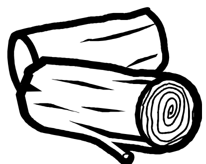
Linear Programming

Examples

a_{ij} : number of units of the i^{th} raw material needed to produce one unit of the j^{th} product.

x_j : the number of units of the j^{th} product produced.

The Resource Allocation Problem

Unit Value	Stock	Raw Material
ρ_1	b_1	
ρ_2	b_2	
ρ_3	b_3	
$\vdots$	$\vdots$	$\vdots$
ρ_m	b_m	

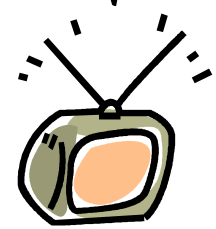
$$c_j = \sigma_j - \sum_{i=1}^{m} \rho_i a_{ij}$$

Maximize $\sum_{j=1}^{n} c_j x_j$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = 1, \dots, m$$

$$x_j \ge 0 \qquad j = 1, \dots, n$$

Product	Unit Market Price	Unit Profit
	σ_1	c_1
	σ_2	c_2
$\vdots$	$\vdots$	$\vdots$
	σ_n	c_n

a_{ij} : number of units of nutrient i in one unit food j.

x_j : number of units of food j in the diet.

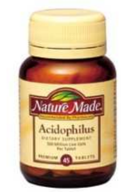
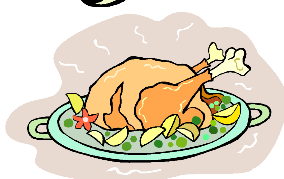
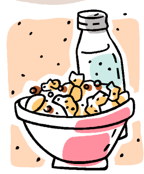
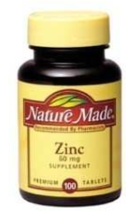
The Diet Problem

Minimize $\sum_{j=1}^{m} c_j x_j$

Subject to

$$\sum_{j=1}^{m} a_{ij} x_j \geq b_i \qquad i = 1, \ldots, n$$

$$x_j \geq 0 \qquad j = 1, \ldots, m$$

Unit Price	Food	Nutrient	Required Units
c_1			b_1
c_2			
c_3			b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_m			b_n

Linear Programming

Forms of LP

The Linear Programming Problem

- An application of linear algebra.
- Its central goal is to maximize or minimize a linear function on a domain defined by linear inequalities and equations.

$x_1, \ldots, x_n$: decision variables

The General Form

Maximize or minimize $c_1 x_1 \quad + c_2 x_2 \quad + \cdots \quad + c_n x_n$

Subject to $a_{11} x_1 \quad + a_{12} x_2 \quad + \cdots \quad + a_{1n} x_n \quad \begin{Bmatrix} \geq \\ = \\ \leq \end{Bmatrix} b_1$

$$\vdots$$

$$a_{m1} x_1 \quad + a_{m2} x_2 \quad + \cdots \quad + a_{mn} x_n \quad \begin{Bmatrix} \geq \\ = \\ \leq \end{Bmatrix} b_m$$

The Canonical Form

"greater than" only

Minimize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \end{aligned}$$

Non-negative $x_1, x_2, \ldots, x_n \geq 0$

This can also be considered canonical.

The Canonical Form

" less than" only

$$
\begin{aligned}
\text{Minimize} \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{Subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m
\end{aligned}
$$

Non-negative $x_1, x_2, \ldots, x_n \ge 0$

The Standard Form

Minimize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

$$x_1, x_2, \ldots, x_n \geq 0$$

Example

$$\begin{array}{lrrrcr}
\text{Maximize} & 3x_1 & +4x_2 & -8x_3 & & \\
\text{Subject to} & 3x_1 & -4x_2 & & \leq & 12 \\
 & x_1 & +2x_2 & +\ x_3 & \geq & 4 \\
 & 4x_1 & -2x_2 & +5x_3 & \leq & 20 \\
 & & & x_1 & \geq & 0 \\
 & & & x_2 & \geq & 0 \\
 & & & x_3 & \geq & 0
\end{array}$$

$$\begin{array}{lrrrcr}
\text{Minimize} & -3x_1 & -4x_2 & +8x_3 & & \\
\text{Subject to} & -3x_1 & +4x_2 & & \geq & -12 \\
 & x_1 & +2x_2 & +\ x_3 & \geq & 4 \\
 & -4x_1 & +2x_2 & -5x_3 & \geq & -20 \\
 & & & x_1 & \geq & 0 \\
 & & & x_2 & \geq & 0 \\
 & & & x_3 & \geq & 0
\end{array}$$

Canonical Form

Example

$$\begin{array}{lrrrcr}
\text{Minimize} & -3x_1 & -4x_2 & +8x_3 & & \\
\text{Subject to} & -3x_1 & +4x_2 & & \geq & -12 \\
& x_1 & +2x_2 & +x_3 & \geq & 4 \\
& -4x_1 & +2x_2 & -5x_3 & \geq & -20 \\
& & & x_1 & \geq & 0 \\
& & & x_2 & \geq & 0 \\
& & & x_3 & \geq & 0
\end{array}$$

$$\begin{array}{lrrrrrrcr}
\text{Minimize} & -3x_1 & -4x_2 & +8x_3 & & & & & \\
\text{Subject to} & -3x_1 & +4x_2 & & -w_1 & & & = & -12 \\
& x_1 & +2x_2 & +x_3 & & -w_2 & & = & 4 \\
& -4x_1 & +2x_2 & -5x_3 & & & -w_3 & = & 20 \\
& & & & & & x_1 & \geq & 0 \\
& & & & & & x_2 & \geq & 0 \\
& & & & & & x_3 & \geq & 0 \\
& & & & & & w_1 & \geq & 0 \\
& & & & & & w_2 & \geq & 0 \\
& & & & & & w_3 & \geq & 0
\end{array}$$

Standard Form

Linear Programming

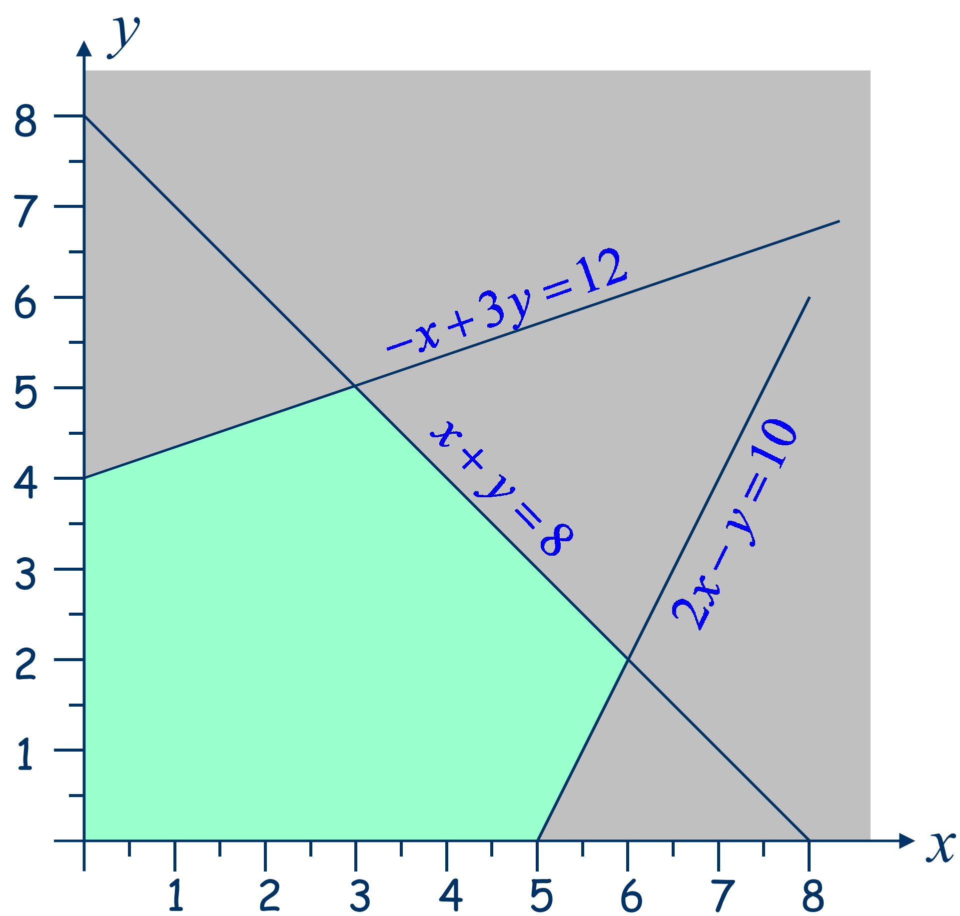
Geometric View of LP

An Example of 2 Decision Variables

Maximize $3x + 2y$

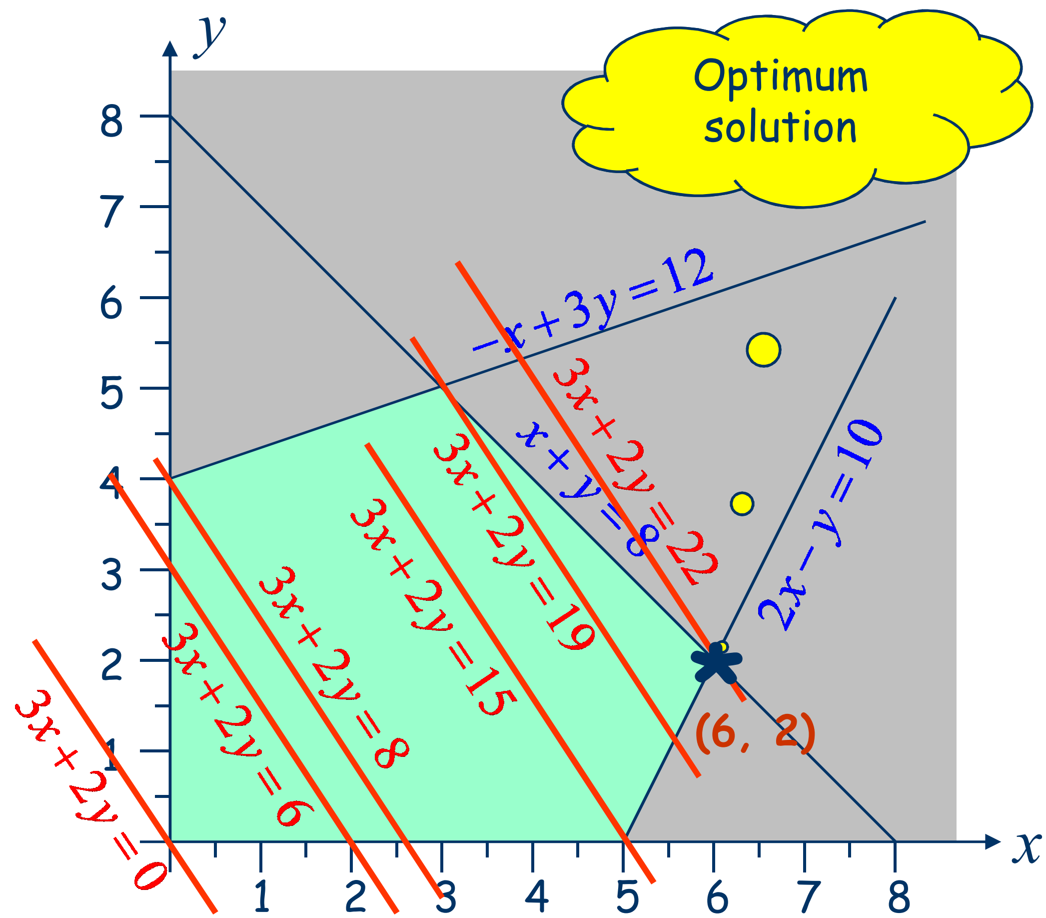
Subject to

$$
\begin{aligned}
-x + 3y &\le 12 \\
x + y &\le 8 \\
2x - y &\le 10 \\
x &\ge 0 \\
y &\ge 0
\end{aligned}
$$



An Example of 2 Decision Variables

$$
\begin{aligned}
\text{Maximize} \quad & 3x + 2y \\
\text{Subject to} \quad & -x + 3y \le 12 \\
& x + y \le 8 \\
& 2x - y \le 10 \\
& x \ge 0 \\
& y \ge 0
\end{aligned}
$$

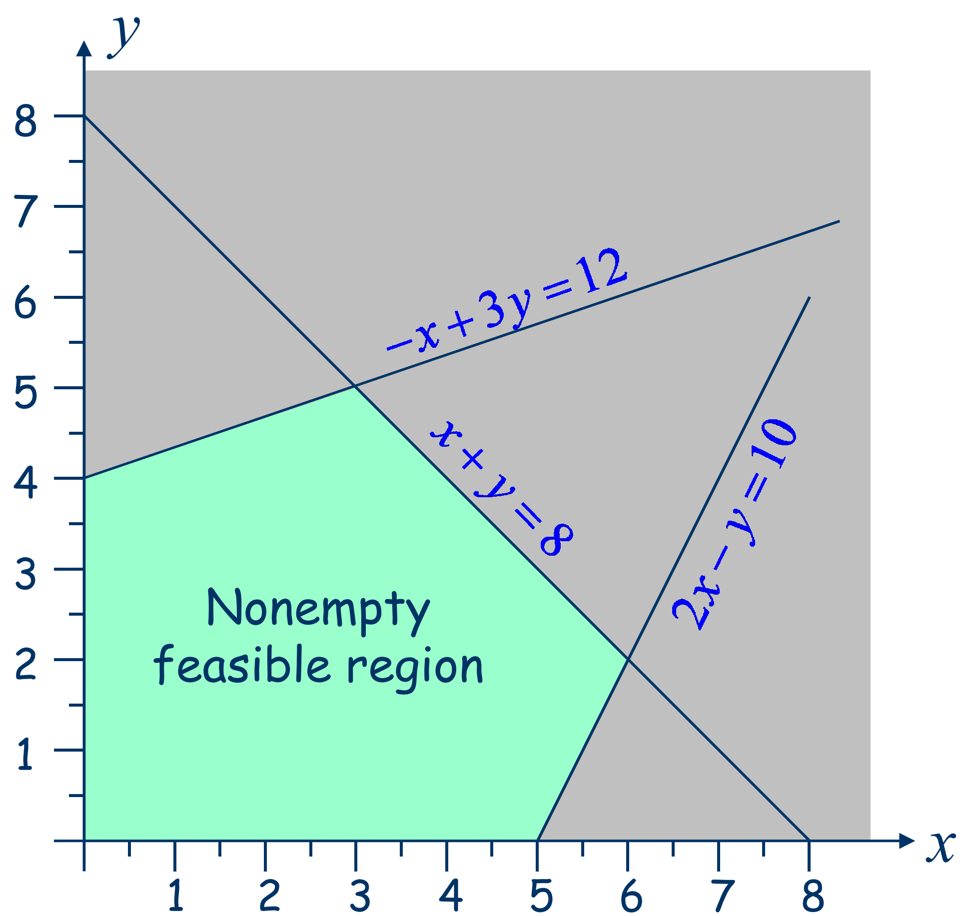


The feasible region determined by a collection of linear inequalities is the collection of points that satisfy all of the inequalities.

Feasible LP Problems

$$\begin{array}{llcrcr} \text{Maximize} & 3x & + & 2y & & \\ \text{Subject to} & -x & + & 3y & \leq & 12 \\ & x & + & y & \leq & 8 \\ & 2x & - & y & \leq & 10 \\ & & & x & \geq & 0 \\ & & & y & \geq & 0 \end{array}$$

A feasible LP problem



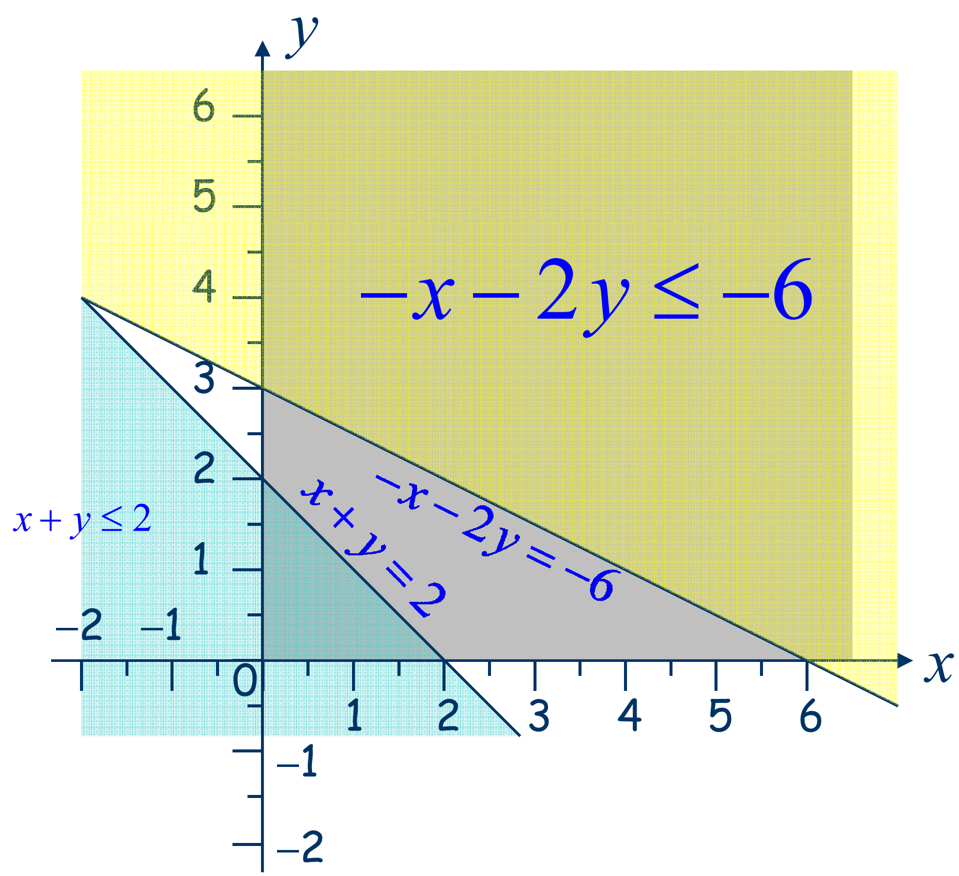
Infeasible LP Problems

Maximize $5x + 4y$

Subject to

$$\begin{aligned} x + y &\le 2 \\ -x - 2y &\le -6 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

An infeasible LP problem

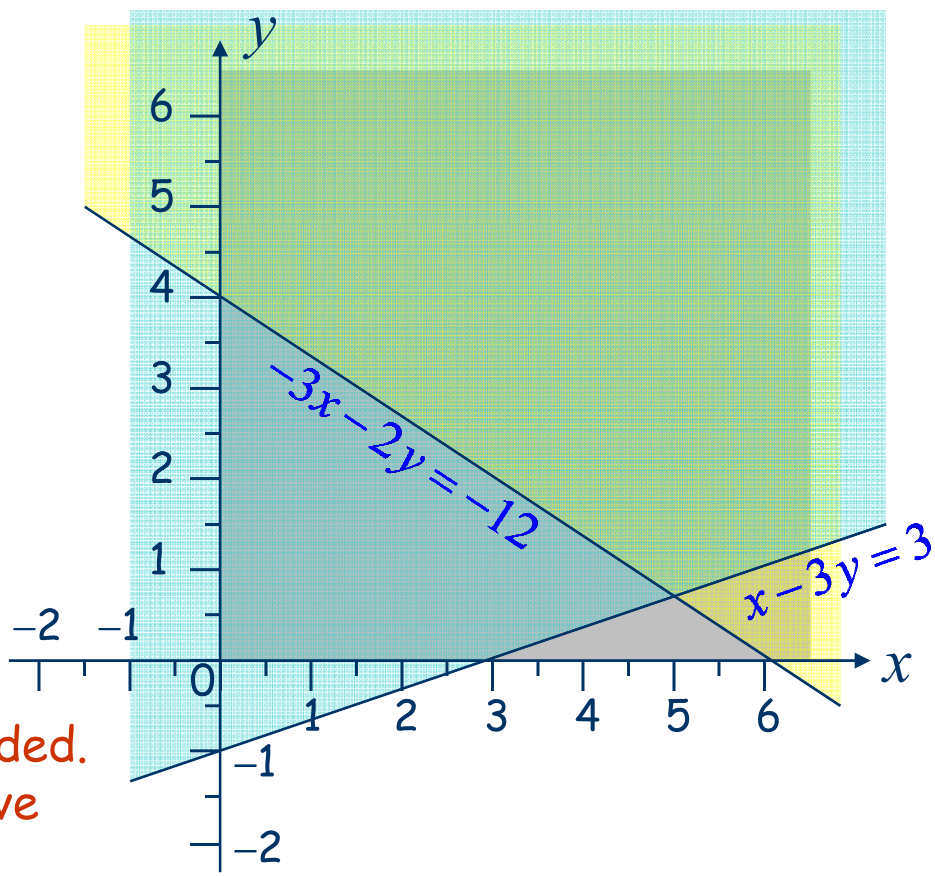


The feasible region is empty

Unbounded LP Problems

$$
\begin{aligned}
\text{Maximize} \quad & 5x + 4y \\
\text{Subject to} \quad & -3x - 2y \le -12 \\
& x - 3y \le 3 \\
& x \ge 0 \\
& y \ge 0
\end{aligned}
$$

An unbounded LP problem



1. The feasible region is unbounded.
2. Some coefficients of objective function is non-negative.

Linear Programming

Intuitive View of the Simplex Method

Two possible ways to help maximization:

1. Increase the variables with positive coefficients.
2. Decrease the variables with negative coefficient.

An Example

Maximize $5x_1 - 4x_2 + 3x_3$ —— Objective Function

$x_2 \rightarrow 0$

Subject to $x_1, x_2, x_3 \geq 0$

An Example

$$
\begin{array}{llllllll}
\text{Maximize} & 5x_1 & - & 4x_2 & + & 3x_3 & & \\
\text{Subject to} & 2x_1 & + & 3x_2 & + & x_3 & \leq & 5 \\
 & 4x_1 & + & x_2 & + & 2x_3 & \leq & 11 \\
 & 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 8 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Converting the LP into Standard Form

$$
\begin{array}{llcrcrcrcl}
\text{Maximize} & 5x_1 & - & 4x_2 & + & 3x_3 & = & \varsigma & & \\
\text{Subject to} & 2x_1 & + & 3x_2 & + & x_3 & \leq & 5 \\
& 4x_1 & + & x_2 & + & 2x_3 & \leq & 11 \\
& 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 8
\end{array}
$$

$$x_1, x_2, x_3 \geq 0$$

$$
\begin{array}{rcrcrcccccccccr}
2x_1 & + & 3x_2 & + & x_3 & + & w_1 & & & & & & & = & 5 \\
4x_1 & + & x_2 & + & 2x_3 & & & + & w_2 & & & & & = & 11 \\
3x_1 & + & 4x_2 & + & 2x_3 & & & & & + & w_3 & & & = & 8 \\
\hline
-5x_1 & + & 4x_2 & - & 3x_3 & & & & & & & + & \varsigma & = & 0
\end{array}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3, \varsigma \geq 0$$

Basic Solutions

$x_1 = 0$ $\quad w_1 = 5$

$x_2 = 0$ $\quad w_2 = 11$ $\quad \varsigma = 0$

$x_3 = 0$ $\quad w_3 = 8$

Such a basic solution is feasible if and only if all b_i's are feasible.

$$\begin{array}{rcrcrcccccccl}
2x_1 & + & 3x_2 & + & x_3 & + & w_1 & & & & & = & 5 \\
4x_1 & + & x_2 & + & 2x_3 & & & + & w_2 & & & = & 11 \\
3x_1 & + & 4x_2 & + & 2x_3 & & & & & + & w_3 & = & 8 \\
\hline
-5x_1 & + & 4x_2 & - & 3x_3 & & & & & + & \varsigma & = & 0
\end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3, \varsigma \geq 0$$

How?

Two possible ways to help maximization:

1. Increase the variables with negate coefficients.
2. Decrease the variables with positive coefficient.

$$x_1 = 0 \qquad w_1 = 5$$

$$x_2 = 0 \qquad w_2 = 11 \qquad \varsigma = 0$$

$$x_3 = 0 \qquad w_3 = 8$$

$$\begin{array}{rcrcrcccccl}
2x_1 & + & 3x_2 & + & x_3 & + & w_1 & & & & = 5 \\
4x_1 & + & x_2 & + & 2x_3 & & & + w_2 & & & = 11 \\
3x_1 & + & 4x_2 & + & 2x_3 & & & & + w_3 & & = 8 \\
\hline
-5x_1 & + & 4x_2 & - & 3x_3 & & & & & + \varsigma & = 0
\end{array}$$

$$x_1, x_2, x_3, w_1, w_2, w_3, \varsigma \geq 0$$

0

Set Up the Initial Tableau

Active	x_1	x_2	x_3	w_1	w_2	w_3	ς	*Ans*
$w_1 = 5$	2	3	1	1	0	0	0	5
$w_2 = 11$	4	1	2	0	1	0	0	11
$w_3 = 8$	3	4	2	0	0	1	0	8
$\varsigma = 0$	–5	4	–3	0	0	0	1	0

$$
\begin{array}{rcrcrcccccccl}
2x_1 & + & 3x_2 & + & x_3 & + & w_1 & & & & & = & 5 \\
4x_1 & + & x_2 & + & 2x_3 & & & + & w_2 & & & = & 11 \\
3x_1 & + & 4x_2 & + & 2x_3 & & & & & + & w_3 & = & 8 \\
\hline
-5x_1 & + & 4x_2 & - & 3x_3 & & & & & + & \varsigma & = & 0
\end{array}
$$

$$x_1, x_2, x_3, w_1, w_2, w_3, \varsigma \geq 0$$

Select the Pivot Column

Active	x_1	x_2	x_3	w_1	w_2	w_3	ζ	*Ans*
$w_1 = 5$	2	3	1	1	0	0	0	5
$w_2 = 11$	4	1	2	0	1	0	0	11
$w_3 = 8$	3	4	2	0	0	1	0	8
$\zeta = 0$	–5	4	–3	0	0	0	1	0

- From bottom row, choose the negative number with the largest magnitude.
- Its column is the pivot column.
  - If there are two candidates, choose either one
- If all the numbers in the bottom row are zero or positive, then you are done, and the basic solution is the optimal solution.

Select the **Pivot** in the Pivot Column

Active	x_1	x_2	x_3	w_1	w_2	w_3	ζ	*Ans*
$w_1 = 5$	2	3	1	1	0	0	0	5
$w_2 = 11$	4	1	2	0	1	0	0	11
$w_3 = 8$	3	4	2	0	0	1	0	8
$\zeta = 0$	–5	4	–3	0	0	0	1	0

- The pivot must always be a positive number.
- For each positive entry b in the pivot column, compute the ratio a/b, where a is the number in the rightmost column in that row. We call this a test ratio.
- Of these ratios, choose the smallest one. The corresponding number b is the pivot.

Clear the Pivot Column

Active	x_1	x_2	x_3	w_1	w_2	w_3	ζ	*Ans*
$w_1 = 5$	2	3	1	1	0	0	0	5
$w_2 = 11$	4	1	2	0	1	0	0	11
$w_3 = 8$	3	4	2	0	0	1	0	8
$\zeta = 0$	–5	4	–3	0	0	0	1	0

Active	x_1	x_2	x_3	w_1	w_2	w_3	ζ	*Ans*	
$x_1 = 2.5$	2	3	1	1	0	0	0	5	
$w_2 = 1$	0	–5	0	–2	1	0	0	1	$R_2 - 2R_1$
$w_3 = 0.5$	0	–0.5	0.5	–1.5	0	1	0	0.5	$R_3 - 1.5R_1$
$\zeta = 12.5$	0	11.5	–0.5	2.5	0	0	1	12.5	$R_4 + 2.5R_1$

Select the Pivot Column

Active	x_1	x_2	x_3	w_1	w_2	w_3	ζ	*Ans*
$x_1 = 2.5$	2	3	1	1	0	0	0	5
$w_2 = 1$	0	–5	0	–2	1	0	0	1
$w_3 = 0.5$	0	–0.5	0.5	–1.5	0	1	0	0.5
ζ=12.5	0	11.5	–0.5	2.5	0	0	1	12.5

Select the **Pivot** in the Pivot Column

Active	x_1	x_2	x_3	w_1	w_2	w_3	ζ	*Ans*
$x_1 = 2.5$	2	3	1	1	0	0	0	5
$w_2 = 1$	0	–5	0	–2	1	0	0	1
$w_3 = 0.5$	0	–0.5	0.5	–1.5	0	1	0	0.5
$\zeta = 12.5$	0	11.5	–0.5	2.5	0	0	1	12.5

Clear the Pivot Column

Active	x_1	x_2	x_3	w_1	w_2	w_3	ζ	Ans
$x_1 = 2.5$	2	3	1	1	0	0	0	5
$w_2 = 1$	0	–5	0	–2	1	0	0	1
$w_3 = 0.5$	0	–0.5	0.5	–1.5	0	1	0	0.5
ζ=12.5	0	11.5	–0.5	2.5	0	0	1	12.5

Active	x_1	x_2	x_3	w_1	w_2	w_3	ζ	Ans	
$x_1 = 2$	2	4	0	4	0	–2	0	4	$R_1 - 2R_3$
$w_2 = 1$	0	–5	0	–2	1	0	0	1	
$x_3 = 1$	0	–0.5	0.5	–1.5	0	1	0	0.5	
ζ=13	0	11	0	1	0	1	1	13	$R_4 + R_3$

The Optimum

$$x_1 = 2$$
$$x_2 = 0$$
$$x_3 = 1$$

Active	x_1	x_2	x_3	w_1	w_2	w_3	ζ	*Ans*
$x_1 = 2$	2	4	0	4	0	–2	0	4
$w_2 = 1$	0	–5	0	–2	1	0	0	1
$x_3 = 1$	0	–0.5	0.5	–1.5	0	1	0	0.5
ζ=13	0	11	0	1	0	1	1	13

$R_4 + R_3$

All of them are positive.

The optimal solution is here

The Optimum

$$x_1 = 2$$
$$x_2 = 0$$
$$x_3 = 1$$

Maximize $5x_1 - 4x_2 + 3x_3$

Subject to

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &\le 5 \\ 4x_1 + x_2 + 2x_3 &\le 11 \\ 3x_1 + 4x_2 + 2x_3 &\le 8 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

The Optimum

$x_1 = 2$

$x_2 = 0$

$x_3 = 1$

Maximize $5x_1 - 4x_2 + 3x_3$ **13**

Subject to

$2x_1 + 3x_2 + x_3 \le 5$ **5**

$4x_1 + x_2 + 2x_3 \le 11$ **10**

$3x_1 + 4x_2 + 2x_3 \le 8$ **8**

$x_1, x_2, x_3 \ge 0$