# 1

# INTRODUCTION TO OPTIMIZATION

## 1.1 INTRODUCTION

Optimization is the act of obtaining the best result under given circumstances. In design, construction, and maintenance of any engineering system, engineers have to take many technological and managerial decisions at several stages. The ultimate goal of all such decisions is either to minimize the effort required or to maximize the desired benefit. Since the effort required or the benefit desired in any practical situation can be expressed as a function of certain decision variables, *optimization* can be defined as the process of finding the conditions that give the maximum or minimum value of a function. It can be seen from Fig. 1.1 that if a point $x^*$ corresponds to the minimum value of function $f(x)$, the same point also corresponds to the maximum value of the negative of the function, $-f(x)$. Thus, without loss of generality, optimization can be taken to mean minimization since the maximum of a function can be found by seeking the minimum of the negative of the same function. There is no single method available for solving all optimization problems efficiently. Hence a number of optimization methods have been developed for solving different types of optimization problems.

The optimum seeking methods are also known as *mathematical programming techniques* and are generally studied as a part of operations research. *Operations research* is a branch of mathematics concerned with the application of scientific methods and techniques to decision making problems and with establishing the best or optimal solutions. Table 1.1 lists various mathematical programming techniques together with other well-defined areas of operations research. The classification given in Table 1.1 is not unique; it is given mainly for convenience.

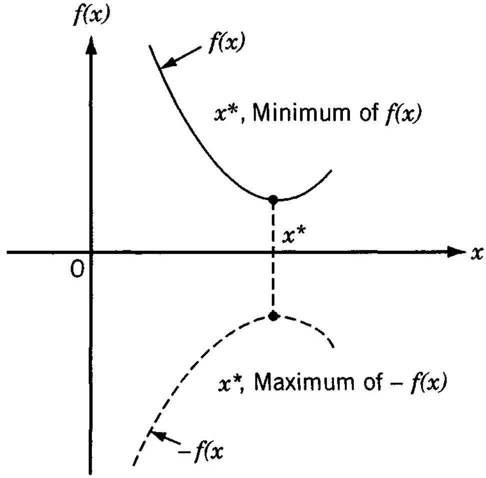


**Figure 1.1** Minimum of $f(x)$ is same as maximum of $-f(x)$.

Mathematical programming techniques are useful in finding the minimum of a function of several variables under a prescribed set of constraints. Stochastic process techniques can be used to analyze problems described by a set of random variables having known probability distributions. Statistical methods enable one to analyze the experimental data and build empirical models to

**TABLE 1.1 Methods of Operations Research**

| Mathematical Programming Techniques | Stochastic Process Techniques | Statistical Methods |
|---|---|---|
| Calculus methods | Statistical decision theory | Regression analysis |
| Calculus of variations | Markov processes | Cluster analysis, pattern recognition |
| Nonlinear programming | Queueing theory | Design of experiments |
| Geometric programming | Renewal theory | Discriminate analysis (factor analysis) |
| Quadratic programming | Simulation methods | |
| Linear programming | Reliability theory | |
| Dynamic programming | | |
| Integer programming | | |
| Stochastic programming | | |
| Separable programming | | |
| Multiobjective programming | | |
| Network methods: CPM and PERT | | |
| Game theory | | |
| Simulated annealing | | |
| Genetic algorithms | | |
| Neural networks | | |

obtain the most accurate representation of the physical situation. This book deals with the theory and application of mathematical programming techniques suitable for the solution of engineering design problems.

## 1.2 HISTORICAL DEVELOPMENT

The existence of optimization methods can be traced to the days of Newton, Lagrange, and Cauchy. The development of differential calculus methods of optimization was possible because of the contributions of Newton and Leibnitz to calculus. The foundations of calculus of variations, which deals with the minimization of functionals, were laid by Bernoulli, Euler, Lagrange, and Weirstrass. The method of optimization for constrained problems, which involves the addition of unknown multipliers, became known by the name of its inventor, Lagrange. Cauchy made the first application of the steepest descent method to solve unconstrained minimization problems. Despite these early contributions, very little progress was made until the middle of the twentieth century, when high-speed digital computers made implementation of the optimization procedures possible and stimulated further research on new methods. Spectacular advances followed, producing a massive literature on optimization techniques. This advancement also resulted in the emergence of several well-defined new areas in optimization theory.

It is interesting to note that the major developments in the area of numerical methods of unconstrained optimization have been made in the United Kingdom only in the 1960s. The development of the simplex method by Dantzig in 1947 for linear programming problems and the annunciation of the principle of optimality in 1957 by Bellman for dynamic programming problems paved the way for development of the methods of constrained optimization. Work by Kuhn and Tucker in 1951 on the necessary and sufficiency conditions for the optimal solution of programming problems laid the foundations for a great deal of later research in nonlinear programming. The contributions of Zoutendijk and Rosen to nonlinear programming during the early 1960s have been very significant. Although no single technique has been found to be universally applicable for nonlinear programming problems, work of Carroll and Fiacco and McCormick allowed many difficult problems to be solved by using the well-known techniques of unconstrained optimization. Geometric programming was developed in the 1960s by Duffin, Zener, and Peterson. Gomory did pioneering work in integer programming, one of the most exciting and rapidly developing areas of optimization. The reason for this is that most real-world applications fall under this category of problems. Dantzig and Charnes and Cooper developed stochastic programming techniques and solved problems by assuming design parameters to be independent and normally distributed.

The desire to optimize more than one objective or goal while satisfying the physical limitations led to the development of multiobjective programming methods. Goal programming is a well-known technique for solving specific types of multiobjective optimization problems. The goal programming was originally proposed for linear problems by Charnes and Cooper in 1961. The foundations of game theory were laid by von Neumann in 1928 and since then the technique has been applied to solve several mathematical economics and military problems. Only during the last few years has game theory been applied to solve engineering design problems.

Simulated annealing, genetic algorithms, and neural network methods represent a new class of mathematical programming techniques that have come into prominence during the last decade. Simulated annealing is analogous to the physical process of annealing of solids. The genetic algorithms are search techniques based on the mechanics of natural selection and natural genetics. Neural network methods are based on solving the problem using the efficient computing power of the network of interconnected "neuron" processors.

## 1.3 ENGINEERING APPLICATIONS OF OPTIMIZATION

Optimization, in its broadest sense, can be applied to solve any engineering problem. To indicate the wide scope of the subject, some typical applications from different engineering disciplines are given below.

1. Design of aircraft and aerospace structures for minimum weight
2. Finding the optimal trajectories of space vehicles
3. Design of civil engineering structures such as frames, foundations, bridges, towers, chimneys, and dams for minimum cost
4. Minimum-weight design of structures for earthquake, wind, and other types of random loading
5. Design of water resources systems for maximum benefit
6. Optimal plastic design of structures
7. Optimum design of linkages, cams, gears, machine tools, and other mechanical components
8. Selection of machining conditions in metal-cutting processes for minimum production cost
9. Design of material handling equipment such as conveyors, trucks, and cranes for minimum cost
10. Design of pumps, turbines, and heat transfer equipment for maximum efficiency
11. Optimum design of electrical machinery such as motors, generators, and transformers
12. Optimum design of electrical networks
13. Shortest route taken by a salesperson visiting various cities during one tour
14. Optimal production planning, controlling, and scheduling

15. Analysis of statistical data and building empirical models from experimental results to obtain the most accurate representation of the physical phenomenon
16. Optimum design of chemical processing equipment and plants
17. Design of optimum pipeline networks for process industries
18. Selection of a site for an industry
19. Planning of maintenance and replacement of equipment to reduce operating costs
20. Inventory control
21. Allocation of resources or services among several activities to maximize the benefit
22. Controlling the waiting and idle times and queueing in production lines to reduce the costs
23. Planning the best strategy to obtain maximum profit in the presence of a competitor
24. Optimum design of control systems

## 1.4 STATEMENT OF AN OPTIMIZATION PROBLEM

An optimization or a mathematical programming problem can be stated as follows.

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X})$$

subject to the constraints

$$\begin{aligned} g_j(\mathbf{X}) &\leq 0, \quad j = 1,2,\ldots,m \\ l_j(\mathbf{X}) &= 0, \quad j = 1,2,\ldots,p \end{aligned} \tag{1.1}$$

where $\mathbf{X}$ is an $n$-dimensional vector called the *design vector*, $f(\mathbf{X})$ is termed the *objective function*, and $g_j$ ($\mathbf{X}$) and $l_j$ ($\mathbf{X}$) are known as *inequality* and *equality* constraints, respectively. The number of variables $n$ and the number of constraints $m$ and/or $p$ need not be related in any way. The problem stated in Eq. (1.1) is called a *constrained optimization problem.*† Some optimization

†In the mathematical programming literature, the equality constraints $l_j(\mathbf{X}) = 0, j = 1,2,\ldots,p$ are often neglected, for simplicity, in the statement of a constrained optimization problem, although several methods are available for handling problems with equality constraints.

problems do not involve any constraints and can be stated as:

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X}) \tag{1.2}$$

Such problems are called *unconstrained optimization problems.*

### 1.4.1 Design Vector

Any engineering system or component is defined by a set of quantities some of which are viewed as variables during the design process. In general, certain quantities are usually fixed at the outset and these are called *preassigned parameters.* All the other quantities are treated as variables in the design process and are called *design* or *decision variables* $x_i$, $i = 1,2,\ldots,n$. The design variables are collectively represented as a design vector $\mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$. As an example, consider the design of the gear pair shown in Fig. 1.2, characterized by its face width $b$, number of teeth $T_1$ and $T_2$, center distance $d$, pressure angle $\psi$, tooth profile, and material. If center distance $d$, pressure angle $\psi$, tooth profile, and material of the gears are fixed in advance, these quantities can be called *preassigned parameters.* The remaining quantities can be collectively represented by a design vector $\mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b \\ T_1 \\ T_2 \end{Bmatrix}$. If there are no restrictions on the choice of $b$, $T_1$, and $T_2$, any set of three numbers will constitute a design for the gear pair. If an $n$-dimensional Cartesian space with each coordinate axis representing a design variable $x_i$ ($i = 1,2,\ldots,n$) is considered, the space is called the *design variable space* or simply, *design space.* Each point in the $n$-dimensional design space is called a *design point* and represents either a possible or an impossible solution to the design problem. In the case of the design of a gear pair, the design point $\begin{Bmatrix} 1.0 \\ 20 \\ 40 \end{Bmatrix}$, for example, represents a possible solution, whereas the design point $\begin{Bmatrix} 1.0 \\ -20 \\ 40.5 \end{Bmatrix}$ represents an impos-

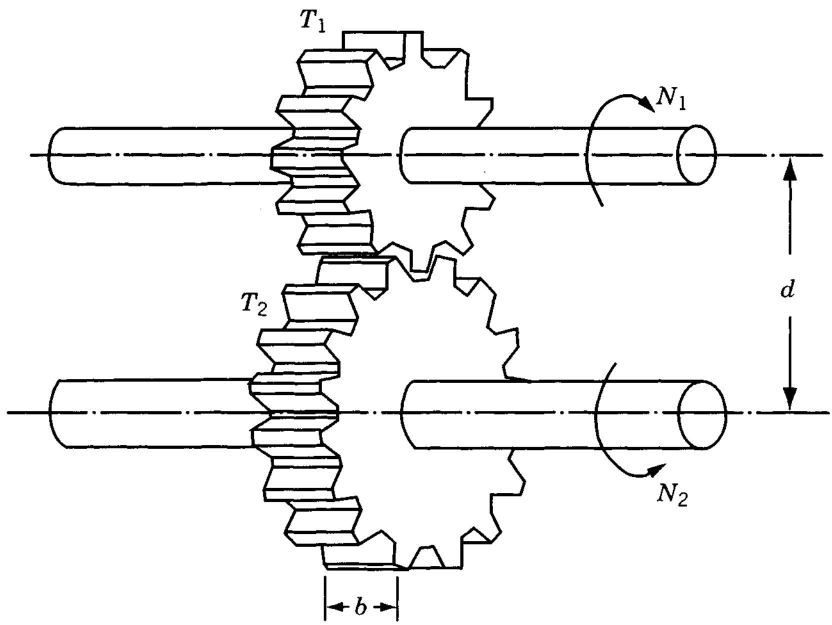


**Figure 1.2** Gear pair in mesh.

sible solution since it is not possible to have either a negative value or a fractional value for the number of teeth.

### 1.4.2 Design Constraints

In many practical problems, the design variables cannot be chosen arbitrarily; rather, they have to satisfy certain specified functional and other requirements. The restrictions that must be satisfied to produce an acceptable design are collectively called *design constraints.* Constraints that represent limitations on the behavior or performance of the system are termed *behavior* or *functional constraints.* Constraints that represent physical limitations on design variables such as availability, fabricability, and transportability are known as *geometric* or *side constraints.* For example, for the gear pair shown in Fig. 1.2, the face width $b$ cannot be taken smaller than a certain value, due to strength requirements. Similarly, the ratio of the numbers of teeth, $T_1/T_2$, is dictated by the speeds of the input and output shafts, $N_1$ and $N_2$. Since these constraints depend on the performance of the gear pair, they are called behavior constraints. The values of $T_1$ and $T_2$ cannot be any real numbers but can only be integers. Further, there can be upper and lower bounds on $T_1$ and $T_2$ due to manufacturing limitations. Since these constraints depend on the physical limitations, they are called side *constraints.*

### 1.4.3 Constraint Surface

For illustration, consider an optimization problem with only inequality constraints $g_j(\mathbf{X}) \leq 0$. The set of values of $\mathbf{X}$ that satisfy the equation $g_j(\mathbf{X}) = 0$ forms a hypersurface in the design space and is called a *constraint surface.* Note that this is an $(n-1)$-dimensional subspace, where $n$ is the number of design variables. The constraint surface divides the design space into two regions: one in which $g_j(\mathbf{X}) < 0$ and the other in which $g_j(\mathbf{X}) > 0$. Thus the points lying on the hypersurface will satisfy the constraint $g_j(\mathbf{X})$ critically, whereas the points lying in the region where $g_j(\mathbf{X}) > 0$ are infeasible or unacceptable, and the points lying in the region where $g_j(\mathbf{X}) < 0$ are feasible or acceptable. The collection of all the constraint surfaces $g_j(\mathbf{X}) = 0$, $j = 1,2,\ldots,m$, which separates the acceptable region is called the *composite constraint surface.*

Figure 1.3 shows a hypothetical two-dimensional design space where the infeasible region is indicated by hatched lines. A design point that lies on one or more than one constraint surface is called a *bound point*, and the associated constraint is called an *active constraint.* Design points that do not lie on any constraint surface are known as *free points.* Depending on whether a particular design point belongs to the acceptable or unacceptable region, it can be iden-

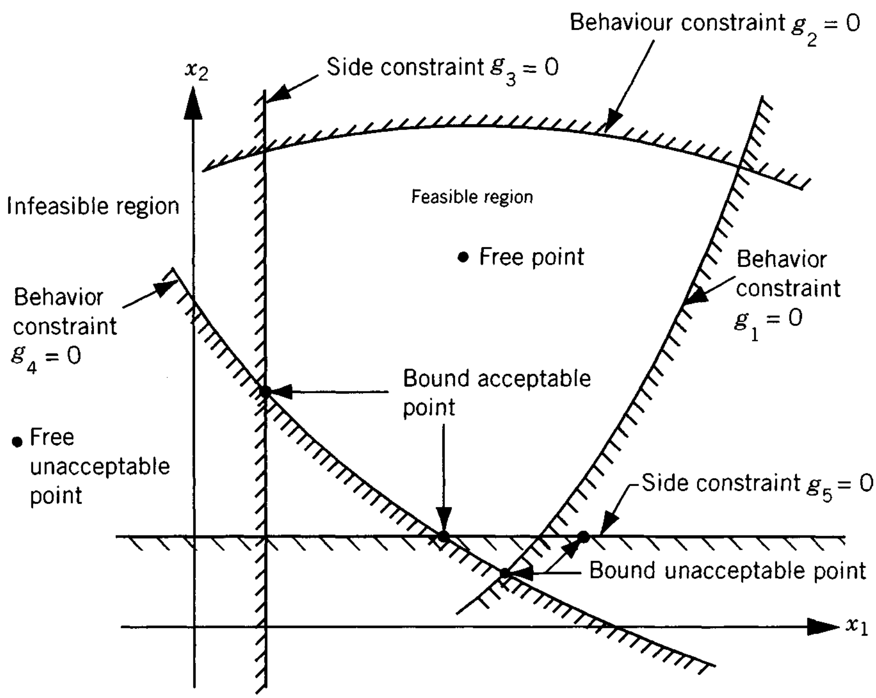


**Figure 1.3** Constraint surfaces in a hypothetical two-dimensional design space.

tified as one of the following four types:

1. Free and acceptable point
2. Free and unacceptable point
3. Bound and acceptable point
4. Bound and unacceptable point

All four types of points are shown in Fig. 1.3.

### 1.4.4 Objective Function

The conventional design procedures aim at finding an acceptable or adequate design which merely satisfies the functional and other requirements of the problem. In general, there will be more than one acceptable design, and the purpose of optimization is to choose the best one of the many acceptable designs available. Thus a criterion has to be chosen for comparing the different alternative acceptable designs and for selecting the best one. The criterion with respect to which the design is optimized, when expressed as a function of the design variables, is known as the *criterion or merit* or *objective function*. The choice of objective function is governed by the nature of problem. The objective function for minimization is generally taken as weight in aircraft and aerospace structural design problems. In civil engineering structural designs, the objective is usually taken as the minimization of cost. The maximization of mechanical efficiency is the obvious choice of an objective in mechanical engineering systems design. Thus the choice of the objective function appears to be straightforward in most design problems. However, there may be cases where the optimization with respect to a particular criterion may lead to results that may not be satisfactory with respect to another criterion. For example, in mechanical design, a gearbox transmitting the maximum power may not have the minimum weight. Similarly, in structural design, the minimum-weight design may not correspond to minimum stress design, and the minimum stress design, again, may not correspond to maximum frequency design. Thus the selection of the objective function can be one of the most important decisions in the whole optimum design process.

In some situations, there may be more than one criterion to be satisfied simultaneously. For example, a gear pair may have to be designed for minimum weight and maximum efficiency while transmitting a specified horsepower. An optimization problem involving multiple objective functions is known as a *multiobjective programming problem.* With multiple objectives there arises a possibility of conflict, and one simple way to handle the problem is to construct an overall objective function as a linear combination of the conflicting multiple objective functions. Thus if $f_1(\mathbf{X})$ and $f_2(\mathbf{X})$ denote two objective functions, construct a new (overall) objective function for optimization as

$$f(\mathbf{X}) = \alpha_1 f_1(\mathbf{X}) + \alpha_2 f_2(\mathbf{X}) \tag{1.3}$$

where $\alpha_1$ and $\alpha_2$ are constants whose values indicate the relative importance of one objective function relative to the other.

### 1.4.5 Objective Function Surfaces

The locus of all points satisfying $f(\mathbf{X}) = c =$ constant forms a hypersurface in the design space, and for each value of $c$ there corresponds a different member of a family of surfaces. These surfaces, called *objective function surfaces*, are shown in a hypothetical two-dimensional design space in Fig. 1.4.

Once the objective function surfaces are drawn along with the constraint surfaces, the optimum point can be determined without much difficulty. But the main problem is that as the number of design variables exceeds two or three, the constraint and objective function surfaces become complex even for visualization and the problem has to be solved purely as a mathematical problem. The following example illustrates the graphical optimization procedure.

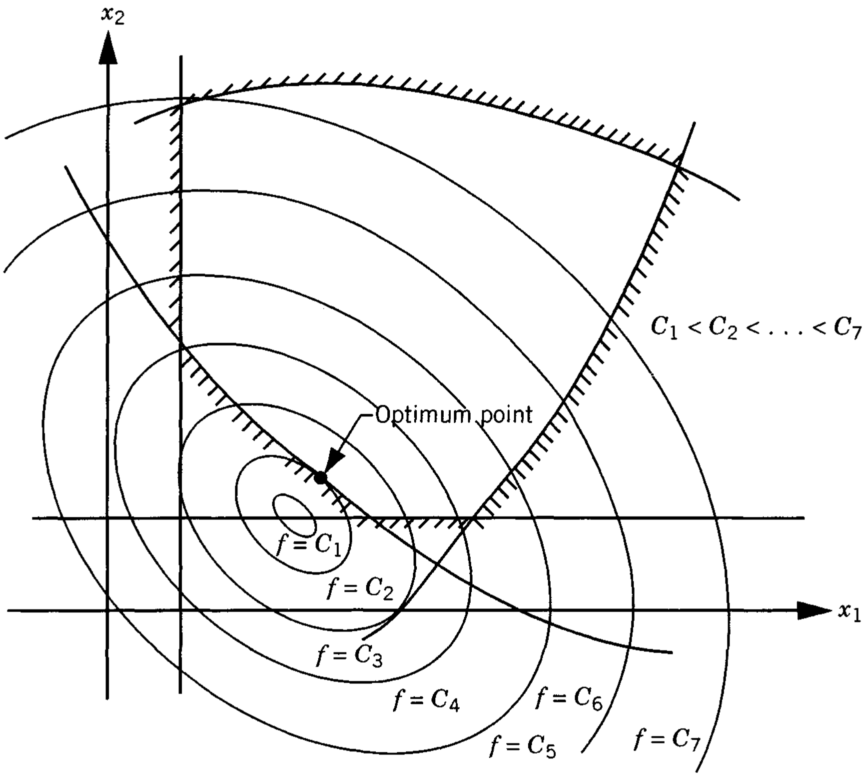


**Figure 1.4** Contours of the objective function.

***Example 1.1*** Design a uniform column of tubular section (Fig. 1.5) to carry a compressive load $P = 2500\ \text{kg}_f$ for minimum cost. The column is made up of a material that has a yield stress ($\sigma_y$) of 500 $\text{kg}_f/\text{cm}^2$, modulus of elasticity ($E$) of $0.85 \times 10^6\ \text{kg}_f/\text{cm}^2$, and density ($\rho$) of 0.0025 $\text{kg}_f/\text{cm}^3$. The length of the column is 250 cm. The stress induced in the column should be less than the buckling stress as well as the yield stress. The mean diameter of the column is restricted to lie between 2 and 14 cm, and columns with thicknesses outside the range 0.2 to 0.8 cm are not available in the market. The cost of the column includes material and construction costs and can be taken as $5W + 2d$, where $W$ is the weight in kilograms force and $d$ is the mean diameter of the column in centimeters.

SOLUTION The design variables are the mean diameter ($d$) and tube thickness ($t$):

$$\mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} d \\ t \end{Bmatrix} \tag{E$_1$}$$

The objective function to be minimized is given by

$$f(\mathbf{X}) = 5W + 2d = 5\rho l \pi\, dt + 2d = 9.82 x_1 x_2 + 2x_1 \tag{E$_2$}$$

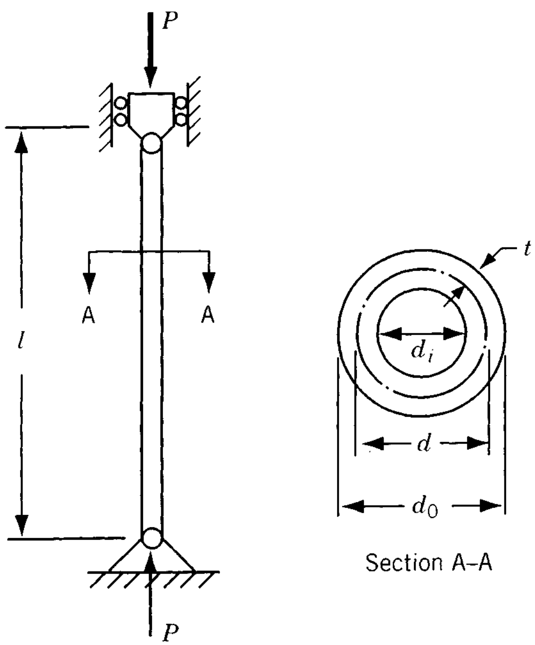


**Figure 1.5** Tubular column under compression.

The behavior constraints can be expressed as

$$\text{stress induced} \leq \text{yield stress}$$
$$\text{stress induced} \leq \text{buckling stress}$$

The induced stress is given by

$$\text{induced stress} = \sigma_i = \frac{P}{\pi\, dt} = \frac{2500}{\pi x_1 x_2} \tag{E$_3$}$$

The buckling stress for a pin-connected column is given by

$$\text{buckling stress} = \sigma_b = \frac{\text{Euler buckling load}}{\text{cross-sectional area}} = \frac{\pi^2 EI}{l^2} \frac{1}{\pi\, dt} \tag{E$_4$}$$

where

$$\begin{aligned} I &= \text{second moment of area of the cross section of the column} \\ &= \frac{\pi}{64}(d_o^4 - d_i^4) \\ &= \frac{\pi}{64}(d_o^2 + d_i^2)(d_o + d_i)(d_o - d_i) = \frac{\pi}{64}[(d+t)^2 + (d-t)^2] \\ &\quad \cdot [(d+t) + (d-t)][(d+t) - (d-t)] \\ &= \frac{\pi}{8} dt(d^2 + t^2) = \frac{\pi}{8} x_1 x_2 (x_1^2 + x_2^2) \end{aligned} \tag{E$_5$}$$

Thus the behavior constraints can be restated as

$$g_1(\mathbf{X}) = \frac{2500}{\pi x_1 x_2} - 500 \leq 0 \tag{E$_6$}$$

$$g_2(\mathbf{X}) = \frac{2500}{\pi x_1 x_2} - \frac{\pi^2 (0.85 \times 10^6)(x_1^2 + x_2^2)}{8(250)^2} \leq 0 \tag{E$_7$}$$

The side constraints are given by

$$2 \leq d \leq 14$$
$$0.2 \leq t \leq 0.8$$

which can be expressed in standard form as

$$g_3(\mathbf{X}) = -x_1 + 2.0 \leq 0 \tag{E$_8$}$$

$$g_4(\mathbf{X}) = x_1 - 14.0 \le 0 \tag{E$_9$}$$

$$g_5(\mathbf{X}) = -x_2 + 0.2 \le 0 \tag{E$_{10}$}$$

$$g_6(\mathbf{X}) = x_2 - 0.8 \le 0 \tag{E$_{11}$}$$

Since there are only two design variables, the problem can be solved graphically as shown below.

First, the constraint surfaces are to be plotted in a two-dimensional design space where the two axes represent the two design variables $x_1$ and $x_2$. To plot the first constraint surface, we have

$$g_1(\mathbf{X}) = \frac{2500}{\pi x_1 x_2} - 500 \le 0$$

that is,

$$x_1 x_2 \ge 1.593$$

Thus the curve $x_1 x_2 = 1.593$ represents the constraint surface $g_1(\mathbf{X}) = 0$. This curve can be plotted by finding several points on the curve. The points on the curve can be found by giving a series of values to $x_1$ and finding the corresponding values of $x_2$ that satisfy the relation $x_1 x_2 = 1.593$:

| $x_1$ | 2.0 | 4.0 | 6.0 | 8.0 | 10.0 | 12.0 | 14.0 |
|---|---|---|---|---|---|---|---|
| $x_2$ | 0.7965 | 0.3983 | 0.2655 | 0.1990 | 0.1593 | 0.1328 | 0.1140 |

These points are plotted and a curve $P_1 Q_1$ passing through all these points is drawn as shown in Fig. 1.6, and the infeasible region, represented by $g_1(\mathbf{X}) > 0$ or $x_1 x_2 < 1.593$, is shown by hatched lines.† Similarly, the second constraint $g_2(\mathbf{X}) \le 0$ can be expressed as $x_1 x_2(x_1^2 + x_2^2) \ge 47.3$ and the points lying on the constraint surface $g_2(\mathbf{X}) = 0$ can be obtained as follows:

[for $x_1 x_2(x_1^2 + x_2^2) = 47.3$]:

| $x_1$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
|---|---|---|---|---|---|---|---|
| $x_2$ | 2.41 | 0.716 | 0.219 | 0.0926 | 0.0473 | 0.0274 | 0.0172 |

These points are plotted as curve $P_2 Q_2$, the feasible region is identified, and the infeasible region is shown by hatched lines as shown in Fig. 1.6. The plotting of side constraints is very simple since they represent straight lines. After plotting all the six constraints, the feasible region can be seen to be given by the bounded area *ABCDEA*.

†The infeasible region can be identified by testing whether the origin lies in the feasible or infeasible region.

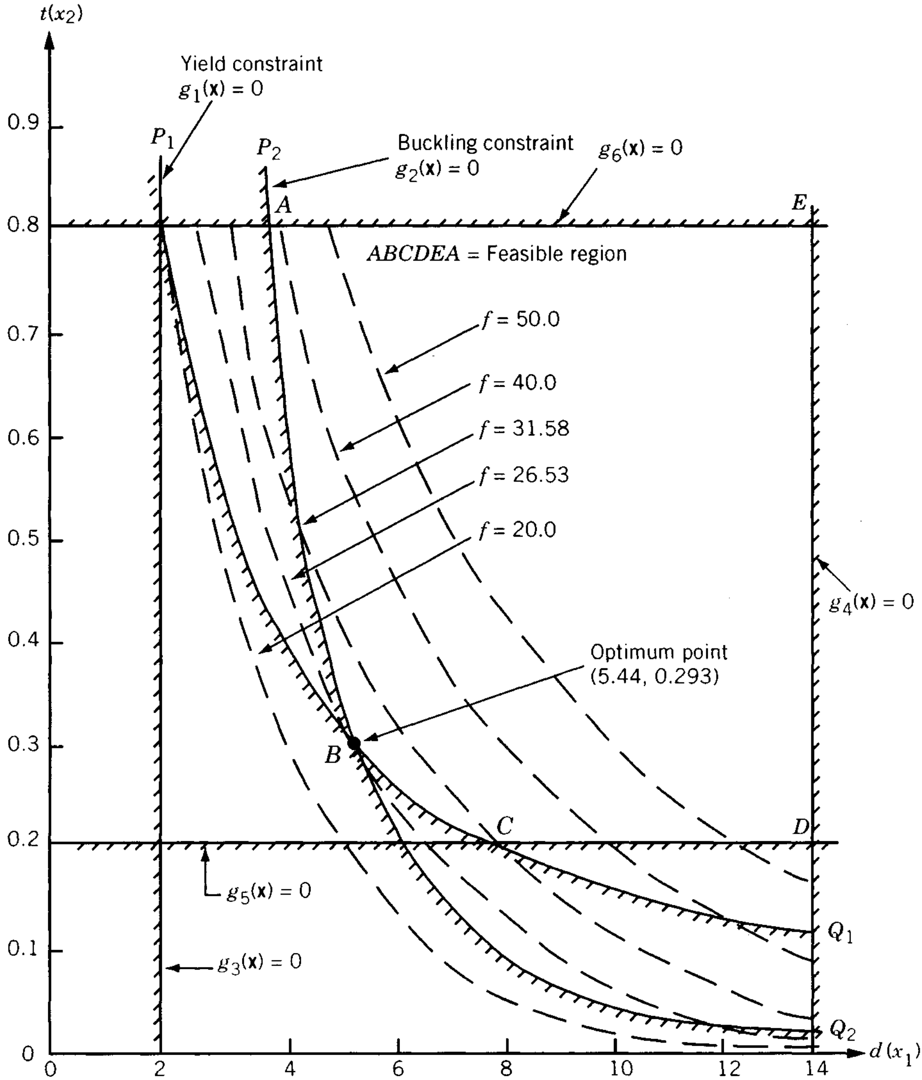


**Figure 1.6** Graphical optimization of Example 1.1.

Next, the contours of the objective function are to be plotted before finding the optimum point. For this, we plot the curves given by

$$f(\mathbf{X}) = 9.82x_1x_2 + 2x_1 = c = \text{constant}$$

for a series of values of $c$. By giving different values to $c$, the contours of $f$ can be plotted with the help of the following points.

For $9.82x_1x_2 + 2x_1 = 50.0$:

| $x_2$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|
| $x_1$ | 16.77 | 12.62 | 10.10 | 8.44 | 7.24 | 6.33 | 5.64 | 5.07 |

For $9.82x_1x_2 + 2x_1 = 40.0$:

| $x_2$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|
| $x_1$ | 13.40 | 10.10 | 8.08 | 6.75 | 5.79 | 5.06 | 4.51 | 4.05 |

For $9.82x_1x_2 + 2x_1 = 31.58$ (passing through the corner point $C$):

| $x_2$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|
| $x_1$ | 10.57 | 7.96 | 6.38 | 5.33 | 4.57 | 4.00 | 3.56 | 3.20 |

For $9.82x_1x_2 + 2x_1 = 26.53$ (passing through the corner point $B$):

| $x_2$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|
| $x_1$ | 8.88 | 6.69 | 5.36 | 4.48 | 3.84 | 3.36 | 2.99 | 2.69 |

For $9.82x_1x_2 = 2x_1 = 20.0$:

| $x_2$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|
| $x_1$ | 6.70 | 5.05 | 4.04 | 3.38 | 2.90 | 2.53 | 2.26 | 2.02 |

These contours are shown in Fig. 1.6 and it can be seen that the objective function cannot be reduced below a value of 26.53 (corresponding to point $B$) without violating some of the constraints. Thus the optimum solution is given by point $B$ with $d^* = x_1^* = 5.44$ cm and $t^* = x_2^* = 0.293$ cm with $f_{\min} = 26.53$.

## 1.5 CLASSIFICATION OF OPTIMIZATION PROBLEMS

Optimization problems can be classified in several ways, as described below.

### 1.5.1 Classification Based on the Existence of Constraints

As indicated earlier, any optimization problem can be classified as constrained or unconstrained, depending on whether or not constraints exist in the problem.

### 1.5.2 Classification Based on the Nature of the Design Variables

Based on the nature of design variables encountered, optimization problems can be classified into two broad categories. In the first category, the problem is to find values to a set of design parameters that make some prescribed function of these parameters minimum subject to certain constraints. For example, the problem of minimum-weight design of a prismatic beam shown in Fig. 1.7*a* subject to a limitation on the maximum deflection can be stated as follows.

$$\text{Find } \mathbf{X} = \begin{Bmatrix} b \\ d \end{Bmatrix} \text{ which minimizes} \qquad f(\mathbf{X}) = \rho l b d \tag{1.4}$$

subject to the constraints

$$\delta_{\text{tip}}(\mathbf{X}) \leq \delta_{\max}$$
$$b \geq 0$$
$$d \geq 0$$

where $\rho$ is the density and $\delta_{\text{tip}}$ is the tip deflection of the beam. Such problems are called *parameter* or *static optimization problems.* In the second category of problems, the objective is to find a set of design parameters, which are all continuous functions of some other parameter, that minimizes an objective function subject to a set of constraints. If the cross-sectional dimensions of the rectangular beam are allowed to vary along its length as shown in Fig. 1.7*b*, the optimization problem can be stated as:

$$\text{Find } \mathbf{X}(t) = \begin{Bmatrix} b(t) \\ d(t) \end{Bmatrix} \text{ which minimizes}$$

$$f[\mathbf{X}(t)] = \rho \int_0^l b(t)\, d(t)\, dt \tag{1.5}$$

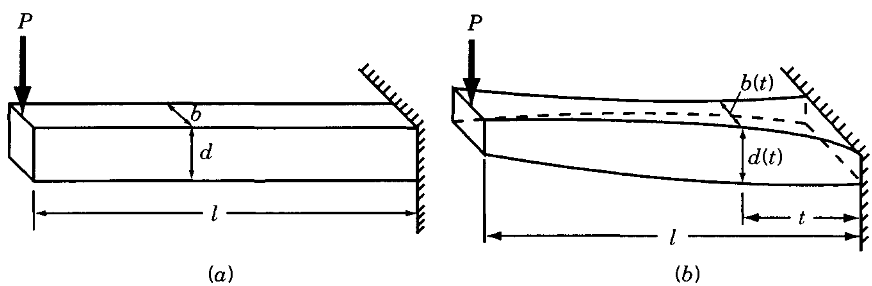


**Figure 1.7** Cantilever beam under concentrated load.

subject to the constraints

$$\begin{aligned} \delta_{\text{tip}}[\mathbf{X}(t)] &\le \delta_{\max}, & 0 \le t \le l \\ b(t) &\ge 0, & 0 \le t \le l \\ d(t) &\ge 0, & 0 \le t \le l \end{aligned}$$

Here the design variables are functions of the length parameter $t$. This type of problem, where each design variable is a function of one or more parameters, is known as a *trajectory* or *dynamic optimization problem* [1.40].

### 1.5.3 Classification Based on the Physical Structure of the Problem

Depending on the physical structure of the problem, optimization problems can be classified as optimal control and nonoptimal control problems.

***Optimal Control Problem.*** An *optimal control* (OC) *problem* is a mathematical programming problem involving a number of stages, where each stage evolves from the preceding stage in a prescribed manner. It is usually described by two types of variables: the control (design) and the state variables. The *control variables* define the system and govern the evolution of the system from one stage to the next, and the *state variables* describe the behavior or status of the system in any stage. The problem is to find a set of control or design variables such that the total objective function (also known as the *performance index*, PI) over all the stages is minimized subject to a set of constraints on the control and state variables. An OC problem can be stated as follows [1.40]:

$$\text{Find } \mathbf{X} \text{ which minimizes } f(\mathbf{X}) = \sum_{i=1}^{l} f_i(x_i,y_i) \tag{1.6}$$

subject to the constraints

$$\begin{aligned} q_i(x_i,y_i) + y_i &= y_{i+1}, & i &= 1,2,\ldots,l \\ \mathbf{g}_j(x_j) &\le 0, & j &= 1,2,\ldots,l \\ \mathbf{h}_k(y_k) &\le 0, & k &= 1,2,\ldots,l \end{aligned}$$

where $x_i$ is the $i$th control variable, $y_i$ the $i$th state variable, and $f_i$ the contribution of the $i$th stage to the total objective function; $\mathbf{g}_j$, $\mathbf{h}_k$, and $q_i$ are functions of $x_j$, $y_k$ and $x_i$ and $y_i$, respectively, and $l$ is the total number of stages. The control and state variables $x_i$ and $y_i$ can be vectors in some cases. The following example serves to illustrate the nature of an optimal control problem.

***Example 1.2*** A rocket is designed to travel a distance of $12s$ in a vertically upward direction [1.32]. The thrust of the rocket can be changed only at the discrete points located at distances of 0, $s$, $2s$, $3s$, . . . , $11s$. If the maximum thrust that can be developed at point $i$ either in the positive or negative direction is restricted to a value of $F_i$, formulate the problem of minimizing the total time of travel under the following assumptions:

1. The rocket travels against the gravitational force.
2. The mass of the rocket reduces in proportion to the distance traveled.
3. The air resistance is proportional to the velocity of the rocket.

SOLUTION Let points (or control points) on the path at which the thrusts of the rocket are changed be numbered as 1, 2, 3, . . . , 13 (Fig. 1.8). Denoting

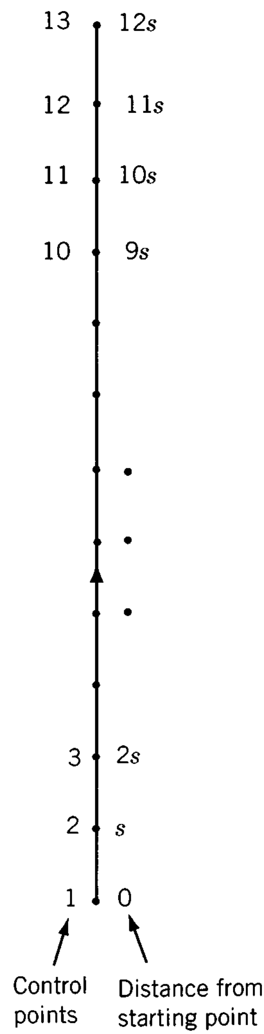


**Figure 1.8** Control points in the path of the rocket.

$x_i$ as the thrust, $v_i$ the velocity, $a_i$ the acceleration, and $m_i$ the mass of the rocket at point $i$, Newton's second law of motion can be applied as

net force on the rocket = mass × acceleration

This can be written as

thrust − gravitational force − air resistance = mass × acceleration

or

$$x_i - m_i g - k_1 v_i = m_i a_i \tag{E$_1$}$$

where the mass $m_i$ can be expressed as

$$m_i = m_{i-1} - k_2 s \tag{E$_2$}$$

and $k_1$ and $k_2$ are constants. Equation (E$_1$) can be used to express the acceleration, $a_i$, as

$$a_i = \frac{x_i}{m_i} - g - \frac{k_1 v_i}{m_i} \tag{E$_3$}$$

If $t_i$ denotes the time taken by the rocket to travel from point $i$ to point $i + 1$, the distance traveled between the points $i$ and $i + 1$ can be expressed as

$$s = v_i t_i + \tfrac{1}{2} a_i t_i^2$$

or

$$\frac{1}{2} t_i^2 \left( \frac{x_i}{m_i} - g - \frac{k_1 v_i}{m_i} \right) + t_i v_i - s = 0 \tag{E$_4$}$$

from which $t_i$ can be determined as

$$t_i = \frac{-v_i \pm \sqrt{v_i^2 + 2s \left( \dfrac{x_i}{m_i} - g - \dfrac{k_1 v_i}{m_i} \right)}}{\dfrac{x_i}{m_i} - g - \dfrac{k_1 v_i}{m_i}} \tag{E$_5$}$$

Of the two values given by Eq. (E$_5$), the positive value has to be chosen for $t_i$. The velocity of the rocket at point $i + 1$, $v_{i+1}$, can be expressed in terms of $v_i$ as (by assuming the acceleration between points $i$ and $i + 1$ to be constant for simplicity)

$$v_{i+1} = v_i + a_i t_i \tag{E$_6$}$$

The substitution of Eqs. ($E_3$) and ($E_5$) into Eq. ($E_6$) leads to

$$v_{i+1} = \sqrt{v_i^2 + 2s\left(\frac{x_i}{m_i} - g - \frac{k_1 v_i}{m_i}\right)} \tag{E$_7$}$$

From an analysis of the problem, the control variables can be identified as the thrusts $x_i$ and the state variables as the velocities, $v_i$. Since the rocket starts at point 1 and stops at point 13,

$$v_1 = v_{13} = 0 \tag{E$_8$}$$

Thus the problem can be stated as an OC problem as

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{12} \end{Bmatrix} \text{ which minimizes}$$

$$f(\mathbf{X}) = \sum_{i=1}^{12} t_i = \sum_{i=1}^{12} \left\{ \frac{-v_i + \sqrt{v_i^2 + 2s\left(\frac{x_i}{m_i} - g - \frac{k_1 v_i}{m_i}\right)}}{\left(\frac{x_i}{m_i} - g - \frac{k_1 v_i}{m_i}\right)} \right\}$$

subject to

$$m_{i+1} = m_i - k_2 s, \qquad i = 1,2,\ldots,12$$

$$v_{i+1} = \sqrt{v_i^2 + 2s\left(\frac{x_i}{m_i} - g - \frac{k_1 v_i}{m_i}\right)}, \qquad i = 1,2,\ldots,12$$

$$|x_i| \le F_i, \qquad i = 1,2,\ldots,12$$

$$v_1 = v_{13} = 0$$

### 1.5.4 Classification Based on the Nature of the Equations Involved

Another important classification of optimization problems is based on the nature of expressions for the objective function and the constraints. According to this classification, optimization problems can be classified as linear, nonlinear, geometric, and quadratic programming problems. This classification is extremely useful from the computational point of view since there are many special methods available for the efficient solution of a particular class of problems. Thus the first task of a designer would be to investigate the class of problem encountered. This will, in many cases, dictate the types of solution procedures to be adopted in solving the problem.

***Nonlinear Programming Problem.*** If any of the functions among the objective and constraint functions in Eq. (1.1) is nonlinear, the problem is called a *nonlinear programming* (NLP) *problem*. This is the most general programming problem and all other problems can be considered as special cases of the NLP problem.

***Example 1.3*** The step-cone pulley shown in Fig. 1.9 is to be designed for transmitting a power of at least 0.75 hp. The speed of the input shaft is 350 rpm and the output speed requirements are 750, 450, 250, and 150 rpm for a fixed center distance of $a$ between the input and output shafts. The tension on the tight side of the belt is to be kept more than twice that on the slack side. The thickness of the belt is $t$ and the coefficient of friction between the belt

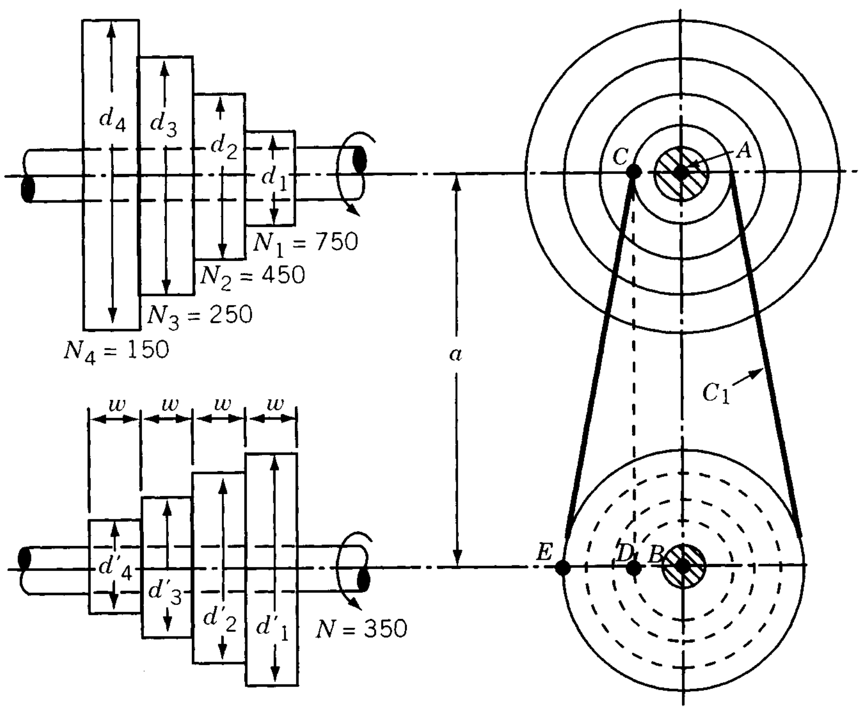


**Figure 1.9** Step-cone pulley.

and the pulleys is $\mu$. Formulate the problem of finding the width and diameters of the steps for minimum weight.

SOLUTION The design vector can be taken as

$$\mathbf{X} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ w \end{Bmatrix}$$

where $d_i$ is the diameter of the $i$th step on the output pulley and $w$ is the width of the belt and the steps. The objective function is the weight of the step-cone pulley system:

$$\begin{aligned} f(\mathbf{X}) &= \rho w \frac{\pi}{4} (d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_1'^2 + d_2'^2 + d_3'^2 + d_4'^2) \\ &= \rho w \frac{\pi}{4} \left\{ d_1^2 \left[ 1 + \left( \frac{750}{350} \right)^2 \right] + d_2^2 \left[ 1 + \left( \frac{450}{350} \right)^2 \right] \right. \\ &\quad \left. + d_3^2 \left[ 1 + \left( \frac{250}{350} \right)^2 \right] + d_4^2 \left[ 1 + \left( \frac{150}{350} \right)^2 \right] \right\} \end{aligned} \tag{E$_1$}$$

where $\rho$ is the density of the pulleys and $d_i'$ is the diameter of the $i$th step on the input pulley.

To have the belt equally tight on each pair of opposite steps, the total length of the belt must be kept constant for all the output speeds. This can be ensured by satisfying the following equality constraints:

$$C_1 - C_2 = 0 \tag{E$_2$}$$

$$C_1 - C_3 = 0 \tag{E$_3$}$$

$$C_1 - C_4 = 0 \tag{E$_4$}$$

where $C_i$ denotes length of the belt needed to obtain output speed $N_i$ ($i = 1,2,3,4$) and is given by [1.66,1.67]

$$C_i \simeq \frac{\pi d_i}{2} \left( 1 + \frac{N_i}{N} \right) + \frac{\left( \dfrac{N_i}{N} - 1 \right)^2 d_i^2}{4a} + 2a$$

where $N$ is the speed of the input shaft and $a$ is the center distance between the shafts. The ratio of tensions in the belt can be expressed as [1.66,1.67]

$$\frac{T_1^i}{T_2^i} = e^{\mu\theta_i}$$

where $T_1^i$ and $T_2^i$ are the tensions on the right and slack sides of the $i$th step, $\mu$ the coefficient of friction, and $\theta_i$ the angle of lap of the belt over the $i$th pulley step. The angle of lap is given by

$$\theta_i = \pi - 2\sin^{-1}\left[\frac{\left(\frac{N_i}{N} - 1\right)d_i}{2a}\right]$$

and hence the constraint on the ratio of tensions becomes

$$\exp\left\{\mu\left[\pi - 2\sin^{-1}\left\{\left(\frac{N_i}{N} - 1\right)\frac{d_i}{2a}\right\}\right]\right\} \geq 2, \qquad i = 1,2,3,4 \qquad (E_5)$$

The limitation on the maximum tension can be expressed as

$$T_1^i \leq stw, \qquad i = 1,2,3,4 \qquad (E_6)$$

where $s$ is the maximum allowable stress in the belt and $t$ is the thickness of the belt. The constraint on the power transmitted can be stated as (using $\mathrm{lb_f}$ for force and ft for linear dimensions)

$$\frac{\left(T_1^i - T_2^i\right)\pi d_i'(350)}{33{,}000} \geq 0.75$$

which can be rewritten, using $T_1^i = stw$ (upper bound used for simplicity) and Eq. ($E_5$), as

$$stw\left(1 - \exp\left[-\mu\left(\pi - 2\sin^{-1}\left\{\left(\frac{N_i}{N} - 1\right)\frac{d_i}{2a}\right\}\right)\right]\right)\pi d_i' \left(\frac{350}{33{,}000}\right) \geq 0.75, \qquad i = 1,2,3,4 \qquad (E_7)$$

Finally, the lower bounds on the design variables can be taken as

$$w \geq 0 \qquad (E_8)$$

$$d_i \geq 0, \qquad i = 1,2,3,4 \qquad (E_9)$$

As the objective function, ($E_1$), and most of the constraints, ($E_2$) to ($E_9$), are nonlinear functions of the design variables $d_1$, $d_2$, $d_3$, $d_4$, and $w$, this problem is a nonlinear programming problem.

***Geometric Programming Problem***

***Definition*** A function $h(\mathbf{X})$ is called a *posynomial* if $h$ can be expressed as the sum of power terms each of the form

$$c_i x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_n^{a_{in}}$$

where $c_i$ and $a_{ij}$ are constants with $c_i > 0$ and $x_j > 0$. Thus a posynomial with $N$ terms can be expressed as

$$h(\mathbf{X}) = c_1 x_1^{a_{11}} x_2^{a_{12}} \cdots x_n^{a_{1n}} + \cdots + c_N x_1^{a_{N1}} x_2^{a_{N2}} \cdots x_n^{a_{Nn}} \qquad (1.7)$$

A *geometric programming* (GMP) *problem* is one in which the objective function and constraints are expressed as posynomials in $\mathbf{X}$. Thus GMP problem can be posed as follows [1.44]:

Find $\mathbf{X}$ which minimizes

$$f(\mathbf{X}) = \sum_{i=1}^{N_0} c_i \left( \prod_{j=1}^{n} x_j^{p_{ij}} \right), \qquad c_i > 0, \quad x_j > 0 \qquad (1.8)$$

subject to

$$g_k(\mathbf{X}) = \sum_{i=1}^{N_k} a_{ik} \left( \prod_{j=1}^{n} x_j^{q_{ijk}} \right) > 0, \qquad a_{ik} > 0, \quad x_j > 0, \quad k = 1,2,\ldots,m$$

where $N_0$ and $N_k$ denote the number of posynomial terms in the objective and $k$th constraint function, respectively.

***Example 1.4*** Four identical helical springs are used to support a milling machine weighing 5000 lb. Formulate the problem of finding the wire diameter ($d$), coil diameter ($D$), and the number of turns ($N$) of each spring (Fig. 1.10) for minimum weight by limiting the deflection to 0.1 in. and the shear stress to 10,000 psi in the spring. In addition, the natural frequency of vibration of the spring is to be greater than 100 Hz. The stiffness of the spring ($k$), the shear stress in the spring ($\tau$), and the natural frequency of vibration of the spring ($f_n$) are given by

$$k = \frac{d^4 G}{8 D^3 N}$$

$$\tau = K_s \frac{8FD}{\pi d^3}$$

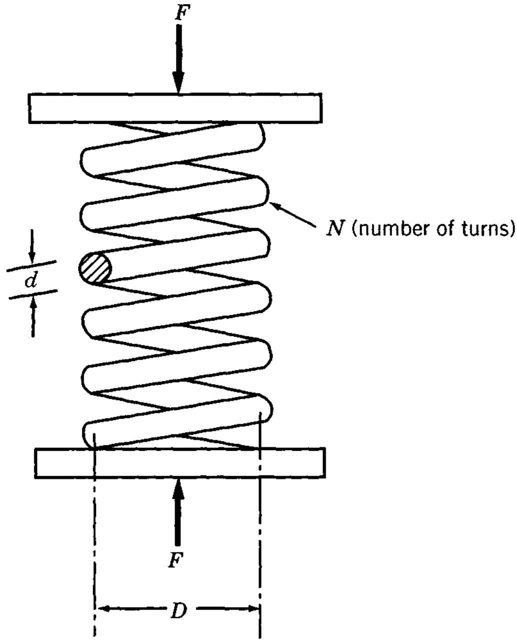


**Figure 1.10** Helical spring.

$$f_n = \frac{1}{2}\sqrt{\frac{kg}{w}} = \frac{1}{2}\sqrt{\frac{d^4G}{8D^3N}\,\frac{g}{\rho(\pi d^2/4)\pi DN}} = \frac{\sqrt{Ggd}}{2\sqrt{2}\rho\pi D^2N}$$

where $G$ is the shear modulus, $F$ the compressive load on the spring, $w$ the weight of the spring, $\rho$ the weight density of the spring, and $K_s$ the shear stress correction factor. Assume that the material is spring steel with $G = 12 \times 10^6$ psi and $\rho = 0.3$ lb/in$^3$, and the shear stress correction factor is $K_s \approx 1.05$.

SOLUTION The design vector is given by

$$\mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} d \\ D \\ N \end{Bmatrix}$$

and the objective function by

$$f(\mathbf{X}) = \text{weight} = \frac{\pi d^2}{4}\,\pi DN\rho \tag{E$_1$}$$

The constraints can be expressed as

$$\text{deflection} = \frac{F}{k} = \frac{8FD^3N}{d^4G} \leq 0.1$$

that is,

$$g_1(\mathbf{X}) = \frac{d^4G}{80FD^3N} > 1 \tag{E$_2$}$$

$$\text{shear stress} = K_s \frac{8FD}{\pi d^3} \leq 10{,}000$$

that is,

$$g_2(\mathbf{X}) = \frac{1250\pi d^3}{K_s FD} > 1 \tag{E$_3$}$$

$$\text{natural frequency} = \frac{\sqrt{Gg}}{2\sqrt{2\rho\pi}} \frac{d}{D^2N} \geq 100$$

that is,

$$g_3(\mathbf{X}) = \frac{\sqrt{Gg}d}{200\sqrt{2\rho\pi}D^2N} > 1 \tag{E$_4$}$$

Since the equality sign is not included (along with the inequality symbol, $>$) in the constraints of Eqs. ($E_2$) to ($E_4$), the design variables are to be restricted to positive values as

$$d > 0, \qquad D > 0, \qquad N > 0 \tag{E$_5$}$$

By substituting the known data, $F$ = weight of the milling machine /4 = 1250 lb, $\rho = 0.3$ lb/in$^3$, $G = 12 \times 10^6$ psi, and $K_s = 1.05$, Eqs. ($E_1$) to ($E_4$) become

$$f(\mathbf{X}) = \tfrac{1}{4}\pi^2(0.3)d^2DN = 0.7402x_1^2x_2x_3 \tag{E$_6$}$$

$$g_1(\mathbf{X}) = \frac{d^4(12 \times 10^6)}{80(1250)D^3N} = 120x_1^4x_2^{-3}x_3^{-1} > 1 \tag{E$_7$}$$

$$g_2(\mathbf{X}) = \frac{1250\pi d^3}{1.05(1250)D} = 2.992x_1^3x_2^{-1} > 1 \tag{E$_8$}$$

$$g_3(\mathbf{X}) = \frac{\sqrt{Gg}d}{200\sqrt{2\rho\pi}D^2N} = 139.8388x_1x_2^{-2}x_3^{-1} > 1 \tag{E$_9$}$$

It can be seen that the objective function, $f(\mathbf{X})$, and the constraint functions, $g_1(\mathbf{X})$ to $g_3(\mathbf{X})$, are posynomials and hence the problem is a GMP problem.

***Quadratic Programming Problem.*** A quadratic programming problem is a nonlinear programming problem with a quadratic objective function and linear constraints. It is usually formulated as follows:

$$F(\mathbf{X}) = c + \sum_{i=1}^{n} q_i x_i + \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij} x_i x_j \tag{1.9}$$

subject to

$$\sum_{i=1}^{n} a_{ij} x_i = b_j, \qquad j = 1,2,\ldots,m$$

$$x_i \geq 0, \qquad i = 1,2,\ldots,n$$

where $c$, $q_i$, $Q_{ij}$, $a_{ij}$, and $b_j$ are constants.

***Example 1.5*** A manufacturing firm produces two products, $A$ and $B$, using two limited resources. The maximum amounts of resources 1 and 2 available per day are 1000 and 250 units, respectively. The production of 1 unit of product $A$ requires 1 unit of resource 1 and 0.2 unit of resource 2, and the production of 1 unit of product $B$ requires 0.5 unit of resource 1 and 0.5 unit of resource 2. The unit costs of resources 1 and 2 are given by the relations $(0.375 - 0.00005u_1)$ and $(0.75 - 0.0001u_2)$, respectively, where $u_i$ denotes the number of units of resource $i$ used ($i = 1, 2$). The selling prices per unit of products $A$ and $B$, $p_A$ and $p_B$, are given by

$$p_A = 2.00 - 0.0005x_A - 0.00015x_B$$

$$p_B = 3.50 - 0.0002x_A - 0.0015x_B$$

where $x_A$ and $x_B$ indicate, respectively, the number of units of products $A$ and $B$ sold. Formulate the problem of maximizing the profit assuming that the firm can sell all the units it manufactures.

SOLUTION Let the design variables be the number of units of products $A$ and $B$ manufactured per day:

$$\mathbf{X} = \begin{Bmatrix} x_A \\ x_B \end{Bmatrix}$$

The requirement of resource 1 per day is $(x_A + 0.5x_B)$ and that of resource 2 is $(0.2x_A + 0.5x_B)$ and the constraints on the resources are

$$x_A + 0.5x_B \leq 1000 \tag{E$_1$}$$

$$0.2x_A + 0.5x_B \leq 250 \tag{E$_2$}$$

The lower bounds on the design variables can be taken as

$$x_A \geq 0 \tag{E$_3$}$$

$$x_B \geq 0 \tag{E$_4$}$$

The total cost of resources 1 and 2 per day is

$$(x_A + 0.5x_B)[0.375 - 0.00005(x_A + 0.5x_B)]$$
$$+ (0.2x_A + 0.5x_B)[0.750 - 0.0001(0.2x_A + 0.5x_B)]$$

and the return per day from the sale of products $A$ and $B$ is

$$x_A(2.00 - 0.0005x_A - 0.00015x_B) + x_B(3.50 - 0.0002x_A - 0.0015x_B)$$

The total profit is given by the total return minus the total cost. Since the objective function to be minimized is the negative of the profit per day, $f(\mathbf{X})$ is given by

$$\begin{aligned} f(\mathbf{X}) = {} & (x_A + 0.5x_B)[0.375 - 0.00005(x_A + 0.5x_B)] \\ & + (0.2x_A + 0.5x_B)[0.750 - 0.0001(0.2x_A + 0.5x_B)] \\ & - x_A(2.00 - 0.0005x_A - 0.00015x_B) \\ & - x_B(3.50 - 0.0002x_A - 0.0015x_B) \end{aligned} \tag{E$_5$}$$

As the objective function [Eq. (E$_5$)] is a quadratic and the constraints [Eqs. (E$_1$) to (E$_4$)] are linear, the problem is a quadratic programming problem.

***Linear Programming Problem.*** If the objective function and all the constraints in Eq. (1.1) are linear functions of the design variables, the mathematical programming problem is called a *linear programming* (LP) *problem*. A linear programming problem is often stated in the following standard form:

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

$$\text{which minimizes } f(\mathbf{X}) = \sum_{i=1}^{n} c_i x_i$$

subject to the constraints (1.10)

$$\sum_{i=1}^{n} a_{ij}x_i = b_j, \qquad j = 1,2,\ldots,m$$

$$x_i \geq 0, \qquad i = 1,2,\ldots,n$$

where $c_i$, $a_{ij}$, and $b_j$ are constants.

***Example 1.6*** A scaffolding system consists of three beams and six ropes as shown in Fig. 1.11. Each of the top ropes $A$ and $B$ can carry a load of $W_1$, each of the middle ropes $C$ and $D$ can carry a load of $W_2$, and each of the bottom ropes $E$ and $F$ can carry a load of $W_3$. If the loads acting on beams 1, 2, and 3 are $x_1$, $x_2$, and $x_3$, respectively, as shown in Fig. 1.11, formulate the problem of finding the maximum load $(x_1 + x_2 + x_3)$ that can be supported by the system. Assume that the weights of the beams 1, 2, and 3 are $w_1$, $w_2$, and $w_3$, respectively, and the weights of the ropes are negligible.

SOLUTION Assuming that the weights of the beams act through their respective middle points, the equations of equilibrium for vertical forces and moments for each of the three beams can be written as:

For beam 3:

$$T_E + T_F = x_3 + w_3$$

$$x_3(3l) + w_3(2l) - T_F(4l) = 0$$

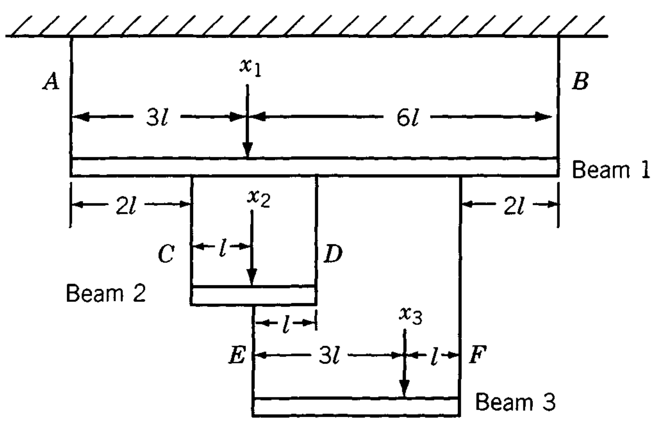


**Figure 1.11** Scaffolding system with three beams.

For beam 2:

$$T_C + T_D - T_E = x_2 + w_2$$

$$x_2(l) + w_2(l) + T_E(l) - T_D(2l) = 0$$

For beam 1:

$$T_A + T_B - T_C - T_D - T_F = x_1 + w_1$$

$$x_1(3l) + w_1(\tfrac{9}{2}l) - T_B(9l) + T_C(2l) + T_D(4l) + T_F(7l) = 0$$

where $T_i$ denotes the tension in rope $i$. The solution of these equations gives

$$\begin{aligned}
T_F &= \tfrac{3}{4}x_3 + \tfrac{1}{2}w_3 \\
T_E &= \tfrac{1}{4}x_3 + \tfrac{1}{2}w_3 \\
T_D &= \tfrac{1}{2}x_2 + \tfrac{1}{8}x_3 + \tfrac{1}{2}w_2 + \tfrac{1}{4}w_3 \\
T_C &= \tfrac{1}{2}x_2 + \tfrac{1}{8}x_3 + \tfrac{1}{2}w_2 + \tfrac{1}{4}w_3 \\
T_B &= \tfrac{1}{3}x_1 + \tfrac{1}{3}x_2 + \tfrac{2}{3}x_3 + \tfrac{1}{2}w_1 + \tfrac{1}{3}w_2 + \tfrac{5}{9}w_3 \\
T_A &= \tfrac{2}{3}x_1 + \tfrac{2}{3}x_2 + \tfrac{1}{3}x_3 + \tfrac{1}{2}w_1 + \tfrac{2}{3}w_2 + \tfrac{4}{9}w_3
\end{aligned}$$

The optimization problem can be formulated by choosing the design vector as

$$\mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

Since the objective is to maximize the total load

$$f(\mathbf{X}) = -(x_1 + x_2 + x_3) \tag{E$_1$}$$

The constraints on the forces in the ropes can be stated as

$$T_A \le W_1 \tag{E$_2$}$$

$$T_B \le W_1 \tag{E$_3$}$$

$$T_C \le W_2 \tag{E$_4$}$$

$$T_D \le W_2 \tag{E$_5$}$$

$$T_E \le W_3 \tag{E$_6$}$$

$$T_F \le W_3 \tag{E$_7$}$$

Finally, the nonnegativity requirement of the design variables can be expressed as

$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$x_3 \geq 0 \qquad (E_8)$$

Since all the equations of the problem ($E_1$) to ($E_8$), are linear functions of $x_1$, $x_2$, and $x_3$, the problem is a linear programming problem.

### 1.5.5 Classification Based on the Permissible Values of the Design Variables

Depending on the values permitted for the design variables, optimization problems can be classified as integer- and real-valued programming problems.

***Integer Programming Problem.*** If some or all of the design variables $x_1, x_2, \ldots, x_n$ of an optimization problem are restricted to take on only integer (or discrete) values, the problem is called an *integer programming problem.* On the other hand, if all the design variables are permitted to take any real value, the optimization problem is called a *real-valued programming problem.* According to this definition, the problems considered in Examples 1.1 to 1.6 are real-valued programming problems.

***Example 1.7*** A cargo load is to be prepared from five types of articles. The weight $w_i$, volume $v_i$, and monetary value $c_i$ of different articles are given below.

| Article Type | $w_i$ | $v_i$ | $c_i$ |
|---|---|---|---|
| 1 | 4 | 9 | 5 |
| 2 | 8 | 7 | 6 |
| 3 | 2 | 4 | 3 |
| 4 | 5 | 3 | 2 |
| 5 | 3 | 8 | 8 |

Find the number of articles $x_i$ selected from the $i$th type ($i = 1,2,3,4,5$), so that the total monetary value of the cargo load is a maximum. The total weight and volume of the cargo cannot exceed the limits of 2000 and 2500 units, respectively.

SOLUTION Let $x_i$ be the number of articles of type $i$ ($i = 1$ to 5) selected. Since it is not possible to load a fraction of an article, the variables $x_i$ can take only integer values.

The objective function to be maximized is given by

$$f(\mathbf{X}) = 5x_1 + 6x_2 + 3x_3 + 2x_4 + 8x_5 \tag{E$_1$}$$

and the constraints by

$$4x_1 + 8x_2 + 2x_3 + 5x_4 + 3x_5 \le 2000 \tag{E$_2$}$$

$$9x_1 + 7x_2 + 4x_3 + 3x_4 + 8x_5 \le 2500 \tag{E$_3$}$$

$$x_i \ge 0 \text{ and integral}, \qquad i = 1,2,\ldots,5 \tag{E$_4$}$$

Since $x_i$ are constrained to be integers, the problem is an integer programming problem.

### 1.5.6 Classification Based on the Deterministic Nature of the Variables

Based on the deterministic nature of the variables involved, optimization problems can be classified as deterministic and stochastic programming problems.

***Stochastic Programming Problem.*** A stochastic programming problem is an optimization problem in which some or all of the parameters (design variables and/or preassigned parameters) are probabilistic (nondeterministic or stochastic). According to this definition, the problems considered in Examples 1.1 to 1.7 are deterministic programming problems.

***Example 1.8*** Formulate the problem of designing a minimum-cost rectangular under-reinforced concrete beam that can carry a bending moment $M$ with a probability of at least 0.95. The costs of concrete, steel, and formwork are given by $C_c$ = \$200/m$^3$, $C_s$ = \$5000/m$^3$ and $C_f$ = \$40/m$^2$ of surface area. The bending moment $M$ is a probabilistic quantity and varies between $1 \times 10^5$ and $2 \times 10^5$ N-m with a uniform probability. The strengths of concrete and steel are also uniformly distributed probabilistic quantities whose lower and upper limits are given by

$$f_c = 25 \text{ and } 35 \text{ MPa}$$

$$f_s = 500 \text{ and } 550 \text{ MPa}$$

Assume that the area of the reinforcing steel and the cross-sectional dimensions of the beam are deterministic quantities.

SOLUTION The breadth $b$ in meters, the depth $d$ in meters, and the area of reinforcing steel $A_s$ in square meters are taken as the design variables $x_1$, $x_2$, and $x_3$, respectively (Fig. 1.12). The cost of the beam per meter length is given

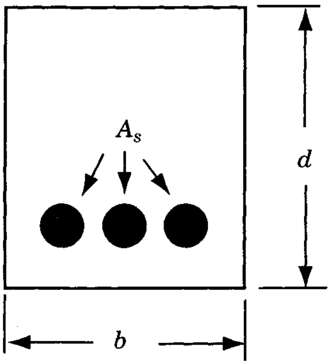


**Figure 1.12** Cross section of a reinforced concrete beam.

by

$$f(\mathbf{X}) = \text{cost of steel} + \text{cost of concrete} + \text{cost of formwork}$$
$$= A_s C_s + (bd - A_s)C_c + 2(b + d)C_f \tag{E$_1$}$$

The resisting moment of the beam section is given by [1.69]

$$M_R = A_s f_s \left( d - 0.59 \frac{A_s f_s}{f_c b} \right)$$

and the constraint on the bending moment can be expressed as [1.70]

$$P[M_R - M \geq 0] = P\left[ A_s f_s \left( d - 0.59 \frac{A_s f_s}{f_c b} \right) - M \geq 0 \right] \geq 0.95 \tag{E$_2$}$$

where $P[\cdot\ \cdot\ \cdot]$ indicates the probability of occurrence of the event $[\cdot\ \cdot\ \cdot]$.

To ensure that the beam remains under-reinforced,† the area of steel is bounded by the balanced steel area $A_s^{(b)}$ as

$$A_s \leq A_s^{(b)} \tag{E$_3$}$$

where

$$A_s^{(b)} = (0.542) \frac{f_c}{f_s} bd \frac{600}{600 + f_s}$$

†If steel area is larger than $A_s^{(b)}$, the beam becomes over-reinforced and failure occurs all of a sudden due to lack of concrete strength. If the beam is under-reinforced, failure occurs due to lack of steel strength and hence it will be gradual.

Since the design variables cannot be negative, we have

$$d \geq 0$$
$$b \geq 0$$
$$A_s \geq 0 \tag{E$_4$}$$

Since the quantities $M$, $f_c$, and $f_s$ are nondeterministic, the problem is a stochastic programming problem.

### 1.5.7 Classification Based on the Separability of the Functions

Optimization problems can be classified as separable and nonseparable programming problems based on the separability of the objective and constraint functions.

*Separable Programming Problem*

***Definition*** A function $f(\mathbf{X})$ is said to be *separable* if it can be expressed as the sum of $n$ single-variable functions, $f_1(x_1), f_2(x_2), \ldots, f_n(x_n)$, that is,

$$f(\mathbf{X}) = \sum_{i=1}^{n} f_i(x_i) \tag{1.11}$$

A separable programming problem is one in which the objective function and the constraints are separable and can be expressed in standard form as:

$$\text{Find } \mathbf{X} \text{ which minimizes } f(\mathbf{X}) = \sum_{i=1}^{n} f_i(x_i) \tag{1.12}$$

subject to

$$g_j(\mathbf{X}) = \sum_{i=1}^{n} g_{ij}(x_i) \leq b_j, \qquad j = 1,2,\ldots,m$$

where $b_j$ is a constant.

***Example 1.9*** A retail store stocks and sells three different models of TV sets. The store cannot afford to have an inventory worth more than \$45,000 at any time. The TV sets are ordered in lots. It costs \$$a_j$ for the store whenever a lot of TV model $j$ is ordered. The cost of one TV set of model $j$ is $c_j$. The demand rate of TV model $j$ is $d_j$ units per year. The rate at which the inventory costs accumulate is known to be proportional to the investment in inventory at any time, with $q_j = 0.5$, denoting the constant of proportionality for TV model $j$.

Each TV set occupies an area of $s_j = 0.40\ \text{m}^2$ and the maximum storage space available is $90\ \text{m}^2$. The data known from the past experience are given below.

| | TV Model $j$ | | |
|---|---|---|---|
| | 1 | 2 | 3 |
| Ordering cost $a_j$ (\$) | 50 | 80 | 100 |
| Unit cost $c_j$ (\$) | 40 | 120 | 80 |
| Demand rate, $d_j$ | 800 | 400 | 1200 |

Formulate the problem of minimizing the average annual cost of ordering and storing the TV sets.

SOLUTION Let $x_j$ denote the number of TV sets of model $j$ ordered in each lot ($j = 1,2,3$). Since the demand rate per year of model $j$ is $d_j$, the number of times the TV model $j$ needs to be ordered is $d_j/x_j$. The cost of ordering TV model $j$ per year is thus $a_j d_j/x_j$, $j = 1,2,3$. The cost of storing TV sets of model $j$ per year is $q_j c_j x_j/2$ since the average level of inventory at any time during the year is equal to $c_j x_j/2$. Thus the objective function (cost of ordering plus storing) can be expressed as

$$f(\mathbf{X}) = \left(\frac{a_1 d_1}{x_1} + \frac{q_1 c_1 x_1}{2}\right) + \left(\frac{a_2 d_2}{x_2} + \frac{q_2 c_2 x_2}{2}\right) + \left(\frac{a_3 d_3}{x_3} + \frac{q_3 c_3 x_3}{2}\right) \tag{E$_1$}$$

where the design vector $\mathbf{X}$ is given by

$$\mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \tag{E$_2$}$$

The constraint on the volume of inventory can be stated as

$$c_1 x_1 + c_2 x_2 + c_3 x_3 \le 45{,}000 \tag{E$_3$}$$

The limitation on the storage area is given by

$$s_1 x_1 + s_2 x_2 + s_3 x_3 \le 90 \tag{E$_4$}$$

Since the design variables cannot be negative, we have

$$x_j \ge 0, \qquad j = 1,2,3 \tag{E$_5$}$$

By substituting the known data, the optimization problem can be stated as follows:
Find **X** which minimizes

$$f(\mathbf{X}) = \left(\frac{40{,}000}{x_1} + 10x_1\right) + \left(\frac{32{,}000}{x_2} + 30x_2\right) + \left(\frac{120{,}000}{x_3} + 20x_3\right) \tag{E$_6$}$$

subject to

$$g_1(\mathbf{X}) = 40x_1 + 120x_2 + 80x_3 \le 45{,}000 \tag{E$_7$}$$

$$g_2(\mathbf{X}) = 0.40(x_1 + x_2 + x_3) \le 90 \tag{E$_8$}$$

$$g_3(\mathbf{X}) = -x_1 \le 0 \tag{E$_9$}$$

$$g_4(\mathbf{X}) = -x_2 \le 0 \tag{E$_{10}$}$$

$$g_5(\mathbf{X}) = -x_3 \le 0 \tag{E$_{11}$}$$

It can be observed that the optimization problem stated in Eqs. (E$_6$) to (E$_{11}$) is a separable programming problem.

### 1.5.8 Classification Based on the Number of Objective Functions

Depending on the number of objective functions to be minimized, optimization problems can be classified as single- and multiobjective programming problems. According to this classification, the problems considered in Examples 1.1 to 1.9 are single objective programming problems.

***Multiobjective Programming Problem.*** A multiobjective programming problem can be stated as follows:

$$\text{Find } \mathbf{X} \text{ which minimizes } f_1(\mathbf{X}),\ f_2(\mathbf{X}), \ldots, f_k(\mathbf{X})$$

subject to (1.13)

$$g_j(\mathbf{X}) \le 0, \qquad j = 1,2,\ldots,m$$

where $f_1, f_2, \ldots, f_k$ denote the objective functions to be minimized simultaneously.

***Example 1.10*** A uniform column of rectangular cross section is to be constructed for supporting a water tank of mass $M$ (Fig. 1.13). It is required (1) to minimize the mass of the column for economy, and (2) to maximize the natural frequency of transverse vibration of the system for avoiding possible

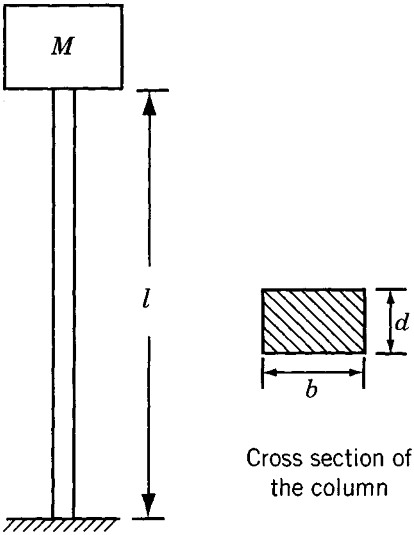


**Figure 1.13** Water tank on a column.

resonance due to wind. Formulate the problem of designing the column to avoid failure due to direct compression and buckling. Assume the permissible compressive stress to be $\sigma_{max}$.

SOLUTION Let $x_1 = b$ and $x_2 = d$ denote the cross-sectional dimensions of the column. The mass of the column ($m$) is given by

$$m = \rho bdl = \rho l x_1 x_2 \tag{E$_1$}$$

where $\rho$ is the density and $l$ is the height of the column. The natural frequency of transverse vibration of the water tank ($\omega$), by treating it as a cantilever beam with a tip mass $M$, can be obtained as [1.68]:

$$\omega = \left[ \frac{3EI}{(M + \frac{33}{140} m) l^3} \right]^{1/2} \tag{E$_2$}$$

where $E$ is the Young's modulus and $I$ is the area moment of inertia of the column given by

$$I = \tfrac{1}{12} bd^3 \tag{E$_3$}$$

The natural frequency of the water tank can be maximized by minimizing $-\omega$. With the help of Eqs. (E$_1$) and (E$_3$), Eq. (E$_2$) can be rewritten as

$$\omega = \left[ \frac{E x_1 x_2^3}{4 l^3 (M + \frac{33}{140} \rho l x_1 x_2)} \right]^{1/2} \tag{E$_4$}$$

The direct compressive stress ($\sigma_c$) in the column due to the weight of the water tank is given by

$$\sigma_c = \frac{Mg}{bd} = \frac{Mg}{x_1 x_2} \tag{E$_5$}$$

and the buckling stress for a fixed–free column ($\sigma_b$) is given by [1.71]

$$\sigma_b = \left(\frac{\pi^2 EI}{4l^2}\right)\frac{1}{bd} = \frac{\pi^2 E x_2^2}{48 l^2} \tag{E$_6$}$$

To avoid failure of the column, the direct stress has to be restricted to be less than $\sigma_{\max}$ and the buckling stress has to be constrained to be greater than the direct compressive stress induced.

Finally, the design variables have to be constrained to be positive. Thus the multiobjective optimization problem can be stated as follows:

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \text{ which minimizes}$$

$$f_1(\mathbf{X}) = \rho l x_1 x_2 \tag{E$_7$}$$

$$f_2(\mathbf{X}) = -\left[\frac{E x_1 x_2^3}{4l^2(M + \frac{33}{140}\rho l x_1 x_2)}\right]^{1/2} \tag{E$_8$}$$

subject to

$$g_1(\mathbf{X}) = \frac{Mg}{x_1 x_2} - \sigma_{\max} \le 0 \tag{E$_9$}$$

$$g_2(\mathbf{X}) = \frac{Mg}{x_1 x_2} - \frac{\pi^2 E x_2^2}{48 l^2} \le 0 \tag{E$_{10}$}$$

$$g_3(\mathbf{X}) = -x_1 \le 0 \tag{E$_{11}$}$$

$$g_4(\mathbf{X}) = -x_2 \le 0 \tag{E$_{12}$}$$

## 1.6 OPTIMIZATION TECHNIQUES

The various techniques available for the solution of different types of optimization problems are given under the heading of mathematical programming techniques in Table 1.1. The classical methods of differential calculus can be used to find the unconstrained maxima and minima of a function of several variables. These methods assume that the function is differentiable twice with