocation model. Lim and Zhou (2002) analyzed a continuous time mean–variance portfolio selection problem with random market coefficients.

Recently, Markovian regime-switching models have been widely applied in economics and finance, since it can give a reasonably good description for some important stylized features of the price dynamics of assets. The applications of Markov-switching time series models to economics and econometrics were introduced by Hamilton (1989). The use of Markovian regime-switching models for portfolio selection has received much attention. For example, in Zhou and Yin (2003), the state of the market model which would affect the parameters in the stock price process was described by Markovian regime switching models with observable regimes. The efficient portfolios were derived explicitly in closed forms for their Markowitz mean–variance portfolio selection model using techniques of stochastic linear–quadratic control. In Elliott, Siu, and Badescu (2010), they modeled the evolution of the state of the economy by a hidden Markov chain model and assumed that the "true" state of the underlying economy is unobservable. An explicit solution was derived in their mean–variance portfolio selection model using the

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stochastic maximum principle. Honda (2003) studied the optimal portfolio choice when the mean returns of a risky asset depend on a hidden Markov chain. Some other works on optimal asset allocation in hidden Markovian regime-switching economy are, for example, Baeuerle and Rieder (2007), Elliott and Siu (2012), Korn, Siu, and Zhang (2011), Sass and Haussmann (2004), Shen and Siu (2015) and Siu (2011, 2012, 2013, 2015, 2016), amongst others.

Value-at-Risk (VaR) is one of the popular risk measures used in market risk management. Informally speaking, VaR describes the maximum expected loss during a given period at a given level of confidence. VaR has been used as a risk constraint in portfolio optimization. For example, Basak and Shapiro (2001) considered the optimal portfolio policies when VaR is imposed as a constraint though they pointed out that the use of the VaR constraint may lead to sub-optimal results. Yiu (2004) derived the VaR constraint for multiple risky assets and a riskless asset, and found that investments in risky assets are reduced when the VaR constraint becomes active. Typically, VaR is derived under the assumption that the parameters, such as interest rate, drift and volatility, in the price dynamics of assets are assumed to be known beforehand so that some standard distributions such as a normal distribution may be applied to compute VaR. However, if these parameters depend on the state of the underlying economy which may switch over time, then the values of these parameters may be uncertain or unknown. The Maximum Value-at-Risk (MVaR) may provide a conservative way to describe risks under this situation. It is defined as the maximum value of the VaRs of the portfolio at different states of the underlying economy in a given time. Yiu, Liu, Siu, and Ching (2010) discussed a utility maximization problem constrained by the MVaR. In their paper, the state of the economy is modeled by an observable Markov chain. While it seems that there is a relatively little work on optimal portfolio allocation in multiple risky assets in a hidden Markovian regime-switching economy using the MVaR as a risk constraint.

In this paper we extend the portfolio allocation model with one risky asset in Yiu et al. (2010) to a more general situation of multiple risky assets. Furthermore, the state of an economy was assumed to be observable in Yiu et al. (2010). However, in practice, the state of the underlying economy may be unobservable or not directly observable. It is assumed that the hidden state of the economy is described by a continuous-time hidden Markov chain. Similar method can be found in Elliott and van der Hoek (1997). The price dynamics of n risky assets follow an n-dimensional Geometric Brownian Motion (GBM) where the value of the drift is supposed to switch over time according to the states of the underlying hidden Markov chain. The MVaR is derived for n risky assets plus one riskless asset and imposed as a constraint. We formulate this optimal portfolio allocation problem as a constrained utility maximization problem. Then using the separation principle in Elliott et al. (2010), the problem can be separated into two problems: a filtering-estimation problem and a constrained stochastic control problem. A robust form of filtering equations is presented to estimate the unknown parameters by applying the gauge transformation technique which is proposed by Clark (1978) and applied in Elliott, Malcolm, and Tsoi (2003); Elliott et al. (2010). In this way, solving the constrained stochastic control problem is converted to solving a Hamilton–Jacobi–Bellman (HJB) equation, where the MVaR constraint is handled by the method of Lagrange multiplier. A numerical algorithm is used to solve the HJB equation for the optimal constrained portfolio numerically.

The rest of this paper is structured as follows. In Section 2, the price dynamics of n risky assets are presented. The optimal portfolio selection problem without constraint is formulated as a maximization of the expected utility over a given period. Section 3 discusses the separation principle. The corresponding MVaR is also derived to describe investment risks. In Section 4, the filters for the hidden states of the economy are presented. In Sections 5 and 6, a numerical algorithm and numerical results are presented. Finally, concluding remarks are given in Section 7.

2. Model dynamics and portfolio allocation problem

In this section, we consider a continuous-time economy with a finite time horizon $\mathcal{T} := [0, T]$. All of the uncertainties are described by a complete probability space $(\Omega, \mathcal{F}, P)$, where P is a real-world probability measure. Let $\mathbf{y}'$ denote the transpose of a matrix or a vector $\mathbf{y}$ and $\mathbb{1}_{m\times n}$ denote an $m \times n$-dimensional matrix whose entries are all equal to one. The model dynamics described are those in standard hidden Markovian regime-switching financial models. Similar models have been used for portfolio selection in the literature (see, for example, Elliott & Siu, 2012; Elliott et al., 2010; Korn et al., 2011; Sass & Haussmann, 2004; Shen & Siu, 2015; Siu, 2011, 2012, 2013, 2015, 2016, and the relevant literature therein).

Let $X := \{X(t)\}_{t\le T}$ be a continuous-time, finite-state Markov chain with state space $\varepsilon := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$, where $\mathbf{e}_i$ is the unit vector in $\mathcal{R}^N$ with one in the ith position and zero elsewhere. This convention of the state space of the chain was adopted in, for example, Elliott, Aggoun, and Moore (1995). Then as in Elliott et al. (1995), a semi-martingale representation for the chain is given as follows:

$$X(t) = X(0) + \int_0^t AX(u)du + M(t),$$

where $A := [a_{ij}]_{N\times N}$ is a time invariant rate matrix of the chain and $\{M(t)|t \in \mathcal{T}\}$ is a martingale under P. The element a_{ij} in A is the instantaneous intensity of the transition of the chain X from State $\mathbf{e}_j$ to State $\mathbf{e}_i$. Here the states of the chain X are interpreted as hidden states of an economy.

We consider an optimal portfolio allocation problem with n risky assets and one riskless asset. Suppose the price process of the riskless asset which is denoted as $S_0 := \{S_0(t)|t \in \mathcal{T}\}$ follows:

$$S_0(t) = \exp(rt) \quad \text{and} \quad S_0(0) = 1.$$

Here the interest rate r is assumed to be a positive constant. The price process of the n risky assets, denoted by $\mathbf{S} = \{\mathbf{S}(t)|t \in \mathcal{T}\}$, satisfies:

$$d\mathbf{S}(t) = D(\mathbf{S}(t))\mu(t)dt + D(\mathbf{S}(t))\boldsymbol{\sigma} dW(t) \quad \text{and} \quad \mathbf{S}(0) = s_0.$$

Here $\{W(t)|t \in \mathcal{T}\}$ is an n-dimensional standard Brownian motion and $D(\mathbf{S}(t))$ is the diagonal matrix of the vector $\mathbf{S}(t) = (S_1(t), \ldots, S_n(t))'$. The volatility $\boldsymbol{\sigma} = (\sigma_{ij})_{n\times n}$ is a constant non-singular matrix. Readers interested in stochastic volatility models can refer to Pham and Quenez (2001). Shen and Siu (2013) investigated the pricing of variance swaps under stochastic interest rate and volatility. They incorporated the stochastic interest rate process and separated it from the volatility process using techniques of forward measure changes. The situation when r and σ are time-varying or stochastic processes will lead to complication in the estimation of the model. In this case, the standard EM algorithm may not work well and the statistical properties such as asymptotic properties of the estimators may not be available. The drift $\mu(t)$ is assumed to depend on the state of the economy and is given as follows:

$$\mu(t) = \boldsymbol{\mu} X(t),$$

where $\boldsymbol{\mu} = (\mu_{ij})$ is an $n \times N$ matrix.

Both the drift process and the Brownian motion are assumed to be unobservable to the investor. The only observable information is the price process $\mathbf{S}$. The importance of incorporating the model uncertainty of the drift is discussed in, for example, Elliott and Siu (2012) and Elliott et al. (2010). Instead of considering the price process directly, we consider the log return process $Y := \{Y(t)|t \in \mathcal{T}\}$. Here $Y(t) = (Y_1(t), Y_2(t), \ldots, Y_n(t))'$ and $Y_i(t) = \ln(S_i(t)/S_i(0))$. It is well-known that by Itô's lemma,

$$dY(t) = g(t)dt + \boldsymbol{\sigma} dW(t),$$

where

$$g(t) = \mu(t) - \frac{1}{2}(\tilde{\sigma})'$$

and

$$\tilde{\sigma} = \left(\sum_{i=1}^{n} \sigma_{1i}^2, \sum_{i=1}^{n} \sigma_{2i}^2, \ldots, \sum_{i=1}^{n} \sigma_{ni}^2\right).$$

Let

$$\mathbf{g} = \boldsymbol{\mu} - \frac{1}{2}(\tilde{\sigma})'\mathbb{1}_{1\times N},$$

then $g(t) = \mathbf{g}X(t)$.

The filtrations generated by the state process and the return process are given, respectively, as follows:

$$\mathcal{F}_t^X = \sigma\{X(u), 0 \le u \le t\} \vee \mathcal{N},$$
$$\mathcal{F}_t^i = \sigma\{Y_i(u), 0 \le u \le t\} \vee \mathcal{N}$$

and

$$\mathcal{F}_t = \mathcal{F}_t^1 \vee \mathcal{F}_t^2 \vee \cdots \vee \mathcal{F}_t^n, \qquad \mathcal{G}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^1 \vee \cdots \vee \mathcal{F}_t^n$$

where $\mathcal{N}$ is the collection of all P-null subsets in $\mathcal{F}$; $\mathcal{E}_1 \vee \mathcal{E}_2$ is the minimal σ-algebra containing the σ-algebras $\mathcal{E}_1$ and $\mathcal{E}_2$. Write

$$\mathbb{F} := \{\mathcal{F}_t, t \in [0, T]\} \quad \text{and} \quad \mathbb{G} := \{\mathcal{G}_t, t \in [0, T]\}.$$

Before presenting the optimization problem, we first define some notations. Let $\pi^i(t)$ be the amount of wealth invested in the ith risky asset at time t and $c(t)$ be the consumption rate of the investor at time t. A consumption–investment strategy $\{(\pi(t), c(t)), t \in \mathcal{T}\}$ is supposed to be an $\mathbb{F}$-adapted process, which satisfies

$$\int_0^t (\|\pi(s)\|^2 + c(s))ds < \infty.$$

The wealth process of the investor is written as $\{V(t), t \in \mathcal{T}\}$ with initial wealth $V(0) = v > 0$ and

$$dV(t) = [\pi'(t)(\boldsymbol{\mu}X(t) - \mathbb{1}_{n\times 1}r) + rV(t) - c(t)]dt + \pi'(t)\boldsymbol{\sigma}dW(t).$$

Denote $U(\cdot,\cdot) : [0, T] \times [0, +\infty) \to \mathcal{R}$ as a utility function, which satisfies the standard property that for each $t \in [0, T]$, $U(t, \cdot)$ is strictly increasing, strictly concave, twice continuously differentiable on $(0, +\infty)$. In addition,

$$\lim_{c\to\infty} U'(t, c) = 0 \quad \text{and} \quad \lim_{c\to 0} U'(t, c) = \infty.$$

The unconstrained optimization problem of the investor with initial wealth $V(0) = v$ and initial state $X(0) = \mathbf{e}_i$ is described as follows:

$$\max_{\pi(t),c(t)} E\left[\int_0^T U(t, c(t))dt \,\middle|\, V(0) = v, X(0) = \mathbf{e}_i\right] \tag{1}$$

subject to the dynamic budget constraint:

$$dV(t) = [\pi'(t)(\boldsymbol{\mu}X(t) - \mathbb{1}_{n\times 1}r) + rV(t) - c(t)]dt + \pi'(t)\boldsymbol{\sigma}dW(t). \tag{2}$$

Recall that the investor can only observe the price dynamics of the assets, and that the processes $X := \{X(t)\}_{t\in\mathcal{T}}$ and $W := \{W(t)\}_{t\in\mathcal{T}}$ are both unobservable. A method to discuss the above optimization problem is to transform it to the one with complete observations using the separation principle, which is well-known in the filtering and control literature.

3. The separation principle and MVaR constraint

In this section, the separation principle and the Maximum Value-at-Risk (MVaR) constraint will be illustrated. The purpose of using the separation principle is to solve the two problems, filtering and stochastic control, separately for partially observed stochastic systems. Several works have successfully employed the separation principle to solve stochastic optimal control problems with partial observations, see for instance Elliott (1982), Elliott et al. (2010) and Karatzas and Zhao (2001). The separation principle has been adopted to discuss optimal asset allocation in hidden Markovian regime-switching models. Some examples include Elliott and Siu (2012), Elliott et al. (2010), Korn et al. (2011), Shen and Siu (2015) and Siu (2011, 2012, 2013, 2015, 2016), amongst others. Here as in some of the literature, after giving filtered estimates of the hidden states of the model, the optimal control problem with complete observations can be solved by using the HJB dynamic programming approach. Write $\{\widehat{\gamma}(t), t \in [0, T]\}$ for the $\mathbb{F}$-optional projection of any integrable, $\mathbb{G}$-adapted process $\{\gamma(t), t \in [0, T]\}$. Then it is well known that $\widehat{\gamma}(t) = \mathrm{E}[\gamma(t)|\mathcal{F}(t)]$, P-a.s., and that $\widehat{\gamma}$ takes into account the measurability in (t, ω).

Before discussing the separation principle, an $\mathcal{F}$-adapted process $\widehat{W}(t)$ on $(\Omega, \mathcal{F}, P)$ is defined by setting

$$\widehat{W}(t) := \int_0^t \boldsymbol{\sigma}^{-1}(dY(s) - \widehat{g}(s)ds), \tag{3}$$

where $\widehat{g}(t) = E[g(t)|\mathcal{F}(t)]$ for any $t \in [0, T]$. Rewriting it gives:

$$\widehat{W}(t) = W(t) + \int_0^t \boldsymbol{\sigma}^{-1}(\mu(s) - \widehat{\mu}(s))ds.$$

In filtering theory, this is called an *innovation process*. Indeed, $\{\widehat{W}(t), t \in [0, T]\}$ is an n-dimensional $(\mathbb{F}, P)$-Brownian motion. For a more detailed discussion, we refer readers to, for example, Karatzas and Zhao (2001) and Liptser and Shiryaev (1977).

Then the dynamics of the wealth process V and the logarithmic return process Y can be expressed in terms of $\widehat{W}$:

$$dV(t) = [\pi'(t)(\widehat{\mu}(t) - \mathbb{1}_{n\times 1}r) + rV(t) - c(t)]dt + \pi'(t)\boldsymbol{\sigma}d\widehat{W}(t)$$

and

$$dY(t) = \widehat{g}(t)dt + \boldsymbol{\sigma}d\widehat{W}(t). \tag{4}$$

Note that all of the processes in the wealth dynamics are $\mathbb{F}$-adapted.

Next, we shall derive the Maximum Value-at-Risk (MVaR) constraint for the captured optimal portfolio problem, using the method in Yiu (2004) and Yiu et al. (2010).

The dynamics of the wealth process in Eq. (2) can be rewritten as follows:

$$dV(t) = \alpha(\theta(t) - V(t))dt + \pi'(t)\boldsymbol{\sigma}d\widehat{W}(t),$$

where

$$\alpha = -r \quad \text{and} \quad \theta(t) = \frac{\pi'(t)(\mu(t) - \mathbb{1}_{n\times 1}r) - c(t)}{-r}.$$

Assume that the investor adjusts his portfolio frequently and the time duration from t to $t + h$ ($h > 0$) is very small. During this small interval, suppose that no trading happens. The decision for the portfolio and the consumption rate is made at the beginning of every time interval. The consumption rate is approximately a constant during the time interval $[t, t+h]$. The hidden state which represents the state of the economy is assumed to be stable and unchanged in the small time interval.

Define a new process $\{Z(t), t \in [0, T]\}$ based on the wealth process as follows:

$$Z(t) = e^{\alpha t}V(t).$$

We let

$$\begin{cases} \alpha := -r; \\ \theta_i(\tau) := \theta_i(t) = \dfrac{\pi'(t)(\boldsymbol{\mu}^i - \mathbb{1}_{n\times 1} r) - c(t)}{-r}; \\ Z(\tau) = e^{\alpha\tau} V(\tau), \quad i = 1, \ldots, N, \ \tau \in [t, t+h]. \end{cases}$$

Here $\boldsymbol{\mu}^i$ means the ith column in $\boldsymbol{\mu}$. Then,

$$V(t+h) = e^{-\alpha h}(V(t) - \theta_i(t)) + \theta_i(t) + \int_t^{t+h} e^{-\alpha(t+h-\tau)} \pi'(t)\boldsymbol{\sigma} dW(\tau).$$

The details can be found in, for example, Yiu (2004). Under the probability measure P, the conditional mean and the conditional covariance are given, respectively, by

$$E[V(t+h)|\mathcal{G}_t, X(t) = \mathbf{e}_i] = \theta_i(t) + e^{-\alpha h}(V(t) - \theta_i(t));$$

and

$$\mathrm{Cov}[V(t+h_1), V(t+h_2)|\mathcal{G}_t, X(t) = \mathbf{e}_i] = \frac{\pi'(t)\Sigma\pi(t)}{2\alpha}(e^{-\alpha|h_1-h_2|} - e^{-\alpha(h_1+h_2)}), \tag{5}$$

where $\Sigma = \boldsymbol{\sigma}\boldsymbol{\sigma}'$. In particular, from Eq. (5),

$$\mathrm{Var}[V(t+h)|\mathcal{G}_t, X(t) = \mathbf{e}_i] = \frac{\pi'(t)\Sigma\pi(t)}{2\alpha}(1 - e^{-2\alpha h}).$$

The net conditional loss of the portfolio over the time interval $[t, t+h]$ given that $X(t) = \mathbf{e}_i$ is defined as

$$\Delta_i V(t, h) = V(t+h) - e^{rh} V(t) \quad \text{for } i = 1, 2, \ldots, N.$$

Then according to the definition of VaR which requires the maximum expected loss during a given period $[t, t+h]$ at a given level of confidence k:

$$P(\Delta_i V(t, h) \le -VaR(t, h, i, k)|\mathcal{G}_t, X(t) = \mathbf{e}_i) = k,$$

we obtain

$$\mathrm{VaR}(t, h, i, k) = -\theta_i(t)(1 - e^{rh}) - \Phi^{-1}(k)\sqrt{\frac{\pi'(t)\Sigma\pi(t)}{2\alpha}(1 - e^{-2\alpha h})}, \tag{6}$$

where $\Phi(x)$ is the cumulative distribution of the standard normal random variable. To simplify the expression, we let

$$\begin{cases} a_1 = -\Phi^{-1}(k)\sqrt{\dfrac{e^{2rh} - 1}{2r}}; \\ a_{2i} = -\dfrac{(\boldsymbol{\mu}^i - \mathbb{1}_{n\times 1} r)'}{r}(e^{rh} - 1); \\ b = \dfrac{e^{rh} - 1}{r}. \end{cases}$$

Then, we have for $i = 1, 2, \ldots, N$.

$$\mathrm{VaR}(t, h, i, k) = a_1\sqrt{\pi'(t)\Sigma\pi(t)} + a'_{2i}\pi(t) + bc(t). \tag{7}$$

As in Yiu et al. (2010), the conditional MVaR of the portfolio with the probability level k over $[t, t+h]$ is given by

$$MVaR(t, h, k) = \max_{i=1,\ldots,N} \mathrm{VaR}(t, h, i, k). \tag{8}$$

The rationale of MVaR is to take account of the worst-case scenario over different states of the economy in evaluating VaR. To control the risks caused by the trading strategies, we ensure that the MVaR is restricted by a given level R, i.e.

$$MVaR(t, h, k) \le R,$$

which is equivalent to

$$a_1\sqrt{\pi'(t)\Sigma\pi(t)} + a'_{2i}\pi(t) + bc(t) \le R, \quad \forall i = 1, 2, \ldots, N.$$

By imposing the MVaR as a constraint, the original portfolio problem presented in Section 2 can be transformed into the following constrained optimization problem with complete observations:

$$\max_{\pi(t), c(t)} E\left[\int_0^T U(t, c(t))dt \,\middle|\, V(0) = v, X(0) = \mathbf{e}_i\right] \tag{9}$$

subject to

$$\begin{cases} dV(t) = [\pi'(t)(\widehat{\boldsymbol{\mu}}(t) - \mathbb{1}_{n\times 1} r(t)) + r(t)V(t) - c(t)]dt \\ \quad + \pi'(t)\boldsymbol{\sigma} d\widehat{W}(t), \\ a_1\sqrt{\pi'(t)\Sigma\pi(t)} + a'_{2i}\pi(t) + bc(t) \le R, \quad \forall i = 1, 2, \ldots, N. \end{cases}$$

4. Filtering and estimation

In this section, we present the robust-filtered estimates for the drift parameters in different states of the economy. We first present the Zakai stochastic differential equation, and then use the gauge transformation technique proposed by Clark (1978) to give the robust unnormalized filter of the hidden state $X(t)$. The filtering results and their derivations presented in this section are standard. These results or some related results can be found in, for example, Elliott (1993), Elliott et al. (1995), Elliott and Siu (2012), Elliott et al. (2010), Korn et al. (2011), Shen and Siu (2015), Siu (2011, 2012, 2013, 2015, 2016) and the relevant references therein. For the sake of completeness, the derivations of the filtering results are placed in Appendix.

The filtering results are derived using the standard reference probability approach. Firstly, a reference probability measure $\bar{P}$ is introduced, under which the observation process $\{Y(t)\}_{t\in\mathcal{T}}$ does not depend on the chain X. Consider the following $\mathbb{G}$-adapted process $\Lambda := \{\Lambda(t)\}_{t\in\mathcal{T}}$ with

$$\Lambda(t) = \exp\left(\int_0^t \langle \boldsymbol{\sigma}^{-1} g(s), \boldsymbol{\sigma}^{-1} dY(s)\rangle - \frac{1}{2}\int_0^t \langle \boldsymbol{\sigma}^{-1} g(s), \boldsymbol{\sigma}^{-1} g(s)\rangle ds\right). \tag{10}$$

A new probability measure $\bar{P}$ is defined by putting

$$\left.\frac{d\bar{P}}{dP}\right|_{\mathcal{G}(t)} := \Lambda^{-1}(t), \quad t \in [0, T]. \tag{11}$$

By Girsanov's Theorem, $\{\boldsymbol{\sigma}^{-1} Y(t), t \in [0, T]\}$ is an n-dimensional standard Brownian motion under the reference probability measure $\bar{P}$.

Consider now the scalar process $\{H_t, t \in [0, T]\}$ of the following form:

$$H_t = H_0 + \int_0^t \alpha_s ds + \int_0^t \beta'_s dM(s) + \int_0^t \delta'_s dW(s)$$

where α, β, δ are $\mathbb{G}$-adapted, square-integrable processes of appropriate dimensions. Write $\bar{E}[\cdot]$ for the expectation taken with respect to the probability measure $\bar{P}$ and $\sigma(H_t)$ for the unnormalized filter of H_t, i.e.,

$$\sigma(H_t) = \bar{E}[\Lambda(t) H_t|\mathcal{F}_t]. \tag{12}$$

The following proposition presents the Zakai stochastic differential equation governing the evolution of the unnormalized filter $\sigma(H_t X(t))$ over time. The proof is given in Appendix.

Proposition 1. *The unnormalized filter $\sigma(H_tX(t))$ is governed by the following stochastic differential equation:*

$$\sigma(H_tX(t)) = \sigma(H_0X(0)) + \int_0^t \sigma(\alpha_sX(s))ds + \int_0^t A\sigma(H_sX(s))ds + \int_0^t \sum_{i,j=1}^N \langle \sigma(\beta_s^jX(s) - \beta_s^iX(s)), \mathbf{e}_i\rangle a_{ji}ds(\mathbf{e}_j - \mathbf{e}_i) + \int_0^t \mathrm{diag}(\mathbf{g}'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}dY(s))\sigma(H_sX(s)) + \sum_{k=1}^n \int_0^t \langle \boldsymbol{\sigma}^{-1}dY(s), e_k\rangle \sigma(\delta_s^kX(s)).$$

From Proposition 1, the recursive equation for $X(t)$ is given by

$$\sigma(X(t)) = \sigma(X(0)) + \int_0^t A\sigma(X(s))ds + \int_0^t \mathrm{diag}(\mathbf{g}'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}dY(s))\sigma(X(s)). \tag{13}$$

In the sequel, we shall estimate $\boldsymbol{\mu}$ and the rate matrix A by developing the filter-based EM algorithm as in, for example, Elliott (1993) and Elliott et al. (1995).

As in Elliott et al. (1995), some quantities, which are useful for deriving the filter-based estimates of unknown parameters, are defined. For each $t \in [0, T]$ and each $i, j = 1, 2, \ldots, N$, let

$$\begin{cases} \mathcal{J}_t^{ij} := \int_0^t \langle X(s_-), \mathbf{e}_i\rangle a_{ji}ds + M_t^{ij}, \\ \mathcal{O}_t^i := \int_0^t \langle X(s), \mathbf{e}_i\rangle ds, \\ \mathcal{T}_t^{ij} := \int_0^t \langle X(s), \mathbf{e}_i\rangle\langle \mathbf{e}_j, dY(s)\rangle \end{cases} \tag{14}$$

where

$$M_t^{ij} = \int_0^t \langle X(s_-), \mathbf{e}_i\rangle\langle \mathbf{e}_j, dM(s)\rangle.$$

Here $\mathcal{J}_t^{ij}$ is the number of jumps from $\mathbf{e}_i$ to $\mathbf{e}_j$ of the chain X in the time interval $[0, t]$, $\mathcal{O}_t^i$ is the amount of time that the chain X stays in $\mathbf{e}_i$ in the time interval $[0, t]$ and $\mathcal{T}_t^{ij}$ is the level integral with respect to the logarithm return process of the jth asset $Y_j(t)$ corresponding to the state $\mathbf{e}_i$ up to time t.

The following lemma gives the recursive equations for the unnormalized filters $\sigma(\mathcal{J}_t^{ij}X(t))$, $\sigma(\mathcal{O}_t^iX(t))$ and $\sigma(\mathcal{T}_t^{ij}X(t))$. The proof is given in Appendix.

Lemma 1. *The quantities are governed by the following stochastic differential equations:*

$$\sigma(\mathcal{J}_t^{ij}X(t)) = \int_0^t \langle \sigma(X(s)), \mathbf{e}_i\rangle a_{ji}\mathbf{e}_jds + \int_0^t A\sigma(\mathcal{J}_s^{ij}X(s))ds + \int_0^t \mathrm{diag}(\mathbf{g}'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}dY(s))\sigma(\mathcal{J}_s^{ij}X(s)); \tag{15}$$

$$\sigma(\mathcal{O}_t^iX(t)) = \int_0^t \langle \sigma(X(s)), \mathbf{e}_i\rangle\mathbf{e}_ids + \int_0^t A\sigma(\mathcal{O}_s^i)X(s)ds + \int \mathrm{diag}(\mathbf{g}'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}dY(s))\sigma(\mathcal{O}_s^iX(s)); \tag{16}$$

$$\sigma(\mathcal{T}_t^{ij}X(t)) = \int_0^t \langle \sigma(X(s)), \mathbf{e}_i\rangle g_{ji}\mathbf{e}_ids + \int_0^t A\sigma(\mathcal{T}_s^{ij}X(s))ds + \int_0^t \mathrm{diag}(\mathbf{g}'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}dY(s))\sigma(\mathcal{T}_s^{ij}X(s)) + \int_0^t dY_j(s)\langle \sigma(X(s)), \mathbf{e}_i\rangle\mathbf{e}_i. \tag{17}$$

Note that the above stochastic differential equations involve stochastic integrals. Then in order to compute the Zakai filter, some numerical approximations to the stochastic integrals need to be introduced. As in Elliott et al. (2010), the gauge transformation technique pioneered by Clark (1978) is employed to transform the stochastic differential equations involving stochastic integrals to linear ordinary differential equations. A transformation matrix is defined before applying the gauge transformation technique. For $i = 1, 2, \ldots, N$, let

$$\phi_i(t) := \exp\left(\mathbf{g}^{i'}(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}Y(t) - \frac{1}{2}\mathbf{g}^{i'}(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}g^it\right)$$

where $\mathbf{g}^i$ represents the ith column of the matrix $\mathbf{g}$. Define the gauge transformation matrix Φ_t as the diagonal matrix $\mathrm{diag}(\phi_1(t), \phi_2(t), \ldots, \phi_N(t))$. Write Φ_t^{-1} for the inverse of Φ_t. Then,

$$d\Phi_t = \Phi_t\mathrm{diag}(\mathbf{g}'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}dY(t)). \tag{18}$$

Denote the transformed quantities as follows:

$$\begin{cases} \bar{\sigma}(X(t)) = \Phi_t^{-1}\sigma(X(t)); \\ \bar{\sigma}(\mathcal{J}_t^{ij}X(t)) = \Phi_t^{-1}\sigma(\mathcal{J}_t^{ij}X(t)); \\ \bar{\sigma}(\mathcal{O}_t^iX(t)) = \Phi_t^{-1}\sigma(\mathcal{O}_t^iX(t)); \\ \bar{\sigma}(\mathcal{T}_t^{ij}X(t)) = \Phi_t^{-1}\sigma(\mathcal{T}_t^{ij}X(t)). \end{cases}$$

The dynamics of these transformed quantities are given in the following proposition. The proof can be found in Appendix.

Proposition 2. *The transformed quantities are:*

$$\begin{cases} d\bar{\sigma}(X(t)) = \Phi_t^{-1}A\Phi_t\bar{\sigma}(X(t))dt; \\ d\bar{\sigma}(\mathcal{J}_t^{ij}X(t)) = \Phi_t^{-1}\langle \Phi_t\bar{\sigma}(X(t)), \mathbf{e}_i\rangle a_{ji}\mathbf{e}_jdt + \Phi_t^{-1}A\Phi_t\bar{\sigma}(\mathcal{J}_t^{ij}X(t))dt; \\ d\bar{\sigma}(\mathcal{O}_t^iX(t)) = \Phi_t^{-1}\langle \Phi_t\bar{\sigma}(X(t)), \mathbf{e}_i\rangle\mathbf{e}_idt + \Phi_t^{-1}A\Phi_t\bar{\sigma}(\mathcal{O}_t^iX(t))dt; \\ d\bar{\sigma}(\mathcal{T}_t^{ij}X(t)) = \Phi_t^{-1}A\Phi_t\bar{\sigma}(\mathcal{T}_t^{ij}X(t))dt + \Phi_t^{-1}\langle \Phi_t\bar{\sigma}(X(s)), \mathbf{e}_i\rangle dY_j(t)\mathbf{e}_i. \end{cases}$$

Denote the model parameters by

$$\theta := \{\mu_{kj}, a_{ij}, k = 1, 2, \ldots, n, i, j = 1, 2, \ldots, N\}.$$

The Filter-based Expectation Maximization (EM) algorithm is employed to estimate the unknown parameters. The key idea of the algorithm is to ensure the observation would be the most probable under the measure P^*, given by another set of approximate parameters

$$\theta^* := \{\mu_{kj}^*, a_{ij}^*, k = 1, 2, \ldots, n, i, j = 1, 2, \ldots, N\}.$$

Interested readers may refer to, for example, Elliott et al. (1995). Let

$$L_t^{ij} = \exp\left[\int_0^t \log\left(\frac{a_{ji}^*}{a_{ji}}\right) d\mathcal{J}_s^{ij} + \int_0^t (a_{ji} - a_{ji}^*)\langle X(s), \mathbf{e}_i\rangle\, ds\right]$$

and

$$L_t^{\mu} = \exp\left[\int_0^t ((\mathbf{g}^* - \mathbf{g})X(s))'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}dY(s) - \frac{1}{2}\int_0^t \left\{(\mathbf{g}^*X(s))' \times (\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}(\mathbf{g}^*X(s)) - (\mathbf{g}X(s))'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}(\mathbf{g}X(s))\right\} ds\right].$$

Define

$$\left.\frac{dP^*}{dP}\right|_{\mathcal{G}_t} = L_t := \prod_{i\neq j} L_t^{ij} L_t^{\mu}. \tag{19}$$

Maximizing the likelihood function $E[\log(L_t)|\mathcal{F}_t]$ over the parameters space yields

$$a_{ji}^* = \frac{\sigma(\mathcal{J}_t^{ij})}{\sigma(\mathcal{O}_t^i)} \quad\text{and}\quad \boldsymbol{\mu}^i = \frac{\sigma(\mathcal{T}_t^i)}{\sigma(\mathcal{O}_t^i)} + \frac{1}{2}\tilde{\sigma}' \tag{20}$$

where $\boldsymbol{\mu}^i$ is the ith column of $\boldsymbol{\mu}$. The unnormalized estimates of $\mathcal{J}_t^{ij}$, $\mathcal{O}_t^i$ and $\mathcal{T}_t^i$ can be determined by taking an inner product with $\mathbb{1}_{N\times 1}$:

$$\begin{cases} \sigma(\mathcal{J}_t^{ij}) = \langle\sigma(\mathcal{J}_t^{ij}X(t)), \mathbb{1}_{N\times1}\rangle; \\ \sigma(\mathcal{O}_t^{i}) = \langle\sigma(\mathcal{O}_t^{i}X(t)), \mathbb{1}_{N\times1}\rangle; \\ \sigma(\mathcal{T}_t^{ij}) = \langle\sigma(\mathcal{T}_t^{ij}X(t)), \mathbb{1}_{N\times1}\rangle; \\ \sigma(\mathcal{T}_t^{i}) = (\sigma(\mathcal{T}_t^{i1}), \ldots, \sigma(\mathcal{T}_t^{in}))'. \end{cases}$$

Since discrete-time observations are used for the estimation, a discretization scheme for the filtering equations is adopted to compute the estimates of the unknown parameters. For the time horizon $[0, T]$, an equidistant time discretization is considered as follows:

$$0 = t_0 < t_1 < \cdots < t_k < \cdots < t_K = T,$$

where $\delta = T/K$ and $t_k = k\delta$. Then

$$\phi_{ik} := \exp\left(\mathbf{g}^{i'}(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}Y(t_k) - \frac{1}{2}\mathbf{g}^{i'}(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}\mathbf{g}^i k\delta\right)$$

and

$$\Phi_k := \mathrm{diag}(\phi_{1k}, \phi_{2k}, \ldots, \phi_{Nk}).$$

It follows that $\Phi_k^{-1} = \mathrm{diag}(\phi_{1k}^{-1}, \phi_{2k}^{-1}, \ldots, \phi_{Nk}^{-1})$. We denote $\Psi_{k,k+1} := \Phi_{k+1}\Phi_k^{-1}$. The discrete-time dynamics for the quantities are stated in Proposition 3, where symbols in the form of $\epsilon(t_k)$ are simplified as ϵ_k. The proof of Proposition 3 can be found in Appendix.

Proposition 3. *The unnormalized filter $\sigma(X(t))$ and the processes $\sigma(\mathcal{J}_k^{ij}X_k)$, $\sigma(\mathcal{O}_k^i)$ and $\sigma(\mathcal{T}_k^{ij})$ are given as follows:*

$$\sigma(X_{k+1}) = \Psi_{k,k+1}(I + \delta A)\sigma(X_k); \tag{21}$$

$$\sigma(\mathcal{J}_{k+1}^{ij}X_{k+1}) = \Psi_{k,k+1}(I + \delta A)\sigma(\mathcal{J}_k^{ij}X_k) + \Psi_{k,k+1}\langle\sigma(X_k), \mathbf{e}_i\rangle a_{ji}\mathbf{e}_j\delta; \tag{22}$$

$$\sigma(\mathcal{O}_{k+1}^{i}X_{k+1}) = \Psi_{k,k+1}(I + \delta A)\sigma(\mathcal{O}_k^{i}X_k) + \Psi_{k,k+1}\langle\sigma(X_k), \mathbf{e}_i\rangle\mathbf{e}_i\delta; \tag{23}$$

$$\sigma(\mathcal{T}_{k+1}^{ij}X_{k+1}) = \Psi_{k,k+1}(I + \delta A)\sigma(\mathcal{T}_k^{ij}X_k) + \big(\delta\Psi_{k,k+1}\langle A\sigma(X_k), \mathbf{e}_i\rangle + \Psi_{k,k+1}\langle\sigma(X_k), \mathbf{e}_i\rangle\big)(\Delta Y_k)_j\mathbf{e}_i. \tag{24}$$

Given the observed returns $\{Y_k\}$, the parameter estimates in Eq. (20) can be computed according to the recursive discrete-time equations in Eqs. (21)–(24).

5. The portfolio allocation problem

In this section, we shall solve the constrained optimization problem in Eq. (9) using the standard HJB dynamic programming approach. Note that the objective function in Eq. (9) is equivalent to

$$\max_{\pi(t),c(t)} E\left[\int_t^T U(s, c(s))ds \,\middle|\, \mathcal{F}_t\right] = \max_{\pi(t),c(t)} E\left[\int_t^T U(s, c(s))ds \,\middle|\, V(t) = v, \mathbf{q}(t) = \mathbf{q}\right]$$

where $\mathbf{q}(t) = \sigma(X(t))$ and the process $(V, \mathbf{q})$ is jointly Markovian with respect to the observed filtration $\mathbb{F}$, (see, for example, Elliott & Siu, 2012). Readers who are interested in solving this kind of stochastic control problem can refer to, for example, Yong and Zhou (1999).

Denote that

$$J(t, v, \mathbf{q}) = \max_{\pi(t),c(t)} E\left[\int_t^T U(s, c(s))ds \,\middle|\, V(t) = v, \mathbf{q}(t) = \mathbf{q}\right].$$

For the partial derivatives, write

$$J_t := \frac{\partial J}{\partial t}, \qquad J_v = \frac{\partial J}{\partial v}, \qquad J_{vv} := \frac{\partial^2 J}{\partial v^2}$$

and

$$D_{\mathbf{q}}J := (J_1, \ldots, J_N)' \in \mathcal{R}^N,$$
$$D_{\mathbf{q}}^2 J := [J_{ij}]_{i,j=1,\ldots,N} \in \mathcal{R}^N \otimes \mathcal{R}^N,$$

where $J_i := \frac{\partial J}{\partial q_i}$ and $J_{ij} := \frac{\partial^2 J}{\partial q_i q_j}$ for each $i, j = 1, 2, \ldots, N$.

$$D_{\mathbf{q}}J_v := \left(\frac{\partial^2 J}{\partial v \partial q_1}, \frac{\partial^2 J}{\partial v \partial q_2}, \ldots, \frac{\partial^2 J}{\partial v \partial q_N}\right) \in \mathcal{R}^N.$$

Assume that the above derivatives exist. This assumption may perhaps be rather strong and may be relaxed using viscosity solutions. Set $\sigma(1) = \langle\sigma(X(t)), 1\rangle$, then standard calculations based on the Bayes' rule give the normalized filter of $X(t)$ as follows:

$$\frac{\sigma(X(t))}{\sigma(1)}.$$

Write π_t for $\pi(t)$ and we denote

$$F(t, v, \mathbf{q}, \pi_t, c(t)) = \left[\pi_t'(\boldsymbol{\mu}\frac{\mathbf{q}}{\sigma(1)} - \mathbb{1}_{n\times1}r) + rv - c(t)\right],$$

$$G(t, v, \mathbf{q}, \pi_t, c(t)) = \pi_t'\Sigma\pi_t, \qquad q_i(t) = \langle\mathbf{q}(t), \mathbf{e}_i\rangle,$$

$$K(t, \mathbf{q}, v) = A\mathbf{q} + \mathrm{diag}\left(\mathbf{g}'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}\mathbf{g}\frac{\mathbf{q}}{\sigma(1)}\right)\mathbf{q},$$

and

$$B(t, v, \mathbf{q}) = \mathrm{diag}(\mathbf{q})(\mathbf{g}'\Sigma^{-1}\mathbf{g})\mathrm{diag}(\mathbf{q}).$$

Then using the standard principle of dynamic programming (see, for example, Fleming & Rishel, 1975), under the smoothness assumption of J, J should satisfy the following HJB equation:

$$J_t + \sup_{\pi_t,c(t)}[U(t, c(t)) + F(t, v, \mathbf{q}, \pi_t, c(t))J_v + \frac{1}{2}J_{vv}G(t, v, \mathbf{q}, \pi_t, c(t)) + \langle D_{\mathbf{q}}J, K(t, \mathbf{q}, v)\rangle + \frac{1}{2}Tr(D_{\mathbf{q}}^2 JB(t, \mathbf{q}, v)) + \langle D_{\mathbf{q}}J_v, \mathrm{diag}(\mathbf{q})\mathbf{g}'\pi_t\rangle] = 0, \tag{25}$$

where $Tr(I)$ denotes the trace of matrix I. The terminal and boundary conditions are given by

$$J(T, v, \mathbf{q}) = 0 \quad\text{and}\quad J(t, 0, \mathbf{q}) = 0.$$

The method of Lagrange multiplier is adopted to handle the MVaR constraint and the Lagrange function is

$$\begin{aligned}\mathcal{L}(\pi_t, c(t), \lambda(v, \mathbf{q}, t)) \\ &= U(t, c(t)) + F(t, v, \mathbf{q}, \pi_t, c(t))J_v + \frac{1}{2}J_{vv}G(t, v, \mathbf{q}, \pi_t, c(t)) \\ &\quad + \langle D_{\mathbf{q}}V, K(t, \mathbf{q}, v)\rangle + \frac{1}{2}Tr(D^2_{\mathbf{q}}VB(t, \mathbf{q}, v)) \\ &\quad + \langle D_{\mathbf{q}}V_v, \mathrm{diag}(\mathbf{q})\mathbf{g}'\pi_t\rangle \\ &\quad - \sum_{i=1}^{N}\lambda_i(v, \mathbf{q}, t)(R - a_1\sqrt{\pi_t'\Sigma\pi_t} - a'_{2i}\pi_t - bc(t)).\end{aligned}$$

Then the first-order necessary conditions of the optimization problem are given by:

$$\begin{aligned}&\left(\boldsymbol{\mu}\frac{\mathbf{q}}{\sigma(1)} - \mathbb{1}_{n\times1}r\right)J_v + \Sigma\pi_t J_{vv} + \mathbf{g}\mathrm{diag}(\mathbf{q})(D_{\mathbf{q}}J_v) \\ &\quad + \sum_{i=1}^{N}\lambda_i(v, \mathbf{q}, t)(a_1\frac{\Sigma\pi_t}{\sqrt{\pi_t'\Sigma\pi_t}} + a_{2i}) = 0,\end{aligned} \tag{26}$$

$$\frac{\partial U}{\partial c} - \frac{\partial J}{\partial v} + \sum_{i=1}^{N}\lambda_i(v, \mathbf{q}, t)b = 0, \tag{27}$$

$$\lambda_i(v, \mathbf{q}, t)(R - a_1\sqrt{\pi_t'\Sigma\pi_t} - a'_{2i}\pi_t - bc(t)) = 0, \tag{28}$$

$$\lambda_i(v, \mathbf{q}, t) \le 0. \tag{29}$$

From Eq. (26), we get

$$\pi_t^{\mathrm{opt}} = \frac{Q_1}{Q_2}, \tag{30}$$

where

$$\begin{cases} Q_1 = -\Sigma^{-1}(\boldsymbol{\mu}\frac{\mathbf{q}}{\sigma(1)} - \mathbb{1}_{n\times1}r)J_v \\ \qquad - \Sigma^{-1}\mathbf{g}\mathrm{diag}(\mathbf{q})(D_{\mathbf{q}}J_v) - \sum_{i=1}^{N}\lambda_i(v, \mathbf{q}, t)\Sigma^{-1}a_{2i}, \\ Q_2 = J_{vv} + \sum_{i=1}^{N}\lambda_i^{\mathrm{opt}}(v, \mathbf{q}, t)a_1/\sqrt{\pi_t^{\mathrm{opt}'}\Sigma\pi_t^{\mathrm{opt}}}. \end{cases}$$

In addition, $c^{\mathrm{opt}}(v, t)$ and $\lambda_i^{\mathrm{opt}}(v, \mathbf{q}, t)$ can be calculated from Eqs. (27) and (28) whenever $\lambda_i^{\mathrm{opt}}(v, \mathbf{q}, t) \ne 0$. Substituting π_t^{opt}, $c^{\mathrm{opt}}(v, t)$ and $\lambda_i^{\mathrm{opt}}(v, \mathbf{q}, t)$ into Eq. (25), we obtain:

$$\begin{aligned}&J_t + U(t, c^{\mathrm{opt}}(t)) + F(t, v, \mathbf{q}, \pi_t^{\mathrm{opt}}, c^{\mathrm{opt}}(t))J_v \\ &\quad + \frac{1}{2}J_{vv}G(t, v, \mathbf{q}, \pi_t^{\mathrm{opt}}, c^{\mathrm{opt}}(t)) + \langle D_{\mathbf{q}}J, K(t, \mathbf{q}, v)\rangle \\ &\quad + \frac{1}{2}Tr(D^2_{\mathbf{q}}JB(t, \mathbf{q}, v)) + \langle D_{\mathbf{q}}J_v, \mathrm{diag}(\mathbf{q})\mathbf{g}'\pi_t^{\mathrm{opt}}\rangle = 0.\end{aligned} \tag{31}$$

The optimal wealth function $J^{\mathrm{opt}}(v, \mathbf{q}, t)$ can be obtained by solving the above equation. Due to the non-linearity in the quantities, the first-order necessary conditions and the partial differential equations will be solved by numerical methods.

6. Numerical methods and computational results

In this section, we shall conduct numerical experiments to provide some analysis for the optimal portfolio. First, the HJB equation is solved numerically by employing an iterative algorithm. Then some comparisons are made between the qualitative behaviors of the optimal portfolio, the optimal consumption rate with and without the MVaR constraint.

6.1. The iterative algorithm

In this subsection, an iterative algorithm is presented under some assumptions. The utility function is defined to be

$$U(t, c(t)) = e^{-\eta t}c(t)^{\gamma}, \quad \eta > 0,\ 0 < \gamma < 1. \tag{32}$$

This is a power utility function discounted by a factor $e^{\eta t}$. Then following the approach in Merton (1971), the value function is assumed to take the following parametric form:

$$J(t, v, \mathbf{q}) = e^{-\eta t}h(t, v)v^{\gamma}. \tag{33}$$

As in Yiu (2004), the derivatives of $h(t, v)$ with respect to v are ignored and the following approximation results are obtained

$$\begin{cases} \frac{\partial J}{\partial v} = e^{-\eta t}h(t, v)\gamma v^{\gamma-1}; \\ \frac{\partial^2 J}{\partial v^2} = e^{-\eta t}h(t, v)\gamma(\gamma-1)v^{\gamma-2}; \\ \frac{\partial J}{\partial t} = e^{-\eta t}\frac{\partial h(t, v)}{\partial t}v^{\gamma} - \eta e^{-\eta t}h(t, v)v^{\gamma}; \\ D_{\mathbf{q}}J = D^2_{\mathbf{q}}J = D_{\mathbf{q}}J_v = \mathbf{0}. \end{cases} \tag{34}$$

Substituting Eqs. (32)–(34) into Eq. (31), then dividing by $e^{-\eta t}v^{\gamma}$ and rearranging the terms, we obtain

$$\frac{\partial h(t, v)}{\partial t} + M(\pi_{\mathrm{opt}}, v, \mathbf{q})h(t, v) + N(c_{\mathrm{opt}}, h(t, v)) = 0, \tag{35}$$

where

$$\begin{cases} M(\pi_{\mathrm{opt}}, v, \mathbf{q}) = \gamma\left[\frac{1}{v}\pi'_{\mathrm{opt}}(\boldsymbol{\mu}\frac{\mathbf{q}}{\sigma(1)} - \mathbb{1}_{n\times1}r) + r\right] \\ \qquad + \frac{1}{2}\frac{\pi'_{\mathrm{opt}}\Sigma\pi_{\mathrm{opt}}}{v^2}\gamma(\gamma-1) - \eta, \\ N(c_{\mathrm{opt}}(t, v), h(t, v)) = \frac{c_{\mathrm{opt}}^{\gamma}}{v^{\gamma}} - \frac{\gamma h(t, v)c_{\mathrm{opt}}}{v}. \end{cases}$$

Denote

$$\Phi = -\frac{\gamma}{1-\gamma} \quad \text{and} \quad g(t, v) = h(t, v)^{1-\Phi},$$

then Eq. (35) becomes

$$\begin{aligned}&\frac{\partial g(t, v)}{\partial t} + (1-\Phi)M(\pi_{\mathrm{opt}}, v, \mathbf{q})g(t, v) \\ &\quad + (1-\Phi)N(c_{\mathrm{opt}}(t, v), g^{1-\gamma}(t, v))g^{\gamma}(t, v) = 0.\end{aligned} \tag{36}$$

The above HJB equation can be solved by employing an iterative algorithm where the computational domain is divided into a grid of $N_t \times N_v$ mesh points. The steps of the iterative algorithm are summarized as follows:

(Step 1) $\lambda^{(0)}_{opt,i} = 0$;

$\pi^{(0)}_{\mathrm{opt}} = -\frac{\Sigma^{-1}(\boldsymbol{\mu}\frac{\mathbf{q}}{\sigma(1)} - \mathbb{1}_{n\times1}r)v}{\gamma-1}$;

$c^{(0)}_{\mathrm{opt}} = vh^{-\frac{1}{1-\gamma}}(0, v)$ and solve Eq. (36). Set $k = 0$.

(Step 2) For each $n = N_t - 1, \ldots, 0$, $v = [0, \Delta v, \ldots, N_v\Delta v]$ and $t = [(N_t-1)\Delta t, \ldots, \Delta t, 0]$,

$$\lambda^{(k+1)}_{opt,i}(R - a_1\sqrt{\pi^{(k)'}_{\mathrm{opt}}\Sigma\pi^{(k)}_{\mathrm{opt}}} - a'_{2i}\pi^{(k)}_{\mathrm{opt}} - bc_{\mathrm{opt}}) = 0;$$

$$\pi^{(k+1)}_{\mathrm{opt}} = \frac{-\Sigma^{-1}\left(\boldsymbol{\mu}\frac{\mathbf{q}}{\sigma(1)} - \mathbb{1}_{n\times1}r\right)J^{(k)}_v - \sum_{i=1}^{N}\lambda^{(k+1)}_{opt,i}\Sigma^{-1}a_{2i}}{J^{(k)}_{vv} + \sum_{i=1}^{N}\lambda^{(k+1)}_{opt,i}a_1/\sqrt{\pi^{(k)'}_{\mathrm{opt}}\Sigma\pi^{(k)}_{\mathrm{opt}}}};$$

and calculate $c_{opt}^{(k+1)}$ from

$$\gamma(c_{\text{opt}}^{(k+1)})^{\gamma-1} = \gamma \upsilon^{\gamma-1} h^{(k)} - e^{\eta t}\left(\sum_{i=1}^{N} \lambda_{opt,i}^{(k+1)} b\right).$$

(Step 3) Then solve

$$g_n^{(k+1)} = g_{n+1}^{(k+1)} + \Delta t(1-\Phi) M(\pi_{\text{opt}}^{(k+1)}, \upsilon, \mathbf{q}) g_{n+1}^{(k+1)} + \Delta t(1-\Phi) N(c_{\text{opt}}^{(k+1)}, (g_n^{(k)})^{1-\gamma})(g_n^{(k)})^{\gamma}.$$

(Step 4) Return to Step 2 with $k = k+1$ until

$$\|J^{(k)} - J^{(k-1)}\| < \varepsilon$$

for some desirable level of accuracy $\varepsilon > 0$.

In the above algorithm, $\mathbf{q}$ is the value of $\mathbf{q}(t)$ which is given by $\sigma(X(t))$. The values of $\sigma(X(t))$, $t \in [0, \Delta t, \ldots, (N_t - 1)\Delta t]$, are calculated using the recursive formula in Eq. (21), which needs to approximately estimate the return at each iteration according to Eq. (4).

6.2. Numerical results

This section considers the optimal portfolio allocation with two risky assets plus one riskless asset in a two-state hidden Markovian regime-switching economy. We collected the weekly data of close prices of the Hang Seng Index from January 2000 to January 2012, which were downloaded from https://hk.finance.yahoo.com/. MATLAB is used for computing the results. Set $T = 12$ years, $\Delta t = \frac{1}{50} \approx 7$ days, $X(0) = \mathbf{e}_1$, $\sigma_{11} = 0.25$, $\sigma_{12} = 0.1$, $\sigma_{21} = 0.1$, and $\sigma_{22} = 0.25$. Then, using the observed asset returns $\{Y(t_k)\}$, the estimates of $\boldsymbol{\mu}$ and A in Eq. (20) can be computed using the recursive discrete-time equations in Eqs. (21)–(24). Fig. 1(a) and (b) show the values of $\boldsymbol{\mu}$ and A at each iteration, respectively. From these two figures, we can see that convergence is achieved when the number of iterations is approximately equal to 15. Furthermore, from these two figures, the estimates of the unknown parameters are given as follows:

$$\boldsymbol{\mu} = \begin{pmatrix} 0.085 & 0.093 \\ 0.071 & 0.062 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -16.3 & 23.1 \\ 16.3 & -23.1 \end{pmatrix}. \tag{37}$$

The estimated parameters in $\boldsymbol{\mu}$ reveal that the value of the drift in Asset 1 (2) increases (decreases) by 9.4% (12.7%) when the state of the economy changes from e_1 to e_2. Using the estimates of model parameters, the iterative algorithm presented in Section 6.1 is used to calculate the optimal portfolio strategy. We choose the terminal year as 4 and $\Delta t = \frac{1}{50} \approx 7$ days. The portfolio value υ is assumed to take values in the range [10, 1000]. We set $\Delta\upsilon = 10$, then $N_\upsilon = 100$. The initial state is assumed to be $X(0) = \mathbf{e}_1$. The parameters in the utility function are chosen as $\eta = 0.2$ and $\gamma = 0.33$. For the MVaR constraint, the maximum loss is limited to be $R = 80$ with a probability level $k = 0.01$. Suppose the volatility $\boldsymbol{\sigma}$ is a constant matrix.

The unconstrained solutions plotted in Fig. 2(a) and (b) reveal that the optimal investment strategy in the two risky assets is approximately $\pi = (1.050\upsilon, -0.362\upsilon)$, where υ is the total portfolio value of an investor. Here the investor is able to maximize his expected utility of consumption under a good control of risk even if the underlying economic states are unknown. The constrained solutions in Fig. 2(a) and (b) show that at each time t, the MVaR constraint becomes active when the portfolio value researches a certain level. Otherwise, if the portfolio value is below the critical value, the optimal strategies should coincide with the unconstrained case. Fig. 2 shows that the investor will increase his/her stock holdings in Asset 1 and decrease his/her stock holdings in Asset 2 when his/her portfolio value goes beyond some values, which indicates that the investor reduces share holdings to fulfill the MVaR constraint. From Fig. 2(a), we find that when time goes closer to the terminal time T, (say from $t = 0.4$ to $t = 3.6$), the corresponding portfolio value of the point where the constraint becomes active would be much smaller. Similar results were found in Yiu (2004).

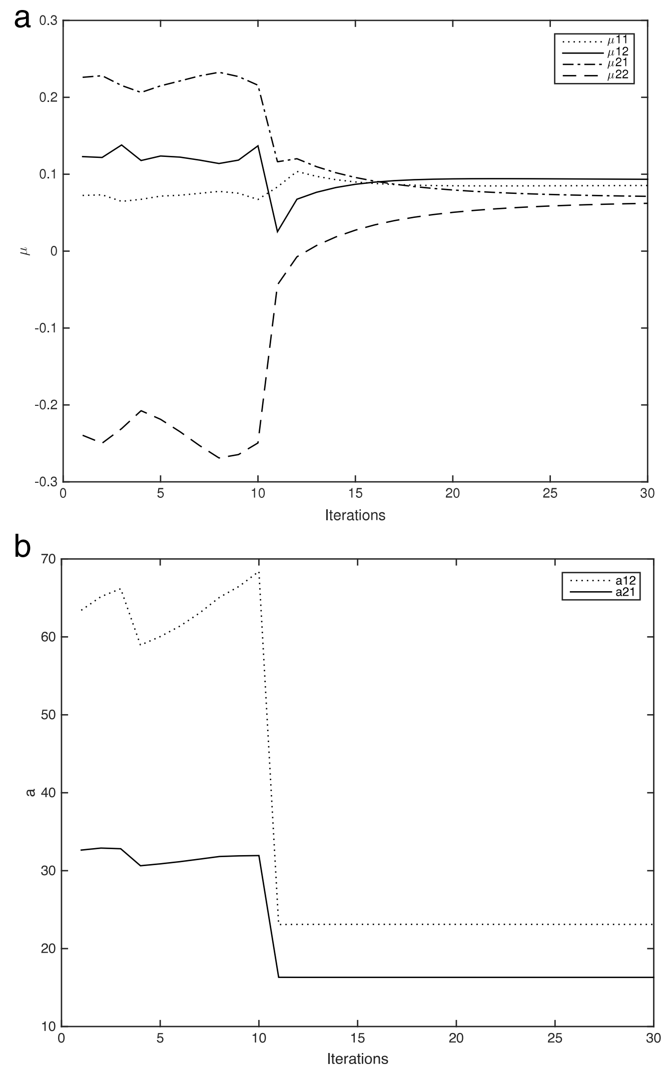

Fig. 1. (a) The estimate of $\boldsymbol{\mu}$ at each iteration, (b) the estimate of the rate matrix A at each iteration.

Fig. 3 shows the changes of the Lagrange multipliers λ_1 and λ_2 with respect to the value of portfolio at three different times. In Fig. 3(b), the values of λ_2 are zeros and keep unchanged, which may indicate that the VaR constraint in state $\mathbf{e}_2$ is inactive in this example. In Fig. 3(a), the VaR constraint in state $\mathbf{e}_1$ becomes active when λ_1 moves into negative values. From Fig. 3(a), we find that the portfolio value gets smaller as the time goes closer to the terminal time (say from $t = 0.4$ to $t = 3.6$), which is consistent with the results depicted in Fig. 2. The MVaR values are plotted in Fig. 4. In order to meet the requirements of risk control, the value of the MVaR is reduced to 80 when it exceeds 80 with the constraint.

The plot of the consumption rates at the two different times (say $t = 0.4$ and $t = 3.6$) against the portfolio values is depicted in Fig. 5. In this example, the curves of consumption rate under both unconstrained and constrained optimization look similar when $t = 0.4$. While in the case when $t = 3.6$, the curves of the consumption rate under the constrained case would be slightly affected when the portfolio value is large enough.

Fig. 6 describes the optimal value function $J(0, \upsilon, \mathbf{e_1})$ along the portfolio value with and without the MVaR constraint. It shows

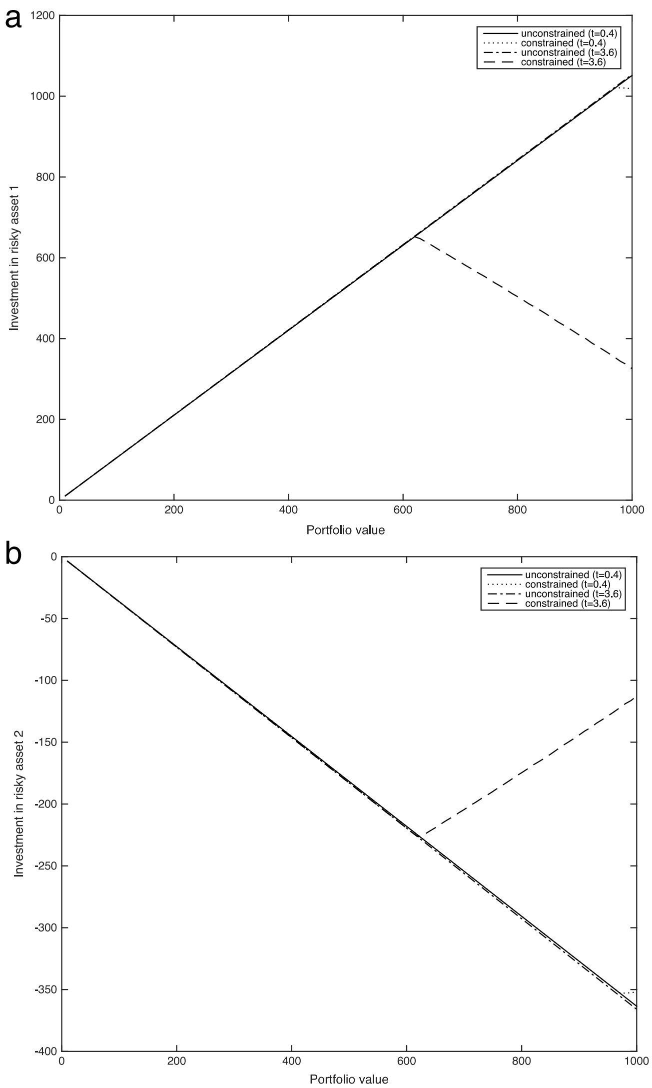

Fig. 2. (a) Optimal investment in Asset 1 with and without the MVaR constraint, (b) optimal investment in Asset 2 with and without the MVaR constraint.

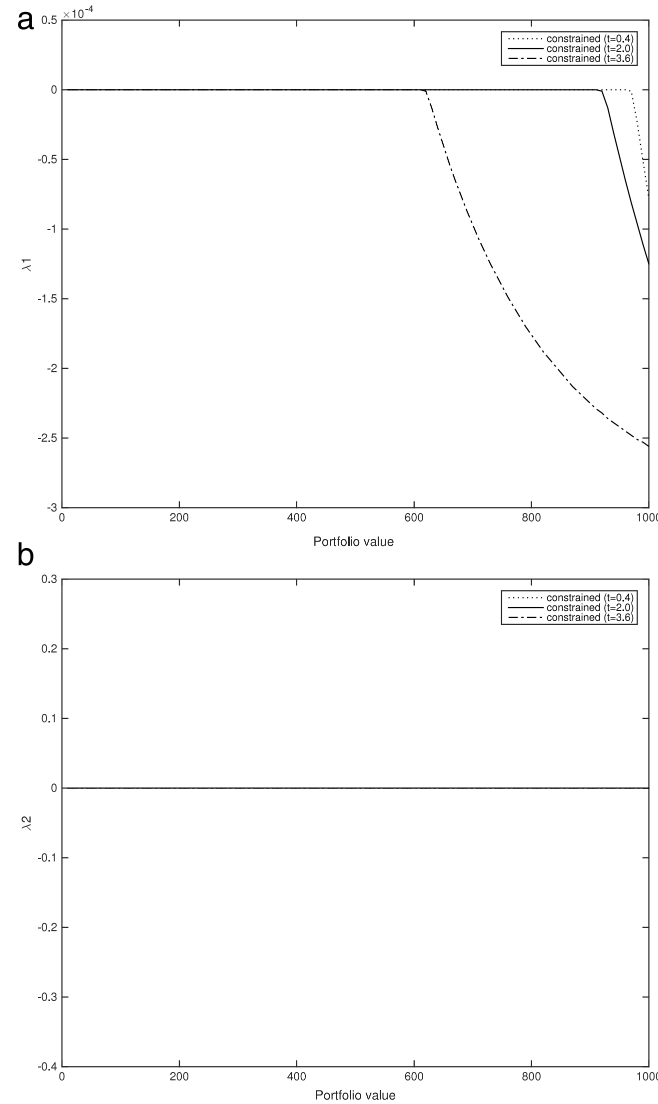

Fig. 3. (a) Lagrange multiplier λ_1 corresponding to $X(t) = \mathbf{e}_1$, (b) Lagrange multiplier λ_2 corresponding to $X(t) = \mathbf{e}_2$.

that when υ increases, a very small decrease occurs when the MVaR constraint is present.

Fig. 7(b) and (a) show the function $h(t, \upsilon)$ over time with and without the MVaR constraint, respectively. In the case without constraint, Fig. 7(a) shows that there is no difference between the curves of $h(t, 100)$ and $h(t, 1000)$ against time. Relative to Fig. 7(a), (b) reveals that $h(t, \upsilon)$ is affected by υ when the MVaR constraint is present, since a little variation is observed between the two curves in Fig. 7(b). When υ is larger, the MVaR constraint is more likely to be active. Therefore, the MVaR constraint may be active when $\upsilon = 1000$ while the MVaR constraint is mostly inactive when $\upsilon = 100$. However, from the figure, only a very little variation of the function $h(t, \upsilon)$ in υ along the time is observed. This may lend some supports to ignore the derivatives of $h(t, \upsilon)$ with respect to υ, which seems to be not affecting the final results very much in this example.

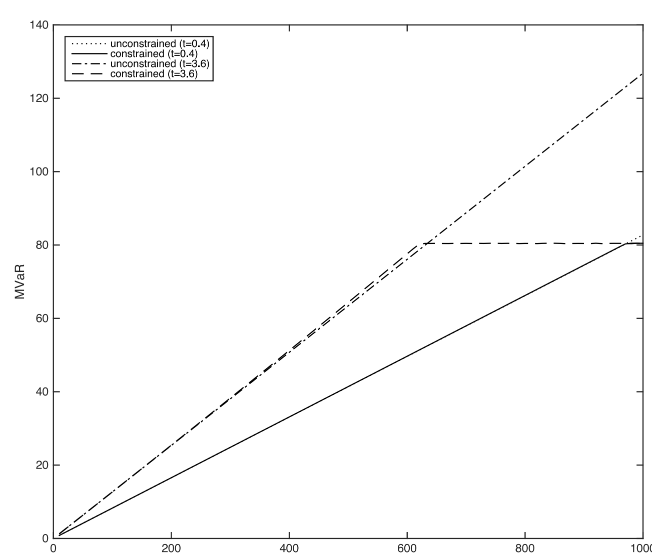

Fig. 4. The MVaR for different portfolio values.

7. Conclusions

This paper studies optimal portfolio problems under a hidden Markov-modulated multi-dimensional diffusion model with a Maximum Value-at-Risk (MVaR) as a constraint. The optimal consumption–investment strategies are determined by maximizing the total expected utility of consumption under the MVaR

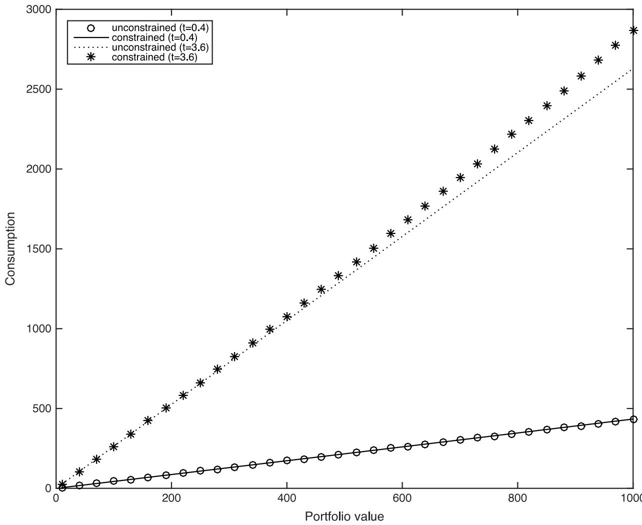

Fig. 5. The consumption rates for different portfolio values.

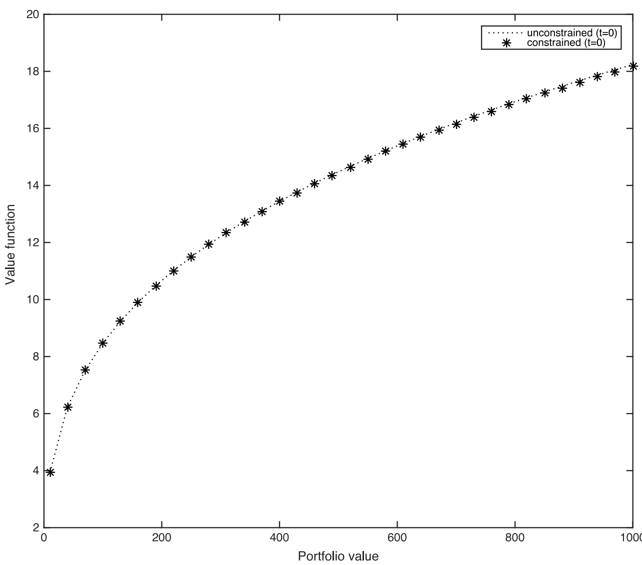

Fig. 6. The optimal total expected utility for different portfolio values.

constraint. The drifts in the price dynamics depend on the state of the economy. The latter is unobservable to the agent and modeled by a hidden Markov chain. The separation principle and the dynamic programming technique have been employed to convert the original problem to solving a HJB-equation where the MVaR constraint is handled by the Lagrange multiplier method. The gauge transformation technique is applied to estimate the unknown parameters. Finally, an iterative numerical method is used to solve the constrained optimal portfolios numerically. From the numerical results, we find that the agent would reduce his/her share holdings of the risky assets. This may provide a way to reduce one's risk exposure so as to fulfill the MVaR constraint. This paper also provides a flexible model for optimal portfolio problems by assuming the underlying economic states are unknown to agents.

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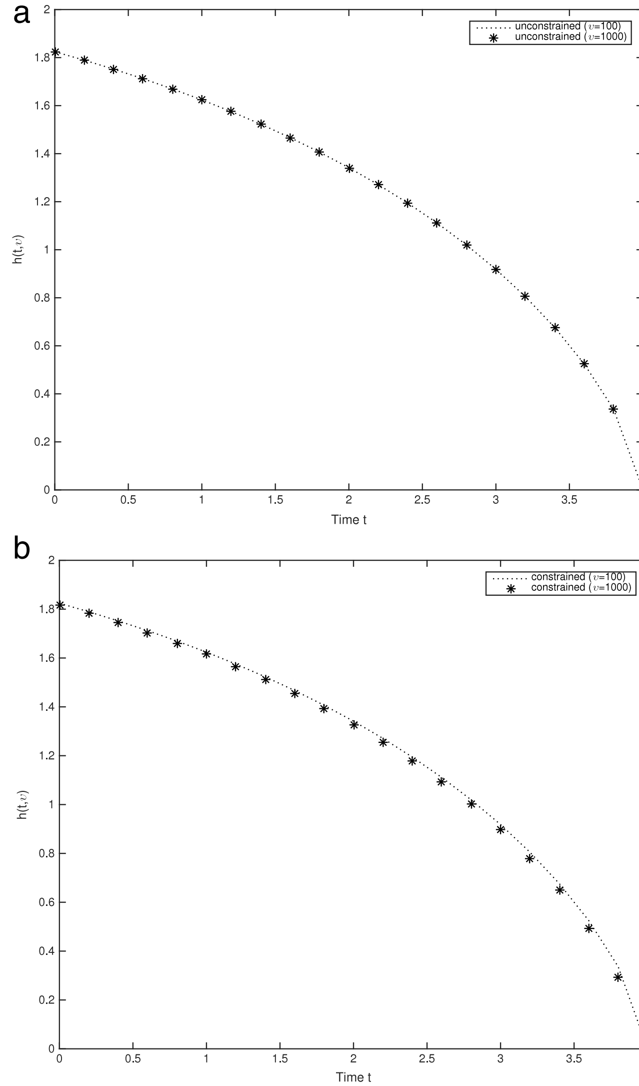

Fig. 7. (a) The values of $h(t, \upsilon)$ at two different portfolio values of $\upsilon = 100$ and $\upsilon = 1000$ without the MVaR constraint. (b) The values of $h(t, \upsilon)$ at two different portfolio values of $\upsilon = 100$ and $\upsilon = 1000$ with the MVaR constraint.

Appendix

The proofs of the results presented in Appendix follow those in the literature, (see, for example, Elliott, 1993 and Elliott et al., 1995).

Proof of Proposition 1. Given that

$$H_t = H_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s' dM_s + \int_0^t \delta_s' dW_s$$

and

$$X_t = X_0 + \int_0^t AX_s ds + M_t,$$

then

$$\begin{aligned} H_t X_t &= H_0 X_0 + \int_0^t \alpha_s X_s ds + \int_0^t X_s \beta_s' dM_s \\ &\quad + \int_0^t X_s \delta_s' dW_s + \int_0^t H_s AX_s ds + \int_0^t H_s dM_s \\ &\quad + \sum_{0<s\le t} \beta_s' \Delta X_s \Delta X_s. \end{aligned}$$

Here

$$\sum_{0<s\le t}\beta_s'\Delta X_s\Delta X_s = \sum_{i,j=1}^N\int_0^t(\beta'^j_s-\beta'^i_s)\langle X_{r-},e_i\rangle\langle e_j,dM_s\rangle(e_j-e_i) + \sum_{i,j=1}^N\int_0^t\langle\beta'^j_sX_s-\beta'^i_sX_s,e_i\rangle a_{ji}ds(e_j-e_i).$$

Set

$$\Lambda_t=\exp\left(\int_0^t\langle\boldsymbol{\sigma}^{-1}g_s,\boldsymbol{\sigma}^{-1}dY_s\rangle-\frac{1}{2}\int_0^t\langle\boldsymbol{\sigma}^{-1}g_s,\boldsymbol{\sigma}^{-1}g_s\rangle ds\right),$$

which implies that

$$d\Lambda_t=\Lambda_t\langle\boldsymbol{\sigma}^{-1}g_t,\boldsymbol{\sigma}^{-1}dY_t\rangle.$$

Then it gives

$$\begin{aligned}\Lambda_tH_tX_t &= H_0X_0+\int_0^t\Lambda_s\alpha_sX_sds+\int_0^t\Lambda_{s-}X_{s-}\beta_s'dM_s\\ &+\int_0^t\Lambda_{s-}X_{s-}\delta_s'\boldsymbol{\sigma}^{-1}dY_s\\ &+\int_0^t\Lambda_sH_sAX_sds+\int_0^t\Lambda_{s-}H_dM_s\\ &+\sum_{i,j=1}^N\int_0^t(\beta'^j_s-\beta'^i_s)\langle\Lambda_{s-}X_{s-},e_i\rangle\langle e_j,dM_s\rangle(e_j-e_i)\\ &+\sum_{i,j=1}^N\int_0^t\langle\Lambda_s\beta'^j_sX_s-\Lambda_s\beta'^i_sX_s,e_i\rangle a_{ji}ds(e_j-e_i)\\ &+\int_0^t\Lambda_s\langle\boldsymbol{\sigma}^{-1}g_s,\boldsymbol{\sigma}^{-1}dY_s\rangle H_sX_s.\end{aligned}$$

Denote $\sigma(H_tX_t)=\bar{E}(\Lambda_tH_tX_t|\mathcal{F}_t)$. Under $\bar{P}$, $\boldsymbol{\sigma}^{-1}Y$ is an n-dimensional standard Brownian motion, thus

$$\begin{aligned}\sigma(H_tX_t) &= \sigma(H_0X_0)+\int_0^t\sigma(\alpha_sX_s)ds+\int_0^tA\sigma(H_sX_s)ds\\ &+\int_0^t\sum_{i,j=1}^N\langle\sigma(\beta'^j_sX_s-\beta'^i_sX_s),e_i\rangle a_{ji}ds(e_j-e_i)\\ &+\int_0^t\sigma(X_s\delta_s'\boldsymbol{\sigma}^{-1}dY_s)\\ &+\int_0^t\sigma(\langle\boldsymbol{\sigma}^{-1}g_s,\boldsymbol{\sigma}^{-1}dY_s\rangle H_sX_s).\end{aligned}$$

Since

$$\sigma(X_s\delta_s'\boldsymbol{\sigma}^{-1}dY_s)=\sum_{k=1}^n(\boldsymbol{\sigma}^{-1}dY_s)_k\sigma(\delta_s^kX_s)$$

and

$$\sigma(\langle\boldsymbol{\sigma}^{-1}g_s,\boldsymbol{\sigma}^{-1}dY_s\rangle H_sX_s)=\mathrm{diag}(\mathbf{g}'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}dY_s)\sigma(H_sX_s),$$

the result follows. □

Proof of Lemma 1. Take $H_t=\mathcal{J}_t^{ij}$, which implies that $H_0=0$, $\alpha_s=\langle X_s,\mathbf{e}_i\rangle a_{ji}$, $\beta_s=\langle X_s,\mathbf{e}_i\rangle e_j$, $\delta_s=0\in\mathcal{R}^n$. Then the Zakai equation for $\sigma(\mathcal{J}_t^{ij}X_t)$ can be obtained by substituting in Proposition 1.

Take $H_t=\mathcal{O}_t^i$, which implies that $H_0=0$, $\alpha_s=\langle X_s,\mathbf{e}_i\rangle a_{ji}$, $\beta_s=0\in\mathcal{R}^N$, $\delta_s=0\in\mathcal{R}^n$. Then the Zakai equation for $\sigma(\mathcal{O}_t^iX_t)$ can be obtained by substituting in Proposition 1.

Take $H_t=\mathcal{T}_t^{ij}$, which implies that $H_0=0$, $\alpha_s=\langle X_s,\mathbf{e}_i\rangle\mathbf{g}_{ji}$, $\beta_s=0\in\mathcal{R}^N$, $\delta_s=\langle X_s,\mathbf{e}_i\rangle(\boldsymbol{\sigma}_{j-row})'$. Then the Zakai equation for $\sigma(\mathcal{T}_t^{ij}X_t)$ can be obtained by substituting in Proposition 1. □

Proof of Proposition 2. For $i=1,2,\ldots,N$, define

$$\phi_i(t):=\exp\left(\mathbf{g}^{i'}(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}Y(t)-\frac{1}{2}\mathbf{g}^{i'}(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}\mathbf{g}^it\right)$$

where $\mathbf{g}^i$ represents the ith column of the matrix $\mathbf{g}$. Note that

$$\begin{aligned}d\Phi_t &= \mathrm{diag}(d\phi_1(t),d\phi_2(t),\ldots,d\phi_N(t))\\ &= \Phi_t\mathrm{diag}(\mathbf{g}'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}dY(t))\end{aligned}$$

and

$$\begin{aligned}d\Phi_t^{-1} &= -\Phi_t^{-2}d\Phi_t+\Phi_t^{-3}d\Phi_t^2\\ &= -\Phi_t^{-1}\mathrm{diag}(\mathbf{g}'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}dY(t))\\ &\quad+\Phi_t^{-1}\mathrm{diag}((\mathbf{g}'\boldsymbol{\sigma}^{-\mathbf{1}'}\boldsymbol{\sigma}^{-1}\mathbf{g})_{ii})dt.\end{aligned}$$

The transformed quantity is defined by

$$\bar{\sigma}(X(t))=\Phi_t^{-1}\sigma(X(t))$$

which satisfies

$$d\bar{\sigma}(X(t))=d\Phi_t^{-1}\sigma(X(t))+\Phi_t^{-1}d\sigma(X(t))+d\Phi_t^{-1}d\sigma(X(t)).$$

The equation for $d\bar{\sigma}(X(t))$ can be obtained by substituting Φ_t^{-1}, $\sigma(X(t))$, $d\Phi_t^{-1}$ and $d\sigma(X(t))$ into the above function. Equations for $d\bar{\sigma}(\mathcal{J}_t^{ij}X(t))$, $d\bar{\sigma}(\mathcal{O}_t^iX(t))$ and $d\bar{\sigma}(\mathcal{T}_t^{ij}X(t))$ are obtained by applying similar method. □

Proof of Proposition 3. For the time interval $[0,T]$, an equidistant time discretization is considered:

$$0=t_0<t_1<\cdots<t_k<\cdots<t_K=T,$$

where $\delta=T/K$ and $t_k=k\delta$. Integrating both sides of the equation for $d\bar{\sigma}(X(t))$ in Proposition 2 from t_k to t_{k+1},

$$\int_{t_k}^{t_{k+1}}d\bar{\sigma}(X(t))=\int_{t_k}^{t_{k+1}}\Phi_t^{-1}A\Phi_t\bar{\sigma}(X(t))dt.$$

The above equation can be transformed to

$$\bar{\sigma}(X_{k+1})-\bar{\sigma}(X_k)=\Phi_k^{-1}A\Phi_k\bar{\sigma}(X_k)\delta$$

where Φ_t^{-1}, Φ_t and $\bar{\sigma}(X(t))$ are assumed to be unchanged during the time interval δ when δ is small enough. By substituting $\bar{\sigma}(X_k)=\Phi_k^{-1}\sigma(X_k)$, then

$$\sigma(X_{k+1})=\Psi_{k,k+1}(I+\delta A)\sigma(X_k).$$

Here $\Psi_{k,k+1}=\Phi_{k+1}\Phi_k^{-1}$. The equations for $\sigma(\mathcal{J}_{k+1}^{ij}X_{k+1})$ and $\sigma(\mathcal{O}_{k+1}^iX_{k+1})$ are obtained by using the similar method.

To derive the equation for $\sigma(\mathcal{T}_t^{ij}X(t))$, according to Proposition 2,

$$\int_{t_k}^{t_{k+1}}d\bar{\sigma}(\mathcal{T}_t^{ij}X(t)) = \int_{t_k}^{t_{k+1}}\Phi_t^{-1}A\sigma(\mathcal{T}_t^{ij}X(t))dt + \int_{t_k}^{t_{k+1}}\Phi_t^{-1}\langle\sigma(X(t)),\mathbf{e}_i\rangle(dY(t))_j\mathbf{e}_i.$$

Here using the function that

$$\begin{aligned}&\int_{t_k}^{t_{k+1}}\Phi_t^{-1}\langle\sigma(X(s)),\mathbf{e}_i\rangle(dY(t))_j\mathbf{e}_i\\ &=\langle\Phi_t^{-1}\sigma(X(s)),\mathbf{e}_i\rangle(Y(t))_j\mathbf{e}_i|_{t_k}^{t_{k+1}}\\ &\quad-\int_{t_k}^{t_{k+1}}(Y(t))_j\langle\Phi_t^{-1}A\sigma(X(t))dt,\mathbf{e}_i\rangle\mathbf{e}_i\\ &=\left(\delta\Phi_t^{-1}\langle A\sigma(X_k),\mathbf{e}_i\rangle+\Phi_t^{-1}\langle\sigma(X_k),\mathbf{e}_i\rangle\right)(\Delta Y_k)_j\mathbf{e}_i,\end{aligned}$$

then the equation for $\sigma(\mathcal{T}_t^{ij}X(t))$ can be obtained by using similar method in deriving $\sigma(X(t))$. □

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