CHAPTER 5

Integration

"There are in this world optimists who feel that any symbol that starts off with an integral sign must necessarily denote something that will have every property that they should like an integral to possess. This of course is quite annoying to us rigorous mathematicians; what is even more annoying is that by doing so they often come up with the right answer."

E. J. McShane
Bulletin of the American Mathematical Society, v. 69, p. 611, 1963

Introduction

The second fundamental problem addressed by calculus is the problem of areas, that is, the problem of determining the area of a region of the plane bounded by various curves. Like the problem of tangents considered in Chapter 2, many practical problems in various disciplines require the evaluation of areas for their solution, and the solution of the problem of areas necessarily involves the notion of limits. On the surface the problem of areas appears unrelated to the problem of tangents. However, we will see that the two problems are very closely related; one is the inverse of the other. Finding an area is equivalent to finding an antiderivative or, as we prefer to say, finding an integral. The relationship between areas and antiderivatives is called the Fundamental Theorem of Calculus. When we have proved it, we will be able to find areas at will, provided only that we can integrate (i.e., antidifferentiate) the various functions we encounter.

We would like to have at our disposal a set of integration rules similar to the differentiation rules developed in Chapter 2. We can find the derivative of any differentiable function using those differentiation rules. Unfortunately, integration is generally more difficult; indeed, some fairly simple functions are not themselves derivatives of simple functions. For example, e^{x^2} is not the derivative of any finite combination of elementary functions. Nevertheless, we will expend some effort in Section 5.6 and Sections 6.1–6.4 to develop techniques for integrating as many functions as possible. Later, in Chapter 6, we will examine how to approximate areas bounded by graphs of functions that we cannot antidifferentiate.

5.1 Sums and Sigma Notation

When we begin calculating areas in the next section, we will often encounter sums of values of functions. We need to have a convenient notation for representing sums of arbitrary (possibly large) numbers of terms, and we need to develop techniques for evaluating some such sums.

We use the symbol $\sum$ to represent a sum; it is an enlarged Greek capital letter S called *sigma*.

DEFINITION 1

Sigma notation

If m and n are integers with $m \le n$, and if f is a function defined at the integers $m, m+1, m+2, \ldots, n$, the symbol $\sum_{i=m}^{n} f(i)$ represents the sum of the values of f at those integers:

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n).$$

The explicit sum appearing on the right side of this equation is the **expansion** of the sum represented in sigma notation on the left side.

EXAMPLE 1 $\displaystyle\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$

The i that appears in the symbol $\sum_{i=m}^{n} f(i)$ is called an **index of summation**. To evaluate $\sum_{i=m}^{n} f(i)$, replace the index i with the integers $m, m+1, \ldots, n$, successively, and sum the results. Observe that the value of the sum does not depend on what we call the index; the index does not appear on the right side of the definition. If we use another letter in place of i in the sum in Example 1, we still get the same value for the sum:

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

The index of summation is a *dummy variable* used to represent an arbitrary point where the function is evaluated to produce a term to be included in the sum. On the other hand, the sum $\sum_{i=m}^{n} f(i)$ does depend on the two numbers m and n, called the **limits of summation**; m is the **lower limit**, and n is the **upper limit**.

EXAMPLE 2 **(Examples of sums using sigma notation)**

$$\sum_{j=1}^{20} j = 1 + 2 + 3 + \cdots + 18 + 19 + 20$$

$$\sum_{i=0}^{n} x^i = x^0 + x^1 + x^2 + \cdots + x^{n-1} + x^n$$

$$\sum_{m=1}^{n} 1 = \underbrace{1 + 1 + 1 + \cdots + 1}_{n \text{ terms}}$$

$$\sum_{k=-2}^{3} \frac{1}{k+7} = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10}$$

Sometimes we use a subscripted variable a_i to denote the ith term of a general sum instead of using the functional notation $f(i)$:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_n.$$

In particular, an **infinite series** is such a sum with infinitely many terms:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots .$$

When no final term follows the $\cdots$, it is understood that the terms go on forever. We will study infinite series in Chapter 9.

When adding finitely many numbers, the order in which they are added is unimportant; any order will give the same sum. If all the numbers have a common factor, then that factor can be removed from each term and multiplied after the sum is evaluated: $ca + cb = c(a + b)$. These laws of arithmetic translate into the following *linearity* rule for finite sums; if A and B are constants, then

$$\sum_{i=m}^{n} \big(Af(i) + Bg(i)\big) = A\sum_{i=m}^{n} f(i) + B\sum_{i=m}^{n} g(i).$$

Both of the sums $\sum_{j=m}^{m+n} f(j)$ and $\sum_{i=0}^{n} f(i+m)$ have the same expansion, namely, $f(m) + f(m+1) + \cdots + f(m+n)$. Therefore, the two sums are equal.

$$\sum_{j=m}^{m+n} f(j) = \sum_{i=0}^{n} f(i+m).$$

This equality can also be derived by substituting $i + m$ for j everywhere j appears on the left side, noting that $i + m = m$ reduces to $i = 0$, and $i + m = m + n$ reduces to $i = n$. It is often convenient to make such a **change of index** in a summation.

EXAMPLE 3 Express $\sum_{j=3}^{17} \sqrt{1 + j^2}$ in the form $\sum_{i=1}^{n} f(i)$.

Solution Let $j = i + 2$. Then $j = 3$ corresponds to $i = 1$ and $j = 17$ corresponds to $i = 15$. Thus,

$$\sum_{j=3}^{17} \sqrt{1 + j^2} = \sum_{i=1}^{15} \sqrt{1 + (i+2)^2}.$$

Evaluating Sums

There is a **closed form** expression for the sum S of the first n positive integers, namely,

$$S = \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

To see this, write the sum forwards and backwards and add the two to get

$$\begin{array}{rcccccccccc} S = & 1 & + & 2 & + & 3 & + \cdots + & (n-1) & + & n \\ S = & n & + & (n-1) & + & (n-2) & + \cdots + & 2 & + & 1 \\ \hline 2S = & (n+1) & + & (n+1) & + & (n+1) & + \cdots + & (n+1) & + & (n+1) = n(n+1) \end{array}$$

The formula for S follows when we divide by 2.

It is not usually this easy to evaluate a general sum in closed form. We can only simplify $\sum_{i=m}^{n} f(i)$ for a small class of functions f. The only such formulas we will need in the next sections are collected in Theorem 1.

THEOREM 1

Summation formulas

(a) $\displaystyle\sum_{i=1}^{n} 1 = \underbrace{1+1+1+\cdots+1}_{n \text{ terms}} = n.$

(b) $\displaystyle\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}.$

(c) $\displaystyle\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}.$

(d) $\displaystyle\sum_{i=1}^{n} r^{i-1} = 1+r+r^2+r^3+\cdots+r^{n-1} = \frac{r^n-1}{r-1}$ if $r \neq 1$.

PROOF Formula (a) is trivial; the sum of n ones is n. One proof of formula (b) was given above.

To prove (c) we write n copies of the identity

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1,$$

one for each value of k from 1 to n, and add them up:

$$\begin{array}{rcrcccccc}
2^3 & - & 1^3 & = & 3\times 1^2 & + & 3\times 1 & + & 1\\
3^3 & - & 2^3 & = & 3\times 2^2 & + & 3\times 2 & + & 1\\
4^3 & - & 3^3 & = & 3\times 3^2 & + & 3\times 3 & + & 1\\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots\\
n^3 & - & (n-1)^3 & = & 3(n-1)^2 & + & 3(n-1) & + & 1\\
(n+1)^3 & - & n^3 & = & 3n^2 & + & 3n & + & 1\\
\hline
(n+1)^3 & - & 1^3 & = & 3\left(\sum_{i=1}^{n} i^2\right) & + & 3\left(\sum_{i=1}^{n} i\right) & + & n\\
 & & & = & 3\left(\sum_{i=1}^{n} i^2\right) & + & \dfrac{3n(n+1)}{2} & + & n.
\end{array}$$

We used formula (b) in the last line. The final equation can be solved for the desired sum to give formula (c). Note the cancellations that occurred when we added up the left sides of the n equations. The term 2^3 in the first line cancelled the -2^3 in the second line, and so on, leaving us with only two terms, the $(n+1)^3$ from the nth line and the -1^3 from the first line:

$$\sum_{k=1}^{n}\left((k+1)^3 - k^3\right) = (n+1)^3 - 1^3.$$

This is an example of what we call a **telescoping sum**. In general, a sum of the form $\sum_{i=m}^{n}\big(f(i+1) - f(i)\big)$ telescopes to the closed form $f(n+1) - f(m)$ because all but the first and last terms cancel out.

To prove formula (d), let $s = \sum_{i=1}^{n} r^{i-1}$ and subtract s from rs:

$$\begin{aligned}(r-1)s = rs - s &= (r + r^2 + r^3 + \cdots + r^n) - (1 + r + r^2 + \cdots + r^{n-1})\\ &= r^n - 1.\end{aligned}$$

The result follows on division by $r-1$.

Other proofs of (b) – (d) are suggested in Exercises 36–38.

EXAMPLE 4 Evaluate $\displaystyle\sum_{k=m+1}^{n} (6k^2 - 4k + 3)$, where $1 \le m < n$.

Solution Using the rules of summation and various summation formulas from Theorem 1, we calculate

$$\begin{aligned}\sum_{k=1}^{n}(6k^2-4k+3) &= 6\sum_{k=1}^{n}k^2-4\sum_{k=1}^{n}k+3\sum_{k=1}^{n}1\\ &= 6\,\frac{n(n+1)(2n+1)}{6}-4\,\frac{n(n+1)}{2}+3n\\ &= 2n^3+n^2+2n\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{k=m+1}^{n}(6k^2-4k+3) &= \sum_{k=1}^{n}(6k^2-4k+3)-\sum_{k=1}^{m}(6k^2-4k+3)\\ &= 2n^3+n^2+2n-2m^3-m^2-2m.\end{aligned}$$

Remark Maple can find closed form expressions for some sums. For example,

```
> sum(i^4, i=1..n); factor(%);
```

$$\frac{1}{5}(n+1)^5-\frac{1}{2}(n+1)^4+\frac{1}{3}(n+1)^3-\frac{1}{30}n-\frac{1}{30}$$

$$\frac{1}{30}n(2n+1)(n+1)(3n^2+3n-1)$$

EXERCISES 5.1

Expand the sums in Exercises 1–6.

1. $\sum_{i=1}^{4} i^3$

2. $\sum_{j=1}^{100} \frac{j}{j+1}$

3. $\sum_{i=1}^{n} 3^i$

4. $\sum_{i=0}^{n-1} \frac{(-1)^i}{i+1}$

5. $\sum_{j=3}^{n} \frac{(-2)^j}{(j-2)^2}$

6. $\sum_{j=1}^{n} \frac{j^2}{n^3}$

Write the sums in Exercises 7–14 using sigma notation. (Note that the answers are not unique.)

7. $5+6+7+8+9$

8. $2+2+2+\cdots+2$ (200 terms)

9. $2^2-3^2+4^2-5^2+\cdots-99^2$

10. $1+2x+3x^2+4x^3+\cdots+100x^{99}$

11. $1+x+x^2+x^3+\cdots+x^n$

12. $1-x+x^2-x^3+\cdots+x^{2n}$

13. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots+\frac{(-1)^{n-1}}{n^2}$

14. $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots+\frac{n}{2^n}$

Express the sums in Exercises 15–16 in the form $\sum_{i=1}^{n} f(i)$.

15. $\sum_{j=0}^{99} \sin(j)$

16. $\sum_{k=-5}^{m} \frac{1}{k^2+1}$

Find closed form values for the sums in Exercises 17–28.

17. $\sum_{i=1}^{n} (i^2+2i)$

18. $\sum_{j=1}^{1,000} (2j+3)$

19. $\sum_{k=1}^{n} (\pi^k-3)$

20. $\sum_{i=1}^{n} (2^i-i^2)$

21. $\sum_{m=1}^{n} \ln m$

22. $\sum_{i=0}^{n} e^{i/n}$

23. The sum in Exercise 8.

24. The sum in Exercise 11.

25. The sum in Exercise 12.

26. The sum in Exercise 10. *Hint:* Differentiate the sum $\sum_{i=0}^{100} x^i$.

27. The sum in Exercise 9. *Hint:* The sum is

$$\sum_{k=1}^{49}\Big((2k)^2-(2k+1)^2\Big)=\sum_{k=1}^{49}(-4k-1).$$

28. The sum in Exercise 14. *Hint:* apply the method of proof of Theorem 1(d) to this sum.

29. Verify the formula for the value of a telescoping sum:

$$\sum_{i=m}^{n}\Big(f(i+1)-f(i)\Big)=f(n+1)-f(m).$$

Why is the word "telescoping" used to describe this sum?

In Exercises 30–32, evaluate the given telescoping sums.

30. $\displaystyle\sum_{n=1}^{10}\left(n^4-(n-1)^4\right)$ **31.** $\displaystyle\sum_{j=1}^{m}(2^j-2^{j-1})$

32. $\displaystyle\sum_{i=m}^{2m}\left(\frac{1}{i}-\frac{1}{i+1}\right)$

33. Show that $\dfrac{1}{j(j+1)}=\dfrac{1}{j}-\dfrac{1}{j+1}$, and hence evaluate $\displaystyle\sum_{j=1}^{n}\frac{1}{j(j+1)}$.

34. Figure 5.1 shows a square of side n subdivided into n^2 smaller squares of side 1. How many small squares are shaded? Obtain the closed form expression for $\sum_{i=1}^{n} i$ by considering the sum of the areas of the shaded squares.

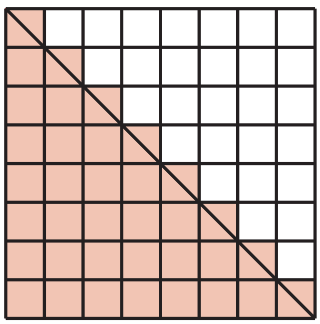


Figure 5.1

35. Write n copies of the identity $(k+1)^2-k^2=2k+1$, one for each integer k from 1 to n, and add them up to obtain the formula

$$\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$$

in a manner similar to the proof of Theorem 1(c).

36. Use mathematical induction to prove Theorem 1(b).

37. Use mathematical induction to prove Theorem 1(c).

38. Use mathematical induction to prove Theorem 1(d).

39. Figure 5.2 shows a square of side $\sum_{i=1}^{n} i=n(n+1)/2$ subdivided into a small square of side 1 and $n-1$ L-shaped regions whose short edges are $2, 3, \ldots, n$. Show that the area of the L-shaped region with short side i is i^3, and hence verify that

$$\sum_{i=1}^{n} i^3=\frac{n^2(n+1)^2}{4}.$$

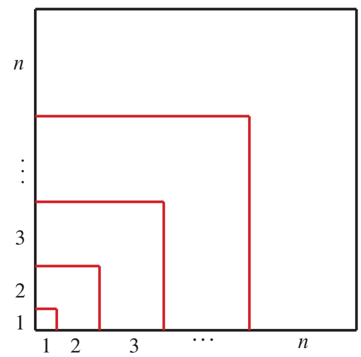


Figure 5.2

40. Write n copies of the identity

$$(k+1)^4-k^4=4k^3+6k^2+4k+1,$$

one for each integer k from 1 to n, and add them up to obtain the formula

$$\sum_{i=1}^{n} i^3=\frac{n^2(n+1)^2}{4}$$

in a manner similar to the proof of Theorem 1(c).

41. Use mathematical induction to verify the formula for the sum of cubes given in Exercise 40.

42. Extend the method of Exercise 40 to find a closed form expression for $\sum_{i=1}^{n} i^4$. You will probably want to use Maple or other computer algebra software to do all the algebra.

43. Use Maple or another computer algebra system to find $\sum_{i=1}^{n} i^k$ for $k=5, 6, 7, 8$. Observe the term involving the highest power of n in each case. Predict the highest-power term in $\sum_{i=1}^{n} i^{10}$ and verify your prediction.

5.2 Areas as Limits of Sums

We began the study of derivatives in Chapter 2 by defining what is meant by a tangent line to a curve at a particular point. We would like to begin the study of integrals by defining what is meant by the **area** of a plane region, but a definition of area is much more difficult to give than a definition of tangency. Let us assume (as we did, for example, in Section 3.3) that we know intuitively what area means and list some of its properties. (See Figure 5.3.)

(i) The area of a plane region is a nonnegative real number of *square units*.

(ii) The area of a rectangle with width w and height h is $A=wh$.

(iii) The areas of congruent plane regions are equal.

(iv) If region S is contained in region R, then the area of S is less than or equal to that of R.

(v) If region R is a union of (finitely many) nonoverlapping regions, then the area of R is the sum of the areas of those regions.

Using these five properties we can calculate the area of any **polygon** (a region bounded by straight line segments). First, we note that properties (iii) and (v) show that the area of a parallelogram is the same as that of a rectangle having the same base width and height. Any triangle can be butted against a congruent copy of itself to form a parallelogram, so a triangle has area half the base width times the height. Finally, any polygon can be subdivided into finitely many nonoverlapping triangles so its area is the sum of the areas of those triangles.

We can't go beyond polygons without taking limits. If a region has a curved boundary, its area can only be approximated by using rectangles or triangles; calculating the exact area requires the evaluation of a limit. We showed how this could be done for a circle in Section 1.1.

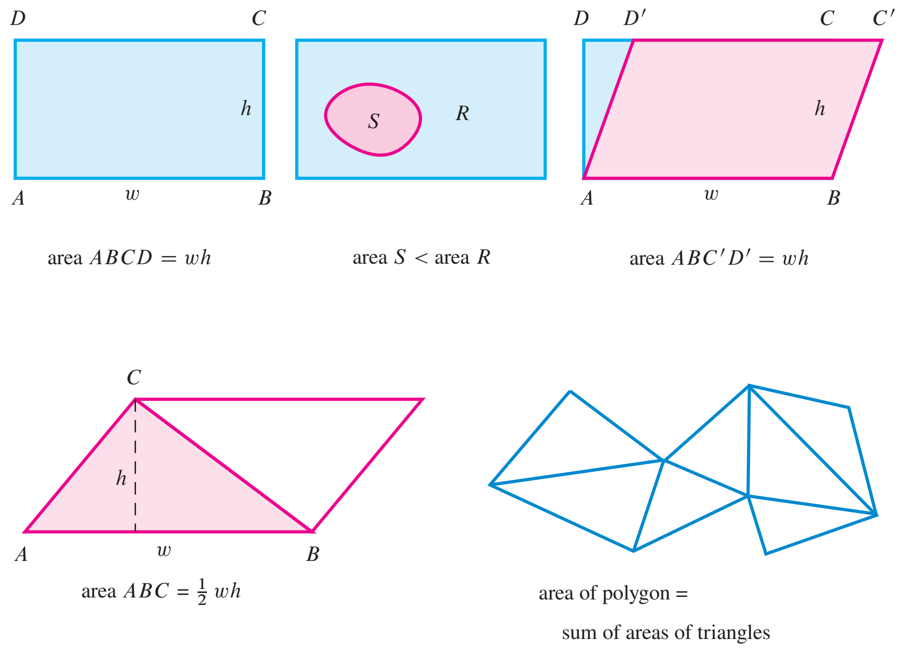


Figure 5.3 Properties of area

The Basic Area Problem

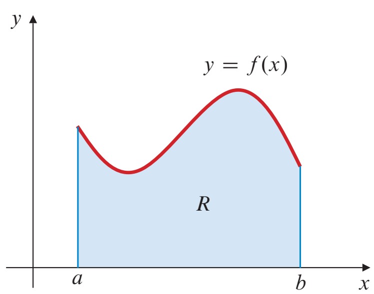


Figure 5.4 The basic area problem: find the area of region R

In this section we are going to consider how to find the area of a region R lying under the graph $y = f(x)$ of a nonnegative-valued, continuous function f, above the x-axis and between the vertical lines $x = a$ and $x = b$, where $a < b$. (See Figure 5.4.) To accomplish this, we proceed as follows. Divide the interval $[a, b]$ into n subintervals by using division points:

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b.$$

Denote by Δx_i the length of the ith subinterval $[x_{i-1}, x_i]$:

$$\Delta x_i = x_i - x_{i-1}, \qquad (i = 1, 2, 3, \ldots, n).$$

Vertically above each subinterval $[x_{i-1}, x_i]$ build a rectangle whose base has length Δx_i and whose height is $f(x_i)$. The area of this rectangle is $f(x_i)\,\Delta x_i$. Form the sum of these areas:

$$S_n = f(x_1)\,\Delta x_1 + f(x_2)\,\Delta x_2 + f(x_3)\,\Delta x_3 + \cdots + f(x_n)\,\Delta x_n = \sum_{i=1}^{n} f(x_i)\,\Delta x_i.$$

The rectangles are shown shaded in Figure 5.5 for a decreasing function f. For an increasing function, the tops of the rectangles would lie above the graph of f rather than below it. Evidently, S_n is an approximation to the area of the region R, and the approximation gets better as n increases, provided we choose the points $a = x_0 < x_1 < \cdots < x_n = b$ in such a way that the width Δx_i of the widest rectangle approaches zero.

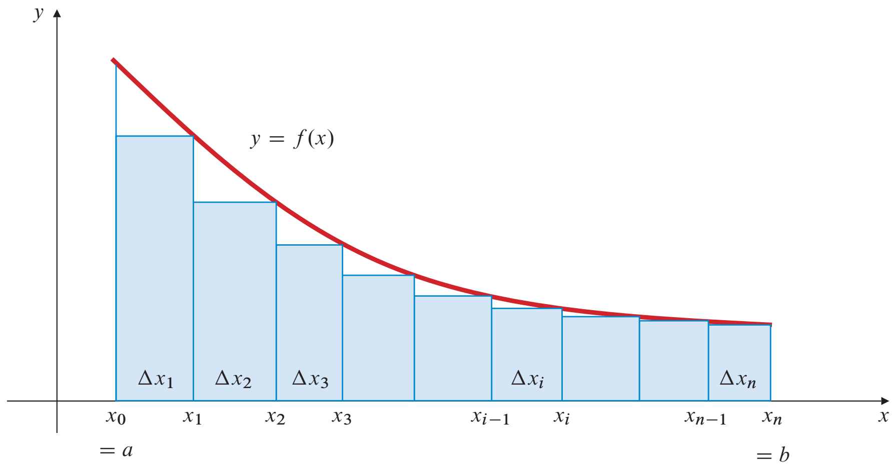
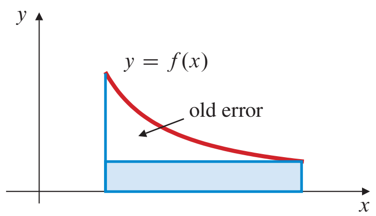


Figure 5.5 Approximating the area under the graph of a decreasing function using rectangles

Observe in Figure 5.6, for example, that subdividing a subinterval into two smaller subintervals reduces the error in the approximation by reducing that part of the area under the curve that is not contained in the rectangles. It is reasonable, therefore, to calculate the area of R by finding the limit of S_n as $n \to \infty$ with the restriction that the largest of the subinterval widths Δx_i must approach zero:

$$\text{Area of } R = \lim_{\substack{n\to\infty \\ \max \Delta x_i \to 0}} S_n.$$



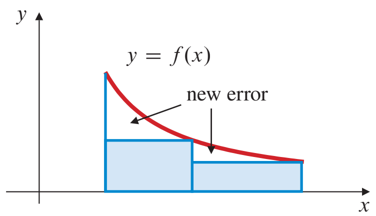


Figure 5.6 Using more rectangles makes the error smaller

Sometimes, but not always, it is useful to choose the points x_i $(0 \le i \le n)$ in $[a, b]$ in such a way that the subinterval lengths Δx_i are all equal. In this case we have

$$\Delta x_i = \Delta x = \frac{b-a}{n}, \qquad x_i = a + i\,\Delta x = a + \frac{i}{n}(b-a).$$

Some Area Calculations

We devote the rest of this section to some examples in which we apply the technique described above for finding areas under graphs of functions by approximating with rectangles. Let us begin with a region for which we already know the area so we can satisfy ourselves that the method does give the correct value.

EXAMPLE 1 Find the area A of the region lying under the straight line $y = x + 1$, above the x-axis, and between the lines $x = 0$ and $x = 2$.

Solution The region is shaded in Figure 5.7(a). It is a *trapezoid* (a four-sided polygon with one pair of parallel sides) and has area 4 square units. (It can be divided into a rectangle and a triangle, each of area 2 square units.) We will calculate the area as a limit of sums of areas of rectangles constructed as described above. Divide the interval $[0, 2]$ into n subintervals *of equal length* by points

$$x_0 = 0,\ x_1 = \frac{2}{n},\ x_2 = \frac{4}{n},\ x_3 = \frac{6}{n},\ \ldots\ x_n = \frac{2n}{n} = 2.$$

The value of $y = x + 1$ at $x = x_i$ is $x_i + 1 = \dfrac{2i}{n} + 1$ and the ith subinterval, $\left[\dfrac{2(i-1)}{n}, \dfrac{2i}{n}\right]$, has length $\Delta x_i = \dfrac{2}{n}$. Observe that $\Delta x_i \to 0$ as $n \to \infty$.

The sum of the areas of the approximating rectangles shown in Figure 5.7(a) is

$$\begin{aligned} S_n &= \sum_{i=1}^{n} \left(\frac{2i}{n} + 1 \right) \frac{2}{n} \\ &= \left(\frac{2}{n} \right) \left[\frac{2}{n} \sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \right] \qquad \text{(Use parts (b) and (a) of Theorem 1.)} \\ &= \left(\frac{2}{n} \right) \left[\frac{2}{n} \frac{n(n+1)}{2} + n \right] \\ &= 2 \frac{n+1}{n} + 2. \end{aligned}$$

Therefore, the required area A is given by

$$A = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(2 \frac{n+1}{n} + 2 \right) = 2 + 2 = 4 \text{ square units.}$$

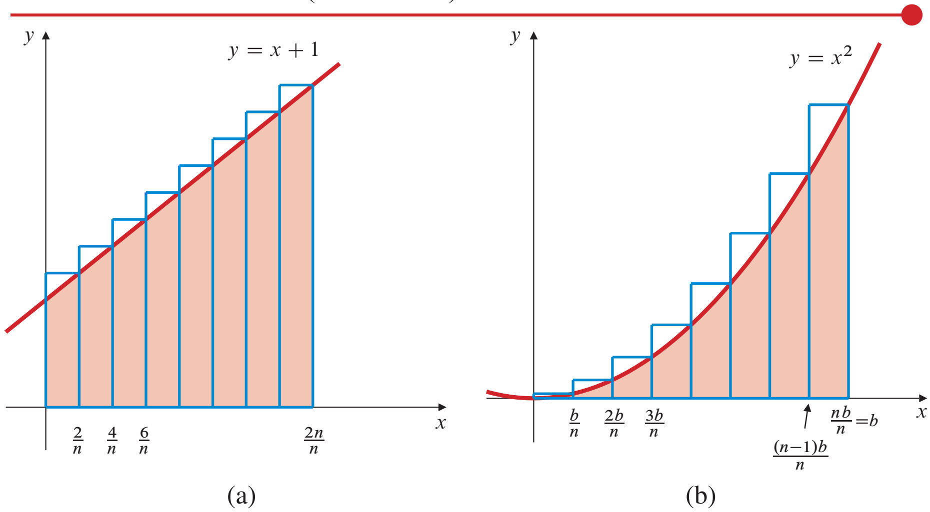


Figure 5.7

(a) The region of Example 1

(b) The region of Example 2

EXAMPLE 2 Find the area of the region bounded by the parabola $y = x^2$ and the straight lines $y = 0$, $x = 0$, and $x = b$, where $b > 0$.

Solution The area A of the region is the limit of the sum S_n of areas of the rectangles shown in Figure 5.7(b). Again we have used equal subintervals, each of length b/n. The height of the ith rectangle is $(ib/n)^2$. Thus,

$$S_n = \sum_{i=1}^{n} \left(\frac{ib}{n} \right)^2 \frac{b}{n} = \frac{b^3}{n^3} \sum_{i=1}^{n} i^2 = \frac{b^3}{n^3} \frac{n(n+1)(2n+1)}{6},$$

by formula (c) of Theorem 1. Hence, the required area is

$$A = \lim_{n \to \infty} S_n = \lim_{n \to \infty} b^3 \frac{(n+1)(2n+1)}{6n^2} = \frac{b^3}{3} \text{ square units.}$$

Finding an area under the graph of $y = x^k$ over an interval I becomes more and more difficult as k increases if we continue to try to subdivide I into subintervals of equal length. (See Exercise 14 at the end of this section for the case $k = 3$.) It is, however, possible to find the area for arbitrary k if we subdivide the interval I into subintervals whose lengths increase in geometric progression. Example 3 illustrates this.

EXAMPLE 3 Let $b > a > 0$, and let k be any real number except -1. Show that the area A of the region bounded by $y = x^k$, $y = 0$, $x = a$, and $x = b$ is

$$A = \frac{b^{k+1} - a^{k+1}}{k+1} \text{ square units.}$$

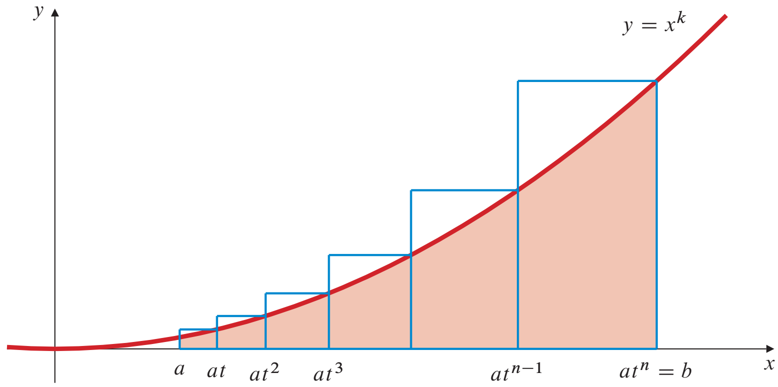


Figure 5.8 For this partition the subinterval lengths increase exponentially

BEWARE! This is a long and rather difficult example. Either skip over it or take your time and check each step carefully.

Solution Let $t = (b/a)^{1/n}$ and let

$$x_0 = a, \; x_1 = at, \; x_2 = at^2, \; x_3 = at^3, \; \ldots \; x_n = at^n = b.$$

These points subdivide the interval $[a, b]$ into n subintervals of which the ith, $[x_{i-1}, x_i]$, has length $\Delta x_i = at^{i-1}(t-1)$. If $f(x) = x^k$, then $f(x_i) = a^k t^{ki}$. The sum of the areas of the rectangles shown in Figure 5.8 is:

$$\begin{aligned}
S_n &= \sum_{i=1}^{n} f(x_i)\,\Delta x_i \\
&= \sum_{i=1}^{n} a^k t^{ki}\, at^{i-1}(t-1) \\
&= a^{k+1}(t-1)\,t^k \sum_{i=1}^{n} t^{(k+1)(i-1)} \\
&= a^{k+1}(t-1)\,t^k \sum_{i=1}^{n} r^{(i-1)} \qquad \text{where } r = t^{k+1} \\
&= a^{k+1}(t-1)\,t^k \frac{r^n - 1}{r - 1} \qquad \text{(by Theorem 1(d))} \\
&= a^{k+1}(t-1)\,t^k \frac{t^{(k+1)n} - 1}{t^{k+1} - 1}.
\end{aligned}$$

Now replace t with its value $(b/a)^{1/n}$ and rearrange factors to obtain

$$\begin{aligned}
S_n &= a^{k+1}\left(\left(\frac{b}{a}\right)^{1/n} - 1\right)\left(\frac{b}{a}\right)^{k/n} \frac{\left(\frac{b}{a}\right)^{k+1} - 1}{\left(\frac{b}{a}\right)^{(k+1)/n} - 1} \\
&= \left(b^{k+1} - a^{k+1}\right) c^{k/n} \frac{c^{1/n} - 1}{c^{(k+1)/n} - 1}, \qquad \text{where } c = \frac{b}{a}.
\end{aligned}$$

Of the three factors in the final line above, the first does not depend on n, and the second, $c^{k/n}$, approaches $c^0 = 1$ as $n \to \infty$. The third factor is an indeterminate form of type $[0/0]$, which we evaluate using l'Hôpital's Rule. First let $u = 1/n$. Then

$$\begin{aligned}
\lim_{n\to\infty} \frac{c^{1/n} - 1}{c^{(k+1)/n} - 1} &= \lim_{u\to 0+} \frac{c^u - 1}{c^{(k+1)u} - 1} \qquad \left[\frac{0}{0}\right] \\
&= \lim_{u\to 0+} \frac{c^u \ln c}{(k+1)\,c^{(k+1)u} \ln c} = \frac{1}{k+1}.
\end{aligned}$$

Therefore, the required area is

$$A = \lim_{n\to\infty} S_n = \left(b^{k+1} - a^{k+1}\right) \times 1 \times \frac{1}{k+1} = \frac{b^{k+1} - a^{k+1}}{k+1} \text{ square units.}$$

As you can see, it can be rather difficult to calculate areas bounded by curves by the methods developed above. Fortunately, there is an easier way, as we will discover in Section 5.5.

Remark For technical reasons it was necessary to assume $a > 0$ in Example 3. The result is also valid for $a = 0$ provided $k > -1$. In this case we have $\lim_{a\to 0+} a^{k+1} = 0$, so the area under $y = x^k$, above $y = 0$, between $x = 0$ and $x = b > 0$ is $A = b^{k+1}/(k+1)$ square units. For $k = 2$ this agrees with the result of Example 2.

EXAMPLE 4 Identify the limit $L = \displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n-i}{n^2}$ as an area, and evaluate it.

Solution We can rewrite the ith term of the sum so that it depends on i/n:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 - \frac{i}{n}\right) \frac{1}{n}.$$

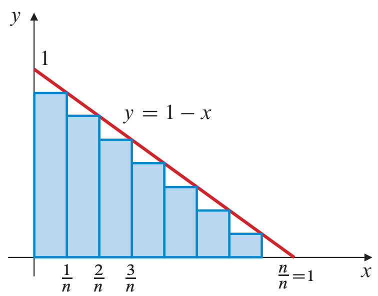


Figure 5.9 Recognizing a sum of areas

The terms now appear to be the areas of rectangles of base $1/n$ and heights $1 - x_i$, $(1 \le i \le n)$, where

$$x_1 = \frac{1}{n}, \quad x_2 = \frac{2}{n}, \quad x_3 = \frac{3}{n}, \quad \ldots, \quad x_n = \frac{n}{n}.$$

Thus, the limit L is the area under the curve $y = 1 - x$ from $x = 0$ to $x = 1$. (See Figure 5.9.) This region is a triangle having area $1/2$ square unit, so $L = 1/2$.

EXERCISES 5.2

Use the techniques of Examples 1 and 2 (with subintervals of equal length) to find the areas of the regions specified in Exercises 1–13.

1. Below $y = 3x$, above $y = 0$, from $x = 0$ to $x = 1$.
2. Below $y = 2x + 1$, above $y = 0$, from $x = 0$ to $x = 3$.
3. Below $y = 2x - 1$, above $y = 0$, from $x = 1$ to $x = 3$.
4. Below $y = 3x + 4$, above $y = 0$, from $x = -1$ to $x = 2$.
5. Below $y = x^2$, above $y = 0$, from $x = 1$ to $x = 3$.
6. Below $y = x^2 + 1$, above $y = 0$, from $x = 0$ to $x = a > 0$.
7. Below $y = x^2 + 2x + 3$, above $y = 0$, from $x = -1$ to $x = 2$.
8. Above $y = x^2 - 1$, below $y = 0$.
9. Above $y = 1 - x$, below $y = 0$, from $x = 2$ to $x = 4$.
10. Above $y = x^2 - 2x$, below $y = 0$.
11. Below $y = 4x - x^2 + 1$, above $y = 1$.
12. Below $y = e^x$, above $y = 0$, from $x = 0$ to $x = b > 0$.
13. Below $y = 2^x$, above $y = 0$, from $x = -1$ to $x = 1$.
14. Use the formula $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$, from Exercises 39–41 of Section 5.1, to find the area of the region lying under $y = x^3$, above the x-axis, and between the vertical lines at $x = 0$ and $x = b > 0$.
15. Use the subdivision of $[a, b]$ given in Example 3 to find the area under $y = 1/x$, above $y = 0$, from $x = a > 0$ to $x = b > a$. Why should your answer not be surprising?

In Exercises 16–19, interpret the given sum S_n as a sum of areas of rectangles approximating the area of a certain region in the plane and hence evaluate $\lim_{n\to\infty} S_n$.

16. $S_n = \displaystyle\sum_{i=1}^{n} \frac{2}{n}\left(1 - \frac{i}{n}\right)$
17. $S_n = \displaystyle\sum_{i=1}^{n} \frac{2}{n}\left(1 - \frac{2i}{n}\right)$
18. $S_n = \displaystyle\sum_{i=1}^{n} \frac{2n + 3i}{n^2}$
19. $S_n = \displaystyle\sum_{j=1}^{n} \frac{1}{n}\sqrt{1 - (j/n)^2}$

5.3 The Definite Integral

In this section we generalize and make more precise the procedure used for finding areas developed in Section 5.2, and we use it to define the *definite integral* of a function f on an interval I. Let us assume, for the time being, that $f(x)$ is defined and continuous on the closed, finite interval $[a, b]$. We no longer assume that the values of f are nonnegative.

Partitions and Riemann Sums

Let P be a finite set of points arranged in order between a and b on the real line, say

$$P = \{x_0, x_1, x_2, x_3, \dots, x_{n-1}, x_n\},$$

where $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$. Such a set P is called a **partition** of $[a, b]$; it divides $[a, b]$ into n subintervals of which the ith is $[x_{i-1}, x_i]$. We call these the subintervals of the partition P. The number n depends on the particular partition, so we write $n = n(P)$. The length of the ith subinterval of P is

$$\Delta x_i = x_i - x_{i-1}, \qquad \text{(for } 1 \le i \le n\text{)},$$

and we call the greatest of these numbers Δx_i the **norm** of the partition P and denote it $\|P\|$:

$$\|P\| = \max_{1 \le i \le n} \Delta x_i.$$

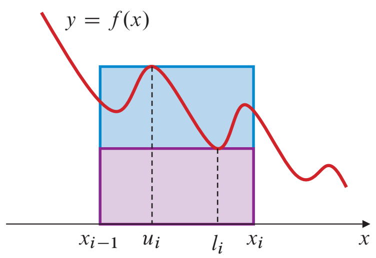


Figure 5.10

Since f is continuous on each subinterval $[x_{i-1}, x_i]$ of P, it takes on maximum and minimum values at points of that interval (by Theorem 8 of Section 1.4). Thus, there are numbers l_i and u_i in $[x_{i-1}, x_i]$ such that

$$f(l_i) \le f(x) \le f(u_i) \qquad \text{whenever } x_{i-1} \le x \le x_i.$$

If $f(x) \ge 0$ on $[a, b]$, then $f(l_i)\,\Delta x_i$ and $f(u_i)\,\Delta x_i$ represent the areas of rectangles having the interval $[x_{i-1}, x_i]$ on the x-axis as base, and having tops passing through the lowest and highest points, respectively, on the graph of f on that interval. (See Figure 5.10.) If A_i is that part of the area under $y = f(x)$ and above the x-axis that lies in the vertical strip between $x = x_{i-1}$ and $x = x_i$, then

$$f(l_i)\,\Delta x_i \le A_i \le f(u_i)\,\Delta x_i.$$

If f can have negative values, then one or both of $f(l_i)\,\Delta x_i$ and $f(u_i)\,\Delta x_i$ can be negative and will then represent the negative of the area of a rectangle lying below the x-axis. In any event, we always have $f(l_i)\,\Delta x_i \le f(u_i)\,\Delta x_i$.

DEFINITION 2

Upper and lower Riemann sums

The **lower (Riemann) sum**, $L(f, P)$, and the **upper (Riemann) sum**, $U(f, P)$, for the function f and the partition P are defined by:

$$\begin{aligned} L(f, P) &= f(l_1)\,\Delta x_1 + f(l_2)\,\Delta x_2 + \cdots + f(l_n)\,\Delta x_n \\ &= \sum_{i=1}^{n} f(l_i)\,\Delta x_i, \\ U(f, P) &= f(u_1)\,\Delta x_1 + f(u_2)\,\Delta x_2 + \cdots + f(u_n)\,\Delta x_n \\ &= \sum_{i=1}^{n} f(u_i)\Delta x_i. \end{aligned}$$

Figure 5.11 illustrates these Riemann sums as sums of *signed* areas of rectangles; any such areas that lie below the x-axis are counted as negative.

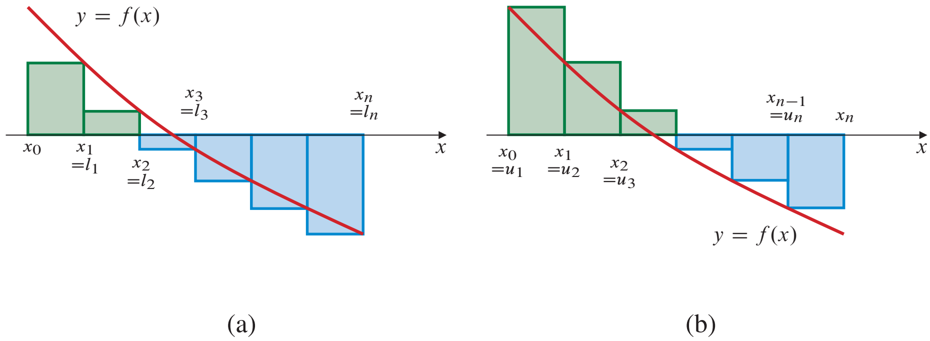


Figure 5.11 (a) A lower Riemann sum and (b) an upper Riemann sum for a decreasing function f. The areas of rectangles shaded in green are counted as positive; those shaded in blue are counted as negative

EXAMPLE 1 Calculate lower and upper Riemann sums for the function $f(x) = 1/x$ on the interval $[1, 2]$, corresponding to the partition P of $[1, 2]$ into four subintervals of equal length.

Solution The partition P consists of the points $x_0 = 1$, $x_1 = 5/4$, $x_2 = 3/2$, $x_3 = 7/4$, and $x_4 = 2$. Since $1/x$ is decreasing on $[1, 2]$, its minimum and maximum values on the ith subinterval $[x_{i-1}, x_i]$ are $1/x_i$ and $1/x_{i-1}$, respectively. Thus, the lower and upper Riemann sums are

$$L(f, P) = \frac{1}{4}\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2}\right) = \frac{533}{840} \approx 0.6345,$$

$$U(f, P) = \frac{1}{4}\left(1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right) = \frac{319}{420} \approx 0.7595.$$

EXAMPLE 2 Calculate the lower and upper Riemann sums for the function $f(x) = x^2$ on the interval $[0, a]$ (where $a > 0$), corresponding to the partition P_n of $[0, a]$ into n subintervals of equal length.

Solution Each subinterval of P_n has length $\Delta x = a/n$, and the division points are given by $x_i = ia/n$ for $i = 0, 1, 2, \ldots, n$. Since x^2 is increasing on $[0, a]$, its minimum and maximum values over the ith subinterval $[x_{i-1}, x_i]$ occur at $l_i = x_{i-1}$ and $u_i = x_i$, respectively. Thus, the lower Riemann sum of f for P_n is

$$\begin{aligned} L(f, P_n) &= \sum_{i=1}^{n} (x_{i-1})^2 \Delta x = \frac{a^3}{n^3} \sum_{i=1}^{n} (i-1)^2 \\ &= \frac{a^3}{n^3} \sum_{j=0}^{n-1} j^2 = \frac{a^3}{n^3}\, \frac{(n-1)n(2(n-1)+1)}{6} = \frac{(n-1)(2n-1)a^3}{6n^2}, \end{aligned}$$

where we have used Theorem 1(c) of Section 5.1 to evaluate the sum of squares. Similarly, the upper Riemann sum is

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} (x_i)^2 \Delta x \\ &= \frac{a^3}{n^3} \sum_{i=1}^{n} i^2 = \frac{a^3}{n^3}\, \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)a^3}{6n^2}. \end{aligned}$$

The Definite Integral

If we calculate $L(f, P)$ and $U(f, P)$ for partitions P having more and more points spaced closer and closer together, we expect that, in the limit, these Riemann sums will converge to a common value that will be the area bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ if $f(x) \ge 0$ on $[a, b]$. This is indeed the case, but we cannot fully prove it yet.

If P_1 and P_2 are two partitions of $[a, b]$ such that every point of P_1 also belongs to P_2, then we say that P_2 is a **refinement** of P_1. It is not difficult to show that in this case

$$L(f, P_1) \le L(f, P_2) \le U(f, P_2) \le U(f, P_1);$$

adding more points to a partition increases the lower sum and decreases the upper sum. (See Exercise 18 at the end of this section.) Given any two partitions, P_1 and P_2, we can form their **common refinement** P, which consists of all of the points of P_1 and P_2. Thus,

$$L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2).$$

Hence, every lower sum is less than or equal to every upper sum. Since the real numbers are complete, there must exist *at least one* real number I such that

$$L(f, P) \le I \le U(f, P) \qquad \text{for every partition } P.$$

If there is *only one* such number, we will call it the definite integral of f on $[a, b]$.

DEFINITION 3

The definite integral

Suppose there is exactly one number I such that for every partition P of $[a, b]$ we have

$$L(f, P) \le I \le U(f, P).$$

Then we say that the function f is **integrable** on $[a, b]$, and we call I the **definite integral** of f on $[a, b]$. The definite integral is denoted by the symbol

$$I = \int_a^b f(x)\,dx.$$

The definite integral of $f(x)$ over $[a, b]$ is a *number*; it is not a function of x. It depends on the numbers a and b and on the particular function f, but not on the variable x (which is a **dummy variable** like the variable i in the sum $\sum_{i=1}^{n} f(i)$). Replacing x with another variable does not change the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt.$$

While we normally write the definite integral of $f(x)$ as

$$\int_a^b f(x)\,dx,$$

it is equally correct to write it as

$$\int_a^b dx\, f(x).$$

This latter form will become quite useful when we deal with multiple integrals in Chapter 14.

The various parts of the symbol $\int_a^b f(x)\,dx$ have their own names:

(i) $\int$ is called the **integral sign**; it resembles the letter S since it represents the limit of a sum.

(ii) a and b are called the **limits of integration**; a is the **lower limit**, b is the **upper limit**.

(iii) The function f is the **integrand**; x is the **variable of integration**.

(iv) dx is the **differential** of x. It replaces Δx in the Riemann sums. If an integrand depends on more than one variable, the differential tells you which one is the variable of integration.

EXAMPLE 3 Show that $f(x) = x^2$ is integrable over the interval $[0, a]$, where $a > 0$, and evaluate $\int_0^a x^2\,dx$.

Solution We evaluate the limits as $n \to \infty$ of the lower and upper sums of f over $[0, a]$ obtained in Example 2 above.

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} \frac{(n-1)(2n-1)a^3}{6n^2} = \frac{a^3}{3},$$
$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \frac{(n+1)(2n+1)a^3}{6n^2} = \frac{a^3}{3}.$$

If $L(f, P_n) \le I \le U(f, P_n)$, we must have $I = a^3/3$. Thus, $f(x) = x^2$ is integrable over $[0, a]$, and

$$\int_0^a f(x)\,dx = \int_0^a x^2\,dx = \frac{a^3}{3}.$$

For all partitions P of $[a, b]$, we have

$$L(f, P) \le \int_a^b f(x)\,dx \le U(f, P).$$

If $f(x) \ge 0$ on $[a, b]$, then the area of the region R bounded by the graph of $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ is A square units, where $A = \int_a^b f(x)\,dx$. If $f(x) \le 0$ on $[a, b]$, the area of R is $-\int_a^b f(x)\,dx$ square units. For general f, $\int_a^b f(x)\,dx$ is the area of that part of R lying above the x-axis minus the area of that part lying below the x-axis. (See Figure 5.12.) You can think of $\int_a^b f(x)\,dx$ as a "sum" of "areas" of infinitely many rectangles with heights $f(x)$ and "infinitesimally small widths" dx; it is a limit of the upper and lower Riemann sums.

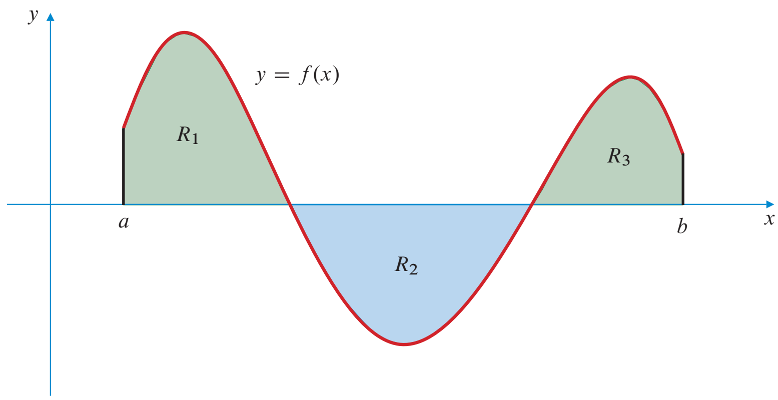


Figure 5.12 $\int_a^b f(x)\,dx$ equals area R_1 − area R_2 + area R_3

General Riemann Sums

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, be a partition of $[a, b]$ having norm $\|P\| = \max_{1\le i\le n} \Delta x_i$. In each subinterval $[x_{i-1}, x_i]$ of P, pick a point c_i (called a *tag*). Let $c = (c_1, c_2, \ldots, c_n)$ denote the set of these tags. The sum

$$R(f, P, c) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i = f(c_1)\,\Delta x_1 + f(c_2)\,\Delta x_2 + f(c_3)\,\Delta x_3 + \cdots + f(c_n)\,\Delta x_n$$

is called the **Riemann sum** of f on $[a, b]$ corresponding to partition P and tags c.

Note in Figure 5.13 that $R(f, P, c)$ is a sum of *signed* areas of rectangles between the x-axis and the curve $y = f(x)$. For any choice of the tags c, the Riemann sum $R(f, P, c)$ satisfies

$$L(f, P) \le R(f, P, c) \le U(f, P).$$

Therefore, if f is integrable on $[a,b]$, then its integral is the limit of such Riemann sums, where the limit is taken as the number $n(P)$ of subintervals of P increases to infinity in such a way that the lengths of all the subintervals approach zero. That is,

$$\lim_{\substack{n(P)\to\infty \\ \|P\|\to 0}} R(f,P,c) = \int_a^b f(x)\,dx.$$

As we will see in Chapter 7, many applications of integration depend on recognizing that a limit of Riemann sums is a definite integral.

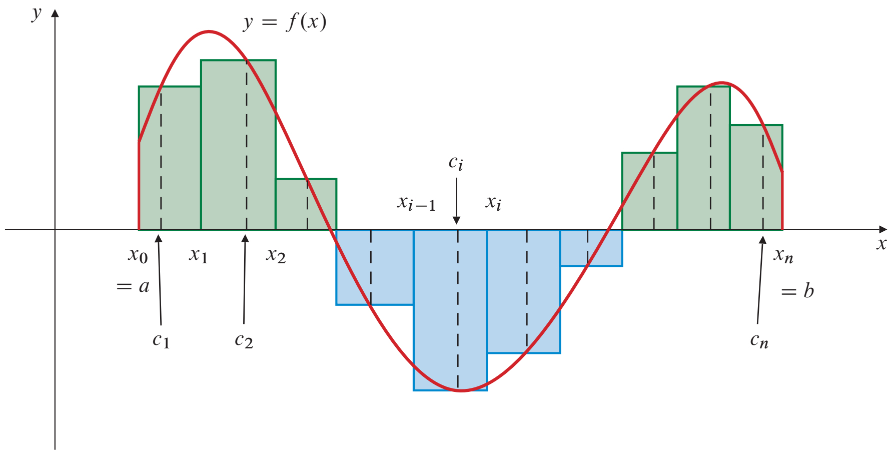


Figure 5.13 The Riemann sum $R(f,P,c)$ is the sum of areas of the rectangles shaded in green minus the sum of the areas of the rectangles shaded in blue

THEOREM 2

If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Remark The assumption that f is continuous in Theorem 2 may seem a bit superfluous since continuity was required throughout the above discussion leading to the definition of the definite integral. We cannot, however, prove this theorem yet. Its proof makes subtle use of the completeness property of the real numbers and is given in Appendix IV in the context of an extended definition of definite integral that is meaningful for a larger class of functions that are not necessarily continuous. (The integral studied in Appendix IV is called the **Riemann integral**.)

We can, however, make the following observation. In order to prove that f is integrable on $[a,b]$, it is sufficient that, for any given positive number ϵ, we should be able to find a partition P of $[a,b]$ for which $U(f,P) - L(f,P) < \epsilon$. This condition prevents there being more than one number I that is both greater than every lower sum and less than every upper sum. It is not difficult to find such a partition if the function f is nondecreasing (or if it is nonincreasing) on $[a,b]$. (See Exercise 17 at the end of this section.) Therefore, nondecreasing and nonincreasing continuous functions are integrable; so, therefore, is any continuous function that is the sum of a nondecreasing and a nonincreasing function. This class of functions includes any continuous functions we are likely to encounter in concrete applications of calculus but, unfortunately, does not include all continuous functions.

Meanwhile, in Sections 5.4 and 6.5 we will extend the definition of the definite integral to certain kinds of functions that are not continuous, or where the interval of integration is not closed or not bounded.

EXAMPLE 4 Express the limit $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(1+\frac{2i-1}{n}\right)^{1/3}$ as a definite integral.

Solution We want to interpret the sum as a Riemann sum for $f(x) = (1+x)^{1/3}$. The factor $2/n$ suggests that the interval of integration has length 2 and is partitioned into n equal subintervals, each of length $2/n$. Let $c_i = (2i-1)/n$ for $i = 1, 2, 3, \ldots, n$. As $n \to \infty$, $c_1 = 1/n \to 0$ and $c_n = (2n-1)/n \to 2$. Thus, the interval is $[0, 2]$, and the points of the partition are $x_i = 2i/n$. Observe that $x_{i-1} = (2i-2)/n < c_i < 2i/n = x_i$ for each i, so that the sum is indeed a Riemann sum for $f(x)$ over $[0, 2]$. Since f is continuous on that interval, it is integrable there, and

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(1 + \frac{2i-1}{n}\right)^{1/3} = \int_0^2 (1+x)^{1/3}\,dx.$$

EXERCISES 5.3

In Exercises 1–6, let P_n denote the partition of the given interval $[a, b]$ into n subintervals of equal length $\Delta x_i = (b-a)/n$. Evaluate $L(f, P_n)$ and $U(f, P_n)$ for the given functions f and the given values of n.

1. $f(x) = x$ on $[0, 2]$, with $n = 8$

2. $f(x) = x^2$ on $[0, 4]$, with $n = 4$

3. $f(x) = e^x$ on $[-2, 2]$, with $n = 4$

4. $f(x) = \ln x$ on $[1, 2]$, with $n = 5$

5. $f(x) = \sin x$ on $[0, \pi]$, with $n = 6$

6. $f(x) = \cos x$ on $[0, 2\pi]$, with $n = 4$

In Exercises 7–10, calculate $L(f, P_n)$ and $U(f, P_n)$ for the given function f over the given interval $[a, b]$, where P_n is the partition of the interval into n subintervals of equal length $\Delta x = (b-a)/n$. Show that

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n).$$

Hence, f is integrable on $[a, b]$. (Why?) What is $\int_a^b f(x)\,dx$?

7. $f(x) = x, \ [a, b] = [0, 1]$

8. $f(x) = 1 - x, \ [a, b] = [0, 2]$

9. $f(x) = x^3, \ [a, b] = [0, 1]$

10. $f(x) = e^x, \ [a, b] = [0, 3]$

In Exercises 11–16, express the given limit as a definite integral.

11. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\sqrt{\frac{i}{n}}$

12. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\sqrt{\frac{i-1}{n}}$

13. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n}\sin\left(\frac{\pi i}{n}\right)$

14. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\ln\left(1 + \frac{2i}{n}\right)$

15. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\tan^{-1}\left(\frac{2i-1}{2n}\right)$

16. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2 + i^2}$

17. If f is continuous and nondecreasing on $[a, b]$, and P_n is the partition of $[a, b]$ into n subintervals of equal length ($\Delta x_i = (b-a)/n$ for $1 \le i \le n$), show that

$$U(f, P_n) - L(f, P_n) = \frac{(b-a)\big(f(b) - f(a)\big)}{n}.$$

Since we can make the right side as small as we please by choosing n large enough, f must be integrable on $[a, b]$.

18. Let $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be a partition of $[a, b]$, and let P' be a refinement of P having one more point, x', satisfying, say, $x_{i-1} < x' < x_i$ for some i between 1 and n. Show that

$$L(f, P) \le L(f, P') \le U(f, P') \le U(f, P)$$

for any continuous function f. (*Hint:* Consider the maximum and minimum values of f on the intervals $[x_{i-1}, x_i]$, $[x_{i-1}, x']$, and $[x', x_i]$.) Hence, deduce that

$$L(f, P) \le L(f, P'') \le U(f, P'') \le U(f, P) \text{ if } P''$$

is *any* refinement of P.

5.4 Properties of the Definite Integral

It is convenient to extend the definition of the definite integral $\int_a^b f(x)\,dx$ to allow $a = b$ and $a > b$ as well as $a < b$. The extension still involves partitions P having $x_0 = a$ and $x_n = b$ with intermediate points occurring in order between these end points, so that if $a = b$, then we must have $\Delta x_i = 0$ for every i, and hence the integral is zero. If $a > b$, we have $\Delta x_i < 0$ for each i, so the integral will be negative for positive functions f and vice versa.

Some of the most important properties of the definite integral are summarized in the following theorem.

THEOREM 3

Let f and g be integrable on an interval containing the points a, b, and c. Then

(a) An integral over an interval of zero length is zero.

$$\int_a^a f(x)\,dx = 0.$$

(b) Reversing the limits of integration changes the sign of the integral.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

(c) An integral depends linearly on the integrand. If A and B are constants, then

$$\int_a^b \big(Af(x) + Bg(x)\big)\,dx = A\int_a^b f(x)\,dx + B\int_a^b g(x)\,dx.$$

(d) An integral depends additively on the interval of integration.

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

(e) If $a \le b$ and $f(x) \le g(x)$ for $a \le x \le b$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

(f) The **triangle inequality** for sums extends to definite integrals. If $a \le b$, then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

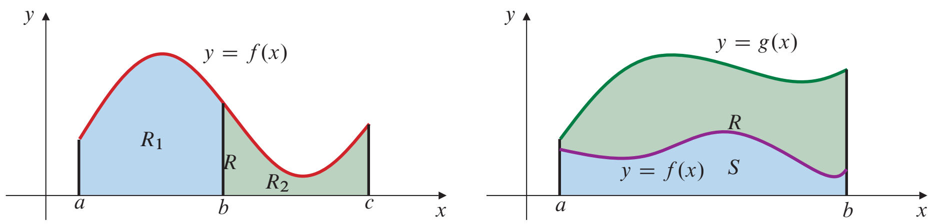
(g) The integral of an odd function over an interval symmetric about zero is zero. If f is an odd function (i.e., $f(-x) = -f(x)$), then

$$\int_{-a}^a f(x)\,dx = 0.$$

(h) The integral of an even function over an interval symmetric about zero is twice the integral over the positive half of the interval. If f is an even function (i.e., $f(-x) = f(x)$), then

$$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx.$$

The proofs of parts (a) and (b) are suggested in the first paragraph of this section. We postpone giving formal proofs of parts (c)–(h) until Appendix IV (see Exercises 5–8 in that Appendix). Nevertheless, all of these results should appear intuitively reasonable if you regard the integrals as representing (signed) areas. For instance, properties (d) and (e) are, respectively, properties (v) and (iv) of areas mentioned in the first paragraph of Section 5.2. (See Figure 5.14.)



area R_1 + area R_2 = area R

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

(a)

area $S \le$ area R

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

(b)

Figure 5.14

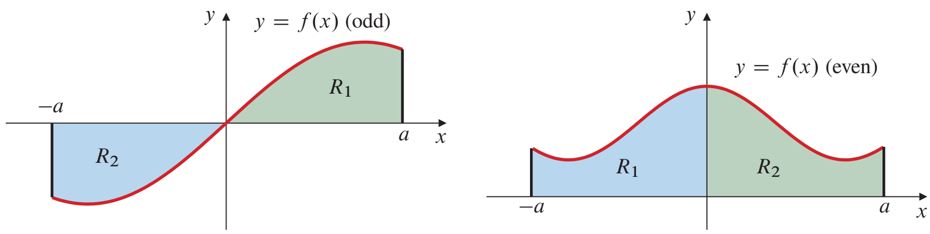
(a) Property (d) of Theorem 3

(b) Property (e) of Theorem 3

Property (f) is a generalization of the triangle inequality for numbers:

$$|x+y| \le |x| + |y|, \quad \text{or more generally,} \quad \left|\sum_{i=1}^{n} x_i\right| \le \sum_{i=1}^{n} |x_i|.$$

It follows from property (e) (assuming that $|f|$ is integrable on $[a,b]$), since $-|f(x)| \le f(x) \le |f(x)|$. The symmetry properties (g) and (h), which are illustrated in Figure 5.15, are particularly useful and should always be kept in mind when evaluating definite integrals because they can save much unnecessary work.



area R_1 − area R_2 = 0

$$\int_{-a}^{a} f(x)\,dx = 0$$

(a)

area R_1 + area R_2 = 2 × area R_2

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

(b)

Figure 5.15

(a) Property (g) of Theorem 3

(b) Property (h) of Theorem 3

As yet we have no easy method for evaluating definite integrals. However, some such integrals can be simplified by using various properties in Theorem 3, and others can be interpreted as known areas.

EXAMPLE 1 Evaluate

$$\text{(a)} \int_{-2}^{2} (2+5x)\,dx, \quad \text{(b)} \int_0^3 (2+x)\,dx, \quad \text{and} \quad \text{(c)} \int_{-3}^{3} \sqrt{9-x^2}\,dx.$$

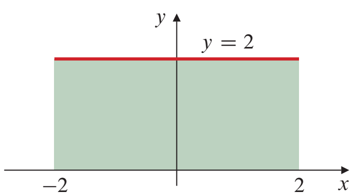


Figure 5.16

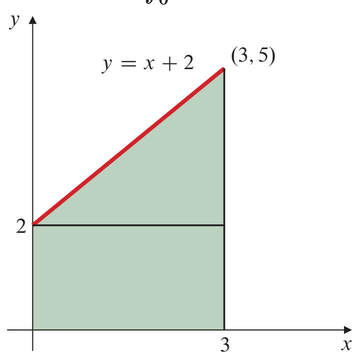


Figure 5.17

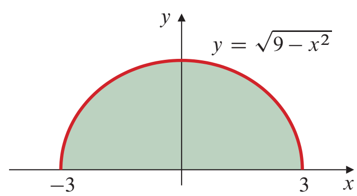


Figure 5.18

Solution See Figures 5.16–5.18.

(a) By the linearity property (c), $\int_{-2}^{2}(2+5x)\,dx = \int_{-2}^{2} 2\,dx + 5\int_{-2}^{2} x\,dx$. The first integral on the right represents the area of a rectangle of width 4 and height 2 (Figure 5.16), so it has value 8. The second integral on the right is 0 because its integrand is odd and the interval is symmetric about 0. Thus,

$$\int_{-2}^{2}(2+5x)\,dx = 8 + 0 = 8.$$

(b) $\int_{0}^{3}(2+x)\,dx$ represents the area of the trapezoid in Figure 5.17. Adding the areas of the rectangle and triangle comprising this trapezoid, we get

$$\int_{0}^{3}(2+x)\,dx = (3\times 2) + \frac{1}{2}(3\times 3) = \frac{21}{2}.$$

While areas are measured in squared units of length, definite integrals are numbers and have no units. Even when you use an area to find an integral, do not quote units for the integral.

(c) $\int_{-3}^{3}\sqrt{9-x^2}\,dx$ represents the area of a semicircle of radius 3 (Figure 5.18), so

$$\int_{-3}^{3}\sqrt{9-x^2}\,dx = \frac{1}{2}\pi(3^2) = \frac{9\pi}{2}.$$

A Mean-Value Theorem for Integrals

Let f be a function continuous on the interval $[a,b]$. Then f assumes a minimum value m and a maximum value M on the interval, say at points $x = l$ and $x = u$, respectively:

$$m = f(l) \le f(x) \le f(u) = M \qquad \text{for all } x \text{ in } [a,b].$$

For the 2-point partition P of $[a,b]$ having $x_0 = a$ and $x_1 = b$, we have

$$m(b-a) = L(f,P) \le \int_a^b f(x)\,dx \le U(f,P) = M(b-a).$$

Therefore,

$$f(l) = m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M = f(u).$$

By the Intermediate-Value Theorem, $f(x)$ must take on every value between the two values $f(l)$ and $f(u)$ at some point between l and u (Figure 5.19). Hence, there is a number c between l and u such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

That is, $\int_a^b f(x)\,dx$ is equal to the area $(b-a)f(c)$ of a rectangle with base width $b-a$ and height $f(c)$ for some c between a and b. This is the Mean-Value Theorem for integrals.

THEOREM 4

The Mean-Value Theorem for integrals

If f is continuous on $[a,b]$, then there exists a point c in $[a,b]$ such that

$$\int_a^b f(x)\,dx = (b-a)f(c).$$

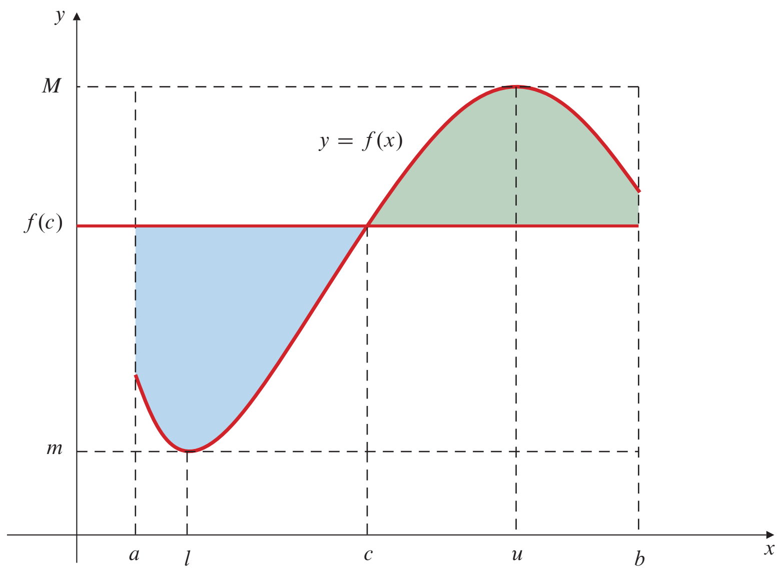


Figure 5.19 Half of the area between $y = f(x)$ and the horizontal line $y = f(c)$ lies above the line, and the other half lies below the line

Observe in Figure 5.19 that the area below the curve $y = f(x)$ and above the line $y = f(c)$ is equal to the area above $y = f(x)$ and below $y = f(c)$. In this sense, $f(c)$ is the average value of the function $f(x)$ on the interval $[a, b]$.

DEFINITION 4

Average value of a function

If f is integrable on $[a, b]$, then the **average value** or **mean value** of f on $[a, b]$, denoted by $\bar{f}$, is

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

EXAMPLE 2 Find the average value of $f(x) = 2x$ on the interval $[1, 5]$.

Solution The average value (see Figure 5.20) is

$$\bar{f} = \frac{1}{5-1} \int_1^5 2x\,dx = \frac{1}{4}\left(4 \times 2 + \frac{1}{2}(4 \times 8)\right) = 6.$$

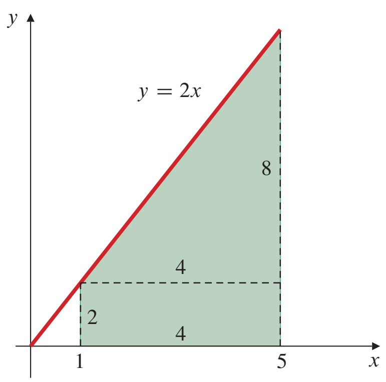


Figure 5.20 $\int_1^5 2x\,dx = 24$

Definite Integrals of Piecewise Continuous Functions

The definition of integrability and the definite integral given above can be extended to a wider class than just continuous functions. One simple but very important extension is to the class of *piecewise continuous functions*.

Consider the graph $y = f(x)$ shown in Figure 5.21(a). Although f is not continuous at all points in $[a, b]$ (it is discontinuous at c_1 and c_2), clearly the region lying under the graph and above the x-axis between $x = a$ and $x = b$ does have an area. We would like to represent this area as

$$\int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{b} f(x)\,dx.$$

This is reasonable because there are continuous functions on $[a, c_1]$, $[c_1, c_2]$, and $[c_2, b]$ equal to $f(x)$ on the corresponding open intervals, (a, c_1), (c_1, c_2), and (c_2, b).

DEFINITION 5

Piecewise continuous functions

Let $c_0 < c_1 < c_2 < \cdots < c_n$ be a finite set of points on the real line. A function f defined on $[c_0, c_n]$ except possibly at some of the points c_i, $(0 \le i \le n)$, is called **piecewise continuous** on that interval if for each i $(1 \le i \le n)$ there exists a function F_i continuous on the *closed* interval $[c_{i-1}, c_i]$ such that

$$f(x) = F_i(x) \qquad \text{on the } \textit{open} \text{ interval} \quad (c_{i-1}, c_i).$$

In this case, we define the definite integral of f from c_0 to c_n to be

$$\int_{c_0}^{c_n} f(x)\,dx = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} F_i(x)\,dx.$$

EXAMPLE 3 Find $\displaystyle\int_0^3 f(x)\,dx$, where $f(x) = \begin{cases} \sqrt{1-x^2} & \text{if } 0 \le x \le 1 \\ 2 & \text{if } 1 < x \le 2 \\ x-2 & \text{if } 2 < x \le 3. \end{cases}$

Solution The value of the integral is the sum of the shaded areas in Figure 5.21(b):

$$\begin{aligned}\int_0^3 f(x)\,dx &= \int_0^1 \sqrt{1-x^2}\,dx + \int_1^2 2\,dx + \int_2^3 (x-2)\,dx \\ &= \left(\frac{1}{4} \times \pi \times 1^2\right) + (2 \times 1) + \left(\frac{1}{2} \times 1 \times 1\right) = \frac{\pi + 10}{4}.\end{aligned}$$

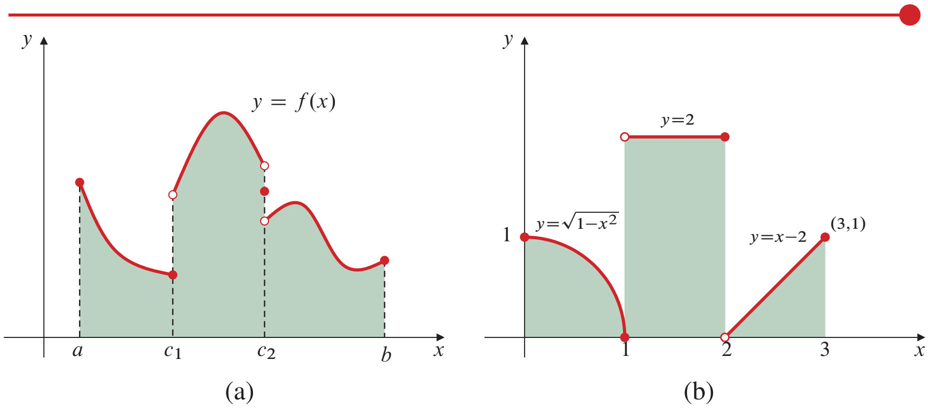


Figure 5.21 Two piecewise continuous functions

EXERCISES 5.4

1. Simplify $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx + \int_c^a f(x)\,dx.$

2. Simplify $\displaystyle\int_0^2 3f(x)\,dx + \int_1^3 3f(x)\,dx - \int_0^3 2f(x)\,dx - \int_1^2 3f(x)\,dx.$

Evaluate the integrals in Exercises 3–16 by using the properties of the definite integral and interpreting integrals as areas.

3. $\displaystyle\int_{-2}^{2} (x+2)\,dx$

4. $\displaystyle\int_0^2 (3x+1)\,dx$

5. $\displaystyle\int_a^b x\,dx$

6. $\displaystyle\int_{-1}^{2} (1-2x)\,dx$

7. $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-t^2}\,dt$

8. $\displaystyle\int_{-\sqrt{2}}^{0} \sqrt{2-x^2}\,dx$

9. $\displaystyle\int_{-\pi}^{\pi} \sin(x^3)\,dx$

10. $\displaystyle\int_{-a}^{a} (a-|s|)\,ds$

11. $\displaystyle\int_{-1}^{1} (u^5 - 3u^3 + \pi)\,du$

12. $\displaystyle\int_0^2 \sqrt{2x-x^2}\,dx$

13. $\displaystyle\int_{-4}^{4} (e^x - e^{-x})\,dx$

14. $\displaystyle\int_{-3}^{3} (2+t)\sqrt{9-t^2}\,dt$

15. $\displaystyle\int_0^1 \sqrt{4-x^2}\,dx$

16. $\displaystyle\int_1^2 \sqrt{4-x^2}\,dx$

Given that $\int_0^a x^2\,dx = \frac{a^3}{3}$, evaluate the integrals in Exercises 17–22.

17. $\int_0^2 6x^2\,dx$

18. $\int_2^3 (x^2-4)\,dx$

19. $\int_{-2}^2 (4-t^2)\,dt$

20. $\int_0^2 (v^2-v)\,dv$

21. $\int_0^1 (x^2+\sqrt{1-x^2})\,dx$

22. $\int_{-6}^6 x^2(2+\sin x)\,dx$

The definition of $\ln x$ as an area in Section 3.3 implies that

$$\int_1^x \frac{1}{t}\,dt = \ln x$$

for $x > 0$. Use this to evaluate the integrals in Exercises 23–26.

23. $\int_1^2 \frac{1}{x}\,dx$

24. $\int_2^4 \frac{1}{t}\,dt$

25. $\int_{1/3}^1 \frac{1}{t}\,dt$

26. $\int_{1/4}^3 \frac{1}{s}\,ds$

Find the average values of the functions in Exercises 27–32 over the given intervals.

27. $f(x) = x+2$ over $[0,4]$

28. $g(x) = x+2$ over $[a,b]$

29. $f(t) = 1+\sin t$ over $[-\pi,\pi]$

30. $k(x) = x^2$ over $[0,3]$

31. $f(x) = \sqrt{4-x^2}$ over $[0,2]$

32. $g(s) = 1/s$ over $[1/2,2]$

Piecewise continuous functions

33. Evaluate $\int_{-1}^2 \operatorname{sgn} x\,dx$. Recall that $\operatorname{sgn} x$ is 1 if $x>0$ and -1 if $x<0$.

34. Find $\int_{-3}^2 f(x)\,dx$, where $f(x) = \begin{cases} 1+x & \text{if } x<0 \\ 2 & \text{if } x \ge 0. \end{cases}$

35. Find $\int_0^2 g(x)\,dx$, where $g(x) = \begin{cases} x^2 & \text{if } 0 \le x \le 1 \\ x & \text{if } 1 < x \le 2. \end{cases}$

36. Evaluate $\int_0^3 |2-x|\,dx$.

37. Evaluate $\int_0^2 \sqrt{4-x^2}\,\operatorname{sgn}(x-1)\,dx$.

38. Evaluate $\int_0^{3.5} \lfloor x \rfloor\,dx$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. (See Example 10 of Section P.5.)

Evaluate the integrals in Exercises 39–40 by inspecting the graphs of the integrands.

39. $\int_{-3}^4 \big(|x+1|-|x-1|+|x+2|\big)\,dx$

40. $\int_0^3 \frac{x^2-x}{|x-1|}\,dx$

41. Find the average value of the function $f(x) = |x+1|\operatorname{sgn} x$ on the interval $[-2,2]$.

42. If $a<b$ and f is continuous on $[a,b]$, show that $\int_a^b \big(f(x)-\bar{f}\big)\,dx = 0$.

43. Suppose that $a<b$ and f is continuous on $[a,b]$. Find the constant k that minimizes the integral $\int_a^b \big(f(x)-k\big)^2\,dx$.

5.5 The Fundamental Theorem of Calculus

In this section we demonstrate the relationship between the definite integral defined in Section 5.3 and the indefinite integral (or general antiderivative) introduced in Section 2.10. A consequence of this relationship is that we will be able to calculate definite integrals of functions whose antiderivatives we can find.

In Section 3.3 we wanted to find a function whose derivative was $1/x$. We solved this problem by defining the desired function ($\ln x$) in terms of the area under the graph of $y = 1/x$. This idea motivates, and is a special case of, the following theorem.

THEOREM 5

The Fundamental Theorem of Calculus

Suppose that the function f is continuous on an interval I containing the point a.

PART I. Let the function F be defined on I by

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is differentiable on I, and $F'(x) = f(x)$ there. Thus, F is an antiderivative of f on I:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

PART II. If $G(x)$ is *any* antiderivative of $f(x)$ on I, so that $G'(x) = f(x)$ on I, then for any b in I, we have

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

PROOF Using the definition of the derivative, we calculate

$$\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)\\
&= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt && \text{by Theorem 3(d)}\\
&= \lim_{h\to 0}\frac{1}{h}hf(c) && \text{for some } c = c(h) \text{ (depending on } h\text{)}\\
& && \text{between } x \text{ and } x+h \text{ (Theorem 4)}\\
&= \lim_{c\to x} f(c) && \text{since } c \to x \text{ as } h \to 0\\
&= f(x) && \text{since } f \text{ is continuous.}
\end{aligned}$$

Also, if $G'(x) = f(x)$, then $F(x) = G(x)+C$ on I for some constant C (by Theorem 13 of Section 2.8). Hence,

$$\int_a^x f(t)\,dt = F(x) = G(x) + C.$$

Let $x = a$ and obtain $0 = G(a) + C$ via Theorem 3(a), so $C = -G(a)$. Now let $x = b$ to get

$$\int_a^b f(t)\,dt = G(b) + C = G(b) - G(a).$$

Of course, we can replace t with x (or any other variable) as the variable of integration on the left-hand side.

Remark You should remember *both* conclusions of the Fundamental Theorem; they are both useful. Part I concerns the derivative of an integral; it tells you how to differentiate a definite integral with respect to its upper limit. Part II concerns the integral of a derivative; it tells you how to evaluate a definite integral if you can find an antiderivative of the integrand.

DEFINITION 6

To facilitate the evaluation of definite integrals using the Fundamental Theorem of Calculus, we define the **evaluation symbol**:

$$F(x)\Big|_a^b = F(b) - F(a).$$

Thus,

$$\int_a^b f(x)\,dx = \left(\int f(x)\,dx\right)\bigg|_a^b,$$

where $\int f(x)\,dx$ denotes the indefinite integral or general antiderivative of f. (See Section 2.10.) When evaluating a definite integral this way, we will omit the constant of integration $(+C)$ from the indefinite integral because it cancels out in the subtraction:

$$(F(x)+C)\bigg|_a^b = F(b)+C-(F(a)+C) = F(b)-F(a) = F(x)\bigg|_a^b.$$

Any antiderivative of f can be used to calculate the definite integral.

EXAMPLE 1 Evaluate (a) $\displaystyle\int_0^a x^2\,dx$ and (b) $\displaystyle\int_{-1}^2 (x^2-3x+2)\,dx$.

Solution

(a) $\displaystyle\int_0^a x^2\,dx = \frac{1}{3}x^3\bigg|_0^a = \frac{1}{3}a^3 - \frac{1}{3}0^3 = \frac{a^3}{3}$ (because $\displaystyle\frac{d}{dx}\frac{x^3}{3} = x^2$).

(b) $$\int_{-1}^2 (x^2-3x+2)\,dx = \left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right)\bigg|_{-1}^2$$
$$= \frac{1}{3}(8) - \frac{3}{2}(4) + 4 - \left(\frac{1}{3}(-1) - \frac{3}{2}(1) + (-2)\right) = \frac{9}{2}.$$

BEWARE! Be careful to keep track of all the minus signs when substituting a negative lower limit.

EXAMPLE 2 Find the area A of the plane region lying above the x-axis and under the curve $y = 3x - x^2$.

Solution We need to find the points where the curve $y = 3x - x^2$ meets the x-axis. These are solutions of the equation

$$0 = 3x - x^2 = x(3-x).$$

The only roots are $x = 0$ and $x = 3$. (See Figure 5.22.) Hence, the area of the region is given by

$$A = \int_0^3 (3x - x^2)\,dx = \left(\frac{3}{2}x^2 - \frac{1}{3}x^3\right)\bigg|_0^3$$
$$= \frac{27}{2} - \frac{27}{3} - (0-0) = \frac{27}{6} = \frac{9}{2} \text{ square units.}$$

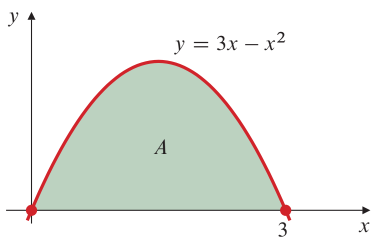


Figure 5.22

EXAMPLE 3 Find the area under the curve $y = \sin x$, above $y = 0$, from $x = 0$ to $x = \pi$.

Solution The required area, illustrated in Figure 5.23, is

$$A = \int_0^\pi \sin x\,dx = -\cos x\bigg|_0^\pi = -\big(-1-(1)\big) = 2 \text{ square units.}$$

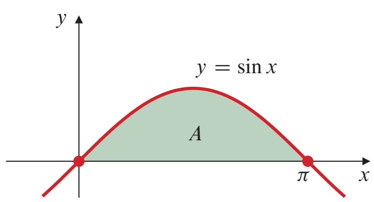


Figure 5.23

Note that while the definite integral is a pure number, an area is a geometric quantity that implicitly involves units. If the units along the x- and y-axes are, for example, metres, the area should be quoted in square metres (m^2). If units of length along the x-axis and y-axis are not specified, areas should be quoted in square units.

EXAMPLE 4 Find the area of the region R lying above the line $y = 1$ and below the curve $y = 5/(x^2 + 1)$.

Solution The region R is shaded in Figure 5.24. To find the intersections of $y = 1$ and $y = 5/(x^2 + 1)$, we must solve these equations simultaneously:

$$1 = \frac{5}{x^2 + 1},$$

so $x^2 + 1 = 5$, $x^2 = 4$, and $x = \pm 2$.

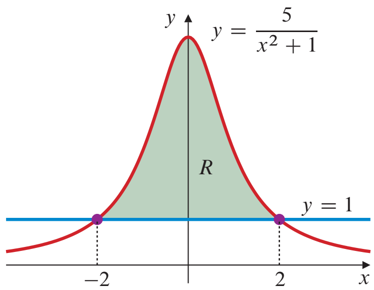


Figure 5.24

The area A of the region R is the area under the curve $y = 5/(x^2 + 1)$ and above the x-axis between $x = -2$ and $x = 2$, minus the area of a rectangle of width 4 and height 1. Since $\tan^{-1} x$ is an antiderivative of $1/(x^2 + 1)$,

$$\begin{aligned} A &= \int_{-2}^{2} \frac{5}{x^2+1}\,dx - 4 = 2\int_0^2 \frac{5}{x^2+1}\,dx - 4 \\ &= 10 \tan^{-1} x \Big|_0^2 - 4 = 10 \tan^{-1} 2 - 4 \text{ square units.} \end{aligned}$$

Observe the use of even symmetry (Theorem 3(h) of Section 5.4) to replace the lower limit of integration by 0. It is easier to substitute 0 into the antiderivative than -2.

EXAMPLE 5 Find the average value of $f(x) = e^{-x} + \cos x$ on the interval $[-\pi/2, 0]$.

Solution The average value is

$$\begin{aligned} \bar{f} &= \frac{1}{0 - \left(-\frac{\pi}{2}\right)} \int_{-(\pi/2)}^{0} (e^{-x} + \cos x)\,dx \\ &= \frac{2}{\pi}(-e^{-x} + \sin x)\Big|_{-(\pi/2)}^{0} \\ &= \frac{2}{\pi}\Big(-1 + 0 + e^{\pi/2} - (-1)\Big) = \frac{2}{\pi} e^{\pi/2}. \end{aligned}$$

Beware of integrals of the form $\int_a^b f(x)\,dx$ where f is not continuous at *all* points in the interval $[a, b]$. The Fundamental Theorem does not apply in such cases.

EXAMPLE 6 We know that $\dfrac{d}{dx} \ln|x| = \dfrac{1}{x}$ if $x \neq 0$. It is *incorrect*, however, to state that

$$\int_{-1}^{1} \frac{dx}{x} = \ln|x|\Big|_{-1}^{1} = 0 - 0 = 0,$$

even though $1/x$ is an odd function. In fact, $1/x$ is undefined and has no limit at $x = 0$, and it is not integrable on $[-1, 0]$ or $[0, 1]$ (Figure 5.25). Observe that

$$\lim_{c\to 0+} \int_c^1 \frac{1}{x}\,dx = \lim_{c\to 0+} -\ln c = \infty,$$

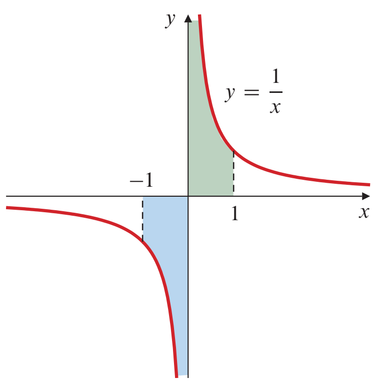


Figure 5.25

so both shaded regions in Figure 5.25 have infinite area. Integrals of this type are called **improper integrals**. We deal with them in Section 6.5.

The following example illustrates, this time using definite integrals, the relationship observed in Example 1 of Section 2.11 between the area under the graph of its velocity and the distance travelled by an object over a time interval.

EXAMPLE 7 An object at rest at time $t = 0$ accelerates at a constant 10 m/s^2 during the time interval $[0, T]$. If $0 \le t_0 \le t_1 \le T$, find the distance travelled by the object in the time interval $[t_0, t_1]$.

Solution Let $v(t)$ denote the velocity of the object at time t, and let $y(t)$ denote the distance travelled by the object during the time interval $[0, t]$, where $0 \le t \le T$. Then $v(0) = 0$ and $y(0) = 0$. Also $v'(t) = 10$ and $y'(t) = v(t)$. Thus,

$$v(t) = v(t) - v(0) = \int_0^t v'(u)\, du = \int_0^t 10\, du = 10u\Big|_0^t = 10t$$

$$y(t) = y(t) - y(0) = \int_0^t y'(u)\, du = \int_0^t v(u)\, du = \int_0^t 10u\, du = 5u^2\Big|_0^t = 5t^2.$$

On the time interval $[t_0, t_1]$, the object has travelled distance

$$y(t_1) - y(t_0) = 5t_1^2 - 5t_0^2 = \int_0^{t_1} v(t)\, dt - \int_0^{t_0} v(t)\, dt = \int_{t_0}^{t_1} v(t)\, dt \text{ m.}$$

Observe that this last integral is the area under the graph of $y = v(t)$ above the interval $[t_0, t_1]$ on the t axis.

We now give some examples illustrating the first conclusion of the Fundamental Theorem.

EXAMPLE 8 Find the derivatives of the following functions:

(a) $F(x) = \int_x^3 e^{-t^2}\, dt$, (b) $G(x) = x^2 \int_{-4}^{5x} e^{-t^2}\, dt$, (c) $H(x) = \int_{x^2}^{x^3} e^{-t^2}\, dt$.

Solution The solutions involve applying the first conclusion of the Fundamental Theorem together with other differentiation rules.

(a) Observe that $F(x) = -\int_3^x e^{-t^2}\, dt$ (by Theorem 3(b)). Therefore, by the Fundamental Theorem, $F'(x) = -e^{-x^2}$.

(b) By the Product Rule and the Chain Rule,

$$\begin{aligned} G'(x) &= 2x \int_{-4}^{5x} e^{-t^2}\, dt + x^2 \frac{d}{dx} \int_{-4}^{5x} e^{-t^2}\, dt \\ &= 2x \int_{-4}^{5x} e^{-t^2}\, dt + x^2 e^{-(5x)^2}(5) \\ &= 2x \int_{-4}^{5x} e^{-t^2}\, dt + 5x^2 e^{-25x^2}. \end{aligned}$$

(c) Split the integral into a difference of two integrals in each of which the variable x appears only in the upper limit:

$$\begin{aligned} H(x) &= \int_0^{x^3} e^{-t^2}\, dt - \int_0^{x^2} e^{-t^2}\, dt \\ H'(x) &= e^{-(x^3)^2}(3x^2) - e^{-(x^2)^2}(2x) \\ &= 3x^2 e^{-x^6} - 2x\, e^{-x^4}. \end{aligned}$$

Parts (b) and (c) of Example 8 are examples of the following formulas that build the Chain Rule into the first conclusion of the Fundamental Theorem:

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f\big(g(x)\big)\,g'(x)$$
$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = f\big(g(x)\big)\,g'(x) - f\big(h(x)\big)\,h'(x)$$

EXAMPLE 9 Solve the **integral equation** $f(x) = 2 + 3\int_4^x f(t)\,dt$.

Solution Differentiate the integral equation to get $f'(x) = 3f(x)$, the DE for exponential growth, having solution $f(x) = Ce^{3x}$. Now put $x = 4$ into the integral equation to get $f(4) = 2$. Hence $2 = Ce^{12}$, so $C = 2e^{-12}$. Therefore, the integral equation has solution $f(x) = 2e^{3x-12}$.

We conclude with an example showing how the Fundamental Theorem can be used to evaluate limits of Riemann sums.

EXAMPLE 10 Evaluate $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \cos\left(\frac{j\pi}{2n}\right)$.

Solution The sum involves values of $\cos x$ at the right endpoints of the n subintervals of the partition

$$0,\quad \frac{\pi}{2n},\quad \frac{2\pi}{2n},\quad \frac{3\pi}{2n},\quad \ldots,\quad \frac{n\pi}{2n}$$

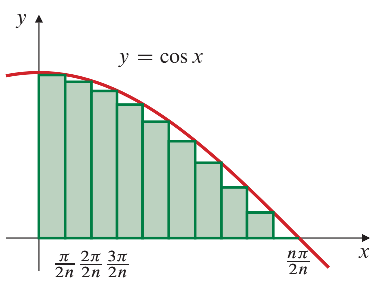


Figure 5.26

of the interval $[0, \pi/2]$. Since each of the subintervals of this partition has length $\pi/(2n)$, and since $\cos x$ is continuous on $[0, \pi/2]$, we have, expressing the limit of a Riemann sum as an integral (see Figure 5.26),

$$\lim_{n\to\infty} \frac{\pi}{2n}\sum_{j=1}^{n} \cos\left(\frac{j\pi}{2n}\right) = \int_0^{\pi/2} \cos x\,dx = \sin x\bigg|_0^{\pi/2} = 1 - 0 = 1.$$

The given sum differs from the Riemann sum above only in that the factor $\pi/2$ is missing. Thus,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \cos\left(\frac{j\pi}{2n}\right) = \frac{2}{\pi}.$$

EXERCISES 5.5

Evaluate the definite integrals in Exercises 1–20.

1. $\int_0^2 x^3\,dx$
2. $\int_0^4 \sqrt{x}\,dx$
3. $\int_{1/2}^1 \frac{1}{x^2}\,dx$
4. $\int_{-2}^{-1} \left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx$
5. $\int_{-1}^2 (3x^2 - 4x + 2)\,dx$
6. $\int_1^2 \left(\frac{2}{x^3} - \frac{x^3}{2}\right) dx$
7. $\int_{-2}^2 (x^2 + 3)^2\,dx$
8. $\int_4^9 \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$
9. $\int_{-\pi/4}^{-\pi/6} \cos x\,dx$
10. $\int_0^{\pi/3} \sec^2\theta\,d\theta$
11. $\int_{\pi/4}^{\pi/3} \sin\theta\,d\theta$
12. $\int_0^{2\pi} (1 + \sin u)\,du$
13. $\int_{-\pi}^{\pi} e^x\,dx$
14. $\int_{-2}^2 (e^x - e^{-x})\,dx$

15. $\displaystyle\int_0^e a^x\,dx \quad (a>0)$

16. $\displaystyle\int_{-1}^1 2^x\,dx$

17. $\displaystyle\int_{-1}^1 \frac{dx}{1+x^2}$

18. $\displaystyle\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$

19. $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt{4-x^2}}$

20. $\displaystyle\int_{-2}^0 \frac{dx}{4+x^2}$

Find the area of the region R specified in Exercises 21–32. It is helpful to make a sketch of the region.

21. Bounded by $y=x^4$, $y=0$, $x=0$, and $x=1$

22. Bounded by $y=1/x$, $y=0$, $x=e$, and $x=e^2$

23. Above $y=x^2-4x$ and below the x-axis

24. Bounded by $y=5-2x-3x^2$, $y=0$, $x=-1$, and $x=1$

25. Bounded by $y=x^2-3x+3$ and $y=1$

26. Below $y=\sqrt{x}$ and above $y=\dfrac{x}{2}$

27. Above $y=x^2$ and to the right of $x=y^2$

28. Above $y=|x|$ and below $y=12-x^2$

29. Bounded by $y=x^{1/3}-x^{1/2}$, $y=0$, $x=0$, and $x=1$

30. Under $y=e^{-x}$ and above $y=0$ from $x=-a$ to $x=0$

31. Below $y=1-\cos x$ and above $y=0$ between two consecutive intersections of these graphs

32. Below $y=x^{-1/3}$ and above $y=0$ from $x=1$ to $x=27$

Find the integrals of the piecewise continuous functions in Exercises 33–34.

33. $\displaystyle\int_0^{3\pi/2} |\cos x|\,dx$

34. $\displaystyle\int_1^3 \frac{\operatorname{sgn}(x-2)}{x^2}\,dx$

In Exercises 35–38, find the average values of the given functions over the intervals specified.

35. $f(x)=1+x+x^2+x^3$ over $[0,2]$

36. $f(x)=e^{3x}$ over $[-2,2]$

37. $f(x)=2^x$ over $[0,1/\ln 2]$

38. $g(t)=\begin{cases}0 & \text{if } 0\le t\le 1\\ 1 & \text{if } 1<t\le 3\end{cases}$ over $[0,3]$

Find the indicated derivatives in Exercises 39–46.

39. $\displaystyle\frac{d}{dx}\int_2^x \frac{\sin t}{t}\,dt$

40. $\displaystyle\frac{d}{dt}\int_t^3 \frac{\sin x}{x}\,dx$

41. $\displaystyle\frac{d}{dx}\int_{x^2}^0 \frac{\sin t}{t}\,dt$

42. $\displaystyle\frac{d}{dx}x^2\int_0^{x^2} \frac{\sin u}{u}\,du$

43. $\displaystyle\frac{d}{dt}\int_{-\pi}^t \frac{\cos y}{1+y^2}\,dy$

44. $\displaystyle\frac{d}{d\theta}\int_{\sin\theta}^{\cos\theta} \frac{1}{1-x^2}\,dx$

45. $\displaystyle\frac{d}{dx}F(\sqrt{x})$, if $\displaystyle F(t)=\int_0^t \cos(x^2)\,dx$

46. $H'(2)$, if $\displaystyle H(x)=3x\int_4^{x^2} e^{-\sqrt{t}}\,dt$

47. Solve the integral equation $\displaystyle f(x)=\pi\left(1+\int_1^x f(t)\,dt\right)$.

48. Solve the integral equation $\displaystyle f(x)=1-\int_0^x f(t)\,dt$.

49. Criticize the following erroneous calculation:

$$\int_{-1}^1 \frac{dx}{x^2} = -\frac{1}{x}\bigg|_{-1}^{1} = -1+\frac{1}{-1} = -2.$$

Exactly where did the error occur? Why is -2 an unreasonable value for the integral?

50. Use a definite integral to define a function $F(x)$ having derivative $\dfrac{\sin x}{1+x^2}$ for all x and satisfying $F(17)=0$.

51. Does the function $\displaystyle F(x)=\int_0^{2x-x^2}\cos\left(\frac{1}{1+t^2}\right)dt$ have a maximum or a minimum value? Justify your answer.

Evaluate the limits in Exercises 52–54.

52. $\displaystyle\lim_{n\to\infty}\frac{1}{n}\left(\left(1+\frac{1}{n}\right)^5+\left(1+\frac{2}{n}\right)^5+\cdots+\left(1+\frac{n}{n}\right)^5\right)$

53. $\displaystyle\lim_{n\to\infty}\frac{\pi}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\cdots+\sin\frac{n\pi}{n}\right)$

54. $\displaystyle\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+4}+\frac{n}{n^2+9}+\cdots+\frac{n}{2n^2}\right)$

5.6 The Method of Substitution

As we have seen, the evaluation of definite integrals is most easily carried out if we can antidifferentiate the integrand. In this section and Sections 6.1–6.4 we develop some *techniques of integration*, that is, methods for finding antiderivatives of functions. Although the techniques we develop can be used for a large class of functions, they will not work for all functions we might want to integrate. If a definite integral involves an integrand whose antiderivative is either impossible or very difficult to find, we may wish, instead, to approximate the definite integral by numerical means. Techniques for doing that will be presented in Sections 6.6–6.8.

Let us begin by assembling a table of some known indefinite integrals. These results have all emerged during our development of differentiation formulas for elementary functions. You should *memorize* them.

Some elementary integrals

1.	$\int 1\,dx = x + C$	**2.**	$\int x\,dx = \frac{1}{2}x^2 + C$
3.	$\int x^2\,dx = \frac{1}{3}x^3 + C$	**4.**	$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$
5.	$\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$	**6.**	$\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C$
7.	$\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C \quad (r \neq -1)$	**8.**	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
9.	$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$	**10.**	$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$
11.	$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$	**12.**	$\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$
13.	$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax + C$	**14.**	$\int \csc ax\cot ax\,dx = -\frac{1}{a}\csc ax + C$
15.	$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C \quad (a > 0)$	**16.**	$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
17.	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$	**18.**	$\int b^{ax}\,dx = \frac{1}{a\ln b}b^{ax} + C$
19.	$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$	**20.**	$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$

Note that formulas 1–6 are special cases of formula 7, which holds on any interval where x^r makes sense. The linearity formula

$$\int (A\,f(x) + B\,g(x))\,dx = A\int f(x)\,dx + B\int g(x)\,dx$$

makes it possible to integrate sums and constant multiples of functions.

EXAMPLE 1 **(Combining elementary integrals)**

(a) $\int (x^4 - 3x^3 + 8x^2 - 6x - 7)\,dx = \frac{x^5}{5} - \frac{3x^4}{4} + \frac{8x^3}{3} - 3x^2 - 7x + C$

(b) $\int \left(5x^{3/5} - \frac{3}{2 + x^2}\right)\,dx = \frac{25}{8}x^{8/5} - \frac{3}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$

(c) $\int (4\cos 5x - 5\sin 3x)\,dx = \frac{4}{5}\sin 5x + \frac{5}{3}\cos 3x + C$

(d) $\int \left(\frac{1}{\pi x} + a^{\pi x}\right)\,dx = \frac{1}{\pi}\ln\lvert x\rvert + \frac{1}{\pi\ln a}a^{\pi x} + C, \quad (a > 0).$

Sometimes it is necessary to manipulate an integrand so that the method can be applied.

EXAMPLE 2

$$\begin{aligned}\int \frac{(x+1)^3}{x}\,dx &= \int \frac{x^3 + 3x^2 + 3x + 1}{x}\,dx \\ &= \int \left(x^2 + 3x + 3 + \frac{1}{x}\right)\,dx \\ &= \frac{1}{3}x^3 + \frac{3}{2}x^2 + 3x + \ln\lvert x\rvert + C.\end{aligned}$$

When an integral cannot be evaluated by inspection, as those in Examples 1–2 can, we require one or more special techniques. The most important of these techniques is the **method of substitution**, the integral version of the Chain Rule. If we rewrite the Chain Rule, $\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big)\, g'(x)$, in integral form, we obtain

$$\int f'\big(g(x)\big)\, g'(x)\, dx = f\big(g(x)\big) + C.$$

Observe that the following formalism would produce this latter formula even if we did not already know it was true:

Let $u = g(x)$. Then $du/dx = g'(x)$, or in differential form, $du = g'(x)\, dx$. Thus,

$$\int f'\,(g(x))\; g'(x)\, dx = \int f'(u)\, du = f(u) + C = f\,(g(x)) + C.$$

EXAMPLE 3 **(Examples of substitution)** Find the indefinite integrals:

(a) $\displaystyle\int \frac{x}{x^2+1}\, dx$, (b) $\displaystyle\int \frac{\sin(3\ln x)}{x}\, dx$, and (c) $\displaystyle\int e^x\sqrt{1+e^x}\, dx$.

Solution

(a) $\displaystyle\int \frac{x}{x^2+1}\, dx$ Let $u = x^2 + 1$.
Then $du = 2x\, dx$ and
$x\, dx = \frac{1}{2}\, du$

$$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C = \ln\sqrt{x^2+1} + C.$$

(Both versions of the final answer are equally acceptable.)

(b) $\displaystyle\int \frac{\sin(3\ln x)}{x}\, dx$ Let $u = 3\ln x$.
Then $du = \dfrac{3}{x}\, dx$

$$= \frac{1}{3}\int \sin u\, du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos(3\ln x) + C.$$

(c) $\displaystyle\int e^x\sqrt{1+e^x}\, dx$ Let $v = 1 + e^x$.
Then $dv = e^x\, dx$

$$= \int v^{1/2}\, dv = \frac{2}{3}v^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C.$$

Sometimes the appropriate substitutions are not as obvious as they were in Example 3, and it may be necessary to manipulate the integrand algebraically to put it into a better form for substitution.

EXAMPLE 4 Evaluate (a) $\displaystyle\int \frac{1}{x^2+4x+5}\, dx$ and (b) $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$.

Solution

(a) $\displaystyle\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1}$ Let $t = x + 2$.
Then $dt = dx$.

$$= \int \frac{dt}{t^2+1}$$

$$= \tan^{-1} t + C = \tan^{-1}(x+2) + C.$$

(b) $$\begin{aligned}\int \frac{dx}{\sqrt{e^{2x}-1}} &= \int \frac{dx}{e^x\sqrt{1-e^{-2x}}} \\ &= \int \frac{e^{-x}\,dx}{\sqrt{1-(e^{-x})^2}} \qquad \text{Let } u = e^{-x}. \text{ Then } du = -e^{-x}\,dx. \\ &= -\int \frac{du}{\sqrt{1-u^2}} \\ &= -\sin^{-1} u + C = -\sin^{-1}(e^{-x}) + C.\end{aligned}$$

The method of substitution cannot be *forced* to work. There is no substitution that will do much good with the integral $\int x(2+x^7)^{1/5}\,dx$, for instance. However, the integral $\int x^6(2+x^7)^{1/5}\,dx$ will yield to the substitution $u = 2+x^7$. The substitution $u = g(x)$ is more likely to work if $g'(x)$ is a factor of the integrand.

The following theorem simplifies the use of the method of substitution in definite integrals.

THEOREM 6

Substitution in a definite integral

Suppose that g is a differentiable function on $[a,b]$ that satisfies $g(a) = A$ and $g(b) = B$. Also suppose that f is continuous on the range of g. Then

$$\int_a^b f\big(g(x)\big)\,g'(x)\,dx = \int_A^B f(u)\,du.$$

PROOF Let F be an antiderivative of f; $F'(u) = f(u)$. Then

$$\frac{d}{dx}F\big(g(x)\big) = F'\big(g(x)\big)\,g'(x) = f\big(g(x)\big)\,g'(x).$$

Thus,

$$\begin{aligned}\int_a^b f\big(g(x)\big)\,g'(x)\,dx &= F\big(g(x)\big)\Big|_a^b = F\big(g(b)\big) - F\big(g(a)\big) \\ &= F(B) - F(A) = F(u)\Big|_A^B = \int_A^B f(u)\,du.\end{aligned}$$

EXAMPLE 5 Evaluate the integral $I = \displaystyle\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\,dx$.

Solution **METHOD I.** Let $u = \sqrt{x+1}$. Then $du = \dfrac{dx}{2\sqrt{x+1}}$. If $x = 0$, then $u = 1$; if $x = 8$, then $u = 3$. Thus,

$$I = 2\int_1^3 \cos u\,du = 2\sin u\Big|_1^3 = 2\sin 3 - 2\sin 1.$$

METHOD II. We use the same substitution as in Method I, but we do not transform the limits of integration from x values to u values. Hence, we must return to the variable x before substituting in the limits:

$$I = 2\int_{x=0}^{x=8} \cos u\,du = 2\sin u\Big|_{x=0}^{x=8} = 2\sin\sqrt{x+1}\Big|_0^8 = 2\sin 3 - 2\sin 1.$$

Note that the limits *must* be written $x = 0$ and $x = 8$ at any stage where the variable is not x. It would have been *wrong* to write

$$I = 2\int_0^8 \cos u\, du$$

because this would imply that u, rather than x, goes from 0 to 8. Method I gives the shorter solution and is therefore preferable. However, in cases where the transformed limits (the u-limits) are very complicated, you might prefer to use Method II.

EXAMPLE 6 Find the area of the region bounded by $y = \left(2 + \sin\frac{x}{2}\right)^2 \cos\frac{x}{2}$, the x-axis, and the lines $x = 0$ and $x = \pi$.

Solution Because $y \geq 0$ when $0 \leq x \leq \pi$, the required area is

$$A = \int_0^{\pi} \left(2 + \sin\frac{x}{2}\right)^2 \cos\frac{x}{2}\, dx \qquad \text{Let } v = 2 + \sin\frac{x}{2}.$$

$$\text{Then } dv = \frac{1}{2}\cos\frac{x}{2}\, dx$$

$$= 2\int_2^3 v^2\, dv = \frac{2}{3}v^3\Big|_2^3 = \frac{2}{3}(27 - 8) = \frac{38}{3} \text{ square units.}$$

Remark The condition that f be continuous on the range of the function $u = g(x)$ (for $a \leq x \leq b$) is essential in Theorem 6. Using the substitution $u = x^2$ in the integral $\int_{-1}^{1} x\csc(x^2)\, dx$ leads to the erroneous conclusion

$$\int_{-1}^{1} x \csc(x^2)\, dx = \frac{1}{2}\int_1^1 \csc u\, du = 0.$$

Although $x\csc(x^2)$ is an odd function, it is not continuous at 0, and it happens that the given integral represents the difference of *infinite* areas. If we assume that f is continuous on an interval containing A and B, then it suffices to know that $u = g(x)$ is one-to-one as well as differentiable. In this case the range of g will lie between A and B, so the condition of Theorem 6 will be satisfied.

Trigonometric Integrals

The method of substitution is often useful for evaluating trigonometric integrals. We begin by listing the integrals of the four trigonometric functions whose integrals we have not yet seen. They arise often in applications and should be memorized.

Integrals of tangent, cotangent, secant, and cosecant

$$\int \tan x\, dx = \ln|\sec x| + C,$$

$$\int \cot x\, dx = \ln|\sin x| + C = -\ln|\csc x| + C,$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C,$$

$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C = \ln|\csc x - \cot x| + C.$$

All of these can, of course, be checked by differentiating the right-hand sides. The first two can be evaluated directly by rewriting $\tan x$ or $\cot x$ in terms of $\sin x$ and $\cos x$ and using an appropriate substitution. For example,

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \qquad \text{Let } u = \cos x. \\ & \qquad\qquad\qquad\quad \text{Then } du = -\sin x\,dx. \\ &= -\int \frac{du}{u} = -\ln|u| + C \\ &= -\ln|\cos x| + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C.\end{aligned}$$

The integral of $\sec x$ can be evaluated by rewriting it in the form

$$\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx$$

and using the substitution $u = \sec x + \tan x$. The integral of $\csc x$ can be evaluated similarly. (Show that the two versions given for that integral are equivalent!)

We now consider integrals of the form

$$\int \sin^m x \cos^n x\,dx.$$

If either m or n is an odd, positive integer, the integral can be done easily by substitution. If, say, $n = 2k + 1$ where k is an integer, then we can use the identity $\sin^2 x + \cos^2 x = 1$ to rewrite the integral in the form

$$\int \sin^m x\,(1 - \sin^2 x)^k \cos x\,dx,$$

which can be integrated using the substitution $u = \sin x$. Similarly, $u = \cos x$ can be used if m is an odd integer.

EXAMPLE 7 Evaluate (a) $\int \sin^3 x \cos^8 x\,dx$ and (b) $\int \cos^5 ax\,dx$.

Solution

(a)
$$\begin{aligned}\int \sin^3 x \cos^8 x\,dx &= \int (1 - \cos^2 x)\cos^8 x \sin x\,dx \qquad \text{Let } u = \cos x, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\quad du = -\sin x\,dx. \\ &= -\int (1 - u^2)u^8\,du = \int (u^{10} - u^8)\,du \\ &= \frac{u^{11}}{11} - \frac{u^9}{9} + C = \frac{1}{11}\cos^{11} x - \frac{1}{9}\cos^9 x + C.\end{aligned}$$

(b)
$$\begin{aligned}\int \cos^5 ax\,dx &= \int (1 - \sin^2 ax)^2 \cos ax\,dx \qquad \text{Let } u = \sin ax, \\ & \qquad\qquad\qquad\qquad\qquad\qquad du = a\cos ax\,dx. \\ &= \frac{1}{a}\int (1 - u^2)^2\,du = \frac{1}{a}\int (1 - 2u^2 + u^4)\,du \\ &= \frac{1}{a}\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + C \\ &= \frac{1}{a}\left(\sin ax - \frac{2}{3}\sin^3 ax + \frac{1}{5}\sin^5 ax\right) + C.\end{aligned}$$

If the powers of $\sin x$ and $\cos x$ are both even, then we can make use of the *double-angle formulas* (see Section P.7):

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \qquad \text{and} \qquad \sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

EXAMPLE 8 **(Integrating even powers of sine and cosine)** Verify the integration formulas

$$\int \cos^2 x\,dx = \frac{1}{2}(x + \sin x \cos x) + C,$$
$$\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x \cos x) + C.$$

These integrals are encountered frequently and are worth remembering.

Solution Each of the integrals follows from the corresponding double-angle identity. We do the first; the second is similar.

$$\begin{aligned}\int \cos^2 x\,dx &= \frac{1}{2}\int (1 + \cos 2x)\,dx \\ &= \frac{x}{2} + \frac{1}{4}\sin 2x + C \\ &= \frac{1}{2}(x + \sin x \cos x) + C \qquad (\text{since } \sin 2x = 2\sin x \cos x).\end{aligned}$$

EXAMPLE 9 Evaluate $\int \sin^4 x\,dx$.

Solution We will have to apply the double-angle formula twice.

$$\begin{aligned}\int \sin^4 x\,dx &= \frac{1}{4}\int (1 - \cos 2x)^2\,dx \\ &= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx \\ &= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{1}{8}\int (1 + \cos 4x)\,dx \\ &= \frac{x}{4} - \frac{1}{4}\sin 2x + \frac{x}{8} + \frac{1}{32}\sin 4x + C \\ &= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C\end{aligned}$$

(Note that there is no point in inserting the constant of integration C until the last integral has been evaluated.)

Using the identities $\sec^2 x = 1 + \tan^2 x$ and $\csc^2 x = 1 + \cot^2 x$ and one of the substitutions $u = \sec x$, $u = \tan x$, $u = \csc x$, or $u = \cot x$, we can evaluate integrals of the form

$$\int \sec^m x \tan^n x\,dx \qquad \text{or} \qquad \int \csc^m x \cot^n x\,dx,$$

unless m is odd and n is even. (If this is the case, these integrals can be handled by integration by parts; see Section 6.1.)

EXAMPLE 10 **(Integrals involving secants and tangents)** Evaluate the following integrals:

(a) $\int \tan^2 x\,dx$, (b) $\int \sec^4 t\,dt$, and (c) $\int \sec^3 x \tan^3 x\,dx$.

Solution

(a) $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C.$

(b) $\int \sec^4 t\,dt = \int (1 + \tan^2 t)\sec^2 t\,dt$ Let $u = \tan t$, $du = \sec^2 t\,dt$.

$= \int (1 + u^2)\,du = u + \frac{1}{3}u^3 + C = \tan t + \frac{1}{3}\tan^3 t + C.$

(c) $\int \sec^3 x \tan^3 x\,dx$

$= \int \sec^2 x\,(\sec^2 x - 1)\sec x \tan x\,dx$ Let $u = \sec x$, $du = \sec x \tan x\,dx$.

$= \int (u^4 - u^2)\,du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.$

EXERCISES 5.6

Evaluate the integrals in Exercises 1–44. Remember to include a constant of integration with the indefinite integrals. Your answers may appear different from those in the Answers section but may still be correct. For example, evaluating $I = \int \sin x \cos x\,dx$ using the substitution $u = \sin x$ leads to $I = \frac{1}{2}\sin^2 x + C$; using $u = \cos x$ leads to $I = -\frac{1}{2}\cos^2 x + C$; and rewriting $I = \frac{1}{2}\int \sin(2x)\,dx$ leads to $I = -\frac{1}{4}\cos(2x) + C$. These answers are all equal except for different choices for the constant of integration C: $\frac{1}{2}\sin^2 x = -\frac{1}{2}\cos^2 + \frac{1}{2} = -\frac{1}{4}\cos(2x) + \frac{1}{4}$.

You can always check your own answer to an indefinite integral by differentiating it to get back to the integrand. This is often easier than comparing your answer with the answer in the back of the book. You may find integrals that you can't do, but you should not make mistakes in those you can do because the answer is so easily checked. (This is a good thing to remember during tests and exams.)

1. $\int e^{5-2x}\,dx$

2. $\int \cos(ax + b)\,dx$

3. $\int \sqrt{3x + 4}\,dx$

4. $\int e^{2x}\sin(e^{2x})\,dx$

5. $\int \frac{x\,dx}{(4x^2 + 1)^5}$

6. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

7. $\int x e^{x^2}\,dx$

8. $\int x^2 2^{x^3+1}\,dx$

9. $\int \frac{\cos x}{4 + \sin^2 x}\,dx$

10. $\int \frac{\sec^2 x}{\sqrt{1 - \tan^2 x}}\,dx$

! **11.** $\int \frac{e^x + 1}{e^x - 1}\,dx$

12. $\int \frac{\ln t}{t}\,dt$

13. $\int \frac{ds}{\sqrt{4 - 5s}}$

14. $\int \frac{x + 1}{\sqrt{x^2 + 2x + 3}}\,dx$

15. $\int \frac{t\,dt}{\sqrt{4 - t^4}}$

16. $\int \frac{x^2\,dx}{2 + x^6}$

! **17.** $\int \frac{dx}{e^x + 1}$

! **18.** $\int \frac{dx}{e^x + e^{-x}}$

19. $\int \tan x \ln\cos x\,dx$

20. $\int \frac{x + 1}{\sqrt{1 - x^2}}\,dx$

21. $\int \frac{dx}{x^2 + 6x + 13}$

22. $\int \frac{dx}{\sqrt{4 + 2x - x^2}}$

23. $\int \sin^3 x \cos^5 x\,dx$

24. $\int \sin^4 t \cos^5 t\,dt$

25. $\int \sin ax \cos^2 ax\,dx$

26. $\int \sin^2 x \cos^2 x\,dx$

27. $\int \sin^6 x\,dx$

28. $\int \cos^4 x\,dx$

29. $\int \sec^5 x \tan x\,dx$

30. $\int \sec^6 x \tan^2 x\,dx$

31. $\int \sqrt{\tan x}\sec^4 x\,dx$

32. $\int \sin^{-2/3} x \cos^3 x\,dx$

33. $\int \cos x \sin^4(\sin x)\,dx$

34. $\int \frac{\sin^3 \ln x \cos^3 \ln x}{x}\,dx$

35. $\int \frac{\sin^2 x}{\cos^4 x}\,dx$

36. $\int \frac{\sin^3 x}{\cos^4 x}\,dx$

37. $\int \csc^5 x \cot^5 x\,dx$

38. $\int \frac{\cos^4 x}{\sin^8 x}\,dx$

39. $\int_0^4 x^3(x^2 + 1)^{-\frac{1}{2}}\,dx$

40. $\int_1^{\sqrt{e}} \frac{\sin(\pi \ln x)}{x}\,dx$

41. $\int_0^{\pi/2} \sin^4 x\,dx$

42. $\int_{\pi/4}^{\pi} \sin^5 x\,dx$

43. $\int_e^{e^2} \frac{dt}{t\ln t}$

44. $\int_{\frac{\pi^2}{16}}^{\frac{\pi^2}{9}} \frac{2^{\sin\sqrt{x}}\cos\sqrt{x}}{\sqrt{x}}\,dx$

! **45.** Use the identities $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$ and $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ to help you evaluate the following:

$$\int_0^{\pi/2} \sqrt{1 + \cos x}\,dx \quad \text{and} \quad \int_0^{\pi/2} \sqrt{1 - \sin x}\,dx$$

46. Find the area of the region bounded by $y = x/(x^2 + 16)$, $y = 0$, $x = 0$, and $x = 2$.

47. Find the area of the region bounded by $y = x/(x^4 + 16)$, $y = 0$, $x = 0$, and $x = 2$.

48. Express the area bounded by the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ as a definite integral. Make a substitution that converts this integral into one representing the area of a circle, and hence evaluate it.

49. Use the addition formulas for $\sin(x \pm y)$ and $\cos(x \pm y)$ from Section P.7 to establish the following identities:

$$\cos x \cos y = \frac{1}{2}\Big(\cos(x - y) + \cos(x + y)\Big),$$
$$\sin x \sin y = \frac{1}{2}\Big(\cos(x - y) - \cos(x + y)\Big),$$
$$\sin x \cos y = \frac{1}{2}\Big(\sin(x + y) + \sin(x - y)\Big).$$

50. Use the identities established in Exercise 49 to calculate the following integrals:
$\int \cos ax \cos bx\, dx$, $\int \sin ax \sin bx\, dx$,
and $\int \sin ax \cos bx\, dx$.

51. If m and n are integers, show that:

(i) $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = 0$ if $m \neq n$,

(ii) $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = 0$ if $m \neq n$,

(iii) $\displaystyle\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = 0$.

52. (Fourier coefficients) Suppose that for some positive integer k,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{k} (a_n \cos nx + b_n \sin nx)$$

holds for all x in $[-\pi, \pi]$. Use the result of Exercise 51 to show that the coefficients a_m $(0 \leq m \leq k)$ and b_m $(1 \leq m \leq k)$, which are called the Fourier coefficients of f on $[-\pi, \pi]$, are given by

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos mx\, dx, \quad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx\, dx.$$

5.7 Areas of Plane Regions

In this section we review and extend the use of definite integrals to represent plane areas. Recall that the integral $\int_a^b f(x)\,dx$ measures the area between the graph of f and the x-axis from $x = a$ to $x = b$, but treats as *negative* any part of this area that lies below the x-axis. (We are assuming that $a < b$.) In order to express the total area bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$, counting all of the area positively, we should integrate the *absolute value* of f (see Figure 5.27):

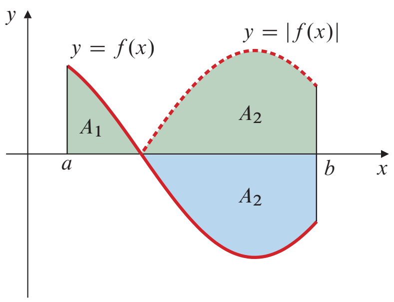


Figure 5.27

$$\int_a^b f(x)\,dx = A_1 - A_2 \qquad \text{and} \qquad \int_a^b |f(x)|\,dx = A_1 + A_2.$$

There is no "rule" for integrating $\int_a^b |f(x)|\,dx$; one must break the integral into a sum of integrals over intervals where $f(x) > 0$ (so $|f(x)| = f(x)$), and intervals where $f(x) < 0$ (so $|f(x)| = -f(x)$).

EXAMPLE 1 The area bounded by $y = \cos x$, $y = 0$, $x = 0$, and $x = 3\pi/2$ (see Figure 5.28) is

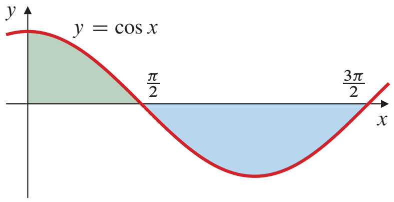


Figure 5.28

$$\begin{aligned} A &= \int_0^{3\pi/2} |\cos x|\,dx \\ &= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{3\pi/2} (-\cos x)\,dx \\ &= \sin x\Big|_0^{\pi/2} - \sin x\Big|_{\pi/2}^{3\pi/2} \\ &= (1 - 0) - (-1 - 1) = 3 \text{ square units.} \end{aligned}$$

Areas Between Two Curves

Suppose that a plane region R is bounded by the graphs of two continuous functions, $y = f(x)$ and $y = g(x)$, and the vertical straight lines $x = a$ and $x = b$, as shown in Figure 5.29(a). Assume that $a < b$ and that $f(x) \le g(x)$ on $[a, b]$, so the graph of f lies below that of g. If $f(x) \ge 0$ on $[a, b]$, then the area A of R is the area above the x-axis and under the graph of g minus the area above the x-axis and under the graph of f:

$$A = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx = \int_a^b \big(g(x) - f(x)\big)\,dx.$$

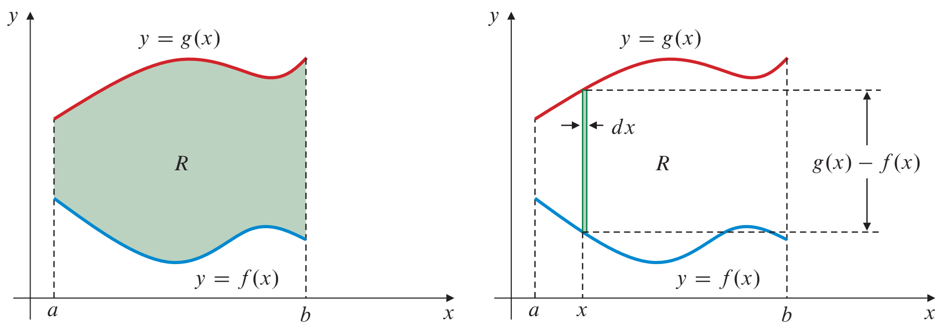


Figure 5.29

(a) The region R lying between two graphs

(b) An area element of the region R

It is useful to regard this formula as expressing A as the "sum" (i.e., the integral) of *infinitely many* **area elements**

$$dA = (g(x) - f(x))\,dx,$$

corresponding to values of x between a and b. Each such area element is the area of an infinitely thin vertical rectangle of width dx and height $g(x) - f(x)$ located at position x (see Figure 5.29(b)). Even if f and g can take on negative values on $[a, b]$, this interpretation and the resulting area formula

$$A = \int_a^b \big(g(x) - f(x)\big)\,dx$$

remain valid, provided that $f(x) \le g(x)$ on $[a, b]$ so that all the area elements dA have positive area. Using integrals to represent a quantity as a *sum* of *differential elements* (i.e., a sum of little bits of the quantity) is a very helpful approach. We will do this often in Chapter 7. Of course, what we are really doing is identifying the integral as a *limit* of a suitable Riemann sum.

More generally, if the restriction $f(x) \le g(x)$ is removed, then the vertical rectangle of width dx at position x extending between the graphs of f and g has height $|f(x) - g(x)|$ and hence area

$$dA = |f(x) - g(x)|\,dx.$$

(See Figure 5.30.) Hence, the total area lying between the graphs $y = f(x)$ and $y = g(x)$ and between the vertical lines $x = a$ and $x = b > a$ is given by

$$A = \int_a^b \big|f(x) - g(x)\big|\,dx.$$

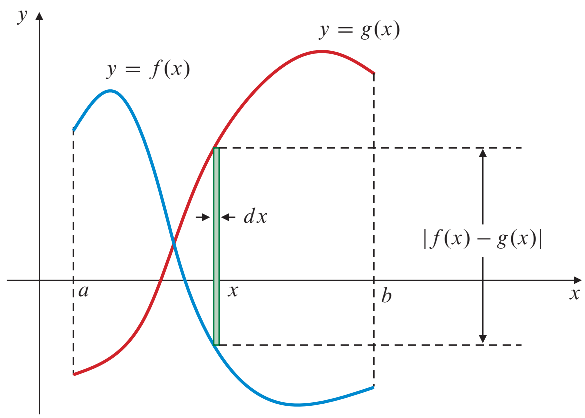


Figure 5.30 An area element for the region between $y = f(x)$ and $y = g(x)$

In order to evaluate this integral, we have to determine the intervals on which $f(x) > g(x)$ or $f(x) < g(x)$, and break the integral into a sum of integrals over each of these intervals.

EXAMPLE 2 Find the area of the bounded, plane region R lying between the curves $y = x^2 - 2x$ and $y = 4 - x^2$.

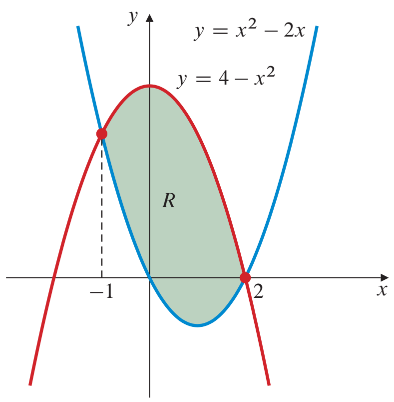


Figure 5.31

Solution First, we must find the intersections of the curves, so we solve the equations simultaneously:

$$\begin{aligned} x^2 - 2x &= y = 4 - x^2 \\ 2x^2 - 2x - 4 &= 0 \\ 2(x-2)(x+1) &= 0 \quad \text{so } x = 2 \text{ or } x = -1. \end{aligned}$$

The curves are sketched in Figure 5.31, and the bounded (finite) region between them is shaded. (A sketch should always be made in problems of this sort.) Since $4 - x^2 \ge x^2 - 2x$ for $-1 \le x \le 2$, the area A of R is given by

$$\begin{aligned} A &= \int_{-1}^{2} \big((4 - x^2) - (x^2 - 2x)\big)\,dx \\ &= \int_{-1}^{2} (4 - 2x^2 + 2x)\,dx \\ &= \left(4x - \frac{2}{3}x^3 + x^2\right)\Bigg|_{-1}^{2} \\ &= 4(2) - \frac{2}{3}(8) + 4 - \left(-4 + \frac{2}{3} + 1\right) = 9 \text{ square units.} \end{aligned}$$

Note that in representing the area as an integral we *must subtract the height y to the lower curve from the height y to the upper curve* to get a positive area element dA. Subtracting the wrong way would have produced a negative value for the area.

EXAMPLE 3 Find the total area A lying between the curves $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = 2\pi$.

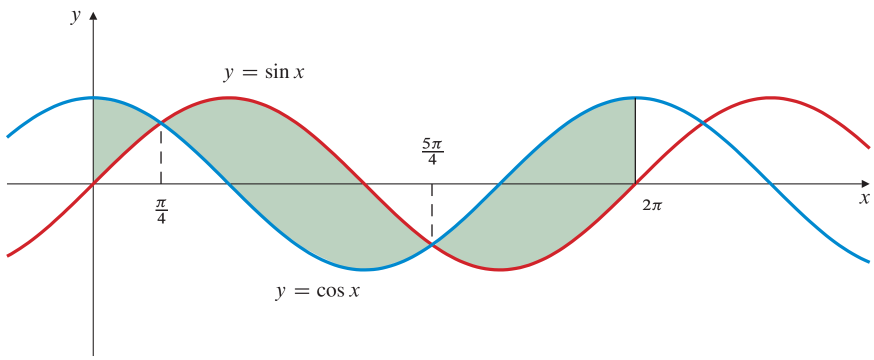


Figure 5.32

Solution The region is shaded in Figure 5.32. Between 0 and 2π the graphs of sine and cosine cross at $x = \pi/4$ and $x = 5\pi/4$. The required area is

$$\begin{aligned} A &= \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx \\ &\quad + \int_{5\pi/4}^{2\pi} (\cos x - \sin x)\,dx \\ &= (\sin x + \cos x)\Big|_0^{\pi/4} - (\cos x + \sin x)\Big|_{\pi/4}^{5\pi/4} + (\sin x + \cos x)\Big|_{5\pi/4}^{2\pi} \\ &= (\sqrt{2} - 1) + (\sqrt{2} + \sqrt{2}) + (1 + \sqrt{2}) = 4\sqrt{2} \text{ square units.} \end{aligned}$$

It is sometimes more convenient to use horizontal area elements instead of vertical ones and integrate over an interval of the y-axis instead of the x-axis. This is usually the case if the region whose area we want to find is bounded by curves whose equations are written in terms of functions of y. In Figure 5.33(a), the region R lying to the right of $x = f(y)$ and to the left of $x = g(y)$, and between the horizontal lines $y = c$ and $y = d > c$, has area element $dA = \big(g(y) - f(y)\big)\,dy$. Its area is

$$A = \int_c^d \big(g(y) - f(y)\big)\,dy.$$

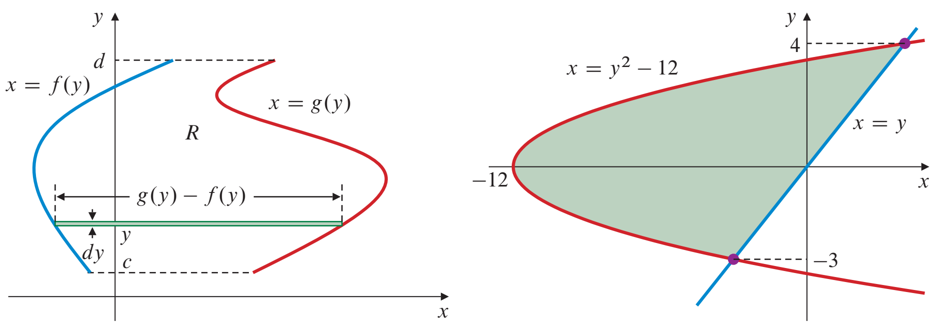


Figure 5.33
(a) A horizontal area element
(b) The finite region bounded by $x = y^2 - 12$ and $x = y$

EXAMPLE 4 Find the area of the plane region lying to the right of the parabola $x = y^2 - 12$ and to the left of the straight line $y = x$, as illustrated in Figure 5.33(b).

Solution For the intersections of the curves:

$$\begin{aligned} &y^2 - 12 = x = y \\ &y^2 - y - 12 = 0 \\ &(y - 4)(y + 3) = 0 \quad \text{so } y = 4 \text{ or } y = -3. \end{aligned}$$

Observe that $y^2 - 12 \le y$ for $-3 \le y \le 4$. Thus, the area is

$$A = \int_{-3}^{4} \big(y - (y^2 - 12)\big)\, dy = \left(\frac{y^2}{2} - \frac{y^3}{3} + 12y\right)\bigg|_{-3}^{4} = \frac{343}{6} \text{ square units.}$$

Of course, the same result could have been obtained by integrating in the x direction, but the integral would have been more complicated:

$$A = \int_{-12}^{-3} \big(\sqrt{12+x} - (-\sqrt{12+x})\big)\, dx + \int_{-3}^{4} \big(\sqrt{12+x} - x\big)\, dx;$$

different integrals are required over the intervals where the region is bounded below by the parabola and by the straight line.

EXERCISES 5.7

In Exercises 1–16, sketch and find the area of the plane region bounded by the given curves.

1. $y = x, \quad y = x^2$

2. $y = \sqrt{x}, \quad y = x^2$

3. $y = x^2 - 5, \quad y = 3 - x^2$

4. $y = x^2 - 2x, \quad y = 6x - x^2$

5. $2y = 4x - x^2, \quad 2y + 3x = 6$

6. $x - y = 7, \quad x = 2y^2 - y + 3$

7. $y = x^3, \quad y = x$

8. $y = x^3, \quad y = x^2$

9. $y = x^3, \quad x = y^2$

10. $x = y^2, \quad x = 2y^2 - y - 2$

11. $y = \dfrac{1}{x}, \quad 2x + 2y = 5$

12. $y = (x^2 - 1)^2, \quad y = 1 - x^2$

13. $y = \dfrac{1}{2}x^2, \quad y = \dfrac{1}{x^2 + 1}$

14. $y = \dfrac{4x}{3 + x^2}, \quad y = 1$

15. $y = \dfrac{4}{x^2}, \quad y = 5 - x^2$

16. $x = y^2 - \pi^2, \quad x = \sin y$

Find the areas of the regions described in Exercises 17–28. It is helpful to sketch the regions before writing an integral to represent the area.

17. Bounded by $y = \sin x$ and $y = \cos x$, and between two consecutive intersections of these curves

18. Bounded by $y = \sin^2 x$ and $y = 1$, and between two consecutive intersections of these curves

19. Bounded by $y = \sin x$ and $y = \sin^2 x$, between $x = 0$ and $x = \pi/2$

20. Bounded by $y = \sin^2 x$ and $y = \cos^2 x$, and between two consecutive intersections of these curves

21. Under $y = 4x/\pi$ and above $y = \tan x$, between $x = 0$ and the first intersection of the curves to the right of $x = 0$

22. Bounded by $y = x^{1/3}$ and the component of $y = \tan(\pi x/4)$ that passes through the origin

23. Bounded by $y = 2$ and the component of $y = \sec x$ that passes through the point $(0, 1)$

24. Bounded by $y = \sqrt{2}\cos(\pi x/4)$ and $y = |x|$

25. Bounded by $y = \sin(\pi x/2)$ and $y = x$

26. Bounded by $y = e^x$ and $y = x + 2$

27. Find the total area enclosed by the curve $y^2 = x^2 - x^4$.

28. Find the area of the closed loop of the curve $y^2 = x^4(2 + x)$ that lies to the left of the origin.

29. Find the area of the finite plane region that is bounded by the curve $y = e^x$, the line $x = 0$, and the tangent line to $y = e^x$ at $x = 1$.

30. Find the area of the finite plane region bounded by the curve $y = x^3$ and the tangent line to that curve at the point $(1, 1)$. *Hint:* Find the other point at which that tangent line meets the curve.

CHAPTER REVIEW

Key Ideas

- **What do the following terms and phrases mean?**
  - ◇ sigma notation
  - ◇ a partition of an interval
  - ◇ a Riemann sum
  - ◇ a definite integral
  - ◇ an indefinite integral
  - ◇ an integrable function
  - ◇ an area element
  - ◇ an evaluation symbol
  - ◇ the triangle inequality for integrals
  - ◇ a piecewise continuous function

◇ the average value of function f on $[a, b]$

◇ the method of substitution

- **State the Mean-Value Theorem for integrals.**
- **State the Fundamental Theorem of Calculus.**
- **List as many properties of the definite integral as you can.**
- **What is the relationship between the definite integral and the indefinite integral of a function f on an interval $[a, b]$?**
- **What is the derivative of $\int_{f(x)}^{g(x)} h(t)\,dt$ with respect to x?**
- **How can the area between the graphs of two functions be calculated?**

Review Exercises

1. Show that $\dfrac{2j+1}{j^2(j+1)^2} = \dfrac{1}{j^2} - \dfrac{1}{(j+1)^2}$; hence evaluate $\displaystyle\sum_{j=1}^{n} \frac{2j+1}{j^2(j+1)^2}$.

2. **(Stacking balls)** A display of golf balls in a sporting goods store is built in the shape of a pyramid with a rectangular base measuring 40 balls long and 30 balls wide. The next layer up is 39 balls by 29 balls, etc. How many balls are in the pyramid?

3. Let $P_n = \{x_0 = 1, x_1, x_2, \ldots, x_n = 3\}$ be a partition of $[1, 3]$ into n subintervals of equal length, and let $f(x) = x^2 - 2x + 3$. Evaluate $\displaystyle\int_1^3 f(x)\,dx$ by finding $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i$.

4. Interpret $R_n = \displaystyle\sum_{i=1}^{n} \frac{1}{n}\sqrt{1 + \frac{i}{n}}$ as a Riemann sum for a certain function f on the interval [0, 1]; hence evaluate $\lim_{n\to\infty} R_n$.

Evaluate the integrals in Exercises 5–8 without using the Fundamental Theorem of Calculus.

5. $\displaystyle\int_{-\pi}^{\pi} (2 - \sin x)\,dx$ **6.** $\displaystyle\int_0^{\sqrt{5}} \sqrt{5 - x^2}\,dx$

7. $\displaystyle\int_1^3 \left(1 - \frac{x}{2}\right) dx$ **8.** $\displaystyle\int_0^{\pi} \cos x\,dx$

Find the average values of the functions in Exercises 9–10 over the indicated intervals.

9. $f(x) = 2 - \sin x^3$ on $[-\pi, \pi]$

10. $h(x) = |x - 2|$ on $[0, 3]$

Find the derivatives of the functions in Exercises 11–14.

11. $f(t) = \displaystyle\int_{13}^{t} \sin(x^2)\,dx$ **12.** $f(x) = \displaystyle\int_{-13}^{\sin x} \sqrt{1 + t^2}\,dt$

13. $g(s) = \displaystyle\int_{4s}^{1} e^{\sin u}\,du$ **14.** $g(\theta) = \displaystyle\int_{e^{\sin\theta}}^{e^{\cos\theta}} \ln x\,dx$

15. Solve the integral equation $2f(x) + 1 = 3\displaystyle\int_x^1 f(t)\,dt$.

16. Use the substitution $x = \pi - u$ to show that

$$\int_0^{\pi} x\,f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$$

for any function f continuous on [0, 1].

Find the areas of the finite plane regions bounded by the indicated graphs in Exercises 17–22.

17. $y = 2 + x - x^2$ and $y = 0$

18. $y = (x - 1)^2$, $y = 0$, and $x = 0$

19. $x = y - y^4$ and $x = 0$ **20.** $y = 4x - x^2$ and $y = 3$

21. $y = \sin x$, $y = \cos 2x$, $x = 0$, and $x = \pi/6$

22. $y = 5 - x^2$ and $y = 4/x^2$

Evaluate the integrals in Exercises 23–30.

23. $\displaystyle\int x^2\cos(2x^3 + 1)\,dx$ **24.** $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

25. $\displaystyle\int_0^4 \sqrt{9t^2 + t^4}\,dt$ **26.** $\displaystyle\int \sin^3(\pi x)\,dx$

27. $\displaystyle\int_0^{\ln 2} \frac{e^u}{4 + e^{2u}}\,du$ **28.** $\displaystyle\int_1^{\sqrt[4]{e}} \frac{\tan^2 \pi \ln x}{x}\,dx$

29. $\displaystyle\int \frac{\sin\sqrt{2s+1}}{\sqrt{2s+1}}\,ds$ **30.** $\displaystyle\int \cos^2\frac{t}{5}\sin^2\frac{t}{5}\,dt$

31. Find the minimum value of $F(x) = \displaystyle\int_0^{x^2-2x} \frac{1}{1+t^2}\,dt$. Does F have a maximum value? Why?

32. Find the maximum value of $\int_a^b (4x - x^2)\,dx$ for intervals $[a, b]$, where $a < b$. How do you know such a maximum value exists?

33. An object moves along the x-axis so that its position at time t is given by the function $x(t)$. In Section 2.11 we defined the average velocity of the object over the time interval $[t_0, t_1]$ to be $v_{\text{av}} = \big(x(t_1) - x(t_0)\big)/(t_1 - t_0)$. Show that v_{av} is, in fact, the average value of the velocity function $v(t) = dx/dt$ over the interval $[t_0, t_1]$.

34. If an object falls from rest under constant gravitational acceleration, show that its average height during the time T of its fall is its height at time $T/\sqrt{3}$.

35. Find two numbers x_1 and x_2 in the interval [0, 1] with $x_1 < x_2$ such that if $f(x)$ is any cubic polynomial (i.e., polynomial of degree 3), then

$$\int_0^1 f(x)\,dx = \frac{f(x_1) + f(x_2)}{2}.$$

Challenging Problems

1. Evaluate the upper and lower Riemann sums, $U(f, P_n)$ and $L(f, P_n)$, for $f(x) = 1/x$ on the interval [1, 2] for the partition P_n with division points $x_i = 2^{i/n}$ for $0 \le i \le n$. Verify that $\lim_{n\to\infty} U(f, P_n) = \ln 2 = \lim_{n\to\infty} L(f, P_n)$.

2. (a) Use the addition formulas for $\cos(a + b)$ and $\cos(a - b)$ to show that

$$\cos\Big((j + \tfrac{1}{2})t\Big) - \cos\Big((j - \tfrac{1}{2})t\Big) = -2\sin(\tfrac{1}{2}t)\,\sin(jt),$$

and hence deduce that if $t/(2\pi)$ is not an integer, then

$$\sum_{j=1}^{n} \sin(jt) = \frac{\cos\frac{t}{2} - \cos\Big((n + \frac{1}{2})t\Big)}{2\sin\frac{t}{2}}.$$

(b) Use the result of part (a) to evaluate $\int_0^{\pi/2} \sin x\,dx$ as a limit of a Riemann sum.

3. (a) Use the method of Problem 2 to show that if $t/(2\pi)$ is not an integer, then

$$\sum_{j=1}^{n} \cos(jt) = \frac{\sin\left((n+\frac{1}{2})t\right) - \sin\frac{t}{2}}{2\sin\frac{t}{2}}.$$

(b) Use the result to part (a) to evaluate $\int_0^{\pi/3} \cos x\,dx$ as a limit of a Riemann sum.

4. Let $f(x) = 1/x^2$ and let $1 = x_0 < x_1 < x_2 < \cdots < x_n = 2$, so that $\{x_0, x_1, x_2, \ldots, x_n\}$ is a partition of $[1, 2]$ into n subintervals. Show that $c_i = \sqrt{x_{i-1}x_i}$ is in the ith subinterval $[x_{i-1}, x_i]$ of the partition, and evaluate the Riemann sum $\sum_{i=1}^{n} f(c_i)\,\Delta x_i$. What does this imply about $\int_1^2 (1/x^2)\,dx$?

5. (a) Use mathematical induction to verify that for every positive integer k, $\sum_{j=1}^{n} j^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + P_{k-1}(n)$, where P_{k-1} is a polynomial of degree at most $k-1$. *Hint:* Start by iterating the identity

$$(j+1)^{k+1} - j^{k+1} = (k+1)j^k + \frac{(k+1)k}{2}j^{k-1} + \text{lower powers of } j$$

for $j = 1, 2, 3, \ldots, k$ and adding.

(b) Deduce from (a) that $\displaystyle\int_0^a x^k\,dx = \frac{a^{k+1}}{k+1}$.

6. Let C be the cubic curve $y = ax^3 + bx^2 + cx + d$, and let P be any point on C. The tangent to C at P meets C again at point Q. The tangent to C at Q meets C again at R. Show that the area between C and the tangent at Q is 16 times the area between C and the tangent at P.

7. Let C be the cubic curve $y = ax^3 + bx^2 + cx + d$, and let P be any point on C. The tangent to C at P meets C again at point Q. Let R be the inflection point of C. Show that R lies between P and Q on C and that QR divides the area between C and its tangent at P in the ratio 16/11.

8. (Double tangents) Let line PQ be tangent to the graph C of the quartic polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ at two distinct points: $P = (p, f(p))$ and $Q = (q, f(q))$. Let $U = (u, f(u))$ and $V = (v, f(v))$ be the other two points where the line tangent to C at $T = ((p+q)/2, f((p+q)/2))$ meets C. If A and B are the two inflection points of C, let R and S be the other two points where AB meets C. (See Figure 5.34. Also see Challenging Problem 17 in Chapter 2 for more background.)

(a) Find the ratio of the area bounded by UV and C to the area bounded by PQ and C.

(b) Show that the area bounded by RS and C is divided at A and B into three parts in the ratio 1 : 2 : 1.

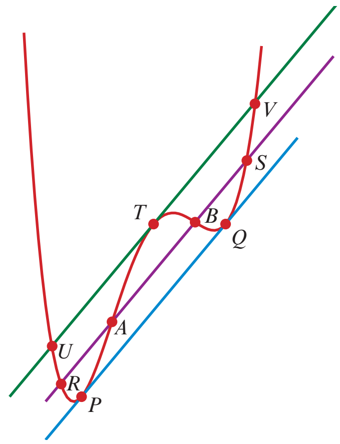


Figure 5.34