Solutions Manual to accompany

AN INTRODUCTION TO MECHANICS

2nd edition

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CONTENTS

1 VECTORS AND KINEMATICS

1.1 *Vector algebra 1*

$\mathbf{A} = (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}) \quad \mathbf{B} = (5\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

(a) $\mathbf{A} + \mathbf{B} = (2 + 5)\hat{\mathbf{i}} + (-3 + 1)\hat{\mathbf{j}} + (7 + 2)\hat{\mathbf{k}} = 7\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 9\hat{\mathbf{k}}$

(b) $\mathbf{A} - \mathbf{B} = (2 - 5)\hat{\mathbf{i}} + (-3 - 1)\hat{\mathbf{j}}(7 - 2)\hat{\mathbf{k}} = -3\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$

(c) $\mathbf{A} \cdot \mathbf{B} = (2)(5) + (-3)(1) + (7)(2) = 21$

(d) $$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -3 & 7 \\ 5 & 1 & 2 \end{vmatrix}$$

$$= -13\hat{\mathbf{i}} + 31\hat{\mathbf{j}} + 17\hat{\mathbf{k}}$$

1.2 *Vector algebra 2*

$\mathbf{A} = (3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 5\hat{\mathbf{k}}) \quad \mathbf{B} = (6\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$

(a) $A^2 = \mathbf{A} \cdot \mathbf{A} = 3^2 + (-2)^2 + 5^2 = 38$

(b) $B^2 = \mathbf{B} \cdot \mathbf{B} = 6^2 + (-7)^2 + 4^2 = 101$

(c) $(\mathbf{A} \cdot \mathbf{B})^2 = [(3)(6) + (-2)(-7) + (5)(4)]^2 = [18 + 14 + 20]^2 = 52^2 = 2704$

1.3 *Cosine and sine by vector algebra*

$\mathbf{A} = (3\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \quad \mathbf{B} = (-2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$

(a)

$$\mathbf{A}\cdot\mathbf{B} = A\,B\cos(\mathbf{A},\mathbf{B})$$
$$\cos(\mathbf{A},\mathbf{B}) = \frac{\mathbf{A}\cdot\mathbf{B}}{A\,B}$$
$$= \frac{(-6+1+1)}{\sqrt{(9+1+1)}\,\sqrt{4+1+1)}} = \frac{-4}{\sqrt{11}\,\sqrt{6}} \approx 0.492$$

(b) *method 1:*

$$|\mathbf{A}\times\mathbf{B}| = A\,B\sin(\mathbf{A},\mathbf{B})$$
$$\sin(\mathbf{A},\mathbf{B}) = \frac{|\mathbf{A}\times\mathbf{B}|}{A\,B}$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 1 & 1 \\ -2 & 1 & 1 \end{vmatrix}$$
$$= (1-1)\hat{\mathbf{i}} - (3+2)\hat{\mathbf{j}} + (3+2)\hat{\mathbf{k}} = -5\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$$
$$|\mathbf{A}\times\mathbf{B}| = \sqrt{5^2+5^2} = 5\sqrt{2}$$
$$\sin(\mathbf{A},\mathbf{B}) = \frac{|\mathbf{A}\times\mathbf{B}|}{A\,B} = \frac{5\sqrt{2}}{\sqrt{11}\,\sqrt{6}} \approx 0.870$$

(c) *method 2 (simpler) – use:*

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin(\mathbf{A},\mathbf{B}) = \sqrt{1-\cos^2(\mathbf{A},\mathbf{B})}$$
$$= \sqrt{1-(0.492)^2} \quad \text{from (a)} \approx 0.871$$

1.4 *Direction cosines*

Note that here α, β, γ stand for direction cosines, not for the angles shown in the figure:
$\theta_x = \cos^{-1}\alpha$,
$\theta_y = \cos^{-1}\beta$,
$\theta_z = \cos^{-1}\gamma$.

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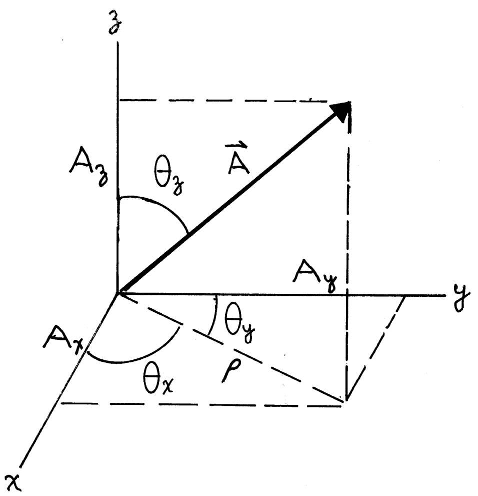


$$
\begin{aligned}
\mathbf{A} &= A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}} + A_z\,\hat{\mathbf{k}} \\
A_x &= \mathbf{A}\cdot\hat{\mathbf{i}} \;= A\;\cos(\mathbf{A},\hat{\mathbf{i}}) \equiv A\,\alpha \\
\alpha &= \cos(\mathbf{A},\hat{\mathbf{i}}) \;= \cos\theta_x.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
A_y &= A\;\cos(\mathbf{A},\hat{\mathbf{j}}) \;\equiv A\,\beta \\
\beta &= \cos(\mathbf{A},\hat{\mathbf{j}}) \;= \cos\theta_y \\
A_z &= A\;\cos(\mathbf{A},\hat{\mathbf{k}}) \;\equiv A\,\gamma \\
\gamma &= \cos(\mathbf{A},\hat{\mathbf{k}}) \;= \cos\theta_z
\end{aligned}
$$

Using these results,

$$
\begin{aligned}
A^2 &= A_x^2 + A_y^2 + A_z^2 \\
&= A^2\,(\alpha^2 + \beta^2 + \gamma^2)
\end{aligned}
$$

from which it follows that

$\alpha^2 + \beta^2 + \gamma^2 = 1$

Another way to see this is

$$A^2 = \rho^2 + A_z^2 \;= A_x^2 + A_y^2 + A_z^2 \;= A^2\,(\alpha^2 + \beta^2 + \gamma^2)$$

and it follows as before that

$\alpha^2 + \beta^2 + \gamma^2 = 1.$

1.5 *Perpendicular vectors*

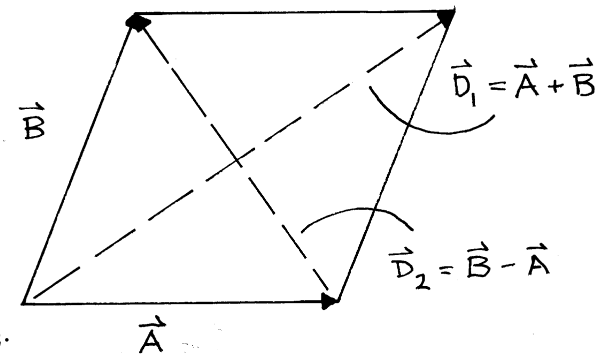
Given $|\mathbf{A}-\mathbf{B}| = |\mathbf{A}+\mathbf{B}|$ with $\mathbf{A}$ and $\mathbf{B}$ nonzero. Evaluate the magnitudes by squaring.

$$
\begin{aligned}
A^2 - 2\,\mathbf{A}\cdot\mathbf{B} + B^2 &= A^2 + 2\,\mathbf{A}\cdot\mathbf{B} + B^2 \\
-2\,\mathbf{A}\cdot\mathbf{B} &= +2\,\mathbf{A}\cdot\mathbf{B}. \\
\mathbf{A}\cdot\mathbf{B} &= 0
\end{aligned}
$$

and it follows that $\mathbf{A} \perp \mathbf{B}$.

1.6 *Diagonals of a parallelogram*

The parallelogram is equilateral, so $A = B$.

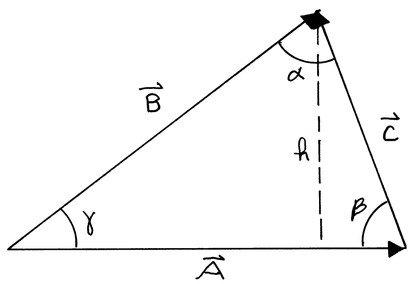


$$\mathbf{D_1} = \mathbf{A} + \mathbf{B}$$
$$\mathbf{D_2} = \mathbf{B} - \mathbf{A}$$
$$\mathbf{D_1} \cdot \mathbf{D_2} = (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{B} - \mathbf{A}) = A^2 - B^2 = 0.$$

Hence $\mathbf{D_1} \cdot \mathbf{D_2} = \mathbf{0}$ and it follows that $\mathbf{D_1} \perp \mathbf{D_2}$.

1.7 *Law of sines*

The area $\mathcal{A}$ of the triangle is



$$\mathcal{A} = \frac{1}{2}A\,h = \frac{1}{2}A\,B\,\sin\gamma = \frac{1}{2}\,|\mathbf{A} \times \mathbf{B}|$$

Similarly,

$$\mathcal{A} = \frac{1}{2}|\mathbf{B} \times \mathbf{C}| = \frac{1}{2}BC\sin\alpha$$
$$\mathcal{A} = \frac{1}{2}|\mathbf{C} \times \mathbf{A}| = \frac{1}{2}AC\sin\beta.$$

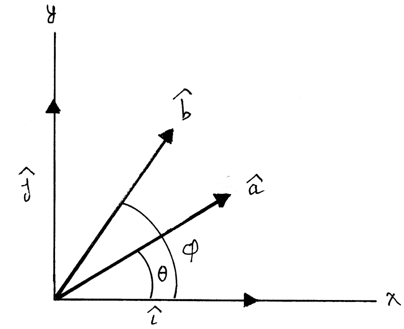
Hence $AB\,\sin\gamma = BC\,\sin\alpha = AC\,\sin\beta$, from which it follows

$$\frac{\sin\gamma}{C} = \frac{\sin\alpha}{A} = \frac{\sin\beta}{B}$$

Introducing the cross product makes the notation convenient, and emphasizes the relation between the cross product and the area of the triangle, but it is not essential for the proof.

1.8 *Vector proof of a trigonometric identity*

Given two unit vectors $\hat{\mathbf{a}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$ and $\hat{\mathbf{b}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}$, with $a = 1, b = 1$. First evaluate their scalar product using components:



$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= ab\ \cos\theta\cos\phi + ab\ \sin\theta\sin\phi \\ &= \cos\theta\cos\phi + \sin\theta\sin\phi\end{aligned}$$

then evaluate their scalar product geometrically.

$$\mathbf{a}\cdot\mathbf{b} = ab\ \cos(\mathbf{a},\mathbf{b})\ = ab\cos(\phi-\theta) = \cos(\phi-\theta)$$

Equating the two results,

$$\cos(\phi-\theta) = \cos\phi\cos\theta + \sin\phi\sin\theta$$

1.9 *Perpendicular unit vector*

Given $\mathbf{A} = (\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}})$ and $\mathbf{B} = (\mathbf{2}\,\hat{\mathbf{i}}+\hat{\mathbf{j}}-\mathbf{3}\,\hat{\mathbf{k}})$, find $\mathbf{C}$ such that $\mathbf{A}\cdot\mathbf{C} = \mathbf{0}$ and $\mathbf{B}\cdot\mathbf{C} = \mathbf{0}$.

$$\begin{aligned}\mathbf{C} &= C_x\,\hat{\mathbf{i}} + C_y\,\hat{\mathbf{j}} + C_z\,\hat{\mathbf{k}} \\ &= C_x(\hat{\mathbf{i}} + (C_y/C_x)\,\hat{\mathbf{j}} + (C_z/C_x)\,\hat{\mathbf{k}}) \\ \mathbf{A}\cdot\mathbf{C} &= C_x(1 + (C_y/C_x) - (C_z/C_x)) = 0 \\ \mathbf{B}\cdot\mathbf{C} &= C_x(2 + (C_y/C_x) - 3(C_z/C_x)) = 0\end{aligned}$$

We have two equations for the two unknowns (C_y/C_x) and (C_z/C_x).

$$\begin{aligned}1 + (C_y/C_x) - (C_z/C_x) &= 0 \\ 2 + (C_y/C_x) - 3(C_z/C_x) &= 0.\end{aligned}$$

The solutions are $(C_y/C_x) = -\frac{1}{2}$ and $(C_z/C_x) = \frac{1}{2}$, so that $\mathbf{C} = \mathbf{C_x}(\hat{\mathbf{i}} - \frac{\mathbf{1}}{\mathbf{2}}\hat{\mathbf{j}} + \frac{\mathbf{1}}{\mathbf{2}}\hat{\mathbf{k}})$. To evaluate C_x, apply the condition that $\mathbf{C}$ is a unit vector.

$$\begin{aligned}C^2 &= \frac{3}{2}C_x^2\ = 1 \\ C_x &= \pm\sqrt{(2/3)} \\ \hat{\mathbf{C}} &= \pm\sqrt{(2/3)}\,(\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}} + \frac{1}{2}\hat{\mathbf{k}})\end{aligned}$$

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which can be written

$$\hat{\mathbf{C}} = \pm\frac{1}{\sqrt{6}}(2\,\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$$

Geometrically, **C** can be perpendicular to both **A** and **B** only if **C** is perpendicular to the plane determined by **A** and **B**. From the standpoint of vector algebra, this implies that $\mathbf{C} \propto \mathbf{A} \times \mathbf{B}$. To prove this, evaluate $\mathbf{A} \times \mathbf{B}$.

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -1 \\ 2 & 1 & -3 \end{vmatrix} \\ &= -2\,\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}} \\ &\propto \mathbf{C}. \end{aligned}$$

1.10 *Perpendicular unit vectors*

Given $\mathbf{A} = \mathbf{3\hat{i}} + \mathbf{4\hat{j}} - \mathbf{4\hat{k}}$, find a unit vector $\hat{\mathbf{B}}$ perpendicular to **A**.

(a)

$$\begin{aligned} \mathbf{B} &= B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} = B_x[\hat{\mathbf{i}} + (B_y/B_x)\hat{\mathbf{j}}] \\ \mathbf{A}\cdot\mathbf{B} &= B_x[3 + 4(B_y/B_x)] = 0 \\ B_y/B_x &= -3/4 \\ \mathbf{B} &= B_x[\hat{\mathbf{i}} - \frac{3}{4}\hat{\mathbf{j}}] \end{aligned}$$

To evaluate B_x, note that **B** is a unit vector, $B^2 = 1$.

$$1 = B_x^2\left[(1)^2 + \left(\frac{3}{4}\right)^2\right] = \left(\frac{25}{16}\right)B_x^2$$

which gives

$$\begin{aligned} B_x &= \pm(4/5) \\ \hat{\mathbf{B}} &= \pm(4/5)(\hat{\mathbf{i}} - (3/4)\hat{\mathbf{j}}) = \pm\frac{1}{5}(4\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}}) \end{aligned}$$

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(b)

$$\begin{aligned}\mathbf{C} &= C_x\,\hat{\mathbf{i}} + C_y\,\hat{\mathbf{j}} + C_z\,\hat{\mathbf{k}}\\ &= C_x[\hat{\mathbf{i}} + (C_y/C_x)\,\hat{\mathbf{j}} + (C_z/C_x)\,\hat{\mathbf{k}}]\end{aligned}$$

$$\mathbf{A}\cdot\mathbf{C} = 0 \;\Rightarrow\; C_x[3 + 4(C_y/C_x) - 4(C_z/C_x)] = 0$$

$$\mathbf{B}\cdot\mathbf{C} = 0 \;\Rightarrow\; \frac{1}{5}C_x[4 - 3(C_y/C_x)] = 0$$

$$C_y/C_x = 4/3 \quad C_z/C_x = 25/12$$

To make **C** a unit vector,

$$C^2 = C_x^2\left[(1)^2 + \left(\frac{4}{3}\right)^2 + \left(\frac{25}{12}\right)^2\right] = 1$$

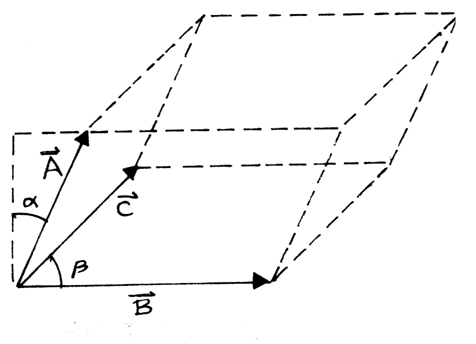
$$C_x \approx \pm 0.348$$

(c) The vector $\mathbf{B}\times\mathbf{C}$ is perpendicular (normal) to the plane defined by **B** and **C**, so we want to prove

$$\mathbf{A} \propto \mathbf{B}\times\mathbf{C}$$

$$\begin{aligned}\mathbf{B}\times\mathbf{C} &= C_x\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{4}{5} & -\frac{3}{5} & 0\\ 1 & \frac{4}{3} & \frac{25}{12}\end{vmatrix}\\ &= C_x\left[-\left(\frac{75}{60}\right)\hat{\mathbf{i}} - \left(\frac{100}{60}\right)\hat{\mathbf{j}} + \left(\frac{25}{15}\right)\hat{\mathbf{k}}\right]\\ &= \left(\frac{5}{12}\right)C_x(-3\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}}) \;\propto \mathbf{A}.\end{aligned}$$

1.11 *Volume of a parallelepiped*



With reference to the sketch, the height is $A\cos\alpha$, so the frontal area is $AB\cos\alpha$. The depth is $C\sin\beta$, so the volume V is

$$V = (AB\cos\alpha)(C\sin\beta) = (A\cos\alpha)(BC\sin\beta) = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$

The same approach can be used starting with a different face.

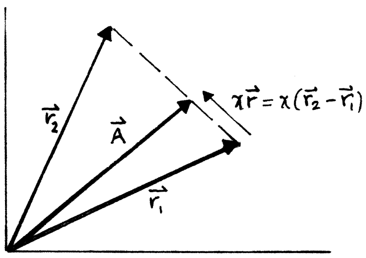
$$V = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \quad V = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$$

Note that **A**, **B**, **C** are arbitrary vectors. This proves the vector identity

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$$

1.12 *Constructing a vector to a point*

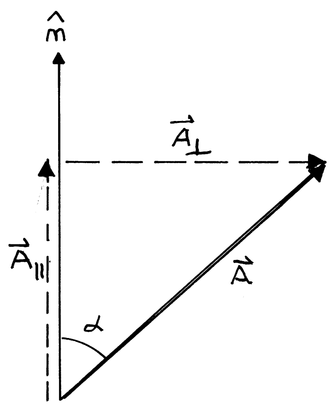
Applying vector addition to the lower triangle in the sketch,



$$\mathbf{A} = \mathbf{r}_1 + x(\mathbf{r}_2 - \mathbf{r}_1)$$
$$= (1 - x)\mathbf{r}_1 + x\mathbf{r}_2$$

1.13 *Expressing one vector in terms of another*

We will express vector $\mathbf{A}$ in terms of a unit vector $\hat{\mathbf{n}}$. As shown in the sketch, we can write $\mathbf{A}$ as the vector sum of a vector $\mathbf{A}_{\parallel}$ parallel to $\hat{\mathbf{n}}$ and a vector $\mathbf{A}_{\perp}$ perpendicular to $\hat{\mathbf{n}}$, so that $\mathbf{A} = \mathbf{A}_{\parallel} + \mathbf{A}_{\perp}$.



$$|\mathbf{A}_{\parallel}| = A\cos\alpha$$

The direction of $\mathbf{A}_{\parallel}$ is along $\hat{\mathbf{n}}$, so it follows that

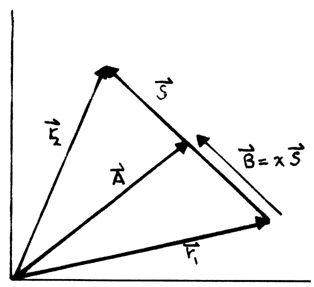
$$\mathbf{A}_{\parallel} = (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}.$$
$$|\mathbf{A}_{\perp}| = A\sin\alpha \ = |\hat{\mathbf{n}}\times\mathbf{A}|$$

The direction of $(\hat{\mathbf{n}}\times\mathbf{A})$ is into the paper, so taking its cross product with $\hat{\mathbf{n}}$ gives a vector $(\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}}$ along $\mathbf{A}_{\perp}$ and with the correct magnitude. Hence

$$\mathbf{A} = (\mathbf{A}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}}$$

1.14 *Two points*

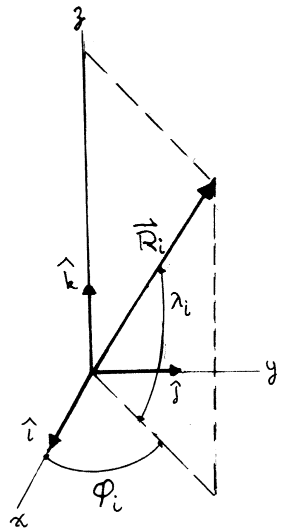
$\mathbf{S} = \mathbf{r}_2 - \mathbf{r}_1 \qquad \mathbf{B} = x\mathbf{S} \qquad \mathbf{A} = \mathbf{r}_1 + \mathbf{B}$



$x = 0$ at $t = 0$; $x = 1$ at $t = T$
so that $x = t/T$, linear in t

$$\mathbf{A} = \mathbf{r}_1 + x\mathbf{S} \ = \mathbf{r}_1 + \frac{t}{T}(\mathbf{r}_2 - \mathbf{r}_1) \ = \left(1 - \frac{t}{T}\right)\mathbf{r}_1 + \frac{t}{T}\mathbf{r}_2$$

1.15 *Great circle*



Consider vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ from the center of a sphere of radius R to points on the surface.
To avoid complications, the sketch shows the geometry of a generic vector $\mathbf{R_i}$ (i = 1 or 2) making angles λ_i and ϕ_i.
The magnitude of $\mathbf{R_i}$ is R, so $R_1 = R_2 = R$.
The coordinates of a point on the surface are

$$\mathbf{R}_i = R\cos\lambda_i\cos\phi_i\,\hat{\mathbf{i}} + R\cos\lambda_i\sin\phi_i\,\hat{\mathbf{j}} + R\sin\lambda_i\,\hat{\mathbf{k}}$$

The angle between two points can be found using the dot product.

$$\theta(1,2) = \arccos\left(\frac{\mathbf{R_1}\cdot\mathbf{R_2}}{R_1R_2}\right) = \arccos\left(\frac{\mathbf{R_1}\cdot\mathbf{R_2}}{R^2}\right)$$

Note that $\theta(1,2)$ is in radians.

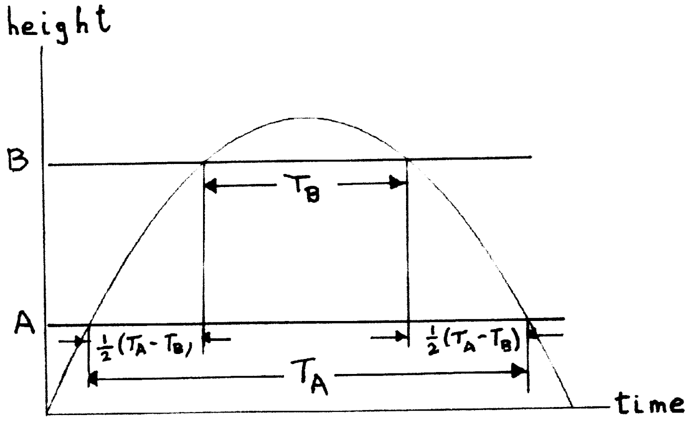
The great circle distance between $\mathbf{R_1}$ and $\mathbf{R_2}$ is $S = R\theta(1,2)$.

$$\mathbf{R_1}\cdot\mathbf{R_2} = R^2(\cos\lambda_1\cos\phi_1\cos\lambda_2\cos\phi_2 + \cos\lambda_1\sin\phi_1\cos\lambda_2\sin\phi_2 + \sin\lambda_1\sin\lambda_2)$$

Hence

$$\begin{aligned}
S &= R\,\theta(1,2) \\
&= R\arccos\left[\cos\lambda_1\cos\lambda_2(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2) + \sin\lambda_1\sin\lambda_2\right] \\
&= R\arccos\left\{\frac{1}{2}\cos(\lambda_1+\lambda_2)\left[\cos(\phi_1-\phi_2)-1\right] + \frac{1}{2}\cos(\lambda_1-\lambda_2)\left[\cos(\phi_1-\phi_2)+1\right]\right\}
\end{aligned}$$

1.16 *Measuring g*



The motion is free fall with uniform acceleration, so the trajectory is a parabola, as shown in the sketch. Take the initial conditions at T=0 to be $z = z_A$ and $v = v_A$. The height z is then

$$z = z_A + v_A T - \frac{1}{2} g T^2$$

The height is again z_A when $T = T_A$.

$$z_A = z_A + v_A T_A - \frac{1}{2} g T_A^2$$

so that

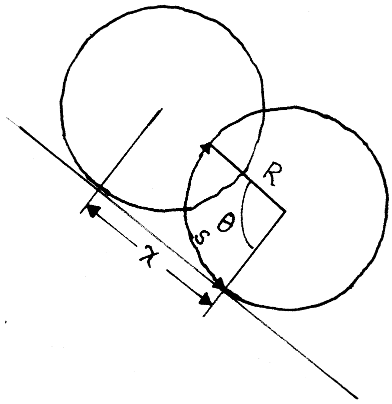
$$0 = v_A T_A - \frac{1}{2} g T_A^2 \Rightarrow v_A = \frac{1}{2} g T_A$$

By the symmetry of the trajectory, the body reaches height z_B for the second time at $T = \frac{1}{2}(T_A + T_B)$.

$$\begin{aligned}
h &= z_B - z_A \\
&= \left[z_A + \frac{1}{2} v_A (T_A + T_B) - \frac{1}{2} g [\frac{1}{2}(T_A + T_B)]^2 \right] - \left[z_A + v_A T_A - \frac{1}{2} g T_A^2 \right] \\
&= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) g T_A (T_A + T_B) - \frac{1}{8} g (T_A + T_B)^2 \\
&= \frac{1}{8} g (T_A^2 - T_B^2) \\
g &= \frac{8h}{T_A^2 - T_B^2}
\end{aligned}$$

1.17 *Rolling drum*

The drum rolls without slipping, so that when it has rotated through an angle θ, it advances down the plane by a distance x equal to the arc length $s = R\theta$ laid down.



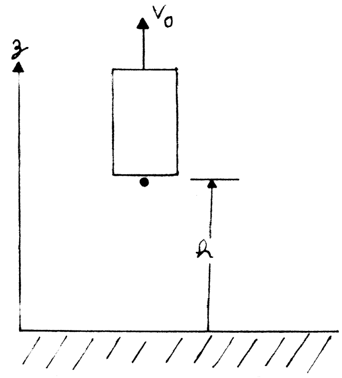
$x = R\theta$

$a = \ddot{x} = R\ddot{\theta} = R\alpha$

so that

$\alpha = \dfrac{a}{R}$

1.18 *Elevator and falling marble*



Starting at $t = 0$, the elevator moves upward with uniform speed v_0, so its height above the ground at time t is $z = v_0 t$.

At time T_1, $h = v_0 T_1$, so that $T_1 = h/v_0$. At the instant T_1 when the marble is released, the marble is at height h and has an instantaneous speed v_0. Its height z at a later time t is then

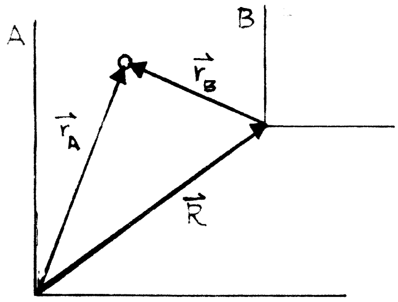
$$z = h + v_0(t - T_1) - \frac{1}{2}g(t - T_1)^2$$

The marble hits the ground $h = 0$ at time $t = T_2$.

$$\begin{aligned}
0 &= h + v_0(T_2 - T_1) - \frac{1}{2}g(T_2 - T_1)^2 \\
&= h + \frac{h}{T_1}(T_2 - T_1) - \frac{1}{2}g(T_2 - T_1)^2 \\
&= h\frac{T_2}{T_1} - \frac{1}{2}g(T_2 - T_1)^2 \\
h &= \frac{1}{2}\frac{T_1}{T_2}g(T_2 - T_1)^2
\end{aligned}$$

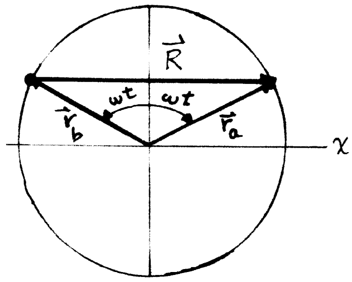
1.19 *Relative velocity*

(a)



$$\mathbf{r_A} = \mathbf{r_B} + \mathbf{R}$$
$$\dot{\mathbf{r}}_\mathbf{A} = \dot{\mathbf{r}}_\mathbf{B} + \dot{\mathbf{R}}$$
$$\mathbf{v_B} = \mathbf{v_A} - \dot{\mathbf{R}}$$

(b)



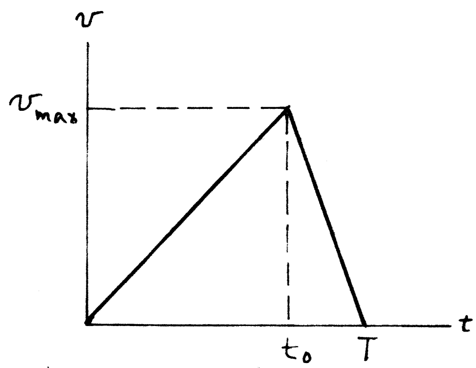
$$\mathbf{R} = 2l\sin(\omega t)\,\hat{\mathbf{i}}$$
$$\dot{\mathbf{R}} = 2l\omega\cos(\omega t)\,\hat{\mathbf{i}}$$

From the result of part (a)

$$\mathbf{v_a} = \mathbf{v_b} + 2l\omega\cos(\omega t)\,\hat{\mathbf{i}}$$

1.20 *Sportscar*

With reference to the sketch, the distance D traveled is the area under the plot of speed vs. time. The goal is to minimize the time while keeping D constant. This involves accelerating with maximum acceleration a_a for time t_0 and then braking with maximum (negative) acceleration a_b to bring the car to rest.



$$v_{max} = a_a t_0 = a_b(T - t_0)$$
$$t_0 = \frac{a_b T}{a_a + a_b}$$
$$D = \frac{1}{2}v_{max}T = \frac{1}{2}a_a t_0 T = \frac{1}{2}\left(\frac{a_a a_b}{a_a + a_b}\right)T^2$$
$$T = \sqrt{\frac{2D(a_a + a_b)}{a_a a_b}}$$
$$a_a = \frac{100\,\text{km/hr}}{3.5\,\text{s}} = \left(\frac{100\,\text{km}}{\text{hr}}\right)\left(\frac{1000\,\text{m}}{1\,\text{km}}\right)\left(\frac{1\,\text{hr}}{3600\,\text{s}}\right)\left(\frac{1}{3.5\,\text{s}}\right) \approx 7.94\text{m/s}^2$$
$$a_b = 0.7g = 0.7(9.80\,\text{m/s}^2) \approx 6.86\,\text{m/s}^2$$
$$T = \sqrt{\frac{(2000\,\text{m})(6.86 + 7.94)\,\text{m/s}^2}{(6.86\,\text{m/s}^2)(7.94\,\text{m/s}^2)}} \approx 23.5\text{s}$$

1.21 *Particle with constant radial velocity*

(a)

$$\mathbf{v} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} = (4.0\,\text{m/s})\,\hat{\mathbf{r}} + (3.0\,\text{m})(2.0\,\text{rad/s})\,\hat{\boldsymbol{\theta}}$$

(Note that radians are dimensionless.)

$$\mathbf{v} = (4.0\,\hat{\mathbf{r}} + 6.0\,\hat{\boldsymbol{\theta}})\,\text{m/s} \quad \text{v} = \sqrt{v_r{}^2 + v_\theta{}^2} = \sqrt{16.0 + 36.0} \approx 7.2\,\text{m/s}$$

(b)

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\boldsymbol{\theta}}$$

$\ddot{r} = 0$ and $\ddot{\theta} = 0$

$$a_r = -r\dot{\theta}^2 = -(3.0\,\text{m})(2.0\,\text{rad/s})^2 = -12.0\,\text{m/s}^2$$
$$a_\theta = 2\dot{r}\dot{\theta} = 2(4.0\,\text{m/s})(2.0\,\text{rad/s}) = 16.0\,\text{m/s}^2$$
$$a = \sqrt{a_r^2 + a_\theta^2} = \sqrt{144.0 + 256.0} = 20.0\,\text{m/s}^2$$

1.22 *Jerk*

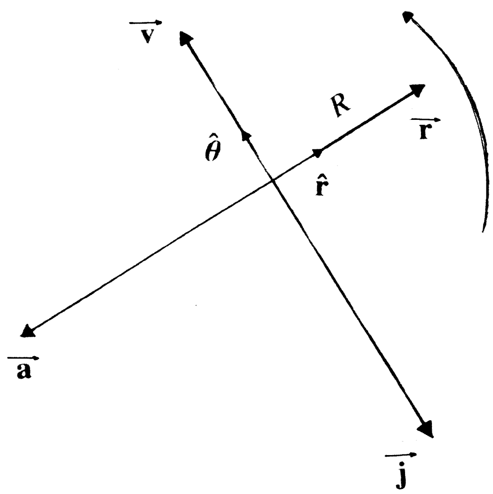
For uniform motion in a circle, $\theta = \omega t$,
where the angular speed ω is constant.

$$\mathbf{r} = r\,\hat{\mathbf{r}} = R\,\hat{\mathbf{r}}$$
$$\mathbf{v} = r\dot{\theta}\,\hat{\boldsymbol{\theta}} = \omega R\,\hat{\boldsymbol{\theta}}$$
$$\mathbf{a} = -r\dot{\theta}^2\,\hat{\mathbf{r}} = -R\omega^2\,\hat{\mathbf{r}}$$

Let $\mathbf{j} \equiv$ jerk.

$$\mathbf{j} = \frac{d\mathbf{a}}{dt} = -R\omega^2\frac{d\mathbf{r}}{dt} = -R\omega^2\hat{\boldsymbol{\theta}}$$

The vector diagram (drawn for $R = 2$ and $\omega = 1.5$) rotates rigidly as the point moves around the circle.



1.23 *Smooth elevator ride*

(a) Let $a(t) \equiv$ *acceleration*

$$a(t) = \frac{1}{2}a_m[1 - \cos(2\pi t/T)] \quad 0 \le t \le T$$
$$a(t) = -\frac{1}{2}a_m[1 - \cos(2\pi t/T)] \quad T \le t \le 2T$$

Let $j(t) \equiv$ *jerk*

$$j(t) = \frac{da}{dt}$$
$$j(t) = a_m(\pi/T)\sin(2\pi t/T) \quad 0 \le t \le T$$
$$j(t) = -a_m(\pi/T)\sin(2\pi t/T) \quad T \le t \le 2T$$

Let $v(t) \equiv$ *speed*

$$\begin{aligned} v(t) &= v(0) + \int_0^t a(t')dt' \quad 0 \le t \le T \\ &= \frac{1}{2}a_m[t - (T/2\pi)\sin(2\pi t/T)] \\ v(t) &= v(T) + \int_T^t a(t')dt' \quad T \le t \le 2T \\ &= \frac{1}{2}a_m T - \frac{1}{2}a_m[(t-T) - (T/2\pi)\sin(2\pi t/T)] \\ &= \frac{1}{2}a_m[(2T - t) + (T/2\pi)\sin 2\pi t/T] \end{aligned}$$

The sketch (in color) shows the jerk $j(t)$ (red), the acceleration $a(t)$ (green), and the speed $v(t)$ (black) versus time t.

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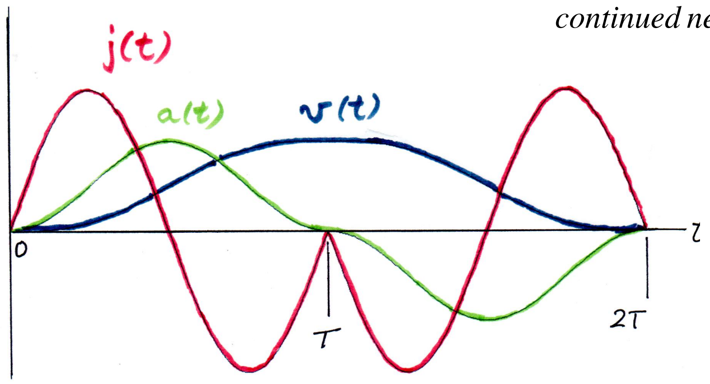

(b) The speed $v(t)$ is the area under the curve of $a(t)$. As the sketch indicates, $v(t)$ increases with time up to $t = T$, and then decreases. The maximum speed v_{max} therefore occurs at $t = T$, so that $v_{max} = v(T)$.

$$\begin{aligned} v_{max} &= v(0) + \int_0^T a(t')dt' \ = \frac{1}{2}a_m \int_0^T [1 - \cos(2\pi t'/T)]dt' \\ &= \frac{1}{2}a_m \ [t' - (T/2\pi)\sin(2\pi t'/T)]|_0^T \ = \frac{1}{2}a_m T \end{aligned}$$

(c) For $t \ll T$, we can use the small angle approximation:

$$\sin\theta = [\theta - \frac{1}{3!}\theta^3 + \ldots]$$

$$\begin{aligned} v(t) &= \int_0^t a(t')dt' \ = \frac{1}{2}a_m[t - (T/2\pi)\sin(2\pi t/T)] \\ &= \frac{a_m}{2}\{t - (T/2\pi)[(2\pi t/T) - \frac{1}{3!}(2\pi t/T)^3 + \ldots\} \\ &\approx \frac{a_m}{2}\{\frac{1}{3!}(2\pi/T)^2 t^3\} \ \approx a_m\left(\frac{\pi^2}{3}\right)\left(\frac{t^3}{T^2}\right) \end{aligned}$$

(d) *direct method:*

Let the distance at time t be $x(t)$.

$$x(t) = \int v(t')dt'$$

where

$$\begin{aligned} v(t) &= \frac{1}{2}\int_0^t a(t')dt' \quad 0 \le t \le T \\ &= \frac{a_m}{2}[t - (T/2\pi)\sin(2\pi t/T)] \quad 0 \le t \le T \\ v(t) &= \int_0^T a(t')dt' + \int_T^t a(t')dt' \quad T \le t \le 2T \\ &= \frac{a_m}{2}[T - t + T + (T/2\pi)\sin(2\pi t/T)] \quad T \le t \le 2T \end{aligned}$$

(Note that $v(2T) = 0$.) Then

$$\begin{aligned} D &= x(2T) \\ &= \frac{a_m}{2}\int_0^T [t' - (T/2\pi)\sin(2\pi t'/T)]dt' + \frac{a_m}{2}\int_T^{2T} [2T - t' + (T/2\pi)\sin(2\pi t'/T)]dt' \\ &= \frac{a_m}{2}T^2 \end{aligned}$$

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(e) *symmetry method:*

By symmetry, the distance from $x(0)$ to $x(T)$ and the distance from $x(T)$ to $x(2T)$ are equal. The distance from $x(0)$ to $x(T)$ is

$$
\begin{aligned}
x(T) &= \int_0^T v(t')dt' \\
&= \frac{a_m}{2}\int_0^T [t - (T/2\pi)\sin(2\pi t'/T)]dt' \\
&= \frac{a_m}{2}\,[t'^2/2 + (T/2\pi)^2\cos(2\pi t'/T)]\Big|_0^T \;=\; \frac{a_m}{4}T^2
\end{aligned}
$$

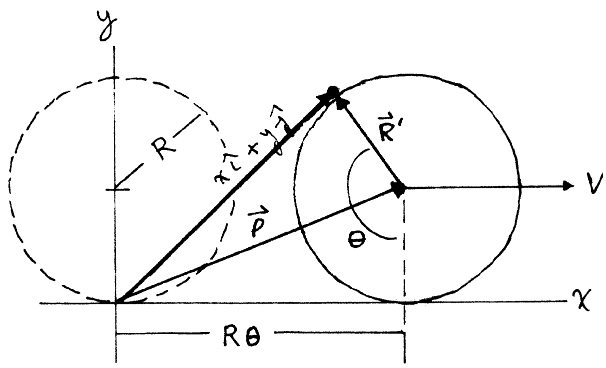
By symmetry

$$D = 2x(T) \;=\; \frac{1}{2}a_m T^2$$

as before.

1.24 *Rolling tire*

Let x, y be the coordinates of the pebble measured from the stationary origin. Let $\boldsymbol{\rho}$ be the vector from the stationary origin to the center of the rolling tire, and let $\mathbf{R}'$ be the vector from the center of the tire to the pebble.



$$
\begin{aligned}
\boldsymbol{\rho} &= R\theta\,\hat{\mathbf{i}} + R\,\hat{\mathbf{j}} \\
\mathbf{R}' &= -R\sin\theta\,\hat{\mathbf{i}} - R\cos\theta\,\hat{\mathbf{j}}
\end{aligned}
$$

From the diagram, the vector from the origin to the pebble is

$$
\begin{aligned}
x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} = \boldsymbol{\rho} + \mathbf{R}' \;&= R\theta\,\hat{\mathbf{i}} + R\,\hat{\mathbf{j}} - R\sin\theta\,\hat{\mathbf{i}} - R\cos\theta\,\hat{\mathbf{j}} \\
x = R\theta - R\sin\theta \quad & \dot{x} = R\dot{\theta} - R\cos\theta\,\dot{\theta} \\
y = R - R\cos\theta \quad & \dot{y} = R\sin\theta\,\dot{\theta}
\end{aligned}
$$

The tire is rolling at constant speed without slipping: $\theta = \omega t = (V/R)t$.

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$\dot{x} = R\omega - R\omega\cos\theta \quad \ddot{x} = R\omega^2\sin\theta$

$\dot{y} = R\omega\sin\theta \quad \ddot{y} = R\omega^2\cos\theta$

Note that

$\ddot{x}\,\hat{\mathbf{i}} + \ddot{y}\,\hat{\mathbf{j}} = \ddot{\boldsymbol{\rho}} + \ddot{\mathbf{R}}' \ = \ddot{\mathbf{R}}'$

The pebble on the tire experiences an inward radial acceleration V^2/R, and from the results for $\ddot{x}$ and $\ddot{y}$

$$\begin{aligned}\sqrt{\ddot{x}^2 + \ddot{y}^2} &= R\omega^2 \\ &= \frac{V^2}{R}\end{aligned}$$

as expected.

This result shows that the acceleration measured in the stationary system is the same as measured in the system moving uniformly along with the tire.

1.25 *Spiraling particle*

(a)

$$r = \frac{\theta}{\pi} \qquad \theta = \frac{\alpha t^2}{2}$$

$$r = \frac{\alpha t^2}{2\pi}$$

$$\dot{r} = \frac{\alpha t}{\pi} \qquad \dot{\theta} = \alpha t$$

$$\ddot{r} = \frac{\alpha}{\pi} \qquad \ddot{\theta} = \alpha$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\boldsymbol{\theta}} \ = \left(\frac{\alpha}{\pi} - \frac{\alpha^3 t^4}{2\pi}\right)\hat{\mathbf{r}} + \left(\frac{5\alpha^2 t^2}{2\pi}\right)\hat{\boldsymbol{\theta}}$$

(b)

$$a_r = \frac{\alpha}{\pi} - \frac{\alpha^3 t^4}{2\pi} = 0 \text{ at time t'}$$

$$\frac{\alpha}{\pi} = \frac{\alpha^3 t'^4}{2\pi} \implies t'^2 = \frac{\sqrt{2}}{\alpha}$$

$$\theta(t') = \frac{\alpha t'^2}{2} = \frac{1}{\sqrt{2}} \text{ rad}$$

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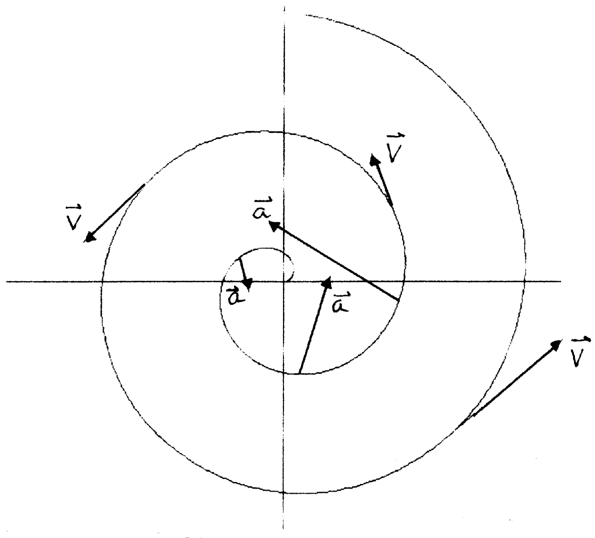
(c)



$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\boldsymbol{\theta}}$$

Using the expression for θ from part (a),

$$\mathbf{a} = \left(\frac{\alpha}{\pi}\right)[(1 - 2\theta^2)\,\hat{\mathbf{r}} + 5\theta\,\hat{\boldsymbol{\theta}}]$$

Setting $|a_r| = |a_\theta|$, then $|1 - 2\theta^2| = |5\theta|$

If $\theta < \frac{1}{\sqrt{2}}$, then $1 - 2\theta^2 = 5\theta$

Because $\theta \geq 0$, the only allowable root is

$$\theta = \frac{-5 + \sqrt{33}}{4} \approx 0.186 \text{ rad} \approx 10^\circ$$

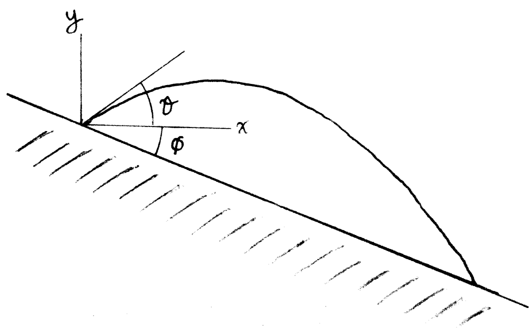
If $\theta > \frac{1}{\sqrt{2}}$, then $2\theta^2 - 1 = 5\theta$

$$\theta = \frac{5 + \sqrt{33}}{4} \approx 2.69 \text{ rad} \approx 154^\circ$$

In the sketch, the velocity vectors are in scale to one another, as are the acceleration vectors.

1.26 *Range on a hill*

The trajectory of the rock is described by coordinates x and y, as shown in the sketch. Let the initial velocity of the rock be v_0 at angle θ.



$$x = (v_0 \cos\theta)\,t \qquad y = (v_0 \sin\theta)\,t - \frac{1}{2}g\,t^2$$

The locus of the hill is $y = -x\tan\phi$

Let the rock land on the hill at time t'.

$$t' = \frac{x}{v_0\cos\theta}$$

The locus of the hill and the trajectory of the rock intersect at t'.

$$-x\tan\phi = x\tan\theta - \frac{1}{2}\left(\frac{g}{{v_0}^2}\right)\left(\frac{x^2}{\cos^2\theta}\right)$$

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Solving for x,

$$x = \left(\frac{2v_0^2}{g}\right)\left[\cos\theta\sin\theta + (\cos^2\theta)\tan\phi\right] = \left(\frac{2v_0^2}{g}\right)\left[\frac{1}{2}\sin 2\theta + (\cos^2\theta)\tan\phi\right]$$

The condition for maximum range is $dx/d\theta = 0$. Note that ϕ is a constant.

$$\frac{dx}{d\theta} = 0 = \cos 2\theta - 2\sin\theta\cos\theta\tan\phi = \cos 2\theta - (\sin 2\theta)\tan\phi$$

$$\cot 2\theta = \tan\phi$$

$$\tan 2\theta = \tan\left(\frac{\pi}{2} - \phi\right)$$

$$\theta = \frac{\pi}{4} - \frac{\phi}{2} \quad \text{for maximum range}$$

The sketch is drawn for the case $\phi = 20°$ and $v_0 = 5.0$ m/s.

1.27 *Peaked roof*

Let the initial speed at $t = 0$ be v_0. A straightforward way to solve this problem is to write the equations of motion in a uniform gravitational field, as follows:

$$x = -h + v_{0x}t \qquad y = v_{0y}t - \frac{1}{2}gt^2$$

$$v_x = v_{0x} \qquad v_y = v_{0y} - gt$$

At time T, he ball is at the peak, where $y = h$ and $v_y = 0$.

$$0 = v_{0y} - gT \;\Rightarrow\; T = \frac{v_{0y}}{g}$$

$$h = v_0yT - \frac{1}{2}gT^2 = \frac{v_{0y}^2}{g} - \frac{1}{2}\frac{v_{0y}^2}{g}$$

$$v_{0y}^2 = 2gh$$

At time T, $x = 0$.

$$0 = -h + v_{0x}T \;\Rightarrow\; v_{0x} = \frac{h}{T} = \frac{\sqrt{gh}}{2}$$

We then have

$$v_0 = \sqrt{v_{0x}^2 + v_{0y}^2} = \sqrt{2 + \frac{1}{2}}\sqrt{gh} = \sqrt{\frac{5}{2}}\sqrt{gh}$$

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A more physical approach is to note that the vertical speed needed to reach the peak is the same as the speed v_{0y} a mass acquires falling a distance h: $v_{0y} = \sqrt{2gh}$. The time T to fall that distance is $T = v_{0y}/g$. The horizontal distance traveled in the time T is

$$h = v_{0x}T = v_{0x}\left(\frac{v_{0y}}{g}\right) = v_{0x}\sqrt{\frac{2h}{g}}$$

$$v_{0x} = \sqrt{\frac{gh}{2}}$$

The initial speed v_0 is therefore

$$v_0 = \sqrt{v_{0x}^2 + v_{0y}^2} = \sqrt{2 + \frac{1}{2}}\sqrt{gh} = \sqrt{\frac{5}{2}}\sqrt{gh}$$

2 NEWTON'S LAWS

2.1 *Time-dependent force*

$$\mathbf{a} = \frac{\mathbf{F}}{m} = \left(\frac{(4t^2\,\hat{\mathbf{i}} - 3t\,\hat{\mathbf{j}})\,\mathrm{N}}{5\,\mathrm{kg}}\right) = \left(\frac{(4t^2\,\hat{\mathbf{i}} - 3t\,\hat{\mathbf{j}})\,\mathrm{kg}\cdot\mathrm{m/s^2}}{5\,\mathrm{kg}}\right) = \left(\frac{4}{5}t^2\,\hat{\mathbf{i}} - \frac{3}{5}t\,\hat{\mathbf{j}}\right)\ \mathrm{m/s^2}$$

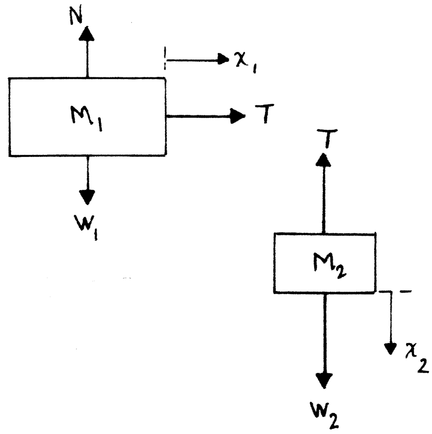
(a) $\mathbf{v}(t) - \mathbf{v}(0) = \int_0^t \mathbf{a}(t')dt' = \left(\frac{4}{15}t^3\hat{\mathbf{i}} - \frac{3}{10}t^2\,\hat{\mathbf{j}}\right)$ m/s

(b) $\mathbf{r}(t) - \mathbf{r}(0) = \int_0^t \mathbf{v}(t')dt' = \left(\frac{1}{15}t^4\,\hat{\mathbf{i}} - \frac{1}{10}t^3\,\hat{\mathbf{j}}\right)$ m

(c) $$\mathbf{r}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{t^4}{15} & -\frac{t^3}{10} & 0 \\ \frac{4t^3}{15} & -\frac{3t^2}{10} & 0 \end{vmatrix}$$

$$= \left(-\frac{t^6}{50}\,\hat{\mathbf{i}} + \frac{2t^6}{75}\,\hat{\mathbf{j}}\right)$$

2.2 *Two blocks and string*



(a) *Step 1: draw the force diagram for each block*
The force diagrams are shown in the sketch. The vertical forces on block M_1 cancel, because M_1 is on the table and has no vertical acceleration. (A constraint.) The tension T is the same at both ends of the string, because the string is massless so the net force on it must be 0.

(b) *Step 2: write the equations of motion for each block*

$$M_1\ddot{x}_1 = T \qquad M_2\ddot{x}_2 = W_2 - T$$

(c) *Step 3: write the constraint equation(s)*
We have already considered the (trivial) constraint condition for the vertical acceleration of M_1. Another constraint is that the length L of the string is fixed.

$$L - x_1 + x_2 = constant \implies -\ddot{x}_1 + \ddot{x}_2 = 0 \implies \ddot{x}_1 = \ddot{x}_2$$

(d) *Step 4: solve*
From the string constraint

$$T = M_1\ddot{x}_1 = M_1\ddot{x}_2 \quad (1)$$

From the equation of motion for M_2, and using $W_2 = M_2g$,

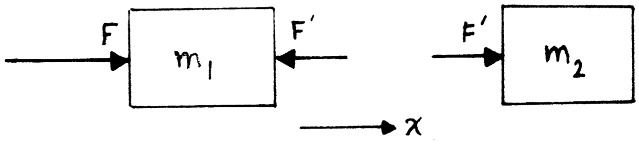
$$M_2g - T = M_2\ddot{x}_2 \quad (2)$$

Combining (1) and (2) gives

$$(M_1 + M_2)\ddot{x}_2 = M_2g \implies \ddot{x}_2 = \frac{M_2g}{(M_1 + M_2)} = \ddot{x}_1$$

2.3 *Two blocks on table*

The blocks have zero vertical acceleration, so the sketch omits the vertical forces. By Newton's third law, the force on m_1 due to m_2 is equal and opposite to the force on m_2 due to m_1.



$$m_1\ddot{x} = F - F' \qquad m_2\ddot{x} = F'$$

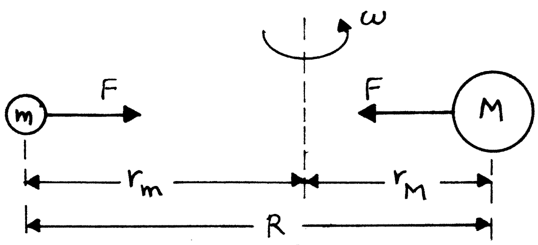
$$\ddot{x} = \frac{F}{(m_1 + m_2)}$$

This result can be found directly by considering a system with mass $(m_1 + m_2)$ acted on by the external force F.

$$F' = m_2\ddot{x} = \frac{m_2}{(m_1 + m_2)}F = \left[\frac{1\,\text{kg}}{(2\,\text{kg} + 1\,\text{kg})}\right](3\,\text{N}) = 1\,\text{N}$$

2.4 *Circling particle and force*

Let a_m be the inward radial acceleration of m, and a_M the inward radial acceleration of M.



$$ma_m = mr_m\omega^2 = F$$

$$Ma_M = Mr_M\omega^2 = F$$

$$r_m = \left(\frac{F}{\omega^2}\right)\left(\frac{1}{m}\right)$$

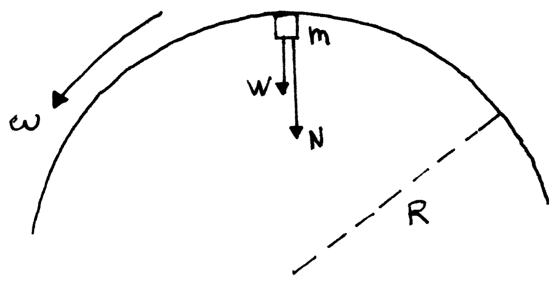
$$r_M = \left(\frac{F}{\omega^2}\right)\left(\frac{1}{M}\right)$$

$$R = r_M + r_M$$

$$= \left(\frac{F}{\omega^2}\right)\left(\frac{1}{m} + \frac{1}{M}\right)$$

2.5 *Concrete mixer*

Consider a small mass m of concrete, momentarily at the top of the rotating drum. Mass m is acted upon by the downward weight force W and by the normal force N exerted by the wall of the drum. Mass m falls away from the drum if $N \le 0$, when $\omega \le \omega_{critical}$.



$$W + N = ma_{radial} = mR\omega^2$$
$$N = mR\omega^2 - W = mR\omega^2 - mg = m(R\omega^2 - g)$$

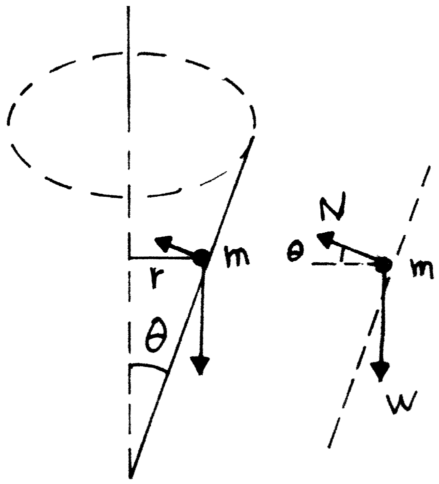
$N = 0$ for $\omega = \omega_{critical}$.

$R\omega^2{}_{critical} - g = 0 \implies \omega_{critical} = \sqrt{g/R}$

For R=0.5 m and g=9.8 m/s^2, $\omega_{critical} = \sqrt{9.8/0.5} = 4.43$ rad/s, or equivalently $4.43/2\pi$ rev/s $\implies$ = (0.705)(60) = 42.3 rpm.

2.6 *Mass in a cone*

From the force diagram,



$N \sin\theta = W \quad (1)$

$N \cos\theta = mr\omega^2 \quad (2)$

Dividing Eq. (1) by Eq. (2) gives

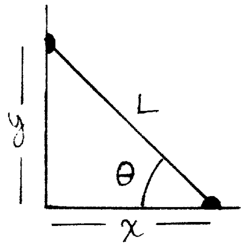
$$\tan\theta = \frac{W}{mr\omega^2} = \frac{mg}{mr\omega^2} = \frac{g}{r\omega^2}$$

The speed v_0 of m around the cone is $v_0 = r\omega$.

$$\tan\theta = \frac{gr}{v_0^2} \implies r = \frac{v_0^2 \tan\theta}{g}$$

2.7 *Leaning pole*

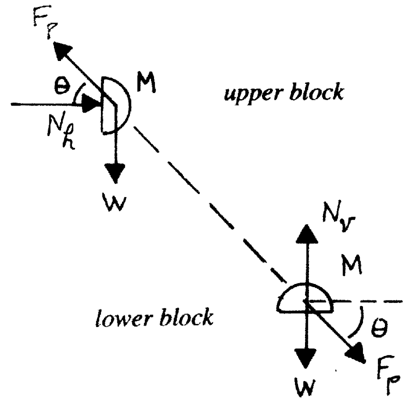
(a)



$$x^2 + y^2 = L^2$$
$$2x\dot{x} + 2y\dot{y} = 0$$
$$\dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y} = 0$$

When the pole is at rest, $\dot{x} = 0$, $\dot{y} = 0$.
At rest, the condition becomes

$$x\ddot{x} + y\ddot{y} = 0$$
$$\ddot{x} = -\frac{y}{x}\ddot{y} = -(\tan\theta)\ddot{y} \quad (1)$$



(b) The pole is taken to be massless, so the net force on the pole must be 0. The pole therefore exerts equal and opposite force F_p on each block, as indicated in the force diagrams. Consider only the equations of motion that do not involve the horizontal normal force N_h exerted on the upper block by the wall, and the vertical normal force N_v exerted on the lower block by the floor.

upper block: $M\ddot{y} = F_p \sin\theta - W \quad (2)$
lower block: $M\ddot{x} = F_p \sin\theta \quad (3)$

Solve Eqs.(1), (2), (3) for the three unknowns F_p, $\ddot{x}$ and $\ddot{y}$. From Eqs. (2) and (3)

$$M\ddot{y} = M\ddot{x}\left(\frac{\sin\theta}{\cos\theta}\right) - W \implies \ddot{y} = (\tan\theta)\ddot{x} - g.$$

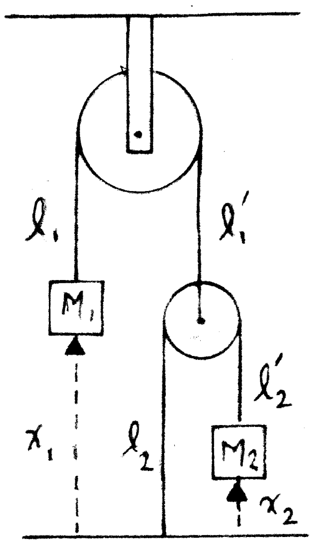
Combining with Eq. (1) yields

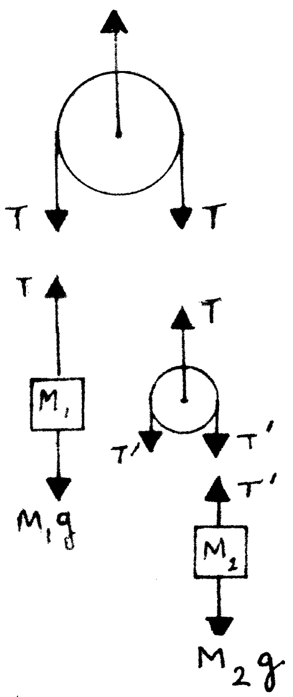
$$\ddot{y} = -(\tan^2\theta)\ddot{y} - g = \frac{-g}{1 + \tan^2\theta}$$

Using the identities $\tan\theta = \sin\theta/\cos\theta$ and $\sin^2\theta + \cos^2\theta = 1$ gives

$$\ddot{y} = -g\cos^2\theta \qquad \ddot{x} = -(\tan\theta)\ddot{y} = g\sin\theta\cos\theta$$

2.8 *Two masses and two pulleys*





constraint:

The fixed length of the string is a constraint.

$$x_1 + l_1 + l_1' + \frac{(x_2 + l_2 + l_2')}{2} = constant \implies \ddot{x}_2 = -2\ddot{x}_1$$

equations of motion:

The vertical force on the upper pulley plays no role in the motion and can be neglected. The lower pulley is taken to be massless, so the net force is 0: $2T' = T$.

$$M_1\ddot{x}_1 = T - M_1 g \implies T = M_1\ddot{x}_1 + M_1 g$$

$$M_2\ddot{x}_2 = T' - M_2 g = \frac{T}{2} - M_2 g$$

Solving,

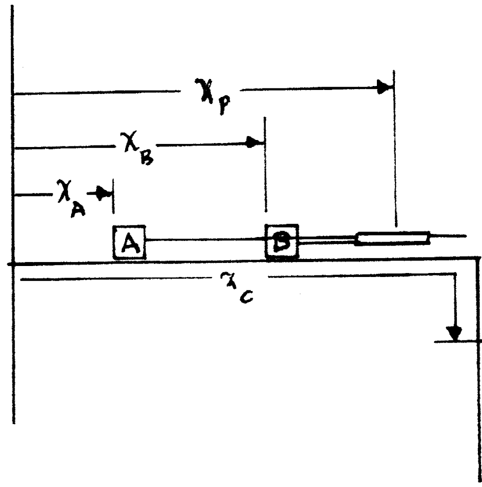
$$-4M_2\ddot{x}_1 = T - 2M_2 g = M_1\ddot{x}_1 + M_1 g - 2M_2 g$$

$$\ddot{x}_1 = \frac{(2M_2 - M_1)g}{(4M_2 + M_1)}$$

The result is reasonable. The weight of M_1 is counterbalanced by twice the weight of M_2; the acceleration of M_2 is twice the rate of M_1. As special cases, if $M_1 \gg M_2$, $\ddot{x}_1 \approx -g$. If $M_2 \gg M_1$, $\ddot{x}_1 \approx g/2$ and $\ddot{x}_2 \approx -g$.

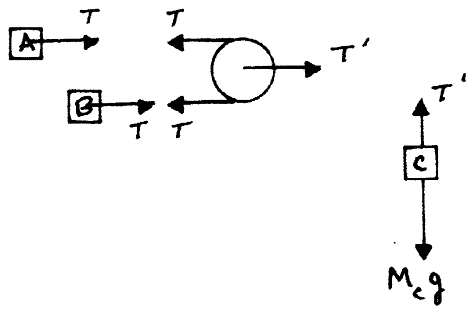
2.9 *Masses on table*

constraints:



$$x_C - x_P = \textit{constant}$$
$$\ddot{x}_C = \ddot{x}_P$$
$$(x_P - x_A) + (x_P - x_B) = \textit{constant}$$
$$\ddot{x}_A + \ddot{x}_B = 2\ddot{x}_P \ = 2\ddot{x}_C \quad (1)$$

The lower sketch shows the forces on the blocks and pulley. The pulley is taken to be massless, so the net force on the pulley is 0: $2T - T' = 0$.



equations of motion:

$$M_A\ddot{x}_A = T \qquad M_B\ddot{x}_B = T \qquad M_C\ddot{x}_C = M_C g - T'$$

solving:

$$\ddot{x}_A = \frac{T}{M_A} \qquad \ddot{x}_B = \frac{T}{M_B} \qquad \ddot{x}_C = g - \frac{2T}{M_C}$$

Using the constraint Eq. (1),

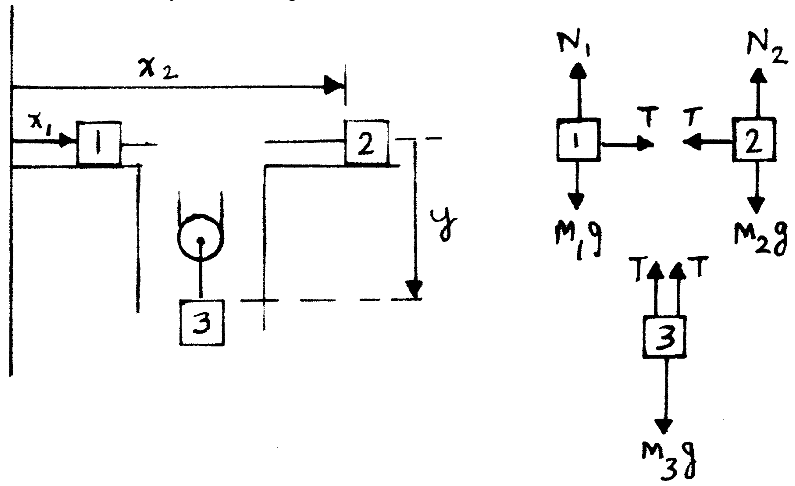
$$\frac{T}{M_A} + \frac{T}{M_B} = 2g - \frac{4T}{M_C}$$
$$T = \frac{2M_A M_B M_C g}{(M_A M_C + M_B M_C + 4M_A M_B)}$$

Then

$$\ddot{x}_A = \frac{2M_B M_C g}{(M_A M_C + M_B M_C + 4M_A M_B)}$$
$$\ddot{x}_B = \frac{2M_A M_C g}{(M_A M_C + M_B M_C + 4M_A M_B)}$$
$$\ddot{x}_C = \frac{(M_A + M_B)M_C g}{(M_A M_C + M_B M_C + 4M_A M_B)}$$

2.10 *Three masses*

(a) *coordinates and force diagrams:*



(b) *constraint:*

$x_2 - x_1 + 2y = $ *length of string* $=$ *constant*

$\ddot{x}_2 - \ddot{x}_1 + 2\ddot{y} = 0$

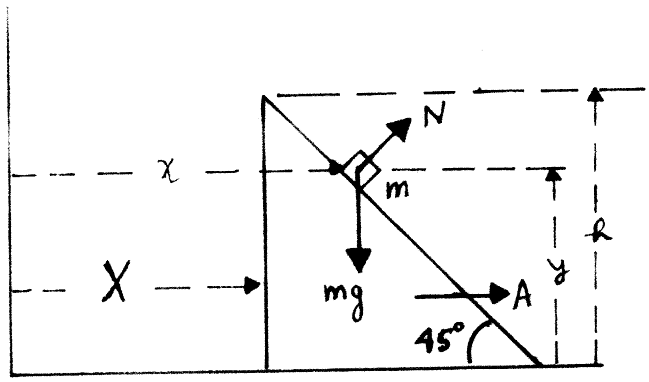
2.11 *Block on wedge*

constraint:

$$\frac{x - X}{h - y} = \tan(45°) = 1$$

$$x - X = h - y$$

$$\ddot{x} = \ddot{X} - \ddot{y} = A - \ddot{y} \qquad (1)$$



equations of motion:

Note that $\cos(45°) = \sin(45°) = 1/\sqrt{2}$

$$m\ddot{x} = N\cos(45°) = \frac{N}{\sqrt{2}} \qquad (2)$$

$$m\ddot{y} = N\sin(45°) - mg = \frac{N}{\sqrt{2}} - mg \qquad (3)$$

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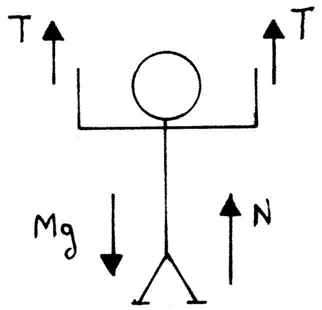
solve: From Eqs. (1), (2), and (3),

$$mA = m\ddot{y} + \frac{N}{\sqrt{2}} = \sqrt{2}N - mg \implies N = \frac{m(A+g)}{\sqrt{2}}$$

$$\ddot{x} = \frac{A+g}{2} \qquad \ddot{y} = \frac{A-g}{2}$$

2.12 *Painter on scaffold*

The constraint is that the painter and the scaffold both accelerate at the same rate a. The equations of motion for the painter and the scaffold are, respectively,

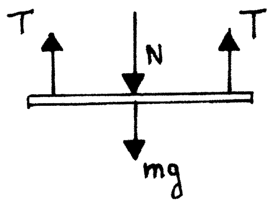


$$Ma = 2T + N - Mg$$

$$ma = 2T - N - mg$$

$$(M+m)a = 4T - (M+m)g$$

$$a = \frac{4T}{(M+m)} - g$$

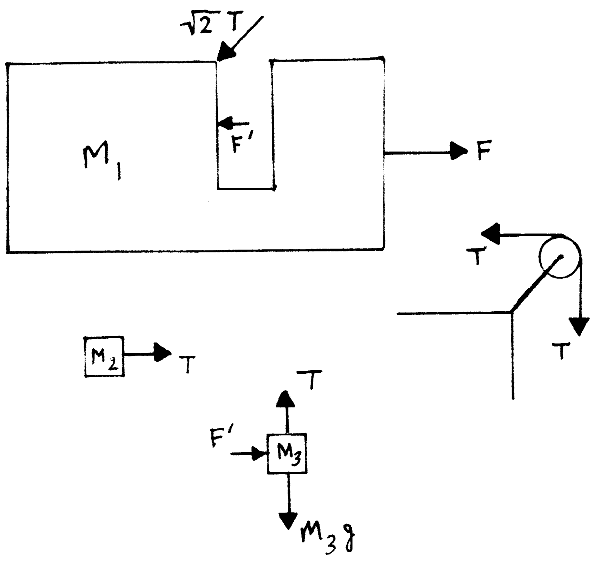


2.13 *Pedagogical machine*

Because M_3 is motionless, all three bodies experience the same horizontal acceleration a. Two further points:

(1) M_1 is pushing on M_3 with force F' to give M_3 the acceleration a. By Newton's third law, M_3 exerts an equal and opposite force F' on M_1, as shown in the force diagram.

(2) A subtle point is that the pulley holder exerts a force on M_1, as illustrated in the sketch. This force has a horizontal component T directed opposite to the applied force F.



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With reference to the force diagrams, the equations of motion are

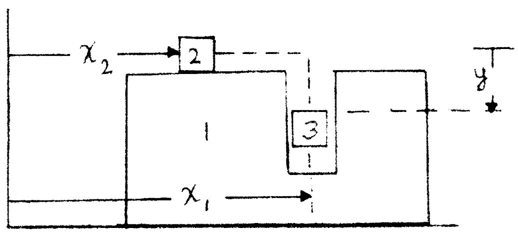
$$M_1a = F - F' - T \qquad M_2a = T \qquad M_3a = F' \qquad M_3g - T = 0$$

To find F, eliminate T, a, and F'.

$$T = M_3g = M_2a \implies a = \left(\frac{M_3}{M_2}\right)g$$

$$F = M_1a + F' + T = (M_1 + M_3)a + M_2a = (M_1 + M_2 + M_3)a = (M_1 + M_2 + M_3)\left(\frac{M_3}{M_2}\right)g$$

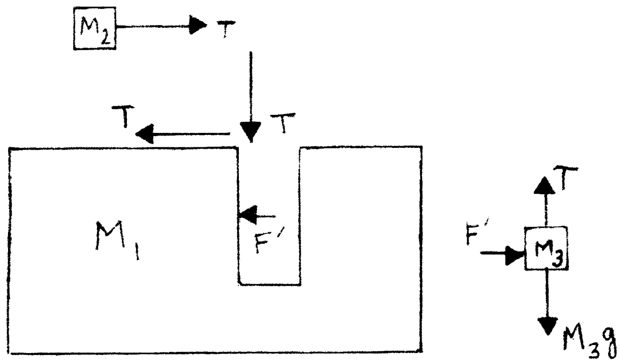
2.14 *Pedagogical machine 2*



constraints:
With reference to the coordinates in the upper sketch, the length of the string is $x_1 - x_2 + y$.

$$x_1 - x_2 + y = constant$$

$$\ddot{x}_1 - \ddot{x}_2 + \ddot{y} = 0 \quad (1)$$



A second constraint is that M_1 and M_3 have equal horizontal acceleration $\ddot{x}_1 = \ddot{x}_3$.
equations of motion:

$$M_1\ddot{x}_1 = -F' - T \qquad M_2\ddot{x}_2 = T \qquad M_3\ddot{x}_3 = F'$$

To solve, eliminate T and F', express $\ddot{x}_2$, $\ddot{x}_3$ and $\ddot{y}$ in terms of $\ddot{x}_1$, and use the constraint Eq. (1).

$$F' = M_3\ddot{x}_3 = M_3\ddot{x}_1$$

$$T = M_2\ddot{x}_2 = -(M_1\ddot{x}_1 + F') = -(M_1 + M_3)\ddot{x}_1 \implies \ddot{x}_2 = \frac{(M_1 + M_3)}{M_2}\ddot{x}_1$$

$$\ddot{y} = g - \frac{T}{M_3} = g - \frac{M_2}{M_3}\ddot{x}_2 = g + \frac{(M_1 + M_3)}{M_3}\ddot{x}_1$$

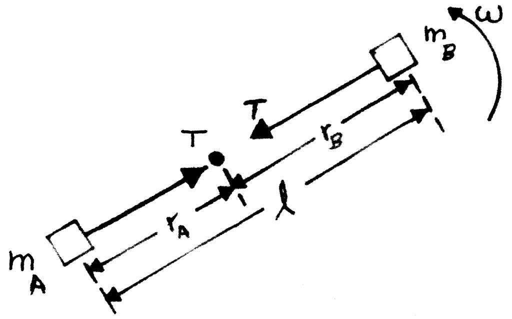
From Eq. (1)

$$0 = \ddot{x}_1 + \frac{(M_1 + M_3)}{M_2}\ddot{x}_1 + g + \frac{(M_1 + M_3)}{M_3}\ddot{x}_1$$

$$\ddot{x}_1 = \frac{-g}{\left(2 + \frac{(M_1+M_3)}{M_2} + \frac{M_1}{M_3}\right)} = \frac{-M_2M_3g}{(2M_2M_3 + M_1M_3 + M_1M_2 + M_3{}^2)}$$

2.15 *Disk with catch*

constraints:



$$r_A + r_B = l$$
$$\ddot{r}_B = -\ddot{r}_A$$

equations of motion:

Because the blocks are constrained by the groove, the tangential motion plays no dynamical role, and is neglected in the force diagram. Note that the force T on each mass is radially inward.

$$-T = m_A(\ddot{r}_A - r_A\omega^2)$$
$$-T = m_B(\ddot{r}_B - r_B\omega^2)$$
$$m_A(\ddot{r}_A - r_A\omega^2) = m_B(\ddot{r}_B - r_B\omega^2) = m_B\left[-\ddot{r}_A + (l - r_A)\omega^2\right]$$
$$\ddot{r}_A = \left(\frac{m_A - m_B}{m_A + m_B}\right) r_A\omega^2 + \frac{m_B\, l\omega^2}{m_A + m_B}$$

2.16 *Planck units*

Using Maxwell's notation, the dimensions of h, G, and c are symbolized as [h], [G], and [c] respectively.

$[h] = ML^2T^{-1}$ $\qquad [G] = M^{-1}L^3T^{-2}$ $\qquad [c] = LT^{-1}$

(a) The Planck length L_p is

$$L_p = h^\alpha G^\beta c^\gamma$$

Converting to dimensions,

$$L = (ML^2T^{-1})^\alpha(M^{-1}L^3T^{-2})^\beta(LT^{-1})^\gamma \ = M^{(\alpha-\beta)}L^{(2\alpha+3\beta+\gamma)}T^{-(\alpha+2\beta+\gamma)}$$

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The fundamental dimensions M, L, T are independent of one another, so the exponents must agree on both sides, leading to the three equations

$$\alpha - \beta = 0 \qquad 2\alpha + 3\beta + \gamma = 1 \qquad -\alpha - 2\beta - \gamma = 0$$

$$\alpha = 1/2 \qquad \beta = 1/2 \qquad \gamma = -3/2$$

$$L_p = \sqrt{\frac{hG}{c^3}} = \sqrt{\frac{(6.6 \times 10^{-34})(6.7 \times 10^{-11})}{(3.0 \times 10^8)^3}} = 4.1 \times 10^{-35}\ \text{m}$$

(b) Proceeding as in (a), with fresh exponents, the Planck mass M_p is

$$M_p = h^\alpha G^\beta c^\gamma \qquad M = M^{(\alpha-\beta)} L^{(2\alpha+3\beta+\gamma)} T^{-(\alpha+2\beta+\gamma)}$$

$$\alpha = 1/2 \qquad \beta = -1/2 \qquad \gamma = 1/2$$

$$M_p = \sqrt{\frac{hc}{G}} = \sqrt{\frac{(6.6 \times 10^{-34})(3.0 \times 10^8)}{6.7 \times 10^{-11}}} = 5.4 \times 10^{-8}\ \text{kg}$$

(c) Proceeding as before, the Planck time T_p is

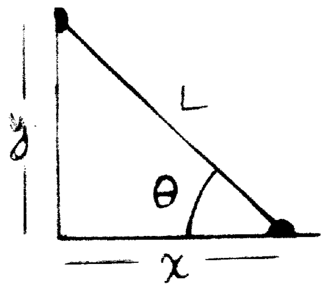
$$T_p = h^\alpha G^\beta c^\gamma \qquad T = M^{(\alpha-\beta)} L^{(2\alpha+3\beta+\gamma)} T^{-(\alpha+2\beta+\gamma)}$$

$$\alpha = 1/2 \qquad \beta = 1/2 \qquad \gamma = -5/2$$

$$T_p = \sqrt{\frac{hG}{c^5}} = \sqrt{\frac{(6.6 \times 10^{-34})(6.7 \times 10^{-11})}{(3.0 \times 10^8)^5}} = 1.3 \times 10^{-43}\ \text{s}$$

3 FORCES AND EQUATIONS OF MOTION

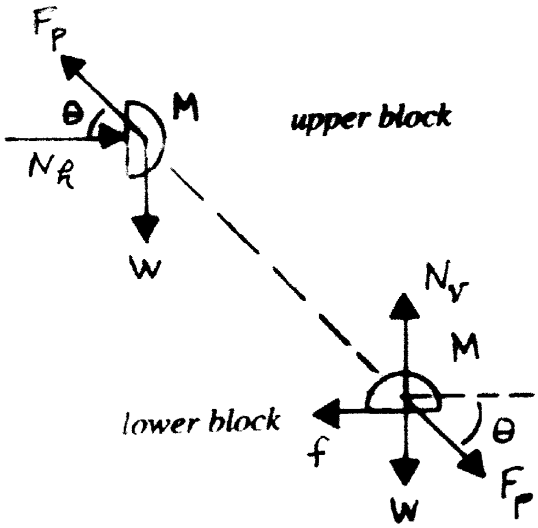
3.1 Leaning pole with friction



constraint:

$$x^2 + y^2 = L^2$$
$$2x\dot{x} + 2y\dot{y} = 0$$
$$\dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y} = 0$$

When the pole is at rest, $\dot{x} = 0$, $\dot{y} = 0$ and the condition becomes



$$x\ddot{x} + y\ddot{y} = 0$$
$$\ddot{x} = -\frac{y}{x}\ddot{y} = -(\tan\theta)\ddot{y} \quad (1)$$

The net force on the massless pole must be 0.
The pole therefore exerts equal and opposite force F_p on each block.

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equations of motion:

upper block:

$$M\ddot{y} = F_p \sin\theta - Mg \quad (2)$$

lower block:

$$N_v - Mg = 0$$

$$\begin{aligned} M\ddot{x} &= F_p \cos\theta - f \\ &= F_p \cos\theta - \mu N_v \\ &= F_p \cos\theta - \mu Mg \quad (3) \end{aligned}$$

We have three equations (1), (2), and (3) for the three unknowns F_p, $\ddot{x}$ and $\ddot{y}$. From Eq. (2),

$$F_p = \frac{M(\ddot{y} + g)}{\sin\theta} \quad (4)$$

From Eq. (3)

$$F_p = \frac{M(\ddot{x} + \mu g)}{\cos\theta}$$

Using Eqs. (4) and (1)

$$\begin{aligned} \ddot{y} + g &= (\tan\theta)(\ddot{x} + \mu g) \\ &= (\tan\theta)[-(\tan\theta)\ddot{y} + \mu g] \\ \ddot{y} &= \frac{(\mu \tan\theta - 1)g}{1 + \tan^2\theta} \end{aligned}$$

With the identities $\tan\theta = \sin\theta / \cos\theta$ and $\sin^2\theta + \cos^2\theta = 1$

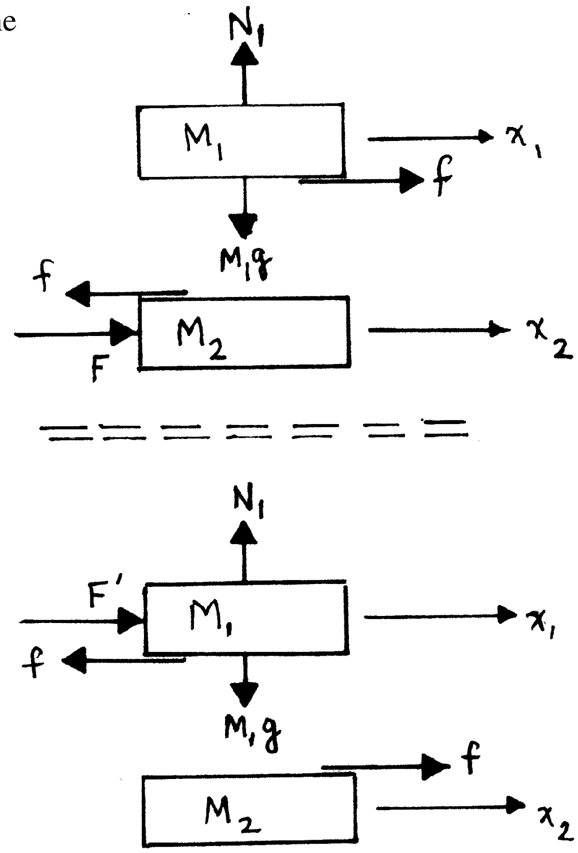
$$\ddot{y} = (\mu \sin\theta\cos\theta - \cos^2\theta)g$$

Then, using Eq. (1),

$$\begin{aligned} \ddot{x} &= -(\tan\theta)\ddot{y} \\ &= (\sin\theta\cos\theta - \mu \sin^2\theta)g \end{aligned}$$

The frictionless case is treated in problem 2.7.

3.2 Sliding blocks with friction



The upper sketch shows the force diagrams for the first part. The vertical forces on M_2 play no dynamical role, and are not shown.

$$M_1\ddot{x}_1 = f \implies \ddot{x}_1 = \frac{f}{M_1}$$

$$M_2\ddot{x}_2 = F - f \implies \ddot{x}_2 = \frac{F}{M_2} - \frac{f}{M_2}$$

The blocks move together if $\ddot{x}_1 = \ddot{x}_2$.

$$\frac{f}{M_1} = \frac{F}{M_2} - \frac{f}{M_2}$$

$$f = \left(\frac{M_1}{M_1 + M_2}\right) F = \left(\frac{4\,\mathrm{kg}}{9\,\mathrm{kg}}\right) 27\,\mathrm{N} = 12\,\mathrm{N}$$

Incidentally, $f = \mu N_1 = \mu M_1 g$ so that $\mu \approx 0.3$

The lower sketch shows the force diagrams for the second part.

$$M_1\ddot{x}_1 = F' - f \implies \ddot{x}_1 = \frac{F'}{M_1} - \frac{f}{M_1}$$

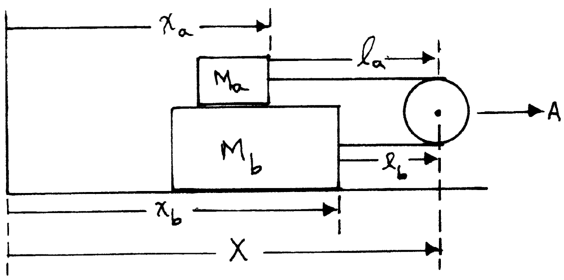
$$M_2\ddot{x}_2 = f \implies \ddot{x}_2 = \frac{f}{M_2}$$

The blocks move together if $x_1 = \ddot{x}_2$.

$$\frac{F'}{M_1} - \frac{f}{M_1} = \frac{f}{M_2}$$

$$F' = \left(\frac{M_1 + M_2}{M_2}\right) f = \left(\frac{M_1}{M_2}\right) F = 21.6\,\mathrm{N}$$

3.3 Stacked blocks and pulley



The upper sketch shows the coordinates and the lower sketches show the force diagrams.

constraints:

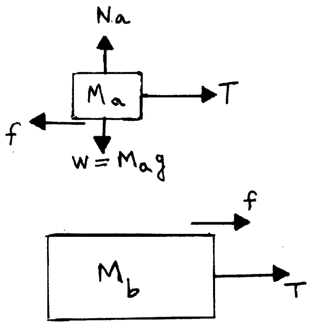
The massless rope has fixed length $l_a + l_b$ $=$*constant* so that $\ddot{l}_b = -\ddot{l}_a$. Also,

$$x_a + l_a = X \implies \ddot{x}_a + \ddot{l}_a = \ddot{X} = A \quad (1a)$$

$$x_b + l_b = X \implies \ddot{x}_b + \ddot{l}_b = \ddot{X} = A \quad (1b)$$

equations of motion:

Vertical forces on M_b are omitted in the sketch, because they play no dynamical role.



$$M_a\ddot{x}_a = T - f \implies \ddot{x}_a = \frac{1}{M_a}(T - f)$$

$$M_b\ddot{x}_b = T + f \implies \ddot{x}_b = \frac{1}{M_b}(T + f)$$

From Eqs. (1a) and (1b)

$$A - \ddot{l}_a = \frac{1}{M_a}(T - f)$$

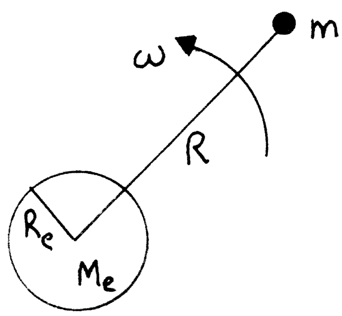
$$A + \ddot{l}_a = \frac{1}{M_b}(T + f)$$

$$2A = \left(\frac{1}{M_a} + \frac{1}{M_b}\right)T + \left(\frac{1}{M_b} - \frac{1}{M_a}\right)f$$

$$T = 2A\left(\frac{M_aM_b}{M_a + M_b}\right) + \left(\frac{M_b - M_a}{M_a + M_b}\right)f = 2A\left(\frac{M_aM_b}{M_a + M_b}\right) + \left(\frac{M_b - M_a}{M_a + M_b}\right)\mu N_a$$

$$= 2A\left(\frac{M_aM_b}{M_a + M_b}\right) + \left(\frac{M_b - M_a}{M_a + M_b}\right)\mu M_a g$$

3.4 Synchronous orbit



$$mR\omega^2 = \frac{GmM_e}{R^2} \implies R^3 = \frac{GM_e}{\omega^2}$$

$$\text{Using}\, g = \frac{GM_e}{R_e^2}$$

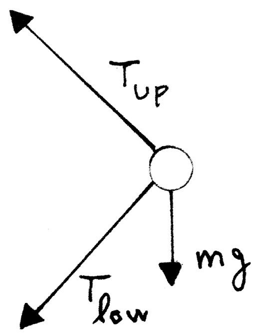
$$\frac{R^3}{R_e^3} = \frac{GM_e g}{GM_e R_e \omega^2}$$

$$R = R_e\left(\frac{g}{R_e\omega^2}\right)^{1/3}$$

$$g = 9.8\,\text{m/s}^2, \quad R_e = 6.4 \times 10^6\ \text{m}, \quad \omega = 2\pi\ \text{rad/day} = 7.3 \times 10^{-5}\ \text{s}^{-1}$$

$$R = 6.6R_e = 4.2 \times 10^7\ \text{m} \approx 26 \times 10^3\text{miles}$$

3.5 Mass and axle



(a) The sketch shows the force diagram.

(b) Note that $\cos 45° = \sin 45° = 1/\sqrt{2}$.
The radial distance of m from the axle is $l/\sqrt{2}$.
vertical equation of motion:

$$\frac{T_{up}}{\sqrt{2}} = mg + \frac{T_{low}}{\sqrt{2}}$$

radial equation of motion:

$$m\frac{l}{\sqrt{2}}\omega^2 = \frac{T_{up} + T_{low}}{\sqrt{2}}$$

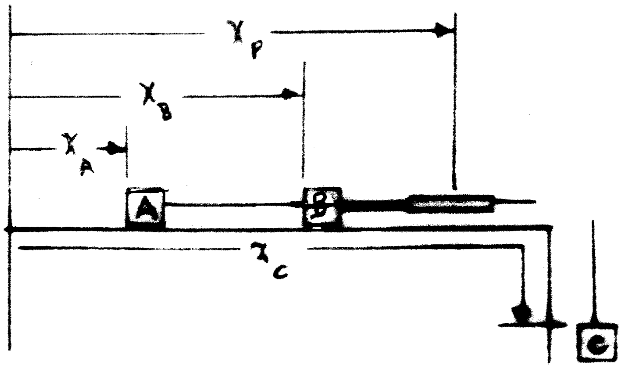
$$T_{low} = \frac{ml\omega^2}{2} - \frac{mg}{\sqrt{2}} \qquad T_{up} = \frac{ml\omega^2}{2} + \frac{mg}{\sqrt{2}}$$

3.6 Tablecloth trick

While the tablecloth is being pulled out from under the glass, the glass is accelerated at a rate a given by $ma = \mu mg$, or $a = \mu g$.If this occurs for time T, the glass reaches speed $v_0 = \mu gT$, and travels a distance $d = \frac{1}{2}\mu gT^2$. The glass is then sliding on the tabletop, and is retarded by a force μmg. It comes to rest in time T after traveling a distance $\frac{1}{2}\mu gT^2$. The total distance traveled by the glass is $D = 2d = \mu gT^2$. We require $D \leq 6$ inches $\approx$ 15 cm. So

$$T^2 \leq \frac{15\,\text{cm}}{\mu g} = \frac{0.15\,\text{m}}{(0.5)(9.8\,\text{m/s}^2)} \implies T \leq 0.17\,\text{s}$$

3.7 Pulleys and rope with friction



(a) *constraints:*

$$x_C - x_P = constant$$
$$(x_P - x_A) + (x_P - x_B) = constant$$

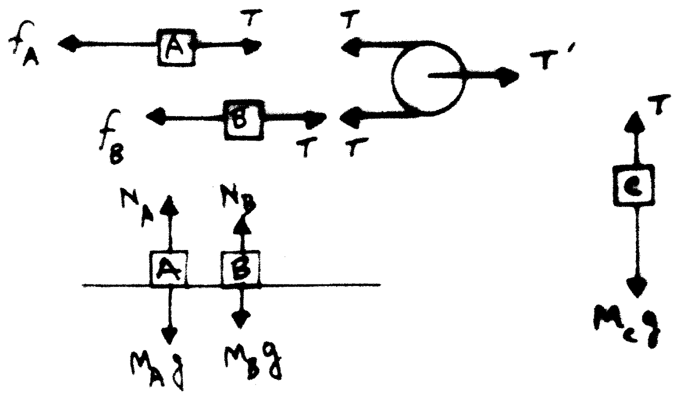
(b) *accelerations:*

$$\ddot{x}_C = \ddot{x}_P$$
$$\ddot{x}_A + \ddot{x}_B = 2\ddot{x}_P = 2\ddot{x}_C$$

(c) *equations of motion:*

$$M_A\ddot{x}_A = T - f_A$$
$$M_B\ddot{x}_B = T - f_B$$
$$N_A - M_Ag = 0 \qquad N_B - M_Bg = 0$$
$$f_A = \mu N_a = \mu M_Ag$$
$$f_B = \mu N_B = \mu M_Bg$$

The pulley is massless, so $T' = 2T$.



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Solving,

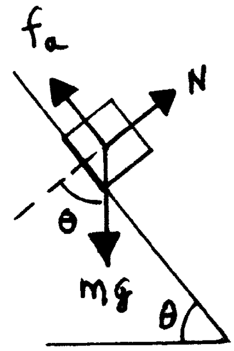
$$T'\left(\frac{1}{M_c}+\frac{1}{4M_A}+\frac{1}{4M_B}\right)=(1+\mu)g$$

$$T'=\left(\frac{4(1+\mu)M_AM_BM_C}{M_AM_C+M_BM_C+4M_AM_B}\right)g$$

$$T=\frac{T'}{2}=\left(\frac{2(1+\mu)M_AM_BM_C}{M_AM_C+M_BM_C+4M_AM_B}\right)g$$

The frictionless case is treated in problem 2.10.

3.8 Block and wedge



(a) The block has 0 acceleration.

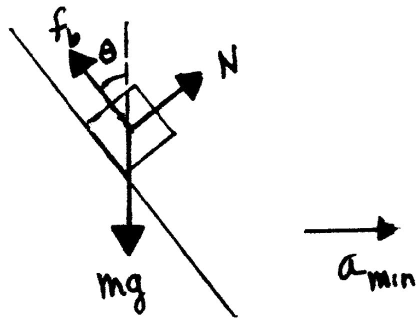
$$N - mg\cos\theta = 0$$

$$mg\sin\theta - f_a = 0$$

$$f_a = \mu N$$

$$mg\sin\theta = \mu N = \mu\, mg\cos\theta$$

$$\mu = \frac{\sin\theta}{\cos\theta} = \tan\theta$$



(b) *Minimum acceleration:*
The block's horizontal acceleration is

$$ma_{min} = N\cos\theta - f_b\sin\theta$$

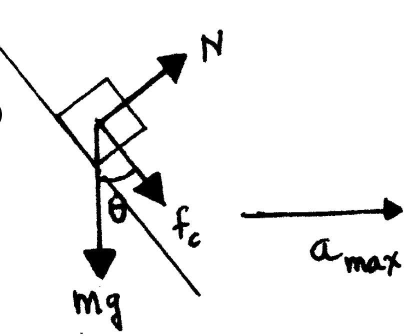
$$f_b \le \mu N$$

In the limit, $f_b = \mu N$

$$ma_{min} = N(\cos\theta - \mu\,\sin\theta) \quad (1)$$

The block has 0 vertical acceleration.

$$N\sin\theta + f_b\cos\theta - mg = 0 \implies N(\sin\theta + \mu\,\cos\theta) = mg \quad (2)$$



Dividing Eq. (1) by Eq. (2) gives

$$a_{min} = \left(\frac{\cos\theta - \mu\,\sin\theta}{\sin\theta + \mu\,\cos\theta}\right)g$$

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(c) *Maximum acceleration:*

$$ma_{max} = N\cos\theta + f_c \sin\theta$$
$$f_c \le \mu N$$

In the limit, $f_c = \mu N \implies ma_{max} = N(\cos\theta + \mu\ \sin\theta) \quad (3)$

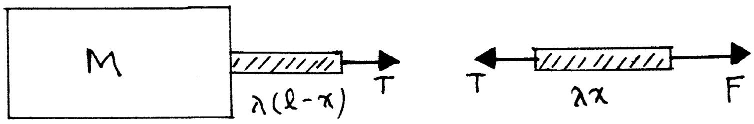
The block has 0 vertical acceleration.

$N\sin\theta - f_c\cos\theta - mg = 0 \implies N(\sin\theta - \mu\ \cos\theta) = mg \quad (4)$

Dividing Eq. (3) by Eq. (4) gives

$$a_{max} = \left(\frac{\cos\theta + \mu\ \sin\theta}{\sin\theta - \mu\ \cos\theta}\right) g$$

3.9 Tension in a rope



The uniform rope has linear mass density $\lambda = m/l$ mass per unit length. The equations of motion are

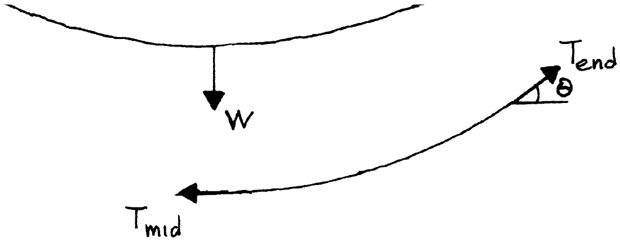
$F - T = (\lambda\, x)\, a \quad (1)$

$T = [M + \lambda\,(l - x)]\, a \quad (2)$

Dividing Eq. (1) by Eq. (2),

$$T = \left[\frac{M + \lambda\,(l - x)}{M + \lambda\, l}\right] F = \left[1 - \frac{m}{M + m}\left(\frac{x}{l}\right)\right] F$$

3.10 Rope and trees



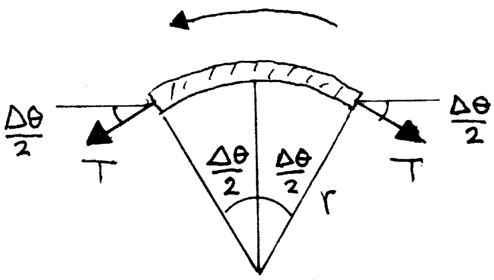
(a)

$$2T_{end}\sin\theta = W \implies T_{end} = \frac{W}{2\sin\theta}$$

(b)

$$T_{mid} = T_{end}\cos\theta = \frac{W\cos\theta}{2\sin\theta} = \frac{W}{2\tan\theta}$$

3.11 Spinning loop



In problems involving small angles, for example in examining a short arc length $\Delta S = R\Delta\theta$, it is helpful to keep the small angle approximations in mind: $\sin\Delta\theta \approx \Delta\theta$ and $\cos\Delta\theta \approx 1 - \Delta\theta^2/2$.

The radially inward force ΔF_r on the arc is

$$\Delta F_r = 2T\sin(\Delta\theta/2) \approx 2T\frac{\Delta\theta}{2} = T\Delta\theta$$

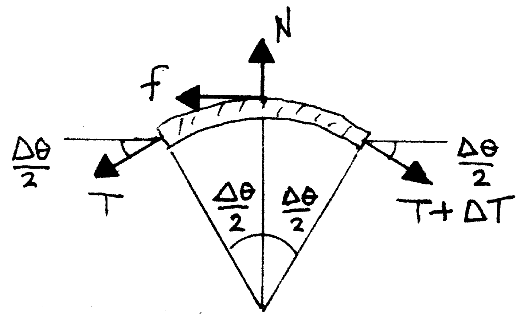
The mass Δm of the arc is

$$\Delta m = M\frac{\Delta\theta}{2\pi}$$

so the radial equation of motion is

$$(\Delta m)r\omega^2 = T\Delta\theta = M\left(\frac{\Delta\theta}{2\pi}\right)\left(\frac{l}{2\pi}\right)\omega^2 \implies T = Ml\left(\frac{\omega}{2\pi}\right)^2$$

3.12 Capstan



The rope is stationary, so the forces are in balance. From the sketch, the vertical equation of motion is

$$0 = N - T\sin(\Delta\theta/2) - (T + \Delta T)\sin(\Delta\theta/2)$$

Because we will be taking the limit, retain only first order terms.

$$N \approx 2T\frac{\Delta\theta}{2} = T\Delta\theta$$

The horizontal equation of motion is

$$0 = (T + \Delta T)\cos(\Delta\theta/2) - f - T\cos(\Delta\theta/2)$$

Using the small angle approximation $\cos x \approx 1 - x^2/2$ and retaining first order terms

$$f \approx \Delta T$$

The maximum friction force is

$$f = \mu N$$

$$\Delta T \approx \mu T\,\Delta\theta$$

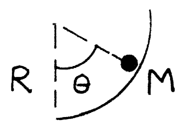
In the limit $\Delta\theta \to 0$,

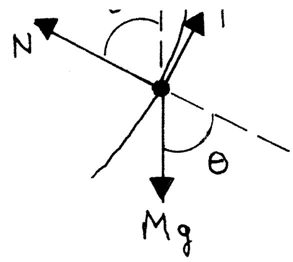
$$\frac{dT}{d\theta} = \mu T$$

Integrating,

$$\int_{T_B}^{T_A} \frac{dt}{T} = \mu \int_0^{\theta_0} d\theta \implies ln\left(\frac{T_A}{T_B}\right) = \mu\,\theta_0 \implies T_A = T_B\, e^{\mu\,\theta_0}$$

3.13 Incomplete loop-the-loop





This problem assumes $r = R = constant$, so that $\dot{r} = 0$, $\ddot{r} = 0$, and $\dot{\theta} = v/R = constant$, so that $\ddot{\theta} = 0$. The radial acceleration is inward, with $\ddot{r} = 0$. The tangential acceleration is 0.

radial equation of motion:

$$M\frac{v^2}{R} = N - Mg\cos\theta \quad (1)$$

tangential equation of motion:

$$f - Mg\sin\theta = 0$$

The car begins to skid when the tangential force $f - Mg\sin\theta \leq 0$. The maximum value of f is μN. The limiting case is $\mu N = Mg\sin\theta$.

Using Eq. (1),

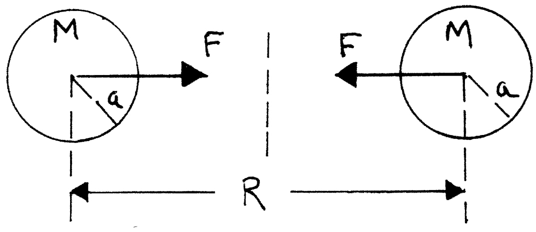
$$M\frac{v^2}{R} = Mg\left(\frac{\sin\theta}{\mu} - \cos\theta\right) \implies \frac{\sin\theta}{\mu} - \cos\theta = \frac{v^2}{Rg} \quad (2)$$

For a flat plane, $R \to \infty$, slipping occurs when $\tan\theta = \mu$, as found in problem 3.8.

The direction of f is a possible source of confusion. Formally, the car would have a tangential acceleration in the reverse direction if f were opposed to the direction of motion. Physically, the car's engine turns the tires, and they exert a friction force on the road opposed to the direction of motion. The road therefore exerts an equal and opposite force; the car is propelled forward by the friction force.

What is the condition for the car to barely make a complete loop ($\theta = \pi$ rad $= 180°$)? According to the result Eq. (2), $v^2/R = g$ when $\theta = \pi$ rad. This means that at the top of the loop, the downward weight force mg in this limiting case is sufficient to account for the radial acceleration v^2/R. It follows that $N = 0$ at the top of the loop, so the car is just parting company with the loop under this condition. If $v^2/R > g$, then $N > 0$, and the car is definitely in contact with the track at the top of the loop.

3.14 Orbiting spheres



Each sphere orbits in a circle of radius $R/2$, and each sphere experiences a radial gravitational attraction F

$$F = \frac{GM^2}{R^2}$$

radial equation of motion:

$$M\frac{R}{2}\omega^2 = \frac{GM^2}{R^2} \implies \omega^2 = \frac{2GM}{R^3}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R^3}{2GM}}$$

Let ρ be the density, $M = \frac{4}{3}\pi a^3\rho$. Make M large to make T small, so that a should be as large as possible. $a_{max} = R/2$ (spheres touching). Then

$$T_{min} = 2\pi\sqrt{\frac{R^3}{2G\rho\,\frac{4}{3}\pi(R/2)^3}} = 2\pi\sqrt{\frac{3}{\pi G\rho}} = \sqrt{\frac{12\pi}{G\rho}}$$

$$G = 6.67\times 10^{-11}\,\text{kg}^{-1}\,\text{m}^3\,\text{s}^{-2}$$

$$\rho = 21.5\text{g cm}^{-3} = 21.5(\text{g cm}^{-3})(10^{-3}\,\text{kg g}^{-1})(10^6\,\text{cm}^3\text{m}^{-3}) \quad = 21.5\times 10^3\,\text{kg m}^{-3}$$

$$T_{min} = 5130\,\text{s}$$

3.15 Tunnel through the Earth

The mass of a sphere of radius $r < R_e$ within a uniform Earth is $M(r) = M_e(r/R_e)^3$. The equation of motion of mass m in a tunnel through the center of the Earth is

$$m\ddot{r} = -m\left(\frac{M_e r^3}{R_e^3}\right)\frac{G}{r^2} \implies \ddot{r} = -\left(\frac{GM_e}{R_e^3}\right)r$$

Using $GM_e/R_e^2 = g$,

$$\ddot{r} + \left(\frac{g}{R_e}\right)r = 0$$

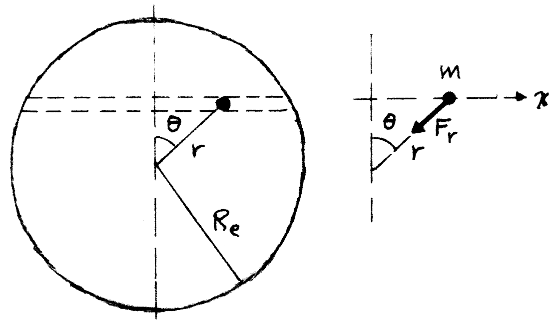
This is the equation for SHM, with frequency ω_{tunnel} and period $T = 2\pi/\omega$.

$$\omega_{tunnel} = \sqrt{\frac{g}{R_e}} \implies T = 2\pi\sqrt{\frac{R_e}{g}} = 2\pi\sqrt{\frac{6.4\times 10^6\,\text{m}}{9.8\,\text{m/s}^2}} = 5080\,\text{s} \approx 85\,\text{min}$$

For a satellite of mass m in circular low Earth orbit, the equation of motion is

$$mR_e\omega_{orbit}^2 = m\frac{GM_e}{R_e^2} = mg \implies \omega_{orbit} = \sqrt{\frac{g}{R_e}} = \omega_{tunnel}$$

3.16 Off-center tunnel



Mass m is at coordinate x along the off-center tunnel. Mass m is gravitationally attracted by the mass of the Earth within the radius r. As shown in problem 3.15, the radial force F_r on m is

$$F_r = -\frac{mgr}{R_e}$$

$$F_x = F_r \sin\theta = -mg\left(\frac{x}{r}\right)\left(\frac{r}{R_e}\right)$$

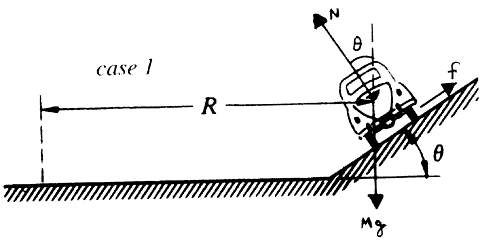
equation of motion along x:

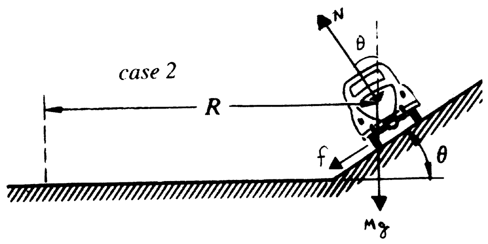
$$m\ddot{x} = -mg\left(\frac{x}{R_e}\right) \implies \ddot{x} + \left(\frac{g}{R_e}\right)x = 0$$

This is the equation for SHM with frequency ω and period $T = 2\pi/\omega$.

$$\omega = \sqrt{\frac{g}{R_e}} = \omega_{tunnel} \quad \text{as in problem 3.15}$$

3.17 Turning car





There are two cases, as the sketches indicate. Keep in mind that the friction force is opposed to the direction of motion. In case 1, the car will tend to slide down the slope if it is moving too slowly, so the friction force f is outward as shown. In case 2, the car will tend to slide up the slope if it is moving too fast, so f is inward.

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case 1:
horizontal equation of motion:

$$\frac{Mv^2}{R} = N\sin\theta - f\cos\theta$$

The maximum friction force is μN.

$$\frac{Mv^2}{R} \geq N(\sin\theta - \mu\cos\theta)$$
$$\frac{Mv2_{min}}{R} = N(\sin\theta - \mu\cos\theta) \quad (1)$$

There is no vertical acceleration if the car is not sliding, so the vertical equation of motion is $N\cos\theta + f\sin\theta - Mg = 0$. In the limit where $f = \mu N$

$$Mg = N(\cos\theta + \mu\sin\theta) \quad (2)$$

Dividing Eq. (1) by Eq. (2),

$$\frac{v_{min}^2}{Rg} = \frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta} \implies v_{min} = \sqrt{Rg\left(\frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}\right)}$$

case 2:
Proceeding as before,

$$M\frac{v^2}{R} \leq N\sin\theta + f\cos\theta$$
$$M\frac{v_{max}^2}{R} = N(\sin\theta + \mu\cos\theta) \quad (3)$$

vertical equation of motion:

$$0 = N\cos\theta - f\sin\theta - Mg = N(\cos\theta - \mu\sin\theta) \quad (4)$$

Dividing Eq. (3) by Eq. (4) leads to

$$v_{max} = \sqrt{Rg\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right)}$$

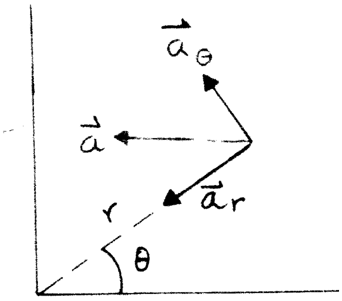
3.18 Car on rotating platform

(a) Acceleration in polar coordinates:

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$$

$$\mathbf{a} = \mathbf{a}_r + \mathbf{a}_\theta$$

(a)



In this problem,

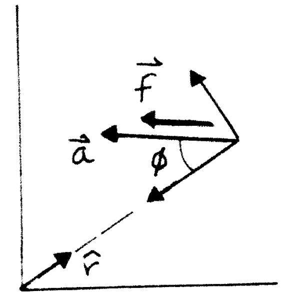
$$r = v_0 t \quad \dot{r} = v_0 \quad \ddot{r} = 0$$

$$\theta = \omega t \quad \dot{\theta} = \omega \quad \ddot{\theta} = 0$$

$$\mathbf{a} = -v_0 t\omega^2\, \hat{\mathbf{r}} + 2v_0\omega\, \hat{\boldsymbol{\theta}}$$

(b) The car starts to skid when

(c)



$$Ma \geq f_{max}$$

$$= \mu W = \mu Mg$$

$$a = \sqrt{a^2{}_r + a^2{}_\theta} = \sqrt{v^2{}_0\, t^2\, \omega^4 + 4v^2{}_0\, \omega^2} \geq \mu g$$

Skidding just starts at t_0, where

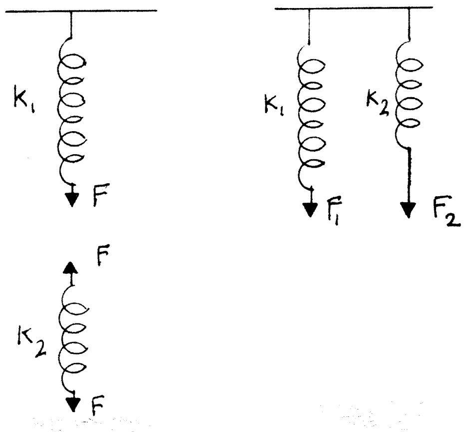
$$(\mu g)^2 = v^2{}_0\, \omega^4 t^2{}_0 + 4v^2{}_0\, \omega^2$$

$$t_0 = \frac{1}{v_0\, \omega^2} \sqrt{(\mu g)^2 - 4v^2{}_0\, \omega^2}$$

Note that if the Coriolis term $2v_0\, \omega$ is $> \mu g$, the car always skids.

(c) The friction force f is directed along the acceleration, at angle ϕ, as shown. When the car begins to skid. it will move backwards along that line.

3.19 Mass and springs



(a) (b)

(a)

$$\Delta x_1 = \frac{F}{k_1} \qquad \Delta x_2 = \frac{F}{k_2}$$

$$\Delta x_{total} = \Delta x_1 + \Delta x_2$$

$$k_{eff} = \frac{F}{\Delta x_{total}} = \frac{1}{\left(\frac{1}{k_1} + \frac{1}{k_2}\right)}$$

$$\omega_a = \sqrt{\frac{k_{eff}}{m}} = \sqrt{\frac{k_1 k_2}{m(k_1 + k_2)}}$$

(b)

Both springs stretch the same amount Δx.

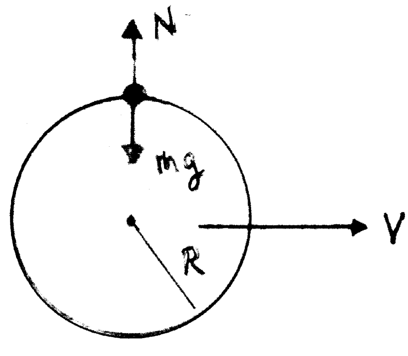
$$F_1 = k_1 \Delta x \qquad F_2 = k_2 \Delta x$$

$$F_{total} = F_1 + F_2 = (k_1 + k_2)\Delta x$$

$$k_{eff} = \frac{F_{total}}{\Delta x} = k_1 + k_2 \implies \omega_b = \sqrt{\frac{(k_1 + k_2)}{m}}$$

3.20 Wheel and pebble

As long as the pebble is in contact with wheel, its speed $V_{pebble} = R\omega = V_{wheel}$, the speed of the wheel's center as it rolls along.



(a)

(a)

$$V_{pebble} = V_{wheel} \equiv V$$

From force diagram (a)

$$m\frac{V^2}{R} = mg - N$$

$N \geq 0$ The pebble flies off when $N = 0$.

$$\frac{mV^2}{R} > mg \implies V > \sqrt{Rg}$$

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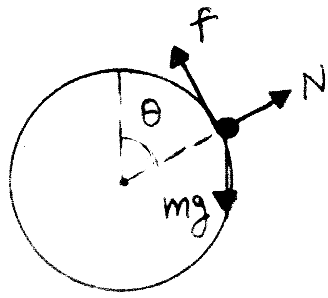
(b)



(b)

While in contact the pebble's radial equation of motion is

$$\frac{mV^2}{R} = mg\cos\theta - N$$

Using the criterion $N \geq 0$,

$$\cos\theta_{max} = \frac{V^2}{Rg}.$$

There is a more stringent criterion based on f: there is no tangential acceleration, so

$$0 = mg\sin\theta - f \implies f = mg\sin\theta \quad (1)$$
$$f \leq \mu N \quad (2)$$
$$N = mg\cos\theta - m\frac{V^2}{R} \quad (3)$$

Combining Eqs. (1), (2), and (3) gives

$$g\sin\theta \leq \mu g\cos\theta - V^2/R$$
$$\sin\theta \leq \cos\theta - \frac{V^2}{Rg} \quad (\text{for } \mu = 1)$$
$$\frac{V^2}{Rg} = \cos(\theta_{max}) - \sin(\theta_{max}) = \sqrt{2}\cos(\theta_{max} + \pi/4)$$
$$\cos(\theta_{max} + \pi/4) = \frac{1}{\sqrt{2}}\frac{V^2}{Rg}$$

3.21 Bead on rod

The radial acceleration is $a_r = \ddot{r} - r\dot{\theta}^2 = \ddot{r} - r\omega^2$

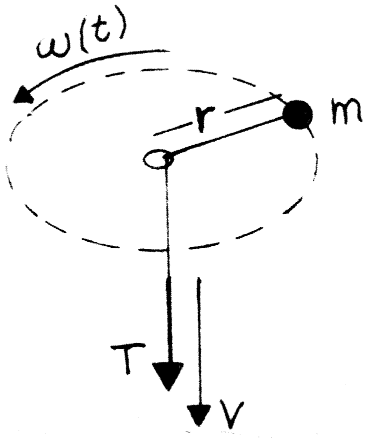
Because the rod is frictionless, $a_r = 0$, so that $\ddot{r} = r\omega^2 \quad (1)$

Given that $r = Ae^{-\gamma t} + Be^{\gamma t}$

$\dot{r} = -\gamma Ae^{-\gamma t} + \gamma Be^{\gamma t} \qquad \ddot{r} = \gamma^2 Ae^{-\gamma t} + \gamma^2 Be^{\gamma t} = \gamma^2 r$

Comparing with Eq. (1) it follows that $\gamma = \omega$.

3.22 Mass, string, and ring



(a) $V = constant$, so that

$$r = r_0 - Vt \quad \dot{r} = -V \quad \ddot{r} = 0$$

$$F_{tangential} = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad (1)$$

Eq. (1) becomes a differential equation for ω.

$$r\frac{d\omega}{dt} + 2\frac{dr}{dt}\omega = 0 \implies \frac{d\omega}{\omega} = -2\frac{dr}{r}$$

$$ln(\omega)|_{\omega_0}^{\omega} = -2ln(r)|_{r_0}^{r}$$

$$ln\left(\frac{\omega}{\omega_0}\right) = -2ln\left(\frac{r}{r_0}\right)$$

$$\frac{\omega}{\omega_0} = \left(\frac{r_0}{r}\right)^2 \implies \omega(t) = \omega_0\frac{{r_0}^2}{(r_0 - Vt)^2}$$

(b)

$$F_{radial} = -T$$

$$m(\ddot{r} - r\dot{\theta}^2) = -T$$

$$T = mr\omega^2$$

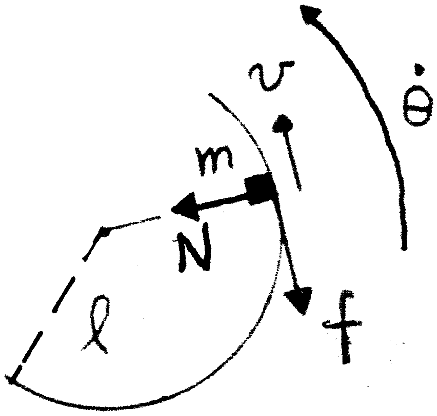
T is a function of time, to keep the end of the string moving at steady rate V.

$$T = m\,\omega_0^2\left(\frac{r_0^4}{r^3}\right) = m\,r_0\,\omega_0^2\left(\frac{r_0}{r}\right)^3 = m\,r_0\,\omega_0^2\left(\frac{r_0}{r_0 - Vt}\right)^3$$

3.23 Mass and ring

(a)

$$-f = F_{tangential} = ma_{tangential} = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$
$$r = l = constant \implies \dot{r} = 0$$
$$\dot{\theta} = \frac{v}{l} \qquad \ddot{\theta} = \left(\frac{1}{l}\right)\frac{dv}{dt}$$
$$\frac{dv}{dt} = -\frac{f}{m}$$
$$N = F_{radial} = mr\dot{\theta}^2 = ml\frac{v^2}{l^2} = m\frac{v^2}{l}$$
$$f = \mu N$$
$$\frac{dv}{dt} = -\frac{\mu v^2}{l} \implies \frac{dv}{v^2} = -\left(\frac{\mu}{l}\right)dt$$
$$\int_{v_0}^{v}\frac{dv}{v^2} = -\frac{\mu}{l}\int_0^t dt$$
$$-\left(\frac{1}{v} - \frac{1}{v_0}\right) = -\frac{\mu t}{l}$$
$$v = \frac{v_0}{(1 + \mu v_0 t/l)}$$



(b)

$$\frac{d\theta}{dt} = \frac{v}{l} = \frac{1}{l}\frac{v_0}{(1 + \mu v_0 t/l)}$$
$$\int_{\theta_0}^{\theta} d\theta = \int_0^t \frac{v_0}{l(1 + \mu v_0 t'/l)}dt'$$
$$\theta - \theta_0 = \frac{1}{\mu}\int\frac{dx}{1 + x} = \frac{1}{\mu}ln(1 + x)$$

where $x = \mu v_0 t/l$

$$\theta(t) = \theta_0 + \frac{1}{\mu}ln\,(1 + \mu v_0 t/l) \quad (1)$$

What is $\theta(t)$ if the ring is frictionless, $\mu = 0$? The solution can be found simply by noting that the block must continue to move with its initial speed v_0, so that $\dot{\theta} = v_0/l = constant$. Then $\theta(t) - \theta_0 = v_0 t/l$. In the frictionless case, θ increases without limit as t increases.

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However, it is worthwhile to describe a general approach to problems of this type. For $\mu = 0$, the result Eq. (1) becomes $\ln(1)/0 = 0/0$, an indeterminate form. To deal with this situation, treat μ as small, and expand the logarithm in Taylor's series.

$ln(1+x) \approx x +$ terms of order x^2 and higher

$$\theta(t) - \theta_0 \approx \frac{\mu\, v_0 t}{\mu\, l} + \text{terms of order } \mu^2 \text{ and higher} \rightarrow \frac{v_0 t}{l} \quad \text{in the limit } \mu \rightarrow 0.$$

3.24 Retarding force

$$m\frac{dv}{dt} = -F = -be^{\alpha v}$$

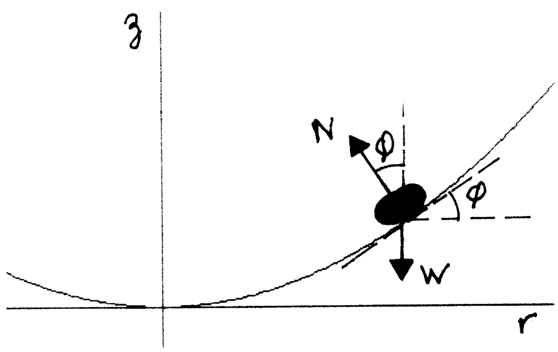
$$e^{-\alpha v}\frac{dv}{dt} = -\frac{b}{m}$$

$$\int_{v_0}^{v} e^{-\alpha v}\, dv = -\frac{b}{m}\int_0^t dt$$

$$-\frac{1}{\alpha}(e^{-\alpha v} - e^{-\alpha v_0}) = -\frac{b}{m}t$$

$$e^{-\alpha v} = \frac{\alpha b}{m}t + e^{-\alpha v_0} \implies v = \frac{1}{\alpha} ln\left(\frac{1}{\alpha bt/m + e^{-\alpha v_0}}\right)$$

3.25 Hovercraft



For stable circular motion

$$N\sin\phi = m\frac{v^2}{r} = mr\left(\frac{2\pi}{T}\right)^2$$

$$N\cos\phi = W = mg$$

$$\tan\phi = \frac{r}{g}\left(\frac{2\pi}{T}\right)^2$$

$$\tan\phi = \frac{dz}{dr} \implies dz = \frac{1}{g}\left(\frac{2\pi}{T}\right)^2 r\,dr$$

$$\int_0^z dz = \frac{1}{g}\left(\frac{2\pi}{T}\right)^2 \int_0^r r\,dr \implies z = \frac{1}{2g}\left(\frac{2\pi}{T}\right)^2 r^2$$

The bowl is a parabola of revolution.

3.26 Viscous force

Consider a force $\mathbf{F} = f(v)\hat{\mathbf{v}}$.

$$\frac{\mathbf{F}}{m} = \mathbf{a} = \frac{d(v\,\hat{\mathbf{v}})}{dt} = \frac{dv}{dt}\hat{\mathbf{v}} + v\frac{d\hat{\mathbf{v}}}{dt} \quad (1)$$

Because $\hat{\mathbf{v}}$ is a unit vector, it cannot change in magnitude, only in direction, as shown earlier in Sec. 1.10.1. In particular, for any vector $\mathbf{A}$ of constant magnitude, $d\mathbf{A}/\mathbf{dt}$ is perpendicular to $\mathbf{A}$. But $\mathbf{F} = f(v)\hat{\mathbf{v}}$ has no component perpendicular to $\mathbf{v}$, so it follows from the equation of motion that $d\hat{\mathbf{v}}/dt = 0$; for a force $\mathbf{F}$ along $\mathbf{v}$, $\hat{\mathbf{v}}$ cannot change either in magnitude or in direction. Hence the force $\mathbf{F}$ cannot alter the direction of motion.

Another approach is to take the dot product of Eq. (1) with $\hat{\mathbf{v}}$. to give

$$\frac{\mathbf{F}\cdot\hat{\mathbf{v}}}{m} = \frac{dv}{dt}\hat{\mathbf{v}}\cdot\hat{\mathbf{v}} + v\hat{\mathbf{v}}\cdot\frac{d\hat{\mathbf{v}}}{dt}$$
$$\hat{\mathbf{v}}\cdot\frac{d\hat{\mathbf{v}}}{dt} = \frac{1}{2}\frac{d(\hat{\mathbf{v}}\cdot\hat{\mathbf{v}})}{dt} = \frac{1}{2}\frac{d(constant)}{dt} = 0$$
$$m\frac{dv}{dt} = \mathbf{F}\cdot\hat{\mathbf{v}} = f(v)$$

For the case $f(v) = -Cv^2$, find $v(t)$ by integration.

$$\frac{dv}{dt} = -\frac{C}{m}v^2$$
$$\int_{v_0}^{v}\frac{dv}{v^2} = -\frac{C}{m}\int_0^t dt \implies \frac{1}{v} = \frac{1}{v_0} + \frac{Ct}{m} = \frac{1}{v_0}\left(1 + \frac{Cv_0 t}{m}\right) = \frac{1}{v_0}\left(1 + \frac{t}{\tau}\right)$$

$\tau = m/(Cv_0)$ is a *characteristic time* for this system – it sets the time scale. A second integration gives $s(t)$, the distance traveled in time t.

$$s = \int v\,dt = \int_0^t \left(\frac{v_0}{1 + t/\tau}\right)dt = v_0\tau\, ln(1 + t/\tau)$$

If $t \ll \tau$, use $ln(1 + x) \approx x$ for $x \ll 1$.

$$s \approx v_0\tau\left(\frac{t}{\tau}\right) \approx v_0 t$$

4 MOMENTUM

4.1 Center of mass of a nonuniform rod

(a)

$$
\begin{aligned}
M &= \int_0^l dm = \int_0^l \lambda\, dx = A \int_0^l \cos\left(\frac{\pi x}{2l}\right) dx \\
&= \frac{2l}{\pi} A\, sin\left(\frac{\pi x}{2l}\right)\Big|_0^l = \frac{2l}{\pi} A \\
A &= \frac{\pi}{2l} M
\end{aligned}
$$

(b)

$$\bar{X} = \frac{1}{M}\int_0^l x\, dm = \frac{1}{M}\int_0^l x\lambda\, dx = \frac{A}{M}\int_0^l x\cos\left(\frac{\pi x}{2l}\right) dx = \frac{\pi}{2l}\int_0^l x\cos\left(\frac{\pi x}{2l}\right) dx$$

Using the substitution

$$u = \frac{\pi x}{2l} \implies \bar{X} = \left(\frac{2l}{\pi}\right)\int_0^{\pi/2} u\cos u\, du \quad \text{and integrate by parts using}$$

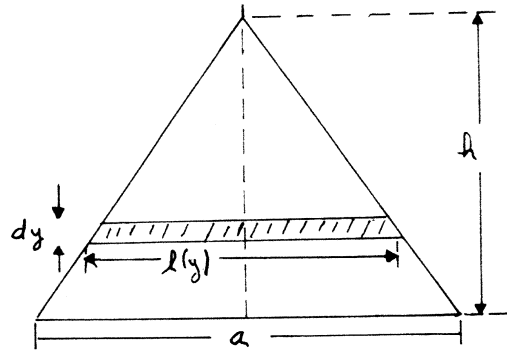
$$u\cos u = \frac{d}{du}(u\sin u) - \sin u$$

$$
\begin{aligned}
\int_0^{\pi/2} u\cos u\, du &= u\sin u|_0^{\pi/2} - \int_0^{\pi/2} \sin u\, du \\
&= (\pi/2 - 0) - (-\cos u)|_0^{\pi/2} = \frac{\pi}{2} - 1
\end{aligned}
$$

$$\bar{X} = \frac{2l}{\pi}\left(\frac{\pi}{2} - 1\right) = l\left(1 - \frac{2}{\pi}\right)$$

For a uniform rod, $\bar{X} = l/2$. This nonuniform rod has greater mass near $x = 0$, so $\bar{X} < l/2$ as expected.

4.2 Center of mass of an equilateral triangle



Method 1: analytical Divide the plate into narrow strips of length $l(y)$ and width dy, as shown. The mass dm of a strip is $dm = \rho l t dy$, where ρ is the density of the plate and t is its thickness. The mass M of the plate is $\rho \times area \times t = \frac{1}{2}\rho a h t$.

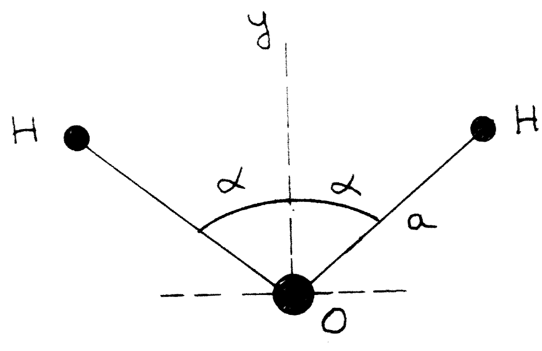
By symmetry, the center of mass is on the y axis.

$$\bar{Y} = \frac{1}{M}\int y\,dm = \frac{2}{\rho a h t}\int \rho t y l(y)\,dy = \frac{2}{ah}\int_0^h ya\left(1-\frac{y}{h}\right)dy = \frac{2}{h}\int_0^h\left(y-\frac{y^2}{h}\right)dy$$
$$= \frac{2}{h}\left(\frac{1}{2}-\frac{1}{3}\right)h^2 = \frac{h}{3}$$

Method 2: geometrical For any uniform triangle, symmetry requires that the center of mass lies on the median line from any vertex to the midpoint of the opposite side. As a simple proof, divide the triangle into strips perpendicular to a median line; the center of mass of the strip is at its center.

According to a theorem from geometry, the three medians of a triangle meet at a point 2/3 the distance from each vertex. In this problem, take the median line that is along the y axis; then the center of mass is 2/3 the distance from the top vertex, so that the center of mass lies at a height $y = h/3$ from the base.

4.3 Center of mass of a water molecule



The center of mass lies on the y axis, by symmetry. Take the origin at the oxygen atom, as shown, so that the y coordinate of the oxygen atom is $y_O = 0$. The y coordinate of each hydrogen atom is
$y_H = a\cos\alpha = 0.097\text{ nm} \times \cos(52.25°) = 0.059\text{ nm}$.

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$$\bar{Y} = \frac{1}{M_{total}}\left(2M_{hydrogen}y_H + M_{oxygen}y_O\right)$$

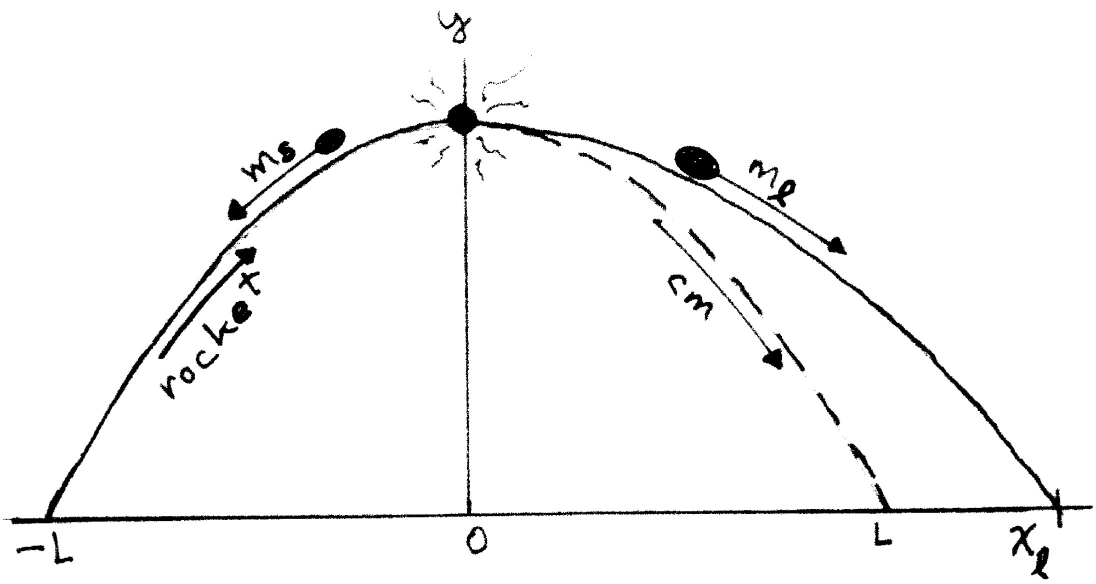
where $M_{total} = 2M_{hydrogen} + M_{oxygen}$.

$M_{hydrogen} = 1$ amu (atomic mass unit) and $M_{oxygen} = 16$ amu.

$$\bar{Y} = \frac{2}{2+16}\,[(2)(0.059) + 16(0.00)]$$
$$= 0.0066\text{nm}$$

The center of mass is very near the massive oxygen atom, as expected.

4.4 Failed rocket



As long as the pieces are in flight, the center of mass continues on the parabolic trajectory. The time to rise is the same as the time to fall, so the center of mass reaches the ground at $x = L$. Let the smaller piece have mass m_s, and the larger piece have mass m_l, as indicated in the sketch.

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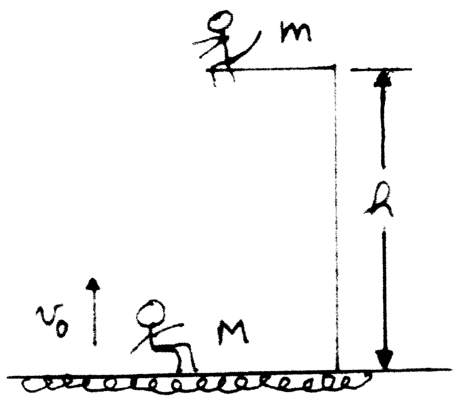
The x coordinate of the center of mass is L when it reaches the ground, so

$$\bar{X} = \frac{m_s x_s + m_l x_l}{m_s + m_l}$$
$$L = \frac{m_s(-L) + m_l x_l}{m_s + m_l} = \frac{-Lm_s + 3x_l m_s}{m_s + 3m_s} = \frac{-L + 3x_l}{4}$$
$$x_l = \frac{5}{3}L$$

in the coordinate system shown in the sketch, or alternatively, from the launch point,

$$x_l = \frac{5}{3}L + L = \frac{8}{3}L$$

4.5 Acrobat and monkey



The acrobat reaches height h at time t.

$$h = -\frac{1}{2}gt^2 + v_0 t$$
$$t = \frac{v_0 - \sqrt{v_0{}^2 - 2gh}}{g}$$

The acrobat's speed v at t is

$$v = -gt + v_0 = \sqrt{v_0{}^2 - 2gh}$$

Vertical momentum is conserved when the acrobat grabs the monkey. The speed v' of the pair just after the collision is

$$(m + M)v' = Mv \implies v' = \frac{M}{m + M}\sqrt{v_0{}^2 - 2gh}$$

The pair rises for a time t' until their speed = 0.

$$-gt' + v' = 0 \implies t' = \frac{v'}{g}$$

At the peak, they are a height h' above the perch.

$$h' = -\frac{1}{2}gt'^2 + v't' = \frac{v'^2}{2g} = \left(\frac{M}{m + M}\right)^2 \left(\frac{v_0{}^2}{2g} - h\right)$$

The total height $h + h'$ is

$$h + h' = \left(\frac{M}{m + M}\right)^2 \frac{v_0{}^2}{2g} + \left[1 - \left(\frac{M}{m + M}\right)^2\right] h$$

4.6 Emergency landing

Let M = mass of plane = $\frac{2500\text{ lb}}{\text{g}}$ m = mass of sandbag = $\frac{250\text{ lb}}{\text{g}}$
v = speed at landing = 120 ft/s $F_{retarding} = F_{friction} + F_{brakes}$
L = distance traveled before coming to rest
Momentum is conserved at the moment the sandbag is picked up. The system's speed then becomes v'.

$$v' = \left(\frac{M}{m+M}\right)v$$

The system slows with uniform acceleration a.

$$a = \frac{F_{retarding}}{(m+M)}$$
$$L = \frac{v'^2}{2a}$$

$F_{friction} = \mu mg = (0.4)(250\text{ lb}) = 100\text{ lb}$ $F_{brakes} = 300\text{ lb}$

$F_{retarding} = 100\text{ lb} + 300\text{ lb} = 400\text{ lb}$

$$\begin{aligned} L &= \frac{v'^2}{2a} = \frac{v^2}{2}\left(\frac{M}{m+M}\right)^2\left(\frac{m+M}{F_{retarding}}\right) = \frac{v^2}{2}\left(\frac{M}{m+M}\right)\left(\frac{M}{F_{retarding}}\right) \\ &= \frac{(120\text{ ft/s})^2}{2} \times \left(\frac{2500}{2750}\right) \times \left(\frac{2500\text{ lb}}{32\text{ ft}/s^2}\right) \times \left(\frac{1}{400\text{ lb}}\right) \approx 1300\text{ ft} \end{aligned}$$

4.7 Blocks and compressed spring

While m_1 is against the wall, m_2 moves according to SHM with $\omega = \sqrt{k/m_2}$.

$x_2 = A\sin(\omega t) + B\cos(\omega t) + C$ $\dot{x}_2 = \omega A\cos(\omega t) - \omega B\sin(\omega t)$

Using the initial conditions $x_2(0) = l/2$ and $\dot{x}_2(0) = 0$, it follows that

$$x_2 = \left(1 - \frac{1}{2}\cos(\omega t)\right)l$$

Until m_1 loses contact with the wall, the coordinate X of the center of mass is

$$X = \frac{m_1x_1 + m_2x_2}{m_1+m_2} = \frac{m_2}{m_1+m_2}\left(1 - \frac{1}{2}\cos(\omega t)\right)l$$

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m_1 loses contact with the wall when $\omega t = \pi/2$; at this instant, $x_2 = l$.

$$\dot{x}_2 = \frac{\omega l}{2} = constant$$

From this time on, the system moves as a whole, with $\dot{x}_1 = \dot{x}_2$. Thus

$$\dot{X} = \frac{m_1\dot{x}_1 + m_2\dot{x}_2}{m_1 + m_2} = \dot{x}_2 = \frac{\omega l}{2}$$

4.8 Jumper

To rise a height h, the initial speed must be $v_0 = \sqrt{2gh}$. The initial momentum is therefore mv_0, and the final momentum is 0. The impulse from the ground is then

$$impulse = mv_0 - 0 = (50\,\text{kg})\sqrt{2 \times 9.8\,\text{m/s}^2 \times 0.8\,\text{m}} = 198\,\text{kg m/s}$$

4.9 Rocket sled

At time t, the system consists of mass M moving with speed v, where $M(0) = M_0$. The momentum $P(t)$ is

$$P(t) = Mv$$

At time $t + \Delta t$, the system (still of total mass M) consists of mass $(M - \Delta M)$ moving with speed $(v + \Delta v)$ and mass ΔM moving with speed $(v - v_0)$. Then

$$\begin{aligned}\Delta P &= P(t + \Delta t) - P(t)\\ &= (M - \Delta M)(v + \Delta v) + \Delta M(v - v_0) - Mv\end{aligned}$$

In the limit $\Delta t \to 0$

$$\frac{dP}{dt} = M\frac{dv}{dt} - v_0\frac{dM}{dt}$$

The friction force on the sled is $-\mu Mg$.

$$\frac{dP}{dt} = -\mu Mg$$

$$M\frac{dv}{dt} - v_0\frac{dM}{dt} = -\mu Mg$$

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The fuel burns at constant rate $dM/dt = -\gamma$, so that $M(t) = M_0 - \gamma\, t$.

$$\frac{dv}{dt} = \frac{v_0\gamma}{(M_0 - \gamma\, t)} - \mu\, g$$

Integrating,

$$\begin{aligned} v(t) &= v_0\gamma \int_0^t \frac{dt'}{(M_0 - \gamma\, t')} - \mu\, gt \\ &= -v_0\, |.ln(M_0 - \gamma\, t')|_0^t - \mu\, gt \\ &= v_0 ln\left(\frac{M_0}{(M_0 - \gamma\, t)}\right) - \mu\, gt \end{aligned}$$

The rocket engine turns off at time t_f when $\gamma t_f = M_0/2$.

$$v(t_f) = v_0\, ln2 - \mu\, g\frac{M_0}{2\gamma}$$

The sled begins to slow for $t > t_f$, so $v(t_f)$ is the maximum speed.

4.10 Rolling freight car with sand

The system consists of the freight car and its contents, with initial mass M_0 at $t = 0$. The bottom is opened at $t = 0$, and the sand runs out at steady rate γ, so that $dm/dt = \gamma$ and $dM/dt = -\gamma$. Then $M(t) = M_0 - \gamma\, t$. To first order, and exact in the limit $\Delta t \to 0$, the mass of sand Δm released in time Δt has at the instant of release the same speed as the freight car, so it does not contribute to the change of the system's momentum. (See Example 4.14.)

$$P(t) = Mv \qquad P(t + \Delta t) = (M - \Delta m)(v + \Delta v) + \Delta m(v + \Delta v)$$

$$\Delta P \approx M\Delta v$$

$$\frac{dP}{dt} = M\frac{dv}{dt} = Ma = F \implies \frac{dv}{dt} = \frac{F}{M} = \frac{F}{M_0 - \gamma t}$$

The sand is all gone at time t_f, so that $t_f = m/\gamma$.

$$\begin{aligned} \int_0^{v(t_f)} dv &= \int_0^{t_f} \frac{F dt'}{M_0 - \gamma t'} \\ v(t_f) &= -\left(\frac{F}{\gamma}\right) ln\left(\frac{M_0 - \gamma t_f}{M_0}\right) = \left(\frac{F}{\gamma}\right) ln\left(\frac{M_0}{M_0 - m}\right) \end{aligned}$$

4.11 Freight car and hopper

The system consists of the freight car of mass M initially at rest, plus the mass of sand m that will fall in by time t; m is also initially at rest, and $m = bt$. The total mass of the car and sand at time t is $M + bt$.

$$P(0) = 0 \qquad P(t) = (M + bt)v$$

$$\textit{impulse} = \Delta P = (M + bt)v = \int_0^t F dt' = F \int_0^t dt' = Ft \implies v = \frac{Ft}{(M + bt)}$$

4.12 Two carts and sand

The system consists of cart A, with mass M and speed v, and the mass Δm moving with speed u. The rate of material flow is b, so that $dm/dt = b$.

$$P(t) = Mv + (\Delta m)u \qquad P(t + \Delta t) = (M + \Delta m)(v + \Delta v)$$

$$\Delta P = M\Delta v + \Delta m(v - u) \implies \frac{dP}{dt} = M\frac{dv}{dt} + \frac{dm}{dt}(v - u)$$

There is no external force on the system, so $dP/dt = 0$. Thus

$$\frac{dv}{dt} = -\frac{dm}{dt}\frac{(v - u)}{M} = \frac{b(u - v)}{M}$$

4.13 Sand sprayer

The system consists of the freight car and load with instantaneous mass M moving with speed v, plus the mass of sand Δm sprayed in during time Δt at rate γ: $\Delta m = \gamma\Delta t$. The speed of Δm is $v + u$, because the locomotive and freight car are moving with the same speed.

$$P(t) = Mv + \Delta m(v + u) \qquad P(t + \Delta t) = M(v + \Delta v) + \Delta m(v + \Delta v)$$

$$\Delta P \approx M\Delta v - (\Delta m)u \qquad \frac{dP}{dt} = M\frac{dv}{dt} - u\frac{dm}{dt} = M\frac{dv}{dt} - u\frac{dM}{dt}$$

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Note that Δm is being added to the freight car, so $dM/dt = +dm/dt > 0$. Because no external force is acting on the system $dP/dt = 0$. Thus

$$dv = u\frac{dM}{M} \implies \int_0^v dv' = u\int_{M_0}^{M_f} \frac{dM'}{M'}$$

where M_0 is the initial mass of the system, and M_f is its mass after 100 s.

$$v = u\, ln\left(\frac{M_f}{M_0}\right)$$

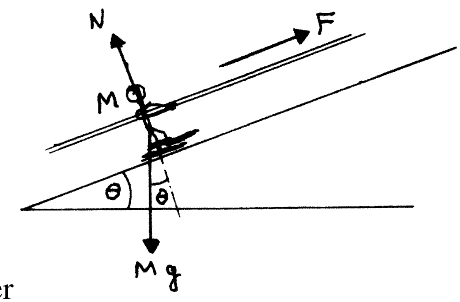
$$M_f = M_0 + \gamma t = 2000\,\text{kg} + (10\,\text{kg/s})(100\,\text{s}) = 3000\,\text{kg}$$

$$v(100\,\text{s}) = (5\,\text{m/s})\, ln\left(\frac{3000\,\text{kg}}{2000\,\text{kg}}\right) = 2.03\,\text{m/s}$$

Comparing these results with the derivation of rocket motion in Sec. 4.8, the system in this problem is seen to be a rocket in ”reverse”, where mass is added instead of being expelled. In Sec. 4.8 $dM/dt < 0$, but in this problem $dM/dt > 0$.

4.14 Ski tow

It is sound practice to solve problems symbolically, introducing numerical values only toward the end, to help maintain numerical accuracy.



Let L be the length of the tow, and let l be the separation between skiers, so that the number of skiers on the tow is L/l. Let t_s be the time interval between skiers grasping the tow. The number of skiers grasping the tow per second is then $\gamma = 1/t_s$.

If W_{tot} is the total weight of the skiers on the tow,

$$\frac{dP}{dt} = F - W_{tot}\sin\theta = Mv\gamma$$

The momentum of the system does not change when a skier leaves the rope, so

$$F - W_{tot}\sin\theta = Mv\gamma$$

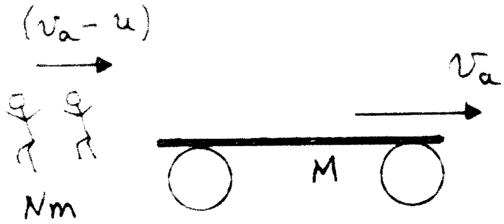
$$F = W_{tot}\sin\theta + Mv\gamma = Mg\left(\frac{L}{l}\right)\sin\theta + Mv\left(\frac{1}{t_s}\right)$$

$$F = (70\,\text{kg})(9.8\,\text{m/s}^2)\left(\frac{100\,\text{m}}{7.5\,\text{m}}\right)\sin(20°) + (70\,\text{kg})(1.5\,\text{m/s})\left(\frac{1}{5\,\text{s}}\right)$$

$$= 3128\,\text{N} + 21\,\text{N} = 3149\,\text{N}$$

4.15 Men and flatcar

In each situation, the flatcar does not accelerate further after the jumper leaves. Just as the jumper leaves, his speed is the final speed of the flatcar minus the speed relative to the flatcar.
The flatcar and its load are initially at rest.

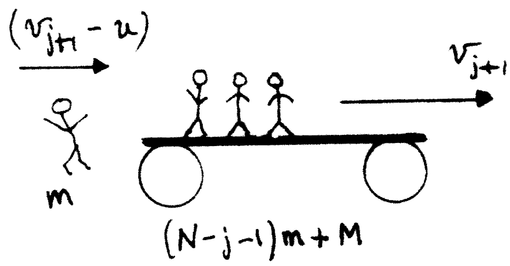


(a)

(a) *N jump at once.*

$$P_{iniitial} = 0$$
$$P_{final} = Mv_a + Nm(v_a - u) = P_{initial} = 0$$
$$v_a = \left(\frac{Nm}{Nm + M}\right) u$$

(b) *one jumps at a time.* Let the speed of the flatcar be v_j after j of N have jumped.



(b)

$$P_{initial} = [(N - j)m + M]v_j$$
$$P_{final} = [(N - j - 1)m + M]v_{j+1} + m(v_{j+1} - u)$$
$$\Delta P = [(N - j - 1)m + M]v_{j+1} + m(v_{j+1} - u) - [(N - j)m + M]v_j$$

There are no external forces, so $\Delta P = 0$.

$$0 = [(N - j)m + M]v_{j+1} - mu - [(N - j)m + M]v_j$$
$$v_{j+1} = \left(\frac{m}{(N - j)m + M}\right) u + v_j$$
$$v_b = \left(\frac{m}{Nm + M} + \frac{m}{(N - 1)m + M} + \ldots \frac{m}{m + M}\right) u$$

(c)

But

$$v_a = \left(\frac{m}{Nm + M} + \frac{m}{Nm + M} + \ldots \frac{m}{Nm + M}\right) u < v_b$$

In the trivial special case $N = 1$, case (a) and case (b) are identical.
Note that case (b) is closely analogous to the derivation of rocket motion in Sec. 4.8. In case (b), however, the expelled mass is in finite packets, one man at a time, while for the rocket the expelled mass is a continuous flow.

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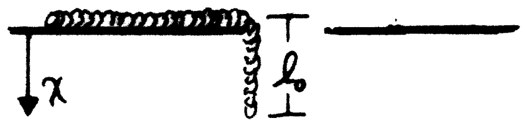
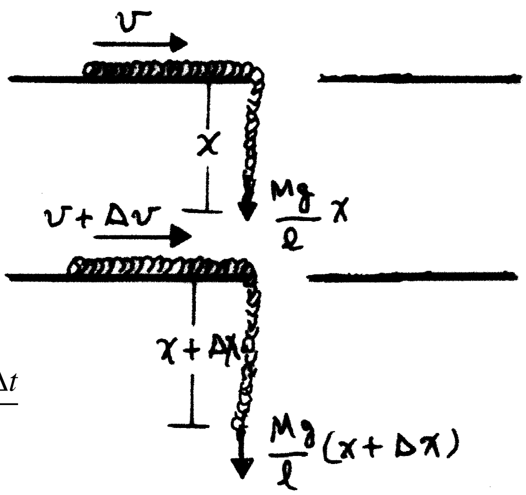
To help understand why the flatcar moves faster in case (b), assume that the mass of the flatcar is small. In this situation, when the men jump together the flatcar moves forward at speed slightly less than u, and the men are moving slowly with respect to the ground. This result is nearly independent of the number of men jumping. Consider now case (b), when the men jump one at a time. The last jumper by himself could cause the forward speed of the flatcar to be close to u, but if there are several jumpers, each previous jumper also contributes to increasing the speed of the flatcar. In case (b), therefore, the final speed of the flatcar could exceed u.

4.16 Rope on table

The rope has total length l and mass M.
At time $t = 0$, the rope is momentarily at rest,
with length $x(0) = l_0$ hanging through the hole.



(a)



The mass of the hanging portion is Mx/l where

$$P_{initial} = Mv$$
$$P_{initial} + \Delta P = M(v + \Delta v)$$
$$F_{external}(t) = \frac{Mg}{l}x$$
$$F_{external}(t + \Delta t) = \frac{Mg}{l}(x + \Delta x)$$
$$\Delta P = M\,\Delta v = \int_t^{t+\Delta t} F_{external}\,dt \approx \frac{Mgx\,\Delta t}{l}$$
$$\frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{gx}{l}$$

The general solution of the differential equation is

$$x(t) = Ae^{\frac{g}{l}t} + Be^{-\frac{g}{l}t}$$

(b)

$$x(0) = A + B = l_0$$
$$\dot{x}(0) = \tfrac{g}{l}A - \tfrac{g}{l}B = 0$$
$$A = B = l_0/2$$
$$x(t) = \frac{l_0}{2}\left(e^{\frac{g}{l}t} + e^{-\frac{g}{l}t}\right) = l_0 \cosh\left(\tfrac{g}{l}t\right)$$

4.17 Solar sail 1

Refer to Example 4.21: the radiation force F_{rad} due to the Sun on area A is

$$F_{rad} = \frac{2S_{Sun}A}{c}$$
$$a_{rad} = \frac{F_{rad}}{m} = \frac{2S_{Sun}A}{mc} \quad (1)$$

m is the mass of the craft, and a_{rad} is the acceleration due to radiation pressure. For the solar sail craft to move outward away from the Sun,

$$a_{rad} \geq g_{Sun}$$
$$\frac{2S_{Sun}A}{mc} \geq g_{Sun}$$
$$A \geq \frac{g_{Sun}mc}{2S_{Sun}}$$

The IKAROS mass was mostly the sail, $m \approx 1.6$ kg. From Example 4.21,

$$g_{Sun} = 5.9 \times 10^{-3}\ \mathrm{m/s^2} \qquad S_{Sun}/c = 4.6 \times 10^{-6}\ \mathrm{kg/ms^2}$$
$$A \geq \frac{(5.9 \times 10^{-3}\ \mathrm{m/s^2})(1.6\ \mathrm{kg})}{24.6 \times 10^{-6}\ \mathrm{kg/ms^2}} \geq 10^3\ \mathrm{m^2}$$

Could such a sail be constructed using the same polyimide film material used for IKAROS? The desired area is 1000/150 = 6.8 times the area of the IKAROS sail, and because the mass of the sail is density x area x thickness = ρAt, the thickness would have to be 6.8 times thinner, or 7.5×10^{-6} m /6.8 = 1.1×10^{-6} m. Constructing a strong sail so extremely large and thin is beyond the limits of current technology.

For further insight into the design issues, using $m \approx \rho At$ in Eq. (1) shows that a_{rad} is essentially independent of the sail area A and depends mainly on ρt, the ”areal density” (mass per unit area) of the sail material.

4.18 Solar sail 2

Assume that the mass m of the craft is essentially the mass of the sail $m = \rho t A$, where ρ is the density of the sail material, t the sail thickness, and A the area. With reference to Example 4.21,

(a)

$$\begin{aligned} a_{rad} &= \frac{(2S/c)A}{m} = \frac{(2S/c)A}{\rho t A} = \frac{2S/c}{\rho t} \\ &= \frac{(2)4.6 \times 10^{-6}\,\text{kg/ms}^2}{(1.4 \times 10^3\,\text{kg/m}^3)(2.5 \times 10^{-5}\,\text{m})} = 2.6 \times 10^{-4}\,\text{m/s}^2 \end{aligned}$$

(b)

$$a_{craft} = a_{rad} - a_{earth} = a_{rad} - g\left(\frac{R_e}{r}\right)^2$$

The craft cannot accelerate outward from the Earth unless it is launched beyond a radius r_{min} such that

$$a_{rad} - g\left(\frac{R_e}{r_{min}}\right)^2 \geq 0 \implies \left(\frac{R_e}{r_{min}}\right)^2 \leq \frac{a_{rad}}{g} = \frac{2.6 \times 10^{-4}}{9.8} = 2.7 \times 10^{-5}$$

$$\frac{R_e}{r_{min}} \leq 5.2 \times 10^{-3} \implies r_{min} \geq 194 R_e$$

(c) If r is so large that the Earth's gravitational attraction can be neglected

$$v = a_{rad} T$$

$$T = \frac{11.2 \times 10^3\,\text{m/s}}{2.6 \times 10^{-4}\,\text{m/s}^2} = 4.3 \times 10^7\,\text{s} \approx 1.4\,\text{years}$$

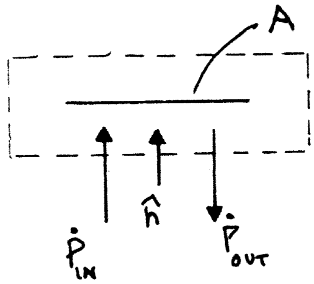
(d) Neglect gravitational forces. The radiation force on the sail is $F_{rad} = 2(S/c)A$. Let the mass of the sail be $m = \rho t A$ and the mass of the payload be $M = 1.0$ kg. For half the original acceleration,

$$\begin{aligned} F_{rad} &= (m + M)\frac{a_{rad}}{2} \\ (m + M)a_{rad} &= (\rho t A + 1.0)a_{rad} = 4(S/c)A \\ A &= \frac{a_{rad}}{4(S/c) - \rho t a_{rad}} \\ &= \frac{2.6 \times 10^{-4}}{(4)4.6 \times 10^{-6} - (1.4 \times 10^3)(2.5 \times 10^{-5})(2.6 \times 10^{-6})} = 28\,\text{m}^2 \end{aligned}$$

A simpler method is to note that the acceleration is halved if the mass M of the payload doubles the mass of the craft. Hence the mass of the sail should be $\rho t A = 1.0$ kg, so $A = 1.0/\rho t = 28\,\text{m}^2$ as before.

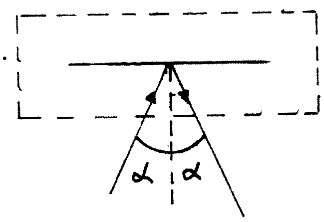
4.19 Tilted mirror

(a) The momentum flow $\dot{\mathbf{P}}$ through a surface of area $\mathbf{A}$ is $\dot{\mathbf{P}} = \mathbf{J} \cdot \mathbf{A}$, where $\mathbf{J}$ is the momentum flux density. Enclose the mirror with a hypothetical surface, as shown in the upper sketch. Taking flow in as positive,



$$\dot{\mathbf{P}}_{net} = \dot{\mathbf{P}}_{in} - \dot{\mathbf{P}}_{out} = 2JA\hat{\mathbf{n}}$$
$$\mathbf{F} = 2JA\hat{\mathbf{n}}$$
$$F = 2JA = 9.2 \times 10^{-6}\,\mathrm{kg \cdot m/s^2} = 9.2 \times 10^{-6}\,\mathrm{N}$$

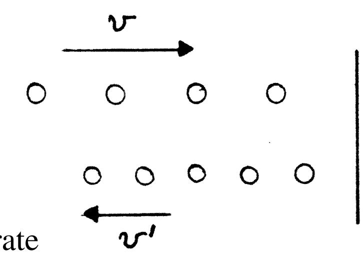
(b) If the mirror is tilted, the reflected beam leaves at the same angle α made by the incident beam, see lower sketch.



$$\mathbf{J}_{in} \cdot \mathbf{A} = JA\,\cos\alpha$$
$$\mathbf{J}_{out} \cdot \mathbf{A} = -JA\,\cos\alpha$$
$$\mathbf{F} = \dot{\mathbf{P}}_{in} - \dot{\mathbf{P}}_{out} = 2JA\cos\alpha\,\hat{\mathbf{n}} = 9.2 \times 10^{-6}\,\cos\alpha\,\hat{\mathbf{n}}$$

4.20 Reflected particle stream

The rate at which incoming particles strike the surface is (number of particles per unit length) x (speed) = λv. If each incoming particle carries momentum mv, the rate $\dot{\mathbf{P}}$ at which momentum arrives at the surface is $\dot{\mathbf{P}}_{in} = (mv)(\lambda v) = \lambda mv^2$.



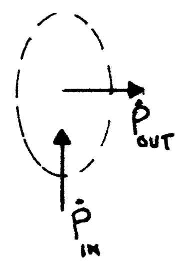
In steady conditions, the rate at which particles leave must equal the rate at which they arrive. If they leave with speed v', with λ' particles per unit length, $\lambda' v' = \lambda v$. The reflected particles carry away momentum in the opposite direction at rate $mv'\lambda' v' = mv'\lambda v$. Hence the total force, which is the difference between the incoming and outgoing rates of momentum, is $\lambda m(v^2 + vv')$.

4.21 Force on a firetruck

A volume of water with mass ΔM moving at velocity $\mathbf{v}$ carries momentum $\Delta\mathbf{P} = \mathbf{v}\Delta\mathbf{M}$. The rate of momentum flow in the stream is then $d\mathbf{P}/dt = \mathbf{v}d\mathbf{M}/dt = K\mathbf{v}$. The vertical component of $\mathbf{v}$ is $v\sin\theta$. From motion under constant gravity, the water ascends to a height $v\sin\theta = \sqrt{2gh}$, so that $v = \sqrt{2gh}/\sin\theta$.

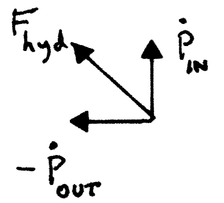
The recoil force is $\mathbf{F} = -\dot{\mathbf{P}} = -K\mathbf{v}$. The magnitude of the force is $|K\mathbf{v}| = K\sqrt{2gh}/\sin\theta$, and its direction is opposite to the flow.

4.22 Fire hydrant



Imagine a hypothetical surface surrounding the hydrant, as shown. The rate of change of momentum within the surface is $\dot{\mathbf{P}} = \dot{\mathbf{P}}_{out} - \dot{\mathbf{P}}_{in}$. The force $\mathbf{F}_{water}$ on the water due to the hydrant is therefore $\mathbf{F}_{water} = \dot{\mathbf{P}}_{out} - \dot{\mathbf{P}}_{in}$.

The force $\mathbf{F}_{hydrant}$ on the hydrant due to the water is equal and opposite: $\mathbf{F}_{hydrant} = -\mathbf{F}_{water} = \dot{\mathbf{P}}_{in} - \dot{\mathbf{P}}_{out}$.



Let ρ be the density of water. Then $|\dot{\mathbf{P}}_{in}| - |\dot{\mathbf{P}}_{out}| = \rho V_0{}^2 A$ where $A = \pi D^2/4$. Hence $F_{hydrant} = \sqrt{2}\rho V_0{}^2 A$, directed upward at 45°, as shown.

4.23 Suspended garbage can

The stream has initial speed v_0, so at height y its speed is $v = \sqrt{v_0{}^2 - 2gy}$. The stream carries mass at a rate $dm/dt \equiv K$. Under steady conditions the rate of mass flow is constant, with an equal amount of mass passing through any horizontal plane per unit time, because water is essentially incompressible. In time interval Δt mass $\Delta m = K\Delta t$ passes through a horizontal surface, transporting momentum $\Delta P = v\Delta m = vK\Delta t$. The upward momentum flux is then $\dot{P} = vK$. (The momentum flux decreases with height, because the downward gravitational force is acting.)

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The maximum height will be reached if the water rebounds elastically from the garbage can. In this case the rate of momentum transfer to the can gives a force $2\dot{P} = 2vK$, double compared to the inelastic case where the water comes to rest without rebounding after colliding with the can. In equilibrium, $2vK = W$ so that

$$v = \frac{W}{2K}$$
$$\sqrt{{v_0}^2 - 2gh} = \frac{W}{2K}$$
$$h = \frac{1}{2g}\left[{v_0}^2 - \left(\frac{W}{2K}\right)^2\right]$$

Note that v_0 must be greater than a minimum value $v_0 \geq W/2K$ to get any lift.

4.24 Growing raindrop

Consider the change in momentum of the drop as it gains mass during the time interval from t to $t + \Delta t$.

$$P(t) = MV \qquad P(t + \Delta t) = (M + \Delta M)(V + \Delta V)$$
$$\Delta P \approx M\Delta V + V\Delta M$$
$$\frac{dP}{dt} = M\frac{dV}{dt} + V\frac{dM}{dt}$$

There is the external gravitational force Mg.

$$M\frac{dV}{dt} + V\frac{dM}{dt} = Mg$$
$$\frac{dM}{dt} = kMV \implies M\frac{dV}{dt} + kMV^2 = Mg \implies \frac{dV}{dt} = g - kV^2$$

The acceleration decreases as the falling drop gains speed, and vanishes at the terminal velocity $V_{terminal} = \sqrt{g/k}$.

4.25 Bowl of water

Let $dM/dt = \sigma A$. The momentum flux is then $v\sigma$, so the force F is $v\sigma A$. In SI units,

$$\sigma = 10^{-3}\ \frac{\text{g}}{\text{cm}^2 \cdot \text{s}} \times \frac{1\ \text{kg}}{10^3\ \text{g}} \times \frac{10^4\ \text{cm}^2}{1\ \text{m}^2} = 10^{-2}\ \frac{\text{kg}}{\text{m}^2 \cdot \text{s}}$$
$$A = 500\ \text{cm}^2 = 5 \times 10^{-2}\ \text{m}^2 \qquad v = 5\ \text{m/s}$$
$$F = v\sigma A = (5\ \text{m/s}) \times \left(10^{-2}\ \frac{\text{kg}}{\text{m}^2 \cdot \text{s}}\right) \times (5 \times 10^{-2}\ \text{m}^2) = 2.5 \times 10^{-3}\ \text{N}$$

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Another approach to this problem is to model the rain as individual droplets arriving with speed v. Let $\mathcal{N}$ be the number of droplets per m^3, and let m_d be the mass of each droplet. Then $vm_d\mathcal{N} \equiv \sigma$. There are $v\mathcal{N}A$ droplets striking the bowl per second. Each droplet brings in momentum m_dv, and runs off with zero momentum, so the force F on the bowl is the change in momentum: F = (droplets arriving per second)(momentum per droplet) $= (v\mathcal{N}A)(m_dv) = v(vm_d\mathcal{N})A = v\sigma A$, as found before.

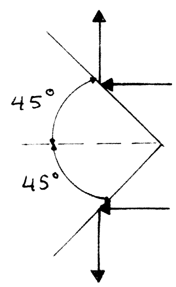
When the bowl is moving upward at speed v', the number of droplets striking the bowl per second is $(v+v')\mathcal{N}A$. Each droplet now strikes the bowl with speed $(v+v')$, so the momentum of each droplet is $(v+v')m_d$. The total momentum delivered per second is

$$\frac{dP}{dt} = (v+v')^2 m_d\mathcal{N}A = \frac{(v+v')^2}{v}\sigma A = \frac{(v+v')^2}{v^2}(v\sigma A)$$

$$F_{moving} = \frac{(5+2)^2}{5^2}F_{static} = \frac{49}{25}F_{static} = 4.9\times 10^{-3}\text{ N}$$

4.26 Rocket in interstellar cloud

Because the collisions are elastic, the particles bounce off the rocket's nose cone transversely to the motion, Hence the reflected particles transfer no net momentum to the rocket.



(a) The rate at which particles strike the rocket is $\mathcal{A}Nv$ where $\mathcal{A}$ is the projected area $\mathcal{A} = \pi R^2$. The incoming momentum of each particle is mv, so the force F on the rocket equals the momentum flux dP/dt

$$F = \frac{dP}{dt} = -\mathcal{A}Nmv^2 \equiv -Av^2$$

(b)

$$M\frac{dv}{dt} = -Av^2 \implies \frac{dv}{v^2} = -\frac{A}{M}dt \implies \int_{v_0}^{v}\frac{dv'}{v'^2} = -\frac{A}{M}\int_0^t dt'$$

$$\frac{1}{v} - \frac{1}{v_0} = \frac{A}{M}t \implies v = \frac{v_0}{\left(1+\frac{A}{M}v_0t\right)}$$

4.27 Exoplanet detection

Consider the Sun - Jupiter system. They rotate about their center of mass given by $M_{Sun}R_{Sun} = M_{Jupiter}R_{Jupiter}$. The distance R_{Sun} of the Sun from the C.M. is

$$R_{Sun} = \frac{M_{Jupiter}}{M_{Sun}} R_{Jupiter} = \frac{1.9 \times 10^{27}\,\mathrm{kg}}{1.99 \times 10^{30}\,\mathrm{kg}}\,(7.8 \times 10^{11}\ \mathrm{m}) = 7.5 \times 10^{8}\ \mathrm{m}$$

The speed v_{Sun} of the Sun as it orbits about the center of mass is then

$$v_{Sun} = \Omega R_{Sun} = \left(\frac{2\pi\,\mathrm{rad}}{4330\,\mathrm{days}} \times \frac{1\ \mathrm{day}}{8.64 \times 10^{4}\ \mathrm{s}}\right) R_{Sun} = (1.68 \times 10^{-8}\ \mathrm{rad/s})\,(7.5 \times 10^{8}\ \mathrm{m}) = 12.6\,\mathrm{m/s}$$

According to current technology, as described in Example 4.6, the effect of Jupiter would be readily detectable.

5 ENERGY

5.1 Loop-the-loop

initial energy: $K_i = 0 \qquad U_i = mgz$

$$E_i = K_i + U_i = 0 + mgz$$

final energy: $K_f = \frac{1}{2}mv^2 \qquad U_f = mg(2R)$

$$E_f = K_f + U_f = \frac{1}{2}mv^2 + mg(2R) = E_i = mgz$$

$$v^2 = 2gz - 4gR \quad (1)$$

At the top of the loop the total downward force is $N + mg$, where N is the normal force exerted by the loop. Using Eq. (1)

$$N + mg = \frac{mv^2}{R} = \frac{2mgz}{R} - 4mg$$

If $N = mg$, then

$$2mg = 2mg\frac{z}{R} - 4mg$$

$$z = 3R$$

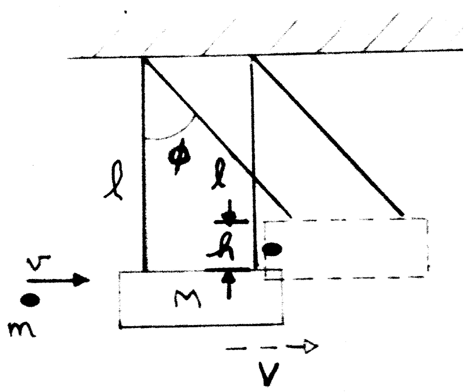
5.2 Block, spring, and friction

$$\begin{aligned}
&\textit{initial energy:} \quad K_i = \tfrac{1}{2}Mv_0^2 \qquad U_i = 0\\
&\qquad\qquad\qquad\quad E_i = K_i + U_i \ = \tfrac{1}{2}Mv_0^2 + 0\\
&\textit{final energy:} \quad K_f = 0 \qquad U_f = \tfrac{1}{2}kl^2\\
&\qquad\qquad\qquad\quad E_f = K_f + U_f \ = 0 + \tfrac{1}{2}kl^2
\end{aligned}$$

The friction force $F_{friction} = \mu N$, where $N = Mg$ is the normal force.

$$\begin{aligned}
E_f - E_i &= \textit{work on the system}\\
&= \int_0^l F_{friction}\, dx' = -\int_0^l \mu Mg\, dx' = -Mg\int_0^l \mu\, dx' = -Mg\int_0^l bx'\, dx'\\
&= -\tfrac{1}{2}Mgbl^2\\
\tfrac{1}{2}kl^2 - \tfrac{1}{2}Mv_0^2 &= -\tfrac{1}{2}Mgbl^2 \implies l^2(k + Mgb) = Mv_0^2\\
l &= v_0\sqrt{\frac{M}{k + Mgb}}
\end{aligned}$$

5.3 Ballistic pendulum



During this collision, linear momentum is conserved but mechanical energy is not conserved.

(a)

Momentum just before and just after collision:

$P_i = mv \qquad P_f = (m + M)V$

Before external forces can act significantly

$$P_i = P_f \implies mv = (m + M)V \implies V = \frac{m}{(m + M)}v$$

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(b)

After the collision, energy is conserved as the block rises.

$$K_i = \frac{1}{2}(m+M)V^2 \quad U_i = 0 \qquad K_f = 0 \quad U_f = (m+M)gh$$

$$E_f = (m+M)gh = E_i = \frac{1}{2}(m+M)V^2 = \frac{1}{2}\frac{m^2}{m+M}v^2$$

$$v^2 = 2\left(\frac{m+M}{m}\right)^2 gh$$

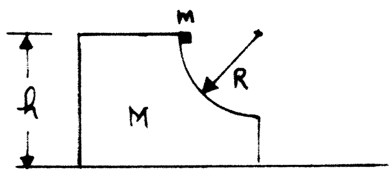
$$h = l(1-\cos\phi)$$

$$v = \left(\frac{m+M}{m}\right)\sqrt{2gl(1-\cos\phi)}$$

Knowing m, M, and l, and measuring ϕ gives the speed v of the bullet.

5.4 Sliding on a circular path

There are no dissipative forces (friction), so in this system both momentum and mechanical energy are conserved.



$$\textit{initial energy:} \quad K_i = 0 \quad U_i = mgh \qquad E_i = 0 + mgh$$

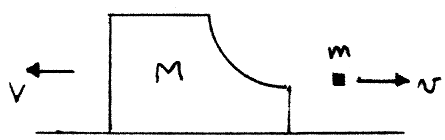
$$\textit{final energy:} \quad K_f = \tfrac{1}{2}mv^2 + \tfrac{1}{2}MV^2 \quad U_f = mg(h-R)$$

$$E_f = \tfrac{1}{2}mv^2 + \tfrac{1}{2}MV^2 + mg(h-R) = E_i = mgh$$

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}MV^2 - mgR = 0$$

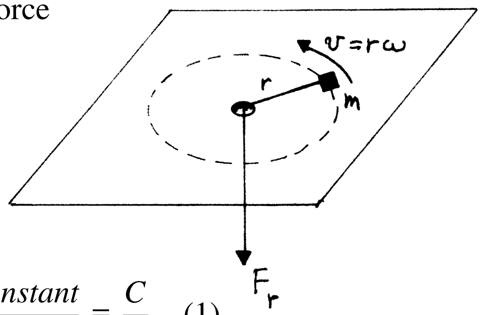
$$P_f = mv - MV = P_i = 0$$

$$v = \sqrt{\left(\frac{M}{M+m}\right)2gR}$$



5.5 Work on a whirling mass

Mechanical energy is conserved (no friction). The applied force F_r is radial, so the tangential equation of motion is $a_\theta = 0$.



$$
\begin{aligned}
a_\theta &= 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \\
\frac{\dot{\omega}}{\omega} &= -\frac{2\dot{r}}{r} \implies \frac{d\omega}{\omega} = \frac{2dr}{r} \\
\int_{\omega_i}^{\omega} \frac{d\omega}{\omega} &= -2\int_{l_1}^{r} \frac{dr}{r} \implies \dot{\theta} = \omega = \frac{{l_1}^2\omega_1}{r^2} = \frac{constant}{r^2} \equiv \frac{C}{r^2} \qquad (1) \\
K^{tangential} &= \frac{1}{2}m(r\omega)^2 = \frac{1}{2}m\frac{C^2}{r^2} \\
K_f^{tangential} - K_i^{tangential} &= \frac{1}{2}\frac{mC^2}{{l_2}^2} - \frac{1}{2}\frac{mC^2}{{l_1}^2} \\
K_f^{radial} - K_i^{radial} &= \frac{1}{2}mv^2_{radial,f} - \frac{1}{2}mv^2_{radial,i} \\
E_f - E_i &= \frac{1}{2}mv^2_{radial,f} + \frac{1}{2}\frac{mC^2}{{l_2}^2} - \frac{1}{2}mv^2_{radial,i} - \frac{1}{2}\frac{mC^2}{{l_1}^2}
\end{aligned}
$$

Now find the work W done by the radial force F_r.

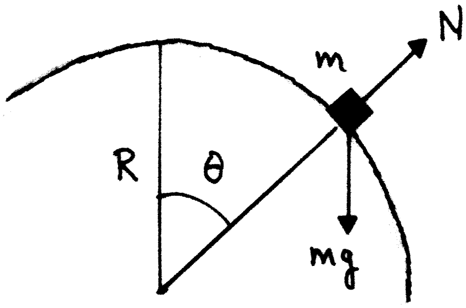
$$
\begin{aligned}
W &= \int_{l_1}^{l_2} F_r\, dr \int_{l_1}^{l_2} ma_r\, dr = m\int_{l_1}^{l_2} (\ddot{r} - r\dot{\theta}^2)\, dr \\
m\int_{l_1}^{l_2} \ddot{r}\, dr &= m\int_{l_1}^{l_2} \frac{d\dot{r}}{dt}\, dr = m\int_{t_i}^{t_f} \frac{d\dot{r}}{dt}\frac{dr}{dt}\, dt = \int_{t_i}^{t_f} \frac{d}{dt}\left(\tfrac{1}{2}m\dot{r}^2\right) dt \\
\int_{t_i}^{t_f} \frac{d}{dt}\left(\tfrac{1}{2}m\dot{r}^2\right) dt &= \frac{1}{2}mv^2_{radial,f} - \frac{1}{2}mv^2_{radial,i}
\end{aligned}
$$

equals the change in mechanical energy due to radial motion. Using Eq. (1),

$$
-m\int_{l_1}^{l_2} r\dot{\theta}^2\, dr = -m\int_{l_1}^{l_2} \frac{C^2}{r^3}\, dr = \frac{1}{2}mC^2\left(\frac{1}{r^2}\right)\Bigg|_{l_1}^{l_2} = \frac{1}{2}\frac{mC^2}{{l_2}^2} - \frac{1}{2}\frac{mC^2}{{l_1}^2}
$$

equals the change in mechanical energy due to tangential motion. Hence $E_f - E_i = W$ as expected.

5.6 Block sliding on a sphere



radial equation: $\frac{mv^2}{R} = mg\cos\theta - N$

Block separates when $N = 0$.

$$\frac{v^2}{R} = g\cos\theta$$

$$K_i = 0 \quad U_i = mgR$$

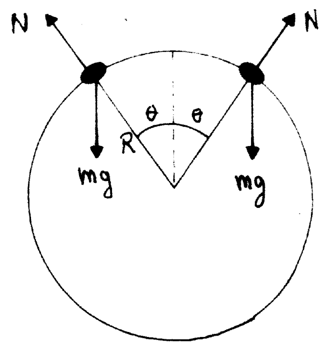
$$K_f = \tfrac{1}{2}mv^2 \quad U_f = mgR\cos\theta$$

$$E_f = \tfrac{1}{2}mv^2 + mgR\cos\theta = E_i = mgR$$

$$\tfrac{1}{2}gR\cos\theta + gR\cos\theta = gR \implies \cos\theta = \frac{2}{3}$$

At separation, the block is a distance $y = R(1 - \cos\theta) = R/3$ below the top.

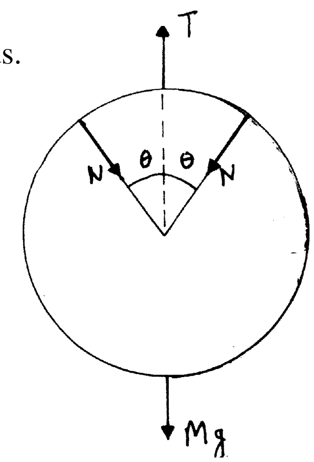
5.7 Beads on hanging ring



Mechanical energy is conserved (no friction).

The upper sketch shows the forces on each bead: the downward weight force mg and the outward radial normal force N exerted by the ring.

The lower sketch shows the forces on the ring: the downward weight force Mg, the upward force T exerted by the thread, and the inward radial forces N exerted by the beads.



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The ring remains stationary if $T \geq 0$ (strings can only pull, not push).

ring equation of motion:

$$T - 2N\cos\theta - Mg = 0$$

bead equation of motion:

$$mg\cos\theta - N = m\frac{v^2}{R}$$

$$K_i = 0 \quad U_i = mgR$$

$$K_f = \tfrac{1}{2}mv^2 \quad U_f = mgR\cos\theta$$

$$E_f = \tfrac{1}{2}mv^2 + mgR\cos\theta = E_i = mgR \implies \frac{mv^2}{R} = 2mg(1-\cos\theta)$$

$$N = mg\cos\theta - \frac{mv^2}{R} = mg(3\cos\theta - 2)$$

for T just 0: $2N\cos\theta_{max} = -Mg$

$$2m(3\cos\theta_{max} - 2)\cos\theta_{max} = -M \quad (1)$$

$$\cos\theta_{max} = \frac{1}{3} \approx 70^\circ$$

From Eq. (1), the ring starts to rise when $2m(3\cos\theta_{max} - 2)\cos\theta_{max} = -M$

$$2m\left[3\left(\frac{1}{3}\right) - 2\right]\left(\frac{1}{3}\right) = -M \implies m \geq \tfrac{3}{2}M$$

To find the angle for any values of m and M, use Eq. (1).

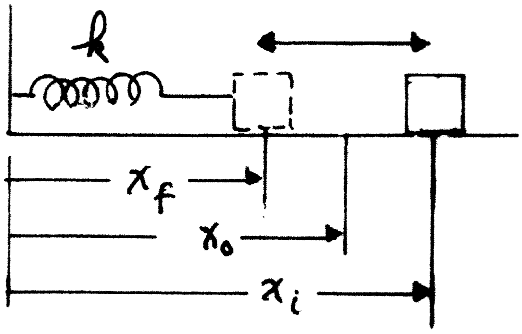
$$3\cos^2\theta - 2\cos\theta + \frac{M}{2m} = 0$$

$$\cos\theta = \frac{1}{3} + \sqrt{\frac{1}{9} - \frac{M}{6m}} \quad (2)$$

The plus sign is chosen because $\cos\theta_{max} = 1/3$, and θ must be $< \theta_{max}$.

Comment: According to Eq. (2), $\cos\theta < 2/3$, or $\theta >\approx 48^\circ$, regardless of the value of M/m. If $m < \frac{3}{2}M$, the argument in Eq. (2) is < 0; the ring will never rise.

5.8 Damped oscillation



Consider one complete cycle. Let x_i be the maximum displacement of the block at the start. It starts from rest, so its kinetic energy is 0. Its potential energy is $\frac{1}{2}k(x_i - x_0)^2$ due to the spring, where x_0 is the unstretched length of the spring. x_0 is halfway between x_i. as shown. If there is no friction, the block returns to x_i after one complete cycle, and the mechanical energy is conserved.

With friction present, the block returns only to $x_i - \Delta x$, so then the potential energy is $\frac{1}{2}k(x_i - \Delta x - x_0)^2$. Mechanical energy is less by an amount ΔE:

(a)

$$\Delta E = \tfrac{1}{2}k[(x_i - \Delta x - x_0)^2 - (x_i - x_0)^2] \approx -2(\tfrac{1}{2})k\Delta x(x_i - x_0)$$

By symmetry of SHM, the block travels $(x_i - x_0)$ in 1/4 cycle, so distance traveled per cycle is $4(x_i - x_0)$. The work W_f of friction is

$$W_f = f \times \textit{distance traveled in one cycle} = 4f(x_i - x_0)$$

By the Work-Energy Theorem, $\Delta E = -W_f$

$$k\Delta x(x_i - x_0) = 4f(x_i - x_0) \implies \Delta x = \frac{4f}{k}$$

so the change in amplitude per cycle is constant, to first order.

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(b)

The block comes to rest after n cycles. The block loses amplitude Δx/cycle.

$$n\Delta x = (x_i - x_0) \implies n = \frac{k}{4f}(x_i - x_0)$$

The result is reasonable; the block makes many cycles if the friction force is weak, or if the spring is stretched quite far at the start.

5.9 Oscillating block

(a) (1) The original period is $T_0 = 2\pi\sqrt{M/k}$. The new mass is $(m + M)$, so the new period is $2\pi\sqrt{(m + M)/k} = T_0\sqrt{(m + M)/M}$.

(2) Because the lump m sticks at the extreme of the motion, the amplitude is unchanged. Note that the lump transfers no horizontal momentum to M.

(3) The mechanical energy is $E = \frac{1}{2}kA_0^2$, where A_0 is the amplitude. Because the amplitude is unchanged, the mechanical energy is also unchanged.

(b) (1) The mass is $(m + M)$, so the new period is $T_0\sqrt{(m + M)/M}$ as in a(1).
(2) In this case, linear momentum is conserved when the putty sticks, but the mechanical energy is not conserved. If V is the speed just before the collision, the speed V' just after the collision is given by $(m + M)V' = MV$. Hence the new mechanical energy E' is

$$E' = \frac{1}{2}(m + M)V'^2 = \frac{1}{2}MV^2\left(\frac{M}{m + M}\right) = \frac{1}{2}kA^2$$

where A is the new amplitude. Hence

$$\frac{1}{2}kA^2 = \frac{1}{2}MV^2\left(\frac{M}{m + M}\right) = \frac{1}{2}kA_0^2\left(\frac{M}{m + M}\right) \implies A = A_0\sqrt{\frac{M}{m + M}}$$

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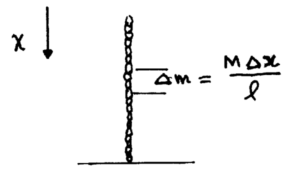
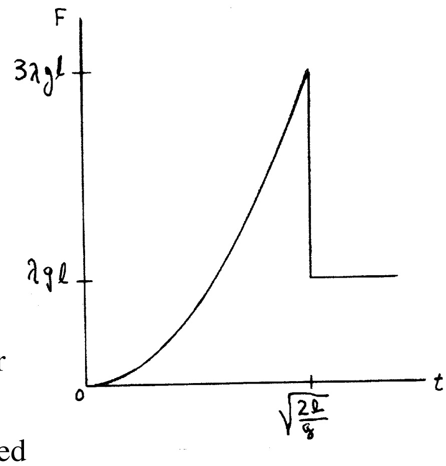
(3) From part b(2),

$$E' = \frac{1}{2}MV^2\left(\frac{M}{m+M}\right) = E\left(\frac{M}{m+M}\right)$$

5.10 Falling chain



The links of the chain fall like free bodies. The mass per unit length of the chain is $\lambda = M/l$. In length Δx, the mass is $\Delta m = \lambda\Delta x$, and if it hits the scale with speed v it carries momentum $\Delta p = v\Delta m = v\lambda\Delta x$. The rate of momentum flow to the scale is $F = dp/dt = \lambda v^2$.



When the top of the chain has fallen a distance x, a length x of the chain is on the pan of the scale, contributing weight λxg. The total force F while the chain is falling is therefore $F = \lambda v^2 + \lambda xg$. While the chain is in free fall, $v^2 = 2gx$. Hence while the chain is falling, $F = 3\lambda gx$. The chain has completely fallen in time t where $l = \frac{1}{2}gt^2$ so that after time $t = \sqrt{2l/g}$, all the chain is on the scale. The scale then reads the chain's full weight λgl, as indicated in the sketch. The idealized sketch assumes that the scale has a very fast response.

5.11 Dropped soldiers

The bale hits the ground with kinetic energy $\frac{1}{2}Mv^2 = Mgh$, where M is the mass of the soldier and h is the altitude of the drop. If the soldier comes to rest in distance s, the average force $\bar{F} = Mgh/s$ by the Work-Energy theorem.

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$\bar{F}$ is the average upward force: $\bar{F} = \bar{F}_{snow} - Mg$.

$$\bar{F}_{snow} = Mg\left(\frac{h}{s} + 1\right) = 180\,\text{lb}\left(\frac{100\,\text{ft}}{2\,\text{ft}} + 1\right) = 9180\,\text{lb}$$

The impact area A is

$$A = 5\,\text{ft}^2 = 5\,\text{ft}^2 \times \frac{144\,\text{in}^2}{1\,\text{ft}^2} = 720\,\text{in}^2$$

The force per square inch P (a pressure) is thus

$$P = \frac{9180}{720} = 12.8\,\text{lb/in}^2$$

The drop should be safe. This problem could also be solved by calculating the impulse and the acceleration, but the energy method used here is more direct. However, the result obtained is not entirely convincing. The assumption that the retarding force is constant is not realistic. For instance, if the compression acts more like a spring force, the peak force would be twice the average force.

5.12 Lennard-Jones potential

$$U = \epsilon\left[\left(\frac{r_0}{r}\right)^{12} - 2\left(\frac{r_0}{r}\right)^{6}\right] \quad (1)$$

Differentiating,

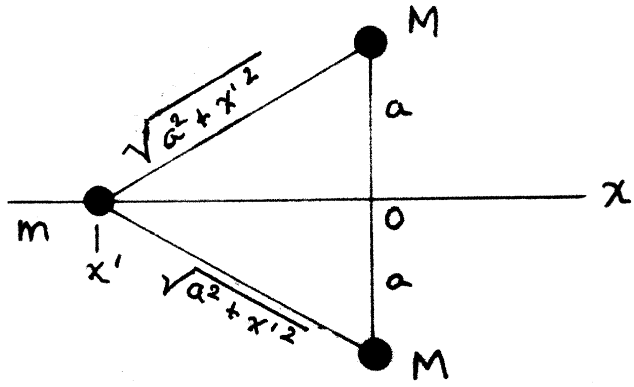
$$\frac{dU}{dr} = \epsilon\left[(-12)\left(\frac{{r_0}^{12}}{r^{13}}\right) + (12)\left(\frac{{r_0}^{6}}{r^{7}}\right)\right] = \frac{-12\epsilon}{r}\left[\left(\frac{r_0}{r}\right)^{12} - \left(\frac{r_0}{r}\right)^{6}\right]$$

$dU/dr = 0$ for $r = r_0$.

Substituting $r = r_0$ in Eq. (1),

$U(r_0) = -\epsilon$

5.13 Bead and gravitating masses



m is initially located at x'. Mechanical energy is conserved. The total gravitational potential energy is the sum of the potential energies of m with each mass M. Using the convention that $U \to 0$ as $r \to \infty$,

(a)

$$U = -\frac{GmM}{r} - \frac{GmM}{r} = -2\frac{GmM}{\sqrt{a^2 + x'^2}}$$

(b)

$$E_i = K_i + U(x' = 3a) = \frac{1}{2}mv_i^2 - 2\frac{GmM}{\sqrt{a^2 + 9a^2}} = \frac{1}{2}mv_i^2 - \frac{2}{\sqrt{10}}\frac{GmM}{a}$$

Let $v(0)$ be the speed of m as it passes the origin.

$$E_f = \frac{1}{2}mv^2(0) - U(x = 0) = \frac{1}{2}mv^2(0) - 2\frac{GmM}{a} = E_i = \frac{1}{2}mv_i^2 - \frac{2}{\sqrt{10}}\frac{GmM}{a}$$

$$\frac{1}{2}mv^2(0) = \frac{1}{2}mv_i^2 + 2\frac{GmM}{a}\left(1 - \frac{1}{\sqrt{10}}\right)$$

$$v(0) \approx \sqrt{v_i^2 + 2.74\frac{GM}{a}}$$

5.14 Particle and two forces

(a)

1) attractive force: $F_a = -B$

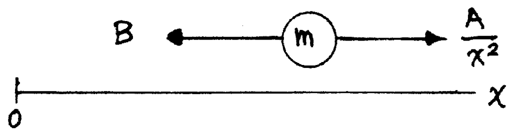
$$U_a(x) - U_a(0) = -\int_0^x F_a\, dx' = Bx'|_0^x \implies U_a(x) = Bx + U_a(0)$$

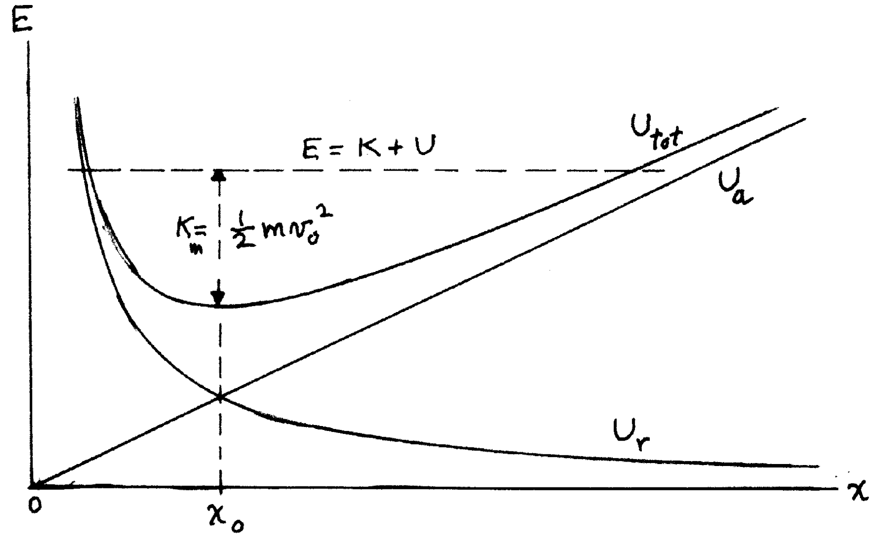
2) repulsive force: $F_r = \dfrac{A}{x^2}$

$$U_r(x) - U_r(\infty) = -\int_\infty^x A\frac{dx'}{x'^2} = \frac{A}{x} \implies U_r(x) = \frac{A}{x}$$

$$U_{total} = Bx + \frac{A}{x}$$

(b)





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(c)

$$\frac{dU_{total}}{dx} = B - \frac{A}{x^2}$$

$$0 = \left.\frac{dU_{total}}{dx}\right|_{x_0} = B - \frac{A}{{x_0}^2} \implies x_0 = \sqrt{\frac{A}{B}}$$

It is easy to prove that for these particular forces, the minimum of U_{total} occurs where $U_a(x_0) = U_r(x_0)$, as shown in the sketch, for any values of A and B.

5.15 Sportscar power

The average power $\bar{\mathcal{P}} = \frac{\Delta E}{\Delta T} = \frac{\frac{1}{2}Mv^2}{\Delta T}$

The units are mixed. Best practice is to work mainly in SI.

$$M = 1800\,\text{lb-mass} \times \frac{1\,\text{kg}}{2.2\,\text{lb-mass}} = 818\,\text{kg}$$

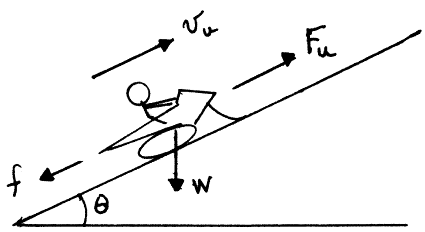
$$v = \frac{60\,\text{miles}}{\text{hour}} \times \frac{1.61 \times 10^3\,\text{m}}{1\,\text{mile}} \times \frac{1\,\text{hour}}{3600\,\text{s}} = 26.8\,\text{m/s}$$

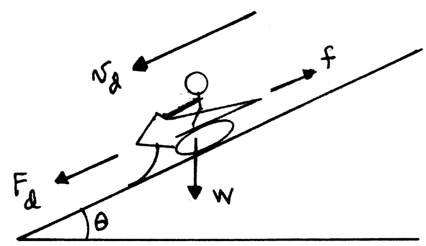
$$\Delta E = \frac{1}{2}Mv^2 = \frac{1}{2} \times (818\,\text{kg}) \times (26.8\,\text{m/s})^2 = 2.94 \times 10^5\,\text{kg} \cdot \text{m}^2/\text{s}^2 = 2.94 \times 10^5\,\text{J}$$

$$\bar{\mathcal{P}} = \frac{2.94 \times 10^5\,\text{J}}{3\,\text{s}} = 9.80 \times 10^4\,\text{J/s} = .80 \times 10^4\,\text{W} = 9.80 \times 10^4\,\text{W} \times \frac{1\,\text{hp}}{746\,\text{W}} = 131\,\text{hp}$$

5.16 Snowmobile and hill

The snowmobile moves at constant speed, which requires the total force to be zero. The forces parallel to the surface are the component of gravity along the slope, the retarding force f, and the propelling force F.





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F is the reaction force from the snow to the propelling force on the snow from the treads. The normal force on the snowmobile has no component along the slope, and is not shown in the sketches. The sketches are not to scale; according to the stated conditions, the slope angle θ is $\approx 1.4°$.

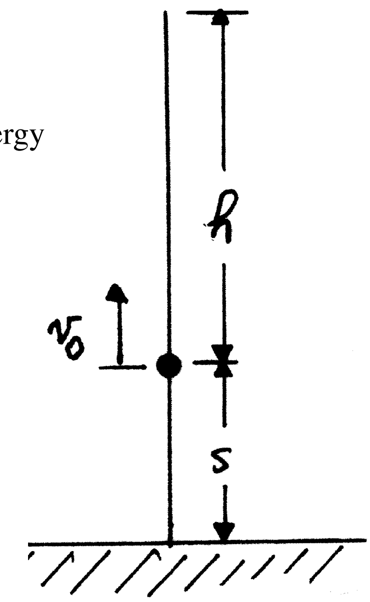
The subscript u stands for *up*, and d for *down*. The power $\mathcal{P}$ delivered by the snowmobile is Fv.

$$F_u = f + W\sin\theta \qquad F_d = f - W\sin\theta$$
$$\mathcal{P}_u = (f + W\sin\theta)\,v_u \qquad \mathcal{P}_d = (f - W\sin\theta)\,v_d$$

The engine's power is constant.

$$\mathcal{P}_u = \mathcal{P}_d \implies v_d = \left(\frac{f + W\sin\theta}{f - W\sin\theta}\right) v_u$$
$$f = 0.05\,W \qquad \sin\theta = \sin(\arctan 1/40) \approx 0.025$$
$$v_d = \left(\frac{(0.05 + 0.025)}{(0.05 - 0.025)}\right) v_u = 3v_u = 45\text{ mph}$$

5.17 Leaper



The leaper applies constant force so the acceleration a is constant. The leaper's center of mass has risen a height s just as the leaper leaves the ground. At this point the speed is v_0, which carries the leaper an additional height h. The mechanical energy E at the top of the leap is $E = Mg(s + h)$.

The time T to reach height s is given by $s = \frac{1}{2}aT^2$ and the speed at that point is $v_0 = aT$, so that $T = 2s/v_0$.

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The average power $\bar{\mathcal{P}}$ is $\bar{\mathcal{P}} = E/T = Mg(s+h)v_0/2s$.

$$v_0 = \sqrt{2gh} \implies \bar{\mathcal{P}} = \frac{Mg}{2}\left(1+\frac{h}{s}\right)\sqrt{2gh} = \frac{Mg^{\frac{3}{2}}}{\sqrt{2}}\left(1+\frac{h}{s}\right)\sqrt{h}$$

$$M = 160\,\text{lb} \times \frac{1\,\text{kg}}{2.2\,\text{lb}} = 72.7\,\text{kg}$$

$$h = 3\,\text{ft} \times \frac{12\,\text{in}}{1\,\text{ft}} \times \frac{2.54\,\text{cm}}{1\,\text{in}} = 91.4\,\text{cm} = 0.91\,\text{m}$$

$$\frac{h}{s} = \frac{3\,\text{ft}}{1.5\,\text{ft}} = 2.0$$

$$\bar{\mathcal{P}} = \frac{72.7}{\sqrt{2}} \times (9.8)^{\frac{3}{2}} \times (1.0+2.0) \times \sqrt{0.91} = 4510\,\text{W} = 4510\,\text{W} \times \frac{1\,\text{hp}}{746\,\text{W}} = 6.0\,\text{hp}$$

The world record for the standing high jump is $\approx 1.6\,\text{m}$. The leaper's jump in this problem is evidently within the realm of human capability.

5.18 Sand and conveyor belt

(a) The momentum change of sand mass Δm is $\Delta p = \Delta m v$. The force F on the belt

$$F = \frac{dp}{dt} = v\frac{dm}{dt}$$

The power $\mathcal{P}$ to drive the belt is

$$\mathcal{P} = Fv = v^2\frac{dm}{dt}$$

(b) The kinetic energy K of mass m on the belt is

$$K = \frac{1}{2}mv^2$$

so the power needed to increase the kinetic energy is

$$\frac{dK}{dt} = \frac{1}{2}\frac{dm}{dt}v^2$$

Half the power to drive the belt goes to giving kinetic energy to the sand.

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Note that energy is dissipated when the sand is abruptly accelerated as it hits the belt. To help understand what happens, consider a simple mechanistic model. Suppose that when mass Δm lands on the belt, it skids a distance d under a constant friction force f. The acceleration a of Δm is $a = f/\Delta m$ so that $v^2 = 2ad = 2(f/\Delta m)d$. The work done by friction is thus $fd = \frac{1}{2}\Delta m\, v^2$ so the power dissipated by friction is $\frac{1}{2}(dm/dt)v^2$, accounting for the other half of the power needed to drive the belt.

5.19 Coil of rope

If a length of rope y is off the ground at any instant, its mass is $m = \lambda y$. The total upward force $F_{tot} = F - mg$, where F is the applied force. The rate of change of the momentum p is, with $dy/dt = v_0$,

(a)

$$\frac{dp}{dt} = v_0 \frac{dm}{dt} = v_0 \lambda \frac{dy}{dt} = v_0^2 \lambda$$

$$F = F_{tot} + mg = \frac{dp}{dt} + mg = v_0^2 \lambda + \lambda y g = \lambda (v_0^2 + yg)$$

(b) The power $\mathcal{P}$ delivered to the rope is

$$\mathcal{P} = F v_0 = \lambda v_0^3 + \lambda y g v_0$$

The rope is uniform, so the raised length y has mass $m = \lambda y$. The length has kinetic energy K.

$$K = \frac{1}{2} m v_0^2 = \frac{1}{2} \lambda y v_0^2$$

The center of mass of the raised portion is at $y/2$, so the potential energy U is

$$U = mg\frac{y}{2} = \frac{1}{2} \lambda g y^2$$

The total mechanical energy E is

$$E = \frac{1}{2} \lambda y v_0^2 + \frac{1}{2} \lambda g y^2$$

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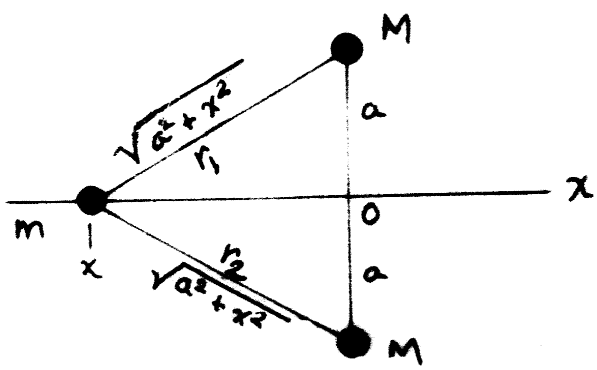
The rate of change of mechanical energy is, using $dy/dt = v_0$,

$$\frac{dE}{dt} = \frac{1}{2}\lambda v_0^3 + \lambda y g v_0$$

Note that $dE/dt < \mathcal{P}$. The kinetic energy of the rope is increasing at only half the rate of the first term in the expression for $\mathcal{P}$. The remainder is dissipated in the sudden acceleration of the rope from rest. Thinking of the rope as a chain, the speed of each link is changed abruptly, in an inelastic process that conserves momentum but not mechanical energy.

6 TOPICS IN DYNAMICS

6.1 Oscillation of bead with gravitating masses



The total gravitational potential energy is the sum of the potential energies of m with each mass M. Using the convention that $U \to 0$ as $r \to \infty$,

$$U(x) = -GmM\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = -\frac{2GmM}{\sqrt{a^2+x^2}}$$

$$\frac{dU}{dx} = \frac{2GmMx}{(a^2+x^2)^{3/2}}$$

$$\frac{d^2U}{dx^2} = \frac{2GmM}{(a^2+x^2)^{3/2}} - \left(\frac{6GmM}{(a^2+x^2)^{5/2}}\right)x^2$$

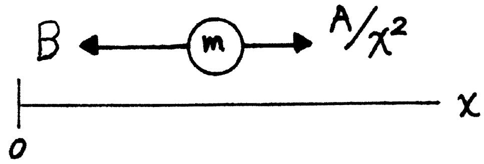
Use a Taylor's series expansion around $x_0 = 0$.

$$\begin{aligned} U(x) &= U(0) + \left.\frac{dU}{dx}\right|_0 (x-0) + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_0 (x-0)^2 + \dots \\ &\approx -\frac{2GmM}{a} + 0 + \frac{1}{2}\left(\frac{2GmM}{a^3}\right)x^2 \end{aligned}$$

For an ideal spring, $U = \frac{1}{2}kx^2$. The effective spring constant is therefore $k = 2GmM/a^3$. The frequency of small oscillations is $\omega = \sqrt{(k/m)} = \sqrt{2GM/a^3}$.

6.2 Oscillation of a particle with two forces

Let x_0 be the location of the potential minimum, where the total force $F_{tot} = 0$. (See the sketch for problem 5.14.)
The total potential energy is the sum of the potential energies due to each force.



1) attractive force: $F_a = -B$

$$U_a(x) - U_a(0) = -\int_0^x F_a\,dx' = Bx'|_0^x \implies U_a(x) = Bx + U_a(0)$$

2) repulsive force: $F_r = \frac{A}{x^2}$

$$U_r(x) - U_r(\infty) = -\int_\infty^x A\frac{dx'}{x'^2} = \frac{A}{x} \implies U_r(x) = \frac{A}{x}$$

$$U_{tot} = Bx + \frac{A}{x} \qquad \frac{dU}{dx} = B - \frac{A}{x^2} \qquad \frac{d^2U}{dx^2} = \frac{2A}{x^3}$$

The potential is a minimum at x_0.

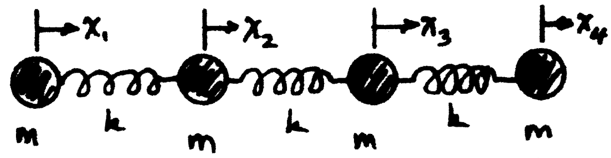
$$0 = \left.\frac{dU}{dx}\right|_{x_0} = B - \frac{A}{{x_0}^2} \implies x_0 = \sqrt{\frac{A}{B}}$$

$$\left.\frac{d^2U}{dx^2}\right|_{x_0} = \frac{2A}{{x_0}^3} = \frac{2A}{(A/B)^{3/2}} = \frac{2B^{3/2}}{A^{1/2}}$$

$$U(x - x_0) = U(x_0) + 0 + \frac{1}{2}\left.\frac{d^2U}{dx^2}\right|_{x_0}(x - x_0)^2 + \ldots$$

so the effective spring constant is $k = (2B^{3/2})/(A^{1/2})$ and the frequency is $\omega = \sqrt{k/m} = \sqrt{(2B^{3/2})(mA^{1/2})}$

6.3 Normal modes and symmetry



As shown in the sketch, four identical masses m are joined by three identical springs of constant k and are constrained to move along the x axis.

In the usual normal mode problem, the coupled equations of motion are solved for the frequencies, from which the relative amplitudes of the normal modes can be found. However, in a problem such as this that has a high degree of symmetry, the normal modes can be guessed, leading to the normal mode frequencies.

The amplitudes are constrained by symmetry, so that $x_1 = \pm x_4$ and $x_2 = \pm x_3$. There are no external forces, so the center of mass must be at rest, leading to the possibilities
$(x_4 = -x_1)$ and $(x_3 = -x_2)$ *(mode A)*
$(x_4 = x_1)$ and $(x_3 = x_2)$ *(mode B)*
In a normal mode, all masses undergo simple harmonic motion with the same frequency ω. For the harmonic motion of each mass, $\ddot{x}_i = \omega^2 x_i$. The equation of motion for mass 1 is, for example,

$$m\ddot{x}_1 = k(x_1 - x_2) \implies \omega^2 x_1 = \frac{k}{m}(x_1 - x_2) = \omega_0{}^2(x_1 - x_2)$$

$$\beta x_1 = (x_1 - x_2)$$

where $\omega_0 = \sqrt{k/m}$ and $\beta = \omega^2/\omega_0{}^2$. Hence

$$\beta x_1 = (x_1 - x_2) \qquad \beta x_2 = (x_2 - x_1 - x_3)$$

$$\beta x_3 = (x_3 - x_2 - x_4) \qquad \beta x_4 = (x_4 - x_3)$$

continued next page $\Longrightarrow$

Consider the modes A for which $x_4 = -x_1$ and $x_3 = -x_2$. The equations of motion reduce to

$\beta x_1 = x_1 - x_2 \qquad (\beta - 1)x_1 = -x_2 \quad (1)$

$\beta x_2 = 2x_2 - x_1 \qquad (\beta - 2)x_2 = -x_1 \quad (2)$

Using Eq. (2) to eliminate x_2 from Eq. (1) leads to a quadratic equation for β: $\beta^2 - 3\beta + 1 = 0$, which has roots $\beta = \frac{1}{2}(3 \pm \sqrt{5})$. The two roots are $\beta \approx 2.618$ and $\beta \approx 0.382$. The corresponding normal mode frequencies are $1.62\omega_0$ and $0.62\omega_0$.

The sketches for modes A show the relative motion for the higher frequency mode ($\beta > 2$) and for the lower frequency mode ($\beta < 1$).

mode A, higher frequency

Proceeding similarly for mode B, where $x_4 = x_1$ and $x_3 = x_2$, the equations of motion reduce to

mode A, lower frequency

$(\beta - 1)x_1 = -x_2 \quad (3)$

$\beta x_2 = -x_1 \quad (4)$

mode B

Equations (3) and (4) lead to a quadratic equation for β: $\beta^2 - \beta - 1 = 0$. The roots are $\beta = \frac{1}{2}(1 \pm \sqrt{5})$. Because $\beta \geq 0$, take the positive sign. The root is $\beta = 1.618$, which gives $\omega \approx 1.27\omega_0$. (As discussed in Example 6.6, only three nontrivial frequencies are expected when there are four equations of motion.)

As the sketch for mode B shows, masses 1 and 4 move together, and 2 and 3 also move together but in the opposite direction.
All the modes conserve momentum, with the center of mass at rest.

6.4 Bouncing ball

The speed after the first collision with the floor is $v_1 = ev_0$. After n collisions, the speed is $v_n = e^n v_0$. The time T to rise to the top of the trajectory is given by $v = gT$, so the time between successive bounces is $2T = 2v/g$. The time T_n for the n^{th} bounce is therefore

$$T_n = 2\frac{v_n}{g} = \left(\frac{2v_0}{g}\right)e^n.$$

The total time $T_{n,total}$ for n bounces is

$$T_{n,total} = \sum_{j=1}^{n} T_i = \left(\frac{2v_0}{g}\right)\sum_{j=1}^{n} e^j$$

As $n \to \infty$, then $v \to 0$. The time T_f to finally come to rest is therefore

$$T_f = \lim_{j\to\infty} T_{n,total} = \left(\frac{2v_0}{g}\right)\sum_{j}^{\infty} e^j = \sum_{k=1}^{\infty} x^k - \sum_{k=1}^{\infty} x^{k+1} = \left(\frac{2v_0}{g}\right)\frac{e}{1-e}$$

The last step makes use of the identity $S = \sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$, provided $x < 1$. Proof:

$$S - xS = \sum_{k=1}^{\infty} x^k - \sum_{k=1}^{\infty} x^{k+1} = \sum_{k=1}^{\infty} x^k - \sum_{k=2}^{\infty} x^k \quad (1)$$

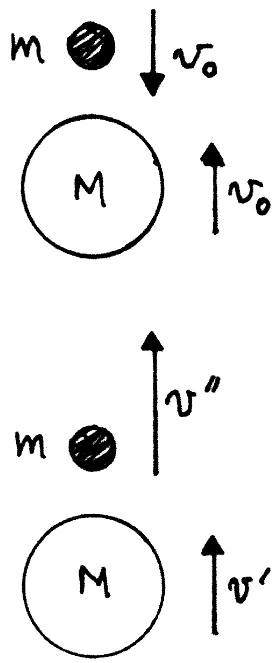
The second sum cancels all of the first sum except for the very first term, for which $k = 1$, so the right hand side of Eq. (1) becomes simply x.

$$S - xS = x \implies S = \frac{x}{1-x} \quad \text{if } x < 1$$

6.5 Marble and superball

After falling from height h, a superball of mass M carrying a marble of mass m hits the floor with speed $v_0 = \sqrt{2gh}$. After the elastic bounce, M moves upward with speed v_0. Here are two methods for finding the upward speed of the marble after it collides with the superball. (To demonstrate the effect, it may be easier to use a coin instead of a marble, but a coin may experience greater air resistance.)

Method 1:
This method is algebraic, using the conservation laws for momentum and for mechanical energy.



The top sketch shows the superball immediately after it has bounced off the floor. A gap (greatly exaggerated in the sketch) is shown between the marble, which is still moving downward with speed v_0, and the superball, which is moving upward with speed v_0.

The lower sketch shows the system immediately after the marble has collided with the superball. The superball's speed is now v' at this instant, and the marble has speed v'' upward.

The initial momentum just before the collision (upper sketch) is P_i, and the final momentum just after (lower sketch) is P_f.

$$P_i = Mv_0 - mv_0 \qquad P_f = Mv' + mv''$$

The external gravitational force has negligible time to act, so $P_f = P_i$.

$$Mv' + mv'' = Mv_0 - mv_0 \implies m(v'' + v_0) = M(v_0 - v') \quad (1)$$

continued next page $\Longrightarrow$

The collision is assumed to be elastic, so mechanical energy is conserved, The change in potential energy is negligible during the short time of the collision, so kinetic energy is the only mechanical energy of interest. K_i is the initial kinetic energy (upper sketch), and K_f is the kinetic energy just after the collision (lower sketch).

$$K_f = -\frac{1}{2}Mv'^2 + \frac{1}{2}mv''^2 = K_i = -\frac{1}{2}Mv^2{}_0 + \frac{1}{2}mv^2{}_0$$

$$\begin{aligned} m(v''^2 - v_0^2) &= M(v_0^2 - v'^2) = M(v_0 - v')(v_0 + v') \\ &\approx 2Mv_0(v_0 - v') \quad (2) \end{aligned}$$

In the last step, $v' \approx v_0$ because $M \gg m$, so $(v_0 + v') \approx 2v_0$. Using Eq. (1) in Eq, (2)

$$m(v''^2 - v_0^2) = m(v'' - v_0)(v'' + v_0) = 2mv_0(v'' + v_0)$$

$$v'' - v_0 = 2v_0 \implies v'' = 3v_0$$

After the collision, the marble flies up a height $h' = (3v_0)^2/(2g) = 9h$.

method 2:
This method is nonalgebraic, and uses simple but sophisticated reasoning.

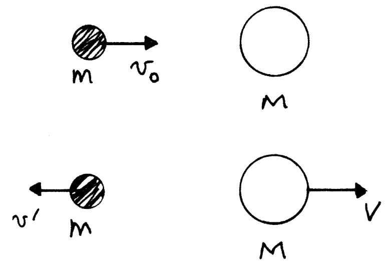
To an observer on M (moving upward), the marble just before the collision appears to be approaching with speed $2v_0$. Because $m \ll M$, the collision resembles a collision with a rigid wall, which reverses the direction of the marble's speed, so after the collision the marble is moving upward with speed $2v_0$ relative to the superball, or $3v_0$ relative to the floor. After the collision, the marble flies up a height $h' = (3v_0)^2/(2g) = 9h$.

6.6 Three car collision

Momentum is conserved in these inelastic collisions, but mechanical energy is not conserved. Each car has mass M.The initial speed of car A is v_0, so its initial kinetic energy is $E_0 = \frac{1}{2}Mv_0^2$. After the first collision, cars A and B move together with speed v_1, so conservation of momentum gives $Mv_0 = 2Mv_1$. Therefore $v_1 = v_0/2$, so the kinetic energy E_1 of A and B is $E_1 = \frac{1}{2}(2M)v_1^2 = (\frac{1}{2})(\frac{1}{2})Mv_0^2 = E_0/2$.

After the second collision, the speed v_2 is given by $3Mv_2 = 2Mv_1 = Mv_0$, so that $v_2 = v_0/3$. The kinetic energy E_2 is $E_2 = \frac{1}{2}(3M)v_2^2 = (\frac{1}{2})(\frac{1}{3})Mv_0^2 = E_0/3$. Hence the mechanical energy lost in the second collision is $E_2 - E_1 = (\frac{1}{2} - \frac{1}{3})E_0 = E_0/6$.

6.7 Proton collision



The proton has mass m, and the unknown particle has mass M. The upper sketch is before the collision, and the lower sketch is after the collision. Both momentum P and mechanical energy (kinetic energy K) are conserved in the elastic collision.

$$P_f = MV - mv' = P_i = mv_0 \implies v_0 = \frac{M}{m}V - v' \quad (1)$$

$$K_f = \frac{1}{2}MV^2 + \frac{1}{2}mv'^2 = K_i = \frac{1}{2}mv_0^2 \implies v_0^2 = \frac{M}{m}V^2 + v'^2 \quad (2)$$

$$E_f = \frac{1}{2}mv'^2 = \frac{4}{9}\left(\frac{1}{2}mv_0^2\right) \implies v' = \frac{2}{3}v_0 \quad (3)$$

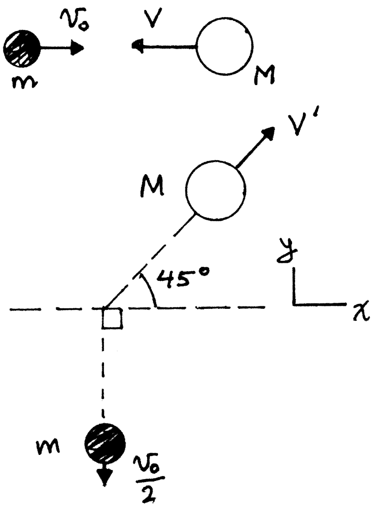
Using Eqs. (1) and (3),

$$V = \frac{5}{3}\frac{m}{M}v_0 \quad (4)$$

Using Eqs. (3) and (4) in Eq. (2),

$$v_0^2 = \frac{M}{m}\frac{25}{9}\left(\frac{m}{M}\right)^2 v_0^2 + \frac{4}{9}v_0^2 \implies \frac{5}{9} = \frac{25}{9}\frac{m}{M} \implies M = 5m$$

6.8 Collision of m and M



The upper sketch shows the system before the collision, and the lower sketch after the collision. Both momentum **P** and mechanical energy (kinetic energy K) are conserved in the elastic collision. **P** has both x and y components.

$$P_{fx} = \frac{MV'}{\sqrt{2}} = P_{ix} = mv_0 - MV$$

$$P_{fy} = \frac{MV'}{\sqrt{2}} - \frac{mv_0}{2} = P_{iy} = 0$$

$$mv_0 - MV = \frac{MV'}{\sqrt{2}} \quad (1)$$

$$0 = \frac{MV'}{\sqrt{2}} - \frac{mv_0}{2} \implies V' = \frac{1}{\sqrt{2}}\frac{m}{M}v_0 \quad (2)$$

From Eqs. (1) and (2)

$$V = \frac{1}{2}\frac{m}{M}v_0 \quad (3)$$

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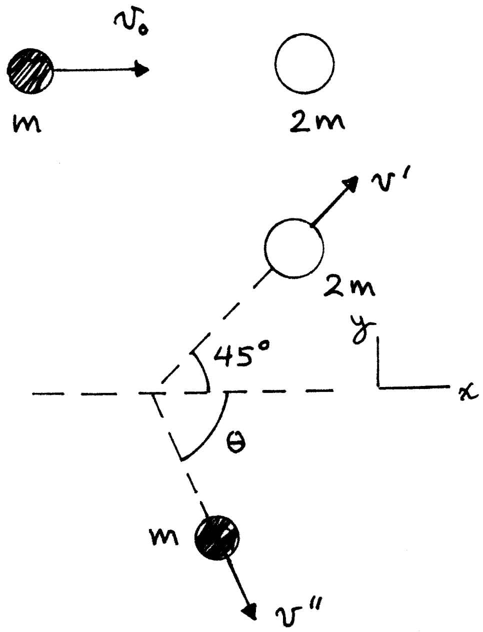
Conservation of mechanical energy:

$$\frac{1}{2}mv_0^2 + \frac{1}{2}MV^2 = \frac{1}{2}m\left(\frac{v_0}{2}\right)^2 + \frac{1}{2}MV'^2$$

Using Eqs. (2) and (3),

$$\frac{3}{4}mv_0^2 + \frac{1}{4}M\left(\frac{m}{M}\right)^2 v_0^2 = \frac{1}{2}M\left(\frac{m}{M}\right)^2 v_0^2 \implies \frac{m}{M} = 3$$

6.9 Collision of m and $2m$



The upper sketch shows the system before the collision, and the lower sketch is after. Both momentum **P** and mechanical energy (kinetic energy K) are conserved in the elastic collision. **P** has both x and y components.

$$P_{fx} = \frac{2mv'}{\sqrt{2}} + mv''\cos\theta = P_{ix} = mv_0$$

$$v_0 = \sqrt{2}v' + v''\cos\theta \quad (1)$$

$$P_{fy} = \frac{2mv'}{\sqrt{2}} - mv''\sin\theta = P_{iy} = 0$$

$$\sqrt{2}v' = v''\sin\theta \quad (2)$$

From Eqs. (1) and (2),

$$v_0 = v''(\sin\theta + \cos\theta) \quad (3)$$

Conservation of mechanical energy:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}(2m)v'^2 + \frac{1}{2}mv''^2 \implies v_0^2 = 2v'^2 + v''^2 \quad (4)$$

Using Eq. (2), Eq. (4) becomes

$$v_0^2 = v''^2(\sin^2\theta + 1) \quad (5)$$

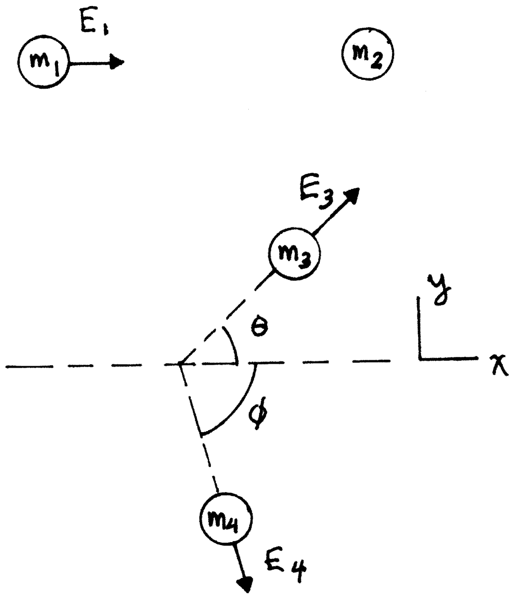
and from Eqs. (3) and (5),

$$\sin^2\theta + 1 = (\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$$

$$\sin\theta = 2\cos\theta \implies \tan\theta = 2 \qquad \theta \approx 63^\circ$$

6.10 Nuclear collision in the *L* system

The upper sketch shows the system before the reaction, and the lower sketch is after. The collision is inelastic, so momentum **P** is conserved, but but mechanical energy E is not. P has both x and y components. Use the convenient form $P = \sqrt{2mE}$.



(a) The energy released is $Q = E_3 + E_4 - E_1$ (1)

(b) conservation of momentum:

$$\sqrt{2m_1E_1} = \sqrt{2m_3E_3}\cos\theta + \sqrt{2m_4E_4}\cos\phi \quad (2)$$

$$0 = \sqrt{2m_3E_3}\sin\theta - \sqrt{2m_4E_4}\sin\phi \quad (3)$$

(c) Now eliminate E_4 and ϕ, using Eqs. (2) and (3). Squaring Eqs. (2) and (3),

$$m_4E_4\cos^2\theta = m_1E_1 + m_3E_3\cos^2\theta - 2\sqrt{m_1m_3E_1E_3}\cos\theta \quad (4)$$

$$m_4E_4\sin^2\phi = m_3E_3\sin^{\theta} \quad (5)$$

Adding Eqs. (4) and (5),

$$m_4E_4 = m_1E_1 + m_3E_3 - 2\sqrt{m_1m_3E_1E_3}\cos\theta$$

$$E_4 = \frac{m_1}{m_4}E_1 + \frac{m_3}{m_4}E_3 - \frac{2}{m_4}\sqrt{m_1m_3E_1E_4}\cos\theta$$

Inserting this expression for E_4 in Eq. (1) gives

$$Q = \left(\frac{m_3}{m_4} + 1\right)E_3 + \left(\frac{m_1}{m_4} - 1\right)E_1 - \frac{2}{m_4}\sqrt{m_1m_3E_1E_3}\cos\theta$$

6.11 Uranium fission

The incoming neutron is slow, and the uranium nucleus is essentially at rest, so take the initial momentum and initial kinetic energy of the neutron and the ^{235}U nucleus to be zero. After fission, the product ^{97}Sr and ^{138}Xe nuclei move apart back-to-back, with equal and opposite momentum P. The total kinetic energy of the fission fragments is $E_t = 170\,\text{MeV}$. Using $E = P^2/2M$,

$$E_{Sr} = \frac{P^2}{2M_{Sr}} \qquad E_{Xe} = \frac{P^2}{2M_{Xe}}$$

$$E_{Sr} + E_{Xe} = E_t$$

$$P^2\left(\frac{1}{2M_{Sr}} + \frac{1}{2M_{Xe}}\right) = E_t \implies P^2 = 2E_t\left(\frac{M_{Sr}M_{Xe}}{M_{Sr} + M_{Xe}}\right)$$

$$E_{Sr} = E_t\left(\frac{M_{Xe}}{M_{Sr} + M_{Xe}}\right) = 170\,\text{MeV} \times \frac{138}{235} = 100\,\text{MeV}$$

$$E_{Xe} = E_t\left(\frac{M_{Sr}}{M_{Sr} + M_{Xe}}\right) = 170\,\text{MeV} \times \frac{97}{235} = 70\,\text{MeV}$$

6.12 Hydrogen fusion

The particles are essentially at rest before the fusion reaction. After fusion, the products have equal and opposite momentum P. Using $E = P^2/2M$,

$$E_{He} = \frac{P^2}{2M_{He}} \qquad E_n = \frac{P^2}{2M_n}$$

The total energy E_{total} released is

$$E_{total} = E_{He} + E_n = 17.6\,\text{MeV}$$

$$P^2\left(\frac{1}{2M_{He}} + \frac{1}{2M_n}\right) = E_t \implies P^2 = 2E_t\left(\frac{M_{He}M_n}{M_{He} + M_n}\right)$$

$$E_{He} = E_t\left(\frac{M_n}{M_{He} + M_n}\right) = 17.6\,\text{MeV} \times \frac{1}{5} = 3.5\,\text{MeV}$$

$$E_n = E_t\left(\frac{M_{He}}{M_{He} + M_n}\right) = 17.6\,\text{MeV} \times \frac{4}{5} = 14.1\,\text{MeV}$$

6.13 Nuclear reaction of α rays with lithium

To high accuracy, the mass of the reactants $M = M_\alpha + M_{Li}$ is equal to the mass of the products $M_n + M_B$, where $M = 4 + 7 = 1 + 10 = 11$ mass units.

Let E_0 be the kinetic energy of the incident α particle in the L system. The reaction collision is inelastic; E_0 supplies the kinetic energy of the products and the reaction energy $Q = 2.8$ MeV.

At threshold, the energy in the C system is just enough to form the products. At threshold, the neutron and the boron are at rest in C, so their energies K_L in the L system are due entirely to the motion of the center of mass, moving with speed V, so $K_L = (1/2)MV^2$.

(a)

$$V = \frac{M_\alpha}{M}v_\alpha \implies K_L = \frac{1}{2}M\left(\frac{M_\alpha}{M}\right)^2 v_\alpha^2 = \frac{1}{2}M_\alpha v_\alpha^2 \times \frac{M_\alpha}{M} = \frac{M_\alpha}{M}E_0 = \frac{4}{11}E_0$$

$$E_{0,threshold} = K_L + Q = \frac{4}{11}E_{0,threshold} + Q = \frac{11}{7}\times 2.8\,\text{MeV} = 4.4\,\text{MeV}$$

At threshold, the neutron is moving with speed V in the L system.

$$E_{n,L} = \frac{1}{2}M_nV^2 = \frac{1}{2}M_n\left(\frac{M_\alpha}{M}\right)v_\alpha^2 = \frac{M_n}{M}E_{0,threshold} = \frac{1}{11}\times 4.4\,\text{MeV} = 0.4\,\text{MeV}$$

(b) For the product neutron moving in the forward direction in the L system, its velocity $\mathbf{v_{n,C}}$ must be either parallel or antiparallel to the center of mass velocity $\mathbf{V}$. If $\mathbf{v}_{n,C}$ is moving parallel to $\mathbf{V}$, the neutron will always be moving in the forward direction in L, and its energy is not restricted. However, if $\mathbf{v}_{n,C}$ is antiparallel to $\mathbf{V}$, the neutron will be moving in the forward direction in L only if $v_{n,C} \leq V$. In this case, the limit for forward motion in L is $v_{n,C} = V$. The speed of the neutron in C is Vin this case, and it is moving antiparallel to $\mathbf{V}$. Because total momentum is always 0 in C, the speed of the boron nucleus is $(M_n/M_B)V$, and it is moving parallel to $\mathbf{V}$. The speeds of the products in the L system, in the limit of forward motion, are

$$v_{n,L} = V - V = 0$$

$$v_{B,L} = \frac{M_n}{M_B}V + V$$

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The kinetic energy of n in L is 0, and the kinetic energy of boron is

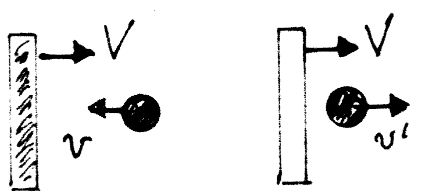
$$\begin{aligned}
K_{B,L} &= \frac{1}{2}M_B\left(1+\frac{M_n}{M_B}\right)^2 V^2 = \frac{1}{2}M_B\left(1+\frac{M_n}{M_B}\right)^2\left(\frac{M_\alpha}{M}\right)^2 v_\alpha^2 = \frac{M_\alpha}{M_B}E_0\\
E_0 &= K_{B,L} + 0 + Q\\
\left(1-\frac{M_\alpha}{M_B}\right)E_0 &= Q\\
E_0 &= \frac{10}{6}\times 2.8\,\text{MeV}
\end{aligned}$$

At threshold,

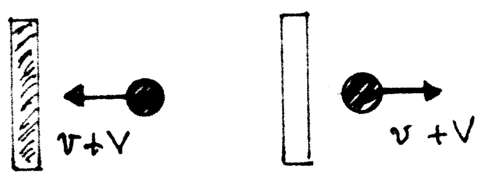
$$\begin{aligned}
E_{0,threshold} &= \frac{11}{7}\times 2.8\,\text{MeV}\\
E_0 - E_{0,threshold} &= \left(\frac{10}{6}-\frac{11}{7}\right)\times 2.8\,\text{MeV} = \frac{4}{42}\times 2.8\,\text{MeV} = 0.27\,\text{MeV}
\end{aligned}$$

6.14 Superball bouncing between walls

This problem is a model for the common observation that when pumping up a tire with a hand pump, the barrel of the pump becomes warmer. Pushing the piston down does work on the gas, raising its temperature by increasing the average speed of the gas molecules.



lab frame



moving frame

(a) The time-average force is the average rate of momentum transfer to a wall. Consider the situation when the walls are stationary. In a single collision (elastic), $\Delta p = 2mv$. The time between collisions is $\Delta T = 2l/v$. The average force $\bar{F}$ is then

$$\bar{F} = \frac{\Delta p}{\Delta T} = \frac{2mv}{2l/v} = \frac{mv^2}{l}$$

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(b) Consider now the case when one wall is moving. To an observer moving with the wall, the superball approaches with speed $v + V$, and leaves with the same speed (elastic collision), as shown in the sketch. Then convert back to the lab frame by adding V.

$$v' = (v + V) + V = v + 2V$$
$$\Delta v = v' - v = 2V$$

The time between collisions is $\Delta T = 2x/v$. In the limit $\Delta T \to 0$,

$$\frac{dv}{dt} = \frac{2V}{2x/v} = \frac{vV}{x}$$
$$\frac{dv}{dx} = \frac{dv/dt}{dx/dt} = -\frac{1}{V}\frac{dv}{dt} \quad \text{(-) sign because } x \text{ decreases with time}$$
$$= -\frac{v}{x}$$
$$\frac{dv}{v} = -\frac{dx}{x} \implies \ln\frac{v}{v_0} = -\ln\frac{x}{x_0} = -\ln\frac{x}{l}$$
$$v = v_0\frac{l}{x} \implies \bar{F} = \frac{2mv}{2x/v} = \frac{mv^2}{x} = \frac{mv_0^2 l^2}{x^3}$$

(c) The work ΔW moving distance Δx in the direction of V is

$$\Delta W = -F\Delta x \quad \text{(-) because } x \text{ is decreasing}$$
$$W = -mv_0^2 l^2 \int_l^x \frac{dx}{x^3} = mv_0^2 l^2 \left(\frac{1}{2x^2}\right)\Bigg|_l^x = \frac{1}{2}mv_0^2\left(\frac{x^2}{l^2} - 1\right)$$

The superball's kinetic energy K is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}mv_0^2\frac{x^2}{l^2}$$
$$\Delta K = K(x) - K(l) = \frac{1}{2}mv_0^2\frac{x^2}{l^2} - \frac{1}{2}mv_0^2 = \frac{1}{2}mv_0^2\left(\frac{x^2}{l^2} - 1\right) = W$$

6.15 Center of mass energy

Let $M = M_a + M_b$. The object of this problem is to prove that for two particles a and b moving with velocities $\mathbf{V}_a$ and $\mathbf{V}_b$,

$$\frac{1}{2}MV^2 + \frac{1}{2}\mu V_r^2 = \frac{1}{2}M_a V_a^2 + \frac{1}{2}M_b V_b^2$$

where $\mathbf{V}$ is the velocity of the center of mass, $\mathbf{V}_r$ is the relative velocity of the two particles, and μ is the reduced mass.

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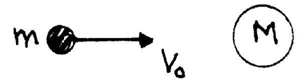
$$
\begin{aligned}
\mathbf{V} &= \frac{M_a\mathbf{V}_a + M_b\mathbf{V}_b}{M} \\
\frac{1}{2}MV^2 &= \frac{1}{2}M\left(\frac{M_aV_a^2 + M_bV_b^2 + 2\mathbf{V}_a\cdot\mathbf{V}_bM_aM_b}{M^2}\right) \\
&= \frac{1}{2}\left(\frac{M_a^2V_a^2 + M_b^2V_b^2 + 2\mathbf{V}_a\cdot\mathbf{V}_bM_aM_b}{M}\right) \qquad (1) \\
\mathbf{V}_r &= \mathbf{V}_a - \mathbf{V}_b \\
\frac{1}{2}\mu V_r^2 &= \frac{M_aM_b}{M}\left(V_a^2 + V_b^2 - 2\mathbf{V}_a\cdot\mathbf{V}_b\right) \qquad (2)
\end{aligned}
$$

Adding Eqs. (1) and (2),

$$
\begin{aligned}
\frac{1}{2}MV^2 + \frac{1}{2}\mu V_r^2 &= \frac{1}{2}\left(\frac{M_a^2 + M_aM_b}{M}\right)V_a^2 + \frac{1}{2}\left(\frac{M_b^2 + M_aM_b}{M}\right)V_b^2 \\
&= \frac{1}{2}M_a\left(\frac{M_a + M_b}{M_a + M_b}\right)V_a^2 + \frac{1}{2}M_b\left(\frac{M_b + M_a}{M_a + M_b}\right)V_b^2 = \frac{1}{2}M_aV_a^2 + \frac{1}{2}M_bV_b^2
\end{aligned}
$$

6.16 Converting between *C* and *L* systems

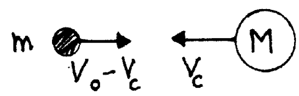
(a) To convert from the *L* to *C*, subtract the center of mass velocity $\mathbf{V}_c$ from every *L* system velocity vector (upper two sketches).



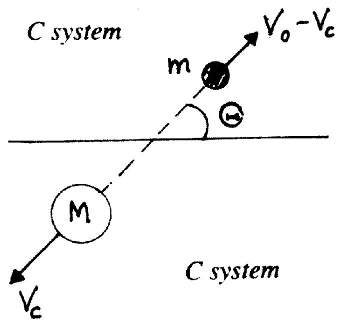
L system

$$\mathbf{V}_c = \frac{mV_0}{m + M}$$

In an elastic collision, the speeds in *C* are unchanged, (third sketch). To convert from *C* to *L*, add $\mathbf{V}_c$ to every velocity vector, as shown for mass *m* (bottom sketch). $\mathbf{v}_f$ is the velocity of *m* in the *L* system after the collision.



C system



$$
\begin{aligned}
v_0 - V_c &= \frac{Mv_0}{m + M} \\
v_f^2 &= V_c^2 + (v_0 - V_c)^2 - 2V_c(v_0 - V_c)\cos(\pi - \Theta) \\
&= \left(\frac{v_0}{m + M}\right)^2\left(m^2 + M^2 + 2mM\cos\Theta\right) \\
v_f &= \left(\frac{v_0}{m + M}\right)\sqrt{m^2 + M^2 + 2mM\cos\Theta}
\end{aligned}
$$

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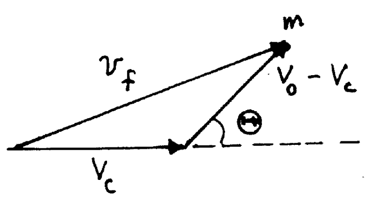


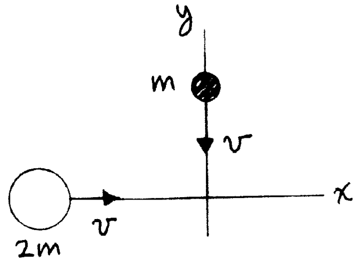
L system

(b)

$$
\begin{aligned}
K_0 &= \frac{1}{2}mv_0^2 \\
K_f &= \frac{1}{2}mv_f^2 \\
&= \frac{1}{2}\left(\frac{mv_0^2}{(m+M)^2}\right)(m^2 + M^2 + 2mM\cos\Theta) \\
\frac{K_0 - K_f}{K_0} &= 1 - \frac{(m^2 + M^2 + 2mM\cos\Theta)}{(m+M)^2} \\
&= \frac{2mM(1-\cos\Theta)}{(m+M)^2}
\end{aligned}
$$

6.17 Colliding balls

The upper sketch is before the collision, and the lower sketch is after the collision.



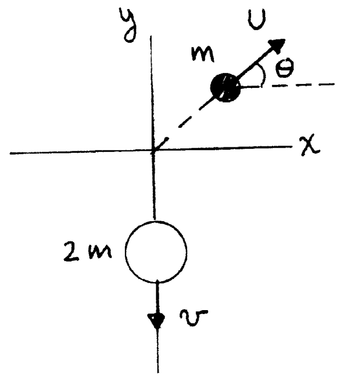
(a)

By conservation of momentum,

$$2mv\,\hat{\mathbf{i}} - mv\,\hat{\mathbf{j}} = mU\cos\theta\,\hat{\mathbf{i}} + mU\sin\theta\,\hat{\mathbf{j}} - 2mv\,\hat{\mathbf{j}}$$

$$2v = U\cos\theta \quad (1)$$

$$v = U\sin\theta \quad (2)$$



Dividing Eq. (2) by Eq. (1),

$$\tan\theta = 1/2 \approx 27^\circ$$

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{5}}$$

$$U = \frac{v}{\sin\theta} = \sqrt{5}\,v \approx 2.2\,v$$

(b)

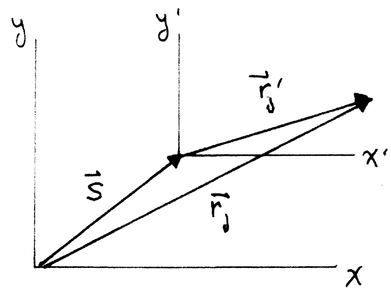
$$
\begin{aligned}
E_i &= \frac{1}{2}(2m)v^2 + \frac{1}{2}mv^2 = \frac{3}{2}mv^2 \\
E_f &= \frac{1}{2}(2m)v^2 + \frac{1}{2}mU^2 = \frac{1}{2}(2m)v^2 + \frac{5}{2}mv^2 = \frac{7}{2}mv^2 \\
&> E_i
\end{aligned}
$$

The collision is superelastic.

7 ANGULAR MOMENTUM AND FIXED AXIS ROTATION

7.1 Origins

(a) $\mathbf{L}$ = angular momentum in $x - y$
$\mathbf{L}'$ = angular momentum in $x' - y'$.



$$\mathbf{L} = \sum \mathbf{r}_j \times \mathbf{p}_j = \sum (\mathbf{S} + \mathbf{r}'_j) \times \mathbf{p}_j$$

$$\mathbf{p}_j = m\dot{\mathbf{r}}_j = m\dot{\mathbf{r}}'_j = \mathbf{p}'_j$$

$$\mathbf{L}' = \sum \mathbf{r}'_j \times \mathbf{p}'_j = \sum (\mathbf{r}_j - \mathbf{S}) \times \mathbf{p}_j = \sum \mathbf{r}_j \times \mathbf{p}_j - \mathbf{S} \times \sum \mathbf{p}_j = \mathbf{L} - \mathbf{S} \times \sum \mathbf{p}_j$$

It is given that $\mathbf{P} = \sum \mathbf{p_j} = 0$, so $\mathbf{L}' = \mathbf{L}$, independent of the origin $\mathbf{S}$.

(b)

$\tau = \sum \mathbf{r_j} \times \mathbf{F_j}$. Because forces are independent of the origin, $\mathbf{F'_j} = \mathbf{F_j}$.

$$\tau' = \sum \mathbf{r}'_j \times \mathbf{F}'_j = \sum (\mathbf{r}_j - \mathbf{S}) \times \mathbf{F}_j = \tau - \mathbf{S} \times \sum \mathbf{F}_j$$

It is given that $\mathbf{F}_{total} = \sum \mathbf{F}_j = 0$ so $\tau' = \tau$

7.2 Drum and sand

The angular momentum of the system at time $t = 0$ is

$$L(0) = (M_A + M_s)a^2\omega_A(0) \quad (1)$$

At time t, the angular momentum is

$$L(t) = (M_A + M_s - \lambda t)a^2\omega_A(0) + (M_B + \lambda t)b^2\omega_B(t) \quad (2)$$

Note that $\omega_A(t) = \omega_A(0)$ because the sand exerts no torque on drum A as it leaves. (To an observer on the drum, the sand appears to fly out radially.) Angular momentum is conserved $L(t) = L(0)$, because the system is isolated – there are no external torques. Equations (1) and (2) then give

$$\omega_B(t) = \frac{\lambda a^2\omega_A(0)t}{(M_B + \lambda t)b^2}$$

At time T given by $\lambda T = M_s$, all of the sand has been transferred from drum A to drum B, and ω_B is then constant. Thus

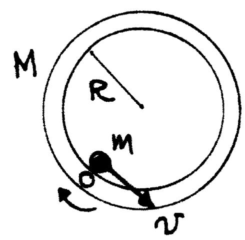
$$\begin{aligned} \omega_B(t \geq T) &= \frac{M_s a^2\omega_A(0)}{(M_B + M_s)b^2} \quad (3) \\ &< \omega_A(0) \end{aligned}$$

It is easy to show that the angular momentum of the system for $t \geq T$ remains equal to $L(0)$. Using Eq. (3) gives

$$\begin{aligned} L(t \geq T) &= M_A a^2\omega_A(0) + (M_B + M_s)b^2\omega_B(t \geq T) \\ &= M_A a^2\omega_A(0) + (M_B + M_s)b^2\frac{M_s a^2\omega_A(0)}{(M_B + M_s)b^2} \\ &= (M_A + M_s)a^2\omega_A(0) \\ &= L(0) \end{aligned}$$

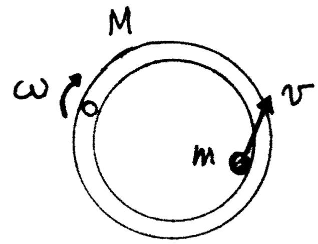
7.3 Ring and bug

(a) With the bug at the origin and the ring at rest, the initial angular momentum is $L(0) = 0$. Angular momentum L about the pivot is conserved, because no external torques are acting on the system. Thus, noting that the ring rotates clockwise as the bug moves counterclockwise,



$$L = L_{bug} - L_{ring}$$

When the bug is halfway around, the ring is rotating at some angular speed ω. The speed of the bug is then $v - 2R\omega$.
The moment of inertia of the ring I_{ring} about the pivot, is, by the parallel axis theorem,



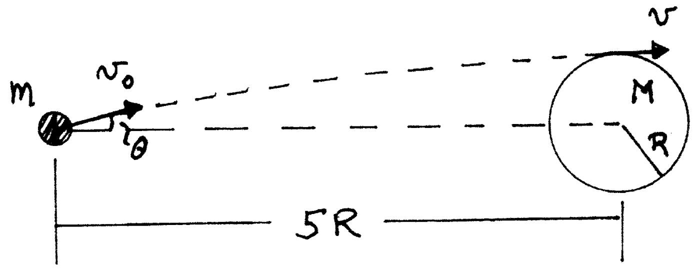
$$\begin{aligned} I_{ring} &= I_0 + MR^2 = MR^2 + MR^2 \\ &= 2MR^2 \end{aligned}$$

The total angular momentum is

$$\begin{aligned} L &= -\omega I_{ring} + m(2R)(v - 2R\omega) \\ &= [-2MR^2 - (2R)^2 m]\omega + 2mRv \\ &= 0 \\ \omega &= \frac{mv}{(M + 2m)R} \end{aligned}$$

(b) When the bug is back at the pivot, $L_{bug} = 0$, and therefore $L_{ring} = 0$ so that $\omega = 0$. The ring is then momentarily at rest, but it is not necessarily back to its initial position.

7.4 Grazing instrument package



Because the gravitational force on m is central, angular momentum about the center of the planet is conserved. At tangential grazing, m is traveling at speed v and its trajectory is perpendicular to R. Hence

$$m(5R)v_0 \sin\theta = mvR \implies v = 5v_0 \sin\theta \quad (1)$$

Another expression for the grazing speed v can be found using conservation of mechanical energy.

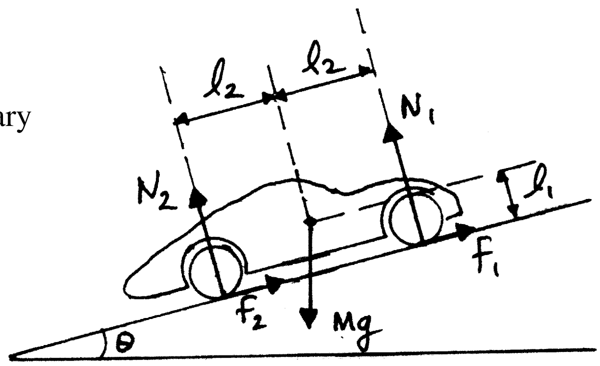
$$\frac{1}{2}mv_0^2 - \frac{GmM}{5R} = \frac{1}{2}mv^2 - \frac{GmM}{R} \implies v^2 = v_0^2 + \frac{8}{5}\frac{GM}{R} \quad (2)$$

Combining Eqs. (1) and (2) to eliminate v,

$$(25\sin^2\theta - 1) = \frac{8}{5}\frac{GM}{Rv_0^2} \implies \sin\theta = \frac{1}{5}\sqrt{1 + \frac{8}{5}\frac{GM}{Rv_0^2}} \quad (3)$$

It is evident from Eq. (3) that $\sin\theta$ must be $> 1/5$ ($\theta > 11.5°$). For example, if θ is too small, the package does not graze the planet but plows into it. On the other hand, if $8GM/5Rv_0^2 > 24$, then according to Eq. (3), $\sin\theta > 1$, an unrealizable value. In this case, the package does not graze, but sails over the planet.

7.5 Car on a hill



The car is in stationary equilibrium, so the total force on the car must be 0, and the total torque about any point must be 0. (Problem 7.1 shows that in a stationary system, if the torque is 0 about some point, it is 0 about any point.)

forces:

$$0 = N_1 + N_2 - Mg\cos\theta$$
$$0 = f_1 + f_2 - Mg\sin\theta \quad (1)$$

torque (about the center of mass):

$$0 = N_1 l_2 + f_1 l_1 + f_2 l_1 - N_2 l_2 \quad (2)$$

Using Eq. (1), Eq. (2) can be written

$$0 = (N_1 - N_2)l_2 + Mgl_1\sin\theta$$

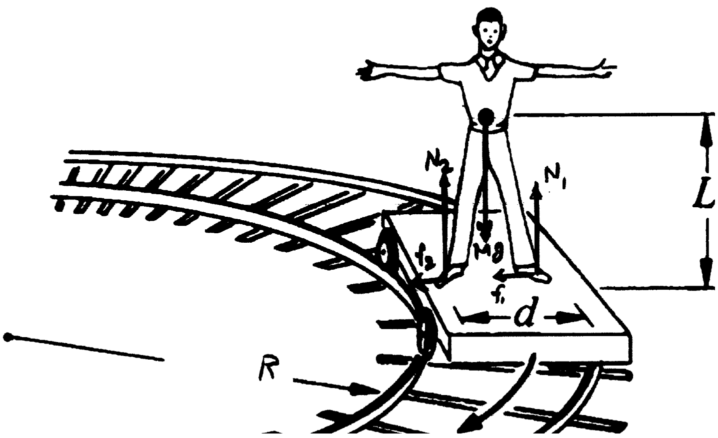
$$N_1 = \frac{1}{2}Mg\left(\cos\theta - \frac{l_1}{l_2}\sin\theta\right) \qquad N_2 = \frac{1}{2}Mg\left(\cos\theta + \frac{l_1}{l_2}\sin\theta\right)$$

$$Mg = 3000\,\text{lb} \qquad \theta = 15^\circ \qquad l_1 = 2\,\text{ft} \qquad l_2 = 4\,\text{ft}$$

$$N_1 = 1500\left(0.966 - \frac{2}{4}0.259\right) = 1255\,\text{lb}$$

$$N_2 = 1500\left(0.966 + \frac{2}{4}0.259\right) = 1643\,\text{lb}$$

7.6 Man on a railroad car



vertical equation of motion:

$$N_1 + N_2 - Mg = 0$$

radial equation of motion:

$$f_1 + f_2 = M\frac{v^2}{R} \quad (1)$$

torque about the center of mass:

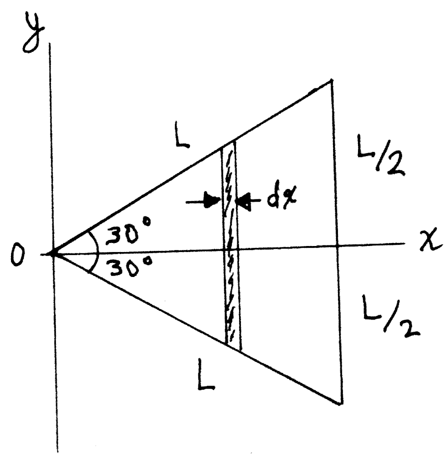
$$-N_1\frac{d}{2} + N_2\frac{d}{2} + (f_1 + f_2)L = 0 \quad (2)$$

Using Eq. (1) in (Eq. (2),

$$\frac{d}{2}(N_1 - N_2) = mv^2\frac{L}{R}$$

outside foot: $N_1 = \frac{1}{2}\left(Mg + \frac{Mv^2}{R}\frac{2L}{d}\right)$ inside foot: $N_2 = \frac{1}{2}\left(Mg - \frac{Mv^2}{R}\frac{2L}{d}\right)$

7.7 Moment of inertia of a triangle



The slanted sides obey the relation

$$y = \pm\frac{x}{\sqrt{3}} \quad (1)$$

The area A of the triangle is

$$A = \frac{1}{2}(\textit{base} \times \textit{height}) = \frac{1}{2}L(L\cos 30°) = \frac{\sqrt{3}}{4}L^2$$

The shaded strip has length $2y$, width dx and mass dm.

$$dm = \frac{2y\,dx}{A}M$$

The moment of inertia of the strip about its center is

$$\frac{1}{12}dm(2y)^2 = \frac{1}{3}y^2 dm$$

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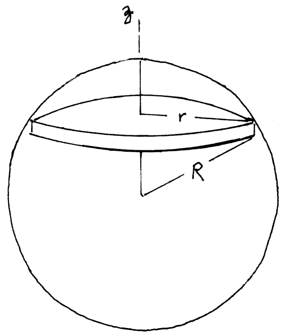
By the parallel axis theorem, the moment of inertia dI_v of the strip about a perpendicular axis through the vertex at the origin is, using Eq. (1),

$$dI_v = x^2\,dm + \frac{1}{3}y^2\,dm = \frac{10}{9}x^2\,dm$$

$$I_v = \frac{10}{9}\int x^2\,dm = \frac{20}{9}\frac{M}{A}\int_0^{\sqrt{3}L/2} x^2 y\,dx = \frac{20}{9\sqrt{3}}\frac{M}{A}\int_0^{\sqrt{3}L/2} x^3\,dx = \frac{20}{9\sqrt{3}}\frac{M}{(\sqrt{3}/2)L^2}\times\frac{1}{4}\left(\frac{\sqrt{3}}{2}L\right)^4$$

$$= \frac{5}{12}ML^2$$

7.8 Moment of inertia of a sphere



Consider the sphere to be a stack of solid disks of radius r and mass dm. The moment of inertia is

$$dI = \frac{1}{2}r^2\,dm \qquad dm = \frac{M}{V}\,dV = \frac{M}{V}(\pi r^2\,dz)$$

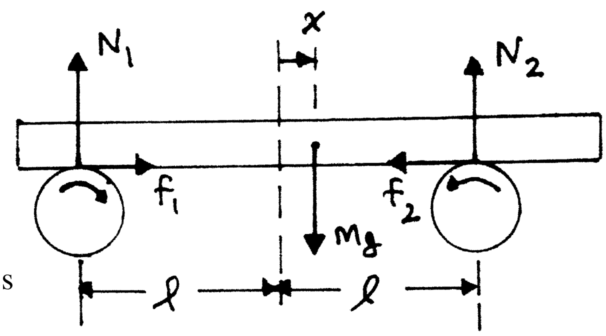
$$= \frac{M}{V}\frac{1}{2}\pi r^4\,dz \implies I = \frac{M}{V}\int \frac{1}{2}\pi r^4\,dz$$

With $r^2 = R^2 - z^2$,

$$I = \frac{M}{V}\int_{-R}^{R}\frac{1}{2}\pi(R^2 - z^2)^2\,dz = \frac{\pi}{2}\frac{M}{V}\int_{-R}^{R}(R^4 - 2R^2z^2 + z^4)\,dz = \frac{\pi}{2}\frac{MR^5}{V}\left(2 - \frac{4}{3} + \frac{2}{5}\right)$$

$$= \frac{\pi}{2}\frac{MR^5}{V}\left(\frac{16}{15}\right) = \frac{\pi}{2}\frac{MR^5}{(4/3)\pi R^3}\left(\frac{16}{15}\right) = \frac{2}{5}MR^2$$

7.9 Bar and rollers



The vertical and horizontal equations of motion:

$$N_1 + N_2 - Mg = 0 \quad (1)$$

$$M\ddot{x} = f_1 - f_2 \quad (2)$$

The torque is 0; take torques about the center of mass

$$-N_1(l + x) + N_2(l - x) = 0 \quad (3)$$

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The bar is thin, so f_1 and f_2 produce negligible torques. Using Eqs. (1) and (3),

$$N_1 = \frac{1}{2}Mg\left(1 - \frac{x}{l}\right) \qquad N_2 = \frac{1}{2}Mg\left(1 + \frac{x}{l}\right)$$

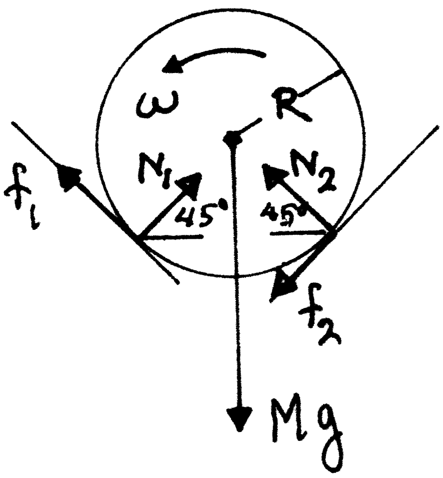
$$f_1 - f_2 = \mu\,(N_1 - N_2) = -\frac{\mu Mg}{l}x$$

so Eq. (2) becomes

$$M\ddot{x} = -\frac{\mu Mg}{l}x \implies \ddot{x} + \frac{\mu g}{l}x = 0$$

This is the equation for SHM, with frequency $\omega = \sqrt{\frac{\mu g}{l}}$

7.10 Cylinder in groove



vertical equation of motion:

$$0 = \frac{N_1}{\sqrt{2}} + \frac{N_2}{\sqrt{2}} + \frac{f_1}{\sqrt{2}} - \frac{f_2}{\sqrt{2}} - Mg \quad (1)$$

horizontal equation of motion:

$$0 = \frac{N_1}{\sqrt{2}} - \frac{N_2}{\sqrt{2}} - \frac{f_1}{\sqrt{2}} - \frac{f_2}{\sqrt{2}} \quad (2)$$

Using the law of friction $f = \mu N$, Eq. (2) gives

$$N_2 = \frac{(1-\mu)}{(1+\mu)}N_1 \quad (3)$$

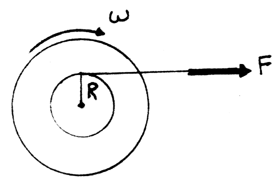
Using Eq. (3) in Eq. (1),

$$N_1 = \frac{Mg}{\sqrt{2}}\left(\frac{1+\mu}{1+\mu^2}\right) \qquad N_2 = \frac{Mg}{\sqrt{2}}\left(\frac{1-\mu}{1+\mu^2}\right)$$

torques about the center of mass:

$$\tau = (f_1 + f_2)R = \mu(N_1 + N_2)R = \sqrt{2}Mg\left(\frac{\mu}{1+\mu^2}\right)R$$

7.11 Wheel and shaft



This problem concerns angular acceleration under constant torque.
Note the exact analogy with linear acceleration under a constant force.

$$\tau = I_o\alpha = I_o\frac{d\omega}{dt}$$

In this system, $\tau = FR$, so $d\omega/dt = FR/I_0$. F is constant, so $\omega = (FR/I_0)t$.

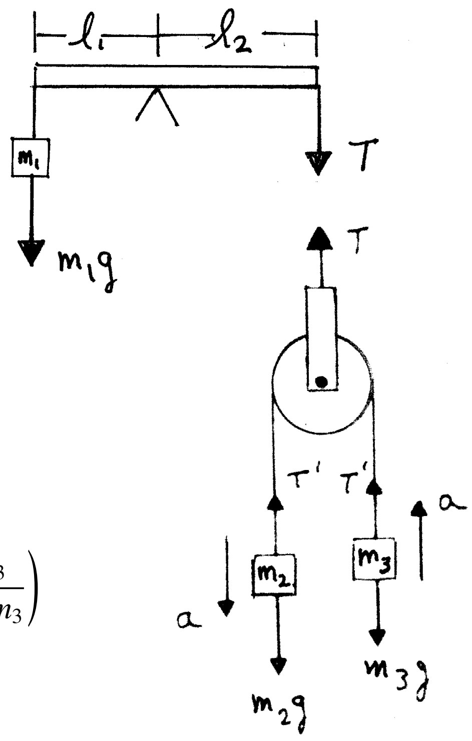
$$\theta = \int \omega\, dt = \frac{1}{2}\left(\frac{FR}{I_0}\right)t^2$$

Let $\omega = \omega_0$ at $t = t_0$, when all the tape of length L has just been unwound.

$$L = R\theta = \frac{1}{2}\left(\frac{FR^2}{I_0}\right)t_0^2 \implies t_0 = \sqrt{\frac{2LI_0}{FR^2}}$$

$$\omega_0 = \frac{FR}{I_0}t_0 = \sqrt{\frac{2LF}{I_0}} \implies I_0 = \frac{2LF}{\omega_0^2}$$

7.12 Beam and Atwood's machine



The beam is in equilibrium, so the total torque = 0. Taking torques about the fulcrum,

$$Tl_2 - m_1gl_1 = 0$$

$$T = m_1g\left(\frac{l_1}{l_2}\right)$$

The equations of motion for m_2 and m_3 are

$$m_2a = m_2g - T' \qquad m_3a = T' - m_3g$$

$$T' = 2g\left(\frac{m_2m_3}{m_2 + m_3}\right)$$

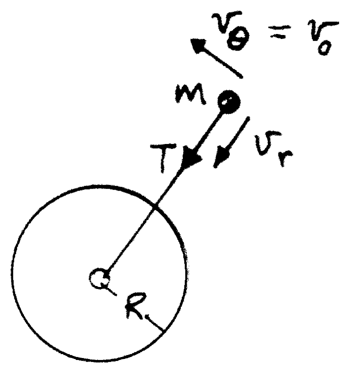
In equilibrium, the pulley does not accelerate.

$$T = 2T' = 4g\left(\frac{m_2m_3}{m_2 + m_3}\right) \implies m_1g\left(\frac{l_1}{l_2}\right) = 4g\left(\frac{m_2m_3}{m_2 + m_3}\right)$$

$$m_1l_1 = 4\left(\frac{m_2m_3}{m_2 + m_3}\right)l_2$$

7.13 Mass and post

(a) Angular momentum about the center of the post is conserved, because the force is radial and cannot exert a torque on m. Initially, m is at distance r from the center, and is moving with tangential velocity v_0, so the initial angular momentum is $L = mv_0r$. At a later time, m is a distance r_f from the center, and has tangential velocity v_f, so that $L = mv_fr_f$.



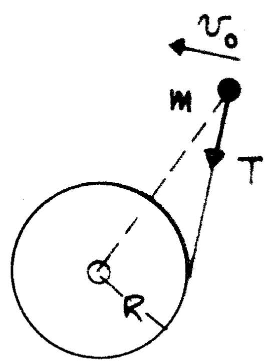
(a)

Momentum p and mechanical energy E are not conserved, because external force is acting on m.

$$p_f = mv_f = p_0\left(\frac{r}{r_f}\right)$$

$$E_f = \frac{1}{2}mv_f^2 = E_0\left(\frac{r^2}{r_f^2}\right)$$

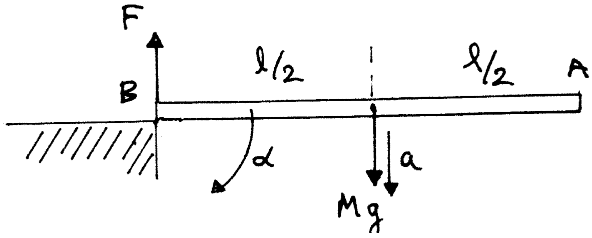
(b) In this case, the force on m is not central, and angular momentum is not conserved. The radius vector $\mathbf{r}$ from the center to m is not perpendicular to $\mathbf{p}$, so $\mathbf{L} = \mathbf{r} \times \mathbf{p} \neq 0$, and furthermore the angle between $\mathbf{r}$ and $\mathbf{p}$ changes during the motion. Momentum is not conserved, because $\int \mathbf{T}\, dt \neq 0$. Mechanical energy is conserved, because the force $\mathbf{T}$ on m is perpendicular to $\mathbf{v}$, and hence does no work. Therefore



(b)

$$\frac{1}{2}mv_f^2 = \frac{1}{2}mv_0^2 \implies v_f = v_0$$

7.14 Stick on table



(a)

$$\tau_B = Mg\left(\frac{l}{2}\right)$$

(b) The moment of inertia of a thin rod of length l about one end is, from Example 7.3, $\frac{1}{3}Ml^2$.

$$\tau_B = I_B\alpha = \frac{1}{3}Ml^2\alpha$$

$$\alpha = \frac{\tau_B}{I_B} = \frac{Mg(l/2)}{(1/3)Ml^2} = \frac{3}{2}\left(\frac{g}{l}\right)$$

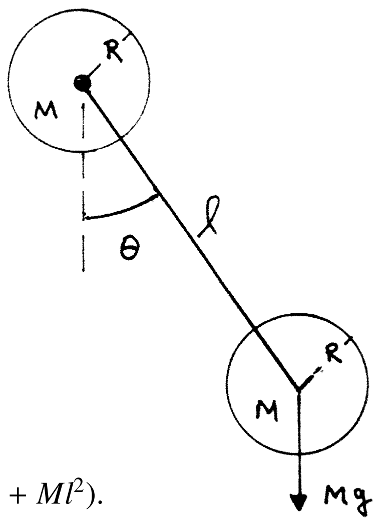
(c)

$$a = \alpha\left(\frac{l}{2}\right) = \frac{3}{4}g$$

(d)

$$Ma = Mg - F \implies F = M(g-a) = \frac{1}{4}Mg$$

7.15 Two-disk pendulum



The upper disk is pivoted about its center.
The torque about the pivot is $\tau_0 = -Mgl\sin\theta$.

equation of motion: $\tau_0 = I_0\alpha$

$$\ddot{\theta} + \frac{Mgl}{I_0}\sin\theta = 0 \implies \ddot{\theta} + \frac{Mgl}{I_0}\theta \approx 0$$

This is the equation for SHM with frequency $\omega = \sqrt{(Mgl)/I_0}$

The moment of inertia of the upper disk about the pivot is $\frac{1}{2}MR^2$.
The moment of inertia of the lower disk about the pivot is $(\frac{1}{2}MR^2 + Ml^2)$.

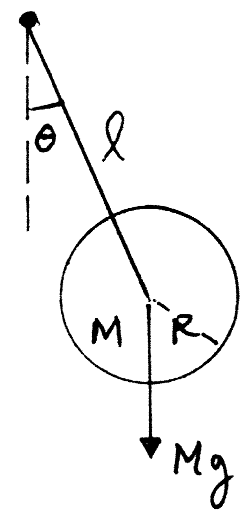
$$I_0 = \frac{1}{2}MR^2 + \left(\frac{1}{2}MR^2 + Ml^2\right) = M(R^2 + l^2) \implies \omega = \sqrt{\frac{gl}{R^2 + l^2}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R^2 + l^2}{gl}}$$

7.16 Disk pendulum

From the result of problem 7.15, the period is

$$\omega = \sqrt{\frac{gl}{I_0}} \implies T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I_0}{gl}}$$



where I_0 is the moment of inertia about the pivot.

$$I_0 = \frac{1}{2}MR^2 + Ml^2 \implies T = 2\pi\sqrt{\frac{\frac{1}{2}R^2 + l^2}{gl}}$$

The minimum period occurs for $dT/dl = 0$.

$$T^2 = \frac{\frac{1}{2}R^2 + l^2}{gl}$$

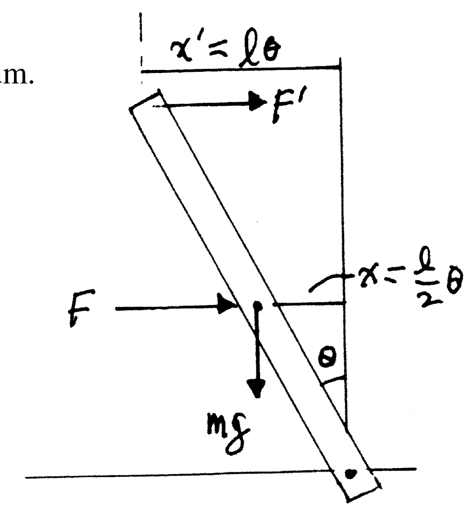
$$0 = 2T\frac{dT}{dl} = 2T\left(\frac{2l}{gl} - \frac{\frac{1}{2}R^2 + l^2}{gl^2}\right)$$

$$2l^2 = \frac{1}{2}R^2 + l^2 \implies l = \frac{R}{\sqrt{2}}$$

The pivot point for minimum period lies within the body of the disk.

7.17 Rod and springs

The sketch shows the rod displaced from equilibrium by angle θ. Both springs act to restore the rod toward equilibrium. Taking torque τ about the pivot,



$$\tau = -F\frac{l}{2} - F'l + mg\frac{l}{2}\sin\theta \approx -F\frac{l}{2} - F'l + mg\frac{l}{2}\theta \quad (1)$$

The spring forces are, in the directions shown,

$$F = kx = k\frac{l}{2}\theta \qquad F' = kx' = kl\theta$$

$$\tau \approx \left[-k\left(\frac{l}{2}\right)^2 - kl^2 + mg\left(\frac{l}{2}\right)\right]\theta$$

The equation of motion is $\tau = I_0\ddot{\theta}$.

$$\ddot{\theta} \approx \left[-k\left(\frac{l^2}{4}\right) - kl^2 + mg\left(\frac{l}{2}\right)\right]\frac{\theta}{I_0}$$

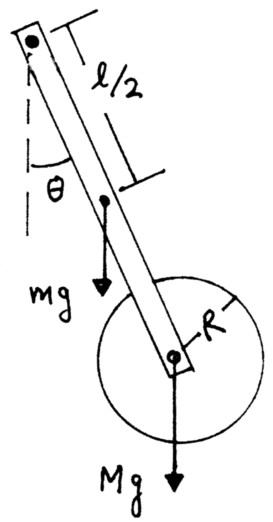
continued next page ⟹

The rod moves according to SHM with frequency ω.

$$\omega = \sqrt{\frac{k\frac{5l^2}{4} - mg\frac{l}{2}}{I_0}} = \sqrt{\frac{k\frac{5l^2}{4} - mg\frac{l}{2}}{\frac{1}{3}ml^2}} = \sqrt{\frac{15}{4}\frac{k}{m} - \frac{3}{2}\frac{g}{l}}$$

As k is decreased, ω also decreases, and finally becomes 0. At this point, the system is no longer stable, and the motion ceases to be harmonic.

7.18 Rod and disk pendulum



The torque τ about the pivot is

$$\tau = \left(-mg\frac{l}{2} - Mgl\right)\sin\theta \approx \left(-mg\frac{l}{2} - Mgl\right)\theta$$

The equation of motion is $\tau = I_{total}\ddot{\theta}$.

$$0 = \ddot{\theta} + \frac{\left(mg\frac{l}{2} + Mgl\right)}{I_{total}}\theta$$

The system moves according to SHM with frequency ω and period T.

$$\omega = \sqrt{\frac{mg\frac{l}{2} + Mgl}{I_{total}}} \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I_{total}}{mg\frac{l}{2} + Mgl}}$$

The moment of inertia I_{total} about the pivot is

$$I_{total} = I_{rod} + I_{disk} = \frac{1}{3}ml^2 + \frac{1}{2}MR^2 + Ml^2$$

$$T = 2\pi\sqrt{\frac{I_{total}}{mg\frac{1}{2} + Mgl}} = 2\pi\sqrt{\frac{\frac{1}{3}ml^2 + \frac{1}{2}MR^2 + Ml^2}{mg\frac{l}{2} + Mgl}} \qquad (1)$$

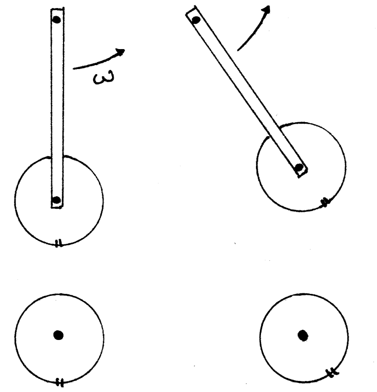
To help understand how the disk's mounting affects the period, consider the contributions to the total angular momentum of the system, using $L_{total} = I_{total}\omega$.

$$L_{total} = \frac{1}{3}ml^2\omega + Ml^2\omega + \frac{1}{2}MR^2\omega$$

The first term on the right is the angular momentum of the rod. The second term is the angular momentum of a mass M concentrated at the end of the rod. The third term is the angular momentum of the disk. All rotate with angular momentum ω.

continued next page $\Longrightarrow$

When the disk is rigidly attached to the rod, the disk rotates, as indicated for the isolated disk in the sketch by the mark on the rim for the isolated disk. The period is T, Eq. (1). When the disk is mounted by a frictionless bearing, it cannot rotate and contributes no rotational angular momentum. The term $\frac{1}{2}MR^2$ should then be omitted from Eq. (1) to give the new period T':



$$T' = 2\pi\sqrt{\frac{\frac{1}{3}ml^2 + Ml^2}{mg\frac{l}{2} + Mgl}} = 2\pi\sqrt{\frac{\frac{2}{3}ml + 2Ml}{mg + 2Mg}}$$

7.19 Disk and coil spring

(a) The restoring torque is $\tau = -C\theta$, so the equation of motion is $I\ddot{\theta} + C\theta = 0$, the equation for SHM. The initial moment of inertia is $\frac{1}{2}MR^2$, so

$$0 = \ddot{\theta} + \frac{C}{I}\theta = \ddot{\theta} + \frac{C}{\frac{1}{2}MR^2}\theta \implies \omega = \sqrt{2C/MR^2}$$

(b) (1) The moment of inertia of the putty ring is MR^2, so the moment of inertia of the disk and putty is now $\frac{3}{2}MR^2$, and the new frequency ω' is

$$\omega' = \sqrt{\frac{2C}{3MR^2}} = \frac{\omega}{\sqrt{3}}$$

(2) At time t_1, just before the putty is dropped,

$$\theta = \theta_0 \sin\omega t_1 = \theta_0 \sin\pi = 0 \qquad \dot{\theta} = \omega\,\theta_0 \cos\omega t_i = \omega\,\theta_0 \cos\pi = -\omega\,\theta_0$$

Immediately before the putty is dropped, the angular momentum L is

$$L = I\omega = \frac{1}{2}MR^2\omega\,\theta_0$$

The amplitude of the initial motion is $\omega\,\theta_0$. The putty ring has angular momentum = 0 before it is dropped. Hence the angular momentum of the system is conserved. The new angular momentum L' equals the initial L.

$$L' = I'\omega' = \frac{3}{2}MR^2\,\omega'\,\theta_0'$$

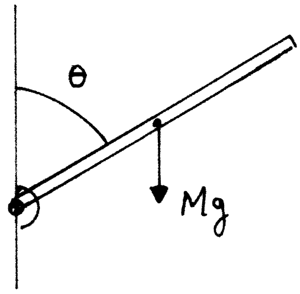
By conservation of momentum,

$$\frac{1}{2}MR^2\,\omega\,\theta_0 = \frac{3}{2}MR^2\,\omega'\,\theta_0' = \frac{3}{2}MR^2\frac{\omega}{\sqrt{3}}\theta_0'$$

so the new amplitude is

$$\theta_0' = \frac{\theta_0}{\sqrt{3}}$$

7.20 Falling plank



For $\theta = 90°$, the torque τ about the pivot is

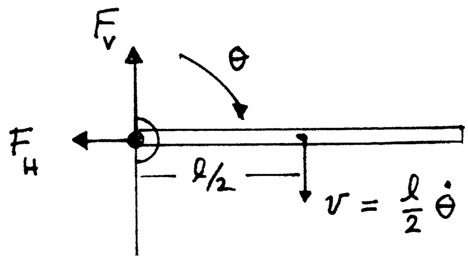
$$\tau = Mg\frac{l}{2} = I\ddot{\theta}(90°)$$

$$\ddot{\theta}(90°) = \left(\frac{Mg\frac{l}{2}}{I}\right) = \left(\frac{Mg\frac{l}{2}}{\frac{1}{3}Ml^2}\right) = \frac{3}{2}\frac{g}{l}$$

tangential equation of motion:

$$Mg - F_V = M\frac{l}{2}\ddot{\theta}(90°) = \frac{3}{4}Mg \implies F_V = \frac{1}{4}Mg$$

radial equation of motion (at 90°*)*:



$$F_H = \frac{Mv^2}{l/2} = M\frac{l}{2}\dot{\theta}^2(90°)$$

Use conservation of mechanical energy to find $\dot{\theta}^2(90°)$. Note that F_V and F_H do no work on the system, because the displacement is 0. Take the gravitational potential energy to be 0 at $\theta = 90°$. Initially, at $\theta = 60°$, the kinetic energy is 0.

$$E(60°) = 0 + Mg\frac{l}{2}\sin 30° = Mg\frac{l}{4}$$

$$E(90°) = \frac{1}{2}I\dot{\theta}^2(90°) \qquad E(90°) = E(60°)$$

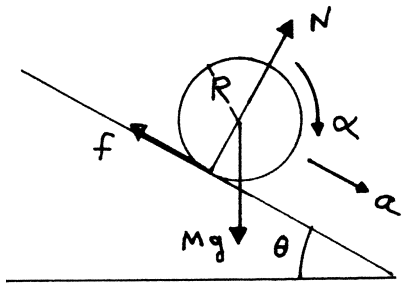
$$\frac{1}{2}I\dot{\theta}^2(90°) = Mg\frac{l}{4} \implies \dot{\theta}^2(90°) = \frac{Mg\frac{l}{2}}{\frac{1}{3}Ml^2} = \frac{3}{2}\frac{g}{l}$$

$$F_H = \frac{3}{4}Mg$$

7.21 Rolling cylinder

Take torques about the center of the cylinder. Only the friction force f contributes.

$$fR = I\alpha = \frac{1}{2}MR^2\alpha$$



For rolling without slipping, $a = R\alpha$, so

$$f = \frac{1}{2}MR\alpha = \frac{1}{2}Ma$$

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equation of motion normal to the plane: $N - Mg\cos\theta = 0$

equation of motion along the plane: $Mg\sin\theta - f = Ma$

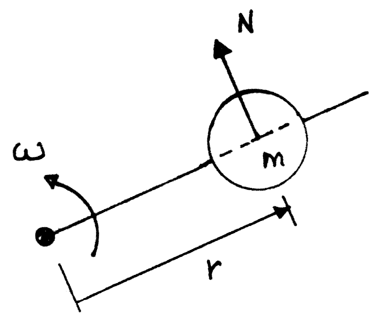
$Mg\sin\theta = Ma + f = \frac{3}{2}Ma \implies g\sin\theta = \frac{3}{2}a$

The strength of f is limited.

$$f \le \mu N = \mu Mg\cos\theta \implies \mu g\cos\theta \ge \frac{1}{2}a = \frac{1}{3}g\sin\theta$$

The condition for rolling without slipping is therefore $\tan\theta \le 3\mu$.

7.22 Bead and rod



(a) *radial equation of motion (ω = constant)*:

$0 = ma_r = m\ddot{r} - mr\omega^2 \implies \ddot{r} - r\omega^2 = 0 \quad (1)$

$r = r_0\, e^{\omega t}$ satisfies Eq. (1), with $r(0) = r_0$. Proof:

$\dot{r} = \omega\, r_0\, e^{\omega t} \qquad \ddot{r} = \omega^2\, r_0\, e^{\omega t} = \omega^2 r$

(b) By Newton's Third Law, the force on the rod is equal and opposite to N, the force on the bead.

tangential equation of motion:

$N = ma_\theta = m(2\dot{r}\omega) = 2m\omega(r_0\,\omega e^{\omega t}) = 2m\omega^2 r_0\, e^{\omega t}$

(c) The power $\mathcal{P} = Fv = \tau\,\omega$. From parts (a) and (b),

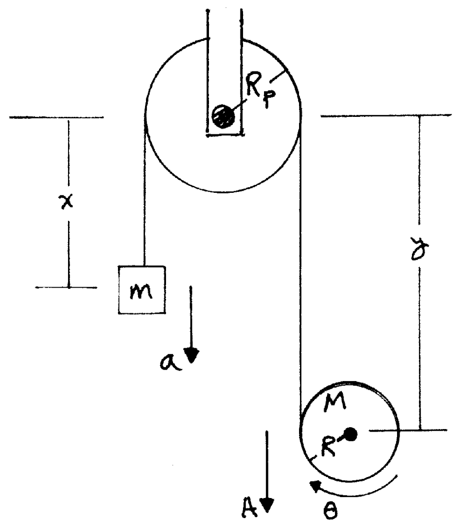
$\mathcal{P} = \tau\,\omega = (Nr)\omega = 2m\omega^3 r_0^2\, e^{2\omega t}$

The kinetic energy K of the bead is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(v_r^2 + v_\theta^2) = \frac{1}{2}m\left(\dot{r}^2 + (r\omega)^2\right) = mr_0^2\,\omega^2 e^{2\omega t}$$

$$\frac{dK}{dt} = 2mr_0^2\omega^3 e^{2\omega t} = \mathcal{P}$$

7.23 Disk, mass, and tape



The tape slips over the stationary pulley.
The length of the tape $x + y + \pi R_p$ increases by $R\theta$ as the tape unwinds from the disk.

(a) *constraint*:

$$x + y = constant + R\theta \implies \ddot{x} + \ddot{y} = R\ddot{\theta}$$

$$R\alpha = a + A$$

(b) *translational equations of motion*:

$$mg - T = ma \qquad Mg - T = MA$$

rotational equation of motion:

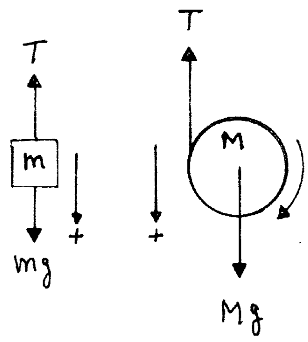
$$TR = I_0\alpha = \frac{1}{2}MR^2\alpha$$

$$T = \frac{1}{2}MR\alpha = \frac{1}{2}M(a + A)$$

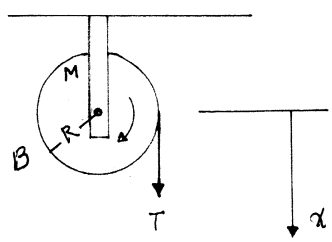
$$a = \left(\frac{3m - M}{3m + M}\right)g$$

$$A = \left(\frac{m + M}{3m + M}\right)g$$

$$R\alpha = \left(\frac{4m}{3m + M}\right)g$$



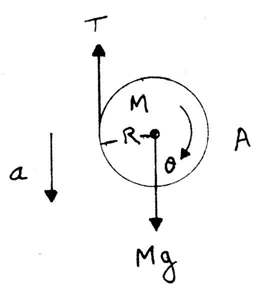
7.24 Two drums



The drums are identical, and they experience the same torque TR. Starting from rest, they consequently rotate through the same angle θ, causing the tape to lengthen by $2R\theta$. Let l_0 be the initial length of the tape.
constraint:

$$x = l_0 + 2R\theta \implies \ddot{x} = a = 2R\alpha$$

equations of motion for drum A:



$$Mg - T = Ma \qquad TR = \frac{1}{2}MR^2\alpha$$

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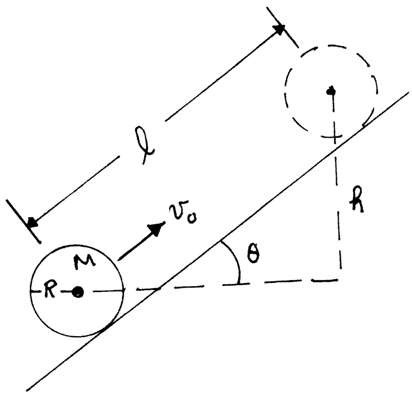
equation of motion for drum B:

$$TR = \frac{1}{2}MR^2\alpha$$
$$T = \frac{1}{2}MR\alpha = \frac{1}{4}Ma$$
$$\frac{1}{4}Ma = Mg - Ma \implies a = \frac{4}{5}g$$

7.25 Rolling marble



Energy methods are a good approach to this problem.

$$E_i = \frac{1}{2}Mv_0^2 + \frac{1}{2}I\dot{\theta}_0^2 = \frac{1}{2}Mv_0^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\dot{\theta}_0^2$$

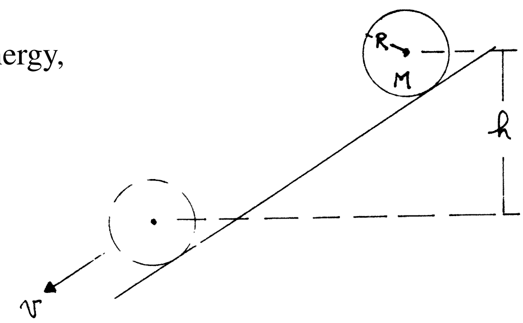
For rolling without slipping, $R\dot{\theta}_0 = v_0$,

$$E_i = \frac{1}{2}Mv_0^2 + \frac{1}{5}Mv_0^2 = \frac{7}{10}Mv_0^2$$

The marble is momentarily stationary at the final height.

$$E_f = 0 + Mgh = Mgl\sin\theta = E_i = \frac{7}{10}Mv_0^2$$
$$l = \frac{7}{10}\left(\frac{Mv_0^2}{Mg\sin\theta}\right) = \frac{7}{10}\left(\frac{v_0^2}{g\sin\theta}\right)$$

7.26 Sphere and cylinder



Energy methods are a good approach to this problem.
At the start, the object has only gravitational potential energy, and it gains kinetic energy as it rolls down the plane.

$$E_f = \frac{1}{2}Mv^2 + \frac{1}{2}I\dot{\theta}_0^2 = E_i = Mgh$$

For rolling without slipping, $R\dot{\theta} = v$.

$$E_f = \frac{1}{2}Mv^2 + \frac{1}{2}I\left(\frac{v^2}{R^2}\right) = \left(\frac{1}{2}M + \frac{1}{2}\frac{I}{R^2}\right)v^2$$
$$v^2 = \left(\frac{Mgh}{\frac{1}{2}M + \frac{1}{2}\frac{I}{R^2}}\right) \quad (1)$$

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From Eq. (1), the sphere is faster because it has the smaller moment of inertia.

$$I_{sphere} = \frac{2}{5}MR^2 < I_{cylinder} = \frac{1}{2}MR^2$$

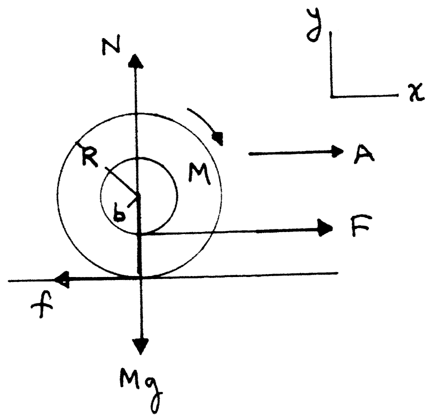
Equation (1) can be written more elegantly using the radius of gyration k (Section 7.7.2). The radius of gyration is given by $I = k^2MR^2$. Writing Eq. (1) in terms of k,

$$v^2 = \left(\frac{Mgh}{\frac{1}{2}M + \frac{1}{2}\frac{I}{R^2}}\right) = \frac{2gh}{1 + k^2}$$

$$k^2{}_{sphere} = \frac{2}{5} < k^2{}_{cylinder} = \frac{1}{2}$$

The sphere is faster, because it has the smaller k.

7.27 Yo-yo on table



translational equation of motion, x:

$$F - f = M\ddot{x} = MA$$

translational equation of motion, y:

$$N - Mg = 0 \implies N = Mg$$

rotational equation of motion:

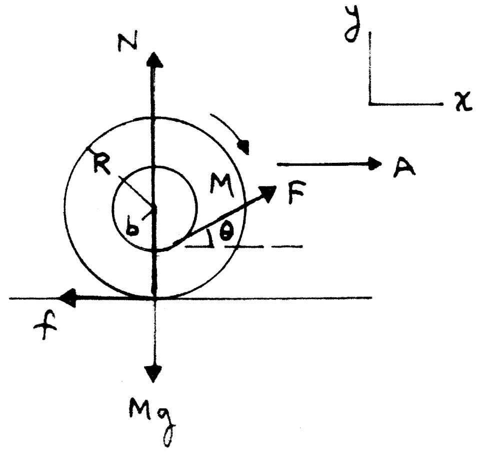
$$Rf - bF = I\ddot{\theta} = \frac{1}{2}MR^2\alpha = \frac{1}{2}MRA$$

Using $A = R\alpha$ for rolling without slipping.

$$f = \frac{1}{3}F\left(1 + \frac{2b}{R}\right) \leq \mu N = \mu Mg$$

$$F_{max} = \frac{3\mu Mg}{1 + \frac{2b}{R}}$$

7.28 Yo-yo pulled at angle



The force F has a fixed value.

equation of motion, y:

$$0 = N + F\sin\theta - Mg \implies N = Mg - F\sin\theta$$

The torque = 0 when

$$0 = Rf - bF \implies f = \frac{bF}{R}$$

$$f \le \mu N = \mu(Mg - F\sin\theta)$$

Using this in Eq. (2),

$$\frac{bF}{R} \le \mu(Mg - F\sin\theta)$$

$$\sin\theta \le \frac{Mg}{F} - \frac{b}{\mu R}$$

$$\sin\theta_{max} = \frac{Mg}{F} - \frac{b}{\mu R} \quad (1)$$

Comments: According to Eq. (1), $|\sin\theta| > 1$ for $\mu \to 0$, an unacceptable result. However, the solution requires that the Yo-yo be on the verge of slipping. For small μ, the Yo-yo always slips. Similarly, the result is unacceptable if $F \to 0$, but for small F, the Yo-yo never slips.

7.29 Yo-yo motion

(a) The equations of motion, either descending or ascending, are, with $A = b\alpha$,

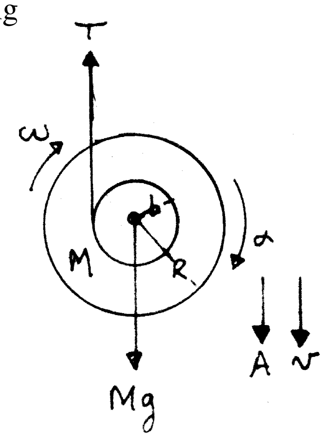
$$Mg - T = MA \quad (1)$$

$$bT = \frac{1}{2}MR^2\alpha = \frac{1}{2}MR^2\frac{A}{b}$$

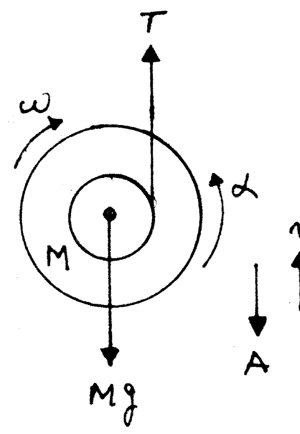
$$MA = \frac{2b^2}{R^2}T$$

$$Mg = T + MA = T + \frac{2b^2}{R^2}T$$

$$T = \left(\frac{MgR^2}{2b^2 + R^2}\right)$$



descending



ascending

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(b) When the Yo-yo reverses direction at the end of the string, the speed changes from v_f downward to v_f upward. The change in momentum is $2Mv_f$ = the impulse $\mathcal{I} = \bar{F}_{string}\Delta t$, where $\bar{F}_{string}$ is the average force exerted by the string and Δt is the time to reverse direction. The Yo-yo is turning at rate ω and it makes a half turn during the reversal, so that $\omega\Delta t = \pi$. Hence

$$\bar{F}_{string} = \frac{2Mv_f}{\Delta t} = \frac{2Mv_f\,\omega}{\pi} = \frac{2M(b\omega)\omega}{\pi} = \frac{2Mb\omega^2}{\pi}$$

Use conservation of mechanical energy to find ω.

$$E_f = \frac{1}{2}Mv_f^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mb^2\omega^2 + \frac{1}{4}MR^2\omega^2 = E_i = Mgh$$

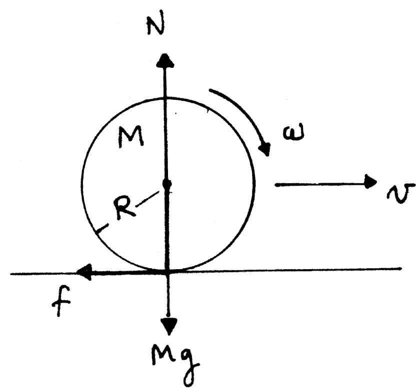
$$Mgh = \frac{1}{4}M\left(2b^2 + R^2\right)\omega^2 \implies \omega^2 = \frac{4gh}{2b^2 + R^2}$$

$$\bar{F}_{string} = \frac{8Mg}{\pi}\left(\frac{bh}{2b^2 + R^2}\right)$$

7.30 Sliding and rolling bowling ball

Method 1 uses the equations of motion, and method 2 uses conservation of angular momentum.

Method 1: *equations of motion*:



$$N = Mg \qquad f = \mu N = \mu Mg$$

$$M\frac{dv}{dt} = -f = -\mu Mg \implies v = v_0 - \mu gt$$

$$Rf = I\frac{d\omega}{dt} \implies \frac{d\omega}{dt} = \frac{Rf}{I} = \frac{\mu MgR}{I}$$

$$= \frac{\mu MgR}{\frac{2}{5}MR^2} \implies \omega = \left(\frac{5\mu g}{2R}\right)t$$

Note that v decreases with time, and ω increases with time. Rolling begins at t_r when $\omega(t_r) = v(t_r)/R$.

$$\omega(t_r) = \frac{5}{2}\frac{\mu g}{R}t_r \implies v(t_r) = \frac{5}{2}\mu gt_r = v_0 - \mu gt_r$$

$$\frac{5}{2}\mu gt_r = v_0 - \mu gt_r \implies t_r = \frac{2}{7}\frac{v_0}{\mu g} \implies v(t_r) = v_0 - \frac{2}{7}v_0 = \frac{5}{7}v_0$$

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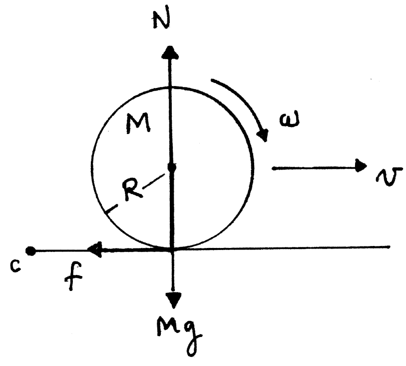
Method 2: conservation of angular momentum:
Take angular momentum about any point on the alley, such as c in the sketch.



$$L_i = Mv_0R \qquad L_f = MvR + I\omega = MvR + \frac{2}{5}MR^2\omega$$

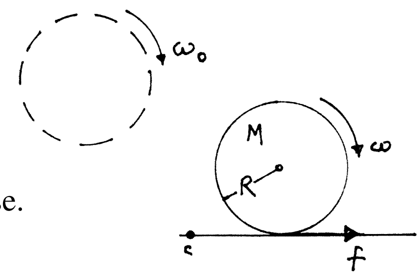
Rolling starts at $t = t_r$ when $\omega(t_r) = v(t_r)/R$.

$$L_f = \left(MR + \frac{2}{5}MR\right)v(t_r) = \frac{7}{5}MRv(t_r) = L_i = Mv_0R$$

$$v(t_r) = \frac{5}{7}v_0$$

7.31 Skidding and rolling cylinder

The approach to this problem uses conservation of angular momentum, similar to Method 2 in problem 7.30.
The friction force f causes the translational speed v to increase and the angular speed ω to decrease.



Taking angular momentum about c in the sketch,

$$L_f = I\omega_f + Mv_fR = L_i = I\omega_0$$

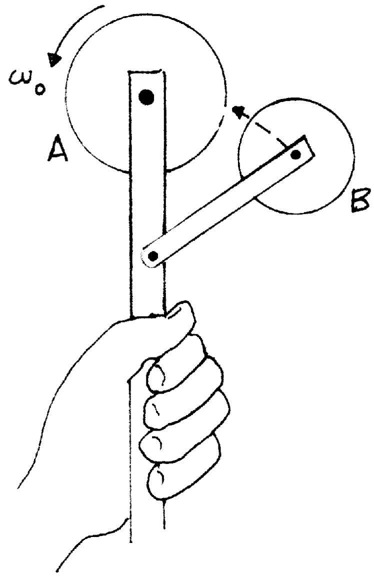
Rolling without skidding begins when $v_f = R\omega_f$.

$$L_f = \frac{1}{2}MR^2\omega_f + MR^2\omega_f = \frac{3}{2}MR^2\omega_f$$

$$L_f = L_i = \frac{1}{2}MR^2\omega_0$$

$$\omega_f = \frac{\omega_0}{3}$$

7.32 Two rubber wheels



As suggested by the lower sketch, the friction force on each wheel produces a torque. To keep the device from rotating, the hand must apply an opposite torque. Thus the angular momentum of the device is not conserved, so analyze the problem.using equations of motion.

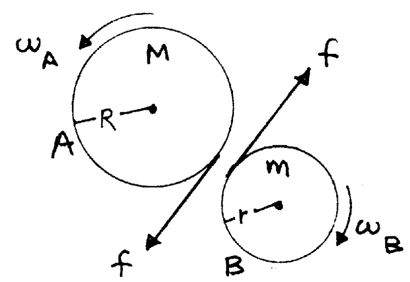
$fR = -I_A\alpha_A \quad (wheel\, A)$

$fr = I_B\alpha_B \quad (wheel\, B)$

$$\omega_A = \omega_0 - \frac{R}{I_A}\int_0^t f\,dt' \qquad \omega_B = \frac{r}{I_B}\int_0^t f\,dt'$$

$$\omega_A = \omega_0 - \frac{R}{I_A}\frac{I_B}{r}\omega_B$$

Sliding continues until the contact points both have the same linear speed $R\omega_A = r\omega_B$. This condition gives

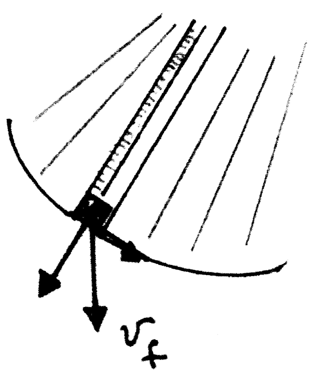


$$\omega_A = \omega_0 - \frac{R^2}{I_A}\frac{I_B}{r^2}\omega_A$$

$I_A = \frac{1}{2}MR^2$ and $I_B = \frac{1}{2}mr^2$. The final angular speed of A is

$$\omega_A = \omega_0 - \frac{m}{M}\omega_A \implies \omega_A = \left(\frac{\omega_0}{1+\frac{m}{M}}\right)$$

7.33 Grooved cone and mass



The initial angular momentum of the rotating cone is along the vertical axis, but there are no external torques, so the angular momentum of the system remains constant in magnitude and direction.

(a) The mass gains angular momentum $mR^2\omega$ as it is carried around with the cone, so the cone must lose angular momentum.

$$L_f = I_0\omega_f + mR^2\omega_f = L_i = I_0\omega_0$$

$$\omega_f = \left(\frac{I_0}{I_0 + mR^2}\right)\omega_0$$

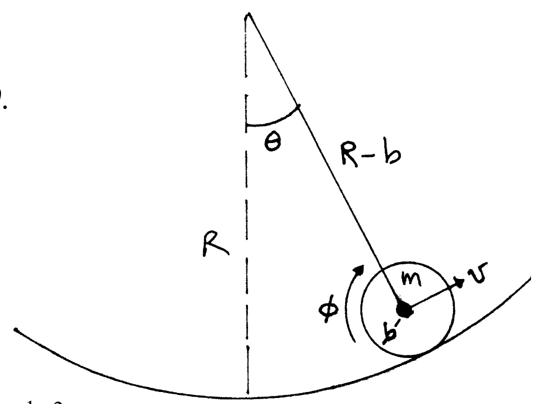
continued next page $\Longrightarrow$

(b) There are no dissipative forces, so mechanical energy is conserved. The mass has gravitational potential energy and kinetic energy, and the cone has rotational kinetic energy.

$$E_f = \frac{1}{2}I_0\omega_f^2 + \frac{1}{2}mv_f^2 = E_i = \frac{1}{2}I_0\omega_0^2 + mgh$$

$$v_f^2 = \frac{I_0}{m}\left(\omega_0^2 - \omega_f^2\right) + 2gh \implies v_f = \sqrt{\frac{I_0\omega_0^2}{m}\left(1 - \frac{I_0}{I_0 + mR^2}\right)^2 + 2gh}$$

7.34 Marble in dish



Energy methods are a good way to solve this problem. One element of this approach is to express all the energy contributions in terms of a single variable, here θ. The marble has gravitational potential energy E_{pot}, translational kinetic energy E_{trans}, and rotational kinetic energy E_{rot}. Take the gravitational potential energy of the marble to be 0 at the bottom of the dish ($\theta = 0$).

$$E_{pot} = mg(R - b)(1 - \cos\theta)$$

Taking $R \gg b$ (the sketch exaggerates b), and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$

$$E_{pot} \approx mgR(1 - \cos\theta) \approx \frac{1}{2}mgR\theta^2$$

$$E_{trans} = \frac{1}{2}mv^2 = \frac{1}{2}m(R\omega)^2 = \frac{1}{2}mR^2\dot{\theta}^2$$

$$E_{rot} = \frac{1}{2}I\dot{\phi}^2 = \frac{1}{2}I\left(\frac{R\dot{\theta}}{b}\right)^2 = \frac{1}{2}I\frac{R^2}{b^2}\dot{\theta}^2$$

$$E_{tot} = E_{pot} + E_{trans} + E_{rot} = \frac{1}{2}mgR\theta^2 + \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}I\frac{R^2}{b^2}\dot{\theta}^2$$

$$= \frac{1}{2}mgR\theta^2 + \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}\left(\frac{2}{5}mb^2\right)\frac{R^2}{b^2}\dot{\theta}^2 = \frac{1}{2}mgR\theta^2 + \frac{1}{2}mR^2\left(1 + \frac{2}{5}\right)\dot{\theta}^2 = \frac{mR}{2}\left(g\theta^2 + \frac{7}{5}R\dot{\theta}^2\right) \quad (1)$$

There are several ways to find the angular frequency of small oscillations from Eq. (1). One way is to note the analogy between Eq. (1) and the mechanical energy of a harmonic oscillator $\frac{1}{2}m'\dot{x}^2 + \frac{1}{2}k'x^2$, which has oscillation frequency $\sqrt{k'/m'}$. From Eq. (1), $k' \sim g$ and $m' \sim \frac{7}{5}R$, so the corresponding angular frequency ω_m of the marble is

$$\omega_m = \sqrt{\frac{5}{7}\frac{g}{R}}$$

continued next page $\Longrightarrow$

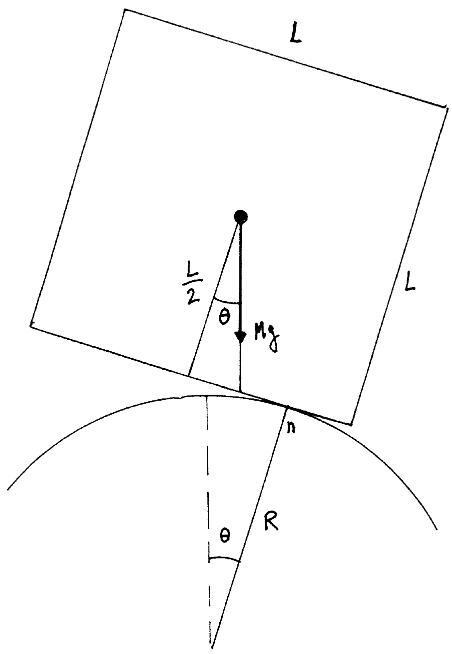
Another approach is to note that Eq. (1) is a quadratic energy form (Sec. 6.2.1), where it is shown that $\omega = \sqrt{A/B}$ for a quadratic energy form $\frac{1}{2}B\dot{q}^2 + \frac{1}{2}Aq^2$, so that

$$\omega_m = \sqrt{\frac{5}{7}\frac{g}{R}}$$

Finally, by conservation of mechanical energy, E_{tot} is constant, so taking the derivative of Eq. (1) with respect to θ gives

$$0 = \frac{dE_{tot}}{d\theta} = \frac{7}{5}R\dot{\theta}\ddot{\theta} + g\theta\dot{\theta} = \frac{7}{5}R\ddot{\theta} + g\theta \implies \omega_m = \sqrt{\frac{5}{7}\frac{g}{R}}$$

7.35 Cube and drum



The cube is rocking, not sliding, on the drum. At the instant shown in the sketch, the pivot point is n. Because the cube is always tangential to the drum, the angle subtended from the center of the drum is equal to the cube's angle of rotation. The cube is stable if the torque created by a small rotation is a negative "restoring" torque tending to bring the cube back to equilibrium. The horizontal displacement of the center of mass must be less than the displacement of n.

The displacement a of the center of mass is

$$a = \frac{L}{2}\sin\theta$$

The displacement b of the contact point is

$$b = R\sin\theta$$

For stability,

$$a < b \implies L < 2R$$

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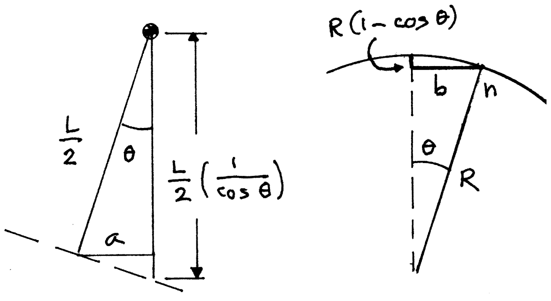
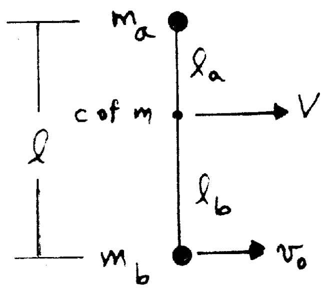
The cube in the sketch is stable. Put descriptively, the cube is stable if the line of the weight vector falls within the $R\theta$ of the cylinder's center,



7.36 Two twirling masses

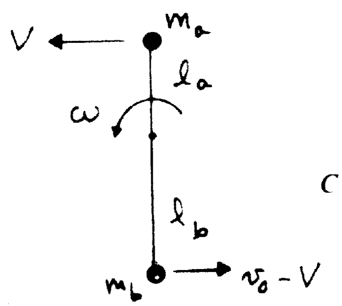


L system

No external forces act horizontally, so the system is isolated in the horizontal plane. The motion is described most naturally as a combination of a uniform translation of the center of mass and a uniform rotation about the center of mass. The speed V of the center of mass is

$$V = \frac{m_a v_a(0) + m_b v_b(0)}{m_a + m_b} = \left(\frac{m_b}{m_a + m_b}\right) v_0$$

$$l_a = \left(\frac{m_b}{m_a + m_b}\right) l \qquad l_b = \left(\frac{m_a}{m_a + m_b}\right) l$$

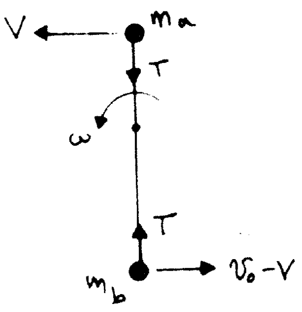


C system

The sketches show the speeds in the C system.

$$\omega = \frac{V}{l_a} = \left(\frac{m_b}{m_a + m_b}\right)\left(\frac{m_a + m_b}{m_b}\right)\frac{v_0}{l} = \frac{v_0}{l}$$

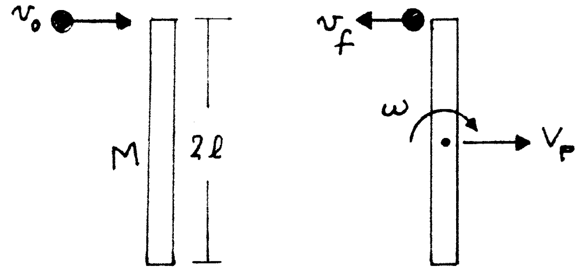
$$\begin{aligned} T &= m_a l_a \omega^2 \\ &= \left(\frac{m_a m_b}{m_a + m_b}\right) l \left(\frac{v_0}{l}\right)^2 \\ &= \left(\frac{m_a m_b}{m_a + m_b}\right)\left(\frac{v_0^2}{l}\right) \end{aligned}$$



C system

7.37 Plank and ball

(a) The system is isolated in the horizontal plane. Linear momentum, angular momentum, and mechanical energy are conserved.



(a)

linear momentum:

$$mv_0 = -mv_f + MV_p$$

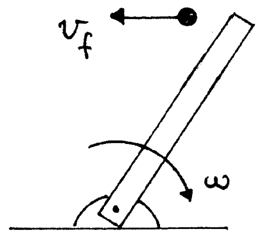
angular momentum:

$$mv_0 l = -mv_f l + I_0\omega$$

mechanical energy:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}MV_p^2 + \frac{1}{2}I_0\omega^2$$

I_0 about the plank's center is $I_0 = \frac{1}{12}M(2l)^2 = \frac{1}{3}Ml^2$.



(b)

Solving for the three unknowns v_f, V_p, and ω,

$$V_p = \frac{m}{M}(v_0 + v_f) \qquad \omega = \frac{ml(v_0 + v_f)}{I_0}$$

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}\frac{m^2}{M}(v_0 + v_f)^2 + \frac{1}{2}\frac{m^2l^2}{I_0}(v_0 + v_f)^2$$

$$v_0^2 - v_f^2 = \frac{4m}{M}(v_0 + v_f)^2$$

$$0 = \left(1 + \frac{4m}{M}\right)v_f^2 + \left(\frac{8mv_0}{M}\right)v_f - \left(1 - \frac{4m}{M}\right)v_0^2$$

$$v_f = \left(\frac{1 - \frac{4m}{M}}{1 + \frac{4m}{M}}\right)v_0$$

(b) Because of the forces at the pivot, linear momentum is not conserved, but angular momentum and mechanical energy are conserved.

angular momentum about the pivot:

$$mv_0(2l) = -mv_f(2l) + I_p\omega \implies \omega = \frac{2ml(v_0 + v_f)}{I_p}$$

$$I_p = \frac{1}{3}M(2l)^2 = \frac{4}{3}Ml^2$$

continued next page $\Longrightarrow$

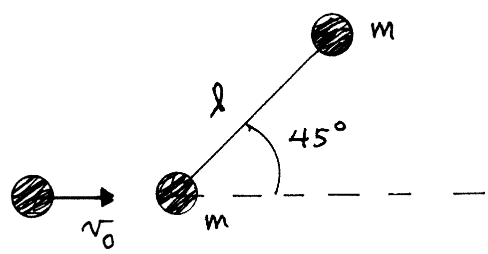
mechanical energy:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}I_p\omega^2$$

$$0 = \left(1 - \frac{3m}{M}\right)v_f^2 + \left(\frac{6mv_0}{M}\right)v_f - \left(1 - \frac{3m}{M}\right)v_0^2 \implies v_f = \left(\frac{1 - \frac{3m}{M}}{1 + \frac{3m}{M}}\right)v_0$$

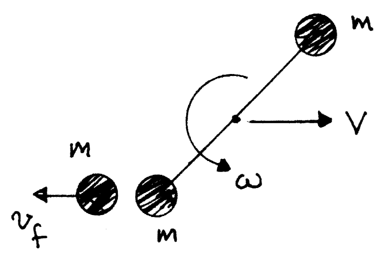
7.38 Collision on a table

The system is isolated in the horizontal plane. Linear momentum, angular momentum, and mechanical energy are conserved.



linear momentum:

$$mv_0 = -mv_f + 2mV$$
$$V = \frac{1}{2}(v_0 + v_f)$$



angular momentum about the rod's midpoint:

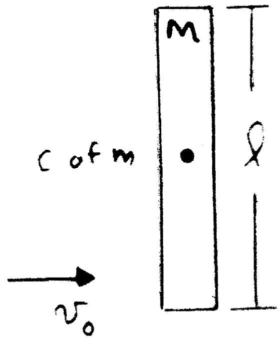
$$mv_0 l\sin(45°) = -mv_f l\sin(45°) + I_0\omega$$
$$v_0 + v_f = \sqrt{2}\frac{I_0}{ml}\omega$$
$$I_0 = 2m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{2}$$
$$v_0 + v_f = \frac{l}{\omega}\sqrt{2}$$

mechanical energy:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}(2m)V^2 + \frac{1}{2}I_0\omega^2$$
$$v_0^2 = v_f^2 + 2V^2 + \frac{l^2\omega^2}{2}$$
$$= \left(\frac{l\omega}{\sqrt{2}} - v_0\right)^2 + \frac{l^2\omega^2}{4} + \frac{l^2\omega^2}{2}$$
$$0 = \frac{5}{4}l\omega - \sqrt{2}v_0$$
$$\omega = \left(\frac{4\sqrt{2}}{5}\right)\frac{v_0}{l}$$

7.39 Child on ice with plank

(a) Linear momentum and angular momentum are conserved in the inelastic collision, but mechanical energy is not. After the inelastic collision, the new center of mass of the child-plank system translates with velocity **v** parallel to the child's initial velocity, by conservation of linear momentum. The system rotates about the new center of mass with angular frequency ω.



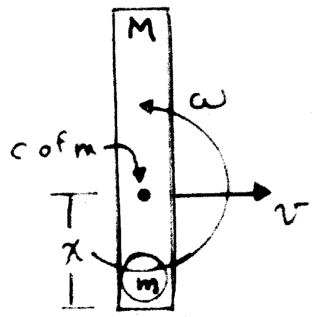
The speed v of the new center of mass is given by

$$v = \left(\frac{m}{m+M}\right) v_0$$

Angular momentum is best calculated about the new center of mass, which is located a distance x from the end of the plank, as indicated in the sketch.

$$x = \left(\frac{M}{m+M}\right)\frac{l}{2}$$

The initial angular momentum about x is $L_i = mv_0x$



(a)

The final angular momentum is $L_f = I_x\omega$.

$$\begin{aligned}
I_x &= I_0 + M\left[\frac{l}{2} - x\right]^2 + mx^2 \\
&= \frac{1}{12}Ml^2 + M\left[\frac{l}{2} - x\right]^2 + mx^2 \\
&= \frac{1}{12}Ml^2 + M\left[\frac{l}{2} - \left(\frac{M}{m+M}\right)\frac{l}{2}\right]^2 + m\left(\frac{M}{m+M}\right)^2\left(\frac{l}{2}\right)^2 \\
&= \frac{1}{12}Ml^2 + \frac{1}{4}\left(\frac{mM}{m+M}\right)l^2 \\
&= \frac{1}{12}Ml^2\frac{(M+4m)}{(m+M)}
\end{aligned}$$

$$L_f = I_x\omega = L_i = mv_0x$$

$$\frac{1}{12}Ml^2\left(\frac{M+4m}{m+M}\right)\omega = mv_0x = mv_0\left(\frac{M}{m+M}\right)\frac{l}{2}$$

$$\omega = 6\left(\frac{m}{M+4m}\right)\frac{v_0}{l}$$

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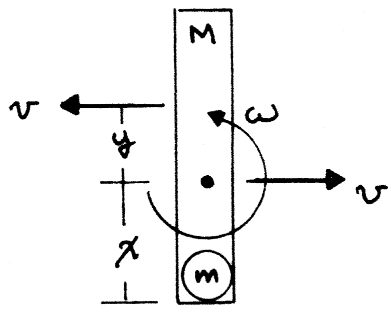
(b) The point instantaneously at rest is moving with speed v (due to the rotation) antiparallel to the translation of the center of mass, as shown in the sketch, so that its instantaneous speed is $v - v = 0$. Let the point be a distance y above the center of mass, so $v = y\omega$.



(b)

$$y = \frac{v}{\omega} = \frac{\left(\frac{mv_0}{m+M}\right)}{\left(\frac{6v_0}{l}\right)\left(\frac{m}{M+4m}\right)} = \frac{l}{6}\left(\frac{M+4m}{M+m}\right)$$

The distance from the child is $y + x$.

$$y + x = \frac{l}{6}\left(\frac{M+4m}{M+m}\right) + \frac{l}{2}\left(\frac{M}{M+m}\right) = \frac{l}{6}\left(\frac{M+4m+3M}{M+m}\right) = \frac{2}{3}l$$

7.40 Toothed wheel and spring

The initial mechanical energy is $\frac{1}{2}kb^2$. Mechanical energy is conserved until $x = l$, but at that point there is an inelastic collision with the track and some mechanical energy is dissipated. Linear momentum is also not conserved in the collision, because of the force exerted at the point of impact.

The force at the point of impact exerts no torque about that point, but the wheel has angular momentum of translation. At $x = l$, the spring force is 0, so the spring exerts no torque. Hence, angular momentum is conserved in the collision. When the wheel engages the toothed track, it is constrained to move with $v = R\omega$.

(a) Before the sliding wheel collides with the track, it is not rotating about its center and therefore has zero angular momentum of rotation. However, it has angular momentum MvR from translation of the axis (refer to Note 7.2 in the text for a summary). Sketch (a) shows the system at the start, and at the instant just before the collision with the track.

Immediately after the collision, the wheel rotates at rate ω_f about the contact point. The moment of inertia about the center of mass (the wheel's center) is $I_0 = MR^2$, and the moment of inertia I_p about the contact point is $I_p = I_0 + MR^2 = 2MR^2$. Just before and after the collision,

$$L_i = MRv_i \qquad E_i = \frac{1}{2}Mv_i^2 = \frac{L_i^2}{2MR^2}$$

$$L_f = 2MR^2\omega_f \qquad E_f = \frac{1}{2}I_p\omega_f^2 = MR^2\omega_f^2 = \frac{L_f^2}{4MR^2}$$

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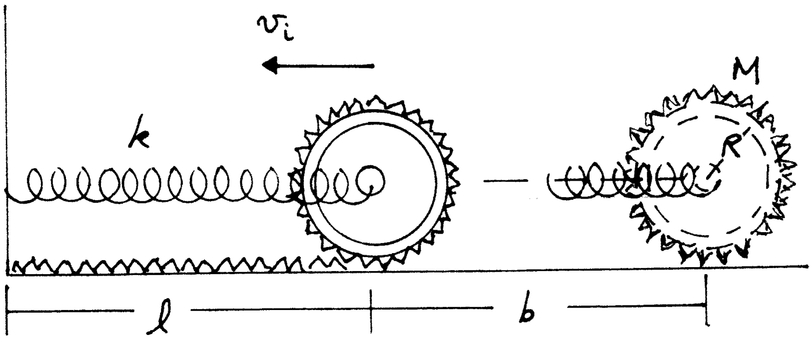
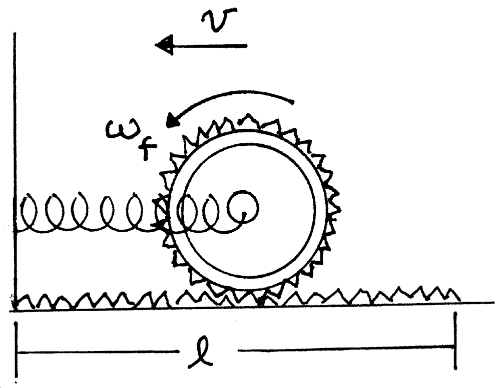


(a)



Because $L_f = L_i$, $E_f = \frac{1}{2}E_i$.
Mechanical energy E_f is conserved following the collision, when the wheel is on the track. The wheel comes to rest when the spring is compressed to b'.

$$\frac{1}{2}kb'^2 = E_f = \frac{1}{2}E_i = \frac{1}{4}kb^2$$

$$b'^2 = \frac{1}{2}b^2 \implies b' = \frac{1}{\sqrt{2}}b$$

The closest approach to the wall is

$$l - b' = l - \frac{1}{\sqrt{2}}b$$

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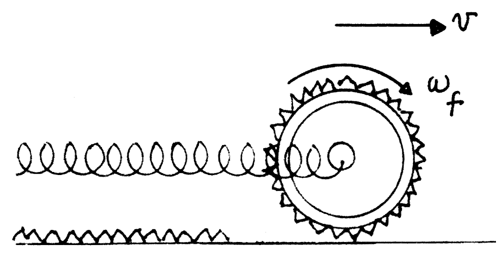
(b) When the wheel returns to the initial point of contact, its velocity and angular velocity are reversed, as shown in sketch (b). Mechanical energy E_f is conserved as the wheel moves off the track onto the smooth surface. The velocity $\mathbf{v}$ changes with time because of the spring force, but the angular velocity remains constant, because there is no torque on the wheel about its center of mass.

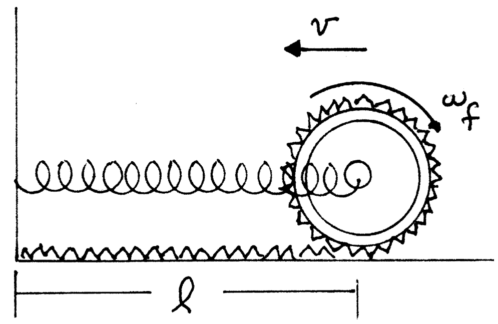


(b)

The wheel moves outward away from the wall until the spring is extended beyond l by an amount b'', where

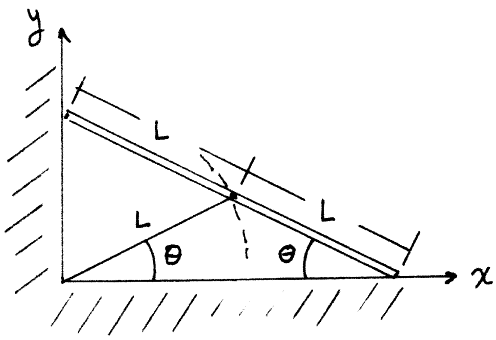
$$\frac{1}{2}kb''^2 = E_f = \frac{1}{2}E_i = \frac{1}{4}kb^2 \implies b'' = \frac{1}{\sqrt{2}}b$$

(c) When the wheel returns to the point of contact for the second time, its angular momentum of translation is $L_{trans} = mv_fR$ and its angular momentum of rotation is $L_{rot} = I_0\omega_f = mR^2\omega_f = mv_fR$. Thus the mechanical energy is equally divided between translation and rotation. As shown is sketch (c), L_{rot} has reversed its direction, so the total angular momentum is 0, and it remains 0 after the second collision. The wheel is therefore at rest, $v = 0$. The second collision has dissipated all the remaining mechanical energy!



(c)

7.41 Leaning plank

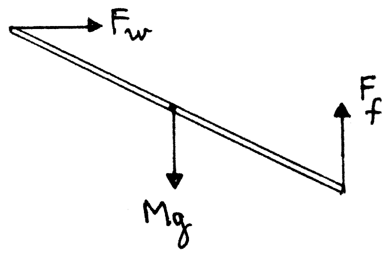


As long as the plank is in contact with the wall, the coordinates of its center of mass are

$$x = L\cos\theta \qquad y = L\sin\theta$$

$$x^2 + y^2 = L^2(\cos^2\theta + \sin^2\theta) = L^2$$

Until contact with the wall is lost, the center of mass moves on a circular path of radius L, as indicated in the upper sketch.
Because the wall and floor are frictionless, the force F_w exerted by the wall on the plank and the force F_f exerted by the floor are normal to the surfaces, as shown in the lower sketch.
The plank loses contact with the wall when $F_w = 0$, or, equivalently, when $\ddot{x} = F_w/M = 0$.



$$x = L\cos\theta \qquad \dot{x} = -L\sin\theta\,\dot{\theta}$$

$$\ddot{x} = -L\cos\theta\,\dot{\theta}^2 - L\sin\theta\,\ddot{\theta} = 0 \implies \dot{\theta}^2 = -\tan\theta\,\ddot{\theta} \quad (1)$$

There are no dissipative forces, so mechanical energy E is conserved. Let y_0 be the initial height of the center of mass above the floor.

$$E_i = Mgy_0 = E_f = Mgy + \frac{1}{2}M(L\dot{\theta})^2 + \frac{1}{2}I_0\,\dot{\theta}^2$$

$$Mgy_0 = MgL\sin\theta + \frac{1}{2}M(L\dot{\theta})^2 + \frac{1}{2}\left(\frac{1}{3}ML^2\right)\dot{\theta}^2 = MgL\sin\theta + \frac{2}{3}ML^2\,\dot{\theta}^2$$

$$y_0 = L\sin\theta + \frac{2}{3}\frac{L^2}{g}\,\dot{\theta}^2 \quad (2)$$

Differentiating Eq. (2),

$$0 = L\cos\theta\,\dot{\theta} + \frac{4}{3}\frac{L^2}{g}\,\dot{\theta}\,\ddot{\theta} \implies \ddot{\theta} = -\frac{3}{4}\frac{g}{L}\cos\theta$$

Using Eq. (1),

$$\dot{\theta}^2 = \frac{3}{4}\frac{g}{L}\sin\theta \quad (3)$$

Substituting Eq. (3) in Eq. (2),

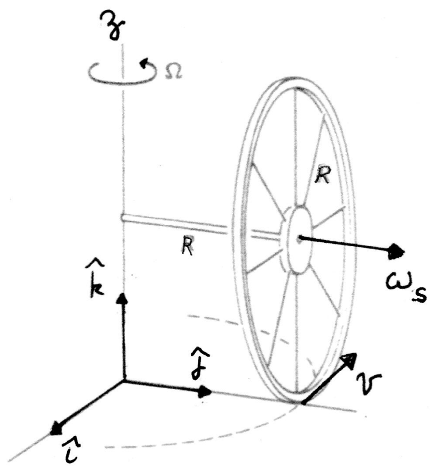
$$y_0 = L\sin\theta + \frac{1}{2}L\sin\theta = \frac{3}{2}L\sin\theta = \frac{3}{2}y$$

so the plank loses contact with the wall at height

$$y = \frac{2}{3}y_0$$

8 RIGID BODY MOTION

8.1 *Rolling hoop*



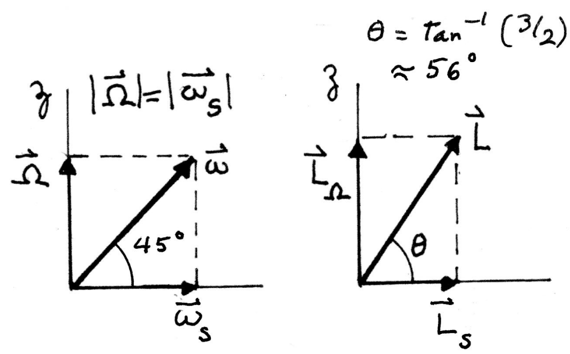
(a)

$$\omega_s = \frac{v}{R} = \frac{\Omega R}{R} = \Omega$$
$$\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\Omega} = \Omega(\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

(b)

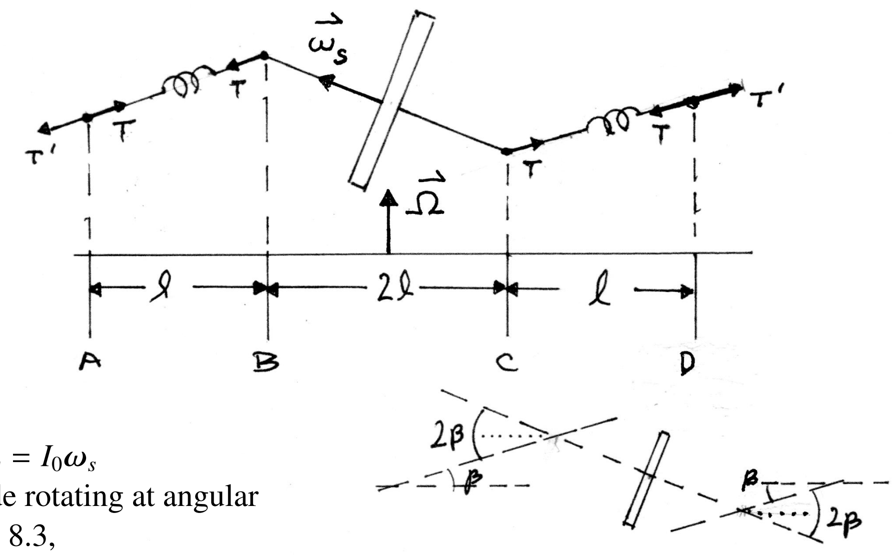
$$\mathbf{L} = \mathbf{L_s} + \mathbf{L}_\omega = I_s\boldsymbol{\omega}_s + I_z\boldsymbol{\Omega}$$
$$I_s = MR^2 \qquad I_z = I_0 + MR^2 = \frac{3}{2}MR^2$$
$$\mathbf{L} = MR^2\left(\boldsymbol{\omega}_s + \frac{3}{2}\boldsymbol{\Omega}\right) = MR^2\,\Omega(\hat{\mathbf{j}} + \tfrac{3}{2}\hat{\mathbf{k}})$$

The lower sketches show that $\boldsymbol{\omega}$ and $\mathbf{L}$ are not parallel. We treated $\mathbf{L}$ as the angular momentum of a body with moment of inertia from the parallel axis theorem. $\mathbf{L}$ can also be viewed as the sum of orbital angular momentum $MR^2\boldsymbol{\Omega}$ plus spin angular momentum $(1/2)MR^2\boldsymbol{\Omega}$.



8.2 *Flywheel on rotating table*

The angle of tilt is assumed to be small so that $\sin x \approx x$. The torque can be calculated from the forces T at either end of the axle (points B, C) or from the forces $T' = T$ at either end of the spring suspension (points A, D).



$$\tau_{BC} = 2l(T \sin 2\beta) \approx 4lT\beta$$
$$\tau_{AD} = 4l(T \sin\beta) \approx 4lT\beta$$

The spin angular momentum $\mathbf{L}_s = I_0\omega_s$ is a vector of constant magnitude rotating at angular speed Ω. From Secs. 1.10.1 and 8.3,

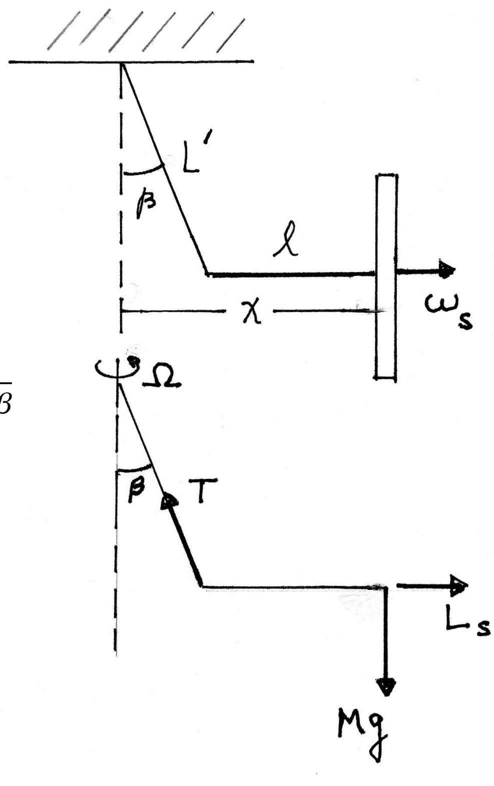
$$\left|\frac{d\mathbf{L}_s}{dt}\right| = \Omega L_s$$

$$\tau = \frac{d\mathbf{L}_s}{dt} \implies 4lT\beta = \Omega L_s \implies \beta = \frac{\Omega L_s}{4lT}$$

8.3 *Suspended gyroscope*

Note that in this problem, β and Ω are both unknown quantities,

Assume β is small, so that $\sin\beta \approx \beta$ and $\cos\beta \approx 1$. Hence $x = l + L' \sin\beta \approx l + L'\beta$.



equations of motion:

$$Mg = T\cos\beta \approx T$$

$$Mx\Omega^2 = T\sin\beta \approx T\beta \implies \Omega^2 = \frac{T\beta}{Mx} = \frac{g\beta}{l + L'\beta}$$

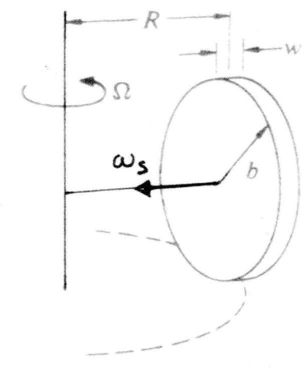
torque :

$$Tl = \dot{L}_s = \Omega I_0 \omega_s \implies \Omega = \frac{Mgl}{I_0\omega_s}$$

$$\frac{g\beta}{l + L'\beta} = \Omega^2 = \left(\frac{Mgl}{I_0\omega_s}\right)^2$$

$$\beta\left(1 - \frac{M^2 g l^2 L'}{I_0^2\omega_s^2}\right) = \frac{M^2 g l^3}{I_0^2 \omega_s^2}$$

8.4 *Grain mill*



F is the force exerted by the pivot. The millstone rolls without slipping, so $\omega_s b = \Omega R$.

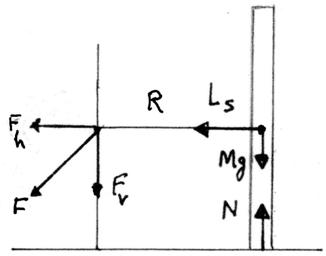
$$L_s = I_0\omega_s = \frac{1}{2}Mb^2\left(\frac{R}{b}\Omega\right) = \frac{1}{2}MbR\Omega$$

$$\frac{dL_s}{dt} = \omega L_s = \frac{1}{2}MbR\Omega^2$$

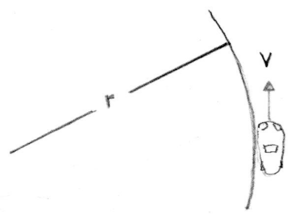
$$\tau_{pivot} = (N - Mg)R = \frac{dL_s}{dt} = \frac{1}{2}MbR\Omega^2$$

$$N = Mg\left(1 + \frac{1}{2}b\frac{\Omega^2}{g}\right)$$

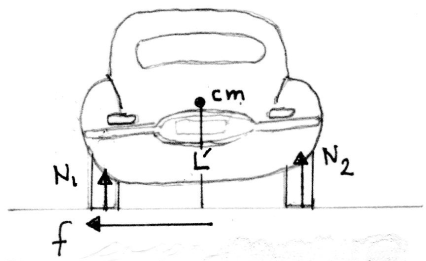
The millstone exerts a downward force $N > Mg$ due to the downward force of the pivot: $N = Mg + F_v$.

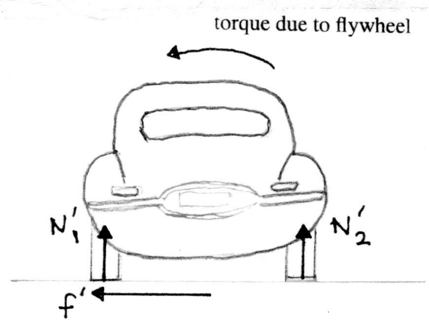


8.5 *Automobile on a curve*



(a) The radial equation of motion is $f = MV^2/r$. Without the flywheel, the torque due to friction f is balanced by the torque due to the unequal loading N_1 and N_2. For equal loading $N_1' = N_2'$, these forces produce no torque. The flywheel thus needs to produce a counterclockwise torque on the car to balance the clockwise torque from f'. The torque *on* the flywheel *by* the car must therefore be clockwise, so that the spin angular momentum $\mathbf{L}_s$ must increase in the forward direction of the car's motion. If the car turns in the opposite direction, both the torque and the direction are reversed, so equal loading remains satisfied. One can also argue that the torque on the total car-flywheel system by the friction force f' must cause $\mathbf{L}_s$ to increase in the forward direction by precessing at the rate $\Omega = V/r$, which can be achieved by mounting the flywheel's axis transverse to the car's motion (parallel to the car's axles). For $\mathbf{L}_s$ in the horizontal plane, the flywheel's disk must be in the vertical plane.





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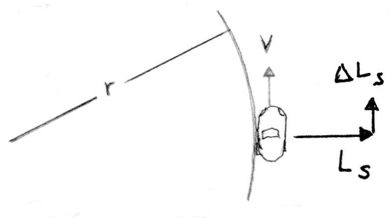


(b) The torque τ_{road} due the road's friction force f' is

$$\tau_{road} = L'f' = L'M\frac{V^2}{r}$$

The torque $\tau_{flywheel}$ due to the flywheel is

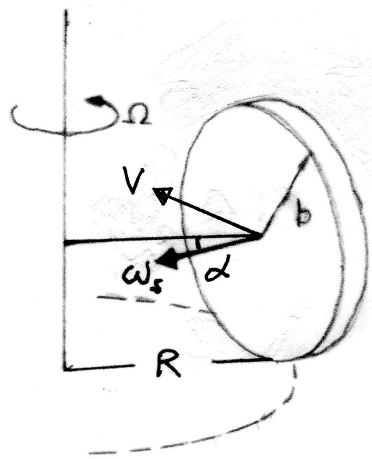
$$\tau_{flywheel} = L_s\Omega = \text{Ł}_s\frac{V}{r} = I_0\omega_s\frac{V}{r} = \frac{1}{2}mR^2\omega_s\frac{V}{r}$$

$$\tau_{road} = \tau_{flywheel} \implies L'M\frac{V^2}{r} = \frac{1}{2}mR^2\omega_s\frac{V}{r}$$

$$\omega_s = 2L'\frac{M}{m}\frac{V}{R^2}$$

The result does not include the radius r of the turn, but ω_s must be kept proportional to V.

8.6 *Rolling coin*

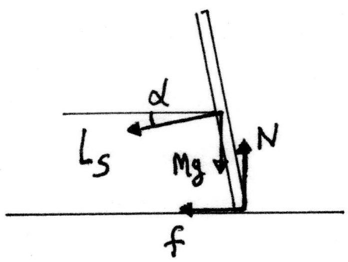


As the coin rolls with speed V around the circle of radius R, it rotates around the vertical at rate $\Omega = V/R$. This rotation is caused by precession of of its spin angular momentum due to the torque induced by the tilt. For rolling without slipping, $V = b\omega_s$, so $\Omega = \omega_s(b/R)$.
The coin is accelerating, so take torques about the center of mass. From the force diagram,

$$\tau_{cm} = fb\cos\alpha - Nb\sin\alpha$$

$$N = Mg \qquad f = \frac{MV^2}{R}$$

The equation of motion for L_s is



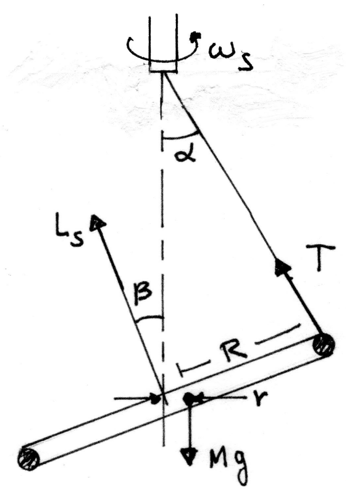
$$\tau_{cm} = \Omega L_s\cos\alpha = \Omega I_0\omega_s\cos\alpha = \omega_s^2\frac{b}{R}I_0\cos\alpha$$

$$= \left(\frac{V}{b}\right)^2\left(\frac{b}{R}\right)\left(\frac{1}{2}Mb^2\right)\cos\alpha = \frac{1}{2}MV^2\left(\frac{b}{R}\right)\cos\alpha$$

$$= MV^2\left(\frac{b}{R}\right)\cos\alpha - Mgb\sin\alpha$$

$$\tan\alpha = \frac{V^2}{Rg}\left(1 - \frac{1}{2}\right) = \frac{1}{2}\frac{V^2}{Rg}$$

8.7 *Suspended hoop*



(a) The spin angular momentum is $L_s = I_0\omega_s = MR^2\omega_s$.
The equation of motion for $\mathbf{L}_s$ is

$$\tau = \omega_s L_s \sin\beta \approx \omega_s L_s \beta \quad \text{(directed out of the paper)}$$

$$L_s = I_0\omega_s = MR^2\omega_s$$

$$T\cos\alpha = Mg \implies \tau = RT\cos\alpha \approx MgR$$

$$MgR = \omega_s L_s \beta = MR^2{\omega_s}^2\beta \implies \beta = \frac{g}{R{\omega_s}^2}$$

(b) equation of motion: (the *cm* moves in a loop of radius *r*)

$$T\sin\alpha = Mr{\omega_s}^2 \implies g\tan\alpha = r{\omega_s}^2$$

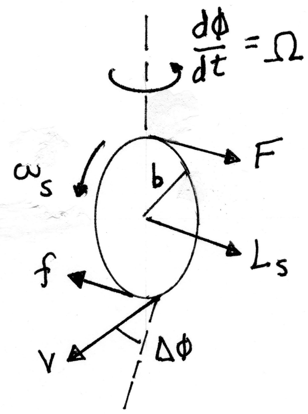
$$r = \frac{g}{{\omega_s}^2}\tan\alpha$$

(c) To gauge the validity of the solution,. compare the torque τ_{cm} needed to spin the center of mass with the torque τ_{hoop} needed to spin the hoop.

$$\tau_{cm} = Mgr \qquad \tau_{hoop} = MgR \implies \frac{\tau_{cm}}{\tau_{hoop}} = \frac{r}{R} = \frac{g\tan\alpha}{R\omega^2}$$

The solution is therefore only valid for large ω, so that τ_{cm} can be neglected. The criterion is equivalent to being able to twirl a lariat vertically as well as horizontally.

8.8 *Deflected hoop*



(a) The force **F** due to the stick and the friction force **f** exert a horizontal torque directed into the paper. The hoop is vertical, so gravity exerts no torque. The blow by the stick is short, so the peak of force **F** is large; **f** can be neglected during the time of impact. The torque τ into the paper is then Fb. The spin speed for rolling without slipping is $\omega_s = V/b$ and the spin angular momentum is $L_s = I_0\omega_s = Mb^2\omega_s = MVb$. The equation of motion for L_s is

$$\tau = L_s\Omega \implies Fb = MVb\Omega$$

where $\Omega = d\Phi/dt$ is the angular speed around the vertical axis.

$$\Omega = \frac{d\Phi}{dt} = \frac{F}{MV} \implies \Delta\Phi = \int \frac{F}{MV}dt' = \frac{1}{MV}\int F dt' = \frac{I}{MV}$$

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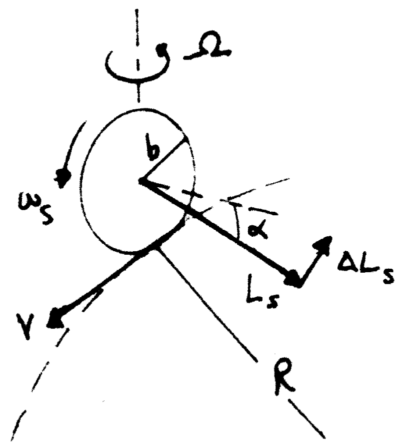
(b) The angular momentum around the vertical axis is

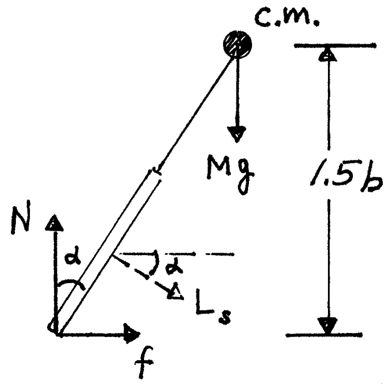
$$L_{vertical} = I_\Phi \Omega = \frac{1}{2} M b^2 \Omega$$

For the solution in (a) to be valid, $L_{vertical} \ll L_s$.

$$\frac{1}{2} M b^2 \frac{F}{MV} \ll MVb \implies F \ll \frac{2MV^2}{b}$$

8.9 *Stability of a bicycle*





The torque τ_h about the center of mass is into the paper.

$$\begin{aligned}\tau_h &= N(1.5b)\tan\alpha - f(1.5b)\\ &= Mg(1.5b)\tan\alpha - (1.5b)\frac{MV^2}{R}\end{aligned}$$

The total spin angular momentum (two wheels) is

$$L_s = 2I_0\omega_s = 2mb^2\frac{V}{b} = 2mbV$$

$$\tau_h = L_h\Omega = L_s\cos\alpha\frac{V}{R}$$

$$Mg(1.5b)\tan\alpha - (1.5b)\frac{MV^2}{R} = 2mb\frac{V^2}{R}\cos\alpha$$

$$\tan\alpha = \frac{V^2}{Rg}\left(1 + \frac{4}{3}\frac{m}{M}\cos\alpha\right)$$

Because $m/M \ll 1$, the second term in parentheses is a small correction and it is adequate to take $\cos\alpha \approx 1$.

$$\tan\alpha \approx \frac{V^2}{Rg}\left(1 + \frac{4}{3}\frac{m}{M}\right)$$

Converting units, using $g = 32$ ft/s^2,

$$V = \left(\frac{20\,\text{miles}}{\text{hour}}\right)\times\left(\frac{5280\,\text{ft}}{\text{mile}}\right)\times\left(\frac{1\,\text{hour}}{3600\,\text{s}}\right) = 29.3\,\text{ft/s}$$

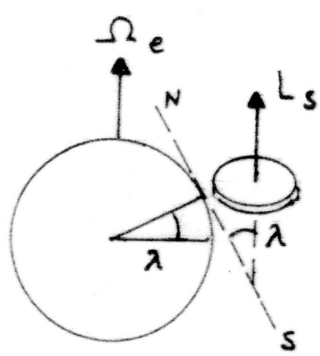
$$\frac{4}{3}\frac{m}{M} = 0.048 \qquad \frac{V^2}{Rg} = 0.268$$

$$\tan\alpha = (0.268)(1.048) = 0.28$$

$$\alpha \approx 16^\circ$$

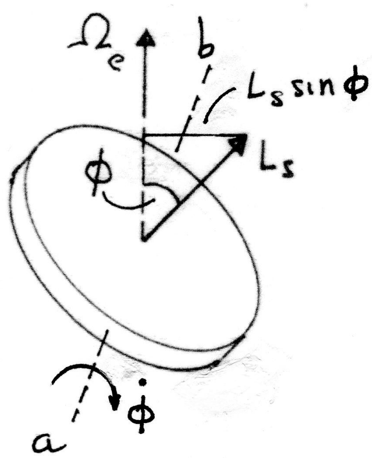
If spin is neglected, the term in m/M should be omitted. Then $\alpha \approx 15^\circ$. The spinning wheels increase the tilt angle by only about a degree, not a substantial effect. Without a rider, M is smaller and α is larger.

8.10 *Measuring latitude with a gyro*



The gyro's disk is spinning with angular speed ω_s.

(a) If the gyro's spin angular momentum $\mathbf{L}_s$ is parallel to the Earth's angular velocity $\mathbf{\Omega}_e$ (upper sketch), $\mathbf{L}_s$ does not change direction as the Earth rotates, and the axis of the gyro remains stationary. Its axis is then at the latitude angle λ with respect to the local horizontal, which lies along the meridian in the north-south (N-S) direction.



(b) The magnitude of $\mathbf{L}_s$ is constant, but there are two different ways to change the angular momentum of the gyro disk. Let I_0 be the moment of inertia about the gyro's spin axis, $L_s = I_0\omega_s$, and let $I_\perp$ be the moment of inertia about the horizontal $a-b$ axis.
Let Φ be the angle between $\mathbf{L}_s$ and $\mathbf{\Omega}_e$, (lower sketch).
(1) rotation of $\mathbf{L}_s$ *about* Ω_e:
The component of $\mathbf{L}_s$ perpendicular to $\mathbf{\Omega}_e$ is $L_s \sin\Phi$, so

$$\textit{rate of change (1)} = L_s \sin\Phi\Omega_e \approx L_s\Omega_e\Phi = I_0\omega_s\Omega_e\Phi$$

(2) rotation of the gyro disk about a-b by Φ:

$$\textit{angular momentum about a-b} : = I_\perp \frac{d\Phi}{dt}$$

$$\textit{rate of change (2)} : = I_\perp \frac{d^2\Phi}{dt^2}$$

The two contributions are parallel to the $a-b$ axis, and add. There is no applied torque, so the net rate of change is 0.

$$I_\perp \frac{d^2\Phi}{dt^2} + (I_0\omega_s\Omega_e)\Phi = 0$$

This is the equation for SHM, with oscillation frequency ω_{osc} and period T_{osc}.

$$\omega_{osc} = \sqrt{\frac{I_0\omega_s\Omega_e}{I_\perp}} \qquad T_{osc} = 2\pi/\omega_{osc}$$

$$I_\perp = I_0/2 \text{ for a thin disk} \implies \omega_{osc} = \sqrt{2\omega_s\Omega_e}$$

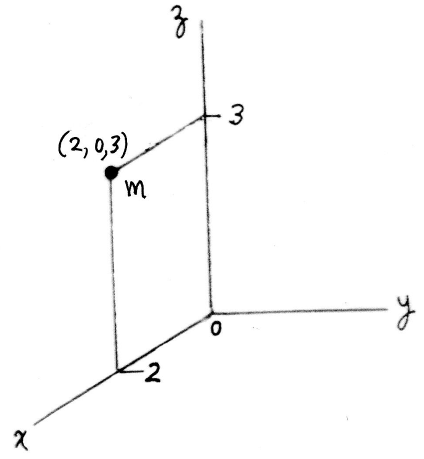
$$\omega_s = 4.0\times10^4\,\text{rpm} = \left(\frac{(2\pi)\,4.0\times10^4\,\text{rad}}{\text{min}}\right)\times\left(\frac{1\,\text{min}}{60\,\text{s}}\right) = 4.19\times10^3\,\text{rad/s}$$

$$\Omega_e = \left(\frac{2\pi\,\text{rad}}{\text{day}}\right)\times\left(\frac{1\,\text{day}}{8.64\times10^4\,\text{s}}\right) = 7.27\times10^{-5}\,\text{rad/s}$$

$$\omega_{osc} = \sqrt{(2)(4.19\times10^3)(7.27\times10^{-5})} = 0.78\,\text{rad/s} \qquad T_{osc} = \frac{2\pi}{\omega_{osc}} = 8.1\,\text{s}$$

8.11 *Tensor of inertia*

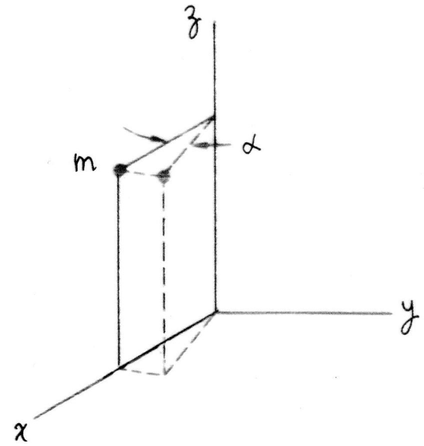
(a)



$$I_{xx} = m(y^2 + z^2) = m((0)^2 + (3)^2) = 9m$$
$$I_{yy} = m(x^2 + z^2) = m((2)^2 + (3)^2) = 13m$$
$$I_{zz} = m(x^2 + y^2) = m((2)^2 + (0)^2) = 4m$$
$$I_{xy} = I_{yx} = -m(xy) = -m(2)(0) = 0$$
$$I_{yz} = I_{zy} = -m(yz) = -m(0)(3) = 0$$
$$I_{xz} = I_{zx} = -m(xz) = -m(2)(3) = 6m$$

In matrix form,

$$\tilde{I} = m\begin{pmatrix} 9 & 0 & -6 \\ 0 & 13 & 0 \\ -6 & 0 & 4 \end{pmatrix}$$



(b) To order α^2,

$$x = 2\cos\alpha \approx 2 - \alpha^2 \qquad y = 2\sin\alpha \approx 2\alpha \qquad z = 3$$

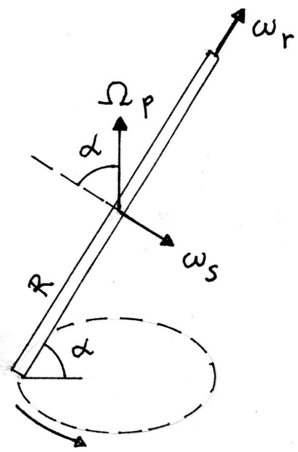
$$\tilde{I}' = m\begin{pmatrix} 9 + 4\alpha^2 & -4\alpha & -6 + 3\alpha^2 \\ -4\alpha & 13 - 4\alpha^2 & -6\alpha \\ -6 + 3\alpha^2 & -6\alpha & 4 \end{pmatrix}$$

Comparing with part (a), note that the moments of inertia (along the main diagonal of the matrix) vary only as α^2, but some of the products of inertia (off-diagonal elements) can vary linearly with α.

When making such approximations, be sure to include all terms up to the highest order retained. For example,

$$I'_{zz} = m[x^2 + y^2] = m[(2 - \alpha^2)^2 + (2\alpha)^2] \approx m[4 - 4\alpha^2 + 4\alpha^2] = 4m$$

8.12 *Euler's disk*



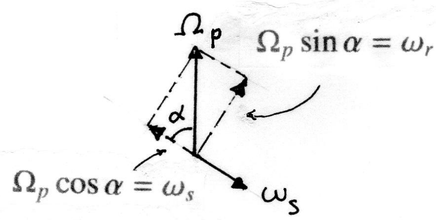
The contact point moves on the surface in a circle of radius $R\cos\alpha$, with speed $V = (R\cos\alpha)(\Omega_p$. The disk is assumed to roll without slipping, so $R\omega_s = V = (R\cos\alpha)\Omega_p$.

equations of motion:

$$0 = N - Mg \implies N = Mg$$

$$f = \frac{MV^2}{R\cos\alpha} = \frac{M(R\cos\alpha)^2\,\Omega_p^2}{R\cos\alpha} = MR\cos\alpha\,\Omega_p^2$$

The total angular velocity is $\mathbf{\Omega}_p + \boldsymbol{\omega}_s = \boldsymbol{\omega}_r$. As shown in the sketches, $\boldsymbol{\omega}_r$ lies along the axis from the contact point to the center of mass. The moment of inertia along this axis is

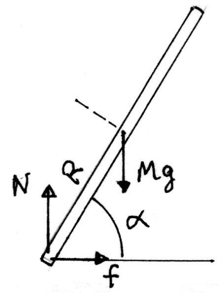


$$I_\perp = \frac{1}{2}I_0 = \frac{1}{4}MR^2$$

The spin angular momentum is

$$L_s = I_\perp\omega_r = \frac{1}{4}MR^2\Omega_p\sin\alpha$$

The horizontal component of the spin angular momentum is



force diagram

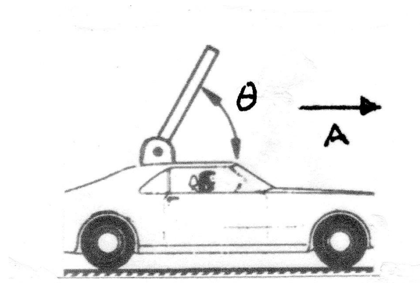
$$L_h = L_s\cos\alpha = \frac{1}{4}MR^2\cos\alpha\sin\alpha\,\Omega_p$$

torque about the cm (positive is into the paper):

$$\begin{aligned}\tau_{cm} &= NR\cos\alpha - fR\sin\alpha = MgR\cos\alpha - MR^2\cos\alpha\sin\alpha\,\Omega_p^2 \\ &= MR\cos\alpha\left(g - R\sin\alpha\,\Omega_p^2\right)\end{aligned}$$

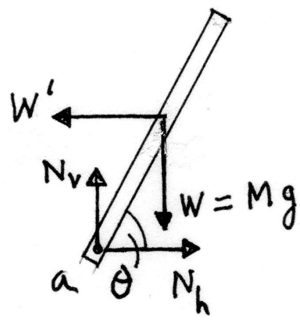
9 NON-INERTIAL SYSTEMS AND FICTITIOUS FORCES

9.1 *Pivoted rod on car*



The force diagram (middle sketch) is in the accelerating system. W' is the fictitious force $\mathbf{W}' = -M\mathbf{A}$, so $\mathbf{W}'$ is drawn opposite to $\mathbf{A}$.
equations of motion:

$$N_v + W = 0 \qquad N_h + W' = 0$$



(a) The torque about the pivot a is

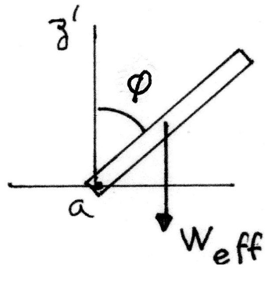
$$\tau_a = \frac{L}{2}\cos\theta\, W - \frac{L}{2}\sin\theta\, W'$$

For equilibrium in the accelerating system, $\tau_a = 0$.

$$0 = \frac{L}{2}\cos\theta\, Mg - \frac{L}{2}\sin\theta\, MA \implies \tan\theta = \frac{g}{A}$$

For equilibrium in any system, the torque about any point must vanish. For example, the torque about the center of mass is

$$\begin{aligned}\tau_{cm} &= \frac{L}{2}\cos\theta\, N_v - \frac{L}{2}\sin\theta\, N_h \\ &= -\frac{L}{2}\cos\theta\, W + \frac{L}{2}\sin\theta\, W' = 0\end{aligned}$$



using the equations of motion.

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(b) Introduce a coordinate system with z' along the equilibrium position of the board. (z' is drawn vertically in the sketch for clarity.)
The effective acceleration due to gravity is

$$g_{eff} = \sqrt{g^2 + A^2}$$

so the effective weight force is

$$W_{eff} = Mg_{eff}$$

For small displacements, the torque is

$$\tau = \frac{L}{2}\sin\phi\, Mg_{eff} \approx \frac{l}{2}Mg_{eff}\,\phi$$

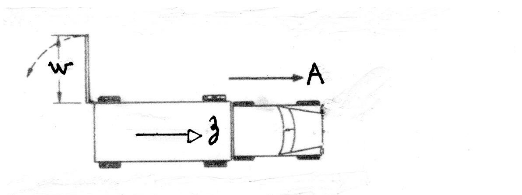
$$I_a\frac{d^2\phi}{dt^2} = \frac{L}{2}Mg_{eff}\,\phi \implies \frac{1}{12}ML^2\frac{d^2\phi}{dt^2} - \frac{L}{2}Mg_{eff}\,\phi = 0 \implies \frac{d^2\phi}{dt^2} - \frac{6g_{eff}}{L}\phi = 0$$

The motion is exponential:

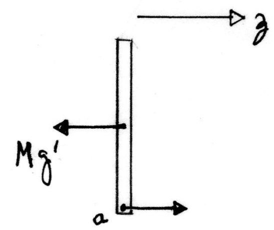
$$\phi = \phi_0 e^{\pm\gamma t} \quad \text{where } \gamma = \sqrt{\frac{6g_{eff}}{L}}$$

9.2 *Truck door*

Consider the motion of the door in a system accelerating with the truck. The door effectively "falls" from rest in a gravitational field $g' = A$, in the direction shown in the sketch. The sketches with z turned 90° shows the door "falling" with weight force Mg'.



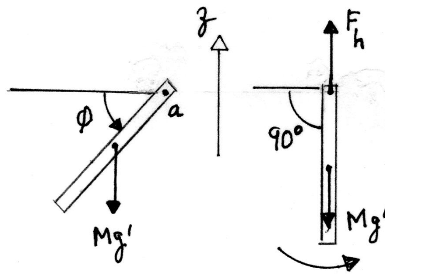
(a) Use conservation of mechanical energy. The door's center of mass falls a distance $w/2$ as it gains rotational kinetic energy.



$$\frac{1}{2}I_a\dot{\phi}^2 = Mg'\frac{w}{2} \implies \frac{1}{2}\left(\frac{1}{3}Mw^2\right)\dot{\phi}^2 = MA\frac{w}{2}$$

$$\dot{\phi} = \sqrt{\frac{3A}{w}}$$

(b)

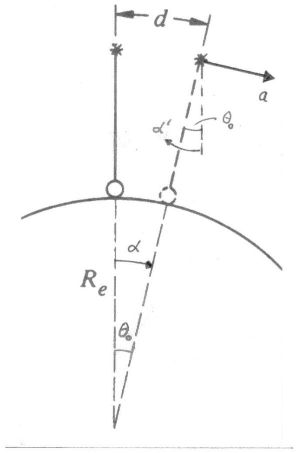


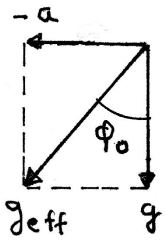
The equation of motion (see force diagram) is

$$F_h - Mg' = M\frac{w}{2}\dot{\phi}^2 = M\frac{w}{2}\left(\frac{3A}{w}\right)$$

$$F_h = Mg' + \frac{3}{2}MA = MA + \frac{3}{2}MA = \frac{5}{2}MA$$

9.3 *Pendulum on moving pivot*

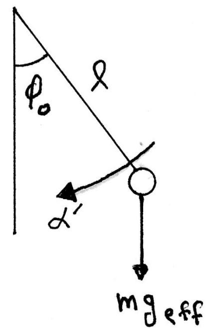




In the sketches, α and α' are angular accelerations in the earthbound system and in the accelerating system respectively.
In the accelerating system, the pendulum swings under an effective gravitational acceleration g_{eff}, where

$$g_{eff} = \sqrt{g^2 + a^2}$$

The pendulum begins to swing from an initial angular displacement $\phi_0 = \arctan a/g$, as indicated in the sketch. Consequently, "down" in the accelerating system differs from "down" in the earthbound system by ϕ_0.
If the simple pendulum has length l and mass m, the torque in the accelerating system is initially



$$\tau' = mg_{eff} l \sin\phi_0$$

$$\alpha' = \frac{\tau'}{I_0} = \frac{mg_{eff} l \sin\phi_0}{ml^2}$$

$$= \frac{g_{eff}\sin\phi_0}{l} = \frac{a}{l}$$

using $g_{eff}\sin\phi_0 = a$ (see the top right-hand sketch)

In order for the pendulum to point continually down (toward the center of the Earth), its angular acceleration α' must be equal and opposite to the angular acceleration α of the point of support with respect to the center of the Earth, $\alpha = a/R_e$,

$$\alpha' = \alpha$$

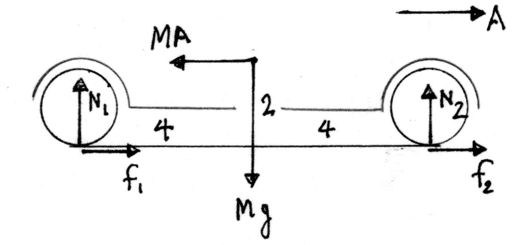
$$\frac{a}{l} = \frac{a}{R_e} \implies l = R_e$$

The period of this pendulum is

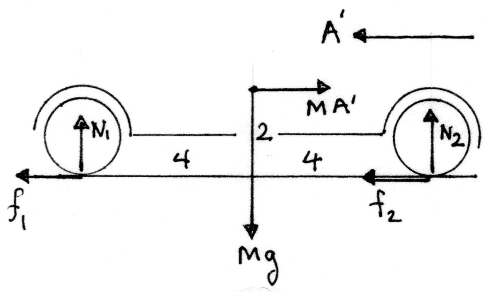
$$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{R_e}{g}} = 84\,\text{minutes}$$

9.4 *Weight on a car's wheels*

In this problem, the forces on the tires due to the road are the normal forces N_1 and N_2, and the friction forces f_1 and f_2. In part (a), MA is the fictitious force, and in part (b), it is MA'. The directions of the fictitious forces are shown in the sketches. English units are used.



(a)



(b)

In both parts, the approach is to take torques about the point of contact of the rear tire with the road. One advantage is that the friction forces do not appear in the torque equations.
The torques are 0, because the car is in horizontal equilibrium as long as N_1 and N_2 are ≥ 0.

(a)

$$\tau = 8N_2 + 2MA - 4Mg = 0$$

For N_2 just 0,

$$A = 2g = 64\,\text{ft/s}^2$$

(b) For $A' = g$ in the direction shown

$$\tau = 8N_2 - 4Mg - 2MA' = 8N_2 - 4Mg - 2Mg = 0$$

$$N_2 = \frac{3}{4}Mg = 2400\,\text{lbs}$$

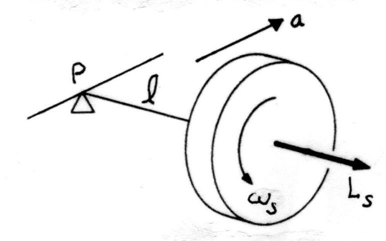
The vertical acceleration = 0, so

$$N_1 + N_2 - Mg = 0 \implies N_1 = Mg - N_2 = 800\,\text{lbs}$$

Under positive acceleration, the rear wheels bear more of the car's weight. Under braking, the front wheels bear more of the weight; when a car is brought to a sudden stop, it can "nose dive".

9.5 *Gyroscope and acceleration*

In the accelerating system, the gyroscope experiences an effective gravitational field -**a**, so there is a fictitious force Ma acting in the direction shown. The torque $\boldsymbol{\tau}_P$ on the gyroscope about the pivot is downward, so the gyroscope precesses as shown.



lab frame

$$\tau_P = Nl = Mal$$

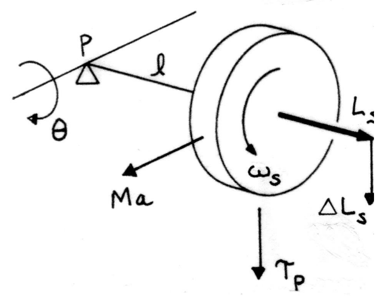
$$|\boldsymbol{\tau}_P| = \left|\frac{d\mathbf{L}_s}{dt}\right| = L_s\,\dot{\theta} = I_0\omega_s\,\dot{\theta}$$

$$\frac{d\theta}{dt} = \frac{Mal}{I_0\omega_s} = \frac{Ml}{I_0\omega_s}\frac{dv}{dt}$$

$$\frac{dv}{dt} = \frac{I_0\omega_s}{Ml}\frac{d\theta}{dt}$$

$$\int_0^v dv' = \int_0^\theta d\theta'$$

$$v = \frac{I_0\omega_s}{Ml}\theta$$



accelerating frame

9.6 *Spinning top in an elevator*

(a) *elevator at rest*

The horizontal component of $\mathbf{L_s}$ is

$$L_{horizontal} = L_s \sin\phi$$

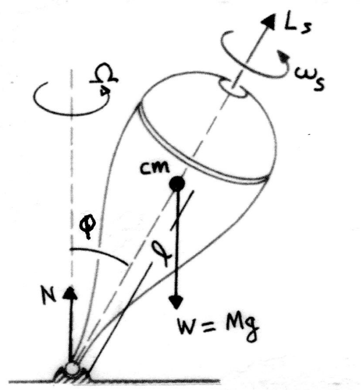
$$\frac{dL_{horizontal}}{dt} = \tau_{cm} = lN\sin\phi = lW\sin\phi$$

$$\frac{dL_{horizontal}}{dt} = L_{horizontal}\Omega = L_s\Omega\sin\phi$$

$$\Omega = \frac{lW}{L_s}$$

The direction of precession is shown in sketch (a).
The same result is obtained using the general result Eq. (9.5)

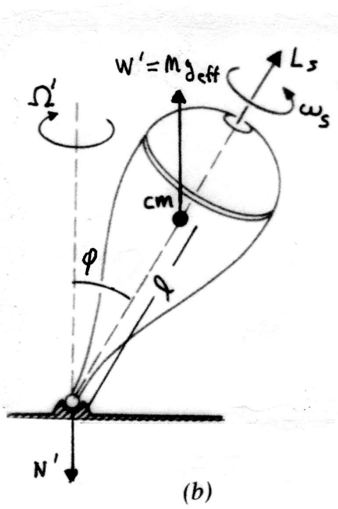
$$\frac{d\mathbf{L}_s}{dt} = \boldsymbol{\Omega}\times\mathbf{L}_s$$



(a)

(b) *elevator accelerating down at rate* $2g$

There is a fictitious force $M(2g)$ acting upward so the effective gravitational acceleration is $g_{eff} = 2g - g = g$ upward.



(b)

continued next page $\Longrightarrow$

In the accelerating system, gravity appears to be reversed, which reverses the direction of τ_{cm}.

$$\Omega' = -\frac{lW}{L_s}$$

As sketch (b) shows, the top consequently reverses its direction of precession in the downward accelerating elevator.

9.7 *Apparent force of gravity*

$$g_{equator} = g - R_e\,\Omega_e^2 \qquad g_{pole} = g$$

$$\Delta g = R_e\,\Omega_e^2 \ \text{ where } \ \Omega_e = \frac{2\pi \text{ rad/day}}{8.64\times 10^4 \text{ s/day}} = 7.27\times 10^{-5}\text{ rad/s}$$

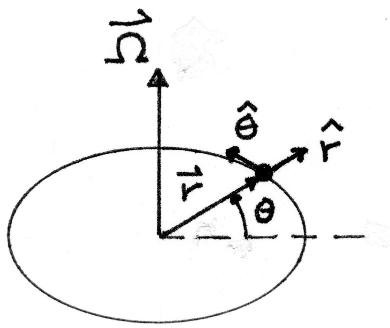
$$\Delta g = (6.37\times 10^6\text{ m})(7.27\times 10^{-5}\text{ rad/s})^2 = 3.37\times 10^{-2}\text{ m/s}^2 \qquad \frac{\Delta g}{g} = 3.44\times 10^{-3}$$

It is interesting to note that the Earth's polar and equatorial radii differ fractionally by nearly the same amount.

$$R_{pol} = 6.357\times 10^6\text{ m} \qquad R_{eq} = 6.378\times 10^6\text{ m} \qquad \frac{\Delta R}{R_e} = \frac{2.1\times 10^4}{6.37\times 10^6} = 3.30\times 10^{-3}$$

9.8 *Velocity in plane polar coordinates*

Consider an inertial frame and a frame instantaneously rotating with the particle. The instantaneous rate of rotation is $\Omega = \dot{\theta}$.



$$\mathbf{v}_{inertial} = \mathbf{v}_{rot} + \mathbf{\Omega}\times\mathbf{r}$$

In the rotating system, the particle has only radial velocity, so

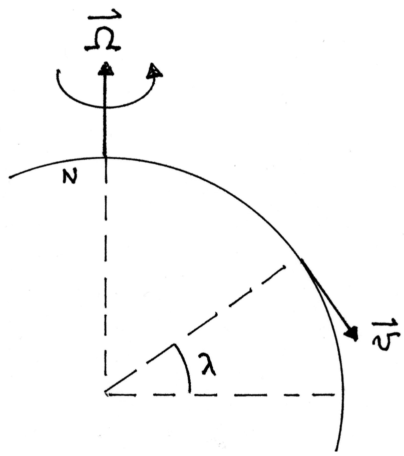
$$\mathbf{v}_{rot} = \dot{r}\hat{\mathbf{r}}$$

$$\mathbf{\Omega}\times\mathbf{r} = \Omega r\hat{\boldsymbol{\theta}} = \dot{\theta} r\hat{\boldsymbol{\theta}}$$

$$\mathbf{v}_{inertial} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

9.9 *Train on tracks*

Let the latitude be λ, for generality.



(a)

$$\mathbf{F}_{Coriolis} = -2M\boldsymbol{\Omega} \times \mathbf{v}$$

$$F_{Coriolis} = 2M\Omega v \sin\lambda$$

$$\frac{F_{Coriolis}}{W} = \frac{2M\Omega v \sin\lambda}{Mg} = \frac{2\Omega v \sin\lambda}{g}$$

$$\Omega = \frac{2\pi \text{ rad/day}}{8.64 \times 10^4 \text{ s/day}} = 7.27 \times 10^{-5} \text{ rad/s}$$

$$v = 60 \text{ mph} = \left(\frac{60 \text{ miles}}{1 \text{ hour}}\right)\left(\frac{5280 \text{ feet}}{1 \text{ mile}}\right)\left(\frac{1 \text{ hour}}{3600 \text{ s}}\right) = 88 \text{ ft/s}$$

$$\sin(60°) = 0.866$$

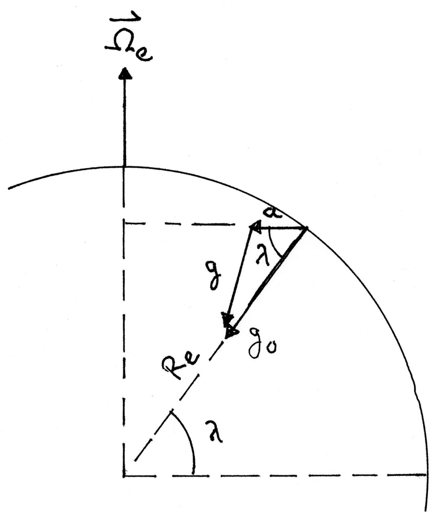
$$W = (400 \text{ tons})(2000 \text{ lbs/ton}) = 8.00 \times 10^5 \text{ lbs}$$

$$F_{Coriolis} = \frac{(2)(7.27 \times 10^{-5} \text{ rad/s})(88 \text{ ft/s})(0.866)(8.00 \times 10^5 \text{ lbs})}{32 \text{ ft/s}^2} = 277 \text{ lbs}$$

(b) From the sketch, $\boldsymbol{\Omega} \times \mathbf{v}$ is directed into the page, toward the east. The Coriolis force on the train is directed toward the west, so the force on the tracks is toward the east. The Coriolis force vanishes at the equator, where $\boldsymbol{\Omega} \times \mathbf{v} = 0$.

9.10 *Apparent gravity versus latitude*

The apparent acceleration of gravity is $\mathbf{g} = \mathbf{g_0} + \mathbf{a}$, where $\mathbf{a}$ is the acceleration of the local reference frame.
For points at rest, $\mathbf{a}$ is radial, directed toward the axis, as shown in the sketch. Using the law of cosines,



$$g^2 = g_0^2 + a^2 - (2ag_0)\cos\lambda$$

$$a = \Omega_e^2 R_e \cos\lambda$$

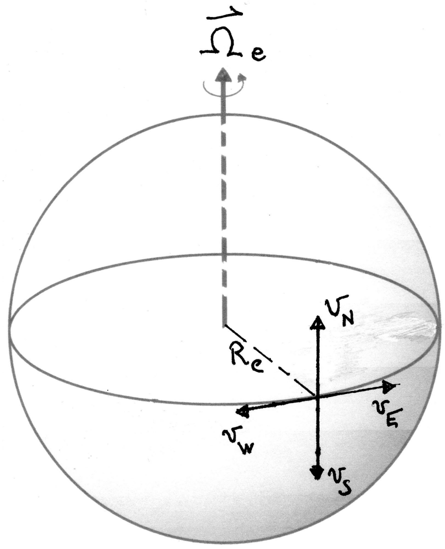
$$g^2 = g_0^2 + (\Omega_e^2 R_e)^2 \cos^2\lambda - 2\Omega_e^2 R_e g_0 \cos^2\lambda$$

$$\text{Let } x \equiv \frac{\Omega_e^2 R_e}{g_0} = \frac{(7.27 \times 10^{-5} \text{ rad/s})^2 (6.37 \times 10^6 \text{ m})}{9.8 \text{ m/s}^2} = 3.44 \times 10^{-3}$$

$$g = g_0\sqrt{1 + x^2\cos^2\lambda - 2x\cos^2\lambda} \approx g_0\sqrt{1 - 2x\cos^2\lambda}$$

because $x \ll 1$. (Note that x is dimensionless.)

9.11 *Racing hydrofoil*



The velocity dependent fictitious force is $\mathbf{F}_{fict} = -2m\mathbf{\Omega}_e \times \mathbf{v}$. The apparent change in gravity is the component of $\mathbf{F}_{fict}/m$ normal to the surface of the Earth.

(a) *East*:

$\mathbf{\Omega}_e \times \mathbf{v}_E$ points radially inward, so F_{fict}/m is radially outward, decreasing g.

$$\frac{\Delta g}{g} = -\frac{F_{frict}}{mg} = -\frac{2\Omega v_E}{g}$$

$$\Omega_e = 7.27 \times 10^{-5}\,\text{rad/s} \qquad v = 200\,\text{mph} = 293\,\text{ft/s}$$

$$\frac{\Delta g}{g} = -\frac{(2)(7.27 \times 10^{-5}\,\text{rad/s})(293\,\text{ft/s})}{32\,\text{ft/s}^2} = -1.33 \times 10^{-3}$$

(b) *West*:

The sign is reversed compared to East, so g is increased.

$$\frac{\Delta g}{g} = +1.33 \times 10^{-3}$$

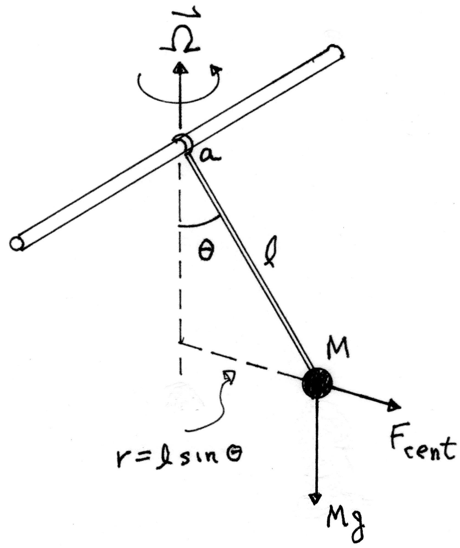
(c) *South*:

$\mathbf{\Omega}_e$ and $\mathbf{v}_S$ are antiparallel, so $\Delta g = 0$.

(d) *North*:

$\mathbf{\Omega}_e$ and $\mathbf{v}_N$ are parallel, so $\Delta g = 0$.

9.12 *Pendulum on rotating platform*



For small amplitude, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$.
In the rotating system, a fictitious centrifugal force F_{cent} acts on M. F_{cent} is directed radially outward from the axis of rotation.

$$F_{cent} = Mr\,\Omega^2 = Ml\,\Omega^2 \sin\theta \approx Ml\,\Omega^2\theta$$

The fictitious Coriolis force in the rotating system acts perpendicular to the plane of swing, and does not play a role in the dynamics of this problem.
Take torques about the pivot point a.

$$\begin{aligned}\tau_a &= Mgl\sin\theta - F_{cent}l\cos\theta\\ &\approx Mgl\theta - F_{cent}l = (Mgl - Ml^2\,\Omega^2)\,\theta\\ &= -I_0\ddot{\theta} = -Ml^2\,\ddot{\theta}\end{aligned}$$

$$l^2\ddot{\theta} + (gl - l^2\,\Omega^2)\,\theta = 0$$

$$\ddot{\theta} + \left(\frac{g}{l} - \Omega^2\right)\theta = 0$$

This is the equation for SHM, with oscillation frequency

$$\omega_{osc} = \sqrt{\frac{g}{l} - \Omega^2}$$

If $\Omega^2 > g/l$, the motion is no longer harmonic, but exponential – M can fly up to a much larger angle. Consider, however, the torque equation without using the small angle approximation.

$$\ddot{\theta} + \left(\frac{g}{l} - \Omega^2\,\cos\theta\right)\sin\theta = 0$$

As θ increases, $\cos\theta$ decreases, so at a sufficiently high angle, the term in parentheses becomes positive, and M undergoes oscillatory motion once again, but about a new equilibrium angle.

10 CENTRAL FORCE MOTION

10.1 *Equations of motion*

$$L = \mu r^2 \dot{\theta} \qquad (10.5) \quad (1)$$

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) \qquad (10.6b) \quad (2)$$

Angular momentum and mechanical energy are both conserved, so their derivatives with respect to time vanish.

$$\text{Differentiating Eq. (1),} \quad \mu(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = \frac{dL}{dt} = 0$$

$$\text{which can be written} \quad \mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \quad (3) \implies = \text{Eq. (10.4b)}$$

$$\text{Differentiating Eq. (2),} \quad \frac{1}{2}\mu(2\dot{r}\ddot{r} + 2r\dot{r}\dot{\theta}^2 + 2r^2\dot{\theta}\ddot{\theta}) + \frac{dU}{dr}\dot{r} = \frac{dE}{dt} = 0$$

$$\text{which can be written} \quad \mu[\dot{r}\ddot{r} + r\dot{\theta}(\dot{r}\dot{\theta} + r\ddot{\theta})] + \frac{dU}{dr}\dot{r} = 0$$

Using Eq. (3) to eliminate $r\ddot{\theta} = -2\dot{r}\dot{\theta}$, and cancelling the common factor $\dot{r}$,

$$\mu(\ddot{r} - r\dot{\theta}^2) + \frac{dU}{dr} = 0$$

Using $dU/dr = -f(r)$, Eq. (10.4a) is obtained.

$$\mu(\ddot{r} - r\dot{\theta}^2) = f(r)$$

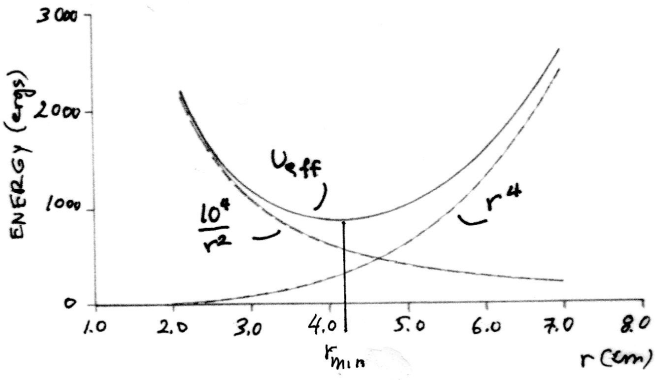
10.2 r^3 *central force*

(a)

$$\mathbf{f}(r) = -Ar^3\,\hat{\mathbf{r}}$$
$$U(r) - U(0) = -\int_0^r f(r')dr' = \frac{1}{4}Ar^4 \qquad \text{taking } U(0) = 0.$$
$$U_{eff}(r) = U(r) + \frac{L^2}{2mr^2} = \frac{1}{4}Ar^4 + \frac{L^2}{2mr^2}$$
$$A = 4.0\,\text{dynes/cm}^3 \quad L = 10^3\,\text{g/cm}^2\cdot\text{s} \quad m = 50.0\,\text{g}$$
$$U_{eff} = \left(r^4 + \frac{1.0\times10^4}{r^2}\right)\text{ergs}$$

(b) Circular motion occurs at r_{min}, the minimum value of U_{eff}, where $dU_{eff}/dr = 0$

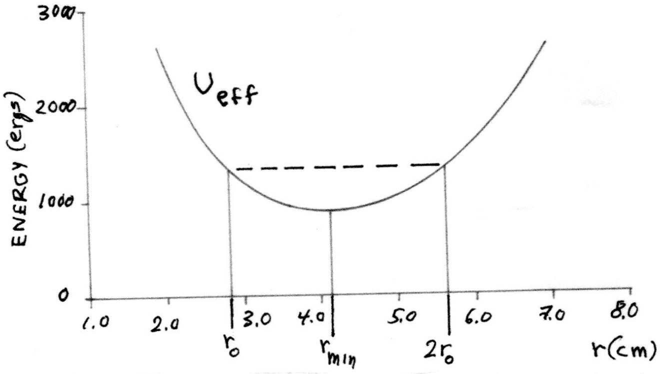
$$\left.\frac{dU_{eff}}{dr}\right|_{r_{min}} = 4r_{min}^3 - \frac{2.0\times10^4}{r_{min}^3} = 0$$
$$r_{min}^6 = 5.0\times10^3\,\text{cm}^6$$
$$r_{min} = 4.1\,\text{cm}$$



(b)

(c)

$$U_{eff}(r_0) = U_{eff}(2r_0)$$
$$\frac{1.0\times10^4}{r_0^2} + r_0^4 = \frac{1.0\times10^4}{(2r_0)^2} + (2r_0)^4$$
$$= \left(\frac{1}{4}\right)\left(\frac{1.0\times10^4}{r_0^2}\right) + 16.0\,r_0^4$$
$$15.0\,r_0^4 = \left(\frac{3}{4}\right)\left(\frac{1.0\times10^4}{r_0^2}\right)$$
$$r_0^6 = \left(\frac{3}{60}\right)\times10^4 = 500$$
$$r_0 = 2.8\,\text{cm}$$



(c)

10.3 *Motion with* $1/r^3$ *central force*

Write the radial force as

$$f(r) = -\frac{2A}{r^3} \quad \text{where } A \text{ is a constant}$$

$$U(r) - U(\infty) = -\int_{\infty}^{r} \left(-\frac{2A}{r'^3}\right) dr' = -\frac{A}{r^2}$$

Take $U(\infty) = 0$.

$$U_{eff} = U(r) + \frac{L^2}{2mr^2} = \left(-A + \frac{L^2}{2m}\right)\frac{1}{r^2}$$

If $A = L^2/(2m)$, then $U_{eff} = 0$; the radial force is 0, so the radial motion is uniform (v is constant).

$$r(t) = r_0 + vt$$

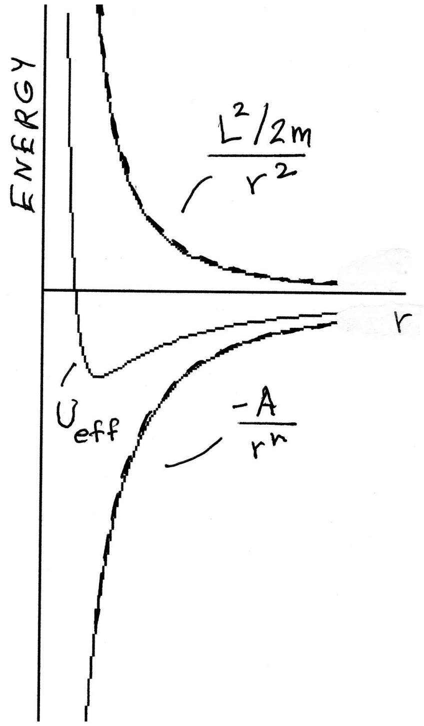
For this case,

$$2mA = L^2 = (mr^2\,\dot{\theta})^2 \implies \dot{\theta} = \sqrt{\frac{2A}{m}}\frac{1}{r^2}$$

$$\begin{aligned}
\int_{\theta_0}^{\theta(t)} d\theta' &= \sqrt{\frac{2A}{m}} \int_{r_0}^{r(t)} \frac{dt}{r'^2} \\
&= \sqrt{\frac{2A}{m}} \int_{r_0}^{r(t)} \frac{dt}{dr'}\frac{dr'}{r'^2} \\
&= \sqrt{\frac{2A}{m}}\frac{1}{v} \int_{r_0}^{r(t)} \frac{dr'}{r'^2} \\
\theta(t) &= \theta_0 + \sqrt{\frac{2A}{m}}\frac{1}{v}\left(\frac{1}{r_0} - \frac{1}{r(t)}\right)
\end{aligned}$$

As $t \to \infty$, $r(t) \to \infty$, so $\theta(t) \to$ *constant*; there is no further rotation.

10.4 *Possible stable circular orbits*



Write U_{eff} as $U_{eff} = U(r) + L^2/(2mr^2) \equiv -A/r^n + B/r^2$ where A is an unspecified force constant and $B = L^2/(2m)$. For a stable circular orbit, U_{eff} must have a minimum at some radius r_0.

$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r_0} = \frac{nA}{{r_0}^{n+1}} - \frac{2B}{{r_0}^3}$$

$$\frac{nA}{{r_0}^{n+1}} = \frac{2B}{{r_0}^3} \quad (1)$$

To be a minimum,

$$\left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} > 0 \implies -\frac{n(n+1)A}{{r_0}^{n+2}} + \frac{6B}{{r_0}^4} > 0$$

Using Eq. (1),

$$-\frac{(n+1)(2B)}{{r_0}^4} + \frac{6B}{{r_0}^4} > 0$$

from which follows $n + 1 \leq 3$ or $n \leq 2$.
The case $n = 0$ does not work, but $n < 0$ is all right for any n.

See Problem 10.2 for the case $U(r) \propto r^4$.

The figure is drawn for $A = 20$, $B = 12$, $n = 1.5$, which gives $r_0 \approx 0.89$.

10.5 *Central spring force*

(a) *radial equation of motion for a circular orbit at r_0*:

$$\frac{mv^2}{r_0} = kr_0$$

mechanical energy:

$$E = \frac{1}{2}Mv_0^2 + \frac{1}{2}kr_0^2$$
$$= \frac{1}{2}kr_0^2 + \frac{1}{2}kr_0^2 = kr_0^2$$

$k = 3.0\,N/m$ and $E = 12.0\,J$

$$E = kr_0^2 \implies 12.0\,\text{J} = (3.0\,\text{N/m})\,r_0^2 \implies r_0 = 2.0\,\text{m}$$

$$\frac{1}{2}Mv_0^2 = \frac{E}{2} = 6.0\,\text{J} \implies v_0^2 = \frac{12.0\,\text{J}}{2.0\,\text{kg}} \implies v_0 = \sqrt{6.0} \approx 2.45\,\text{m/s}$$

continued next page $\Longrightarrow$

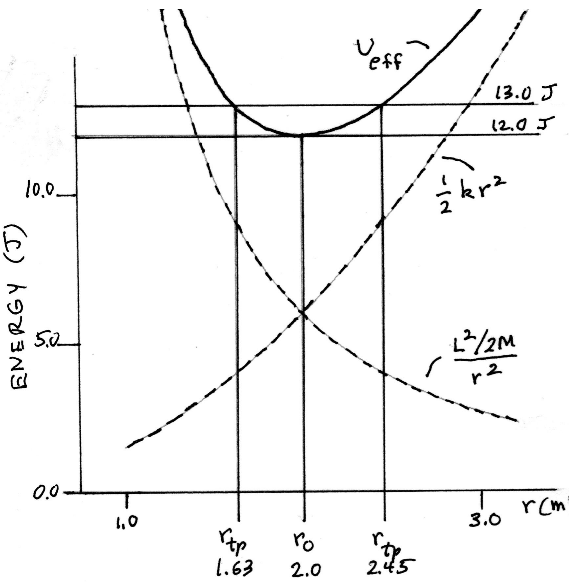
For this particular force, r_0 occurs where the two curves intersect, as shown in the figure below.

$$\frac{dU_{eff}}{dr} = -\frac{L^2/M}{r_0^3} + kr_0 = 0 \implies r_0^4 = \frac{L^2}{kM}$$

The curves intersect at

$$\frac{L^2}{2Mr_0^2} = \frac{1}{2}kr_0^2 \implies r_0^4 = \frac{L^2}{kM}$$

The radial blow changes the total mechanical energy but not the angular momentum = Mv_0r_0, so U_{eff} is unchanged, as indicated in the figure. The radial blow increases E by



$$\Delta E = \frac{1}{2}M\dot{r}^2 = \frac{1}{2}(2.0\,\text{kg})(1.0\,\text{m/s})^2 = 1.0\,\text{J}$$

$$E_f = E_i + \Delta E = 12.0 + 1.0 = 13.0\,\text{J}$$

(c)

$$U_{eff} = \frac{1}{2}kr^2 + \frac{L^2}{2Mr^2}$$

$$= \frac{1}{2}kr^2 + \frac{1}{2}\frac{(Mv_0r_0)^2}{2Mr^2} \quad (1)$$

At the turning points, the kinetic energy is 0, $E = U_{eff}$, and $r = r_{tp}$.

(b) With $U_{eff} = 13.0$ J, $M = 2.0$ kg, $v_0 = \sqrt{6.0}$ m/s, $k = 3.0$ N/m,

$$U_{eff} = \frac{1}{2}kr_{tp}^2 + \frac{1}{2}\frac{Mv_0^2r_0^2}{r_{tp}^2}$$

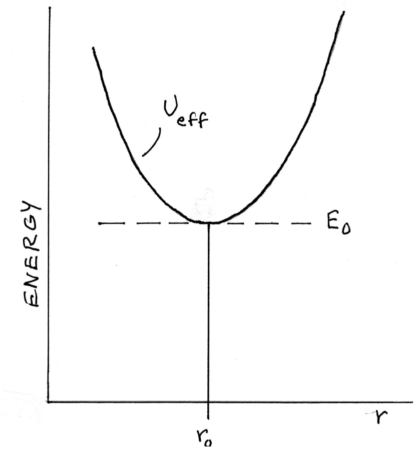
$$13.0 = \frac{1}{2}(3.0)\,r_{tp}^2 + \frac{1}{2}\frac{(2.0)(6.0)(2.0)^2}{r_{tp}^2}$$

$$0 = \frac{3}{2}\,r_{tp}^4 - 13.0\,r_{tp}^2 + 24.0$$

$$r_{tp}^2 = \frac{13.0 \pm \sqrt{169.0 - 144.0}}{3.0} = \frac{13.0 \pm 5.0}{3.0}$$

$$r_{tp} = 1.63\,\text{m},\ 2.45\,\text{m}$$

10.6 $\mathbf{r^4}$ *central force*



$$\mathbf{F} = -Kr^4\,\hat{\mathbf{r}}$$
$$U(r) = K\int_0^r r'^4\,dr' = \frac{1}{5}Kr^5$$
$$U_{eff} = U(r) + \frac{l^2}{2mr^2} = \frac{1}{5}Kr^5 + \frac{l^2}{2mr^2}$$

For circular motion at $r = r_0$,

$$0 = \left.\frac{dU_{eff}}{dr}\right|_{r_0} = Kr_0^4 - \left(\frac{l^2}{m}\right)\frac{1}{r_0^3} \implies r_0 = \left(\frac{l^2}{Km}\right)^{\frac{1}{7}}$$
$$E_0 = U_{eff}(r_0) = \frac{1}{5}Kr_0^5 + \left(\frac{l^2}{2m}\right)\frac{1}{r_0^2}$$

From Sec. 10.3.3, U_{eff} includes the kinetic energy due to tangential motion; for circular motion, there is no kinetic energy due to radial motion.

$$E_0 = \frac{1}{5}K\left(\frac{l^2}{Km}\right)^{\frac{5}{7}} + \left(\frac{l^2}{2m}\right)\left(\frac{Km}{l^2}\right)^{\frac{2}{7}}\left(\frac{Km}{l^2}\right)^{\frac{5}{7}}\left(\frac{l^2}{Km}\right)^{\frac{5}{7}} = \frac{7}{10}K\left(\frac{l^2}{Km}\right)^{\frac{5}{7}}$$

To find the oscillation frequency ω, follow the approach in Sec. 6.2, and expand U_{eff} about r_0. Let $r \ll r_0$ be a small displacement.

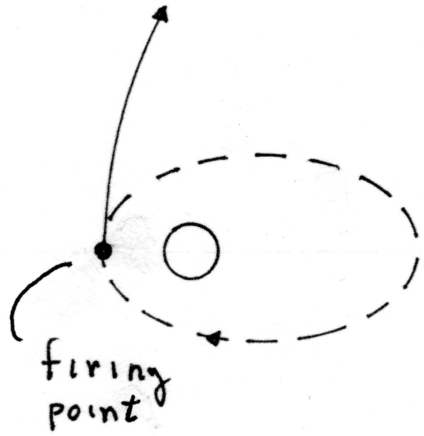
$$U_{eff} = U_{eff}(r_0) + 0 + \frac{1}{2}\left.\frac{d^2U_{eff}}{dr^2}\right|_{r_0} r^2 + \ldots \qquad \approx U_{eff}(r_0) + \frac{1}{2}\left(4Kr_0^3 + \frac{3l^2}{mr_0^4}\right)r^2$$
$$= U_{eff}(r_0) + \frac{1}{2}\left(4K\left(\frac{l^2}{Km}\right)^{\frac{3}{7}} + \left(\frac{3l^2}{m}\right)\left(\frac{Km}{l^2}\right)^{\frac{4}{7}}\left(\frac{Km}{l^2}\right)^{\frac{3}{7}}\left(\frac{l^2}{Km}\right)^{\frac{3}{7}}\right)r^2$$
$$= U_{eff}(r_0) + \frac{1}{2}\left(7K\left(\frac{l^2}{Km}\right)^{\frac{3}{7}}\right)r^2$$

The effective spring constant k is therefore

$$k = 7K\left(\frac{l^2}{Km}\right)^{\frac{3}{7}}$$
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{7K\left(\frac{l^2}{Km}\right)^{\frac{3}{7}}}{m}}$$

10.7 *Transfer to escape*

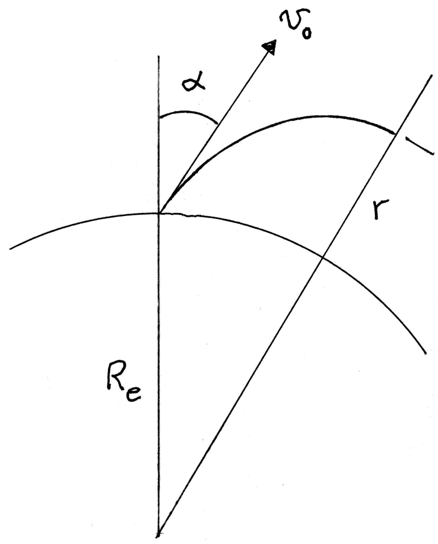
The mechanical energy in an elliptic orbit is < 0. For escape, the energy must be increased to ≥ 0. The satellite does not move far during the brief firing time, so the potential energy is essentially unchanged. The change in kinetic energy is



$$\Delta K = \frac{1}{2}m(\mathbf{v}_i + \Delta\mathbf{v})^2 - \frac{1}{2}mv_i^2$$
$$= m\,(\mathbf{v}_i \cdot \Delta\mathbf{v}) + \frac{1}{2}m\,\Delta v^2$$

The change is largest if $\Delta\mathbf{v} \parallel \mathbf{v}_i$ and when $\mathbf{v}_i$ is largest. These conditions are satisfied best at the closest point (perigee).

10.8 *Projectile rise*



$$E_i = U(R_e) + \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mR_e}$$

where $l = mv_0 \sin\alpha\, R_e$.

$$E_i = -\frac{GM_e m}{R_e} + \frac{1}{2}mv_0^2$$

At the top of the trajectory, $\dot{r} = 0$.

$$E_f = -\frac{GM_e m}{r} + \frac{l^2}{2mr^2}$$
$$= -\frac{GM_e m}{r} + \frac{1}{2}mv_0^2 \sin^2\alpha \left(\frac{R_e}{r}\right)^2$$

The rocket is in free flight, so $E_i = E_f$.

$$-\frac{GM_e m}{R_e} + \frac{1}{2}mv_0^2 = -\frac{GM_e m}{r} + \frac{1}{2}mv_0^2 \sin^2\alpha \left(\frac{R_e}{r}\right)^2$$
$$-\frac{GM_e}{R_e} + \frac{1}{2}v_0^2 = -\frac{GM_e}{r} + \frac{1}{2}v_0^2 \sin^2\alpha \left(\frac{R_e}{r}\right)^2 \quad (1)$$

It is given that $v_0 = \sqrt{\frac{GM_e}{R_e}}$ so Eq. (1) becomes, with $x \equiv r/R_e$,

$$-1 = -\frac{2}{x} + \frac{\sin^2\alpha}{x^2} \implies 0 = x^2 - 2x + \sin^2\alpha$$
$$x = \frac{2 \pm \sqrt{4 - 4\sin^2\alpha}}{2} \implies r = R_e(1 \pm \cos\alpha)$$

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Take the + sign, because $r > R_e$. so that $r = R_e(1 + \cos\alpha)$ The rocket rises to a height $R_e \cos\alpha$ above the Earth's surface.

10.9 *Halley's comet*

The period of an elliptic orbit depends only on the major axis A and the mass of the attractor. In this problem, the reduced mass μ is very nearly the mass of the comet. Using Eq. (10.31),

$$U(r) = -\frac{C}{r} = -\frac{GM_{Sun}\,\mu}{r}$$

$$\frac{\mu}{C} = \frac{1}{GM_{Sun}} \implies A = \left(\frac{2}{\pi^2}T^2 GM_{Sun}\right)^{\frac{1}{3}}$$

$$T = (76\,\text{years})\left(\frac{3.16\times 10^7\,\text{s}}{\text{year}}\right) = 2.40\times 10^9\,\text{s}$$

$$G = 6.67\times 10^{-11}\,\text{m}^3\cdot\text{kg}^{-1}\cdot\text{s}^{-2} \qquad M_{Sun} = 1.99\times 10^{30}\,\text{kg}$$

$$A = 5.37\times 10^{12}\,\text{m}$$

For comparison, the diameter of the Earth's orbit is 2.98×10^{11} m.
From Eq. (10.21), the equation of an elliptic orbit is

$$r = \frac{r_0}{1-\epsilon\cos\theta}$$

$$r_{min} \equiv r_{perihelion} = \frac{r_0}{1+\epsilon}$$

$$r_{max} \equiv r_{aphelion} = \frac{r_0}{1-\epsilon}$$

$$A = r_{perihelion} + r_{aphelion} = r_0\left(\frac{1}{1+\epsilon} + \frac{1}{1-\epsilon}\right) = \frac{2r_0}{1-\epsilon^2}$$

$$r_0 = \frac{1}{2}A(1-\epsilon^2) = \frac{1}{2}(5.37\times 10^{12}\,\text{m})(1-(0.967)^2) = 1.74\times 10^{11}\,\text{m}$$

(a)

$$r_{perihelion} = \frac{1.74\times 10^{11}}{1.967} = 8.86\times 10^{10}\,\text{m}$$

$$r_{aphelion} = \frac{1.74\times 10^{11}}{1-0.967} = 5.27\times 10^{12}\,\text{m}$$

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(b) The comet's maximum speed v_{max} occurs at perihelion, as it must because of the law of equal areas. Here are two ways of finding v_{max}.
method 1: angular momentum $L = \mu\, v_{max} r_{perihelion}$

From Eq. (10.19),

$$L^2 = r_o \mu C = r_0 \mu^2 GM_{Sun}$$

$$v_{max} = \frac{L}{\mu\, r_{perihelion}} = \frac{\sqrt{r_0 GM_{Sun}}}{r_{perihelion}} = \frac{\sqrt{(1.74 \times 10^{11})(6.67 \times 10^{-11})(1.99 \times 10^{30})}}{8.86 \times 10^{10}} = 5.42 \times 10^4 \text{ m/s}$$

method 2: Eq. (10.30)

$$\begin{aligned} v_{max}^2 &= \frac{2C}{\mu}\left(\frac{1}{r_{perihelion}} - \frac{1}{A}\right) \\ &= (2GM_{Sun})\left(\frac{A - r_{perihelion}}{Ar_{perihelion}}\right) = (2GM_{Sun})\left(\frac{r_{aphelion}}{Ar_{perihelion}}\right) \\ &= (2)(6.67 \times 10^{-11})(1.99 \times 10^{30})\left(\frac{5.27 \times 10^{12}}{(5.37 \times 10^{12})(8.86 \times 10^{10})}\right) = 2.94 \times 10^9 \end{aligned}$$

$$v_{max} = 5.42 \times 10^4 \text{ m/s}$$

10.10 *Satellite with air friction*

For m in a circular orbit under an attractive force C/r^2,

$$\frac{mv^2}{r} = \frac{C}{r^2} \implies K = \frac{1}{2}mv^2 = \frac{1}{2}\frac{C}{r} = -\frac{1}{2}U(r)$$

(a)

$$E = K + U = -\frac{1}{2}U(r) + U(r) = \frac{1}{2}U(r) = -\frac{1}{2}\frac{C}{r} = -K$$

(b) The energy loss per revolution ΔE due to friction is $-2\pi rf$.

$$\Delta E = \left(\frac{dE}{dr}\right)\Delta r \implies \Delta r = \frac{\Delta E}{dE/dr} = \frac{-2\pi rf}{C/(2r^2)} = -\frac{4\pi r^3 f}{C}$$

The radius of the orbit decreases because of friction.

(c) In the circular orbit, $E = -K$.

$$\Delta K = -\Delta E = +2\pi rf$$

Friction causes the total mechanical energy E to decrease. Because $K = -E$, a decrease in the total energy is accompanied by an increase in the kinetic energy. Friction causes the satellite's speed to increase!

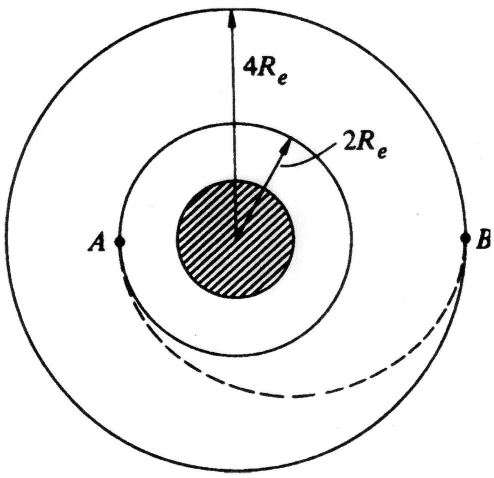
10.11 *Mass of the Moon*

Let the major axis be A, and the period T. Kepler's Third Law gives

$$
\begin{aligned}
T^2 &= \frac{\pi^2}{2}\frac{\mu}{C}A^3 = \frac{\pi^2}{2}\frac{1}{GM_{Moon}}A^3 \\
M_{Moon} &= \frac{\pi^2}{2}\left(\frac{A^3}{T^2}\right)\left(\frac{1}{G}\right) \\
A &= (1861 + 1839)\,\text{km} = 3.70 \times 10^3\,\text{m} \\
T &= (119\,\text{minutes})\left(\frac{1\,\text{minute}}{60\,\text{s}}\right) = 7.14 \times 10^3\,\text{s} \\
G &= 6.67 \times 10^{-11}\,\text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2} \\
M_{Moon} &= \frac{\pi^2}{2}\left[\frac{(3.70 \times 10^6\,\text{m})^3}{(7.14 \times 10^3\,\text{s})^2}\right]\left[\frac{1}{6.67 \times 10^{-11}\,\text{m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}}\right)] \\
&= 7.35 \times 10^{22}\,\text{kg}
\end{aligned}
$$

10.12 *Hohmann transfer orbit*

Let v^c be the speed in a circular orbit, and let v^e be the speed in the elliptic transfer orbit.
The total mechanical energy E of a mass m moving in the Earth's gravitational field is



$$
E = \frac{1}{2}mv^2 - \frac{GmM_e}{r} = \frac{1}{2}mv^2 - \frac{mgR_e^2}{r}
$$

The equation of motion in a circular orbit is

$$
\begin{aligned}
\frac{m(v^c)^2}{r} &= \frac{mgR_e^2}{r^2} \\
E &= \frac{1}{2}\frac{mgR_e^2}{r} - \frac{mgR_e^2}{r} = -\frac{1}{2}\frac{mgR_e^2}{r}
\end{aligned}
$$

Note that $\sqrt{gR_e}$ has the dimensions of velocity.

$$
\sqrt{gR_e} = \sqrt{(9.8\,\text{m} \cdot \text{s}^{-2})(6.4 \times 10^6\,\text{m})} = 7.92 \times 10^3\,\text{m/s}
$$

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(a)

$$r_A = 2R_e \implies E_A = -\frac{1}{2}\left(\frac{mgR_e^2}{2R_e}\right) = -\frac{1}{4}mgR_e$$
$$r_B = 4R_e \implies E_B = -\frac{1}{2}\left(\frac{mgR_e^2}{4R_e}\right) = -\frac{1}{8}mgR_e$$
$$\Delta E = \left(-\frac{1}{8} + \frac{1}{4}\right)mgR_e = \frac{1}{8}mgR_e$$
$$= \frac{1}{8}(3 \times 10^3 \text{ kg})(7.92 \times 10^3 \text{ m/s})^2$$
$$= 2.35 \times 10^{10} \text{ J}$$

(b) The transfer orbit is a semi-ellipse with perigee at A and apogee at B. The major axis A of the ellipse is

$$A = r_A + r_B = 2R_e + 4R_e = 6R_e$$

The energy E' is

$$E' = -\frac{mgR_e^2}{A} = -\frac{1}{6}mgR_e$$

$K = -E$ in a circular orbit, so

$$K_A = \frac{1}{2}m(v_A^c)^2 = -E_A = \frac{1}{4}mgR_e$$
$$v_A^c = \sqrt{\frac{gR_e}{2}}$$

The speed at A for transfer is

$$K'_A + U_A = E'$$
$$\frac{1}{2}m(v_A^e)^2 = -\frac{1}{6}mgR_e + \frac{1}{2}mgR_e = \frac{1}{3}mgR_e$$
$$v_A^e = \sqrt{\frac{2gR_e}{3}}$$
$$\Delta v_A = v_A^e - v_A^c = \sqrt{\frac{2gR_e}{3}} - \sqrt{\frac{gR_e}{2}} = (0.109)(7.92 \times 10^3 \text{ m/s}) = 864 \text{ m/s}$$

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The initial speed at B is

$$\frac{1}{2}m(v_B^e)^2 = E' - U_B = -\frac{1}{6}mgR_e + \frac{mgR_e^2}{4R_e} = \frac{1}{12}mgR_e$$

$$v_B^e = \sqrt{\frac{gR_e}{6}}$$

The final speed at B in the new circular orbit is

$$\frac{1}{2}m(v_B^c)^2 = \frac{1}{2}\frac{GmM_e}{R_B} = \frac{1}{8}mgR_e$$

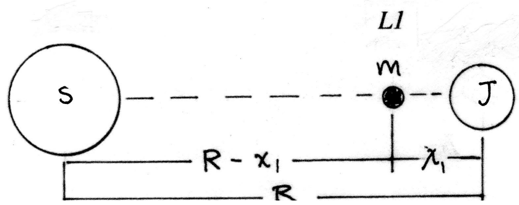
$$v_B^c = \frac{1}{2}\sqrt{gR_e}$$

The change in speed at B is

$$\Delta v_B = v_B^c - v_B^e = \left(\frac{1}{2} - \sqrt{\frac{1}{6}}\right)\sqrt{gR_e} = 727\text{ m/s}$$

This problem can also be solved readily using Eq. (10.30).

10.13 *Lagrange point L1*



Take the center of mass at the center of the Sun, because $M_{Sun} \gg M_J$ and the asteroid's mass m is small. Jupiter rotates about the Sun at angular speed Ω. The asteroid also rotates about the Sun at rate Ω, so the three bodies remain in line, the characteristic behavior at a Lagrange point.

$$M_J R\Omega^2 = \frac{GM_{Sun}M_J}{R^2} \implies \Omega^2 = \frac{GM_{Sun}}{R^3}$$

(a) In the rotating system, the forces on m are the real gravitational forces of the Sun and Jupiter, and the fictitious force $m(R - x_1)\Omega^2$. In equilibrium, the total force on m is 0.

$$-\frac{GmM_{Sun}}{(R-x_1)^2} + \frac{GmM_J}{x_1^2} + m(R-x_1)\Omega^2 = 0$$

$$-\frac{M_{Sun}}{(R-x_1)^2} + \frac{M_J}{x_1^2} + (R-x_1)\frac{M_{Sun}}{R^3} = 0 \quad (1)$$

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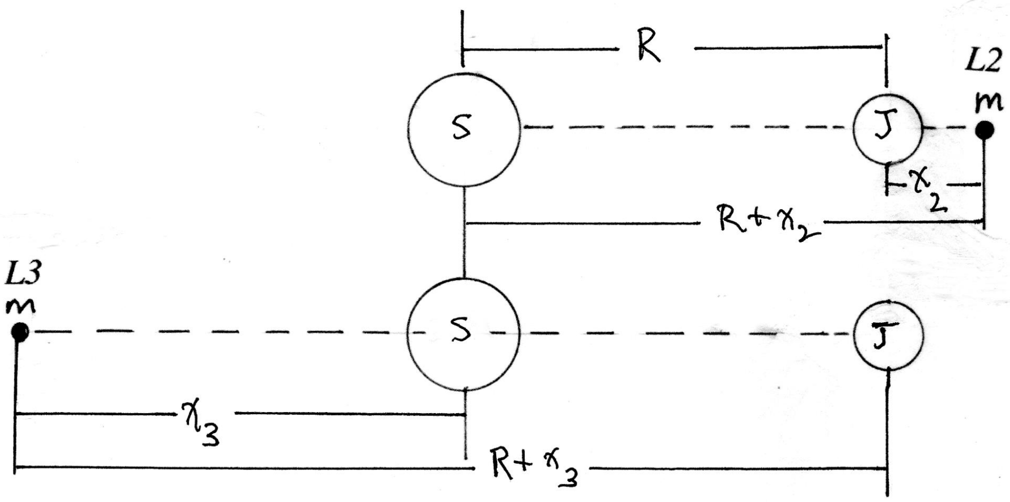
(b) *data*:

$$R = 7.78 \times 10^{11}\,\text{m} \quad x_1 = 5.31 \times 10^{10}\,\text{m} \quad R - x_1 = 7.25 \times 10^{11}\,\text{m}$$
$$M_{Sun} = 1.99 \times 10^{30}\,\text{kg} \quad M_J = 1.90 \times 10^{27}\,\text{kg}$$

Inserting in Eq. (1),

$$-\frac{1.99 \times 10^{30}}{(7.25 \times 10^{11})^2} + \frac{1.90 \times 10^{27}}{(5.31 \times 10^{10})^2} + \frac{(7.25 \times 10^{11})(1.90 \times 10^{30})}{(7.78 \times 10^{11})^3} \stackrel{?}{=} 0$$
$$-0.02 \times 10^6 \stackrel{?}{=} 0$$

The calculated result is consistent with 0, within the numerical accuracy of the data and the approximations.



(c) As required for a Lagrange point, all three bodies rotate about their center of mass at the same angular speed Ω. Neglecting perturbations, the configuration is therefore unchanging during the rotation.
In the rotating system, m is acted upon by the real gravitational forces of the Sun and Jupiter, and by the fictitious centrifugal force $mr\Omega^2$. The forces are in balance at equilibrium, with fixed Ω, so a larger gravitational force leads to a larger orbital radius r for the asteroid to increase the centrifugal force for balance.
Lagrange point L2: Both the Sun and Jupiter exert an additive inward gravitational force on m, so the asteroid's orbit is somewhat outside R as indicated, with $R + x_2 > R$, or $x_2 > 0$.
Lagrange point L3: At L3, m is on the Sun-Jupiter line on the opposite side from Jupiter. If Jupiter were not present, the radius of the asteroid's orbit would be R. However, Jupiter adds a moderate gravitational force, so $x_3 > R$.

10.14 *Speed of S2 around Sgr A**

For a body in elliptic orbit about an attractor, the distance of closest approach has been termed the *periapse*, and the farthest distance the *apoapse*.
The orbiting body's fastest speed occurs at the periapse, as a consequence of the Law of Equal Areas. Let A be the major axis.

$$r_{periapse} = \frac{r_0}{1+\epsilon} \qquad r_{apoapse} = \frac{r_0}{1-\epsilon}$$
$$A = r_{periapse} + r_{apoapse} = \frac{2r_0}{1-\epsilon^2} \implies r_0 = \frac{1}{2}A(1-\epsilon^2)$$

Using Eq. (10.30), the fastest speed of the orbiting star S2 is given by

$$\begin{aligned} v_{max}^2 &= 2C\mu\left(\frac{1}{r_{periapse}} - \frac{1}{A}\right) \\ &= 2GM_{SgrA^*}\left(\frac{1+\epsilon}{r_0} - \frac{1}{A}\right) = \frac{2GM_{SgrA^*}}{A}\left(\frac{2(1+\epsilon)}{1-\epsilon^2} - 1\right) \\ &= \frac{2GM_{SgrA^*}}{A}\left(\frac{1+\epsilon}{1-\epsilon}\right) \end{aligned}$$

data:

$$G = 6.67\times10^{-11}\,\mathrm{m}^3\cdot\mathrm{kg}^{-1}\cdot\mathrm{s}^{-2} \qquad \epsilon \approx 0.87 \qquad A = 2.9\times10^{14}\,\mathrm{m}$$
$$M_{SgrA^*} = 4\times10^6 M_{Sun} = 8\times10^{36}\,\mathrm{kg}$$
$$v_{max} \approx 7300\,\mathrm{km/s}$$

which is approximately 250 times the speed of the Earth around the Sun.
With a similar calculation, the minimum speed v_{min} of S2 (at apoapse) is

$$v_{min}^2 = \frac{2GM_{SgrA^*}}{A}\left(\frac{1-\epsilon}{1+\epsilon}\right)$$
$$v_{min} \approx 500\,\mathrm{km/s}$$

also many times faster than the Earth.

10.15 *Sun-Earth mass ratio*

From the statement of Kepler's third law in Eq. (10.31),

$$T^2 = \frac{\pi^2 \mu}{2C} A^3 = \frac{\pi^2 \mu}{2GmM} A^3$$

where m is the mass of the satellite and M is the mass of the attractor. For the cases in this problem, where $m \ll M$, $\mu = m$ to good accuracy.

$$\frac{A^3}{T^2} = \frac{2GM}{\pi^2}$$

Taking ratios for $M = M_{Sun}$ and $M = M_{Earth}$,

$$\frac{(A^3/T^2)_{Sun}}{(A^3/T^2)_{Earth}} = \frac{M_{Sun}}{M_{Earth}}$$

From Table 10.1,

$$\left(\frac{A^3}{T^2}\right)_{Sun} = 2.69 \times 10^{10}\ \mathrm{km}^3 \cdot \mathrm{s}^{-2}$$

From the Earth satellite data in the problem,

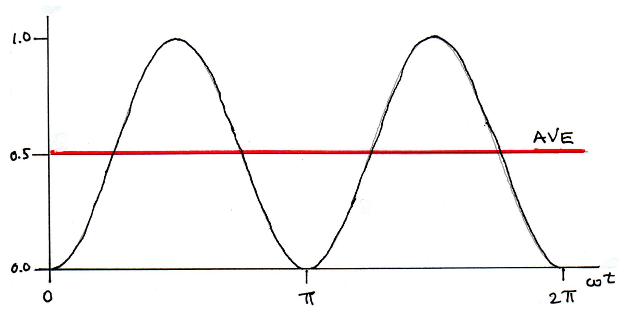
$$\left(\frac{A^3}{T^2}\right)_{Earth} = 8.07 \times 10^{4}\ \mathrm{km}^3 \cdot \mathrm{s}^{-2}$$

$$\frac{M_{Sun}}{M_{Earth}} = 3.33 \times 10^5$$

11 THE HARMONIC OSCILLATOR

11.1 *Time average of* $\sin^2$

Formally:



$$\langle \sin^2 \rangle_{ave} = \frac{1}{2\pi}\int_0^{2\pi} \sin^2 u \, du$$

$$\cos(u+u) = \cos^2 u - \sin^2 u = 1 - 2\sin^2 u$$

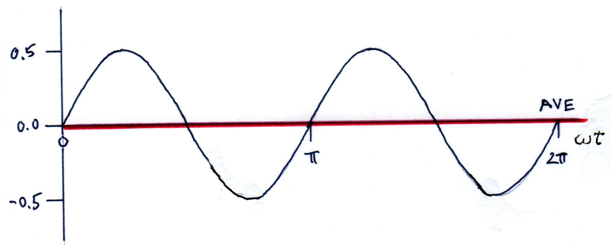
$$\sin^2 u = \frac{1}{2}(1 - \cos 2u)$$

$$\langle \sin^2 \rangle_{ave} = \frac{1}{2}\left(\frac{1}{2\pi}\int_0^{2\pi} (1 - \cos 2u)\, du\right) = \frac{1}{2}$$

Qualitatively: Note from the sketch that $\sin^2$ is symmetric about the value $1/2$.

11.2 *Time average of* $\sin \times \cos$

Formally:



$$\langle \sin \times \cos \rangle_{ave} = \frac{1}{2\pi}\int_0^{2\pi} \sin u \, \cos u \, du$$

$$\sin u \, \cos u = \frac{1}{2}\sin(2u)$$

$$\langle \sin \times \cos \rangle_{ave} = \frac{1}{4\pi}\int_0^{2\pi} \sin(2u)\, du = 0$$

Qualitatively: This is a product of an odd function $\sin u = -\sin(-u)$ and an even function $\cos u = +\cos(-u)$, so the area under the product vanishes for a full period.

11.3 *Damped mass and spring*

$$\omega = \sqrt{\frac{k}{m}}$$

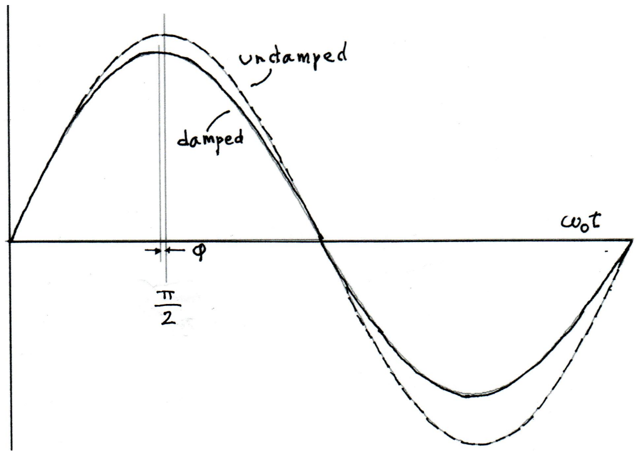
$$k = m\omega^2 = (0.3\,\text{kg})\left[(2\,\text{cycles}\cdot\text{s}^{-1})(2\pi\,\text{radians}\cdot\ \text{cycle}^{-1})\right]^2$$

$$= 47.4\,\text{N}\cdot\text{m}^{-1}$$

$$Q = \frac{\omega}{\gamma} \implies \gamma = \frac{\omega}{Q}$$

$$= \frac{4\pi\,\text{rad}\cdot\text{s}^{-1}}{60} = 0.21\,\text{s}^{-1}$$

11.4 *Phase shift in a damped oscillator*



Let x_u be the coordinate for the undamped case, and x_d for the damped case.

$$x_u = x_0 \sin(\omega_0 t)$$

$$x_d = x_0 e^{-\frac{\gamma}{2}t}\sin(\omega_0 t)$$

Maximum of x_u is at t_{um}:

$$\frac{dx_u}{dt} = \omega_0 x_0 \cos(\omega_0 t_{um}) = 0$$

$$t_{um} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

Maximum of x_d is at t_{dm}:

$$\frac{dx_d}{dt} = x_0 e^{-\frac{\gamma}{2}t}\left[\omega_0\cos(\omega_0 t_{dm}) - \frac{\gamma}{2}\sin(\omega_0 t_{dm})\right] = 0$$

$$\frac{\cos(\omega_0 t_{dm})}{\sin(\omega_0 t_{dm})} = \frac{\gamma}{2\omega_0} = \frac{2}{Q} \quad (1)$$

For $Q \gg 1$, the condition is satisfied near $\pi/2$, $3\pi/2$, …. For example,

$$\omega_0 t_{dm} \approx \frac{\pi}{2} - \phi$$

$$\cos(\pi/2 - \phi) = \sin\phi \approx \phi$$

$$\sin(\pi/2 - \phi) = \cos\phi \approx 1$$

If $Q \gg 1$ the condition in Eq. (1) is then satisfied for

$$\phi = \frac{1}{2Q}$$

For clarity, the value of Q in the figure is not very large.

11.5 *Logarithmic decrement*

$$x = x_0 e^{-\frac{\gamma}{2}t} \sin(\omega_0 t)$$

Let two successive positive maxima occur at t_1 and t_2. Let R be the ratio x_1/x_2.

$$R = \frac{x_1}{x_2} = \frac{x_0 e^{-\frac{\gamma}{2}t} \sin(\omega_0 t_1)}{x_0 e^{-\frac{\gamma}{2}t} \sin(\omega_0 t_2)}$$

$$\omega_0 t_2 = \omega_0 t_1 + 2\pi \implies \sin(\omega_0 t_2) = \sin(\omega_0 t_1)$$

$$R = e^{-\frac{\gamma}{2}(t_1 - t_2)} = e^{\frac{\gamma}{2}\left(\frac{2\pi}{\omega_0}\right)} = e^{\frac{\gamma\pi}{\omega_0}} = e^{\frac{\pi}{Q}}$$

$$\delta = \ln R = \frac{\pi}{Q}$$

11.6 *Logarithmic decrement of a damped oscillator*

$$k = m\omega_0^2 = (5\,\text{kg})\left[(0.5\,\text{cycles}\cdot\text{s}^{-1})(2\pi\,\text{rad}\cdot\text{cycle}^{-1})\right]^2 = 49.3\,\text{N}\cdot\text{m}^{-1}$$

From the preceding problem (problem 11.5),

$$\delta = \frac{\pi}{Q} = \frac{\pi\gamma}{\omega_0}$$

$$\gamma = \frac{\delta\,\omega_0}{\pi} = \frac{(0.02)(1.0\,\pi\,\text{rad}\cdot\text{s}^{-1})}{\pi} = 0.02\,\text{s}^{-1}$$

The damping constant $b = \gamma m$ is defined in Sec. 11.3.

$$b = \gamma m = (0.02\,\text{s}^{-1})(5.0\,\text{kg}) = 0.10\,\text{kg}\cdot\text{s}^{-1} = 0.10\,\text{N}\cdot\text{s}\cdot\text{m}^{-1}$$

11.7 *Critically damped oscillator*

(a) The equation of motion is

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0 \implies \ddot{x} + 2\omega_0\dot{x} + \omega_0^2 x = 0 \quad (1)$$

$$x = (A + Bt)e^{-\omega_0 t} \qquad \dot{x} = [B - \omega_0(A + Bt)]\,e^{-\omega_0 t} \qquad \ddot{x} = \left[-2B\omega_0 + \omega_0^2(A + Bt)\right]e^{-\omega_0 t}$$

Substituting in Eq. (1), omitting the common factor $e^{-\omega_0 t}$,

$$0 \stackrel{?}{=} -2B\omega_0 + \omega_0^2(A + Bt) + 2\omega_0\,[B - \omega_0(A + Bt)] + \omega_0^2(A + Bt) = 0$$

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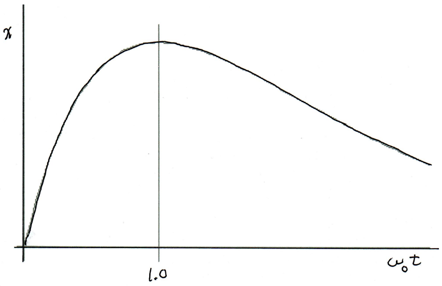
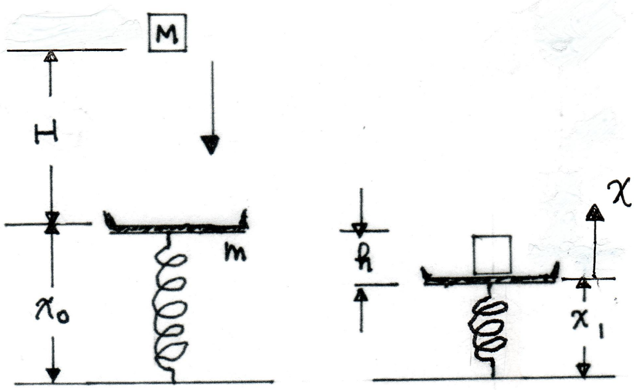
(b) The initial conditions are



$$x(0) = 0 \implies A = 0$$
$$\dot{x}(0) = \frac{I}{m} \implies B = \frac{I}{m}$$
$$x = \frac{I}{m} t e^{-\omega_0 t}$$
$$0 = \left.\frac{dx}{dt}\right|_{t_{max}} = \frac{I}{m}(1 - \omega_0 t_{max})\, e^{-\omega_0 t} \implies t_{max} = \frac{1.0}{\omega_0}$$

11.8 *Scale spring constant*

Let m be the mass of the empty pan, and let M be the mass of the falling block.



(a)

Before M lands,

$$kx_0 = -mg$$

After M lands and is at rest,

$$kx_1 = -(m + M)g$$
$$k(x_1 - x_0) = -Mg$$
$$k = \frac{Mg}{h} = \frac{(10.0\,\mathrm{kg})(9.8\,\mathrm{m \cdot s^{-2}})}{0.10\,\mathrm{m}} = 980\,\mathrm{N \cdot m^{-1}}$$

(b) Let $x \equiv x(t)$ be the coordinate of the pan, measured from final equilibrium x_1. The equation of motion for a critically damped oscillator is given in Eq. (11.43) in Note 11.2:

$$x = (A + Bt)te^{-\frac{\gamma}{2}t} \equiv (A + Bt)e^{-\omega_0 t}$$

Qualitatively, the pan moves down below x_1 after the collision and then moves upward back to the equilibrium position x_1, according to the behavior for a critically damped oscillator (no oscillations).
The initial conditions are

$$x|_{t=0} = x_0 - x_1 = h \implies A = h = 0.10\,\mathrm{m}$$
$$\dot{x}(0) = v_0 = \left[B - \omega_0 (A + Bt)e^{-\omega_0 t}\right]_{t=0} = B - \omega_0 A$$
$$B = v_0 + \omega_0 h$$
$$\omega_0 = \sqrt{\frac{k}{m + M}} = \sqrt{\frac{980}{10.0 + 2.0}} = 9.04\,\mathrm{s^{-1}}$$

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Mechanical energy of M is conserved until it strikes the pan, so its speed V immediately before the collision is

$$\frac{1}{2}MV^2 = MgH \implies V = \sqrt{2gH} = \sqrt{(2)(9.8\,\mathrm{m \cdot s^{-2}})(0.50\,\mathrm{m})} = 3.13\,\mathrm{m/s}$$

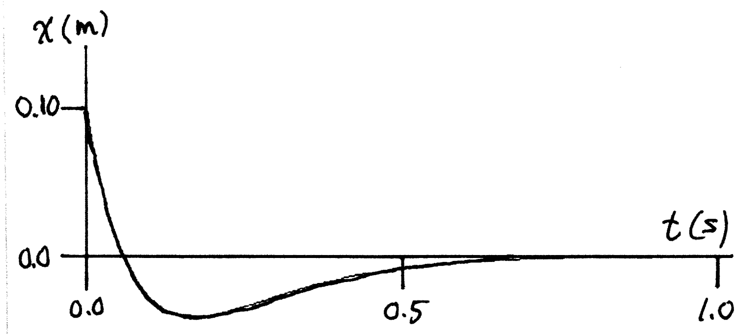
To find v_0, use conservation of momentum, because mechanical energy is not conserved in the collision of M with the pan.

$$-MV = (m+M)v_0 \implies v_0 = -\left(\frac{M}{m+M}\right)V = -\left(\frac{10.0}{10.0+2.0}\right)(3.13) = -2.61\,\mathrm{m/s}$$

$$B = v_0 + \omega_0 h = -2.61 + (9.04)(0.10) = -1.71\,\mathrm{m/s}$$

$$x = (0.10 - 1.71t)e^{-9.04t} \quad (1)$$

The figure is a graph of the result, Eq. (1).



11.9 *Velocity and driving force in phase*

From Sec. (11.4), the driving force F_d is

$$F_d = F_0 \cos(\omega t) \quad (1)$$

From Eqs. (11.26) and (11.28), the motion is

$$x = X_0 \cos(\omega t + \phi)$$

$$\phi = \arctan\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right) \quad (2)$$

The velocity is

$$v = \frac{dx}{dt} = -\omega X_0 \sin(\omega t + \phi) \quad (3)$$

Comparing Eqs. (1) and (3), the condition for v and F_d to be in phase is

$$-\sin(\omega t + \phi) = \cos(\omega t) \quad (4)$$

Using $-\sin(\omega t + \phi) = -\sin(\omega t)\cos\phi - \cos(\omega t)\sin\phi$

condition (4) requires $\sin\phi = -1$ and $\cos\phi = 0$

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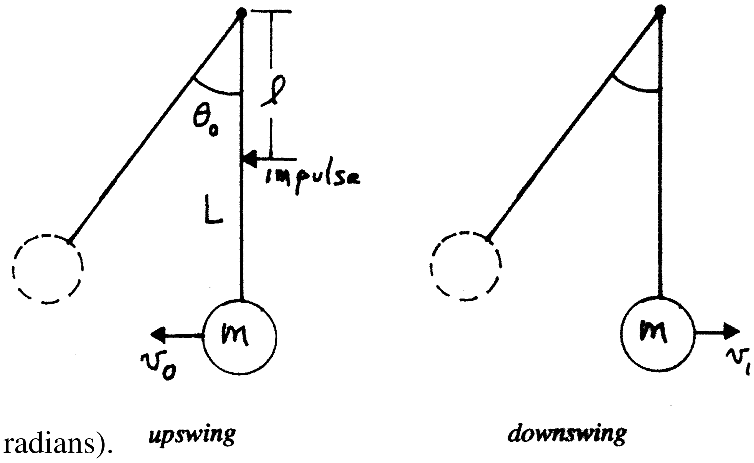
These requirements are both satisfied for

$$\phi = -\frac{\pi}{2} \implies \tan\phi \to -\infty$$

From Eq. (2), the tangent goes to infinity when $\omega = \omega_0$, at the resonance frequency of the undamped oscillator.

11.10 *Grandfather clock*

The pendulum loses mechanical energy ΔE as it swings, due to friction. As indicated in the sketch, its speed speed decreases slightly from v_0 to v_1 during a half cycle. The escapement provides an impulse every period to make up for the loss.



(a)

$$\Delta E = \frac{1}{2}mv_0^2 - \frac{1}{2}mv_1^2$$

ΔE is the energy lost per half cycle (π radians).

From Eq. (11.23),

$$Q = \frac{\text{energy of the oscillator}}{\text{energy dissipated per radian}}$$

$$= \frac{\frac{1}{2}mv_0^2}{\Delta E/\pi} = \frac{\pi(\frac{1}{2}mv_0^2)}{\frac{1}{2}m(v_0^2 - v_1^2)} = \frac{\pi v_0^2}{v_0^2 - v_1^2} = \frac{\pi v_0^2}{(v_0 - v_1)(v_0 + v_1)} \approx \frac{\pi v_0^2}{2v_0(v_0 - v_1)}$$

$$\Delta v = v_0 - v_1 = \frac{\pi v_0}{2Q}$$

The required impulse I is

$$I = m\Delta v = \frac{\pi v_0}{2Q}$$

The pendulum motion is

$$\theta = \theta_0 \sin(\omega t) \quad \text{where } \omega = \sqrt{\frac{g}{L}}$$

The speed v_0 at the beginning of the upswing is

$$v_0 = L\dot\theta = L(\omega\,\theta_0) = \sqrt{gL}\,\theta_0 \implies I = m\left(\frac{\pi\theta_0}{2Q}\right)\sqrt{gL}$$

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(b) The impulse I produces a change in speed $\Delta v = I/m$, so that the energy is increased by ΔE_I.

$$\Delta E_I = \frac{1}{2}m(v + \Delta v)^2 - \frac{1}{2}mv^2 = mv\Delta v + \frac{1}{2}m(\Delta v)^2 = Iv + \frac{I^2}{2m}$$

The point in the cycle where the impulse acts can vary due to mechanical imperfections. To minimize this effect, the impulse should be applied when v is not changing to first order with respect to θ, which is at the bottom of the swing. Proof: The energy equation is

$$E = \frac{1}{2}mv^2 + mgL(1 - \cos\theta) \quad (1)$$

To find where $dv/d\theta = 0$, differentiate Eq. (1), where $E = constant$.

$$0 = mv\frac{dv}{d\theta} + mgL\ \sin\theta = 0 + mgl\ \sin\theta \implies \sin\theta = 0$$

so $\theta = 0$, that is, at the bottom of the swing.

11.11 *Average stored energy*

The energy dissipated by a damped driven oscillator is the mechanical energy converted to heat by the viscous retarding force $F_v = bv$. The motion is

$$x = A\cos(\omega t + \phi) \qquad v = -A\omega\ \sin(\omega t + \phi)$$

The rate of energy dissipation is the power P. Brackets $\langle\ \rangle$ denote time averages.

$$P = vF_v = bv^2A^2\omega^2\ \sin^2(\omega t + \phi) \implies \langle P\rangle = \frac{1}{2}bA^2\omega^2$$

The oscillator traverses 1 radian in time $t_r = 1/\omega$. ΔE dissipated during t_r is

$$\Delta E = \langle P\rangle\, t_r = \frac{\langle P\rangle}{\omega} = \frac{1}{2}\omega\, bA^2$$

For a lightly damped oscillator,

$$\omega \approx \omega_0 = \sqrt{\frac{k}{m}}$$

The average energy $\langle E\rangle$ of the oscillator is

$$\langle E\rangle = \frac{1}{2}k\langle x^2\rangle + \frac{1}{2}m\langle v^2\rangle = \frac{1}{2}k\left(\frac{1}{2}A^2\right) + \frac{1}{2}m\left(\frac{1}{2}\omega^2 A^2\right) \approx \frac{1}{2}k\left(\frac{1}{2}A^2\right) + \frac{1}{2}m\left(\frac{1}{2}\omega_0^2 A^2\right) = \frac{1}{2}kA^2$$

$$\frac{\langle E\rangle}{\Delta E} = \frac{\frac{1}{2}kA^2}{\frac{1}{2}\omega\, bA^2} = \frac{k}{\omega\, b} \approx \frac{k}{\omega_0\, b} = \frac{k}{\omega_0\gamma m} = \frac{\omega_0}{\gamma} = Q$$

using $\gamma = b/m$, Eq. (11.9).

11.12 *Cuckoo clock*

A cuckoo clock is a pendulum clock with a mechanism that causes a bird model to pop out to announce the hour.
Let time averages be denoted by brackets $\langle\ \rangle$.
Let l be the length of the pendulum, and let θ_0 be the amplitude of swing. The mass of the pendulum is m, and the mass of the falling weight is M.
The power to the clock by the descending weight compensates the power dissipated by friction. If the weight descends a distance L in time T_d,

$$\langle P\rangle = \frac{MgL}{T_d}$$

The energy dissipated per radian is

$$\langle \Delta E\rangle = \frac{\langle P\rangle}{\omega}$$

The gravitational potential energy is $mgl(1 - \cos\theta) \approx mgl(\theta^2/2)$, so the average stored energy is

$$\langle E\rangle = \frac{1}{2}m\langle v^2\rangle + \frac{1}{2}mgl\langle\theta^2\rangle$$

The average kinetic and potential energies are equal.

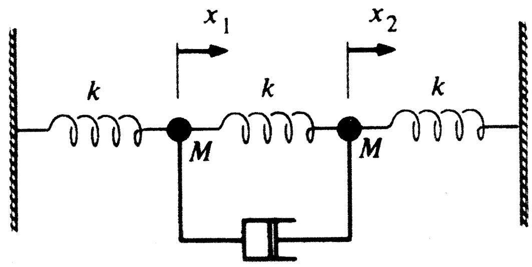
$$\begin{aligned}
\langle E\rangle &= \frac{1}{2}mgl\,\theta_0^2\\
Q &= \frac{\langle E\rangle}{\langle \Delta E\rangle} = \left(\frac{\frac{1}{2}mgl\,\theta_0^2}{\langle P\rangle}\right)\omega = \left(\frac{\frac{1}{2}mgl\,\theta_0^2}{MgL}\right)\omega\, T_d\\
&= \frac{1}{2}\left(\frac{m}{M}\right)\left(\frac{l}{L}\right)\theta_0^2\,\omega T_d\\
&= \frac{1}{2}\left(\frac{0.01\,\mathrm{kg}}{0.2\,\mathrm{kg}}\right)\left(\frac{0.25\,\mathrm{m}}{2.0\,\mathrm{m}}\right)(0.2\,\mathrm{rad})^2(2\pi\,\mathrm{rad/s})(8.64\times10^4\,\mathrm{s})\\
&= 68
\end{aligned}$$

The energy E_d to run the clock for 24 hours is

$$E_d = MgL = (0.2\,\mathrm{kg})(9.8\,\mathrm{m}\cdot\mathrm{s}^{-2})(2\,\mathrm{m}) = 3.9\,\mathrm{J}$$

Therefore a 1J battery could run the clock for only a little over 6 hours.

11.13 *Two masses and three springs*



(a) *equations of motion:*

$$M\ddot{x}_1 = -kx_1 - k(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) \quad (1)$$
$$M\ddot{x}_2 = -kx_2 - k(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1) \quad (2)$$

(b) Adding Eqs. (1) and (2),

$$M(\ddot{x}_1 + \ddot{x}_2) = -k(x_1 + x_2)$$
$$y_1 = x_1 + x_2 \implies M\ddot{y}_1 + ky_1 = 0 \quad (3)$$

Subtracting Eq. (2) from Eq. (1),

$$M(\ddot{x}_1 - \ddot{x}_2) = -k(x_1 - x_2) - 2k(x_1 - x_2) - 2b(\dot{x}_1 - \dot{x}_2)$$
$$y_2 = x_1 - x_2 \implies M\ddot{y}_2 = -3ky_2 - 2b\dot{y}_2 \quad (4)$$

Equations (3) and (4) each have only one dependent variable, y_1 or y_2, so in principle they can be solved directly (numerically if necessary).

(c) For mode y_2 Eq. (4), the motion is a damped transient and eventually becomes negligibly small, so that at long times $y_2 = x_1 - x_2 = 0$ and $x_1 = x_2$. The equation of motion for y_1 Eq. (3) is the equation for free undamped SHM. At long times, x_1 and x_2 move together so that their separation is constant and there is therefore no damping.
To evaluate the initial conditions, express y_1 and y_2 in terms of x_1 and x_2.

$$y_1(0) = x_1(0) + x_2(0) = 0 + 0 \implies y_1(0) = 0$$
$$\dot{y}_1(0) = \dot{x}_1(0) + \dot{x}_2(0) = v_0 + 0 \implies \dot{y}_1(0) = v_0$$
$$y_2(0) = x_1(0) - x_2(0) = 0 - 0 \implies y_2(0) = 0$$
$$\dot{y}_2(0) = \dot{x}_1(0) - \dot{x}_2(0) = v_0 - 0 \implies \dot{y}_2(0) = v_0$$

The solution of Eq. (3) is

$$y_1 = A\sin(\omega t) + B\cos(\omega t) \quad \text{where } \omega = \sqrt{\frac{k}{M}}$$
$$y_1(0) = B = 0$$
$$\dot{y}_1 = \omega A\cos(\omega t)$$
$$\dot{y}_1(0) = \omega A = v_0 \implies y_1 = A\sin(\omega t) = \frac{v_0}{\omega}\sin(\omega t)$$

At long times, $y_2 \to 0$, so $x_1 - x_2 \to 0$, hence $x_1 = x_2$ at long times. Then $y_1 = x_1 + x_2 = 2x_1$, so

$$x_1 = \frac{v_0}{2\omega}\sin(\omega t) \quad \text{at long times}$$

11.14 *Motion of a driven damped oscillator*

(a) *equation of motion for a forced damped oscillator:*

$$\ddot{x}_a + \gamma\,\dot{x}_a + \omega_0^2\,x_a = \left(\frac{F_0}{m}\right)\cos(\omega\,t) \quad (1)$$

The steady-state solution to Eq. (1) is

$$x_a = X_0\cos(\omega\,t + \phi) \quad (1a)$$

where X_0 and ϕ are defined in Eqs. (11.29) and (11.30). Note that X_0 and ϕ have defined values, and therefore do not depend on the initial conditions.

equation of motion for a free damped oscillator:

$$\ddot{x}_b + \gamma\,\dot{x}_b + \omega_0^2\,x_b = 0 \quad (2)$$

Let

$$\begin{aligned} x(t) &= x_a(t) + x_b(t) \\ \ddot{x} + \gamma\,\dot{x} + \omega_0^2\,x &= \ddot{x}_a + \gamma\,\dot{x}_a + \omega_0^2\,x_a + \ddot{x}_b + \gamma\,\dot{x}_b + \omega_0^2\,x_b \\ &= \left(\frac{F_0}{m}\right)\cos(\omega\,t) + 0 \end{aligned}$$

so $x(t)$ also satisfies Eq. (1). This is a consequence of the linearity of Eqs. (1) and (2).

(b) From Eqs. (11.12) and (11.13), the motion for the free damped oscillator is

$$x_b = X_b e^{-\frac{\gamma}{2}t}\cos(\omega_1\,t + \psi) \quad (2a)$$

where Eq. (2a) is the solution to Eq. (2), and where X_b and ψ are to be determined by the initial conditions.
In Eq. (2a),

$$\omega_1 = \sqrt{\omega_0^2 - (\gamma/2)^2} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

$$x = x_a + x_b = X_0\cos(\omega\,t + \phi) + X_b e^{-\frac{\gamma}{2}t}\cos(\omega_1\,t + \psi) \quad (3a)$$

$$\dot{x} = -\omega\,X_0\sin(\omega\,t + \phi) + X_b e^{-\frac{\gamma}{2}t}\; -\left[\frac{\gamma}{2}\cos(\omega_1\,t + \psi) + \omega_1\,\sin(\omega_1\,t + \psi)\right] \quad (3b)$$

Applying the initial conditions,

$$x(0) = X_0\cos\phi + X_b\cos\psi = 0 \quad (4a)$$

$$v(0) = \dot{x}(0) = -\omega\,X_0\,\sin\phi - X_b\left[\frac{\gamma}{2}\cos\psi + \omega_1\,\sin\psi\right] = 0 \quad (4b)$$

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Solving. To simplify, let

$$C \equiv X_b \cos\psi \qquad \mathcal{S} \equiv X_b \sin\psi$$

Equation (4a) becomes

$$C = -X_0 \cos\phi$$

Equation (4b) becomes

$$-\omega X_0 \sin\phi - \frac{\gamma}{2}C - \omega_1 \mathcal{S} = 0$$

$$\mathcal{S} = \frac{X_0}{\omega_1}\left[-\omega \sin\phi + \frac{\gamma}{2}\cos\phi\right]$$

With this notation, Eqs. (3a) and (3b) can be written

$$x = X_0 \cos(\omega t + \phi) + e^{-\frac{\gamma}{2}t}\,[C\cos(\omega_1 t) - \mathcal{S}\sin(\omega_1 t)] \qquad (5a)$$

$$\dot{x} = -\omega X_0 \sin(\omega t + \phi) - e^{-\frac{\gamma}{2}t}\,[(\gamma/2)C\cos(\omega_1 t) - (\gamma/2)\mathcal{S}\sin(\omega_1 t) + \omega_1 C \sin(\omega_1 t) + \omega_1 \mathcal{S}\cos(\omega_1 t)] \qquad (5b)$$

(c) At resonance, $\omega = \omega_0$ and from Eqs. (11.29) and (11.30),

$$X_0 = \frac{F_0}{m\omega_0\gamma} \qquad \phi = \arctan(\infty) = \frac{\pi}{2}$$

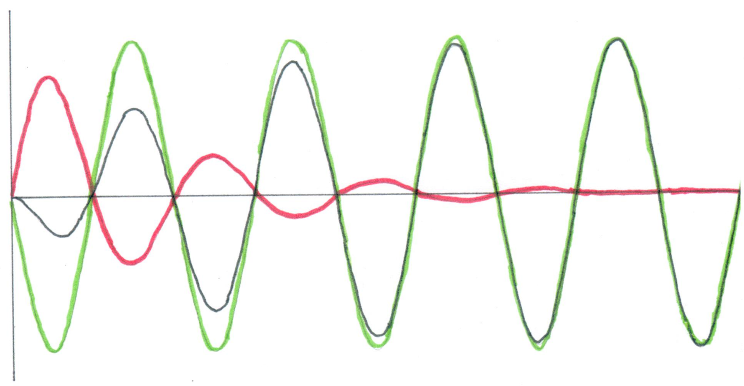
so that

$$C = -X_0\cos(\pi/2) = 0 \qquad \text{and} \qquad \mathcal{S} = -\frac{\omega_0}{\omega_1}X_0 = -\frac{F_0}{m\omega_1\gamma}$$

At resonance, Eq. (5a) becomes

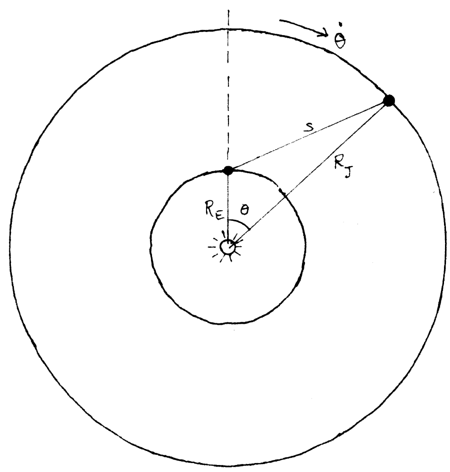
$$x = -\left(\frac{F_0}{m\omega_0\gamma}\right)\sin(\omega_0 t) + \left(\frac{F_0}{m\omega_1\gamma}\right)e^{-\frac{\gamma}{2}t}\sin(\omega_1 t)$$

The figure shows the response for the steady-state forced damped oscillator (green), the free damped oscillator transient (red), and the net behavior (black). The transient eventually becomes negligibly small, leaving only the steady-state behavior.
The figure is drawn for $\omega_0 = 2.0\,\text{s}^{-1}$ and $\gamma = 0.70\,\text{s}^{-1}$, so that $\omega_1 = 1.97\,\text{s}^{-1}$.



12 THE SPECIAL THEORY OF RELATIVITY

12.1 *Maxwell's proposal*



The orbits are essentially circular. The rate of revolution is $\dot{\theta} = 2\pi/T$, where T is the period. In a coordinate system rotating with the Earth, the apparent rate of Jupiter's revolution is

$$\begin{aligned}\dot{\theta} &= 2\pi\left(\frac{1}{T_J} - \frac{1}{T_E}\right)\\ &= 2\pi\left(\frac{1}{11.9\,\text{year}} - \frac{1}{1.0\,\text{year}}\right)\left(\frac{1}{3.15\times 10^7\,\text{s/year}}\right)\\ &= -1.8\times 10^{-7}\,\text{rad}\cdot\text{s}^{-1}\end{aligned}$$

Differentiate the law of cosines to find $2s\dot{s}$.

$$\begin{aligned}s^2 &= R_J^2 + R_E^2 - 2R_jR_E\cos\theta\\ \frac{d(s^2)}{dt} &= 2s\dot{s} = 2R_JR_E\sin\theta\,\dot{\theta}\\ (\Delta T)_{max} &\approx \frac{2R_JR_E\,\dot{\theta}}{c^2} = \frac{(2)(7.8\times 10^{11}\,\text{m})(1.5\times 10^{11}\,\text{m})(1.8\times 10^{-7}\,\text{s}^{-1})}{(3.0\times 10^8\,\text{m}\cdot\text{s}^{-1})^2}\\ &\approx 0.5\,\text{s}\end{aligned}$$

ΔT is for the extremes of s. Minimum s occurs for $\theta = 0°$, when Jupiter, the Earth, and the Sun are in line, with the Earth on the side of its orbit nearest Jupiter. Maximum s occurs a little over 6 months later, for $\theta = 180°$.

12.2 *Refined Michelson-Morley interferometer*

As derived in Sec. 12.3, the fringe shift N is

$$N = 2\frac{l}{\lambda}\left(\frac{v^2}{c^2}\right)$$

The additional factor of 2 arises because the improved interferometer could be rotated to be either with or against the Earth's motion, doubling the possible observable fringe shift.
The upper limit to v for a fringe shift no larger than N is therefore

$$v \le c\sqrt{\frac{N\lambda}{2l}}$$

$$v \le (3.0\times 10^8\ \text{m/s})\sqrt{\frac{(0.01)(590\times 10^{-9}\ \text{m})}{(2)(11.0\ \text{m})}} = 4.9\times 10^3\ \text{m/s}$$

Compared to the speed of the Earth in its orbit, this limit is smaller by a factor of

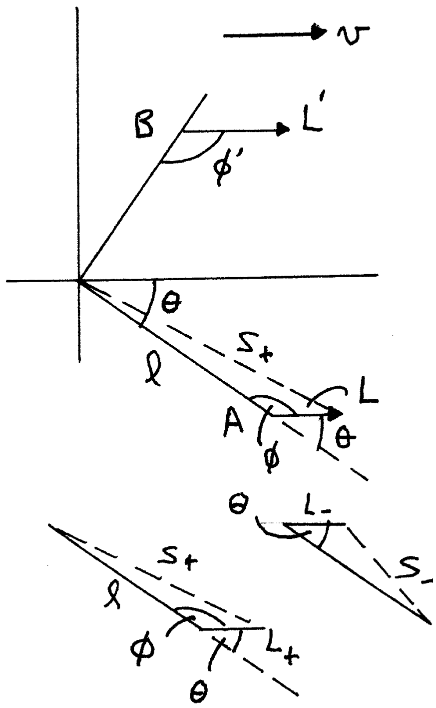
$$\frac{3.0\times 10^4\ \text{m/s}}{4.9\times 10^3\ \text{m/s}} = 6.1$$

12.3 *Skewed Michelson-Morley apparatus*

In the upper sketch, the apparatus is moving to the right with speed v. The time for light to go from the origin to the end of arm A and return is

$$\tau_+ + \tau_- = \frac{S_+}{c} + \frac{S_-}{c}$$

$$L_\pm = v\tau_\pm$$



From the lower sketches,

$$\begin{aligned}
s_+^2 &= l^2 + L_+^2 - 2lL_+\cos\phi \\
&= l^2 + L_+^2 - 2lL_+\cos(\pi-\theta) = l^2 + L_+^2 + 2lL_+\cos\theta \\
c^2\tau_+^2 &= l^2 + v^2\tau_+^2 + 2lv\tau_+\cos\theta \\
0 &= \left(1-\frac{v^2}{c^2}\right)\tau_+^2 - \left(\frac{2lv\cos\theta}{c^2}\right)\tau_+ - \frac{l^2}{c^2} \\
\tau_+ &= \frac{1}{2\left(1-\frac{v^2}{c^2}\right)}\left[\frac{2lv}{c^2}\cos\theta \pm \frac{2l}{c}\sqrt{1+\frac{v^2}{c^2}(\cos^2\theta - 1)}\right]
\end{aligned}$$

Take the + root, because $\tau_+ = 2l/c$ when $v = 0$.

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The calculation for τ_- follows the same algebra but replacing $\cos\theta$ with $-\cos\theta$.

$$\tau_- = \frac{1}{2\left(1-\frac{v^2}{c^2}\right)}\left[-\frac{2lv}{c^2}\cos\theta + \frac{2l}{c}\sqrt{1+\frac{v^2}{c^2}(\cos^2\theta-1)}\right]$$

$$\tau_A \equiv \tau_+ + \tau_- = \frac{1}{2\left(1-\frac{v^2}{c^2}\right)}\left[\frac{4l}{c}\sqrt{1+\frac{v^2}{c^2}(\cos^2\theta-1)}\right]$$

Retain only terms up to order v^2/c^2, and use $\sqrt{1+x} = 1 + x/2 + \dots$

$$\tau_A \approx \left(1+\frac{v^2}{c^2}\right)\frac{2l}{c}\left[1+\frac{1}{2}\frac{v^2}{c^2}(\cos^2\theta-1)\right]$$

The analysis for arm B is similar, but with $\phi' = \pi/2+\theta$, so that $\cos\phi' = -\sin\theta$.

$$\tau_B \approx \left(1+\frac{v^2}{c^2}\right)\frac{2l}{c}\left[1+\frac{1}{2}\frac{v^2}{c^2}(\sin^2\theta-1)\right]$$

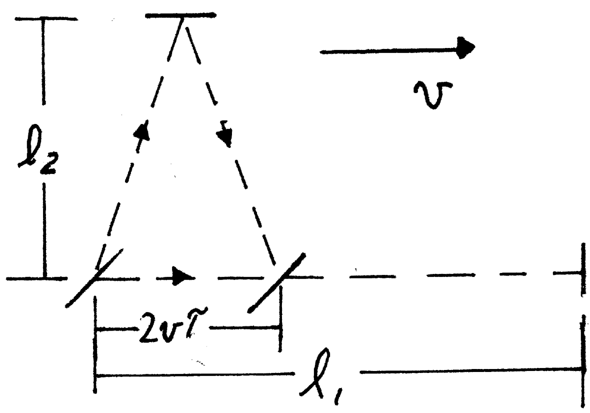
The difference in time τ_{diff} for the two paths is, to order v^2/c^2,

$$\tau_{diff} = \tau_A - \tau_B = \frac{l}{c}\frac{v^2}{c^2}(\cos^2\theta - \sin^2\theta)$$

This agrees with the result in Sec. 12.3 when $\theta = 0$. It also shows that the fringe shift changes sign when the apparatus is rotated by 90°.

12.4 *Asymmetric Michelson-Morley interferometer*

Assume, as Michelson did, that the speed of light adds vectorially to the speed of the interferometer through the ether. The time τ_1 for light to travel out and back along l_1 parallel to $\mathbf{v}$ is



$$\tau_1 = \frac{l_1}{c+v} + \frac{l_1}{c-v} = \frac{2l_1 c}{c^2-v^2}$$
$$= \frac{2l_1}{c}\left(\frac{1}{1-\frac{v^2}{c^2}}\right)$$
$$\approx \frac{2l_1}{c}\left(1-\frac{v^2}{c^2}\right)$$

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Let τ be the time to travel to the upper mirror. During this time, the lower mirror advances a distance $v\tau$. The time τ_2 for the total round trip along arm l_2, which is perpendicular to $\mathbf{v}$, is

$$\tau_2 = 2\tau = \frac{2}{c}\sqrt{l_2^2 + (v\tau)^2}$$

$$4\tau^2 = \frac{4}{c^2}\left(l_2^2 + v^2\tau^2\right) \implies \tau = \left(\frac{l_2}{c}\right)\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\tau_2 = 2\tau = 2\frac{l_2}{c}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 2\frac{l_2}{c}\left(1+\frac{1}{2}\frac{v^2}{c^2}\right)$$

The time difference is

$$\tau_1 - \tau_2 = 2\frac{l_1}{c}\left(1+\frac{v^2}{c^2}\right) - 2\frac{l_2}{c}\left(1+\frac{1}{2}\frac{v^2}{c^2}\right)$$

If the interferometer is rotated by 90°, the arms are reversed, so that the change ΔT in delay is

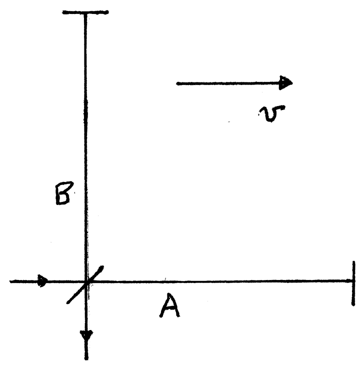
$$\begin{aligned}
\Delta T &= (\tau_1' - \tau_2') - (\tau_1 - \tau_2) = (\tau_1' - \tau_1) - (\tau_2' - \tau_2) \\
&= 2\frac{l_1}{c}\left[\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) - \left(1+\frac{v^2}{c^2}\right)\right] - 2\frac{l_2}{c}\left[\left(1+\frac{v^2}{c^2}\right) - \left(1+\frac{1}{2}\frac{v^2}{c^2}\right)\right] \\
&= -\left(\frac{l_1+l_2}{c}\right)\frac{v^2}{c^2}
\end{aligned}$$

The fringe shift N is therefore

$$N = \frac{c}{\lambda}\,|\Delta T| = \left(\frac{l_1+l_2}{\lambda}\right)\frac{v^2}{c^2}$$

12.5 *Lorentz-FitzGerald contraction*

Assume that both arms have the same rest length l_0, and that during the fringe shift measurement, the lengths are l_A and l_B.
The times to traverse the arms are



$$\tau_A = 2\frac{l_A}{c}\left(1+\frac{v^2}{c^2}\right)$$

$$\tau_B = 2\frac{l_B}{c}\left(1+\frac{1}{2}\frac{v^2}{c^2}\right)$$

$$\Delta\tau = \tau_A - \tau_B = \frac{2}{c}(l_A - l_B) + \frac{2}{c}\frac{v^2}{c^2}(l_A - \tfrac{1}{2}l_B)$$

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Following Lorentz and FitzGerald, take

$$l_A = l_0\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) \qquad \text{and} \quad l_B = l_0$$

$$\Delta\tau = -\frac{2}{c}l_0\left(\frac{v^2}{2c^2}\right) + \frac{2}{c}\frac{v^2}{c^2}\left[l_0\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) - \frac{1}{2}l_0\right]$$

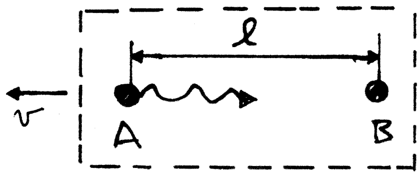
$$\approx 0 \text{ to order } \left(\frac{v}{c}\right)^4$$

There is no fringe shift to order v^2/c^2. For the Earth's orbital motion $v = 3.0 \times 10^4$ m/s, $(v/c)^2 = 10^{-8}$. Terms to order $(v/c)^4 = 10^{-16}$ were too small to be detected in the Michelson-Morley interferometer.

12.6 *One-way test of the consistency of c*

(a) *event a:*
At t=0, a pulse is sent from A and arrives at B at time t_a.

$$t_a = \frac{l}{c - v} \approx \frac{l}{c}\left(1 + \frac{v}{c}\right)$$

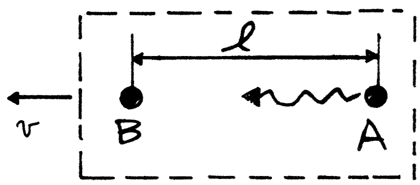


(a)

event b:
At time T, a pulse is sent from B and arrives at A at time t_b.

$$t_b = T + \frac{l}{c + v} \approx T + \frac{l}{c}\left(1 - \frac{v}{c}\right)$$

$$t_b - t_a - T = \frac{2l}{c}\left(\frac{v}{c}\right)$$



(b)

(b) Let l be the distance from the ground to the satellite, $l = 5.6R_e - R_e = 4.6R_e$. Then

$$\Delta T = \left(\frac{(2)(4.6)(6.4 \times 10^6\,\text{m})}{3.0 \times 10^8\,\text{m/s}}\right)\frac{v}{c} = 0.17\frac{v}{c}\,\text{s}$$

The orientation with respect to the ether would be opposite after 12 hours. With an accuracy of 1 part in 10^{16}, the clock could detect a change δT of

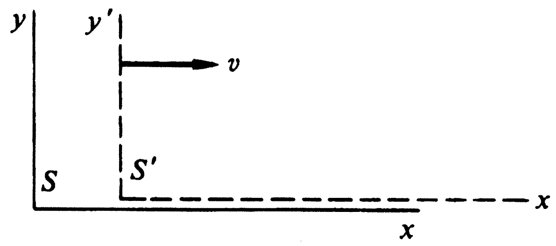
$$\delta T = (12\,\text{hours})(3600\,\text{s/hour}) \times 10^{-16} = 4.3 \times 10^{-12}\,\text{s in a period of 12 hours.}$$

$$v \geq \frac{c\,\delta T}{0.17\,\text{s}} = \frac{(3.0 \times 10^8\,\text{m/s})(4.3 \times 10^{-12}\,\text{s})}{0.17\,\text{s}}$$

The minimum detectable v would be

$$v_{min} = 7.6 \times 10^{-3}\,\text{m/s}$$

12.7 *Four events*



$$\frac{v}{c} = 0.6$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - (0.6)^2}} = 1.25$$

$$x' = \gamma(x - vt)$$

$$t' = \gamma\left(t - \frac{xv}{c^2}\right)$$

(a)

$$x = 4\,\text{m} \qquad t = 0\,\text{s}$$

$$x' = 1.25[4 - 0] = 5\,\text{m}$$

$$t' = 1.25\left[0 - \frac{(4\,\text{m})(0.6c)}{c^2}\right] = -1 \times 10^8\,\text{s}$$

(b)

$$x = 4\,\text{m} \qquad t = 1\,\text{s}$$

$$x' = 1.25[4 - (0.6c)(1)] = 1.25[4 - (0.6)(3 \times 10^8] \approx -2.25 \times 10^8\,\text{m}$$

$$t' = 1.25\left[1 - \frac{(4)(0.6c)}{c^2}\right] \approx 1.25\,\text{s}$$

(c)

$$x = 1.8 \times 10^8\,\text{m} \qquad t = 1\,\text{s}$$

$$x' = 1.25[1.8 \times 10^8 - (0.6c)(1)] = 0\,\text{m}$$

$$t' = 1.25\left[1 - \frac{(1.8 \times 10^8\,\text{m})(0.6c)}{c^2}\right] = 0.8\,\text{s}$$

(d)

$$x = 10^9\,\text{m} \qquad t = 2\,\text{s}$$

$$x' = 1.25[10^9 - (0.6c)(2)] = 8 \times 10^8\,\text{m}$$

$$t' = 1.25\left[2 - \frac{(10^9)(0.6c)}{c^2}\right] = 0\,\text{s}$$

12.8 *Relative velocity of* S *and* S'

Given

$$x = 9 \times 10^8 \text{ m} \qquad x' = 3 \times 10^8 \text{ m} \qquad t' = 1 \text{ s}$$

From Eq. (12.4a)

$$x = \gamma(x' + vt')$$

It is convenient to express all lengths in units of c.

$$\frac{x}{c} = \gamma\left(\frac{x'}{c} + \frac{v}{c}t'\right)$$

so that

$$3 = \gamma\left[1 + \frac{v}{c}(1)\right] \quad (1)$$

Let $r \equiv v^2/c^2$ so that

$$\gamma = \frac{1}{\sqrt{1 - r^2}}$$

and Eq. (1) becomes

$$3 = \frac{1}{\sqrt{1-r^2}}(1 + r) \implies 9(1 - r^2) = (1 + r)^2$$

$$0 = 10r^2 + 2r - 8 \implies r = \frac{-2 \pm \sqrt{4 + 320}}{20} = \frac{-2 \pm 18}{20}$$

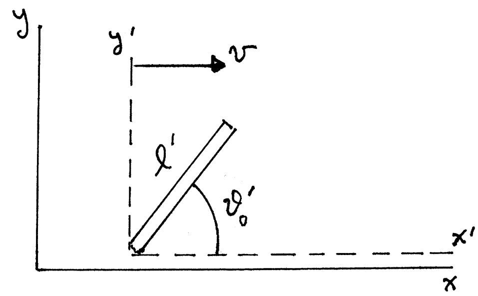
The roots are

$$r = \frac{4}{5}, -1$$

The root -1 is rejected, because γ is undefined for $v^2/c^2 = 1$, and also because the relative speed of observers must be less than c. Hence

$$v = \frac{4}{5}c = 2.4 \times 10^8 \text{ m/s}$$

12.9 *Rotated rod*



In this solution, primed quantities refer to the moving $x' - y'$ frame, and unprimed quantities to the stationary $x - y$ frame.
Let $x' - y'$ be the rod's rest frame, as shown in the sketch. In its rest frame, the rod makes angle θ'_0 with x' axis. The lower end of the rod is at the origin of the $x' - y'$ frame, and the upper end is at

$$x' = l' \cos\theta'_0 \qquad y' = l' \sin\theta'_0$$

As observed in the $x - y$ frame, the rod is moving to the right with speed v, as shown. Let the measurements be made at $t = t' = 0$, when the origins coincide. It is tempting, but *wrong*, to use $x = \gamma x'$. The ends must be measured at the same time in the $x - y$ frame, so it is necessary to use the full transformation $x' = \gamma(x - vt)$, and $y' = y$.
Then at $t = 0$, the lower end is located at

$$0 = x' = \gamma(x - 0) \implies x = 0$$
$$0 = y' = y \implies y = 0$$

At $t = 0$, the upper end in $x - y$ is located at

$$x' = l' \cos\theta'_0 = \gamma(x - 0) \implies \gamma x = x' = l' \cos\theta'_0$$
$$y' = l' \sin\theta'_0 = y$$

The angle θ_0 in the $x - y$ frame is therefore

$$\theta_0 = \arctan\left(\frac{y}{x}\right) = \arctan\left(\gamma\frac{l' \sin\theta'_0}{l' \cos\theta'_0}\right) = \arctan\left(\gamma \tan\theta'_0\right)$$

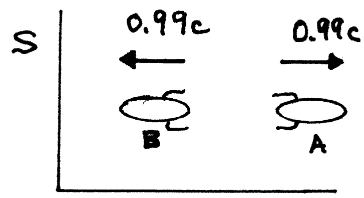
For $v \to c$, then $\gamma \to \infty$ so $\theta_0 \to \pi/2$.
In the $x - y$ frame, the length l of the rod is

$$l = \sqrt{x^2 + y^2} = l' \sqrt{\frac{\cos^2\theta'_0}{\gamma^2} + \sin^2\theta'_0}$$

For $v \to c$, then $\gamma \to \infty$ and $l = l' \sin\theta'_0$.

12.10 *Relative speed*

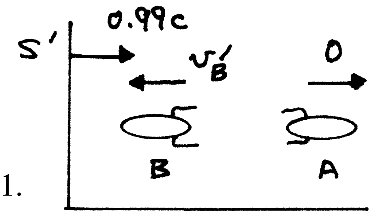
The sketches show the spaceships in the observer's frame S, and in frame S' moving with spaceship A.



In frame S

$v_A = v_B = 0.99c \equiv v$

In frame S' moving with A, $v'_A = 0$.



$$v'_B = \frac{v_A + v_B}{1 + \frac{v_A v_B}{c^2}} = \frac{2v}{1 + \frac{v^2}{c^2}}$$

v nearly equals c, so for accuracy let $v = (1 - x)c$ where $x = 0.01$.

$$v'_B = \frac{2(1-x)c}{1 + (1-x)^2} = \frac{(1-x)c}{1 - (x - x^2/2)}$$

The result is exact so far, but because $x \ll 1$, expand in Taylor's series $(1-u)^{-1} = 1 + u + u^2 + \ldots$

$$\frac{(1-x)c}{1 - (x - x^2/2)} \approx (1-x)c\left[1 + (x - x^2/2) + (x - x^2/2)^2 + \ldots\right]$$

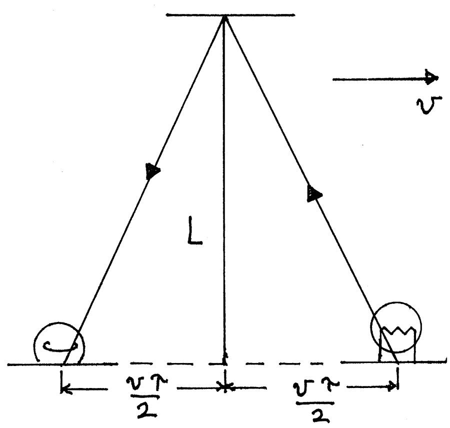
Retaining only terms up to x^2,

$$v'_B = \frac{(1-x)c}{1 - (x - x^2/2)} \approx (1 - x^2/2)c = \left[1 - \frac{(0.01)^2}{2}\right]c = (1 - 5 \times 10^{-5})c = 0.99995c$$

Note the importance of including all terms up to the desired highest order.

12.11 *Time dilation*

The total time of travel τ is



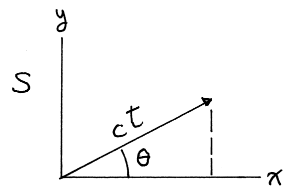
$$\tau = \frac{2\sqrt{L^2 + (\frac{v\tau}{2})^2}}{c}$$

$$c^2\tau^2 = 4\left[L^2 + \left(\frac{v\tau}{2}\right)^2\right]$$

$$\tau^2(c^2 - v^2) = 4L^2 \implies \tau = \frac{2L}{c}\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{\tau_0}{\sqrt{1 - v^2/c^2}}$$

12.12 *Headlight effect*

(a) In the S' frame, the coordinates x', y', t' of the light pulse after $t' = 1$ s are



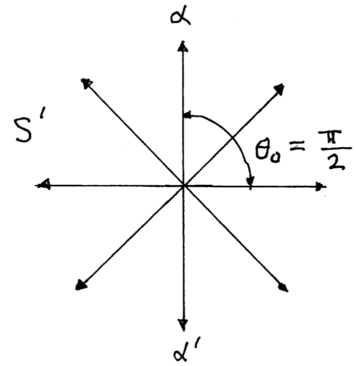
$$x' = c\cos\theta_0 \qquad y' = c\sin\theta_0 \qquad t' = 1\,\mathrm{s}$$

In S, the coordinates x, y, t are

$$x = \gamma(x' + vt') = \gamma(c\cos\theta_0 + v) \qquad y = y' \qquad t = \gamma\left(t' + \frac{vx'}{c^2}\right) = \gamma\left(1 + \frac{v\cos\theta_0}{c}\right)$$

$$\cos\theta = \frac{x}{ct} = \frac{\gamma(c\cos\theta_0 + v)}{\gamma(c + v\cos\theta_0)} = \frac{\cos\theta_0 + \frac{v}{c}}{1 + \frac{v}{c}\cos\theta_0}$$

(b) In S', half the radiation is emitted in the forward hemisphere, limiting θ_0 to $\pm\pi/2$ so that $\cos\theta_0 = 0$ at the limits. In S, using the result from (a), θ is limited to



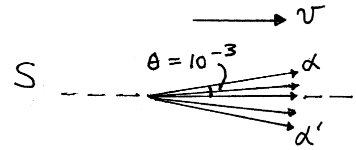
$$\cos\theta = \pm\frac{0 + v/c}{1 + 0} = \pm\frac{v}{c}$$

$$v = c\cos\theta$$

Let θ be the half-angle of the cone, as shown.

$$\theta \ll 1 \implies \cos\theta \approx 1 - \frac{1}{2}\theta^2 = 1 - \frac{1}{2}(10^{-3})^2$$

$$v = (1 - 5 \times 10^{-7})c$$



12.13 *Traveling twin*

From Sec. 12.11, the difference ΔT between the twins' ages is

$$\Delta T = \frac{1}{2}T_0\frac{v^2}{c^2}$$

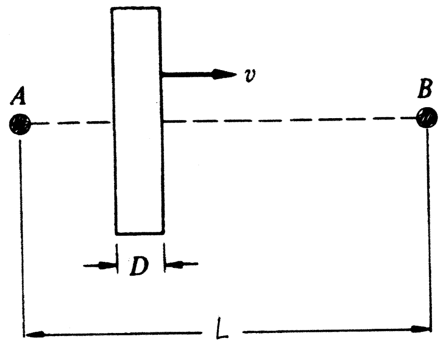
where T_0 is the total time for the voyage and where v is the speed of the craft, assumed constant except at turnaround.

The distance to α Centauri is 4.3 light years. At a speed $v = c/5$, the craft travels 1 light year in 5 years, so the round trip takes

$$T_0 = (2)(4.3)(5) = 43\,\text{years}$$

$$\Delta T = \frac{1}{2}(43\,\text{years})\left(\frac{1}{5}\right)^2 = 0.86\,\text{year} \approx 10\,\text{months}$$

12.14 *Moving glass slab*



For the trip from A to B, let t_0 be the time spent traveling outside the glass and t_g be the time spent traveling through the glass. To find t_g, consider two events in the S' rest frame of the slab.

event 1: light enters the slab at $t'_1 = 0, x'_1 = 0$.

event 2: light leaves at $t'_2 = nD/c, x'_2 = D$.

In the S lab frame, light enters the slab at x_1 and time t_1, and leaves at x_2 at time t_2. The Lorentz transformation gives

$$x_1 = \gamma(x'_1 + vt'_1) = (0 + 0) = 0 \qquad t_1 = \gamma(t'_1 + vx'_1/c^2) = (0 + 0) = 0$$

$$x_2 = \gamma(x'_2 + vt'_2) = \gamma\left(D + nD\frac{v}{c}\right) = \gamma D\left(1 + n\frac{v}{c}\right)$$

$$t_2 = \gamma(t'_2 + vx'_2/c^2) = \gamma\left(\frac{nD}{c} + \frac{vD}{c^2}\right) = \gamma\frac{D}{c}\left(n + \frac{v}{c}\right)$$

The distance traveled outside the slab is $L - x_2$, so

$$t_0 = \frac{(L - x_2)}{c}$$

$$\begin{aligned} T &= t_0 + t_g = t_0 + t_2 - t_1 \\ &= \frac{L}{c} - \gamma\frac{D}{c}\left(1 + n\frac{v}{c}\right) + \gamma\frac{D}{c}\left(n + \frac{v}{c}\right) = \frac{L}{c} + \gamma\frac{D}{c}\left[n - 1 + (1 - n)\frac{v}{c}\right] \\ &= \frac{L}{c} + \frac{D}{c}\frac{1}{\sqrt{1 - v^2/c^2}}(n - 1)\left(1 - \frac{v}{c}\right) = \frac{L}{c} + \frac{D}{c}(n - 1)\sqrt{\frac{1 - v/c}{1 + v/c}} \end{aligned}$$

12.15 *Doppler shift of a hydrogen spectral line*

Light from a receding source is shifted toward the red (longer wavelength), and light from an approaching source is shifted toward the blue (shorter wavelength).

(a) From Eq. (12.12),

$$\nu' = \nu\sqrt{\frac{1 - v/c}{1 + v/c}} \implies \frac{v}{c} = \frac{3 \times 10^6 \text{ m/s}}{3 \times 10^8 \text{ m/s}} = 0.01$$

$$\lambda' = \frac{c}{\nu'} = \lambda\sqrt{\frac{1 + v/c}{1 - v/c}} \approx \lambda\sqrt{1 + 2v/c} \approx \lambda(1 + v/c) = 1.01\lambda$$

$$\lambda' - \lambda = \lambda\frac{v}{c} = 0.01\lambda = 6.6 \times 10^{-9} \text{ m}$$

$$\lambda' = (656.1 + 6.6) \times 10^{-9} = 662.7 \times 10^{-9} \text{ m}$$

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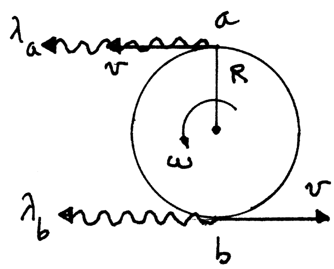


The difference in observed wavelengths from the advancing limb a and the receding limb b is, from (a),

$$\lambda_a - \lambda_b = 2\frac{v}{c}\lambda$$

$$\frac{v}{c} = \frac{(\lambda_a - \lambda_b)}{2\lambda} = \frac{9.0 \times 10^{-12}\ \text{m}}{(2)(656 \times 10^{-9}\ \text{m})} = 6.86 \times 10^{-6}$$

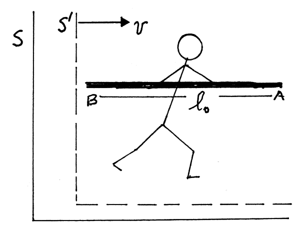
(b) $$v = (6.86 \times 10^{-6})(3 \times 10^{8}\ \text{m/s}) = 2.06 \times 10^{3}\ \text{m/s}$$

$$v = \omega R = \omega(D/2) \implies \omega = \frac{2v}{D} = \frac{4.12 \times 10^{3}\ \text{m/s}}{1.4 \times 10^{9}\ \text{m}} = 2.94 \times 10^{-6}\ \text{s}^{-1}$$

The period T is

$$T = \frac{2\pi}{\omega} = (2.13 \times 10^{6}\ \text{s})\left(\frac{1\ \text{day}}{8.64 \times 10^{4}\ \text{s}}\right) \approx 25\ \text{days}$$

12.16 *Pole-vaulter paradox*



Unprimed quantities refer to frame S, and primed quantities to frame S'. For $v/c = \sqrt{3}/2$, $\gamma = 2$.

(a) *the farmer's point of view (frame S):*
Let end B of the pole be at x_B at time t. At the same time t, the farmer (in frame S) observes end A at x_A. Let l be the length of the pole in S, so that $l = x_A - x_B$. From the Lorentz transformation Eq. (12.13),

$$x'_A - x'_B = l_0 = \gamma(x_A - vt) - \gamma(x_B - vt) = \gamma(x_A - x_B) = \gamma l$$

$l = l_0/\gamma = l_0/2$, so that the pole easily fits inside the barn.

(b) *the runner's point of view (frame S'):*
The length of the pole in S' is l_0, so if end A is at the rear door, the end B is projecting out the front door by $l_0 - (3/4)l_0 = l_0/4$.

(c) From the Lorentz transformation Eq. (12.14),

$$x'_B = \gamma(x_B - vt) \qquad t'_B = \gamma\left(t - \frac{vx_B}{c^2}\right)$$

$$x'_A = \gamma(x_A - vt) \qquad t'_A = \gamma\left(t - \frac{vx_A}{c^2}\right)$$

$$x'_A - x'_B = l_0$$

$$t'_A - t'_B = -\gamma\left(\frac{vx_A}{c^2}\right) + \gamma\left(\frac{vx_B}{c^2}\right) = -\gamma\left(\frac{vl}{c^2}\right) = \frac{vl_0}{c^2}$$

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In the runner's system S', the two ends of the pole are not inside the barn at the same instant. From the runner's point of view, event A occurs before the farmer shuts the door.

Looked at more closely, at the instant the front door is closed, the runner observes that end A of his pole is at $x'_A = l_0$, so that it is already outside the rear door. How does this event, call it C, look to the farmer in frame S?

$$x_C = \gamma(x'_A + vt') - \gamma(x'_B + vt') = \gamma l_0 = 2l_0$$
$$t_C = \gamma\left(t' + \frac{vx'_A}{c^2}\right) - \gamma\left(t' + \frac{vx'_B}{c^2}\right) = 2l_0\frac{v}{c^2}$$

Because $t_C > 0$, the farmer observes that event C occurs after the door is shut. So the farmer and the pole-vaulter are both correct; the bet can't be settled until they agree on whose frame is to be used.

12.17 *Transformation of acceleration*

Take differentials of the equation for the relativistic addition of velocities, Eq. (12.8a). Note that v, the relative velocity of the S and S' frames, is a constant.

$$u = \frac{(u' + v)}{\left(1 + \frac{u'v}{c^2}\right)}$$
$$du = \frac{du'}{\left(1 + \frac{u'v}{c^2}\right)} - \frac{(u' + v)}{\left(1 + \frac{u'v}{c^2}\right)^2}\frac{v}{c^2}du' = \frac{du'}{\left(1 + \frac{u'v}{c^2}\right)^2}\left[1 - \frac{v^2}{c^2}\right] = \frac{du'}{\left(1 + \frac{u'v}{c^2}\right)^2}\frac{1}{\gamma^2} \quad (1)$$

From the Lorentz transformation,

$$dt = \gamma\left(dt' + v\frac{dx'}{c^2}\right) = \gamma\, dt'\left(1 + \frac{u'v}{c^2}\right) \quad (2)$$

Dividing Eq. (1) by Eq. (2),

$$a = \frac{du}{dt} = \frac{du'}{dt'}\frac{1}{\gamma^3}\frac{1}{\left(1 + \frac{u'v}{c^2}\right)^3}$$

In the rest frame, $du'/dt' = a'$ and $u' = 0$.

$$a = \frac{a'}{\gamma^3}$$

12.18 *The consequences of endless acceleration*

(a) If the acceleration in S' is a_0, the acceleration a in S is, with reference to problem 12.17,

$$a = \frac{dv}{dt} = \frac{a_0}{\gamma^3} = a_0\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}$$

$$\int \frac{dv}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} = \int a_0\, dt$$

Let $v = xc$.

$$\int_0^x \frac{dx}{(1 - x^2)^{\frac{3}{2}}} = \int_0^t \frac{a_0}{c}\, dt$$

$$\frac{x}{\sqrt{1 - x^2}} = \frac{a_0\, t}{c} \implies \frac{v}{\sqrt{1 - v^2/c^2}} = a_0\, t$$

$$v = \frac{a_0\, t}{\sqrt{1 + (\frac{a_0\, t}{c})^2}}$$

Note that for $t \to \infty$, $v \to c$.

(b) Let $u_0 = a_0\, t$ so that

$$v = \frac{u_0}{\sqrt{1 + (\frac{u_0}{c})^2}}$$

$$u_0 = 10^{-3}\text{c}: \quad v = \frac{10^{-3}c}{\sqrt{1 + 10^{-6}}} \approx 10^{-3}c$$

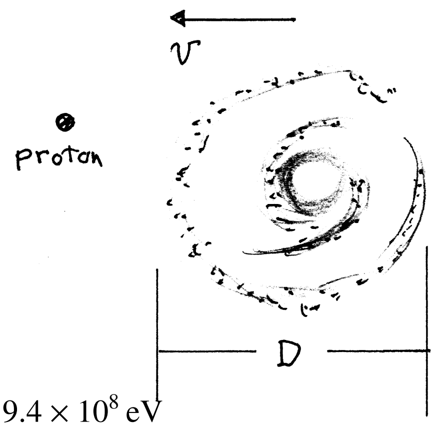
$$u_0 = \text{c}: \quad v = \frac{c}{\sqrt{2}} \approx 0.71c$$

$$u_0 = 10^{3}\text{c}: \quad v = \frac{10^{3}c}{\sqrt{1 + 10^{6}}} = \frac{c}{\sqrt{1 + 10^{-6}}} \approx (1 - 5 \times 10^{-7})\, c$$

High-energy particle accelerators can accelerate particles to energies far greater than their rest-mass energies, but the particle's speed is always less than c, if only slightly.

13 RELATIVISTIC DYNAMICS

13.1 *Energetic proton*



(a) In a frame moving with the proton, the galaxy is approaching at speed v and has thickness $D = D_0/\gamma$. The proton has such high energy that v is very nearly c, to the accuracy of this solution. The time T to traverse the galaxy is

$$T = \frac{D}{v} = \frac{D_0}{\gamma v} \simeq \frac{D_0}{\gamma c}$$

$$E = \gamma m_0 c^2 \implies \gamma = \frac{E}{m_0 c^2}$$

$$m_0 c^2 = \left(1.67 \times 10^{-27}\,\text{kg}\right)\left(3 \times 10^8\,\text{m/s}\right)^2 \left(\frac{1\,\text{eV}}{1.6 \times 10^{-19}\,\text{J}}\right) = 9.4 \times 10^8\,\text{eV}$$

$$\gamma = \frac{3 \times 10^{20}\,\text{eV}}{9.4 \times 10^8\,\text{eV}} = 3.2 \times 10^{11}$$

$$D_0 = (10^5\,\text{light years})(3 \times 10^8\,\text{m/s})\left(\frac{3 \times 10^7\,\text{s}}{1\,\text{year}}\right) = 9 \times 10^{20}\,\text{m}$$

$$T = \frac{9 \times 10^{20}\,\text{m}}{(3 \times 10^{11})(3 \times 10^8\,\text{m/s})} = 10\,\text{s}$$

The photon is traveling at the speed of light, so $\gamma \to \infty$, and $T_{photon} = 0$.

(b)

$$E_{baseball} = \frac{1}{2}Mv^2 = \frac{1}{2}(0.145\,\text{kg})\left[\left(\frac{100\,\text{miles}}{1\,\text{hour}}\right)\left(\frac{1610\,\text{m}}{1\,\text{mile}}\right)\left(\frac{1\,\text{hour}}{3600\,\text{s}}\right)\right]^2 = 145\,\text{J}$$

$$E_{proton} = (3 \times 10^{20}\,\text{eV})\left(\frac{1.6 \times 10^{-19}\,\text{J}}{1\,\text{eV}}\right) = 48\,\text{J}$$

13.2 *Onset of relativistic effects*

(a)

$$K_{rel} = (\gamma - 1)m_0c^2 = \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right)m_0c^2 \approx \left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4}\right)m_0c^2$$

using

$$\frac{1}{\sqrt{1-x}} \approx 1 + \frac{1}{2}x + \frac{3}{8}x^2$$

$$K_{rel} \approx \left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4}\right)m_0c^2 = \frac{1}{2}m_0v^2 + \frac{1}{2}m_0v^2\left(\frac{3}{4}\frac{v^2}{c^2}\right)$$

$$\frac{K_{rel}}{K_{cl}} \approx 1 + \frac{3}{4}\frac{v^2}{c^2}$$

For

$$\frac{K_{rel}}{K_{cl}} = 1.1 \implies \frac{3}{4}\frac{v^2}{c^2} = 0.1 \implies \frac{v^2}{c^2} = 0.133$$

$$\gamma = \frac{1}{\sqrt{1 - .133}} = 1.074$$

(b)

(1) $K_{electron} = (\gamma - 1)m_0c^2 = (0.074)(0.51\,\text{MeV}) = 0.038\,\text{MeV} = 38\,\text{keV}$

(2) $K_{proton} = (\gamma - 1)m_0c^2 = (0.074)(930\,\text{MeV}) = 69\,\text{MeV}$

13.3 *Momentum and energy*

$$K = (\gamma - 1)m_0c^2$$

$$\frac{dK}{dv} = m_0c^2\frac{d}{dv}\left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right) = m_0c^2\left[\frac{1}{(1 - v^2/c^2)^{\frac{3}{2}}}\frac{v}{c^2}\right]$$

$$dK = m_0\, v\, \gamma^3 dv$$

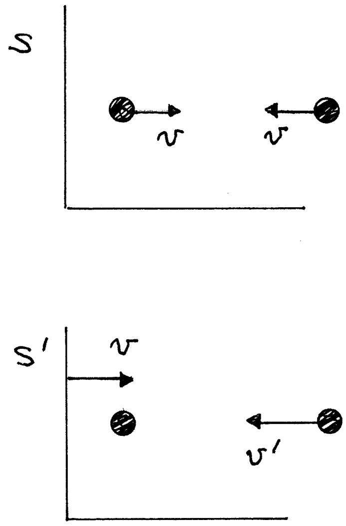
$$\mathbf{p} = \gamma m_0 \mathbf{v}$$

$$d\mathbf{p} = \gamma m_0\, d\mathbf{v} + m_0\mathbf{v}\frac{d}{dv}\left(\frac{1}{\sqrt{1 - v^2/c^2}}\right)dv = \gamma\, m_0\, d\mathbf{v} + \gamma^3 m_0\mathbf{v}\frac{v}{c^2}dv$$

$$\begin{aligned}\mathbf{v}\cdot d\mathbf{p} &= \gamma m_0 v dv + \gamma^3 m_0\frac{v^3}{c^2}dv = \gamma m_0 v dv\left(1 + \gamma^2\frac{v^2}{c^2}\right)\\ &= \gamma m_0 v dv\left(1 + \frac{v^2/c^2}{1 - v^2/c^2}\right) = \gamma m_0 v dv\left(\frac{1}{1 - v^2/c^2}\right)\\ &= m_0 v \gamma^3 dv = dK\end{aligned}$$

13.4 *Particles approaching head-on*

The left-hand particle is at rest in S'. From Eq. (12.9) for the relativistic addition of velocities,



$$v' = \frac{2v}{1 + v^2/c^2} \qquad E' = \gamma m_0 c^2$$

$$\gamma = \frac{1}{\sqrt{1 - v'^2/c^2}} = \frac{1}{\sqrt{1 - \frac{4v^2}{(1+v^2/c^2)^2}}} = \frac{1 + v^2/c^2}{\sqrt{1 + 2v^2/c^2 + v^4/c^4 - 4v^2/c^2}}$$

$$= \frac{1 + v^2/c^2}{1 - v^2/c^2}$$

$$E' = m_0 c^2 \left(\frac{1 + v^2/c^2}{1 - v^2/c^2} \right)$$

13.5 *Speed of a composite particle after an inelastic collision*

before collision

$$p = \gamma m_0 v$$
$$E = \gamma m_0 c^2 + Mc^2$$

after collision

$$p' = \gamma' M v'$$
$$E' = \gamma' M' c^2$$

Momentum and total energy are conserved, so

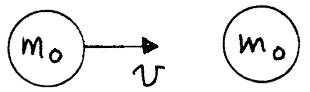
$$\gamma m_0 v = \gamma' M' v' \quad (1)$$
$$\gamma m_0 c^2 = \gamma' M' c^2 \quad (2)$$

Dividing Eq. (1) by Eq. (2),

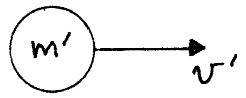
$$\frac{\gamma m_0 v}{\gamma m_0 c^2 + Mc^2} = \frac{v'}{c^2}$$

$$v' = \frac{\gamma m_0 v}{\gamma m_0 + M}$$

13.6 *Rest mass of a composite particle*



before collision



after collision

$$p_i = \gamma\, m_0\, v \qquad\qquad p_f = \gamma'\, m_0'\, v'$$
$$E_i = \gamma\, m_0\, c^2 + m_0\, c^2 \qquad\qquad E_f = \gamma'\, m_0'\, c^2$$
$$K = x m_0\, c^2 = (\gamma - 1) m_0\, c^2 \implies \gamma = x + 1$$

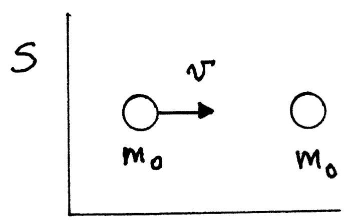
Momentum and total energy are both conserved.

$$\gamma\, m_0\, v = (x+1) m_0\, v = \gamma'\, m_0'\, v' \quad (1)$$
$$(\gamma + 1) m_0 c^2 = (x+2) m_0 c^2 = \gamma'\, m_0'\, c^2 \quad (2)$$

Dividing Eq. (1) by Eq. (2),

$$v' = \frac{(x+1)v}{(x+2)}$$
$$\gamma' = \frac{1}{\sqrt{1 - v'^2/c^2}} = \frac{(x+2)}{\sqrt{(x+2)^2 - (x+1)^2 + 1}} = \frac{(x+2)}{\sqrt{2x+4}} = \frac{\sqrt{x+2}}{\sqrt{2}}$$
$$(\gamma + 1) m_0 c^2 = \gamma'\, m_0' c^2$$
$$m_0' = \frac{(\gamma+1) m_0}{\gamma'} = \frac{\sqrt{2}(x+2) m_0}{\sqrt{x+2}} = \sqrt{2}\,\sqrt{(x+2)}\, m_0$$

13.7 *Zero momentum frame*

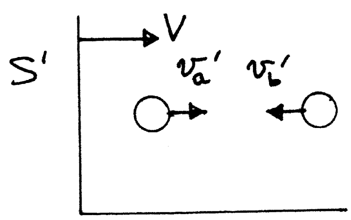


In S', the speeds of the particles are

$$v_a' = \frac{v - V}{1 - vV/c^2} \qquad v_b' = V \text{ to the left, as shown}$$

The corresponding momenta are

$$p_a' = \gamma_a'\, m_0 v_a' \qquad p_b' = \gamma_b'\, m_0 v_b'$$
$$0 = p_a' - p_b' = \frac{v_a'}{\sqrt{1 - v_a'^2/c^2}} - \frac{v_b'}{\sqrt{1 - v_b'^2/c^2}}$$



The particles have equal mass, so by symmetry they must have equal and opposite velocities in S' to give zero net momentum.

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Because their speeds are equal, it follows that $\gamma'_a = \gamma'_b$, so that the condition for zero momentum becomes

$$v'_a = v'_b \implies \frac{v - V}{1 - vV/c^2} = V$$

$$0 = \left(\frac{v}{c^2}\right) V^2 - 2V + v$$

$$V = \frac{2 \pm \sqrt{4 - 4v^2/c^2}}{2v/c^2} = \frac{1 \pm \sqrt{1 - v^2/c^2}}{v/c^2}$$

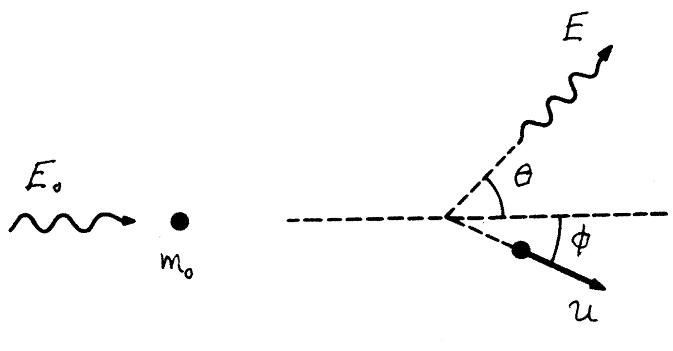
For $v \ll c$, the negative sign correctly gives

$$V \approx \left(\frac{c^2}{v}\right)\left(\frac{1}{2}\frac{v^2}{c^2}\right) = \frac{v}{2}$$

as expected. Hence

$$V = \left(\frac{c^2}{v}\right)\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right)$$

13.8 *Final velocity of a scattered particle*



before collision ***after collision***

$$E_i = E_0 + m_0c^2 \qquad E_f = E + \gamma m_0c^2$$

There are 3 unknowns, θ, ϕ, and u. However, if only u is to be found, the most direct route is to use conservation of total energy, which does not involve the unknown angles.

$$E_0 + m_0c^2 = E + \frac{m_0c^2}{\sqrt{1 - u^2/c^2}}$$

$$\frac{E_0 - E}{m_0c^2} + 1 = \frac{1}{\sqrt{1 - u^2/c^2}}$$

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Let $x = (E_0 - E)/(m_0c^2)$.

$$1 - \frac{u^2}{c^2} = \frac{1}{(x+1)^2}$$

$$\frac{u^2}{c^2} = 1 - \frac{1}{(x+1)^2} = \frac{x^2+2x}{(x+1)^2}$$

$$u = \left(\frac{\sqrt{x^2+2x}}{x+1}\right)c \quad \text{where } x = \frac{E_0 - E}{m_0c^2}$$

13.9 *The force of sunlight*

(a) R_e is the radius of the Earth. Let S be the solar constant.

$$F_{solar} = \frac{S}{c} \times \pi R_e^2$$

As discussed in Example 4.21, the momentum flux density = S/c.

$$F_{solar} = \frac{1.4 \times 10^3\ \text{W/m}^2}{3 \times 10^8\ \text{m/s}} \pi (6.4 \times 10^6\ \text{m})^2 = 6.0 \times 10^8\ \text{N}$$

R_E is the distance from the Sun to the Earth.

$$F_{gravity} = \frac{GM_{Earth}M_{Sun}}{R_E^2} = \frac{(6.7 \times 10^{-11}\ \text{N} \cdot \text{m}^2 \cdot \text{kg}^{-2})(6.0 \times 10^{24}\ \text{kg})(2.0 \times 10^{30}\ \text{kg})}{(1.5 \times 10^{11})^2} = 3.6 \times 10^{22}\ \text{N}$$

$$F_{gravity} \approx 6 \times 10^{13}\ F_{solar}$$

(b) Both the pressure of light and the gravitational attraction of the Sun fall off as the square of the distance, so consider the balance of forces on a particle at the Earth's orbit. Let the density of the particle be $\rho = 5.0 \times 10^3$ kg/m^3, its mass m, and its radius r.

$$F_{rad} = \frac{S}{c} \times \pi r^2$$

$$F_{gravity} = ma_{Sun} = \frac{4}{3}\pi r^3 \rho\, a_{Sun}$$

To escape, $F_{rad} > F_{gravity}$.

$$\frac{S}{c}\pi r^2 > \frac{4}{3}\pi r^3 \rho\, a_{Sun}$$

The acceleration at the Earth's orbit due to the Sun is

$$a_{Sun} = 6 \times 10^{-3}\ \text{m/s}^2$$

$$r < \frac{3}{4}\left(\frac{S}{c}\right)\left(\frac{1}{\rho\, a_{Sun}}\right) = \frac{3}{4}\left(\frac{1.4 \times 10^3}{3 \times 10^8}\right)\left(\frac{1}{(5 \times 10^3)(6 \times 10^{-3})}\right)$$

$$r < 1.2 \times 10^{-7}\ \text{m}$$

13.10 *Levitation by laser light*

Let the density of the particle be $\rho = 2.7 \times 10^3$ kg/m^3, and let r be its radius.
If the sphere is large compared to the spot of laser light, assume that the light is reflected, doubling the force.

$$F = 2\left(\frac{\text{power}}{c}\right) = \frac{2 \times 10^3 \text{ W}}{3 \times 10^8 \text{ m/s}} = 6.7 \times 10^{-6} \text{ N}$$

For levitation,

$$F \geq mg = \frac{4}{3}\pi r^3 \rho g$$

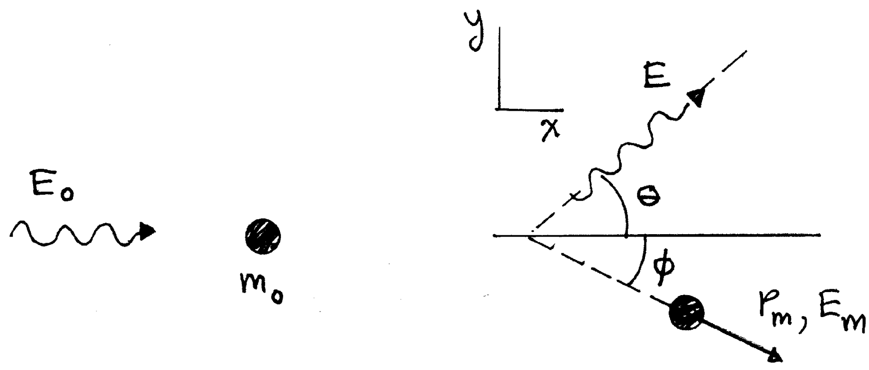
At equilibrium,

$$6.7 \times 10^{-6} \text{ N} = \left(\frac{4}{3}\pi r^3 \text{ m}^3\right)(2.7 \times 10^3 \text{ kg} \cdot \text{m}^{-3})(9.8 \text{ m} \cdot \text{s}^2) = 1.1 \times 10^5 r^3$$

$$r^3 = \frac{6.7 \times 10^{-6} \text{ N}}{1.1 \times 10^5 \text{ N} \cdot \text{m}^{-3}} = 6.1 \times 10^{-11} \text{ m}^3$$

$$r = 3.9 \times 10^{-4} \text{ m}$$

13.11 *Photon-particle scattering*



before collision ***after collision***

x direction: $(p_i)_x = \dfrac{E_0}{c}$ $(p_f)_x = \dfrac{E\cos\theta}{c} + p_m \cos\phi$

y direction: $(p_i)_y = 0$ $(p_f)_y = \dfrac{E\sin\theta}{c} - p_m \sin\phi$

$E_i = E_0 + m_0c^2$ $E_f = E + E_m$

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By conservation of momentum,

$$\frac{E_0}{c} = \frac{E\cos\theta}{c} + p_m\cos\phi \quad (1)$$

$$\frac{E\sin\theta}{c} = p_m\sin\phi \quad (2)$$

Dividing Eq. (1) by Eq. (2),

$$\cot\phi = \frac{E_0}{E\sin\theta} - \cot\theta \quad (3)$$

By conservation of total energy,

$$E_0 + m_0c^2 = E + E_m \quad (4)$$

$$(E_0 - E + m_0c^2)^2 = E_m^2 = (p_mc)^2 + (m_0c^2)^2 \quad (5)$$

From Eqs. (1) and (2),

$$p_mc\cos\phi = E_0 - E\cos\theta \qquad p_mc\sin\phi = E\sin\theta$$

Squaring and adding,

$$(p_mc)^2 = (E_0 - E\cos\theta)^2 + E^2\sin^2\theta = E_0^2 + E^2 - 2E_0E\cos\theta$$

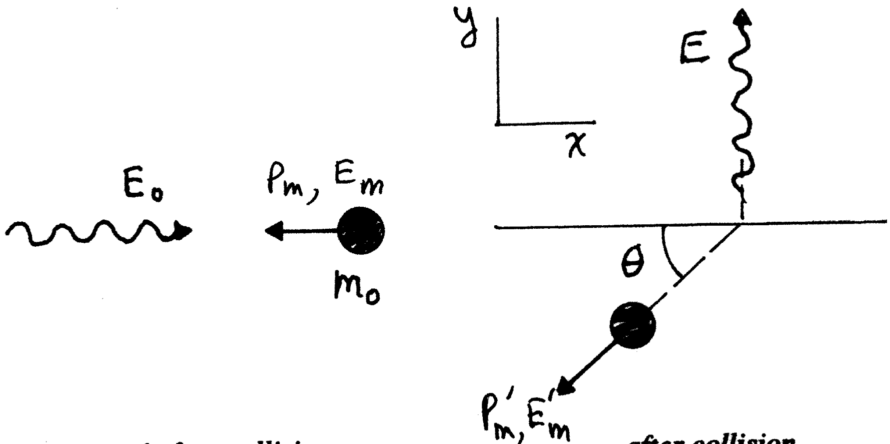
Using this in Eq. (5),

$$E_0E(1-\cos\theta) = m_0c^2(E_0 - E) \implies \frac{E_0}{E} = \frac{E_0}{m_0c^2}(1-\cos\theta) + 1$$

Using this expression for E_0/E in Eq. (4),

$$\begin{aligned}\cot\phi &= \left(\frac{E_0}{m_0c^2}\right)\frac{(1-\cos\theta)}{\sin\theta} + \frac{1}{\sin\theta} - \cot\theta = \left(1 + \frac{E_0}{m_0c^2}\right)\frac{(1-\cos\theta)}{\sin\theta}\\ &= \left(1 + \frac{E_0}{m_0c^2}\right)\tan(\theta/2)\end{aligned}$$

13.12 *Photon-electron collision*



before collision *after collision*

x direction:	$(p_i)_x = \dfrac{E_0}{c} - p_m$	$(p_f)_x = -p'_m \cos\theta$
y direction:	$(p_i)_y = 0$	$(p_f)_y = \dfrac{E}{c} - p'_m \sin\theta$
	$E_i = E_0 + E_m$	$E_f = E + E'_m$

(a) By conservation of momentum,

$$E_0 - cp_m = -cp'_m \cos\theta \quad (1)$$
$$E = cp'_m \sin\theta \quad (2)$$

Squaring and adding Eqs. (1)and (2),

$$E_0^2 - 2E_0\,cp_m + (cp_m)^2 + E^2 = (cp'_m)^2 \quad (3)$$

By conservation of total energy, $E'_m = E_0 + E_m - E \quad (4)$

Use

$$(cp_m)^2 = E_m^2 - (m_0c^2)^2 \qquad \text{and} \qquad (cp'_m)^2 = E'^2_m - (m_0c^2)^2$$

in Eq. (3) to give, with Eq. (4),

$$E_0^2 - 2E_0\,cp_m + E_m^2 + E^2 = E'^2_m = (E_0 + E_m - E)^2 \implies E_0(E_m + cp_m) = E(E_0 + E_m) \quad (5)$$
$$p_m = \gamma m_0 v = \left(\frac{v}{c^2}\right)\gamma m_0 c^2 = \left(\frac{v}{c^2}\right)E_m$$

so Eq. (5) becomes

$$E = \frac{E_0 E_m(1 + v/c)}{E_0 + E_m} = \frac{E_0(1 + v/c)}{1 + E_0/E_m}$$

continued next page $\Longrightarrow$

(b) The line broadening $\Delta\lambda$ is

$$\Delta\lambda = \lambda - \lambda_0$$

Using $E = h\nu = hc/\lambda$,

$$\Delta\lambda = \frac{hc}{E} - \frac{hc}{E_0} \implies \frac{\Delta\lambda}{hc} = \frac{1}{E} - \frac{1}{E_0} \approx \frac{E_0 - E}{E_0^2}$$

From the result of part (a),

$$\frac{E_0 - E}{E_0^2} = \frac{1}{E_0}\left[1 - \frac{(1 + v/c)}{1 + E_0/E_m}\right]$$

$$E_m = \gamma m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} \approx m_0 c^2$$

$$\frac{\Delta\lambda}{hc} = \frac{\lambda_0}{hc}\left[1 - \frac{(1 + v/c)}{1 + h/\lambda_0 m_0 c}\right] = \frac{\lambda_0}{hc}\left[1 - \frac{(1 + v/c)}{1 + \lambda_C/\lambda_0}\right]$$

$$\Delta\lambda = 0.711 \times 10^{-10}\left[1 - \frac{(1 + 6 \times 10^{-3})}{1 + \frac{2.426\times10^{-12}}{0.711\times10^{-10}}}\right] = 0.711 \times 10^{-10}\left[1 - \frac{(1 + 6 \times 10^{-3})}{(1 + 3.4 \times 10^{-2})}\right]$$

$$\approx 0.711 \times 10^{-10}\left[1 - (1 + 6 \times 10^{-3})(1 - 3.4 \times 10^{-2})\right]$$

$$= (0.711 \times 10^{-10})(-6 \times 10^{-3} + 3.4 \times 10^{-2})$$

$$= 2.0 \times 10^{-12}\,\text{m} = (2.0 \times 10^{-12}\,\text{m})\left(\frac{1\,\mathring{A}}{1 \times 10^{-10}\,\text{m}}\right)$$

$$= 0.020\,\mathring{A}$$

consistent with the data plot in Example 13.6.

14 SPACETIME PHYSICS

14.1 *Pi meson decay*

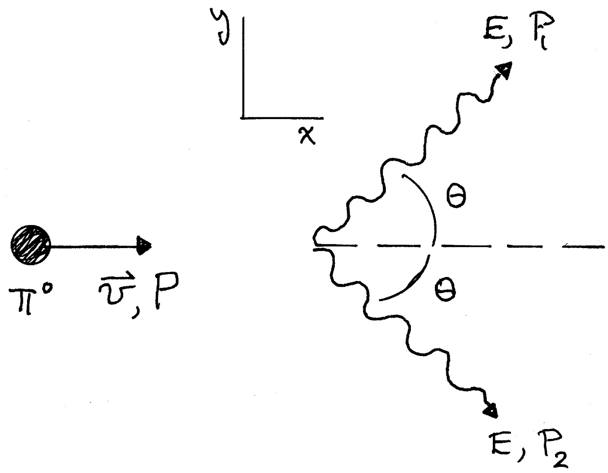
Momentum $\mathbf{P}$ of the π^0:

$\mathbf{P} = \gamma m_0(v, 0, 0, c)$

Momentum $\mathbf{P_i}$ of the photons:

$$\mathbf{P_1} = \frac{E}{c}(\cos\theta, \sin\theta, 0, 1)$$

$$\mathbf{P_2} = \frac{E}{c}(\cos\theta, -\sin\theta, 0, 1)$$



(a)

$$\mathbf{P} = \mathbf{P_1} + \mathbf{P_2}$$

Equating the time-like components

$$\gamma m_0 c = \frac{2E}{c}$$

$$\gamma = \frac{2E}{m_0 c^2} \quad (1)$$

$$= \frac{2 \times 100\,\text{keV}}{135\,\text{keV}} = 1.48$$

$$\frac{1}{1 - v^2/c^2} = (1.48)^2 \implies \frac{v}{c} = \sqrt{1 - 1/(1.48)^2} = 0.74$$

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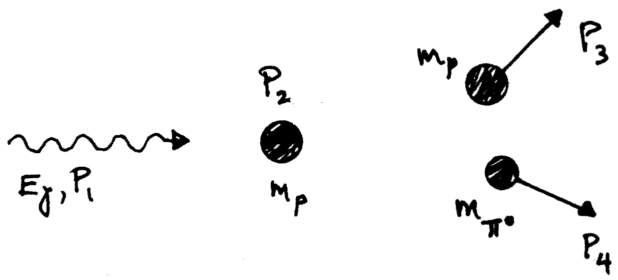
(b) Equating the x components

$$\gamma m_0 v = \frac{2E}{c}\cos\theta$$
$$\gamma\frac{v}{c} = \frac{2E}{m_0c^2}\cos\theta = \gamma\cos\theta \quad \text{using Eq. (1)}$$
$$\cos\theta = \frac{v}{c} = 0.74 \implies \theta = 42^\circ$$

14.2 *Threshold for pi meson production*



before collision *after collision*

E_γ is the energy of the γ ray. $\mathbf{p}_3$ and $\mathbf{p}_4$ are 3-vectors, and $\mathbf{P}_3$ and $\mathbf{P}_4$ are 4-vectors. Comparing components of vectors requires evaluating them in the same coordinate system. However, the norms of 4-vectors are scalars, independent of coordinate system, permitting the 4-vectors to be expressed in any convenient coordinate system. In this problem, the center of mass system is convenient.

conservation of 4-momentum:

$$\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_3 + \mathbf{P}_4$$
$$\mathbf{P}_1 = \frac{E_\gamma}{c}(1,0,0,1)$$
$$\mathbf{P}_2 = m_pc(0,0,0,1)$$
$$\mathbf{P}_3 + \mathbf{P}_4 = (\mathbf{p}_3 + \mathbf{p}_4, (m_p + m_{\pi^0})c) = (0,0,0,(m_p + m_{\pi^0})c)$$

Equating norms,

$$|\mathbf{P}_1 + \mathbf{P}_2|^2 = P_1^2 + 2\mathbf{P}_1\cdot\mathbf{P}_2 + P_2^2 = 0 - 2m_pE_\gamma - (m_pc)^2$$
$$|\mathbf{P}_3 + \mathbf{P}_4|^2 = -[(m_p + m_{\pi^0})c]^2$$

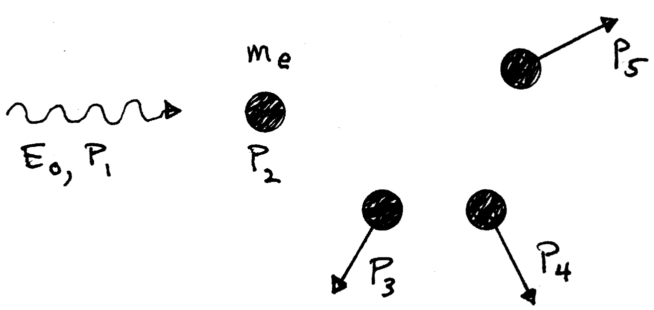
continued next page $\Longrightarrow$

Consequently, the minimum value of E_γ satisfies

$$2m_pE_\gamma + (m_pc)^2 = [(m_p + m_{\pi^0})c]^2 \implies E_\gamma = \frac{1}{2m_p}(2m_pm_{\pi^0} + m_{\pi^0}^2)c^2$$

$$E_\gamma = m_{\pi^0}c^2\left(1 + \frac{m_{\pi^0}c^2}{2m_pc^2}\right)$$
$$= (135\,\text{MeV})\left(1 + \frac{135\,\text{MeV}}{2\times 938\,\text{MeV}}\right) = 145\,\text{MeV}$$

14.3 *Threshold for pair production by a photon*



before collision ***after collision***

At threshold, all of the incident photon's energy goes to promoting the reaction. Above threshold, there would be a residual photon of lower energy among the reaction products.
Let m_e be the rest mass of each particle.

$$\mathbf{P}_1 = \frac{E_0}{c}(1, 0, 0, 1)$$
$$\mathbf{P}_2 = m_ec(0, 0, 0, 1)$$
$$|\mathbf{P}_1 + \mathbf{P}_2|^2 = P_1^2 + 2\mathbf{P}_1\cdot\mathbf{P}_2 + P_2^2 = 0 - 2m_eE_0 - (m_ec)^2$$

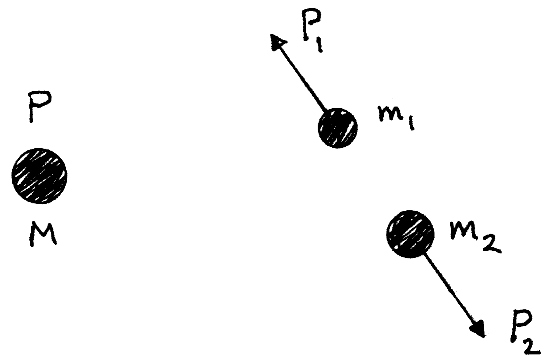
Evaluate the 4-momenta of the three products in the center of mass frame.

$$|\mathbf{P}_3 + \mathbf{P}_4 + \mathbf{P}_5|^2 = -(\mathbf{p}_3 + \mathbf{p}_4 + \mathbf{p}_5, 3m_ec^2)^2 = -(3m_ec)^2$$

conservation of 4-momentum:

$$\mathbf{P}_1 + \mathbf{P}_2 = \mathbf{P}_3 + \mathbf{P}_4 + \mathbf{P}_5$$
$$2m_eE_0 + (m_ec)^2 = (3m_ec)^2 = 9(m_ec)^2$$
$$E_0 = 4m_ec^2 = 4\times(0.51\,\text{MeV})$$
$$= 2.04\,\text{MeV}$$

14.4 *Particle decay*



before decay ***after decay***

The mass symbols M, m_1, and m_2 all refer to rest mass.

$$\begin{aligned}\mathbf{P} &= M(0,0,0,c)\\ \mathbf{P}_1 &= (\mathbf{p}, E_1/c)\\ \mathbf{P}_2 &= (-\mathbf{p}, E_2/c)\end{aligned}$$

Note that 3-momentum and 4-momentum are both conserved.

$$\begin{aligned}\mathbf{P} &= \mathbf{P}_1 + \mathbf{P}_2\\ |\mathbf{P}|^2 &= P_1^2 + 2\mathbf{P}_1\cdot\mathbf{P}_2 + P_2^2\\ -(Mc)^2 &= -(m_1c)^2 - 2\left(p^2 + \frac{E_1E_2}{c^2}\right) - (m_2c)^2 \quad (1)\end{aligned}$$

Use

$$p^2 = p_1^2 = \left(\frac{E_1}{c}\right)^2 - m_1c^2 \quad \text{and} \quad E_1 + E_2 = Mc^2 \implies E_2 = Mc^2 - E_1$$

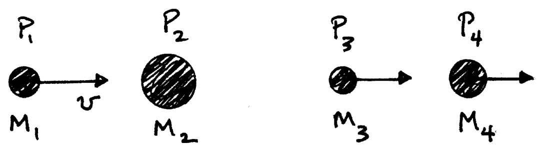
Equation (1) then becomes

$$\begin{aligned}(Mc)^2 &= -(m_1c)^2 + (m_2c)^2 + 2E_1M\\ E_1 &= \left(\frac{M^2 + m_1^2 - m_2^2}{2M}\right)c^2\end{aligned}$$

To find E_2, simply interchange the subscripts.

$$E_2 = \left(\frac{M^2 + m_2^2 - m_1^2}{2M}\right)c^2$$

14.5 *Threshold for nuclear reaction*



before collision *after collision*

The symbols M_1, M_2, M_3, and M_4 all refer to rest mass.

$$\mathbf{P}_1 = \gamma M_1(v, 0, 0, c)$$
$$\mathbf{P}_2 = M_2(0, 0, 0, c)$$

In the center of mass system,

$$|\mathbf{P}_3 + \mathbf{P}_4|^2 = -(M_3 + M_4)^2 c^2$$

Use $\quad \mathbf{P}_1 + \mathbf{P}_2|^2 = |\mathbf{P}_3 + \mathbf{P}_4|^2 \quad$ and $\quad (M_3 + M_4)c^2 = (M_1 + M_2)c^2 + Q$

$$P_1^2 + 2\mathbf{P}_1 \cdot \mathbf{P}_2 + P_2^2 = |\mathbf{P}_3 + \mathbf{P}_4|^2$$

$$-M_1^2 c^2 - 2\gamma M_1 M_2 c^2 - M_2^2 c^2 = -(M_3 + M_4)^2 c^2 = -\left(M_1 + M_2 + \frac{Q}{c^2}\right)^2 c^2$$

$$M_1^2 + 2\gamma M_1 M_2 + M_2^2 = \left(M_1 + M_2 + \frac{Q}{c^2}\right)^2 = M_1^2 + M_2^2 + 2(M_1 + M_2)\frac{Q}{c^2} + \frac{Q^2}{c^2}$$

$$2(\gamma - 1)M_1 M_2 = 2(M_1 + M_2)\frac{Q}{c^2} + \frac{Q^2}{c^4}$$

$$\gamma - 1 = \left(\frac{M_1 + M_2}{M_1 M_2}\right)\frac{Q}{c^2} + \left(\frac{1}{2M_1 M_2}\right)\frac{Q^2}{c^4}$$

$$K_1 = (\gamma - 1)M_1 c^2 = \left(\frac{M_1 + M_2}{M_2}\right)Q + \left(\frac{1}{2M_2 c^2}\right)Q^2$$

14.6 *Photon-propelled rocket*

The initial 4-momentum of the rocket is $\mathbf{P}_i = M_0(0, 0, 0, c)$. When the rocket has acclerated to speed v and its rest mass has decreased to M_f, the rocket's 4-momentum is $\mathbf{P}_f = \gamma M_f(v, 0, 0, c)$.

(a) By conservation of 3-momentum, the momentum p_{ex} of the exhaust is

$$p_{ex} = -\gamma M_f v$$

The exhaust consists of photons, for which $p = E/c$. The 4-momentum of the exhaust is therefore

$$\mathbf{P}_{ex} = (\mathbf{p}_{ex}, E/c) = \gamma M_f(-v, 0, 0, v)$$

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(b) By conservation of 4-momentum,

$$\begin{aligned} \mathbf{P}_i &= \mathbf{P}_f + \mathbf{P}_{ex} \\ M_0(0,0,0,c) &= \gamma M_f(v,0,0,c) + \gamma M_f(-v,0,0,v) \end{aligned}$$

Because all the 4-vectors refer to the same frame (in this case, the laboratory frame), it is correct to equate corresponding components. The 4th component gives

$$\begin{aligned} M_0 c &= \gamma M_f c + \gamma M_f v \\ \frac{M_0}{M_f} &= \gamma\left(1 + \frac{v}{c}\right) \\ \text{Let} \quad \mu &\equiv \frac{M_0}{M_f} \\ \mu^2 &= \left(\frac{1}{1 - v^2/c^2}\right)\left(1 + \frac{v}{c}\right)^2 = \frac{1 + v/c}{1 - v/c} \\ v &= \left(\frac{\mu^2 - 1}{\mu^2 + 1}\right)c \end{aligned}$$

14.7 *Four-acceleration*

Consider motion only along the $x - axis$, so that the 4-velocity is

$$\begin{aligned} \mathbf{U} &= \gamma(u,0,0,c) \\ \mathbf{A} &= \frac{d\mathbf{U}}{d\tau} = \gamma\frac{d\mathbf{U}}{dt} = \gamma\frac{d}{dt}[\gamma(u,0,0,c)] \\ &= \gamma^2(du/dt,0,0,0) + \gamma\frac{d\gamma}{dt}(u,0,0,c) \end{aligned}$$

where $\quad \dfrac{du}{dt} = a \quad$ and $\quad \dfrac{d\gamma}{dt} = \dfrac{d}{dt}\dfrac{1}{\sqrt{1 - u^2/c^2}} = \gamma^3\dfrac{u}{c^2}\dfrac{du}{dt} = \gamma^3\dfrac{au}{c^2}$

$$\begin{aligned} \mathbf{A} &= \gamma^2(a,0,0,0) + \gamma^4\frac{au}{c^2}(u,0,0,c) \\ &= \left(\gamma^2 a + \gamma^4\frac{au^2}{c}, 0, 0, \gamma^4\frac{au}{c}\right) \\ &= \left[\gamma^2 a\left(1 + \gamma^2\frac{u^2}{c^2}\right), 0, 0, \gamma^4\frac{au}{c}\right] = \left[\gamma^2 a\left(1 + \frac{u^2/c^2}{1 - u^2/c^2}\right), 0, 0, \gamma^4\frac{au}{c}\right] \\ &= \gamma^4 a\left[1, 0, 0, \frac{u}{c}\right] \end{aligned}$$

The norm of $\mathbf{A}$ is

$$|\mathbf{A}|^2 = \gamma^8 a^2\left(1 - \frac{u^2}{c^2}\right) = \gamma^6 a^2$$

14.8 *A wave in spacetime*

Let u be the relative velocity of frames S and S', to minimize confusion with the wave's frequency ν.

(a) To demonstrate that $\mathbf{K} = 2\pi(1/\lambda, 0, 0, \nu/c)$ is a 4-vector, show that its dot product with a known 4-vector is a scalar, and that its norm is Lorentz invariant. Take the trial 4-vector to be the displacement $\mathbf{X} = (x, 0, 0, ct)$.

$$\mathbf{K}\cdot\mathbf{X} = 2\pi\left(\frac{x}{\lambda} - \nu t\right)$$

This dot product is the phase of the wave. Different observers must agree on the phase, for instance a point in spacetime where the amplitude vanishes. Because $\mathbf{X}$ transforms according to the Lorentz transformation, $\mathbf{K}$ must also.

The norm is

$$|\mathbf{K}|^2 = (2\pi)^2\left[\frac{1}{\lambda^2} - \left(\frac{\nu}{c}\right)^2\right]$$

The norm is a Lorentz invariant, which implies that $\lambda\,\nu = c$; in other words, the wave travels at the speed of light, as postulated.

(b) Using the notation of Section 14.5,

$$\mathbf{K} = (a_1, a_2, a_3, a_4) = 2\pi\left(\frac{1}{\lambda}, 0, 0, \frac{\nu}{c}\right)$$

In the S' system, Eq. (14.13) gives

$$a_4' = \gamma\left(a_4 - \frac{u}{c}a_1\right) \implies 2\pi\frac{\nu'}{c} = \gamma 2\pi\left[\frac{\nu}{c} - \frac{u}{c}\left(\frac{1}{\lambda}\right)\right]$$

$$\nu' = \gamma\left(\nu - \frac{u}{\lambda}\right) = \gamma\,\nu\left(1 - \frac{u}{c}\right) = \nu\frac{(1-u/c)}{\sqrt{(1-u^2/c^2)}} = \nu\sqrt{\left(\frac{1-u/c}{1+u/c}\right)}$$

$$\nu = \nu'\sqrt{\left(\frac{1+u/c}{1-u/c}\right)}$$

in agreement with Eq. (12.12) for the longitudinal Doppler shift.

(c) For propagation along the y axis,

$$\mathbf{K} = 2\pi\left(0, \frac{1}{\lambda}, 0, \frac{\nu}{c}\right)$$

In the S' frame,

$$\frac{\nu'}{c} = \gamma\frac{\nu}{c} \implies \nu = \frac{\nu'}{\gamma} = \nu'\sqrt{1-u^2/c^2}$$

in agreement with Eq. (12.13) for the transverse Doppler shift, with $\theta = \pi/2$.