How Big Is a Point?

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Euclid's *Elements* has no preface or introduction, no statement of objectives; it offers no motivation or commentary. It opens abruptly with a list of twenty-three "Definitions" at the beginning of Book I. Of these, the first is [2, p. 153].

Definition 1. A **point** is that which has no part.

In Greek mathematics the "parts" of a figure are what we would call its "dimensions"; so what Euclid is saying is that a point has neither length, nor width, nor thickness.

At this juncture your common sense might object.

"How can that be?" I imagine someone saying. "I can understand how a point's diameter—I'm thinking of a point as a tiny ball—might be so small that we can ignore it in practice, the way a chemist would ignore the diameter of an atom. But if a point were truly to have no size at all, how could even an infinite number of them make up a line segment one meter long? No matter how many zeroes you add up, I can't see how the total could be anything but zero."

I can't see how the total could be anything but zero, either, but it doesn't bother me as much as it probably does you.

"I should think it would bother you more. You're the mathematician, and it's a mathematical argument."

Not really, though at first it might *seem* to be a mathematical argument. It's really more intuitive than logical.

"How can you call it intuitive? Look. We have a line segment one meter long—"

Yes.

"—and this segment is made up of points, laid end to end—"

Be careful. If points have no size, how could they have 'ends'? And what would it mean, then, for points to be 'laid end to end'? Do you see what I'm getting at?

"Sort of . . . but wouldn't that mean the notion of points without size has gotten us into hot water sooner than I had thought?"

Intuitive hot water. We can't conjure up a detailed image of how, exactly, points make up a line. But not *logical* hot water, at least not obviously. We have run up against a failure of our power to imagine, which makes the discussion a little strange; but as yet there is no clear-cut contradiction.

"... we have a line segment one meter long. It is *somehow* made up of points—we won't worry how—"

Good. That's *exactly* the attitude a mathematician would—

"But! You're saying each of those points has a length *zero*—"

That's part of Euclid's Definition 1, yes.

"But if each of these points contributes *zero* to the one-meter length, then the entire segment must have length zero as well! *There's* your contradiction."

From the fact that each point has length zero, why does it follow that the entire segment has length zero?

[Frustrated silence.]

It must seem like I'm quibbling, but honestly I'm not. This issue is very subtle, very difficult to disentangle. I'd like to persuade you that the position Euclid takes in Definition 1 is the *only* one logically open to him. I need your help, though. Tell me: why, in your mind, from the fact that each point has length zero, does it follow that the entire segment has length zero? Say it as carefully as you can.

"... because the segment is made up of points *exclusively*. All of its qualities must derive from those of the points. Its length, in particular, must come from the lengths of the points.

"I still want to say the length of the segment is simply the *sum* of the lengths of the points, because I feel you *were* quibbling when you objected to my description of the points as 'laid end to end.' They are, in *some* fashion, 'lined up' to form the segment. Its length must be the sum of theirs and is, therefore, zero.

"But even if your objection has substance, the fact remains that the length of the segment is *somehow* produced by the lengths of its points, and I can see no way of mathematically combining a lot of zeroes to get anything other than another zero."

OK. Good.

Let me begin by saying that my own common sense objects to Euclid's Definition 1 just as strenuously as yours does, and has done so continually since I first studied geometry in high school. But I have learned to ignore it.

That may seem odd—how could one possibly ignore one's own common sense? Einstein once said

> ... common sense is, as a matter of fact, nothing more than layers of preconceived notions stored in our memories and emotions for the most part before age eighteen.

My own view is not so harsh, but I do think of common sense as consisting of *less* than a person's entire intellectual apparatus, and so limited. Without attempting to define 'common sense' precisely, let me say that, when I use the term in a mathematical context, I understand it to include one's powers of intuition and imagination, but not those of logic or computation. Of course the exercise of common sense can *involve* logic or computation, as when a leap of intuition or imagination is used as a premise in a deduction or as a datum for a computation. But I see logic and computation as essentially *distinct* from common sense, because the raw material on which logic and computation operate can have other sources as well—a list of axioms, for example—in which case the conclusions are independent of intuition and imagination, and can even be opposed to them.

What I have in mind, in other words, is a distinction between two types of reasoning: 'common sense,' which accepts material—be the amount ever so tiny—

from the intuition and imagination; and 'mathematical reasoning,' which accepts none whatsoever. And while the history of mathematics teaches us that what one generation considers the soul of mathematical reasoning may be jettisoned by a later generation as hopelessly intuitive, making it sometimes appear that mathematical reasoning is more a goal than an accomplishment, it is the struggle for precisely that goal that has been, since the time of the Pythagoreans, the hallmark of mathematics.

Common sense can be ignored because it is not the only mode of thinking we have. And in mathematics it *must* be ignored when it conflicts with logic or computation.

"May I butt in?"

Of course. I got carried away.

"I think I see where you're headed. You're going to say that my position is based on common sense, that it conflicts with logic, and that Euclid's position is the only alternative. Am I right?"

Yes.

"Then I have two problems. First, I don't see how my position is based on intuition. Second, I don't see how it leads to a logical contradiction—in fact, it seems to show that *Euclid*'s position leads to a contradiction."

You base your position on intuition when you insist that the length of a segment is some mathematical combination—you lean toward the sum—of the lengths of its constituent points. But there are some things that mathematics simply can't do!—and one of them is to combine a collection of quantities as numerous as the points of a line segment. Simple addition, for example, combines only a *finite number* of terms. The same is true of multiplication. To be honest, there *is* a mathematical method, which you have probably encountered, called the 'theory of infinite series,' by means of which infinite collections of terms can, sometimes, be 'added.' There is also a theory of 'infinite products,' which extends multiplication. But the 'infinity' of terms in an infinite series or product is what mathematicians call a 'countable' infinity, which is much smaller than the 'uncountable' infinity of points in a line segment.

There is a notion in calculus—the 'definite integral'—which for a long time was thought to give the sum of uncountably many terms. To this day, in fact, mathematicians find it useful to so *interpret* it. But it was seen as *really* a sum only so long as it was loosely defined. Once a rigorous definition was formulated in the 19th century, mathematicians decided that the definite integral could be regarded as a 'sum' in only an intuitive sense.

"But even if there's no mathematically rigorous way, at present, of adding enough terms, someday someone might invent one."

Even if someone already has, it wouldn't matter. All I'm trying to do, remember, is point to where your reasoning was based on intuition. When you spoke of the lengths of points as 'mathematically combining' into the length of the segment, the arithmetical operations you referred to simply did not exist—as far as you knew. Thus you could only have been reasoning by analogy with the operations you did know. It was, therefore, and regardless of what mathematical news the future might bring, an intuitive argument.

But let me approach this issue from a different tack. Consider a rainbow—a complex phenomenon made up of water droplets suspended in the air, the sun, and an observer, all positioned relative to one another in a certain way. The rainbow is the result of all these factors acting in concert, and it is difficult to assign individual responsibilities. Which factor causes the green band? Which determines the diameter? We suspect such questions are inappropriate, because if we remove any one of the contributing factors—droplets, sun, observer, or their geometric arrangement—the entire rainbow disappears.

Perhaps a line segment is like a rainbow, only with two constituents instead of four: the points and their arrangement. Who's to say where its length comes from? The points all by themselves? But what if they were strewn randomly about the plane, like paint spatters, instead of neatly 'lined' up? It seems their arrangement contributes something to the phenomenon of 'length,' too. But then an attempt to account for the length of the segment solely in terms of the points would be doomed. Because arithmetical operations take no account of arrangement, that's precisely what you were trying to do.

"Aha!"

I know it smells like a contradiction when all those zeroes don't 'add' up to 1 meter. But arithmetic is inadequate to the situation, because it ignores the geometrical aspect.

Euclid himself, by the way, probably wouldn't have had as much arithmetical trouble with this as we do today. In his day the only numbers were positive; there was no zero. So when he said a point had no length, he *meant* exactly that, that it is improper to speak of a point as having length, as opposed to today's sense of 'it has a length, but the length is zero.' To him a point had no length as a water droplet has no color, and trying to deduce the length of a segment solely from the nonexistent lengths of its points would probably have seemed as fruitless to him as trying to deduce the colors of a rainbow, without optics, from the nonexistent colors of its droplets.

"Interesting . . . . Well, I have to admit my position *was* based on more than just logic after all. But how does it *conflict* with logic?"

Do you know the famous story about the Pythagoreans and $\sqrt{2}$?

"Yes. The early Pythagoreans intuited that you can always find a line segment that is a 'common measure' of two other line segments, meaning that its length will divide into each of theirs exactly a whole number of times. It follows from this that the quotient of two lengths will always be a rational number, for

$$\frac{AB}{CD} = \frac{AB/XY}{CD/XY} = \frac{m}{n}$$

where XY is a common measure of AB and CD and consequently m and n are integers. This means that $\sqrt{2}$ is rational, because they knew it is the quotient of the lengths of a square's diagonal and side. But then one of the later Pythagoreans came up with an argument, based on logic and computation, that $\sqrt{2}$ is actually an irrational number."

Yes, and a beautiful argument it was, one that every mathematics student should know by heart. But tell me—when you first heard the story, what was your reaction to the Pythagoreans' original, common-sense position that two line segments always have a common measure?

"At first I didn't know what to think, the whole question was new to me. Then I decided that physically, at least, it was true. I reasoned that if AB and CD were sticks, for example, a carpenter could probably make a third stick XY that laid down some whole number of times would be, for all practical purposes, the same length as AB, and that laid down some other whole number of times would be, for all practical purposes, the same length as CD. But geometric line segments aren't sticks, and what is close enough for practical purposes isn't the same as mathematical exactness, so in the end I had my doubts. Though their position seemed plausible enough, I never had strong feelings that it was right."

But *their* feelings about its correctness were very strong. By all accounts they were devastated when the logical argument showed they had been wrong. And the reason for this is probably that their common-sense position had *not* come directly from intuition—intuitively plausible though it is—but was rather a *conclusion* they had deduced from something else.

"What was the 'something else'?"

Something more directly intuited. Something they found utterly compelling.

"Well?"

Something about points.

"You're kidding."

No. There is persuasive evidence (e.g. [1], pp. 24–25) that to the early Pythagoreans points were tiny balls whose diameters, though extremely small, were not infinitesimal or zero. All along you've been an early Pythagorean!

"You mean that if points have positive diameters, we can deduce that two segments always have a common measure?"

Almost. We need one more assumption, but a very natural one. Tell me—back when you felt so strongly that points had positive diameters, were you picturing all those diameters as equal, or as varying from point to point?

" . . . as equal. There was no reason why they should vary."

That's just what the Pythagoreans would have assumed. And for the very same reason!—which later Greeks called 'the principle of sufficient reason': variation occurs only when there is a reason sufficient to account for it.

All right then. Every point has the same positive diameter, which we can call 'd.' At last we have a situation our imaginations can cope with. Mine responds with the image sketched here, where I imagine a microscope has been aimed at a tiny portion of segment AB to show its fine structure. Is this something like what you had in mind?

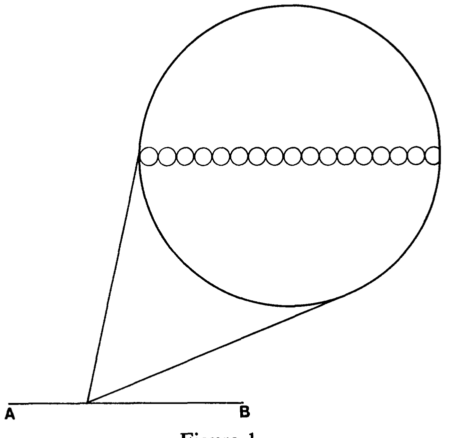

Figure 1.

"It is exactly what I had in mind."

Then consider AB/d, which for short I will call 'm.' Being the quotient of two positive, finite numbers, m is itself a positive, finite number. What does m represent? On the level of common sense.

"The number of points in AB."

I agree. Divide the length of AB by the length of one point, and you get the number of points. (With Figure 1 to look at, the use of simple arithmetic seems perfectly all right, even to an old quibbler like me!) So we see that the positive, finite number m, because it is the number of points in AB, is in fact a *whole* number.

Letting CD be any other line segment, we can reason that CD is likewise made up of points with diameter d, and that $n = CD/d$ is a positive whole number, too. So a minute 'line segment' consisting of only a single point is a common measure of AB and CD.

It now follows exactly as before that the quotient of two lengths is always rational, because

$$\frac{AB}{CD} = \frac{AB/d}{CD/d} = \frac{m}{n}$$

and m and n are whole numbers.

"So Euclid said a point 'has no part' because $\sqrt{2}$ is irrational?"

Yes. And mathematicians ever since have supported his decision. The average modern mathematician wants points to be alike as much as the Greeks did, and this allows only the two alternatives we have considered; points all have the same positive diameter d, or they all have diameter zero. As we have seen, choosing the first makes the conclusion that two segments always have a common measure inescapable, at least on the level of common sense. And while it might be possible—I don't know if it is—to choose the first alternative and yet, by logical contortions, avoid that conclusion, to do so would violate common sense to a much greater extent than points without size do. So the second alternative is still the one that is chosen.

I'm glad we've had this conversation, because it has brought into sharp focus what the standard of mathematical reasoning implies: that the objects with which mathematics deals not only are *nonsensible*—they do not physically exist—but also are at times opposed to common sense and to that degree *nonsensical*. Logic produces a 'mysticism' of its own!

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