Chapter 11: Rolling, Torque, and Angular Momentum

- ✓ **Rolling**
- ✓ **Torque**
- ✓ **Angular Momentum**
- ✓ **Newton’s Second Law in Angular Form**
- ✓ **Conservation of Angular Momentum**

Chapter 11: Rolling, Torque, and Angular Momentum

Session 25:

- ✓ **Newton's Second Law in Angular Form**
- ✓ **Conservation of Angular Momentum**
- ✓ **Examples**

Newton's Second Law in Angular Form

$$\boxed{\vec{\tau}_{net} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}_{net} = \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt}}$$

$$\boxed{\vec{l} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = m(\vec{\mathbf{r}} \times \vec{\mathbf{v}})}$$

$$\frac{d\vec{l}}{dt} = \frac{d}{dt}(\vec{\mathbf{r}} \times \vec{\mathbf{p}}) = m\frac{d}{dt}(\vec{\mathbf{r}} \times \vec{\mathbf{v}}) = m(\frac{d\vec{\mathbf{r}}}{dt} \times \vec{\mathbf{v}} + \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{v}}}{dt})$$

$$\frac{d}{dt}\left(\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right) = \frac{d\vec{\mathbf{A}}}{dt} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \frac{d\vec{\mathbf{B}}}{dt}$$

$$\frac{d\vec{\mathbf{r}}}{dt} \times \vec{\mathbf{v}} = \vec{\mathbf{v}} \times \vec{\mathbf{v}} = 0 \quad \Rightarrow \quad \frac{d\vec{l}}{dt} = m\vec{\mathbf{r}} \times \frac{d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt} = \vec{\tau}_{net}$$

Single Particle:

$$\boxed{\vec{\tau}_{net} = \frac{d\vec{l}}{dt}, \quad if\ \vec{\tau}_{net} = 0 \ \Rightarrow \ \vec{l} = \text{constant}}$$

System of Particles:

$$\boxed{\vec{\tau}_{net} = \frac{d\vec{L}_{tot}}{dt}, \quad if\ \vec{\tau}_{net} = 0 \ \Rightarrow \ \vec{L}_{tot} = \text{constant}}$$

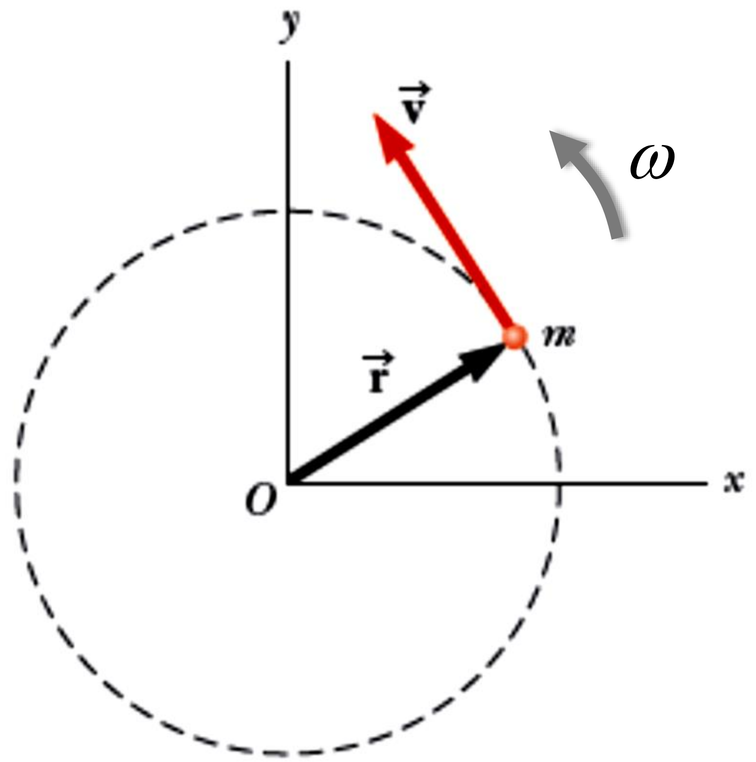
Conservation of Angular Momentum

Conservation of Angular Momentum

Single Particle:

$$\vec{l} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = m(\vec{\mathbf{r}} \times \vec{\mathbf{v}})$$

$$\boxed{\left|\vec{l}\right| = mr^2\omega = I\omega}$$

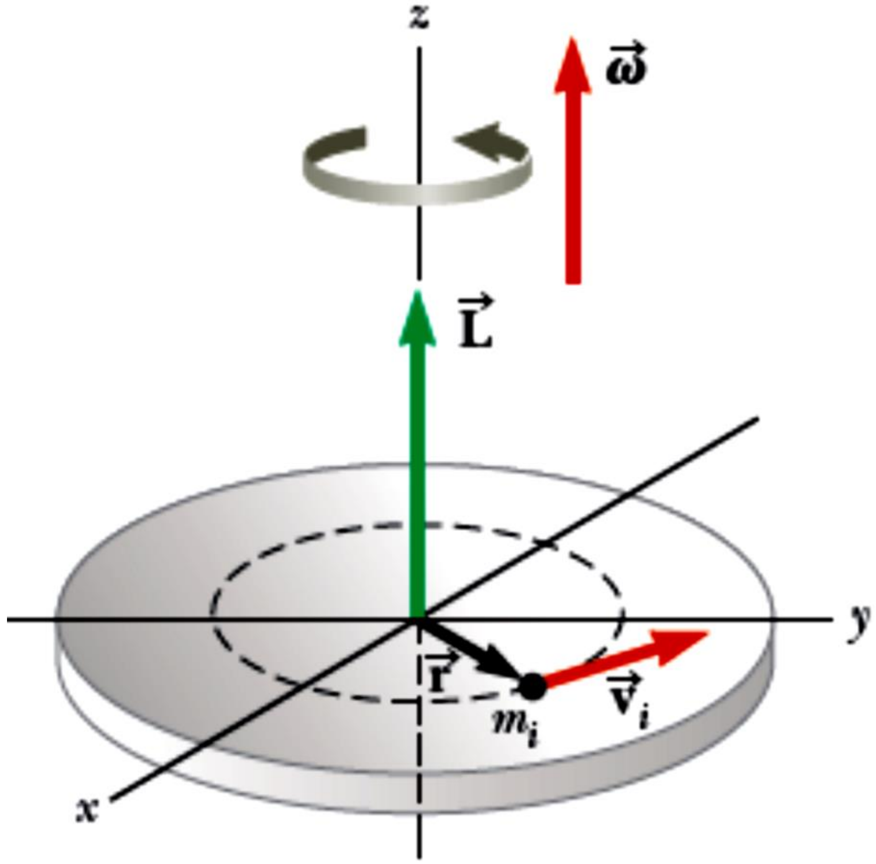


System of Particles:

$$\boxed{\vec{\mathbf{L}}_{tot} = \vec{l}_1 + \vec{l}_2 + \ldots + \vec{l}_n = \sum_{i=1}^{n} \vec{l}_i}$$

Rigid Body:

$$\boxed{\vec{\mathbf{L}} = I\vec{\omega}}$$



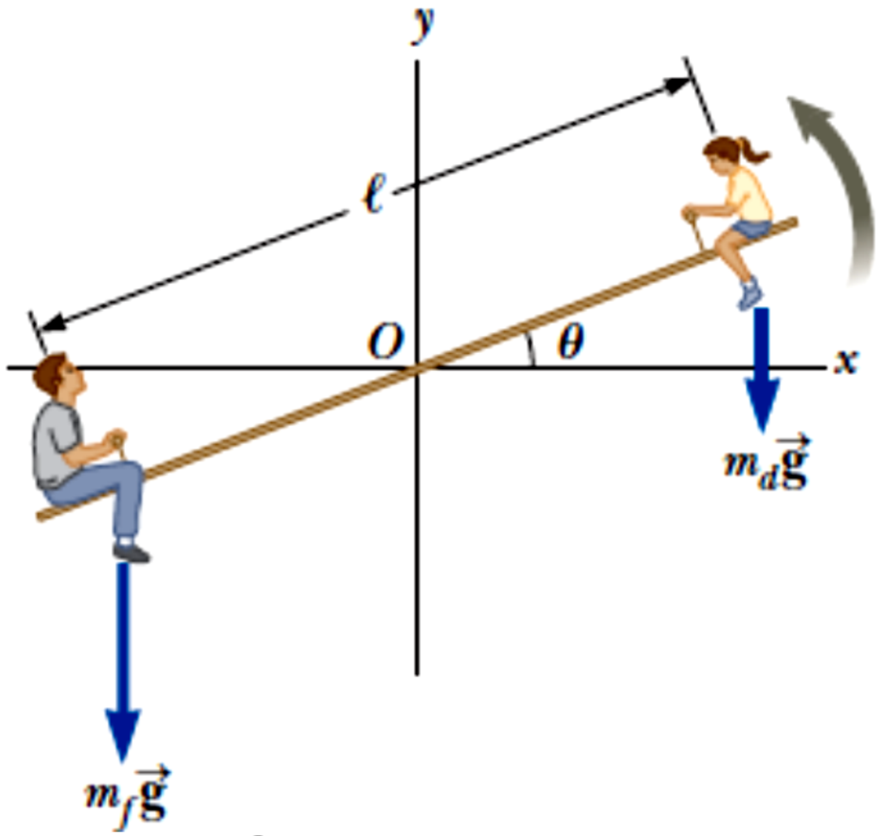
$$\boxed{\vec{\tau}_{net} = \frac{d\vec{L}}{dt}, \quad if\ \vec{\tau}_{net} = 0 \ \Rightarrow\ \vec{L} = \text{constant}}$$

Conservation of Angular Momentum

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau}\ (= \vec{r} \times \vec{F})$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{\ell}\ (= \vec{r} \times \vec{p})$
Linear momentum[b]	$\vec{P}\ (= \Sigma \vec{p}_i)$	Angular momentum[b]	$\vec{L}\ (= \Sigma \vec{\ell}_i)$
Linear momentum[b]	$\vec{P} = M\vec{v}_{com}$	Angular momentum[c]	$L = I\omega$
Newton's second law[b]	$\vec{F}_{net} = \dfrac{d\vec{P}}{dt}$	Newton's second law[b]	$\vec{\tau}_{net} = \dfrac{d\vec{L}}{dt}$
Conservation law[d]	$\vec{P}$ = a constant	Conservation law[d]	$\vec{L}$ = a constant

Ex 11 (10): A father of mass $\mathbf{m_f}$ and his daughter of mass $\mathbf{m_d}$ sit on opposite ends of a seesaw at equal distances from the pivot at the center. The seesaw is modeled as a rigid rod of mass **M** and length *l*, and is pivoted without friction. At a given moment, the combination rotates in a vertical plane with an angular speed $\boldsymbol{\omega}$. (a) Find an expression for the magnitude of the system's angular momentum. (b) Find an expression for the magnitude of the angular acceleration of the system when the seesaw makes an angle θ with the horizontal

$$\boxed{\vec{\tau}_{net} = \frac{d\vec{L}}{dt}} \qquad \mathbf{L} = I\omega$$



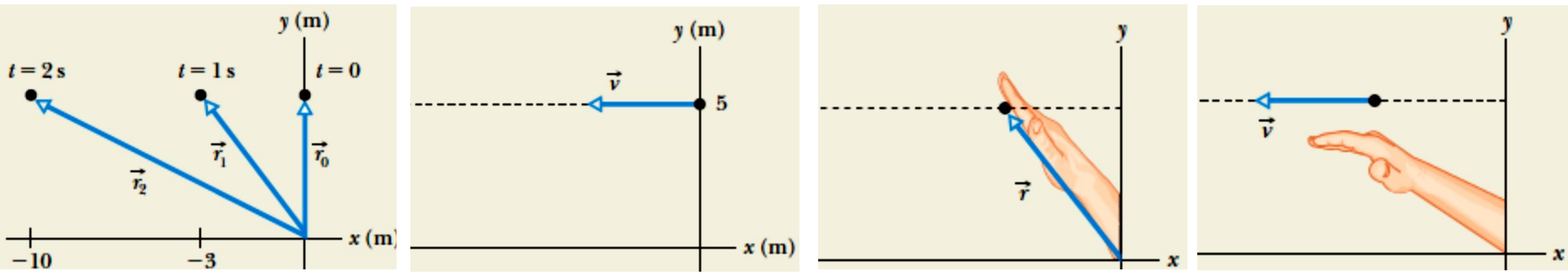
$$I = \frac{1}{12}Ml^2 + m_f\left(\frac{l}{2}\right)^2 + m_d\left(\frac{l}{2}\right)^2 = \frac{l^2}{4}\left(\frac{M}{3} + m_f + m_d\right)$$

$$\boxed{\mathbf{L} = \frac{l^2}{4}\left(\frac{M}{3} + m_f + m_d\right)\omega}$$

$$\boxed{\tau_{net} = m_f g\frac{l}{2}\cos\theta - m_d g\frac{l}{2}\cos\theta} \Rightarrow \frac{1}{2}(m_f - m_d)gl\cos\theta = \frac{l^2}{4}\left(\frac{M}{3} + m_f + m_d\right)\frac{d\omega}{dt}$$

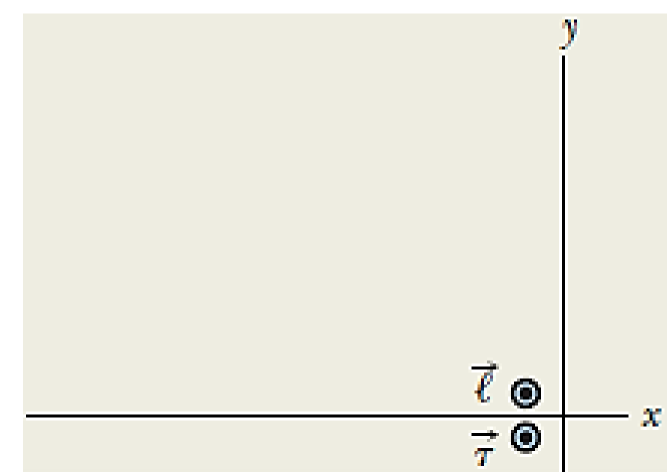
$$\boxed{\alpha = \frac{2(m_f - m_d)g\cos\theta}{l\left(\frac{M}{3} + m_f + m_d\right)}}$$

Ex 12: Figure below shows a freeze-frame of a **0.500 kg** particle moving along a straight line with a position vector given by $\vec{r} = (-2t^2 - t)\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}}$, with **r** in meters and **t** in seconds, starting at t = 0. The position vector points from the origin to the particle. In unit-vector notation, find expressions for the **angular momentum of the particle and the torque** acting on the particle, both with respect to the origin. Justify their algebraic signs in terms of the particle's motion.



$$\boxed{\vec{l} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = m(\vec{\mathbf{r}} \times \vec{\mathbf{v}})}$$

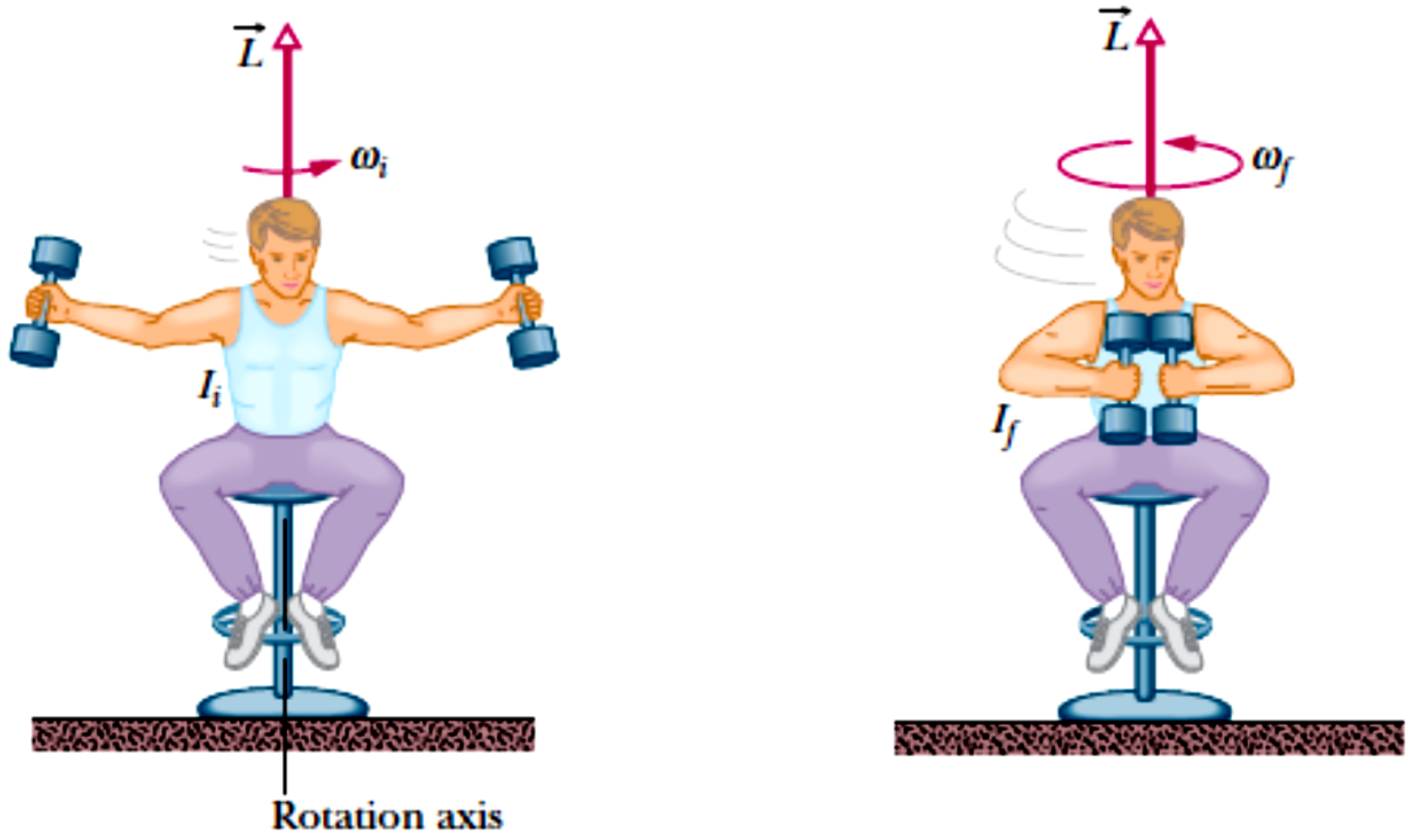
$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}\left[(-2t^2 - t)\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}}\right] = (-4t - 1)\,\hat{\mathbf{i}}$$



$$\boxed{\vec{l} = 0.5\left[(-2t^2 - t)\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}}\right] \times (-4t - 1)\,\hat{\mathbf{i}} = 2.5(4t + 1)(-\hat{\mathbf{j}} \times \hat{\mathbf{i}}) = 2.5(4t + 1)\,\hat{\mathbf{k}} \;\; (kg.m^2 / s)}$$

$$\boxed{\vec{\tau}_{net} = \frac{d\vec{l}}{dt} = \frac{d}{dt}\left[2.5(4t + 1)\,\hat{\mathbf{k}}\right] = 10\,\hat{\mathbf{k}} \;\; (kg.m^2 / s^2 \equiv N.m)}$$

Ex 13:



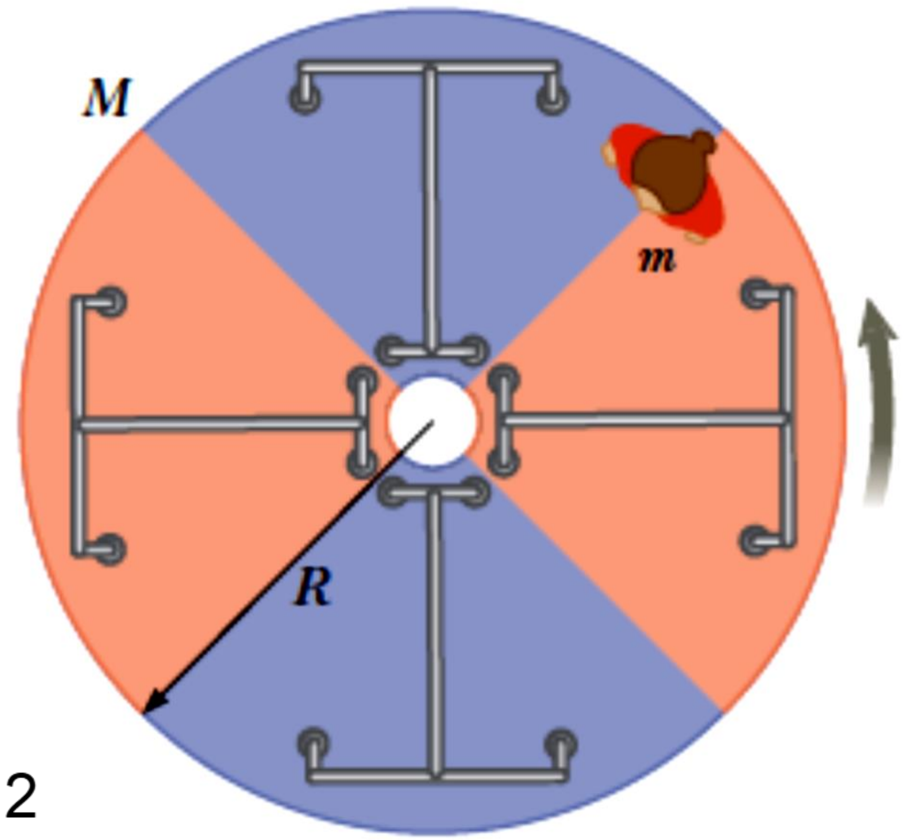
No net external torque acts on the **system consisting of the student, stool, and dumbbells**. Thus, **the angular momentum of that system** about the rotation axis must **remain constant**

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}, \; if \; \vec{\tau}_{net} = 0 \;\Rightarrow\; \vec{L} = \text{constant} \;\Rightarrow L_i = L_f$$

$$I_i \omega_i = I_f \omega_f$$

$$\omega_f = \frac{I_i}{I_f}\omega_i \;\; ; \; I_f < I_i \Rightarrow \omega_f > \omega_i$$

Ex 14: A horizontal platform in the shape of a **circular disk** rotates freely in a horizontal plane about a frictionless, vertical axle. The platform has a mass **M = 100 kg** and a radius **R = 2 m**. A student whose mass is **m = 60 kg** walks slowly from the rim of the disk toward its center. If the angular speed of the system is **2 rad/s** when the student is at the rim, what is the angular speed when she reaches a point **r = 0.50 m** from the center?



$$\boxed{\vec{\tau}_{net} = \frac{d\vec{L}}{dt} = 0 \;\Rightarrow\; \vec{L} = \text{constant} \quad \Rightarrow L_i = L_f}$$

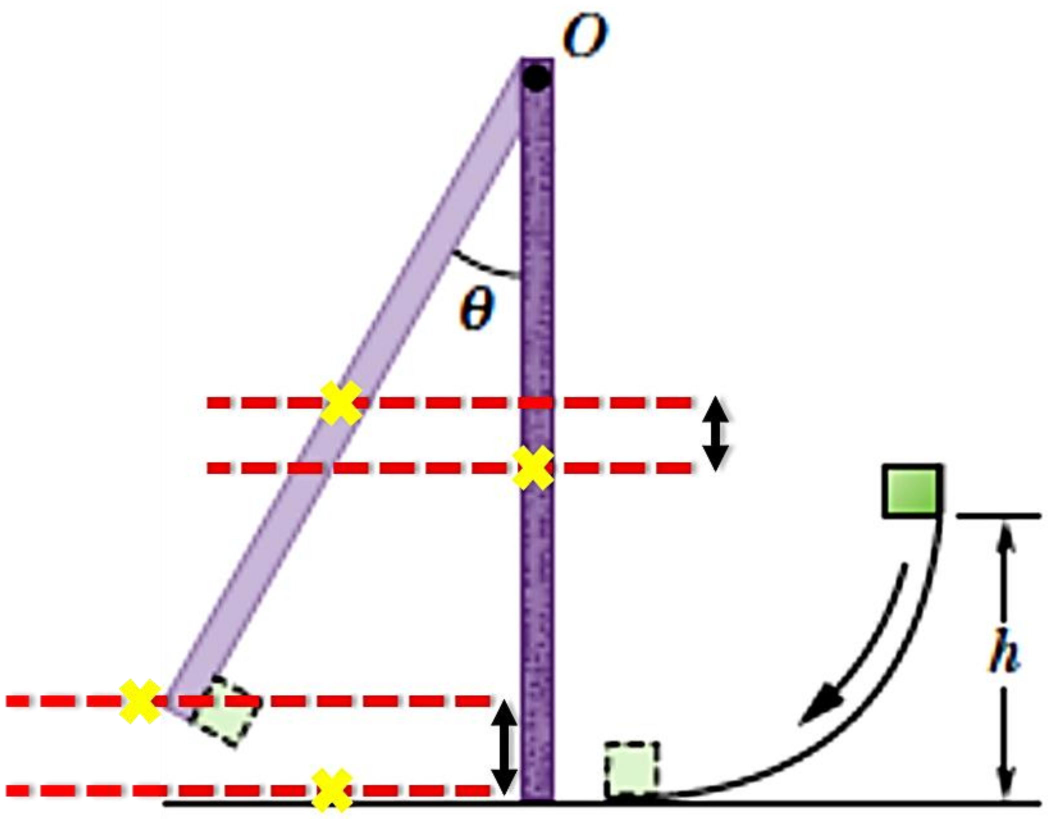
$$I_i\omega_i = I_f\omega_f$$

$$I_i = \frac{1}{2}MR^2 + mR^2 \qquad I_f = \frac{1}{2}MR^2 + mr^2$$

$$\boxed{\omega_f = \frac{I_i}{I_f}\omega_i = \left(\frac{\frac{1}{2}MR^2 + mR^2}{\frac{1}{2}MR^2 + mr^2}\right)\omega_i = 4.1 \quad (rad/s)}$$

Ex 15: (Problem 11. 66 Halliday)

In Fig. 11-58, a small **m=50 g** block slides down a frictionless surface through height **h = 20 cm** and then **sticks to a uniform rod** of mass **M = 100 g** and length **d = 40 cm**. The rod pivots about point O through angle **θ** before momentarily stopping. Find **θ**.



$$mgh = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{2gh}$$

$$\text{collision}: \ \vec{\tau}_{net} = \frac{d\vec{L}}{dt} = 0 \Rightarrow L_i = L_f$$

$$mvd = I_{tot}\omega \qquad I_{tot} = \frac{1}{3}Md^2 + md^2$$

$$\omega = \frac{mv}{(\frac{1}{3}M+m)d} = \frac{m\sqrt{2gh}}{(\frac{1}{3}M+m)d} = 2.97 \ (rad/s)$$

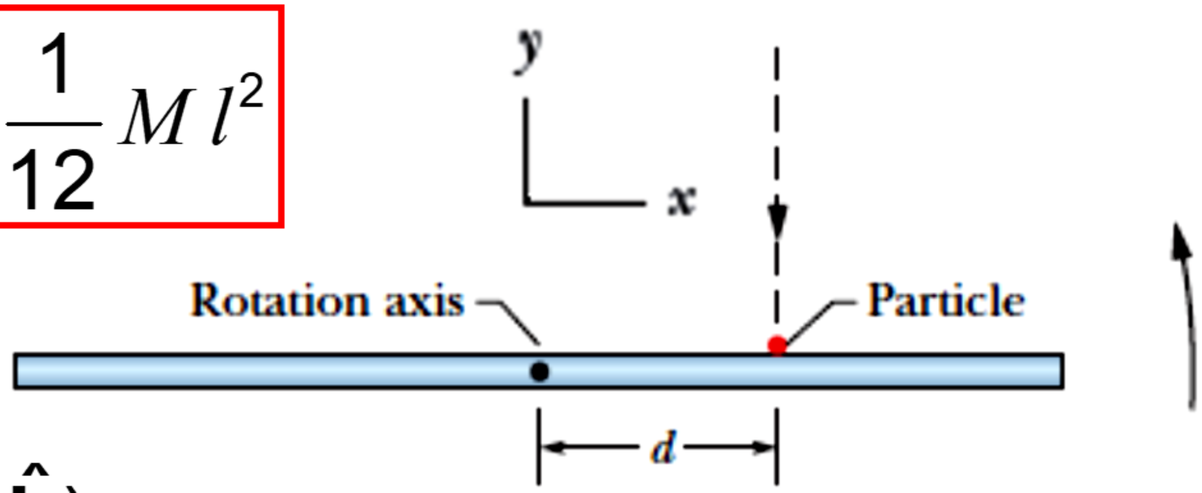
$$\frac{1}{2}I_{tot}\omega^2 = mg(d - d\cos\theta) + Mg(\frac{d}{2} - \frac{d}{2}\cos\theta) \Rightarrow (1-\cos\theta) = \frac{\frac{1}{2}I_{tot}\omega^2}{mgd + Mg\frac{d}{2}} = 0.15$$

$$\cos\theta = 0.85 \Rightarrow \boxed{\theta = \cos^{-1}(0.85) \approx 32^\circ}$$

Ex 16: (Problem 11. 67 Halliday)

Figure 11-59 is an overhead view of a thin uniform rod of length **0.600 m** and mass **M** rotating horizontally at **80 rad/s** counterclockwise about an axis through its center. A particle of mass **M/3** and traveling horizontally at speed **40 m/s** hits the rod and **sticks**. The particle's path is perpendicular to the rod at the instant of the hit, at a distance **d** from the rod's center. (a) At what value of **d** are rod and particle stationary after the hit? (b) In which direction do rod and particle rotate if **d** is greater than this value?

$$\boxed{I_{\text{rod}} = \frac{1}{12} M\, l^2}$$



$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L}_{\text{i}} = \vec{L}_{\text{f}}$$

$$\vec{L}_{\text{i}} = \vec{L}_{\text{i, particle}} + \vec{L}_{\text{i, rod}} = \frac{M}{3} vd(-\hat{\mathbf{k}}) + I_{rod}\,\omega_{\text{i}}(+\hat{\mathbf{k}})$$

$$\vec{L}_{\text{f}} = I_{tot}\,\omega_{\text{f}}(+\hat{\mathbf{k}}) = \left[\frac{M}{3} d^2 + I_{rod}\right]\omega_{\text{f}}(+\hat{\mathbf{k}})$$

$$\boxed{\omega_{\text{f}} = \frac{-\frac{M}{3} vd + \frac{1}{12} M\, l^2 \omega_{\text{i}}}{\frac{M}{3} d^2 + \frac{1}{12} M\, l^2}}$$

$$\boxed{\omega_{\text{f}} = 0 \Rightarrow -\frac{M}{3} vd + \frac{1}{12} M\, l^2 \omega_{\text{i}} = 0 \Rightarrow d = \frac{l^2 \omega_{\text{i}}}{4v} = 0.18\ m}$$

$$\boxed{if\ d > 0.18\ m \quad \Rightarrow \omega_{\text{f}} < 0 \Rightarrow \text{clockwise rotation}}$$