Utrecht University

Master Thesis

Correlated games on simple networks

Louis Leestemaker

Abstract

In this thesis, the equilibrium response strategies for correlated two- and three-agent Battle of the Sexes as well as Pure Coordination game networks are analyzed. Response strategy is a recently proposed addition to the concept of correlated equilibria, in which the agents can choose whether or not to follow the instructions from the correlation device. A classification is proposed of the different Nash equilibria in the response strategy based on stability under fluctuations and whether or not they are realizable with any correlation device. For the two agent networks, the entire phase space is mapped out. For the three agent networks, choices were made for which equilibria to consider. A unique mapping from any realization of the game-theoretical model to the Ising model from physics is defined and analyzed. However, there are some serious complications in this mapping which may limit the use for the analysis of network structures in game theory.

supervised by:
Prof. Dr. Ir. H.T.C. Stoof and Prof. Dr. Ir. V. Buskens

December 18, 2019

Acknowledgements

First and foremost, I want to thank Henk Stoof, Vincent Buskens and Adriana Correia for their supervision while working on this thesis. With their original insights they were able to help me with any difficulties faced. Also, their positive positive spirit has made this project a lot of fun to work on. Furthermore, I would like to thank Joris Broere and Kevin Peters for the useful discussions.

This project largely builds on the earlier work of the aforementioned people from two entirely different research fields. Together, they have been able to formulate an interesting new approach to game theory. It has been exciting to be able to contribute to a better understanding of the response theory of correlated games at such an early stage.

Contents

Chapter 1

Introduction

Game theory is a well established theory to model behaviour with applications in a diverse range of research fields. These include amongst others social sciences, economics, political science and biology [18, 16, 11, 10]. Increasingly often, game theory is used in combination with network theory to predict the behaviour of agents on complex network structures. Such studies have already lead to interesting insights, but there is not always enough theoretical knowledge about the equilibrium behaviour of the agents on a network to explain the observed results. On the one hand, this is because standard game theory, which is still most often used, does not consider outcomes with correlations between the agents [7, 15]. This simplifying assumption does not allow for all possible outcomes, where agents may have access to information about the other's strategy due to negotiation or experience. On the other hand, the lack of theoretical understanding can be explained by the fact that the complexity of analysis is a limiting factor; as the system size increases, the number of interactions between the agents increases rapidly, which makes the results quickly too complicated to work out. The first issue has lead to a number of extensions of the standard game theory, most prominent of which is the correlation device [2]. This is a public signal which tells the agents what to do, forcing correlated outcomes on the system. A recent addition to this framework has been proposed by Correia and Stoof, who introduced the concept of response probabilities [6]. These response probabilities allow the agents to choose to a certain extent whether or not they want to obey the instructions from the correlation device, which gives the agents freedom to choose their actions. They applied this framework to a two-agent 'Snowdrift Game' network, and found new interesting equilibrium strategies. A second contribution of Correia and Stoof is their proposal of an analogy between game theory on a network and the Ising model from physics. This is a new angle to approach the problem, which could potentially be of use for the analysis of compicated systems[1]. This framework was applied to study a two-agent 'Battle of the Sexes' game by Peters, in which he was able to analyze the system within the response strategy as well as define a concrete mapping to the Ising model [14].

[1] Another attempt has recently been made to solve game theory problems using the Ising model, but with an entirely different approach [1].

The results presented in this thesis add to the current knowledge about correlated games in four different ways. First of all, analysis of two-agent Battle of the Sexes within response strategy is expanded upon, since we found that not all equilibrium strategies were found until now. With this new knowledge about the equilibrium strategies it became clear that there are different combinations between responses and correlation devices in equilibrium, which lead to identical observable outcomes. This degeneracy of the system suggests that, if we are interested in the observable outcomes and the correlation device is considered to be implicit, a choice can be made to prevent double outcomes and thereby reduce the amount of analysis. Secondly, the Pure Coordination is in this thesis analyzed within the response strategy of correlated games. The analysis of both two-agent systems is presented in Chapter 3. The third contribution is the analysis of three-agent response strategy,of Battle of the Sexes-like networks. The resulting equilibrium conditions for different network structures and correlation devices lead to interesting new insights, which are presented in Chapter 4. The number of degrees of freedom in three-agent systems is already so high that mapping out all equilibrium responses is a lot of work. Therefore, a choice was made for which types of equilibria to work out in detail. Because in the analysis of the two-agent system it was found that many equilibrium solutions are unstable, it was chosen only to focus on the stable ones. Furthermore, since there are multiple combinations between devices and responses mapping to the same observable results, the choice was made to limit our attention to just part of the correlation device phase space. As an example of the unstable correlated equilibria, one equilibrium response strategy is worked out. The result of this analysis is a classification of the different response strategies, which can easily be generalized to any network structure. A last contribution is that an exact mapping between a general two or three-agent system and the Ising model is defined, which is presented in Chapter 5. In order to make the Ising model analogy work for a general three-agent game, the inclusion of three-body interactions were necessary. Moreover, it was found that the the details of the specific network structure are not captured in the Ising model analogy. Before any of the new results are presented, Chapter 2 introduces the various concepts from game theory in an accessible manner.

Chapter 2

Game theory

In this chapter, an introduction to game theory, in particular the Battle of the Sexes (BoS) game and Pure Coordination (PC) game, is provided which is sufficient to understand the content of this thesis. First, a general introduction is given to explain the various concepts used in this thesis. Consequently, both games are analysed within standard game theory. In the last section, some important further remarks are made.

2.1 Introduction

Game theory is the theory of mathematically modeling situations in which multiple agents influence each other by their actions. Since people are usually interested in their own gain, the main question to consider in these models is: what is the optimal, or rational, way of behaving in the situation as given by the model? Although von Neumann was credited with first introducing the discipline of game theory, it was John Nash who in 1950 formulated a consistent theory to find the equilibria in game-theoretical problems [13, 12]. Over the years, his results have been extended to useful applications in many different fields of research.

When a game-theoretical model is made of a certain situation, a specific gain should be assigned for each agent at each possible outcome. This is how the preference of one situation over another is introduced in the mathematics. Usually, this is done by means of a payoff table, of which Table 2.1 is a generalized example. In this example there are two agents, 1 and 2, who both have two options, C and D. For each outcome two parameters are shown, which specify the outcome of that specific result. The parameter left of the comma is the outcome for agent 1, whereas the parameter right of the comma is the outcome for agent 2. Depending on the chosen values, it is more beneficial for the agents to end up in certain outcomes than in others. The interests of the players might not always align, which makes even these simple examples interesting to analyze. Depending on the parameter values, the type of game changes. For each different type of two-agent game, there is an extensive body of literature covering the analysis under different circumstances and conditions. The games have gotten well known names such as 'Prisoners Dilemma', 'Stag Hunt' and 'Game of Chicken' [15, 7]. These particular games will not be discussed in this introduction, since they are outside of the scope of this thesis. The focus of this thesis is on 'Battle of the Sexes' (BoS) and 'the Pure Coordination game' (PC). Before we are ready to discuss these in detail, the different concepts from game theory which are used in this thesis are introduced in a formal as well as more intuitive manner.

Table 2.1: Payoff table - general

1 \ 2	C	D
C	(A,a)	(B,b)
D	(C,c)	(D,d)

Nash Equilibria

Loosely stated, a Nash equilibrium is a situation in which none of the agents has an incentive to change his or her strategy, given that all other agents play a certain strategy. Such a situation is an equilibrium because, if this situation is reached, no one can improve their expected outcomes just by their own actions. This does not necessarily mean that the equilibrium is objectively the best for all agents; the strategies of the agents could be such that everyone has a sub-optimal result, but still the system is in a deadlock. A more formal definition of a Nash equilibrium can be obtained with the notion of the *strategy space*, which is defined as the space of all possible set of strategies. The pure strategy space S_i is the set of all strategies such that each player plays a certain option with probability 1 and the other option with probability 0. If this is the strategy space that you allow, there can only be definitive outcomes. A slightly looser variant is the mixed strategy space Σ_i, in which the agents $i \in \{1, .., N\}$ are able to play a strategy where they choose one of the outcomes $\mu \in \{C, D\}$ with a certain probability P^i_μ. The payoff functions, which give each agent a certain gain for each possible outcome, are denoted by u_i. In terms of these, the formal definition of a Nash equilibrium in the mixed strategy space can be stated as follows [7]:

A game in strategic (or normal) form has three elements: the *set of players*, the *strategy space* for each player and the *payoff functions*. The notation $-i$ is used to indicate the agents other than i. A Nash equilibrium is a strategy profile $(\sigma_1^*, ...\sigma_N^*)$ such that:

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \quad \forall i, \forall s_i \in S_i. \tag{2.1}$$

A correlation device

Although the pure and mixed strategy spaces are very interesting, they are too restrictive to account for all possible outcomes that may be observed in reality. In particular, it is not possible to get correlated outcomes: since the agents have to choose their strategies independently, the final probability distribution is always product separable ($p_{\mu\nu} = P^1_\mu P^2_\nu$). In order to allow for correlations, the *correlation device* has been introduced as an extra component [2]. The correlation device $p_{\mu\nu}$ is a public signal, which forces the agents into a certain strategy. A classic example of a correlation device is a traffic light, which regulates whether agents drive (option 1) or stop (option 2). Since a correlation device can induce any type of probability distribution over the different outcomes, the exploration of *correlated equilibria* is now possible. Formally, a correlation device is defined by the space of all possible outcomes Ω, the information partition for player i H_i (the information that agent i has regarding the outcome) and a probability distribution over the state phase p. The situation is a correlated equilibrium if no agent can improve his or her expected outcome by unilaterally deviating from the correlation device. The definition of a correlated equilibrium in terms of these can be formulated as [7]:

A strategy profile $(s_1^*, s_2^*, .., s_i^*)$ is a correlated equilibrium if:

$$\sum_{\omega\in\Omega} p(\omega)u_i(s_i^*(\omega), s_{-i}^*(\omega)) \geq \sum_{\omega\in\Omega} p(\omega)u_i(s_i(\omega), s_{-i}^*(\omega)) \quad \forall i, \forall s_i \in S_i. \tag{2.2}$$

Response strategy of correlated games

The addition of Correia and Stoof to this framework is the introduction of *response strategies*, which give the agents the freedom to choose whether to follow the instructions by the correlation device or not [6]. More concretely, a set of response probabilities $P^i_{F\mu}$ are added to the previous framework. First, the framework is defined for two agents. Therefore, in these and subsequent expressions, $i \in \{1, 2\}$. From the response probabilities it directly follows that the probability that advice μ is not followed equals $P^i_{F\mu} = 1 - P^i_{F\mu}$. The response probabilities induce a mapping from the initial probabilities $p_{\mu\nu}$ given by the correlation device to *renormalized* probabilities $p^R_{\mu\nu}$, which are defined as

$$p^R_{\mu\nu} = \sum_{\mu'\nu'} P^1_{\mu\leftarrow\mu'} P^2_{\nu\leftarrow\nu'} p_{\mu'\nu'}, \tag{2.3}$$

where the expression $P^i_{\mu\leftarrow\mu'}$ is defined as

$$P^i_{\mu\leftarrow\mu'} = \delta_{\mu\mu'} P^i_{F\mu} + (1 - \delta_{\mu\mu'}) P^i_{NF\mu}. \tag{2.4}$$

With the introduction of the response strategies, there is both public information, in the form of the correlation device, and private information, in the form of the response probabilities. In this correlated game, the expected outcome for agent i is given by

$$\langle u^i \rangle = \sum_{\mu,\nu} p^R_{\mu\nu} u^i_{\mu\nu} = \sum_{\mu',\nu'} \sum_{\mu,\nu} P_{\mu\leftarrow\mu'} P_{\nu\leftarrow\nu'} p_{\mu\nu} u^i_{\mu\nu}. \tag{2.5}$$

An equilibrium in this framework is a situation where neither agent can improve its own expected payoff by changing his or her own response probabilities, given the strategy of the other agent. In more formal language this comes down to the statement in the box below.

A two-agent response strategy profile $(P^{1*}_{FC}, P^{1*}_{FD}, P^{2*}_{FC}, P^{2*}_{FD})$ is an equilibrium if for all i and for all $P^i_{\mu\leftarrow\mu'}$ the following condition is true:

$$\sum_{\omega,\omega'} u^i(\omega) P^{i*}_{\mu\leftarrow\mu'} P^{-i*}_{\nu\leftarrow\nu'} p(\omega') \geq \sum_{\omega,\omega'} u^i(\omega) P^{i}_{\mu\leftarrow\mu'} P^{-i*}_{\nu\leftarrow\nu'} p(\omega') \quad \omega, \omega' \in \Omega. \tag{2.6}$$

The generalization of this definition from two agents to N agents is clear. Index $i \in \{1, .., N\}$ now denotes the set of agent, $\omega \in \Omega$ denotes a particular outcome from the set of possible outcomes Ω.

A N agent response strategy profile $(P^{1*}_{FC}, P^{1*}_{FD}, ..., P^{N*}_{FC}, P^{N*}_{FD})$ is an equilibrium if the following condition is true:

$$\sum_{\omega,\omega'} u^i(\omega) P^{1*}_{\mu\leftarrow\mu'} \cdot .... \cdot P^{N*}_{\nu\leftarrow\nu'} p(\omega') \geq \sum_{\omega,\omega'} u^i(\omega) P^{1}_{\mu\leftarrow\mu'} \cdot .... \cdot P^{N*}_{\nu\leftarrow\nu'} p(\omega'), \quad \forall i, \forall P^i_{\mu\leftarrow\mu'} \quad \omega, \omega' \in \Omega. \tag{2.7}$$

To denote a specific set of response strategies, the following notation is often used throughout this thesis: $(P^1_{FC}, P^1_{FD}, ..., P^N_{FC}, P^N_{FD})$. The expected outcome, as given in 2.5, is an expression which is linear in the response strategy of the agents. Therefore so-called *slope coefficients* C^i_μ can be defined, which are a function of the correlation device and the response strategy of the other agents:

$$\langle u^i \rangle = C^i_C(P^1_{FC}, ..., P^N_{FD}) P^i_{FC} + C^i_D(P^1_{FC}, ..., P^N_{FD}) P^i_{FD} + C^i(P^1_{FC}, ..., P^N_{FD}). \tag{2.8}$$

In this equation, the coefficients are dependent on the response probabilities of all *other* agents. Given the response strategy of the other agents, the slope coefficients determine whether a strategy for agent i is optimal. For each response probability, there are three situations where you cannot improve your outcome by changing strategy:

1. $C^i_\mu > 0$, and $P^i_{F\mu} = 1$.
2. $C^i_\mu < 0$, and $P^i_{F\mu} = 0$.
3. $C^i_\mu = 0$.

In the first two cases, deviating from the strategy results in a strictly worse outcome. In the last case, you are indifferent between outcomes. For each combination of equilibrium slope conditions it can be calculated whether there are correlation devices for which that strategy results in a Nash equilibrium.

Coordination games v.s. anti-coordination games

In two-agent strategic forms games, a distinction can be made between coordination games and anti-coordination games. A coordination game is a game in which the pure Nash equilibria are on the outcomes where both agents play the same move (they coordinate). An anti-coordination game is a game in which the pure Nash equilibria are the outcomes where both agents play a different move (they anti-coordinate). In terms of the generalized payoff matrix in Table 2.1, a coordination game satisfies $A > B$, $D > C$, $a > c$ and $d > b$. An anti-coordination game satisfies $B > A$, $C > D$, $c > a$ and $b > d$.

2.2 Battle of the Sexes

With use of the theory provided in the previous section, the two-agent Battle of the Sexes game can be analyzed in some detail. Traditionally, Battle of the Sexes is explained with the example of a husband and wife, who (without communicating) individually have to decide whether to go to the ballet (preferred by the woman) or the baseball (preferred by the man). Although this is the situation to which the game owns its name, in the rest of this chapter the more gender-neutral version will be adopted where two friends want to go to the movies together. Agent 1 prefers to go to the comedy, whereas agent 2 prefers the drama. This results in an asymmetric payoff function: the payoff for going to your preferred movie together is assumed to be 1 and the payoff for going to the other movie together is assumed to be $0 < S < 1$. Since both agents do not like going to the movies alone, the situations where they end up at different movies have payoff 0[1]. This situation is shown in Table 2.2. It follows from the definition that this is a coordination game, which already tells us that the preferable outcomes are such that the agents coordinate their strategies.

[1]There also is a variant of BoS where payoff $0 < t < S$ is assigned to ending up alone at your preferred movie. Since this variant is not used in this thesis, it is not further explained.

Table 2.2: Payoff table - Battle of the Sexes

1 \ 2	C	D
C	(1,S)	(0,0)
D	(0,0)	(S,1)

Nash equilibria

Within the pure or mixed strategy space, the Nash equilibria for two-agent Battle of the Sexes are well known. In the pure strategy space there are two Nash equilibria:

1. Both agents go to the comedy (play C).
2. Both agents go to the drama (play D).

These are equilibrium solutions since, if one of the agents changes its strategy while the other agent does not, the agent who changed his strategy (as well as the other) is worse off. Because the expected payoff of both Nash equilibria is different for each agent, it does matter in which equilibrium you end up: even though in both cases you have no incentive to change strategies, one of them is more beneficial for you than for the other.

Within the mixed strategy space, the pure equilibria as described above are present as well (the probabilities are either 0 or 1). However, there now is a *mixed equilibrium* as well, in which both agents are indifferent between their choices: the other agent chooses his strategy such that the expected payoff when you play C is equal to the expected payoff when you play D. This situation is obtained for the strategy profile

$$\begin{aligned} P_C^1 &= \frac{1}{1+S}, \qquad & P_C^2 &= \frac{S}{1+S}, \\ P_D^1 &= \frac{S}{1+S}, \qquad & P_D^2 &= \frac{1}{1+S}, \end{aligned} \tag{2.9}$$

with a corresponding expected payoff for either agent equal to $\frac{2S}{(S+1)^2}$, which is lower than the payoff of either of the pure Nash equilibria. The equilibrium strategies can be conveniently shown in a plane, with P_C^1 and P_C^2 on the axes. In Figure 2.1, the pure Nash equilibria are shown as the coloured dots and the mixed equilibrium is where the lines cross for $S = .5$. As can be seen, the mixed equilibrium is asymmetric between the agents, which reflects their different preferences.

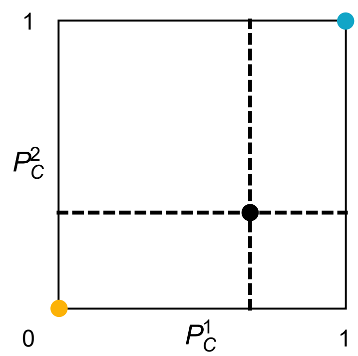


Figure 2.1: The equilibrium strategies in the mixed strategy space for Battle of the Sexes, for $S = .5$.

Correlated equilibria

A concrete example of correlated equilibria in Battle of the Sexes is worked out here, to make the somewhat abstract formal definition more intuitive. Consider the situation in which the agents let some random process (possibly a coin flip) decide beforehand where they will go: if the outcome is heads (H), this will send both of them to D with some probability $p(H) = p_{DD} = A$. If the outcome is tails (T), this will send them to C with probability $p(T) = p_{CC} = 1 - A$. In this situation $\Omega = \{H, T\}$, since these are the possible outcomes. The information partition is such that $\{H_1\} = \{H_2\} = \{\{H\}, \{T\}\}$, which reflects that after the coin flip both agents have information about the outcome. Since in this scenario for any A the probabilities p_{CC} and p_{DD} add up to one, the off-diagonal components must be zero. Therefore, the agents never have incentive to unilaterally to deviate (since the system is fully correlated, and deviating would induce mismatched outcomes): the system is in correlated equilibrium. The expected outcome $\langle u_i \rangle$ for the agents in this case is given by

$$\langle u_1 \rangle = A + (1 - A)S \quad \text{and} \quad \langle u_2 \rangle = SA + (1 - A). \tag{2.10}$$

This situation is always better for both agents than the mixed equilibrium. Depending on the assignment of A one agent might benefit more than the other.

2.3 Pure Coordination game

The other two-agent game considered in this thesis is the Pure Coordination game, which is very similar to Battle of the sexes. The payoff for this game is shown in Table 2.3.

Table 2.3: Payoff table - Pure Coordination

1 \ 2	C	D
C	(1,1)	(0,0)
D	(0,0)	(S,S)

The pure equilibria in this game again are the situations where both agents play C or D with probability 1. The expected outcome for both agents now however is equal, and both are best off in the pure C equilibrium. The mixed equilibrium now is the strategy

$$\begin{aligned} P_C^1 &= \frac{S}{1+S}, \qquad P_C^2 = \frac{S}{1+S}, \\ P_D^1 &= \frac{1}{1+S}, \qquad P_D^2 = \frac{1}{1+S}. \end{aligned} \tag{2.11}$$

A similar figure as for Battle of the Sexes is shown below, where now the mixed equilibrium is on the diagonal since the agents have a symmetric payoff matrix.

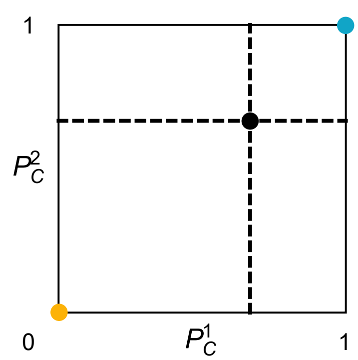


Figure 2.2: The equilibrium strategies in the mixed strategy space for the Pure Coordination game, for $S = .5$.

Of course it is here also possible to construct correlated equilibria. With the same coin flip correlation device as in the example from the previous section, the outcomes are now also in correlated equilibrium, for the same reasons as before: the device always sends both agents to the same outcome, and therefore it is never rational to unilaterally disagree with the device. The expected outcomes for the agents now are

$$\langle u_i \rangle = A + (1 - A)S. \tag{2.12}$$

2.4 Further remarks

With the definitions of equilibria as stated above, there still are many different variants of the Battle of the Sexes game. Depending on the application, this can influence the analysis or interpretation of the results. For example, it is not yet specified how 'the game is played'. In one interpretation, both agents need to make their choice simultaneously. They therefore do not know what the other will play (unless the correlation device tells them what the other will do). On the other hand, there are also situations conceivable where the agents need to play their move after each other. In this case, the latter agent to move might know what the first agent did. This difference of course has an influence on the analysis of the expected behaviour. In this thesis, it will always be assumed that the agents need to make their choice simultaneously; they therefore have no information about the others except what the correlation device tells them. Another important issue is whether you regard the games to be played once, which is often called a 'one-shot' game, or in iterations. In one-shot games there is no information about the other agents. However, in iterative games agents have the ability to learn from past rounds, effectively gaining information about the expected strategies of others. This expectation about the behaviour based on previous experience could be interpreted as an implicit correlation device, where there is no actual device giving advice to the agents but it is rather a means to capture the believes of what the agents are likely to do.

In principle all these different variants of the same Battle of the Sexes game have the same Nash equilibria, but what differs between different applications is which ones can be expected to be realized. The most basic type of equilibrium is the Nash (or correlated) equilibrium, since the equilibrium analysis does not distinguish between different kinds of equilibria yet. However, this means that not all equilibria are necessarily stable or realizable. In Chapter 3 all equilibria are mapped out for Battle of the Sexes as well as the Pure Coordination game. However, in a three-agent network, working out everything is too much. Therefore, the choice was made to focus on those equilibria which are *stable* in iterative versions of the game. There are many different definitions of stability for different purposes, such as Evolutionary Stable States, Stochastic Stability and Strategic Stability to name a few [18, 17, 9]. The exact interpretation of stability used in this thesis, which is closely related to Evolutionary Stable States, is defined now.

Stability

When the game is played in iterations, agents have the ability to converge to an equilibrium. This is because they can choose at each round which move to play, and by doing so they can use the information from passed rounds. Therefore, if they are not in an equilibrium, the strategies of the agents will change every round until an equilibrium is reached. If they are in an equilibrium situation, it can be expected that they only stay in that equilibrium if they do not want to change strategies due to small deviations in the strategies of the other agents. More precisely, it is assumed that a strategy is stable if, when any one of the agents deviates with an infinitesimal amount δ from the equilibrium strategy, none of the other agents has incentive to deviate from the equilibrium. To demonstrate how this works precisely, consider the pure C equilibrium in Battle of the Sexes in the mixed strategy space. Here $P_C^1 = P_C^2 = 1$. However, if there is a fluctuation such that agent 1 now has strategy $P_C^1 = 1 - \delta$, then what is the best strategy for agent 2 (who prefers D)? His expected outcome when playing C now is $S(1-\delta)$, whereas when playing D it is δ. If the value of the fluctuation is small enough, he will still be rational by always playing C. It follows that it is best for agent 1 to play C with probability 1 again in the next round of the game. We now only considered one of the possible infinitesimal deviations, but from similar analysis it follows that the equilibrium is stable under any possible infinitesimal fluctuation. We now consider the mixed equilibrium from Equation 2.9 with a fluctuation such that $P_C^1 = \frac{1}{1+S} - \delta$. The expected outcome for agent 2 when playing C in this case is $S(\frac{1}{1+S} - \delta)$, and his expected outcome when playing D is $(\frac{S}{1+S} + \delta)$. It can be concluded that he is not indifferent anymore and that, if he uses this knowledge in the next iteration of the game, he will play outcome D with probability 1. This in turn causes the original strategy of agent 1 not to be optimal anymore, until he decides to play outcome D with probability 1 as well. Hence the mixed equilibrium is not stable under infinitesimal deviations[2].

[2]This analysis is focused on games played in iterations. In one-shot games, the concept of infinitesimal fluctuations might be less applicable. Also, there might in one-shot games be other motivations to argue why some equilibria might be more stable than others. For example, in the Pure Coordination game both agents prefer the same outcome, which might be used as argument to say that the only rational choice is to both go to the preferred option with probability 1. Since in this thesis we want the results to be applicable in iterative games, all equilibria are considered stable if, once you are in the equilibrium, no infinitesimal fluctuations can induce the agents to change strategies.

Chapter 3

Two agents

The simplest network structures consist of two connected agents. All networks consist of pairs of interaction agents, which makes the two-agent case the building block for the analysis of any more complex situation. There are two possible two-agent interactions in a BoS-like network: when two different type agents interact, the payoff matrix is as in Battle of the Sexes. If two equal agents interact, the payoff is like the Pure Coordination game. The results of both games are worked out in this chapter.

Whenever 2D plots are made, the assumption is made that $p_{CD} = p_{DC} = \frac{1}{2}(1 - p_{CC} - p_{DD})$. This is convenient since it allows plotting on a plane. Also, unless stated otherwise explicitly, the plots are made with value $S = .5$. The equations in derivations in the text however are of the general form without constraints on the correlation device. The role of the determinant ($D[p_{\mu\nu}] = p_{CC}p_{DD} - p_{DC}p_{CD}$) of the correlation device is important to understand the results. The analysis can be subdivided into the case where the determinant is positive, which means that the correlation device induces positive correlations on the system, and the case where the determinant is negative, which means that the correlation device induces negative correlations. Since Battle of the Sexes is a coordination game, the agents benefit from playing similar strategies if the correlations induced by the device are positive (positive determinant). On the other hand, if the correlations in the device are negative (negative determinant), the agents should play opposite strategies to make sure that the renormalized correlations are positive again. This insight suggests that the correlation device phase space can be divided in two parts, as is shown in Figure 3.1.

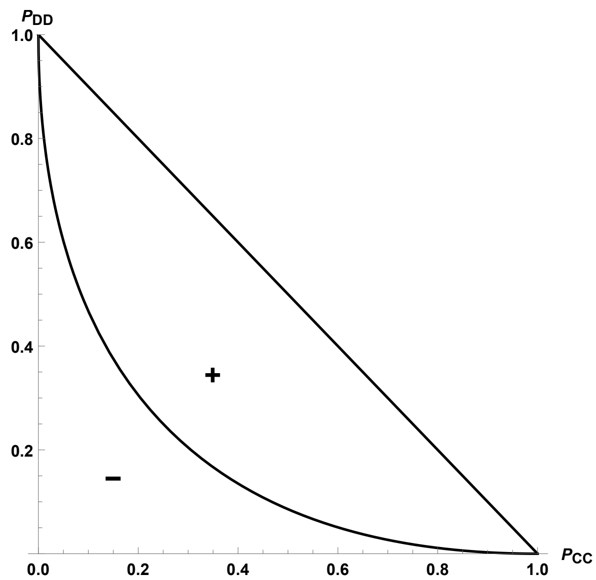


Figure 3.1: Above the curved line, for $p_{CD} = p_{DC}$, the determinant of the correlation device is positive. Below the line, it is negative. Since in coordination games positive correlations are good, it follows that below the curved line the agents should play opposite strategies, whereas above the line they should play similar strategies.

3.1 Battle of the Sexes

In this section, Battle of the Sexes for two agents is worked out in detail. This analysis has large overlap with the work of Peters [14]. Since there are some significant additions to this work, and the results are of importance to the other chapters, it was chosen to include the entire analysis here so the discussion stands on its own.

Battle of the Sexes as defined in this chapter is characterized by the payoff matrix shown in Table 3.1. The results for the variant where the preference of the players is reversed can easily be obtained by switching the roles of outcome C and outcome D, and therefore only analyzing one of these cases suffices.

Table 3.1: Payoff table - Battle of the Sexes

1 \ 2	C	D
C	(1,S)	(0,0)
D	(0,0)	(S,1)

For this payoff matrix, the slope coefficients as defined in Equation 2.8 for agent 1 are:

$$\begin{cases} C_C^1(P_{FC}^2, P_{FD}^2) = & p_{CC}((S+1)P_{FC}^2 - S) - p_{CD}((S+1)P_{FD}^2 - 1) \\ C_D^1(P_{FC}^2, P_{FD}^2) = & p_{DD}((S+1)P_{FD}^2 - 1) - p_{DC}((S+1)P_{FC}^2 - S) \\ C^1(P_{FC}^2, P_{FD}^2) = & Sp_{CC}(1 - P_{FC}^2) + Sp_{CD}P_{FD}^2 + p_{DC}P_{FC}^2 - p_{DD}(P_{FD}^2 - 1) \end{cases} . \quad (3.1)$$

For agent 2 they are:

$$\begin{cases} C_C^2(P_{FC}^1, P_{FD}^1) = & p_{CC}((S+1)P_{FC}^1 - 1) - p_{DC}((S+1)P_{FD}^1 - S) \\ C_D^2(P_{FC}^1, P_{FD}^1) = & p_{DD}((S+1)P_{FD}^1 - S) - p_{CD}((S+1)P_{FC}^1 - 1) \\ C^2(P_{FC}^1, P_{FD}^1) = & -p_{CC}(P_{FC}^1 - 1) + Sp_{CD}P_{FC}^1 + p_{DC}P_{FD}^1 + Sp_{DD}(1 - P_{FD}^1) \end{cases} . \quad (3.2)$$

As was explained in the previous chapter, Nash equilibria are situations where neither of the agents can improve the expected outcome by changing his or her own strategy, given the strategy of the other. In terms of the slope coefficients, this suggests the following method to determine for what kind of correlation devices a certain set of response strategies is an equilibrium:

1. Choose a set of response strategies $P_{F\mu}^i$.
2. For what correlation device values is agent 1 optimal, given $P_{F\mu}^2$?
3. For what correlation device values is agent 2 optimal, given $P_{F\mu}^1$?
4. Is there overlap in the range of correlation device values?

By repeating this procedure for all possible response probabilities, every Nash equilibrium is found. Since the constant coefficients C^i do not influence the optimal response strategies, these are not important for this analysis.

Slope analysis

In this section, all the different slope conditions which result in Nash equilibria are worked out. Since the response strategy has two degrees of freedom per agent (P_{FC}^i and P_{FD}^i), which each have three possible type of slope coefficients (positive, negative or zero), there are in principle 81 possible types of response strategies (always follow, never follow or 'sometimes follow') which each might be a Nash equilibrium for some correlation devices[1]. However, many of these strategies do not result in a Nash equilibrium except in trivial cases. In this section, only the ones which do produce equilibria are worked out in detail. So which are those? First of all, it can easily be shown that if one agent makes the other indifferent for a certain choice, the other should do the

[1] Even though there are infinitely many options to choose a response strategy $0 < P_{F\mu}^i < 1$, these can for the purpose of equilibrium analysis be regarded as one case, since all options are optimal iff the relevant slope coefficient equals zero.

same for the situation to be an equilibrium. This effectively means that both agents should have the same amount of slope conditions which equal zero. Using the notation $(P^1_{FC}, P^1_{FD}, P^2_{FC}, P^2_{FD})$ to denote a response strategy, we can immediately infer from this that for example $(P^{1*}_{FC}, 0, 0, 0)$ and $(P^{1*}_{FC}, 1, P^{2*}_{FC}, P^{2*}_{FD})$ cannot be in equilibrium[2]. This limits the amount of cases to consider to 41. The next thing to realize is that if all slope coefficients are non-zero, which means that no-one is indifferent between choices, there cannot be equilibria such as $(1, 1, 1, 0)$ or $(0, 0, 1, 0)$: there always either needs to be the same amount of 0's and 1's, or just 0's or just 1's[3]. This limits the amount of cases to 25. It was chosen to not include a proof why the excluded cases do not lead to equilibria, since this is easy to verify. Of the 25 remaining cases, there are just 2 which do not result in equilibria: $(1, 0, 0, 1)$ and $(0, 1, 1, 0)$. These are the exact opposite of the pure equilibria, and thereby the worst possible strategy (expected outcome equals zero). The 23 cases which do result in equilibria are worked out in this chapter.

Pure equilibria

Irrespective of what the values of the correlation device are, it is always possible to realize the pure Nash equilibria. These equilibria, which were already discussed in Chapter 2, are the cases where both agents consistently play the same move. There are two pure Nash equilibria, one where each agent always plays option C and one where each agent always plays option D.

The response strategy corresponding to each player always playing C is $(1, 0, 1, 0)$. The expected outcome for the agents in this case is

$$\langle u_1 \rangle = 1 \quad \text{and} \quad \langle u_2 \rangle = S. \tag{3.3}$$

The response strategy corresponding to each player always playing D is $(0, 1, 0, 1)$. The expected outcome for the agents in this case is

$$\langle u_1 \rangle = S \quad \text{and} \quad \langle u_2 \rangle = 1. \tag{3.4}$$

Note that, no matter what correlation device is chosen, the pure equilibria can be reached. This is a consequence of the fact that these are uncorrelated equilibria (in the

[2] For example, consider the strategy $(P^{1*}_{FC}, 0, 0, 0)$. In this case, $C^1_C(0, 0) = -Sp_{CC} + p_{CD}$, so in order to make this slope condition equal to zero, we must have that $p_{CC} = -p_{CD}$. Therefore, both must equal zero. It then follows from the slope coefficients for agent 2 that the only consistent choice is $p_{DD} = 1$, all other indices zero. For this choice, any P^{1*}_{FC} would satisfy the conditions. However, since this is only possible for one point in the correlation device space, and $(0, 0, 0, 0)$ is also an equilibrium here, this equilibrium solution is discarded. Similar analysis shows that any of the other cases with an uneven amount of 'zero' slope conditions does not result in equilibria for a finite range of the correlation device phase space.

[3] Consider for example the strategy $(1, 1, 1, 0)$. In that case, $C^1_D(1, 0) = -p_{DD} - p_{DC}$, for which $P^2_{FD} = 1$ only is an optimal choice if $p_{DD} = p_{DC} = 0$. If this is the case, then $C^2_D(1, 1) = -Sp_{CD}$, from which it follows that $p_{CD} = 0$. Hence the only consistent correlation device is $p_{DD} = 1$, all other indices zero. However, since this only is only one point in phase space, and $(1, 1, 1, 1)$ is also an equilibrium here, this equilibrium solution is discarded. Similar analysis for the other examples shows that these cases do not result in equilibria for a finite range of the correlation device phase space.

sense that the pure equilibria are product separable solutions), which do not require any information from the device. On the other hand, the pure equilibria in a sense are also maximally correlated: the agents always end up together. This is reflected in Figure 3.1 by the fact that the pure equilibria, when you imagine the renormalized probabilities on the axes, are the points in phase space where the curved line of uncorrelated solutions and the straight line of maximally correlated solutions ($p^R_{DD} = 1 - p^R_{CC}$) meet.

Mixed equilibrium

The mixed equilibrium, which also is well known from standard game theory, is found when each agent makes the other agent indifferent between its choices. This translates to slope conditions

$$\begin{cases} C^1_\mu(P^{2*}_{FC}, P^{2*}_{FD}) = & 0 \\ C^2_\mu(P^{1*}_{FC}, P^{1*}_{FD}) = & 0 \end{cases} \quad \forall \mu \in \{C, D\}. \tag{3.5}$$

Solving these four equations gives the following set of response probabilities, which result in a Nash equilibrium for any correlation device:

$$\begin{aligned} P^1_{FC} &= \frac{1}{S+1}, \qquad & P^2_{FC} &= \frac{S}{S+1}, \\ P^1_{FD} &= \frac{S}{S+1}, \qquad & P^2_{FD} &= \frac{1}{S+1}. \end{aligned} \tag{3.6}$$

Irrespective of the correlation device values, this results in uncorrelated probabilities for each agent to play a certain move

$$\begin{aligned} P^1_C &= \frac{1}{S+1}, \qquad & P^2_C &= \frac{S}{S+1}, \\ P^1_D &= \frac{S}{S+1}, \qquad & P^2_D &= \frac{1}{S+1}, \end{aligned} \tag{3.7}$$

with corresponding expected outcome for both agents

$$\langle u_i \rangle = \frac{2S}{(S+1)^2}. \tag{3.8}$$

Note that first of all this mixed equilibrium can always be realized, since it is uncorrelated. Furthermore, the expected outcome is worse than both of the pure equilibria.

Correlated equilibria

In the previously discussed cases the correlation device did not really matter, since the outcomes were independent of the specific correlation device. In all other cases, which will be discussed now, the correlation device is of importance.

The first equilibrium which we consider is $(1, 1, 1, 1)$, which means that both agents always follow the device. This corresponds to slope conditions

$$\begin{cases} C^1_\mu(1,1) > & 0 \\ C^2_\mu(1,1) > & 0 \end{cases}. \tag{3.9}$$

The constraints for agent 1 are satisfied for a range of correlation device values as given by the constraints

$$\begin{aligned} p_{CC} >& Sp_{CD}, \\ p_{DD} >& \frac{p_{DC}}{S}. \end{aligned} \tag{3.10}$$

These conditions correspond to the to the red area in Figure 3.2a. Similarly, agent 2 is only optimized if the correlation device satisfies

$$\begin{aligned} p_{CC} >& \frac{p_{DC}}{S}, \\ p_{DD} >& Sp_{CD}, \end{aligned} \tag{3.11}$$

which corresponds to the blue area in Figure 3.2a.

The expected outcome for each agent in this equilibrium is

$$\langle u_1 \rangle = p_{CC} + Sp_{DD} \quad \text{and} \quad \langle u_2 \rangle = Sp_{CC} + p_{DD}. \tag{3.12}$$

The next equilibrium which we consider is $(0, 0, 0, 0)$, no-one ever follows the device. This corresponds to slope conditions

$$\begin{cases} C^1_\mu(0,0) < & 0 \\ C^2_\mu(0,0) < & 0 \end{cases}, \tag{3.13}$$

which for agent 1 to be optimized give the conditions

$$\begin{aligned} p_{CC} >& \frac{p_{CD}}{S}, \\ p_{DD} >& Sp_{DC}. \end{aligned} \tag{3.14}$$

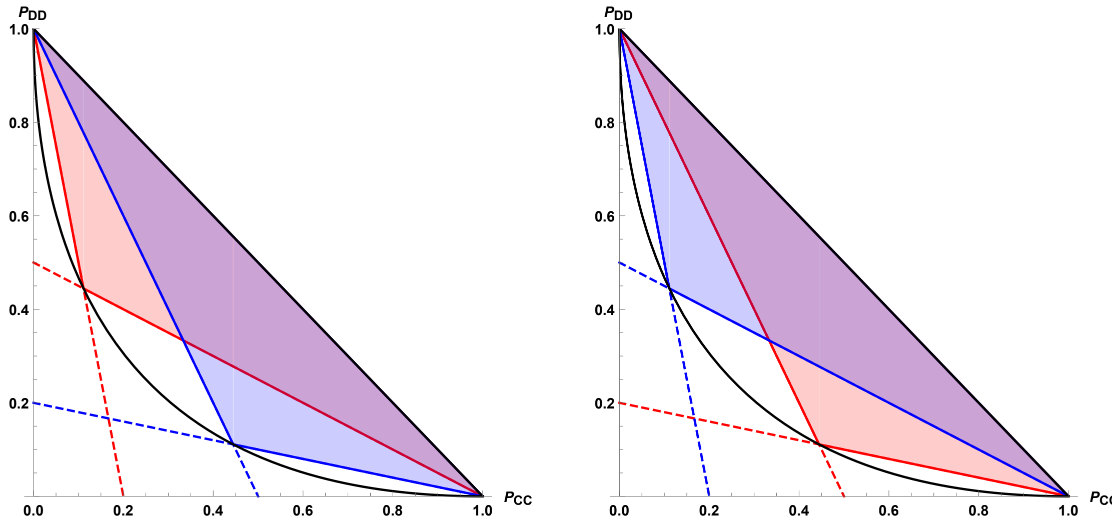
Those conditions correspond to the to the red area in Figure 3.2b. Similarly, the slope conditions for agent 2 translate to constraints on the correlation device such that

$$\begin{aligned} p_{CC} >& Sp_{DC}, \\ p_{DD} >& \frac{p_{CD}}{S}, \end{aligned} \tag{3.15}$$

which correspond to the blue area in Figure 3.2b.

The expected outcome for the agents in this equilibrium respectively is

$$\langle u_1 \rangle = p_{DD} + Sp_{CC} \quad \text{and} \quad \langle u_2 \rangle = Sp_{DD} + p_{CC}. \tag{3.16}$$



(a) **(b)**

Figure 3.2: In this figure, the equilibrium conditions for the response strategies **(a):** $(1,1,1,1)$ and **(b):** $(0,0,0,0)$ are shown, for $S = .5$. The equilibrium conditions for agent 1 are marked in red, whereas the equilibrium conditions for agent 2 are marked in blue. Where these areas overlap, both agents are optimized, which makes the strategy a Nash equilibrium.

The cases where one of the agents always follows the device and the other agent never follows the device are discussed next. When agent 1 always follows and agent 2 never follows the response strategy is $(1,1,0,0)$. This gives slope coefficients

$$\begin{cases} C^1_\mu(0,0) > & 0 \\ C^2_\mu(1,1) < & 0 \end{cases}, \tag{3.17}$$

which for agent 1 result in constraints on the correlation device such that

$$\begin{aligned} p_{CC} <& \frac{p_{CD}}{S}, \\ p_{DD} <& S p_{DC}. \end{aligned} \tag{3.18}$$

For agent agent 2 the corresponding constraints are

$$\begin{aligned} p_{CC} <& \frac{p_{DC}}{S}, \\ p_{DD} <& S p_{CD}. \end{aligned} \tag{3.19}$$

The resulting equilibrium region is shown in Figure 3.3a. The expected outcome for each agent in this equilibrium is

$$\langle u_1 \rangle = p_{CD} + S p_{DC} \quad \text{and} \quad \langle u_2 \rangle = S p_{CD} + p_{DC}. \tag{3.20}$$

Under the assumption that $p_{CD} = p_{DC}$, the expected outcome is equal for both agents. Also, the areas of 3.18 and 3.19 precisely overlap within this constraint[4].

When agent 2 always follows, and agent 1 never follows, the response strategy is $(0, 0, 1, 1)$. The equilibrium conditions on the slope coefficients are

$$\begin{cases} C_\mu^1(1,1) < & 0 \\ C_\mu^2(0,0) > & 0 \end{cases}, \tag{3.21}$$

which for agent 1 give restraints on the correlation device such that

$$\begin{aligned} p_{CC} <& S p_{CD}, \\ p_{DD} <& \frac{p_{DC}}{S}. \end{aligned} \tag{3.22}$$

For agent 2 the constraints are

$$\begin{aligned} p_{CC} <& S p_{DC}, \\ p_{DD} <& \frac{p_{CD}}{S}. \end{aligned} \tag{3.23}$$

The region defined by these constraints is shown in Figure 3.3b. The expected outcome for each agent in this Nash equilibrium is

$$\langle u_1 \rangle = p_{DC} + S p_{CD} \quad \text{and} \quad \langle u_2 \rangle = S p_{DC} + p_{CD}. \tag{3.24}$$

[4]Note that this would not be the case if $p_{CD} \neq p_{DC}$. Not too much value should be given to the symmetries/asymmetries in the plane of plotting, since it would look very different if another plane is chosen.

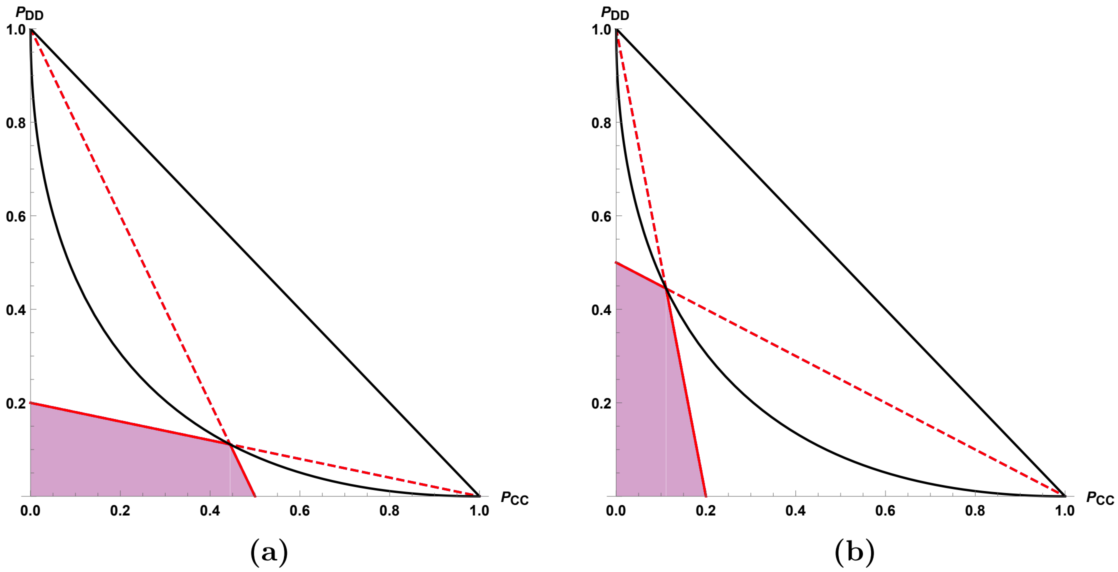


Figure 3.3: In this figure, the equilibrium conditions for the response strategies **(a):** $(1, 1, 0, 0)$ and **(b):** $(0, 0, 1, 1)$ are shown, for $S = .5$. The equilibrium conditions for agent 1 are marked in red, whereas the equilibrium conditions for agent 2 are marked in blue. In the overlapping region, the strategy is a Nash equilibrium. In both sub-figures the equilibrium conditions for the agents precisely coincide, which indicates that in this plane they are optimized for the same correlation device values.

The remaining cases to consider are the ones where the agents make each other indifferent for one outcome, but maximize their own gain on the other outcome. The cases where the agents have 'similar' strategies are worked out first. There are eight of these which result in Nash equilibria, and these are now discussed.

The first case is when both agents always follow C, but sometimes follow D: $(1, P^{1*}_{FD}, 1, P^{2*}_{FD})$. This corresponds to slope conditions

$$\begin{cases} C^1_C(1, P^{2*}_{FD}) > & 0 \\ C^1_D(1, P^{2*}_{FD}) = & 0 \end{cases} \text{and} \quad \begin{cases} C^2_C(1, P^{1*}_{FD}) > & 0 \\ C^2_D(1, P^{1*}_{FD}) = & 0 \end{cases}. \tag{3.25}$$

Solving the two equalities fixes the values of the response probabilities:

$$\begin{aligned} P^{1*}_{FD} =& \frac{S}{S+1}(1 + \frac{p_{CD}}{p_{DD}}), \\ P^{2*}_{FD} =& \frac{1}{S+1}(1 + \frac{p_{DC}}{p_{DD}}). \end{aligned} \tag{3.26}$$

Both inequalities result in the condition

$$p_{CC}p_{DD} > p_{CD}p_{DC}, \tag{3.27}$$

which means that the determinant of the device has to be positive. The last set of constraints comes from the requirement that both probabilities in Equation 3.26 are no smaller than zero and no larger than one:

$$\begin{aligned} p_{DD} &> S p_{CD}, \\ p_{DD} &> \frac{p_{DC}}{S}. \end{aligned} \tag{3.28}$$

The area for which this response strategy results in a Nash equilibrium is shown in Figure 3.4a.

The second case to consider is that both agents never follow D, and sometimes play C: $(P_{FC}^{1*}, 0, P_{FC}^{2*}, 0)$. This corresponds to slope conditions

$$\begin{cases} C_C^1(P_{FC}^{2*}, 0) = & 0 \\ C_D^1(P_{FC}^{2*}, 0) > & 0 \end{cases} \text{and} \quad \begin{cases} C_C^2(P_{FC}^{1*}, 0) = & 0 \\ C_D^2(P_{FC}^{1*}, 0) > & 0 \end{cases}, \tag{3.29}$$

with corresponding equilibrium response probabilities

$$\begin{aligned} P_{FC}^{1*} &= \frac{1}{S+1}(1 - S\frac{p_{DC}}{p_{CC}}), \\ P_{FC}^{2*} &= \frac{1}{S+1}(S - \frac{p_{CD}}{p_{CC}}). \end{aligned} \tag{3.30}$$

The constraints on the correlation device are

$$\begin{aligned} p_{DD} &> \frac{p_{CD} p_{DC}}{p_{CC}}, \\ p_{CC} &> S p_{DC}, \\ p_{CC} &> \frac{p_{CD}}{S}, \end{aligned} \tag{3.31}$$

which is shown in Figure 3.4b.

The third case is when both agents always follow D, but sometimes follow C: $(P_{FC}^{1*}, 1, P_{FC}^{2*}, 1)$. This corresponds to slope conditions

$$\begin{cases} C_C^1(P_{FC}^{2*}, 1) = & 0 \\ C_D^1(P_{FC}^{2*}, 1) > & 0 \end{cases} \text{and} \quad \begin{cases} C_C^2(P_{FC}^{1*}, 1) = & 0 \\ C_D^2(P_{FC}^{1*}, 1) > & 0 \end{cases}, \tag{3.32}$$

which leads to expressions for the equilibrium response probabilities such that

$$\begin{aligned} P_{FC}^{1*} &= \frac{1}{S+1}(1 + \frac{p_{DC}}{p_{CC}}), \\ P_{FC}^{2*} &= \frac{S}{S+1}(1 + \frac{p_{CD}}{p_{CC}}). \end{aligned} \tag{3.33}$$

The constraints on the correlation device are

$$\begin{aligned} p_{DD} &> \frac{p_{CD} p_{DC}}{p_{CC}}, \\ p_{CC} &> \frac{p_{DC}}{S}, \\ p_{CC} &> S p_{CD}, \end{aligned} \tag{3.34}$$

which is shown in Figure 3.4c.

The fourth case to consider is when both agents never follow C, but sometimes follow D: $(0, P_{FD}^{1*}, 0, P_{FD}^{2*})$. This corresponds to slope conditions

$$\begin{cases} C_C^1(0, P_{FD}^{2*}) < & 0 \\ C_D^1(0, P_{FD}^{2*}) = & 0 \end{cases} \text{ and } \begin{cases} C_C^2(0, P_{FD}^{1*}) < & 0 \\ C_D^2(0, P_{FD}^{1*}) = & 0 \end{cases}, \tag{3.35}$$

with corresponding equilibrium response probabilities

$$\begin{aligned} P_{FD}^{1*} =& \frac{1}{S+1}(S - \frac{p_{CD}}{p_{DD}}), \\ P_{FD}^{2*} =& \frac{1}{S+1}(1 - S\frac{p_{DC}}{p_{DD}}). \end{aligned} \tag{3.36}$$

The constraints on the correlation device are

$$\begin{aligned} p_{DD} >& \frac{p_{CD}p_{DC}}{p_{CC}}, \\ p_{DD} >& \frac{p_{CD}}{S}, \\ p_{DD} >& Sp_{DC}, \end{aligned} \tag{3.37}$$

which is shown in Figure 3.4d.

The fifth case to consider is $(1, P_{FD}^{1*}, P_{FC}^{2*}, 1)$. This corresponds to slope conditions:

$$\begin{cases} C_C^1(P_{FC}^{2*}, 1) > & 0 \\ C_D^1(P_{FC}^{2*}, 1) = & 0 \end{cases} \text{ and } \begin{cases} C_C^2(1, P_{FD}^{1*}) = & 0 \\ C_D^2(1, P_{FD}^{1*}) > & 0 \end{cases}, \tag{3.38}$$

with equilibrium response probabilities

$$\begin{aligned} P_{FD}^{1*} =& \frac{S}{S+1}(1 + \frac{p_{CC}}{p_{DC}}), \\ P_{FC}^{2*} =& \frac{S}{S+1}(1 + \frac{p_{DD}}{p_{DC}}). \end{aligned} \tag{3.39}$$

The corresponding constraints on the correlation device are

$$\begin{aligned} p_{DD} >& \frac{p_{CD}p_{DC}}{p_{CC}}, \\ p_{CC} <& \frac{p_{DC}}{S}, \\ p_{DD} <& \frac{p_{DC}}{S}, \end{aligned} \tag{3.40}$$

which are shown in Figure 3.4e.

The sixth case to consider is $(0, P_{FD}^{1*}, P_{FC}^{2*}, 0)$. This corresponds to slope conditions

$$\begin{cases} C_C^1(P_{FC}^{2*}, 0) < & 0 \\ C_D^1(P_{FC}^{2*}, 0) = & 0 \end{cases} \text{ and } \begin{cases} C_C^2(0, P_{FD}^{1*}) = & 0 \\ C_D^2(0, P_{FD}^{1*}) < & 0 \end{cases}, \tag{3.41}$$

with corresponding response probabilities

$$\begin{aligned} P_{FD}^{1*} =& \frac{1}{S+1}(S - \frac{p_{CC}}{p_{DC}}), \\ P_{FC}^{2*} =& \frac{1}{S+1}(S - \frac{p_{DD}}{p_{DC}}). \end{aligned} \tag{3.42}$$

The constraints on the correlation device are

$$\begin{aligned} p_{DD} >& \frac{p_{CD}p_{DC}}{p_{CC}}, \\ p_{CC} <& Sp_{DC}, \\ p_{DD} <& Sp_{DC}, \end{aligned} \tag{3.43}$$

which results in the area shown in Figure 3.4f. From this figure it can be seen that in the $p_{CD} = p_{DC}$ plane, there is no area where the conditions for the agents overlap. However, for a general correlation device this equilibrium can be realized, which is why it is included in the analysis.

The seventh case to consider is $(P_{FC}^{1*}, 1, 1, P_{FD}^{2*})$. This corresponds to slope conditions

$$\begin{cases} C_C^1(1, P_{FD}^{2*}) = & 0 \\ C_D^1(1, P_{FD}^{2*}) > & 0 \end{cases} \text{ and } \begin{cases} C_C^2(P_{FC}^{1*}, 1) > & 0 \\ C_D^2(P_{FC}^{1*}, 1) = & 0 \end{cases}, \tag{3.44}$$

with equilibrium response probabilities

$$\begin{aligned} P_{FC}^{1*} =& \frac{1}{S+1}(1 + \frac{p_{DD}}{p_{CD}}), \\ P_{FD}^{2*} =& \frac{1}{S+1}(1 + \frac{p_{CC}}{p_{CD}}). \end{aligned} \tag{3.45}$$

The corresponding constraints on the correlation device are

$$\begin{aligned} p_{DD} >& \frac{p_{CD}p_{DC}}{p_{CC}}, \\ p_{CC} <& Sp_{CD}, \\ p_{DD} <& Sp_{CD}, \end{aligned} \tag{3.46}$$

which are shown in Figure 3.4g. In this case as well, there is no overlapping area in the plane of plotting.

The last case to consider is $(P_{FC}^{1*}, 0, 0, P_{FD}^{2*})$. This corresponds to slope conditions

$$\begin{cases} C_C^1(0, P_{FD}^{2*}) = & 0 \\ C_D^1(0, P_{FD}^{2*}) < & 0 \end{cases} \text{ and } \begin{cases} C_C^1(P_{FC}^{1*}, 0) < & 0 \\ C_D^1(P_{FC}^{1*}, 0) = & 0 \end{cases}, \tag{3.47}$$

with equilibrium response probabilities

$$\begin{aligned} P_{FC}^{1*} =& \frac{1}{S+1}(1 - S\frac{p_{DD}}{p_{CD}}), \\ P_{FD}^{2*} =& \frac{1}{S+1}(1 - S\frac{p_{CC}}{p_{CD}}). \end{aligned} \tag{3.48}$$

The corresponding constraints on the correlation device are

$$
\begin{aligned}
p_{DD} &> \frac{p_{CD} p_{DC}}{p_{CC}}, \\
p_{CC} &< \frac{p_{CD}}{S}, \\
p_{DD} &< \frac{p_{CD}}{S},
\end{aligned}
\tag{3.49}
$$

which are shown in Figure 3.4h.

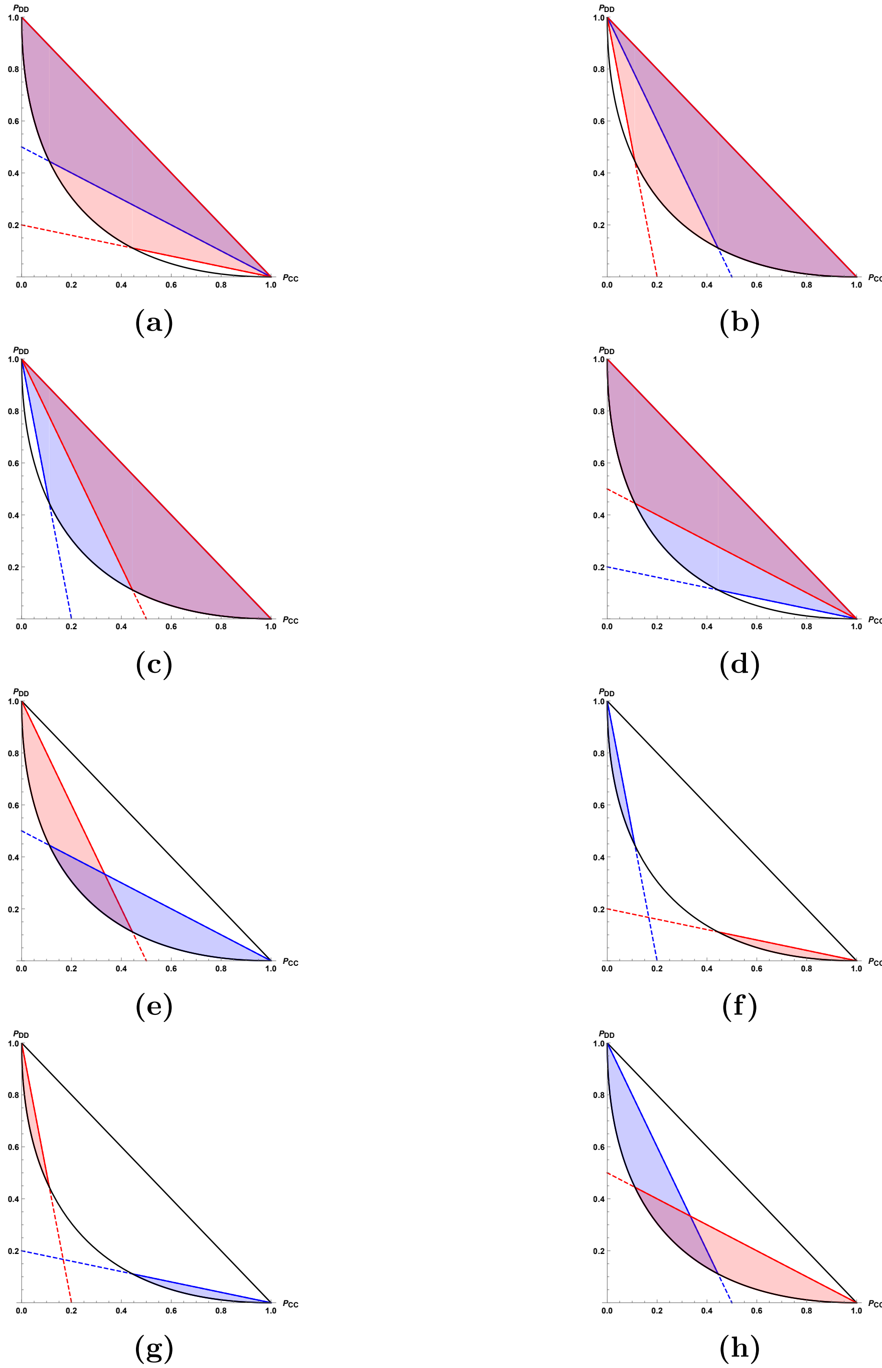


Figure 3.4: In this figure, the equilibrium conditions for eight response strategies are shown for $S = .5$. The derivation of these is shown in the preceding text. The equilibrium regions for agent 1 are shown in red, whereas for agent 2 they are shown in blue. Where they overlap, the situation is a Nash equilibrium. **a):** $(1, P^{1*}_{FD}, 1, P^{2*}_{FD})$, **b):** $(P^{1*}_{FC}, 0, P^{2*}_{FC}, 0)$, **c):** $(P^{1*}_{FC}, 1, P^{2*}_{FC}, 1)$, **d):** $(0, P^{1*}_{FD}, 0, P^{2*}_{FD})$, **e):** $(1, P^{1*}_{FD}, P^{2*}_{FC}, 1)$, **f):** $(0, P^{1*}_{FD}, P^{2*}_{FC}, 0)$, **g):** $(P^{1*}_{FC}, 1, 1, P^{2*}_{FD})$ and **h):** $(P^{1*}_{FC}, 0, 0, P^{2*}_{FD})$.

In the remaining eight cases to consider, both agents again make the other indifferent for one of the choices while optimizing the other choice. However, now the strategy of each agent in these cases is opposite, which again results in Nash equilibria for correlation devices in the lower half of the $p_{CC} - p_{DD}$ plane.

The first of these cases is $(1, P^{1*}_{FD}, 0, P^{2*}_{FD})$, for which the equilibrium slope conditions are

$$\begin{cases} C^1_C(0, P^{2*}_{FD}) > & 0 \\ C^1_D(0, P^{2*}_{FD}) = & 0 \end{cases} \quad \text{and} \quad \begin{cases} C^2_C(1, P^{1*}_{FD}) < & 0 \\ C^2_D(1, P^{1*}_{FD}) = & 0 \end{cases}, \tag{3.50}$$

with associated equilibrium response probabilities

$$\begin{aligned} P^{1*}_{FD} &= \frac{S}{S+1}(1 + \frac{p_{CD}}{p_{DD}}), \\ P^{2*}_{FD} &= \frac{1}{S+1}(1 - S\frac{p_{DC}}{p_{DD}}). \end{aligned} \tag{3.51}$$

The corresponding equilibrium conditions are

$$\begin{aligned} p_{DD} &< \frac{p_{CD}p_{DC}}{p_{CC}}, \\ p_{DD} &> Sp_{CD}, \\ p_{DD} &> Sp_{DC}, \end{aligned} \tag{3.52}$$

which are shown in Figure 3.5a.

The second of these cases is $(P^{1*}_{FC}, 0, P^{2*}_{FC}, 1)$, for which the equilibrium slope conditions are

$$\begin{cases} C^1_C(P^{2*}_{FC}, 1) = & 0 \\ C^1_D(P^{2*}_{FC}, 1) < & 0 \end{cases} \quad \text{and} \quad \begin{cases} C^2_C(P^{1*}_{FC}, 0) = & 0 \\ C^2_D(P^{1*}_{FC}, 0) > & 0 \end{cases}, \tag{3.53}$$

with the associated equilibrium response probabilities

$$\begin{aligned} P^{1*}_{FC} &= \frac{1}{S+1}(1 - S\frac{p_{DC}}{p_{CC}}), \\ P^{2*}_{FC} &= \frac{S}{S+1}(1 + \frac{p_{CD}}{p_{CC}}). \end{aligned} \tag{3.54}$$

The corresponding equilibrium conditions are

$$\begin{aligned} p_{CC} &> Sp_{CD}, \\ p_{CC} &> Sp_{DC}, \end{aligned} \tag{3.55}$$

which are shown in Figure 3.5b.

The third of these cases is $(0, P^{1*}_{FD}, 1, P^{2*}_{FD})$, for which the equilibrium slope conditions are

$$\begin{cases} C^1_C(1, P^{2*}_{FD}) < & 0 \\ C^1_D(1, P^{2*}_{FD}) = & 0 \end{cases} \quad \text{and} \quad \begin{cases} C^2_C(0, P^{1*}_{FD}) > & 0 \\ C^2_D(0, P^{1*}_{FD}) = & 0 \end{cases}, \tag{3.56}$$

with associated equilibrium response probabilities

$$\begin{aligned}P_{FD}^{1*} =& \frac{1}{S+1}(S - \frac{p_{CD}}{p_{DD}}),\\ P_{FD}^{2*} =& \frac{1}{S+1}(1 + \frac{p_{DC}}{p_{DD}}).\end{aligned} \tag{3.57}$$

The corresponding equilibrium conditions are

$$\begin{aligned}p_{DD} >& \frac{p_{CD}p_{DC}}{p_{CC}},\\ p_{DD} >& \frac{p_{CD}}{S},\\ p_{DD} >& \frac{p_{DC}}{S},\end{aligned} \tag{3.58}$$

which are shown in Figure 3.5c.

The fourth case is $(P_{FC}^{1*}, 1, P_{FC}^{2*}, 0)$, for which the equilibrium slope conditions are

$$\begin{cases} C_C^1(P_{FC}^{2*}, 0) = & 0\\ C_D^1(P_{FC}^{2*}, 0) > & 0\end{cases} \text{ and } \begin{cases} C_C^2(P_{FC}^{1*}, 1) = & 0\\ C_D^2(P_{FC}^{1*}, 1) < & 0\end{cases}, \tag{3.59}$$

with associated equilibrium response probabilities

$$\begin{aligned}P_{FD}^{1*} =& \frac{1}{S+1}(1 + \frac{p_{DC}}{p_{CC}}),\\ P_{FD}^{2*} =& \frac{1}{S+1}(S - \frac{p_{CD}}{p_{CC}}).\end{aligned} \tag{3.60}$$

The corresponding equilibrium conditions are

$$\begin{aligned}p_{DD} >& \frac{p_{CD}p_{DC}}{p_{CC}},\\ p_{CC} >& \frac{p_{CD}}{S},\\ p_{CC} >& \frac{p_{DC}}{S},\end{aligned} \tag{3.61}$$

which are shown in Figure 3.5d.

The fifth case is $(P_{FC}^{1*}, 0, 1, P_{FD}^{2*})$, for which the equilibrium slope conditions are

$$\begin{cases} C_C^1(1, P_{FD}^{2*}) = & 0\\ C_D^1(1, P_{FD}^{2*}) < & 0\end{cases} \text{ and } \begin{cases} C_C^2(P_{FC}^{1*}, 0) > & 0\\ C_D^2(P_{FC}^{1*}, 0) = & 0\end{cases}, \tag{3.62}$$

with the associated equilibrium response probabilities

$$\begin{aligned}P_{FC}^{1*} =& \frac{1}{S+1}(1 - S\frac{p_{DD}}{p_{CD}}),\\ P_{FD}^{2*} =& \frac{1}{S+1}(1 + \frac{p_{CC}}{p_{CD}}).\end{aligned} \tag{3.63}$$

The equilibrium conditions are

$$\begin{aligned} p_{DD} <& \frac{p_{CD}p_{DC}}{p_{CC}}, \\ p_{DD} <& \frac{p_{DC}}{S}, \\ p_{CC} <& S p_{DC}, \end{aligned} \tag{3.64}$$

which are shown in Figure 3.5e.

The sixth case is $(P_{FC}^{1*}, 1, 0, P_{FD}^{2*})$, for which the equilibrium slope conditions are

$$\begin{cases} C_C^1(0, P_{FD}^{2*}) = & 0 \\ C_D^1(0, P_{FD}^{2*}) > & 0 \end{cases} \quad \text{and} \quad \begin{cases} C_C^2(P_{FC}^{1*}, 1) < & 0 \\ C_D^2(P_{FC}^{1*}, 1) = & 0 \end{cases}, \tag{3.65}$$

with the associated equilibrium response probabilities

$$\begin{aligned} P_{FC}^{1*} =& \frac{1}{S+1}(1 + \frac{p_{DD}}{p_{CD}}), \\ P_{FD}^{2*} =& \frac{1}{S+1}(1 - S\frac{p_{CC}}{p_{CD}}). \end{aligned} \tag{3.66}$$

The equilibrium conditions are

$$\begin{aligned} p_{DD} <& \frac{p_{CD}p_{DC}}{p_{CC}}, \\ p_{DD} <& S p_{CD}, \\ p_{CC} <& \frac{p_{CD}}{S}, \end{aligned} \tag{3.67}$$

which are shown in Figure 3.5f.

The seventh case is $(1, P_{FD}^{1*}, P_{FC}^{2*}, 0)$, for which the equilibrium slope conditions are

$$\begin{cases} C_C^1(P_{FC}^{2*}, 0) > & 0 \\ C_D^1(P_{FC}^{2*}, 0) = & 0 \end{cases} \quad \text{and} \quad \begin{cases} C_C^2(1, P_{FD}^{1*}) = & 0 \\ C_D^2(1, P_{FD}^{1*}) < & 0 \end{cases}, \tag{3.68}$$

with the associated equilibrium response probabilities

$$\begin{aligned} P_{FD}^{1*} =& \frac{S}{S+1}(1 + \frac{p_{CC}}{p_{DC}}), \\ P_{FC}^{2*} =& \frac{1}{S+1}(S - \frac{p_{DD}}{p_{DC}}). \end{aligned} \tag{3.69}$$

The equilibrium conditions are

$$\begin{aligned} p_{DD} <& \frac{p_{CD}p_{DC}}{p_{CC}}, \\ p_{DD} <& S p_{DC}, \\ p_{CC} <& \frac{p_{DC}}{S}, \end{aligned} \tag{3.70}$$

which are shown in Figure 3.5g.

The last case is $(0, P_{FD}^{1*}, P_{FC}^{2*}, 1)$, for which the equilibrium slope conditions are

$$\begin{cases} C_C^1(P_{FC}^{2*}, 1) < & 0 \\ C_D^1(P_{FC}^{2*}, 1) = & 0 \end{cases} \quad \text{and} \quad \begin{cases} C_C^2(0, P_{FD}^{1*}) = & 0 \\ C_D^2(0, P_{FD}^{1*}) > & 0 \end{cases}, \tag{3.71}$$

with the associated equilibrium response probabilities

$$\begin{aligned} P_{FD}^{1*} =& \frac{1}{S+1}(S - \frac{p_{CC}}{p_{DC}}), \\ P_{FC}^{2*} =& \frac{S}{S+1}(1 + \frac{p_{DD}}{p_{DC}}). \end{aligned} \tag{3.72}$$

The equilibrium constraints are

$$\begin{aligned} p_{DD} <& \frac{p_{CD}p_{DC}}{p_{CC}}, \\ p_{CC} <& Sp_{DC}, \\ p_{DD} <& \frac{p_{DC}}{S}, \end{aligned} \tag{3.73}$$

which are shown in Figure 3.5h.

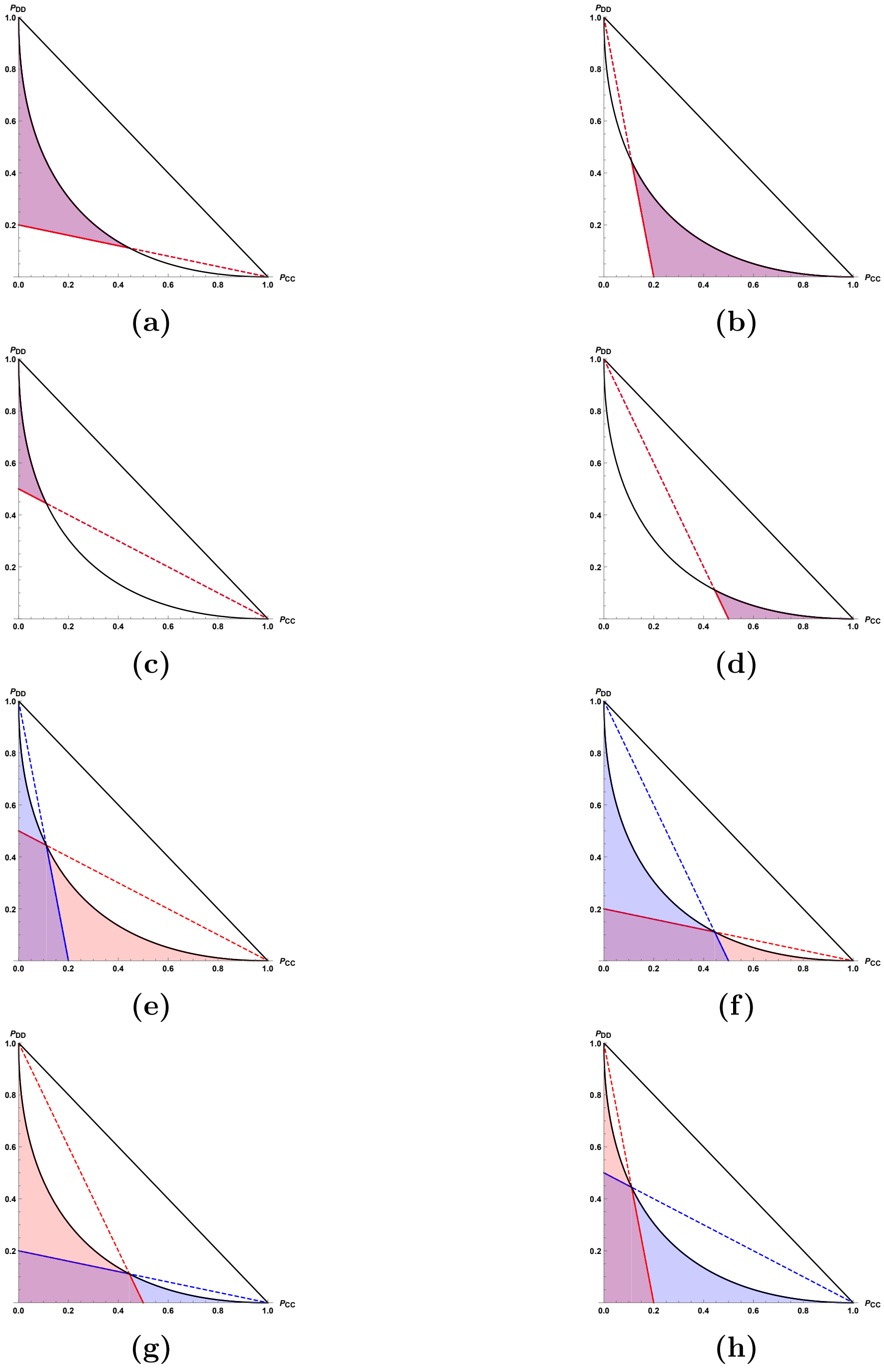


Figure 3.5: In this figure, the equilibrium conditions for eight response strategies are shown for $S = .5$. The derivation of these is shown in the preceding text. The equilibrium regions for agent 1 are shown in red, whereas for agent 2 they are shown in blue. Where they overlap, the situation is a Nash equilibrium. **a):** $(1, P_{FD}^{1*}, 0, P_{FD}^{2*})$, **b):** $(P_{FC}^{1*}, 0, P_{FC}^{2*}, 1)$, **c):** $(0, P_{FD}^{1*}, 1, P_{FD}^{2*})$, **d):** $(P_{FC}^{1*}, 1, P_{FC}^{2*}, 0)$, **e):** $(P_{FC}^{1*}, 0, 1, P_{FD}^{2*})$, **f):** $(P_{FC}^{1*}, 1, 0, P_{FD}^{2*})$, **g):** $(1, P_{FD}^{1*}, P_{FC}^{2*}, 0)$ and **h):** $(0, P_{FD}^{1*}, P_{FC}^{2*}, 1)$.

Discussion of results

Now that all equilibria are worked out, some further discussion of these results is in place. As is clear from the analysis in the previous section, each set of slope conditions has a certain area in the phase space of the correlation device where the equilibrium conditions are satisfied. Since these area's sometimes overlap, it is useful to have an oversight of which Nash equilibria are realized where. In order to do so, the $p_{CC} - p_{DD}$ plane is divided into ten regions, in each of which a different set of equilibria is realized. This is shown in Figure 3.6. In Table 3.2 below, for each of these areas it is shown which Nash equilibria are realized.

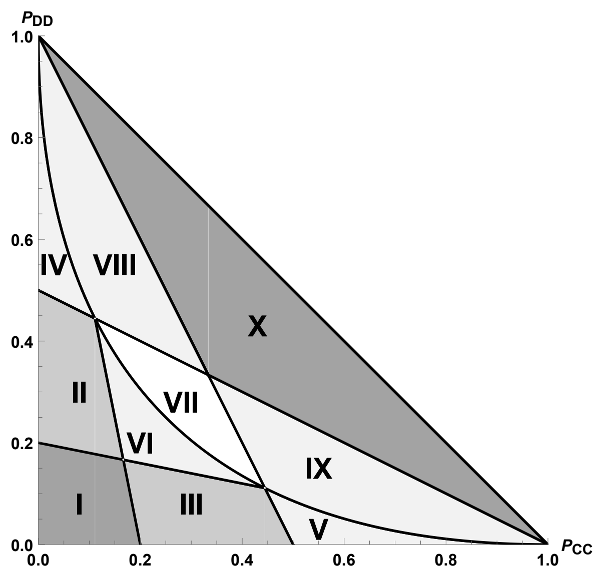


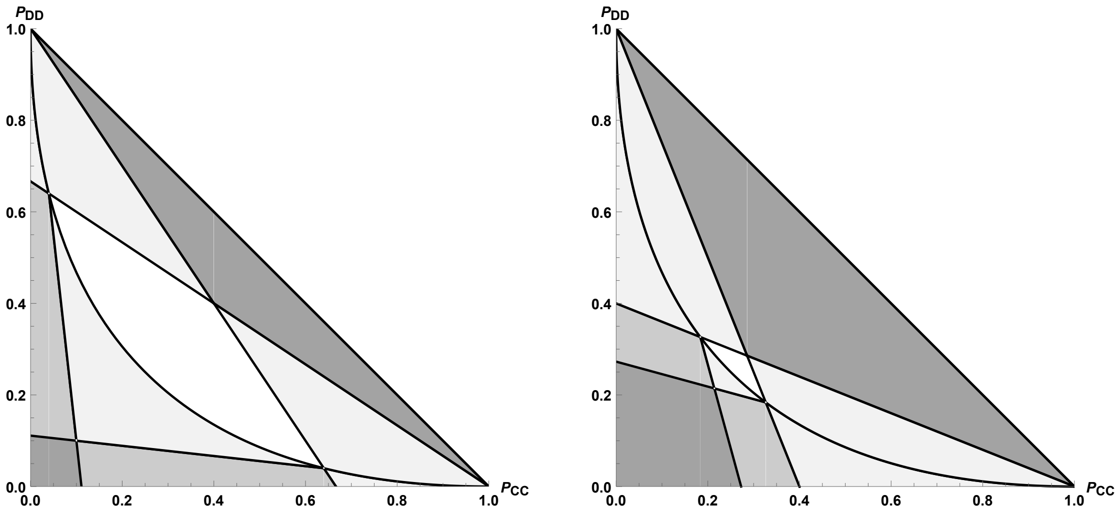
Figure 3.6: In this figure, each region corresponds to an area where a specific combination of response strategies is optimal, for $S = .5$. In Table 3.2 it is for each region specified which response strategies are Nash equilibria.

Table 3.2: In this table, an overview is shown of which equilibria are realized in which regions of Figure 3.6.

N.E.	I	II	III	IV	V	VI	VII	VIII	IX	X
(1,0,1,0)	x	x	x	x	x	x	x	x	x	x
(0,1,0,1)	x	x	x	x	x	x	x	x	x	x
$(P_{FC}^{1*},P_{FD}^{1*},P_{FC}^{2*},P_{FD}^{2*})$	x	x	x	x	x	x	x	x	x	x
(1,1,1,1)										x
(0,0,0,0)										x
(1,1,0,0)	x		x							
(0,0,1,1)	x	x								
$(1,P_{FD}^{1*},1,P_{FD}^{2*})$								x		x
$(P_{FC}^{1*},0,P_{FC}^{2*},0)$									x	x
$(P_{FC}^{1*},1,P_{FC}^{2*},1)$									x	x
$(0,P_{FD}^{1*},0,P_{FD}^{2*})$								x		x
$(1,P_{FD}^{1*},P_{FC}^{2*},1)$							x			
$(0,P_{FD}^{1*},P_{FC}^{2*},0)$										
$(P_{FC}^{1*},1,1,P_{FD}^{2*})$										
$(P_{FC}^{1*},0,0,P_{FD}^{2*})$							x			
$(1,P_{FD}^{1*},0,P_{FD}^{2*})$		x		x		x				
$(P_{FC}^{1*},0,P_{FC}^{2*},1)$			x		x	x				
$(0,P_{FD}^{1*},1,P_{FD}^{2*})$				x						
$(P_{FC}^{1*},1,P_{FC}^{2*},0)$					x					
$(P_{FC}^{1*},0,1,P_{FD}^{2*})$	x	x								
$(P_{FC}^{1*},1,0,P_{FD}^{2*})$	x		x							
$(1,P_{FD}^{1*},P_{FC}^{2*},0)$	x		x							
$(0,P_{FD}^{1*},P_{FC}^{2*},1)$	x	x								

Importance of parameter S

All previous figures were generated for the value $S = .5$. Changing the value of S (or alternatively increasing the value 1 for the prefered choice) changes these results somewhat. A similar Figure as 3.6 with different values for S is generated in Figure 3.7. It can be seen in Figure 3.7a that as the difference in preference between the agents becomes larger, the area's where correlated equilibria are possible become smaller. When the agents become more alike, as in Figure 3.7b, the area for which correlated equilibria are possible becomes larger. If we would increase S to a value larger than 1, the lines would cross, effectively swapping the roles of the agents. This of course makes sense, because then S is the preferred outcome.

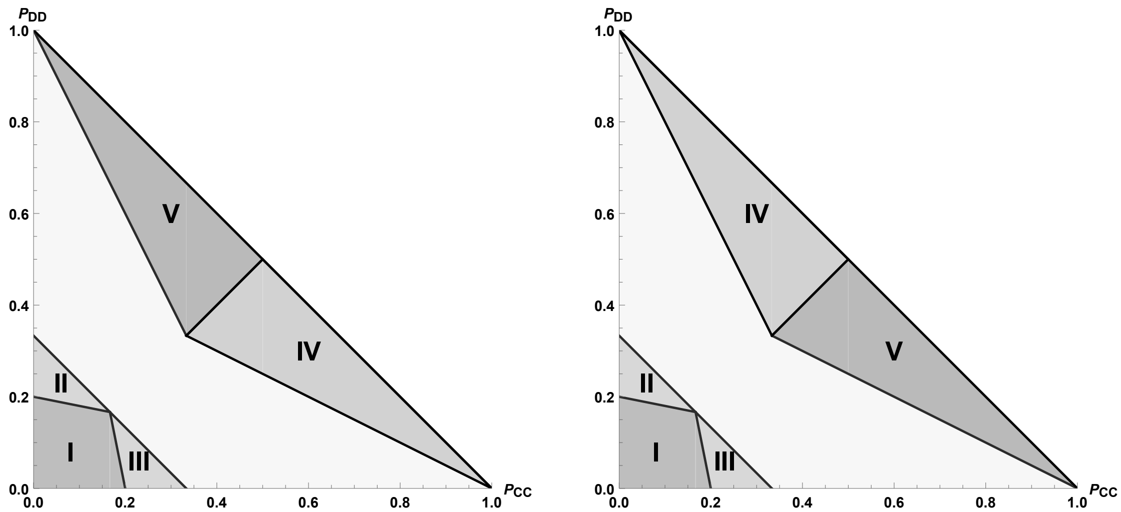


(a) **(b)**

Figure 3.7: These two sub-figures are a variation on Figure 3.6, with different values of S. In sub-figure **a):** $S = .25$. In sub-figure **b):** $S = .75$. Although the precise equilibrium areas change with different values of S, the same set of equilibrium strategies is realized in each region. However, if S would be chosen bigger than 1, the lines would cross.

Expected payoff

The payoff is what agents want to optimize, and therefore it is interesting to compare the expected payoffs for each agent in the different Nash Equilibria. Since there are always multiple Nash equilibria which can be realized for a specific correlation device, one useful question is what the optimal equilibrium is for both agents. First of all, the agents do not always agree on what is the best outcome. For example, the pure C equilibrium is best for agent 1, whereas the pure D equilibrium is best for agent 2. These are the absolute best possible expected payoffs, since this cannot be achieved for both agents at the same time, it is interesting to see what correlated strategies give expected payoffs which are higher than the worst pure equilibrium ($\langle u \rangle > S$). The mixed equilibrium definitely is not better than the worst pure equilibrium, in fact it has the lowest expected payoff of any equilibrium solution. The other equilibria have expected outcomes which are in between these extremes. Also, it is clear from the results that anywhere in the phase space the equilibria without zero slope conditions have better payoffs than the Nash equilibria with zero slope conditions. In Figure 3.8, the regions where there are correlated equilibria with higher expected payoff than the worst pure equilibria are shown. In region **I**, both $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$ are optimal. In region **II** just $(0, 0, 1, 1)$ is optimal and in region **III** just $(1, 1, 0, 0)$ is optimal. In region **IV** $(0, 0, 0, 0)$ is optimal and in region **V** $(1, 1, 1, 1)$ is optimal, but notice that these regions are interchanged when comparing the sub-figures for the two agents. This reflects that in these cases the agents do not agree on which of the two strategies results in the better expected outcome. In the area in the middle, without a number, the worst pure Nash equilibrium is better than all other equilibria.



(a) **(b)**

Figure 3.8: In this figure, each region corresponds to an area where a specific response strategy results in higher payoff than the worst pure equilibrium. **a)** shows the results for agent 1, and **b)** shows the results for agent 2. This figure was generated with the value $S = .5$.

Stability

By definition, equilibria are situations in which neither agent has incentive to deviate from the chosen strategy. However, not all Nash Equilibria are observed in actual experiments [5, 4]. There are probably multiple reasons why some are reached in practice while others are not. One important factor is the stability of the equilibrium. In this section, the stability of the different Nash equilibria under infinitesimal fluctuations is analyzed in accordance with the definition of stability stated in Chapter 2.

As an example, lets consider the pure equilibrium $(1, 0, 1, 0)$, where agent 1 has a small deviation from the equilibrium such that $P^1_{FC} \to 1 - \delta$. If this happens, then the slope coefficients for agent 2, which were given by Equation 3.2, become

$$\begin{cases} C^2_C(1-\delta, 0) = S(p_{CC} + p_{DC}) - \delta(S+1)p_{CC} \\ C^2_D(1-\delta, 0) = -S(p_{DD} + p_{DC}) + \delta(S+1)p_{CD} \end{cases} . \tag{3.74}$$

For a deviation δ which is small compared to the correlation device values, this will not cause agent 2 to want to change his strategy. Since agent 2 does not change his strategy, it would be optimal for agent 1 to go back to its initial strategy $(1, 0, 1, 0)$. In the situation where the game is played in multiple rounds, it can therefore be expected that the system returns to equilibrium after such a deviation. It can easily be shown that this same analysis is valid for any other infinitesimal deviation from this equilibrium, and therefore it can be concluded that the equilibrium is *stable*. For the other pure equilibrium the analysis is identical, and therefore it can immediately be concluded that this also is a stable equilibrium.

So how about the mixed equilibrium? If agent one deviates from the equilibrium such that $P^1_{FC} \to \frac{1}{S+1} - \delta$, this has the following effect on the slope conditions for agent 2:

$$\begin{cases} C^2_C(P^{1*}_{FC}, P^{1*}_{FD}) = -\delta(S+1)p_{CC} \\ C^2_D(P^{1*}_{FC}, P^{1*}_{FD}) = \delta(S+1)p_{CD} \end{cases} . \tag{3.75}$$

Since the slope coefficients are now non-zero, agent 1 is not indifferent between the outcomes anymore. In fact, if he or she would know the strategy of agent 2, it would be optimal to never follow C and always follow D. This takes the system out of the equilibrium, and therefore it can be concluded that the mixed equilibrium is *unstable*. In fact, if due to the deviation from the mixed equilibrium agent 1 does play $(0, 1)$, it would be optimal for agent 2 to adopt the same strategy, such that the new situation is the pure equilibrium $(0, 1, 0, 1)$. This equilibrium is stable, as was shown above.

Now lets see what happens if the initial equilibrium was $(1, 1, 1, 1)$, which implies a correlation device where the equilibrium conditions are satisfied. If $P^1_{FC} \to 1 - \delta$, the slope coefficients for agent 2 become

$$\begin{cases} C^2_C(1-\delta, 1) = Sp_{CC} - p_{DC} - \delta(S+1)p_{CC} \\ C^2_D(1-\delta, 1) = p_{DD} - Sp_{DC} + \delta(S+1)p_{CD} \end{cases} . \tag{3.76}$$

As long as δ is infinitesimal, and the correlation device is such that the equilibrium conditions are satisfied by a finite margin (meaning that $p_{CC} > \frac{p_{DC}}{S-\delta(S+1)}$ and $p_{DD} > \frac{Sp_{DC}}{1+\delta(S+1)}$), agent 2 has no incentive to deviate from the strategy. Just like in the pure equilibrium case discussed above, it is optimal for both agents to return to the original equilibrium. The same arguments apply for any other infinitesimal deviation from the equilibrium: it is *stable*. The same holds for the equilibria $(0, 0, 0, 0)$, $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$. The calculations are not shown in detail here, since this would be repetitive and the results are easy to verify.

The last example which is worked out is $(1, P^{1*}_{FD}, 1, P^{2*}_{FD})$. Consider the case where $P^1_{FC} \to 1 - \delta$. The slope coefficients for agent 2 now become

$$\begin{cases} C^2_C(P^1_{FC}, P^1_{FD}) = & p_{CC} - \delta(S+1)p_{CC} - \frac{p_{CD}p_{DC}}{p_{DD}} \\ C^2_D(P^1_{FC}, P^1_{FD}) = & \delta(1+S)p_{CD} \end{cases} . \tag{3.77}$$

As long as the deviation is small, C^2_C does not make agent 2 change his strategy, since it is still positive. However, C^2_D now has become positive. This does make agent 2 want to change his or her strategy: always following D now is optimal. This is an indication that the equilibrium is *unstable*. If agent 1 can react to the strategy of agent 2, the best response would be to always follow D as well, which causes the system to end up in the equilibrium 'always follow', which as shown above is stable. Without explicitly showing the calculations, it was found with similar arguments that all remaining equilibria are unstable as well.

It can be concluded that, as soon as there are slope coefficients which are zero in the equilibrium strategy, the equilibrium is unstable. This is a property which can immediately be generalized to any more complex network structure. One intuitive explanation for this is that there is less 'room for error' in these cases, since you have to *exactly* make the other agent indifferent and the other agent as well *exactly* has to make you indifferent. Even a small deviation is enough to make the agents prefer a particular outcome. However, in the cases where you maximize your own output, you only can make the agents want to go away from the equilibrium by a deviation which is sufficiently large. It can in any real situations definitely be expected that such fluctuations occur. First of all, human behaviour is never perfect, so any equilibrium that is observed should be stable enough to survive small deviations from the optimum. Even if the human behaviour would be perfect, it is never possible to exactly play the mixed strategy. This is because you can only play one of the outcomes and there are statistical fluctuations when you choose the moves with a certain probability. For example, if you want to play C 50% of the times, you might still play D 100 times in a row. This probably affects the perception of your strategy by the other agents, which means that fluctuations are simply unavoidable.

Results as a mapping

The various equilibria are calculated in this chapter by first considering a given response strategy, and then calculating the conditions under which this is an equilibrium. Another interesting way to get to the equilibria could be to consider a given correlation device, and consider every possible mapping from the original probabilities to the renormalized probabilities: $p_{\mu\nu} \rightarrow p^R_{\mu\nu}$. Such a mapping is a Nash equilibrium if renormalizing these values again would be an equilibrium for the strategy 'always follow'[5]. An intuition behind this is that an equilibrium is a situation where no-one wants to deviate unilaterally from his or her strategy; therefore, if given the chance to renormalize again, 'always follow' should be an equilibrium. So if you know for which values of the phase space always following the device is an equilibrium, you know which mappings are Nash equilibria. In this section, the different equilibria are shown as mappings from $p_{\mu\nu}$ to $p^R_{\mu\nu}$. This is done by considering a range of possible initial correlations on a 3D grid within the area where the response strategy is an equilibrium. The reason to show this in 3D is first of all that three dimensions are needed to plot the entire phase space. A second reason is that, even when you choose the correlation device to be symmetric, in the renormalized phase space the symmetry of the correlation device is broken, which brings the points out of the $p_{CC} - p_{DD}$ plane. Therefore, the whole phase space is actually needed to contain all information. In the plots, the three axes are p_{CC}, p_{DD} and p_{CD} (which together fix the value of p_{DC}). Lines are drawn from

[5] In Nash his own words: 'A self-countering n-tuple is called an equilibrium point.' [12]

the points in correlation device phase space to the renormalized phase space. The renormalized points in phase space are indicated by a point. The volume in the $3D$ plot where 'always follow' is a Nash equilibrium is indicated in yellow. Hence, it should be the case that all equilibrium mappings map to this region. In Figure 3.9, the pure D equilibrium and the mixed equilibrium are shown. As was already obvious from the previous sections, in which the analysis was shown, these Nash equilibria exist for any correlation device, and the renormalized values are independent of the correlation device. Both of these points lay on the edge of the volume where 'always follow' is an equilibrium solution.

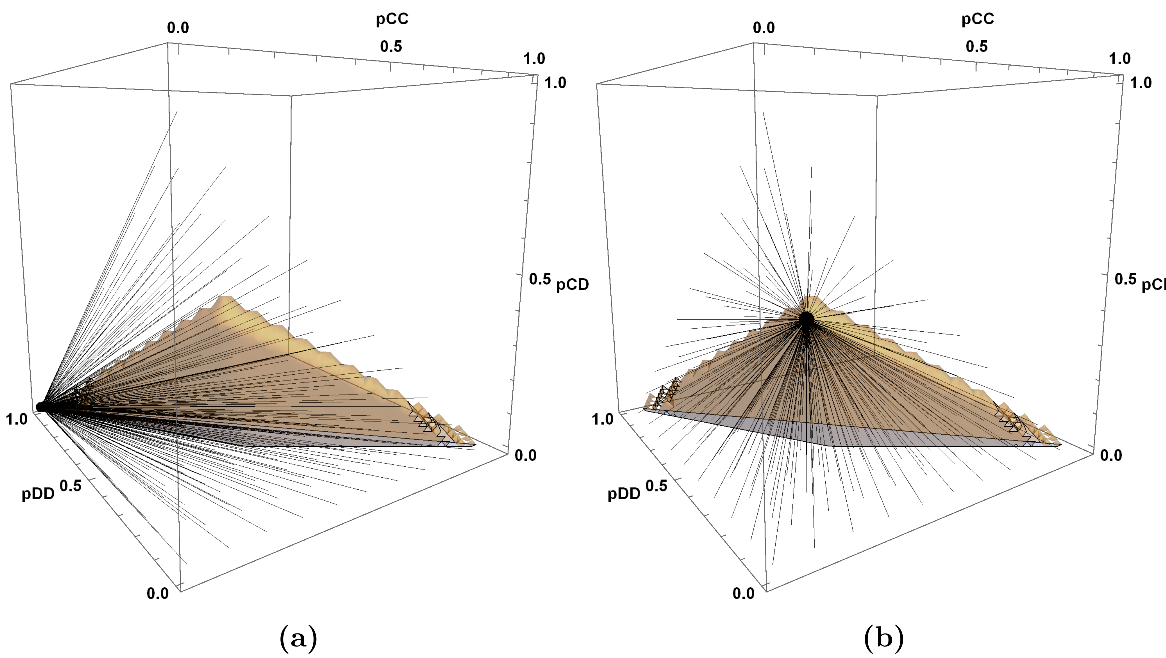
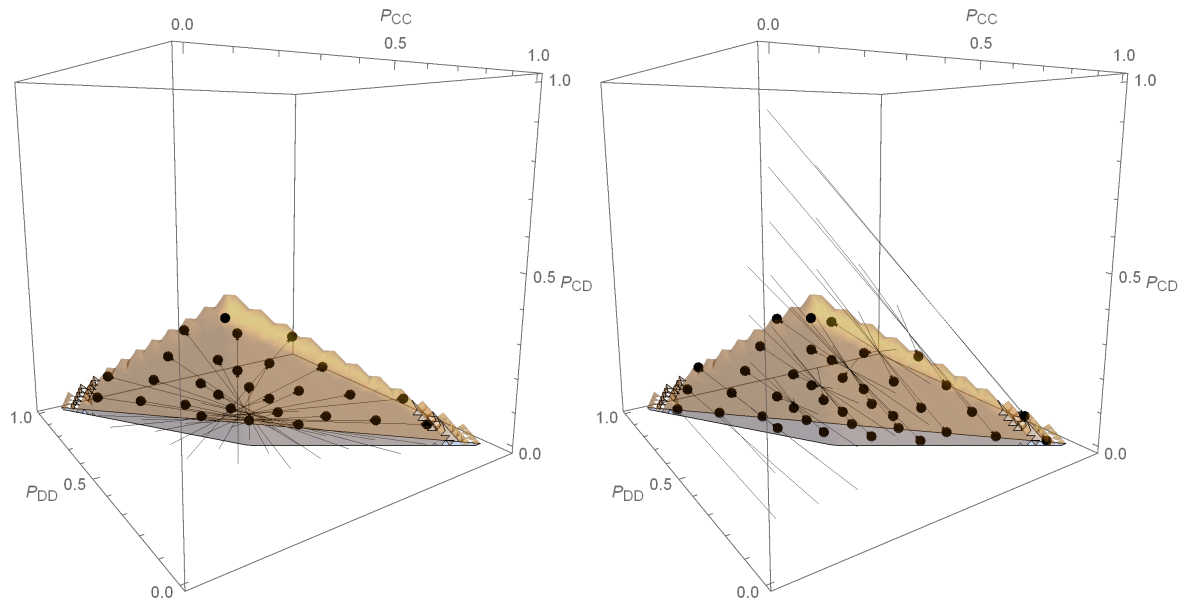


Figure 3.9: In these sub-figures, the equilibrium strategies are shown as a mapping from correlation device space to renormalized space. The lines originate in the point in correlation device space, and end as a point in the renormalized space. The yellow volume indicates the region where strategy $(1, 1, 1, 1)$ is an equilibrium: all equilibrium strategies should map to this region. In **a)** $(0, 1, 0, 1)$ is shown and in **b)** the mixed equilibrium is shown, for $S = .5$.

There are four response probabilities that map into the interior of the volume where 'always follow' is an equilibrium. These are $(1, 1, 1, 1)$, $(0, 0, 0, 0)$, $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$. Two of these are shown in Figure 3.10. These 3D plots show what is going on a lot better than Figures 3.2 and 3.3, since the information on the $p_{CD} = p_{DC}$ plane only tells part of the story. Most importantly, it becomes clear that there are exactly four ways to construct an equilibrium mapping to each part of the volume.



(a) **(b)**

Figure 3.10: In these sub-figures, two examples of the 'class 3' equilibrium strategies are shown as a mapping from correlation device space to renormalized space. The lines originate in the point in correlation device space, and end as a point in the renormalized space. The yellow volume indicates the region where strategy $(1,1,1,1)$ is an equilibrium: all equilibrium strategies should map to this region. In **a)** $(0,0,0,0)$ is shown and in **b)** $(1,1,0,0)$ is shown, for $S = .5$. It can be seen that the lines map to the entire interior of the equilibrium volume.

The remaining sixteen equilibrium response probabilities map to the boundaries of the volume, just like the mixed equilibrium. To be precise, they map to four different lines on the boundary, as is shown in Figure 3.11. These lines cross at the mixed equilibrium. The twelve cases which are not shown in the figure map to exactly the same lines, but from a different original region. These figures also once more highlight the difference in stability between certain Nash equilibria: the ones which were considered stable are the ones which map to the interior of the volume of the equilibrium region, whereas the others map to the boundaries. From the boundary, an infinitesimal deviation can be enough to destroy the equilibrium. On the other hand, it takes a larger deviation (and therefore more effort) to bring the system out of equilibrium if it is within the volume.

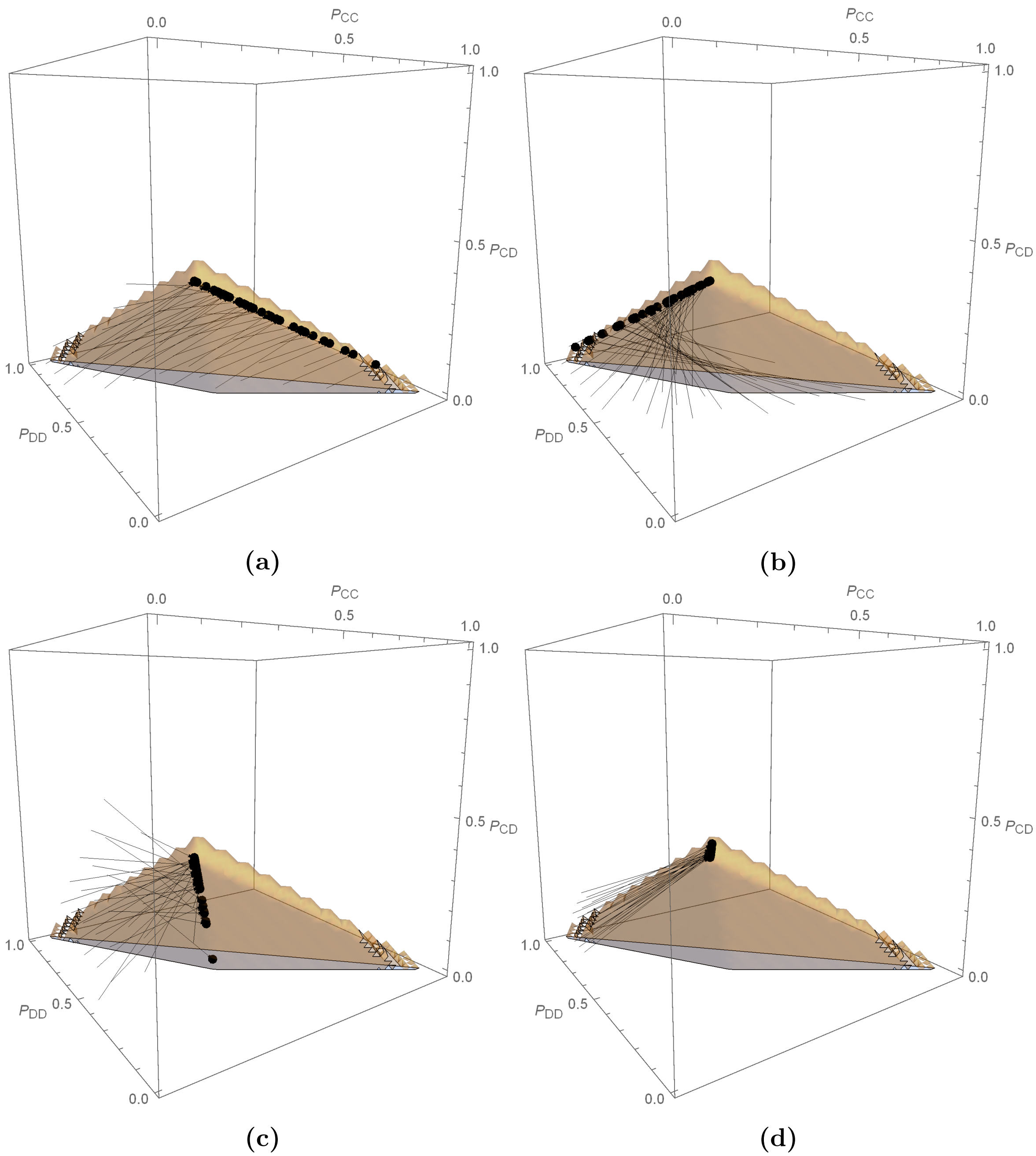
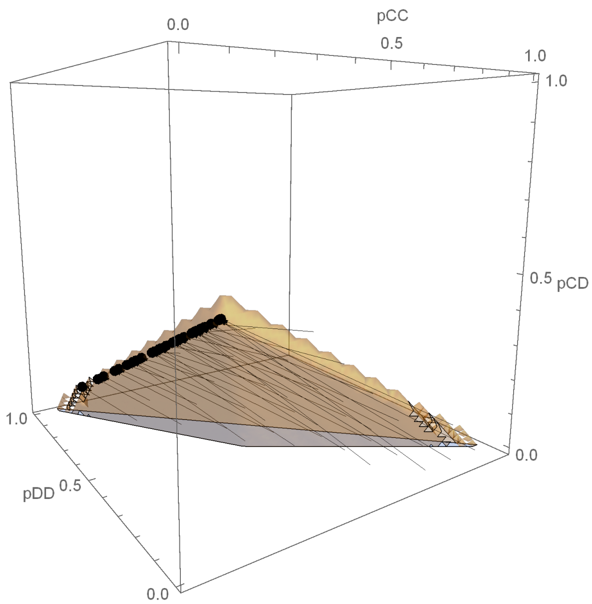
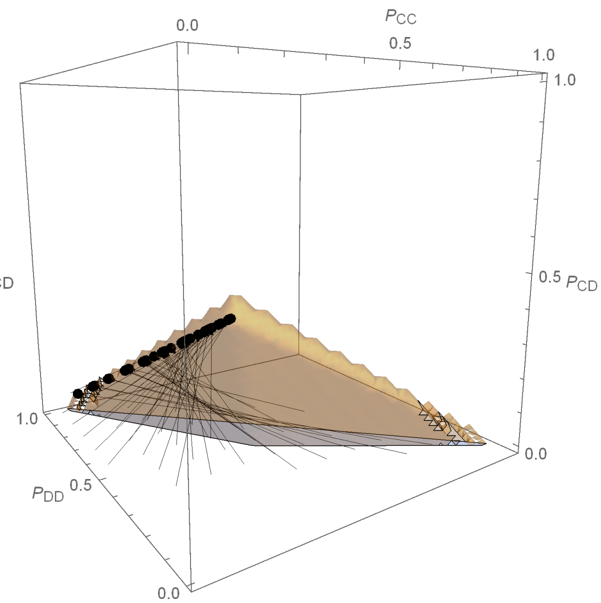


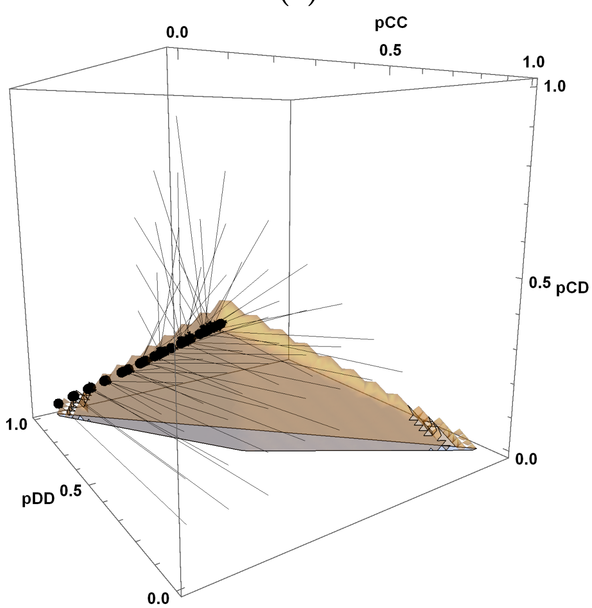
Figure 3.11: In these sub-figures, four examples of the 'class 4' equilibrium strategies are shown as a mapping from correlation device space to renormalized space. The lines originate in the point in correlation device space, and end as a point in the renormalized space. The yellow volume indicates the region where strategy $(1, 1, 1, 1)$ is an equilibrium: all equilibrium strategies should map to this region. **a):** $(1, P^{1*}_{FD}, 1, P^{2}_{FD})$, **b)**: $(0, P^{1*}_{FD}, 0, P^{2*}_{FD})$, **c)**: $(1, P^{1*}_{FD}, 0, P^{2*}_{FD})$ and **d)**: $(0, P^{1*}_{FD}, 1, P^{2*}_{FD})$. It can be seen that these mappings are to four lines on the boundary of the equilibrium volume.



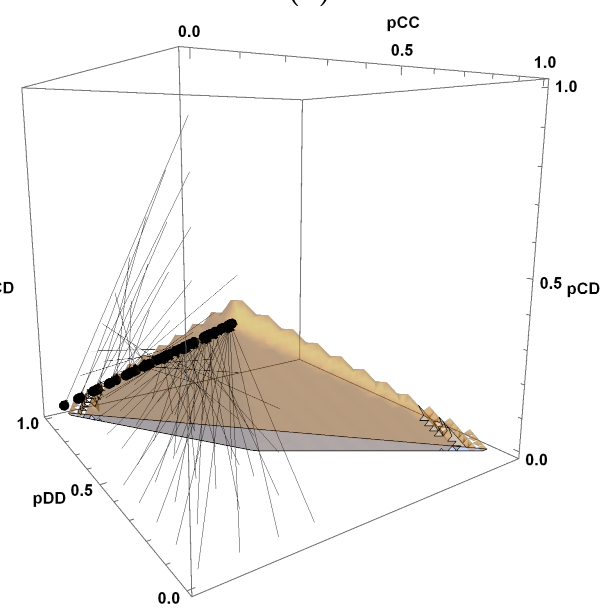
(a)



(b)



(c)



(d)

Figure 3.12: In these sub-figures, four examples of the 'class 4' equilibrium strategies are shown as a mapping from correlation device space to renormalized space. The lines originate in the point in correlation device space, and end as a point in the renormalized space. The yellow volume indicates the region where strategy $(1, 1, 1, 1)$ is an equilibrium: all equilibrium strategies should map to this region. **a)**: $(P^1_{FC}, 1, P^2_{FC}, 1)$, **b)**: $(0, P^1_{FD}, 0, P^2_{FD})$, **c)**: $(P^1_{FC}, 1, 0, P^2_{FD})$ and **d)**: $(0, P^1_{FD}, P^2_{FC}, 1)$. It can be seen that these four equilibrium strategies map to the same line on the boundary volume. This illustrates the fact that there are always four ways to realize the same correlated equilibria.

Symmetries of the mapping

By constructing the figures in which the process of optimization is shown as a mapping from correlation device space to renormalized space, it became clear that each equilibrium situation can be reached by four distinct combinations between equilibrium mappings and correlation devices. In the following short discussion, this observation is translated into a more concrete statement. Some investigation revealed that the following four general combinations between correlation device and response probabilities map to the same point in phase space:

1. $p_{\mu\nu} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and strategy (A, B, C, D).

2. $p_{\mu\nu} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$ and strategy $(A, B, 1-D, 1-C)$.

3. $p_{\mu\nu} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ and strategy $(1-B, 1-A, C, D)$.

4. $p_{\mu\nu} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$ and strategy $(1-B, 1-A, 1-D, 1-C)$.

Compared to the first of these enumerated options, the second and the third have a minus sign in the correlation device determinant. With this knowledge, it is possible to map out the entire phase space by only considering correlation devices such that $p_{CC}p_{DD} \geq p_{CD}p_{DC}$. The mappings to case 2 and 3 can be constructed from these results. Similarly, by only considering devices such that $p_{CC} + p_{DC} \geq p_{DD} + p_{CD}$, the mapping to case 4 can be constructed. For any equilibrium solution in the part of the phase space defined by these two constraints, the corresponding solutions for the different parts of the phase space can be found. For example, assume that you start from equilibrium $(P^{1*}_{FC}, 1, P^{2*}_{FC}, 1)$. You now know that $(0, 1-P^{1*}_{FC}, 0, 1-P^{2*}_{FC})$ is a Nash equilibrium for correlation devices transformed as in case 4 above. From Equation 3.33 and Equation 3.36 it can be verified that indeed this transformation is as expected. Similarly, it follows that the response strategy $(P^{1*}_{FC}, 1, 0, 1-P^{2*}_{FC})$ should be an equilibrium for correlation devices which are transformed as in case 2 above. Again, this is in accordance with Equation 3.63. The last transformation, which results in the equilibrium $(0, 1-P^{1*}_{FC}, P^{2*}_{FD}, 1)$, indeed is in accordance with Equation 3.72. An interesting observation is that both pure equilibria as well as the mixed equilibrium are invariant under these transformations.

Classification of the Equilibria.

Since there are quite a few Nash equilbria, it is helpful to group the ones with the same properties together. For the classification which is used in the rest of this thesis, the following two questions were used:

1. Is the realization of the N.E. dependent on the correlation device?
2. Is the N.E. stable?

With these two questions, we end up with four different classes, which are shown in Table 3.3. This classification will be used in the rest of this thesis to conveniently distinguish between the equilibria with similar properties. The last column of this table indicates precisely which equilibria are related in the way as described in the previous section. Equilibria with the same letter can be derived from each other by symmetry.

Table 3.3: Classification of the Nash Equilibria in two-agent Battle of the Sexes.

N.E.	Always possible?	Stable?	Class	Symmetry
(1,0,1,0)	Y	Y	1	A
(0,1,0,1)	Y	Y	1	B
$(P^{1*}_{FC},P^{1*}_{FD},P^{2*}_{FC},P^{2*}_{FD})$	Y	N	2	C
(1,1,1,1)	N	Y	3	D
(0,0,0,0)	N	Y	3	D
(1,1,0,0)	N	Y	3	D
(0,0,1,1)	N	Y	3	D
$(1,P^{1*}_{FD},1,P^{2*}_{FD})$	N	N	4	E
$(P^{1*}_{FC},0,P^{2*}_{FC},0)$	N	N	4	E
$(P^{1*}_{FC},1,P^{2*}_{FC},1)$	N	N	4	F
$(0,P^{1*}_{FD},0,P^{2*}_{FD})$	N	N	4	F
$(1,P^{1*}_{FD},P^{2*}_{FC},1)$	N	N	4	G
$(0,P^{1*}_{FD},P^{2*}_{FC},0)$	N	N	4	H
$(P^{1*}_{FC},1,1,P^{2*}_{FD})$	N	N	4	H
$(P^{1*}_{FC},0,0,P^{2*}_{FD})$	N	N	4	G
$(1,P^{1*}_{FD},0,P^{2*}_{FD})$	N	N	4	G
$(P^{1*}_{FC},0,P^{2*}_{FC},1)$	N	N	4	G
$(0,P^{1*}_{FD},1,P^{2*}_{FD})$	N	N	4	H
$(P^{1*}_{FC},1,P^{2*}_{FC},0)$	N	N	4	H
$(P^{1*}_{FC},0,1,P^{2*}_{FD})$	N	N	4	E
$(P^{1*}_{FC},1,0,P^{2*}_{FD})$	N	N	4	F
$(1,P^{1*}_{FD},P^{2*}_{FC},0)$	N	N	4	E
$(0,P^{1*}_{FD},P^{2*}_{FC},1)$	N	N	4	F

3.2 Pure coordination game

The analysis of the Pure Coordination game is very similar to Battle of The Sexes, which was extensively discussed before. However, now the payoff matrix is symmetric in the agents, which already suggests the following symmetry in the Nash equilibria: if a certain response strategy is an equilibrium, then by swapping the role of the agents you get an equilibrium as well. In this section, with a similar structure as before, the results for the pure coordination game are presented. The results are formulated more compactly than in the previous section, since the derivations follow the same steps. Also, unless stated otherwise, the figures assume the constraint $p_{CD} = p_{DC}$, but the equations in the text are general. The results are worked out for the case where both agents prefer outcome C, as shown in 3.4, but the opposite case can of course directly be inferred from from these results.

Table 3.4: Payoff table - Pure coordination game

1 \ 2	C	D
C	(1,1)	(0,0)
D	(0,0)	(S,S)

For this payoff matrix, the slope coefficients for agent 1, as defined in Equation 2.8, are:

$$\begin{cases} C^1_C(P^2_{FC}, P^2_{FD}) = & p_{CC}((S+1)P^2_{FC} - S) - p_{CD}((S+1)P^2_{FD} - 1) \\ C^1_D(P^2_{FC}, P^2_{FD}) = & p_{DD}((S+1)P^2_{FD} - 1) - p_{DC}((S+1)P^2_{FC} - S) \\ C^1(P^2_{FC}, P^2_{FD}) = & Sp_{CC}(1 - P^2_{FC}) + Sp_{CD}P^2_{FD} + p_{DC}P^2_{FC} - p_{DD}(P^2_{FD} - 1) \end{cases} \tag{3.78}$$

For agent 2 the coefficient are almost equal, except that the indices for the correlation device components are swapped:

$$\begin{cases} C^2_C(P^1_{FC}, P^1_{FD}) = & p_{CC}((S+1)P^1_{FC} - S) - p_{DC}((S+1)P^1_{FD} - 1) \\ C^2_D(P^1_{FC}, P^1_{FD}) = & p_{DD}((S+1)P^1_{FD} - 1) - p_{CD}((S+1)P^1_{FC} - S) \\ C^2(P^1_{FC}, P^1_{FD}) = & Sp_{CC}(1 - P^1_{FC}) + Sp_{DC}P^1_{FD} + p_{CD}P^1_{FC} - p_{DD}(P^1_{FD} - 1) \end{cases} \tag{3.79}$$

The equilibrium solutions are worked out in the following section.

Slope analysis

Since the slope coefficients are very similar to Battle of the Sexes, it is to be expected that the same combinations of slope coefficients lead to equilibrium solutions here. This indeed turns out to be the case. However, the conditions on the correlation device are different. In this analysis, the classification of Nash equilibria as introduced in Table 3.3 can be used.

Class 1: pure equilibria

The two pure equilibria $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$ exist in this game as well, with the respective expected payoffs

$$\langle u_i \rangle = 1 \quad \text{and} \quad \langle u_i \rangle = S. \tag{3.80}$$

From these expressions it can be seen that the outcomes are symmetric for the agents now, and that one of the pure Nash equilibria is preferable for both agents.

Class 2: Mixed equilibrium

The mixed equilibrium is found when each agent makes the other agent indifferent between its choices. This translates to the slope conditions

$$\begin{cases} C^i_\mu(P^{-i*}_{F\mu}) = & 0 \\ C^i_\mu(P^{-i*}_{F\mu}) = & 0 \end{cases}, \tag{3.81}$$

which are achieved by the response strategies

$$\begin{aligned} P^i_{FC} &= \frac{1}{S+1}, \\ P^i_{FD} &= \frac{S}{S+1}. \end{aligned} \tag{3.82}$$

Irrespective of the correlation device values, this results in the following uncorrelated probabilities for each agent to play a certain move:

$$\begin{aligned} P^i_C &= \frac{1}{S+1}, \\ P^i_D &= \frac{S}{S+1}. \end{aligned} \tag{3.83}$$

The corresponding expected outcome for the agents is

$$\langle u_i \rangle = \frac{2S}{(S+1)^2}. \tag{3.84}$$

Class 3: stable correlated equilibria

The first case set of slope conditions is that both agents always follow: $(1, 1, 1, 1)$. The equilibrium conditions for this case are

$$\begin{cases} C^1_\mu(1,1) > 0 \\ C^2_\mu(1,1) > 0 \end{cases}, \tag{3.85}$$

which result in the following conditions for both agents (on left for agent 1, and on right for agent 2)

$$\begin{aligned} p_{CC} &> S p_{CD}, & p_{CC} &> S p_{DC}, \\ p_{DD} &> \frac{p_{DC}}{S}, & p_{DD} &> \frac{p_{CD}}{S}, \end{aligned} \tag{3.86}$$

which are shown in Figure 3.13a.
When the agents play strategy $(0,0,0,0)$, with the equilibrium conditions

$$\begin{cases} C^1_\mu(0,0) < 0 \\ C^2_\mu(0,0) < 0 \end{cases}, \tag{3.87}$$

the following conditions are found for both agents (on left for agent 1, and on right for agent 2):

$$\begin{aligned} p_{CC} &> \frac{p_{CD}}{S}, & p_{CC} &> \frac{p_{DC}}{S}, \\ p_{DD} &> S p_{DC}, & p_{DD} &> S p_{CD}. \end{aligned} \tag{3.88}$$

The area which these conditions define is shown in Figure 3.13b.

The third case is $(1,1,0,0)$, with slope coefficients

$$\begin{cases} C^1_\mu(0,0) > 0 \\ C^2_\mu(1,1) < 0 \end{cases}, \tag{3.89}$$

which result in the following conditions for both agents (once more on left for agent 1, and on right for agent 2):

$$\begin{aligned} p_{CC} &< \frac{p_{CD}}{S}, & p_{CC} &< S p_{DC}, \\ p_{DD} &< S p_{DC}, & p_{DD} &< \frac{p_{CD}}{S}. \end{aligned} \tag{3.90}$$

Note that now for these equilibrium conditions the area's for the agents differ in the plotting plane[6]. The area which these conditions define is shown in Figure 3.13c.

The last case is $(0,0,1,1)$, with the equilibrium conditions

$$\begin{cases} C^1_\mu(1,1) < 0, \\ C^2_\mu(0,0) > 0, \end{cases} \tag{3.91}$$

which result in the following conditions for both agents (once more on left for agent 1, and on right for agent 2):

$$\begin{aligned} p_{CC} &< S p_{CD}, & p_{CC} &< \frac{p_{DC}}{S}, \\ p_{DD} &< \frac{p_{DC}}{S}, & p_{DD} &< S p_{CD}. \end{aligned} \tag{3.92}$$

[6]Again, this is mainly due to the plane of plotting.

The area defined by these conditions is shown in Figure 3.13d. These figures are very alike the results for two-agent BoS, but it seems as if the upper and lower half of the graph are exchanged. This is the result of the fact that in Battle of the Sexes, there is an asymmetry between the agents, whereas for the Pure Coordination game there is not. For this reason, opposite strategies in combination with a symmetric correlation device cancel this asymmetry for Battle of the Sexes (leading to equal conditions for the agents), whereas equal strategies with a symmetric correlation device lead to equal conditions in the Pure Coordination game.

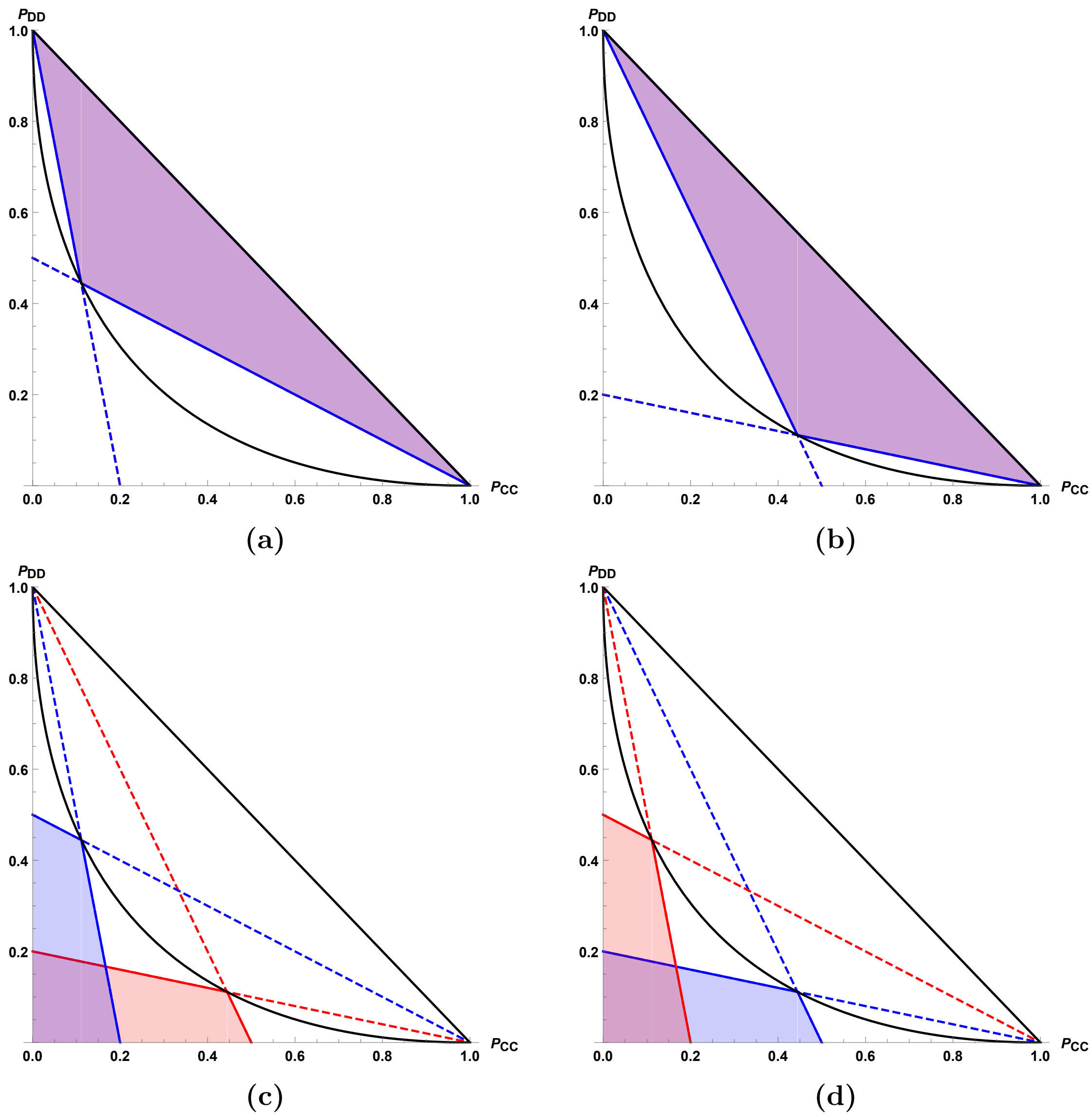


Figure 3.13: In this figure, the equilibrium conditions for the response strategies **a):** $(1,1,1,1)$, **b):** $(0,0,0,0)$, **c):** $(1,1,0,0)$ and **d):** $(0,0,1,1)$ are shown, for $S = .5$. The equilibrium conditions for agent 1 are marked in red, whereas the equilibrium conditions for agent 2 are marked in blue. In the overlapping region, the strategy is a Nash equilibrium.

Class 4: unstable correlated equilibria

For the first equilibrium which we consider, $(1, P^{1*}_{FD}, 1, P^{2*}_{FD})$, the exact expressions for the equilibrium response strategies are

$$\begin{aligned} P^{1*}_{FD} &= \frac{1}{S+1}(1+\frac{p_{CD}}{p_{DD}}), \\ P^{2*}_{FD} &= \frac{1}{S+1}(1+\frac{p_{DC}}{p_{DD}}), \end{aligned} \tag{3.93}$$

with corresponding equilibrium constraints for the device:

$$\begin{aligned} p_{CC}p_{DD} &> p_{CD}p_{DC}, \\ p_{DD} &> \frac{p_{CD}}{S}, \\ p_{DD} &> \frac{p_{DC}}{S}. \end{aligned} \tag{3.94}$$

Here, as well as in the remaining cases, the first constraint stated comes from solving the inequality slope conditions, the middle constraint ensures that the response probabilities of agent 1 are between zero and one, and the lower constraint ensures that the response probabilities of agent 1 are between zero and one. The conditions are shown in Figure 3.14a.

The second case is $(P^{1*}_{FC}, 0, P^{2*}_{FC}, 0)$, which results in

$$\begin{aligned} P^{1*}_{FD} &= \frac{1}{S+1}(S-\frac{p_{DC}}{p_{CC}}), \\ P^{2*}_{FD} &= \frac{1}{S+1}(S-\frac{p_{CD}}{p_{CC}}), \end{aligned} \tag{3.95}$$

with corresponding equilibrium constraints

$$\begin{aligned} p_{CC}p_{DD} &> p_{CD}p_{DC}, \\ p_{CC} &> \frac{p_{DC}}{S}, \\ p_{CC} &> \frac{p_{CD}}{S}. \end{aligned} \tag{3.96}$$

The conditions are shown in Figure 3.14b.

The third case is $(P^{1*}_{FC}, 1, P^{2*}_{FC}, 1)$, which results in

$$\begin{aligned} P^{1*}_{FD} &= \frac{S}{S+1}(1+\frac{p_{DC}}{p_{CC}}), \\ P^{2*}_{FD} &= \frac{S}{S+1}(1+\frac{p_{CD}}{p_{CC}}), \end{aligned} \tag{3.97}$$

with corresponding equilibrium constraints

$$\begin{aligned} p_{CC}p_{DD} &> p_{CD}p_{DC}, \\ p_{CC} &> Sp_{DC}, \\ p_{CC} &> Sp_{CD}. \end{aligned} \tag{3.98}$$

The conditions are shown in Figure 3.14c.

The fourth case is $(0, P_{FD}^{1*}, 0, P_{FD}^{2*})$, which results in

$$\begin{aligned} P_{FD}^{1*} &= \frac{1}{S+1}(1 - S\frac{p_{CD}}{p_{DD}}), \\ P_{FD}^{2*} &= \frac{1}{S+1}(1 - S\frac{p_{DC}}{p_{DD}}), \end{aligned} \tag{3.99}$$

with corresponding equilibrium constraints

$$\begin{aligned} p_{CC}p_{DD} &> p_{CD}p_{DC}, \\ p_{DD} &> Sp_{CD}, \\ p_{DD} &> Sp_{DC}. \end{aligned} \tag{3.100}$$

The conditions are shown in Figure 3.14d.

The fifth case is $(1, P_{FD}^{1*}, P_{FC}^{2*}, 1)$, which results in

$$\begin{aligned} P_{FD}^{1*} &= \frac{1}{S+1}(1 + \frac{p_{CC}}{p_{DC}}), \\ P_{FC}^{2*} &= \frac{S}{S+1}(1 + \frac{p_{DD}}{p_{DC}}), \end{aligned} \tag{3.101}$$

with corresponding equilibrium constraints

$$\begin{aligned} p_{CC}p_{DD} &> p_{CD}p_{DC}, \\ p_{CC} &< Sp_{DC}, \\ p_{DD} &< \frac{p_{DC}}{S}. \end{aligned} \tag{3.102}$$

The conditions are shown in Figure 3.14e. In this case as well as the following three cases, the equilibrium conditions for agents 1 and 2 do not overlap within the plane of plotting.

The sixth case is $(0, P_{FD}^{1*}, P_{FC}^{2*}, 0)$, which results in

$$\begin{aligned} P_{FD}^{1*} &= \frac{1}{S+1}(1 - S\frac{p_{CC}}{p_{DC}}), \\ P_{FC}^{2*} &= \frac{1}{S+1}(S - \frac{p_{DD}}{p_{DC}}), \end{aligned} \tag{3.103}$$

with corresponding equilibrium constraints

$$\begin{aligned} p_{CC}p_{DD} &> p_{CD}p_{DC}, \\ p_{CC} &< \frac{p_{DC}}{S}, \\ p_{DD} &< Sp_{DC}. \end{aligned} \tag{3.104}$$

The conditions are shown in Figure 3.14f.

The seventh case is $(P_{FC}^{1*}, 1, 1, P_{FD}^{2*})$, which results in

$$\begin{aligned} P_{FD}^{1*} &= \frac{S}{S+1}(1 + \frac{p_{DD}}{p_{CD}}), \\ P_{FC}^{2*} &= \frac{1}{S+1}(S - \frac{p_{CC}}{p_{CD}}), \end{aligned} \tag{3.105}$$

with corresponding equilibrium constraints

$$\begin{aligned} p_{CC}p_{DD} &> p_{CD}p_{DC}, \\ p_{DD} &< \frac{p_{CD}}{S}, \\ p_{CC} &< Sp_{CD}. \end{aligned} \tag{3.106}$$

The conditions are shown in Figure 3.14g.

The last case which gives an equilibrium for positive correlation devices is $(P_{FC}^{1*}, 0, 0, P_{FD}^{2*})$, which results in

$$\begin{aligned} P_{FD}^{1*} &= \frac{1}{S+1}(S - \frac{p_{DD}}{p_{CD}}), \\ P_{FC}^{2*} &= \frac{1}{S+1}(1 - \frac{p_{CC}}{p_{CD}}), \end{aligned} \tag{3.107}$$

with corresponding equilibrium constraints

$$\begin{aligned} p_{CC}p_{DD} &> p_{CD}p_{DC}, \\ p_{DD} &< Sp_{CD}, \\ p_{CC} &< \frac{p_{CD}}{S}. \end{aligned} \tag{3.108}$$

The conditions are shown in Figure 3.14h.

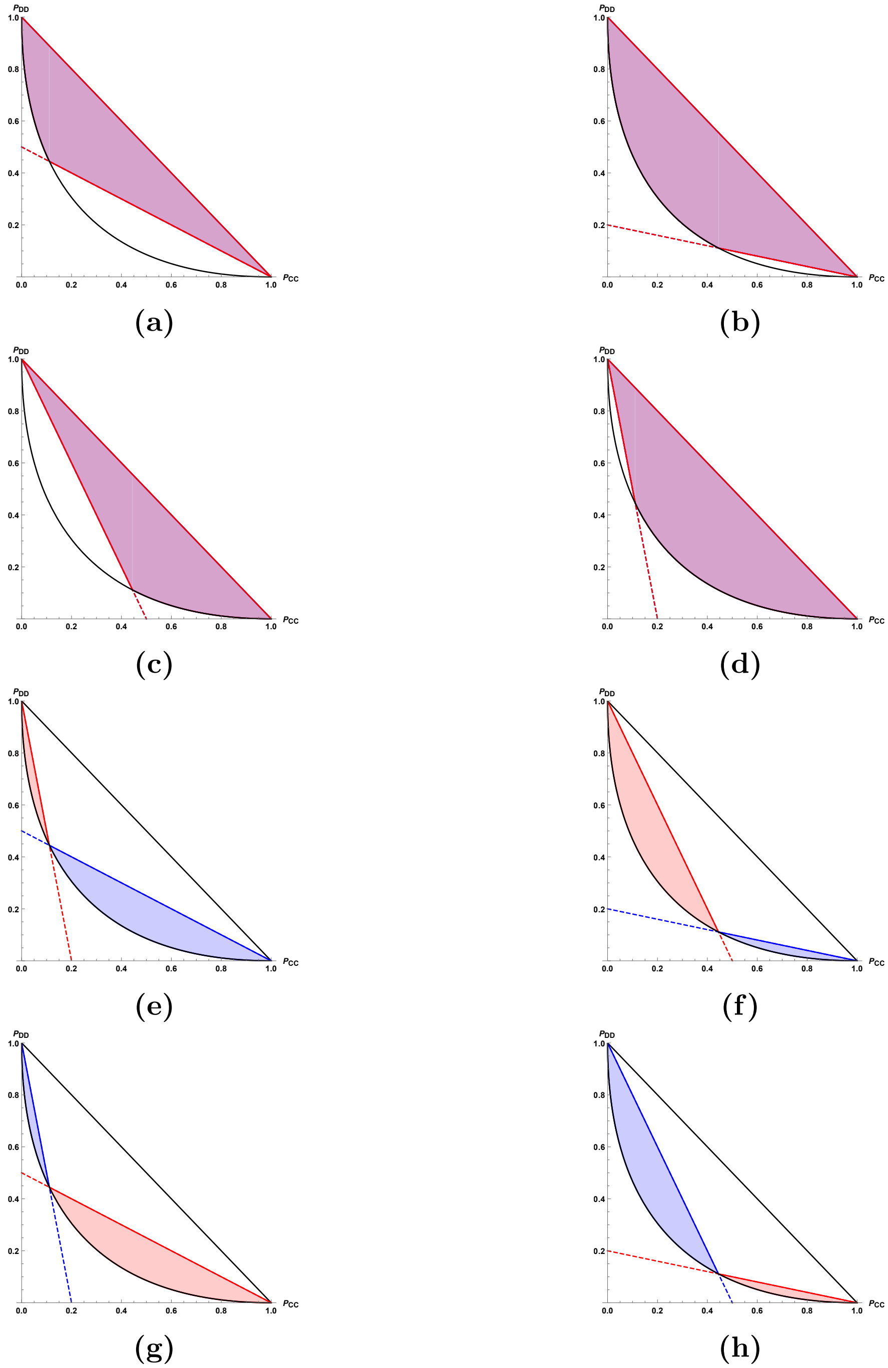


Figure 3.14: In this figure, the equilibrium conditions for eight response strategies are shown for $S = .5$. The derivation of these is shown in the preceding text. The equilibrium regions for agent 1 are shown in red, whereas for agent 2 they are shown in blue. Where they overlap, the situation is a Nash equilibrium. **a):** $(1, P^{1*}_{FD}, 1, P^{2*}_{FD})$, **b):** $(P^{1*}_{FC}, 0, P^{2*}_{FC}, 0)$, **c):** $(P^{1*}_{FC}, 1, P^{2*}_{FC}, 1)$, **d):** $(0, P^{1*}_{FD}, 0, P^{2*}_{FD})$, **e):** $(1, P^{1*}_{FD}, P^{2*}_{FC}, 1)$, **f):** $(0, P^{1*}_{FD}, P^{2*}_{FC}, 0)$, **g):** $(P^{1*}_{FC}, 1, 1, P^{2*}_{FD})$ and **h):** $(P^{1*}_{FC}, 0, 0, P^{2*}_{FD})$.

The remaining eight cases are those where the agents play opposite strategies. The first one which is worked out here is $(1, P_{FD}^{1*}, 0, P_{FD}^{2*})$, which has equilibrium response strategies

$$P_{FD}^{1*} = \frac{1}{S+1}(1 + \frac{p_{CD}}{p_{DD}}),$$
$$P_{FD}^{2*} = \frac{1}{S+1}(1 - S\frac{p_{DC}}{p_{DD}}), \tag{3.109}$$

with equilibrium constraints

$$p_{CC}p_{DD} < p_{CD}p_{DC},$$
$$p_{DD} > \frac{p_{CD}}{S}, \tag{3.110}$$
$$p_{DD} > Sp_{DC}.$$

These are shown in Figure 3.15a.

The second set of responses is $(P_{FC}^{1*}, 0, P_{FC}^{2*}, 1)$, which has equilibrium response strategies

$$P_{FD}^{1*} = \frac{1}{S+1}(S - \frac{p_{DC}}{p_{CC}}),$$
$$P_{FD}^{2*} = \frac{S}{S+1}(1 + \frac{p_{CD}}{p_{CC}}), \tag{3.111}$$

with equilibrium constraints

$$p_{CC}p_{DD} < p_{CD}p_{DC},$$
$$p_{CC} > \frac{p_{DC}}{S}, \tag{3.112}$$
$$p_{CC} > Sp_{CD}.$$

These are shown in Figure 3.15b.

The third case is $(0, P_{FD}^{1*}, 1, P_{FD}^{2*})$, with the equilibrium response strategies

$$P_{FD}^{1*} = \frac{1}{S+1}(1 - S\frac{p_{CD}}{p_{DD}}),$$
$$P_{FD}^{2*} = \frac{1}{S+1}(1 + \frac{p_{DC}}{p_{DD}}), \tag{3.113}$$

which lead to the equilibrium constraints

$$p_{CC}p_{DD} < p_{CD}p_{DC},$$
$$p_{DD} > Sp_{CD}, \tag{3.114}$$
$$p_{DD} > \frac{p_{DC}}{S}.$$

These are shown in Figure 3.15c.

The fourth set of slope conditions is $(P_{FC}^{1*}, 1, P_{FD}^{2*}, 0)$, where

$$\begin{aligned} P_{FD}^{1*} &= \frac{S}{S+1}(1+\frac{p_{DC}}{p_{CC}}), \\ P_{FD}^{2*} &= \frac{1}{S+1}(S-\frac{p_{CD}}{p_{CC}}), \end{aligned} \tag{3.115}$$

which lead to the equilibrium constraints

$$\begin{aligned} p_{CC}p_{DD} &< p_{CD}p_{DC}, \\ p_{CC} &> Sp_{DC}, \\ p_{CC} &> \frac{p_{CD}}{S}. \end{aligned} \tag{3.116}$$

These are shown in Figure 3.15d.

The fifth set of slope conditions is $(P_{FC}^{1*}, 0, 1, P_{FD}^{2*})$, where

$$\begin{aligned} P_{FC}^{1*} &= \frac{1}{S+1}(S-\frac{p_{DD}}{p_{CD}}), \\ P_{FD}^{2*} &= \frac{1}{S+1}(1+\frac{p_{CC}}{p_{CD}}), \end{aligned} \tag{3.117}$$

which lead to the equilibrium constraints

$$\begin{aligned} p_{CC}p_{DD} &< p_{CD}p_{DC}, \\ p_{DD} &< Sp_{CD}, \\ p_{CC} &< Sp_{CD}. \end{aligned} \tag{3.118}$$

These are shown in Figure 3.15e.

The sixth set of slope conditions is $(P_{FC}^{1*}, 1, 0, P_{FD}^{2*})$, where

$$\begin{aligned} P_{FC}^{1*} &= \frac{S}{S+1}(1+\frac{p_{DD}}{p_{CD}}), \\ P_{FD}^{2*} &= \frac{1}{S+1}(1-S\frac{p_{CC}}{p_{CD}}), \end{aligned} \tag{3.119}$$

which lead to the equilibrium constraints

$$\begin{aligned} p_{CC}p_{DD} &< p_{CD}p_{DC}, \\ p_{DD} &< \frac{p_{CD}}{S}, \\ p_{CC} &< \frac{p_{CD}}{S}. \end{aligned} \tag{3.120}$$

These are shown in Figure 3.15f.

The seventh set of slope conditions is $(1, P_{FD}^{1*}, P_{FC}^{2*}, 0)$, where

$$\begin{aligned} P_{FC}^{1*} &= \frac{1}{S+1}(1+\frac{p_{CC}}{p_{DC}}), \\ P_{FD}^{2*} &= \frac{1}{S+1}(S-\frac{p_{DD}}{p_{DC}}), \end{aligned} \tag{3.121}$$

which lead to the equilibrium constraints

$$\begin{aligned} p_{CC}p_{DD} &< p_{CD}p_{DC}, \\ p_{CC} &< Sp_{DC}, \\ p_{DD} &< Sp_{DC}. \end{aligned} \tag{3.122}$$

These are shown in Figure 3.15g.

The last set of slope conditions is $(0, P_{FD}^{1*}, P_{FC}^{2*}, 1)$, where

$$\begin{aligned} P_{FC}^{1*} &= \frac{1}{S+1}(1-S\frac{p_{CC}}{p_{DC}}), \\ P_{FD}^{2*} &= \frac{S}{S+1}(1+\frac{p_{DD}}{p_{DC}}), \end{aligned} \tag{3.123}$$

which lead to the equilibrium constraints

$$\begin{aligned} p_{CC}p_{DD} &< p_{CD}p_{DC}, \\ p_{CC} &< \frac{p_{DC}}{S}, \\ p_{DD} &< \frac{p_{DC}}{S}. \end{aligned} \tag{3.124}$$

These are shown in Figure 3.15h.

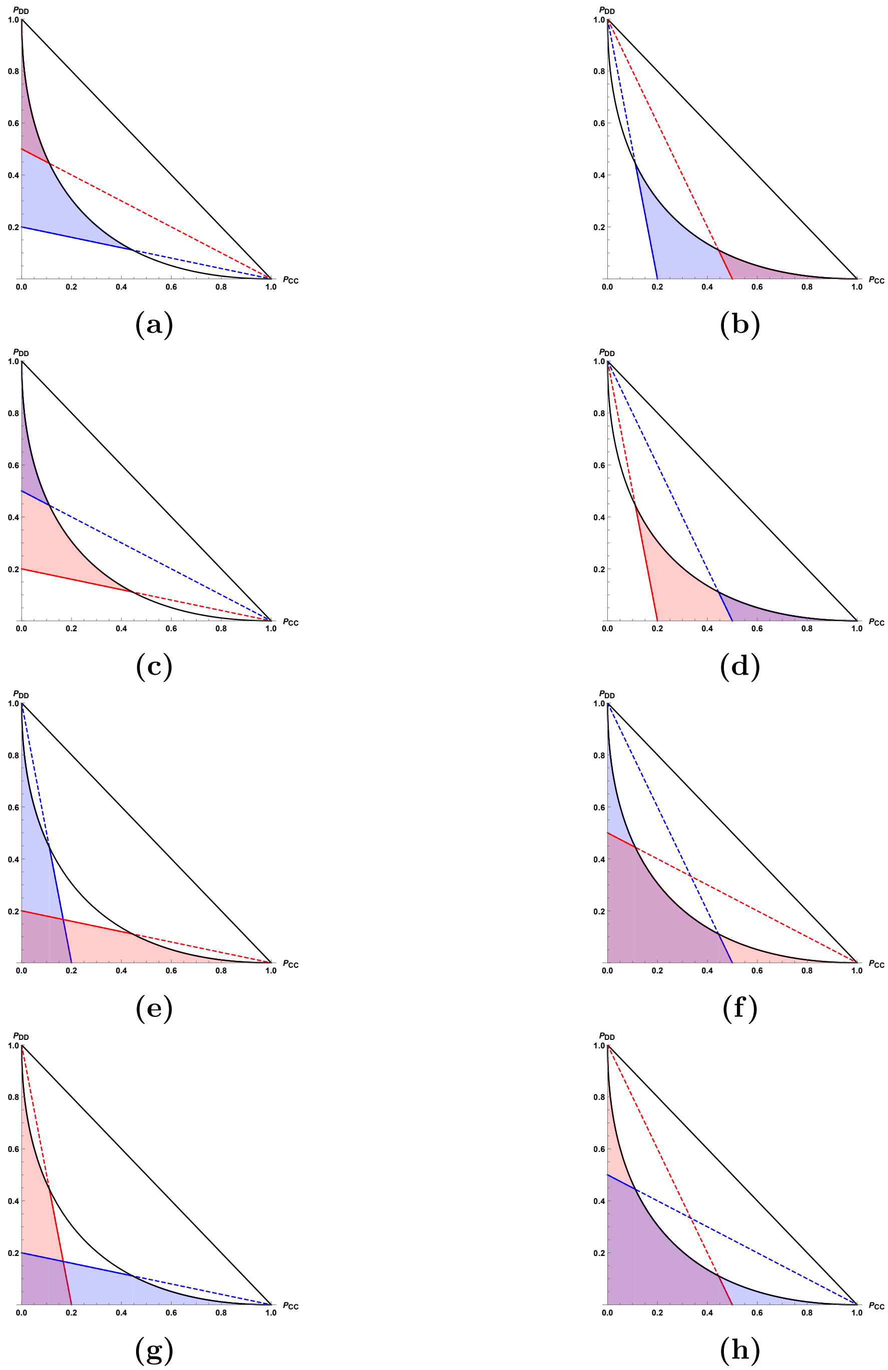


Figure 3.15: In this figure, the equilibrium conditions for eight response strategies are shown for $S = .5$. The derivation of these is shown in the preceding text. The equilibrium regions for agent 1 are shown in red, whereas for agent 2 they are shown in blue. Where they overlap, the situation is a Nash equilibrium. **a):** $(1, P^{1*}_{FD}, 0, P^{2*}_{FD})$, **b):** $(P^{1*}_{FC}, 0, P^{2*}_{FC}, 1)$, **c):** $(0, P^{1*}_{FD}, 1, P^{2*}_{FD})$, **d):** $(P^{1*}_{FC}, 1, P^{2*}_{FC}, 0)$, **e):** $(P^{1*}_{FC}, 0, 1, P^{2*}_{FD})$, **f):** $(P^{1*}_{FC}, 1, 0, P^{2*}_{FD})$, **g):** $(1, P^{1*}_{FD}, P^{2*}_{FC}, 0)$ and **h):** $(0, P^{1*}_{FD}, P^{2*}_{FC}, 1)$.

Discussion of results

In this section, the results are discussed in some detail. However, this discussion is not as extensive as for the case of the two-agent Battle of the Sexes, since there is large overlap in the conclusions. One of the differences is the area in which the different response strategies are optimal. Table 3.5 shows what Nash equilibria are realized in what region of Figure 3.16. Most other analysis, for example concerning the classification or symmetries in the system, is exactly equal to the analysis for Battle of the Sexes. Therefore, these arguments are not repeated here. However, it was found to be useful to show some of the figures of the mapping from correlation device space to renormalized space. These are included in the following discussion.

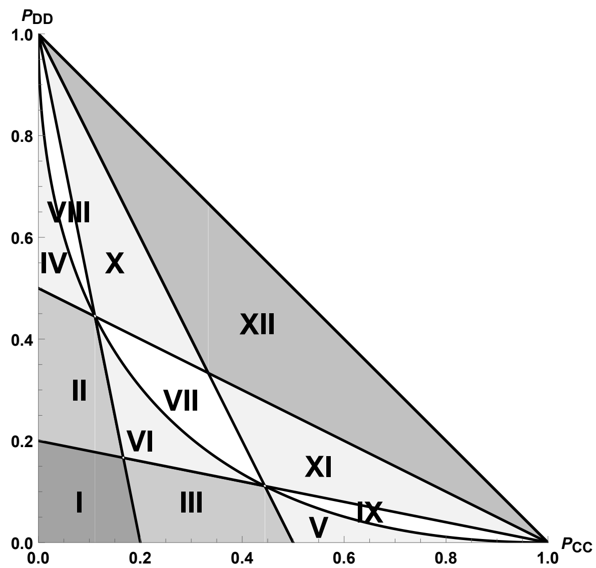


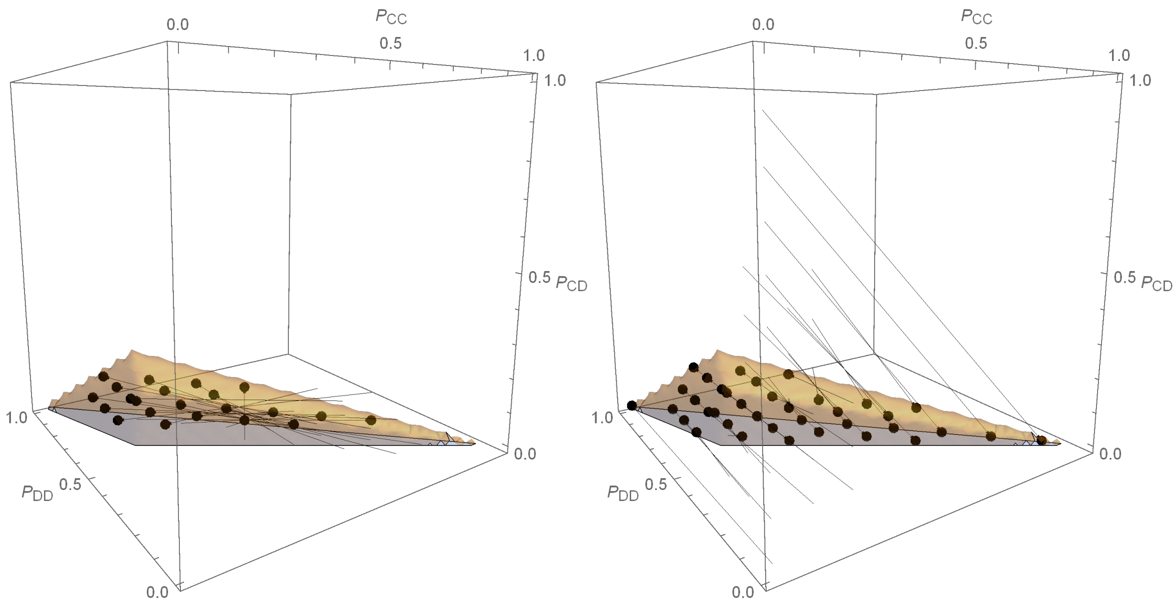
Figure 3.16: In this figure, each region corresponds to an area where a specific combination of response strategies is optimal, for $S = .5$. In Table 3.5 it is for each region specified which response strategies are Nash equilibria.

Table 3.5: In this table, an overview is shown of which Nash equilibria are realized in which regions of Figure 3.16.

N.E.	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
(1,0,1,0)	x	x	x	x	x	x	x	x	x	x	x	x
(0,1,0,1)	x	x	x	x	x	x	x	x	x	x	x	x
$(P_{FC}^{1*}, P_{FD}^{1*}, P_{FC}^{2*}, P_{FD}^{2*})$	x	x	x	x	x	x	x	x	x	x	x	x
(1,1,1,1)										x		x
(0,0,0,0)											x	x
(1,1,0,0)	x											
(0,0,1,1)	x											
$(1, P_{FD}^{1*}, 1, P_{FD}^{2*})$								x		x		x
$(P_{FC}^{1*}, 0, P_{FC}^{2*}, 0)$							x	x		x	x	x
$(P_{FC}^{1*}, 1, P_{FC}^{2*}, 1)$									x	x		x
$(0, P_{FD}^{1*}, 0, P_{FD}^{2*})$									x	x	x	x
$(1, P_{FD}^{1*}, P_{FC}^{2*}, 1)$												
$(0, P_{FD}^{1*}, P_{FC}^{2*}, 0)$												
$(P_{FC}^{1*}, 1, 1, P_{FD}^{2*})$												
$(P_{FC}^{1*}, 0, 0, P_{FD}^{2*})$												
$(1, P_{FD}^{1*}, 0, P_{FD}^{2*})$				x								
$(P_{FC}^{1*}, 0, P_{FC}^{2*}, 1)$					x							
$(0, P_{FD}^{1*}, 1, P_{FD}^{2*})$				x								
$(P_{FC}^{1*}, 1, P_{FC}^{2*}, 0)$					x							
$(P_{FC}^{1*}, 0, 1, P_{FD}^{2*})$	x											
$(P_{FC}^{1*}, 1, 0, P_{FD}^{2*})$	x	x	x			x						
$(1, P_{FD}^{1*}, P_{FC}^{2*}, 0)$	x											
$(0, P_{FD}^{1*}, P_{FC}^{2*}, 1)$	x	x	x			x						

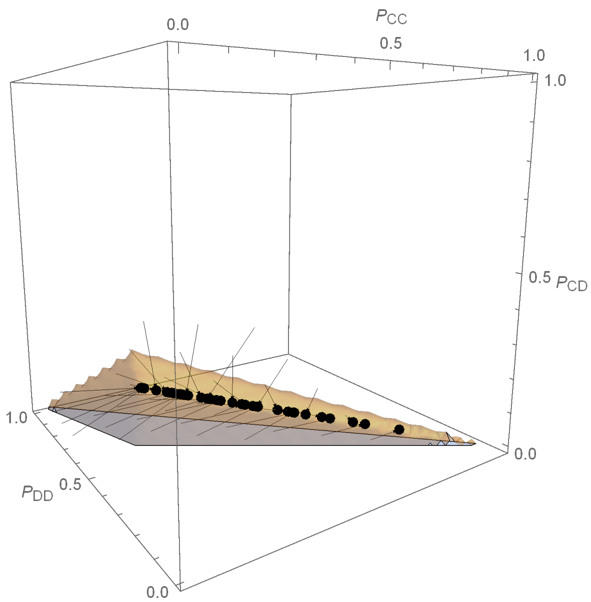
Results as mapping

The results can once again be plotted as a mapping from the 3D correlation device phase space to the renormalized phase space. The volume in this phase space for which 'always follow' is a Nash equilibrium is now slightly different, more specifically the bulk is shifted to one side, which is due to the different payoff table compared to Battle of the Sexes. More specifically, the different but opposite preferences in Battle of the Sexes result in a symmetrical equilibrium volume. In the Pure Coordination game, the agents are equal with an asymmetry in the payoff between both outcomes: this results in the asymmetrical equilibrium volume. Of the four Nash equilibria which map to the inside of the volume, two are shown in Figure 3.17. Four examples of the Nash equilibria which map to the boundary of the volume are shown in Figure 3.18.

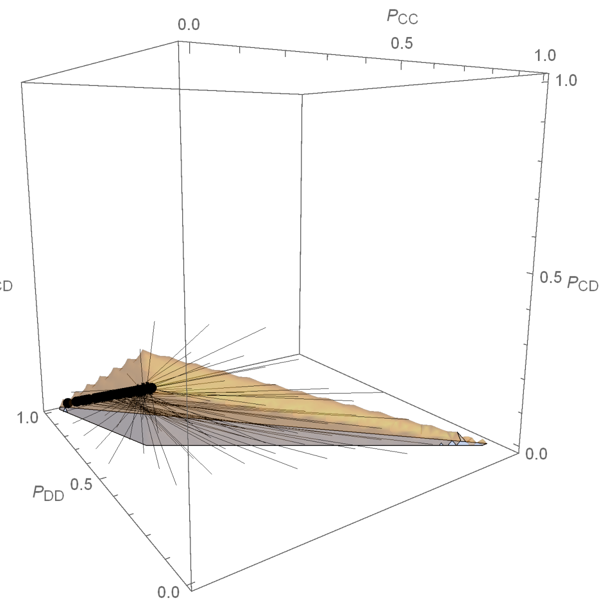


(a) **(b)**

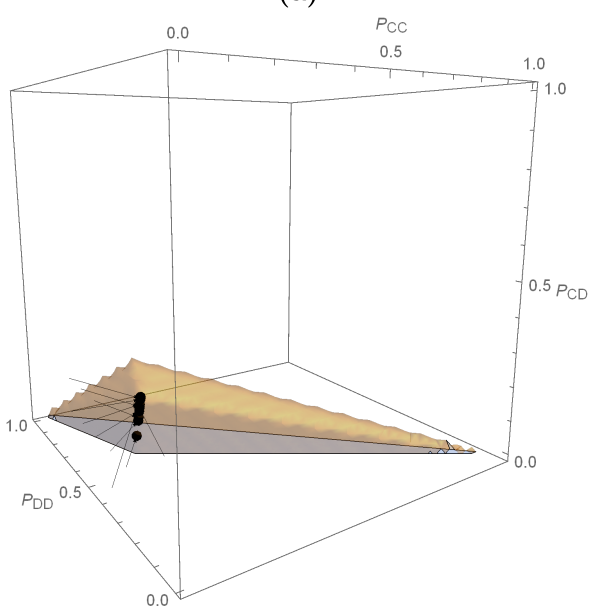
Figure 3.17: In these sub-figures, two examples of the 'class 3' equilibrium strategies are shown as a mapping from correlation device space to renormalized space. The lines originate in the point in correlation device space, and end as a point in the renormalized space. The yellow volume indicates the region where strategy $(1, 1, 1, 1)$ is an equilibrium: all equilibrium strategies should map to this region. In **a)** $(0, 0, 0, 0)$ is shown and in **b)** $(1, 1, 0, 0)$ is shown, for $S = .5$. It can be seen that the lines map to the entire interior of the equilibrium volume.



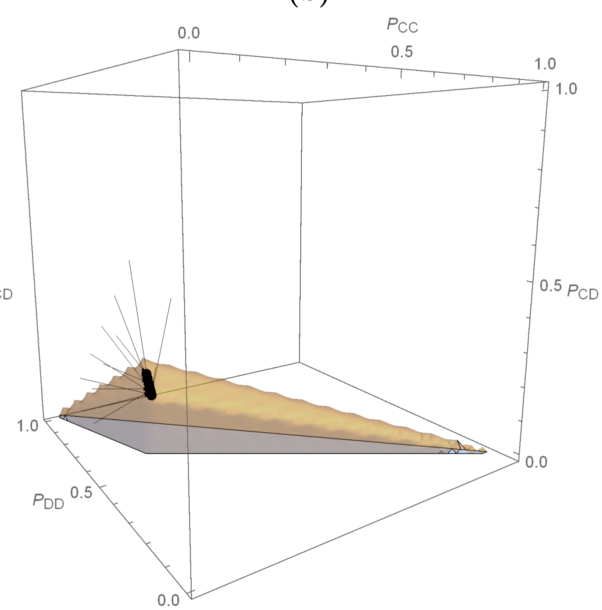
(a)



(b)



(c)



(d)

Figure 3.18: In these sub-figures, four examples of the 'class 4' equilibrium strategies are shown as a mapping from correlation device space to renormalized space. The lines originate in the point in correlation device space, and end as a point in the renormalized space. The yellow volume indicates the region where strategy $(1, 1, 1, 1)$ is an equilibrium: all equilibrium strategies should map to this region. **a):** $(1, P^{1*}_{FD}, 1, P^{2}_{FD})$, **b)**: $(0, P^{1*}_{FD}, 0, P^{2*}_{FD})$, **c)**: $(1, P^{1*}_{FD}, 0, P^{2*}_{FD})$ and **d)**: $(0, P^{1*}_{FD}, 1, P^{2*}_{FD})$. It can be seen that these mappings are to four lines on the boundary of the equilibrium volume.

Chapter 4

Three agents

With the analysis of two-agent Battle of the Sexes as well as the two-agent Pure Coordination game at hand, the three-agent networks can be worked out. The biggest difference compared to the two-agent games is that the network structure becomes an important factor. This makes the analysis more interesting; depending on the connections between the agents, certain situations may or may not be in equilibrium. On the other hand, since the phase space of the correlation device as well as the response probability phase space are larger, even the analysis for one of those network structures is a lot more work than in the two-agent case. There are now six independent response probabilities, which all could have three kinds of slope conditions. Therefore, there are now 3^6 different possible types of slope conditions. Even when dismissing all the combinations which would definitely not result in non-trivial equilibria, the resulting amount of cases that can produce equilibria is still so much that it is not feasible to work out everything. However, the analysis of the two-agent system showed that there are two possible arguments for not having to consider all cases:

1. Not all Nash equilibria are stable.
2. Multiple combinations of correlation devices and responses produce the same outcomes.

The first of these points can be used to justify focusing on the stable solutions. This eliminates all cases where there are slope coefficients which equal zero, which already helps a lot. If we focus our attention to just those which are stable, the amount of analysis is reduced to a more feasible amount. For this reason the choice was made to in this chapter present the results for the stable Nash equilibria, with just one worked out example of the many possible unstable solutions. The second point suggests that we can make a choice for what correlation device values to consider without dismissing any possible renormalized outcomes. For this reason, it was chosen that the correlation devices are such that they positively correlate all of the agents. Negatively correlated outcomes on the renormalized level can still be obtained if the agents play opposite strategies.

All possible three-agent networks are shown in Figure 4.1. In each square box representing an agent, the letter specifies the preferred outcome of the agent. Besides the preference of each agent, it now should also be specified which agents are connected. If two agents are connected, there is a familiar two-agent payoff matrix between them. If there is no connection, the agents are indifferent with respect to each other. The total payoff of an agent is the sum of the different two-agent interactions. As an example to illustrate what sort of situation might correspond to a particular network structure, consider the case of three persons going to the movies. One of them (agent 2), is friends with both others, who do not know each other. Assuming that everyone is only interested in going to the movies with their friends, there now is a link between agent 2 and both others. If, for example, the agent with two friends prefers movie D, and the other two prefer movie C, the situation is the one depicted in Figure 4.1g.

In this chapter, the networks shown in Figures 4.1f, 4.1h and 4.1j are worked out in detail. The results for for the networks in which each agent has the opposite preference can immediately be inferred from these results: the roles of C and D are interchanged. The networks shown in Figures 4.1a, 4.1b, 4.1c and 4.1d are not worked out since they are homogeneous and therefore less interesting.

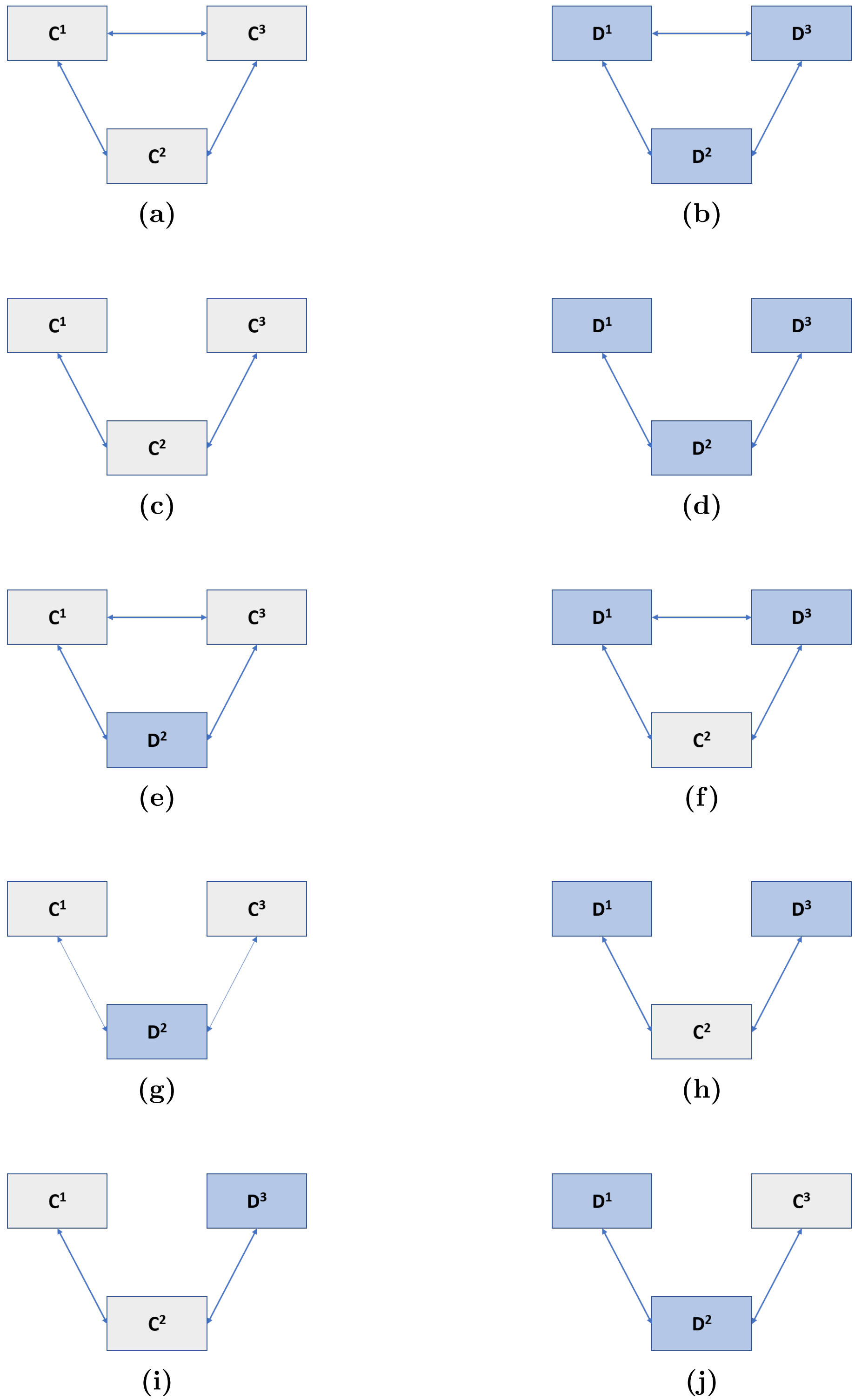


Figure 4.1: All possible three agent network structures.

4.1 General analysis and assumptions

Before the equilibrium solutions of the different network structures are worked out, new definitions and assumptions are necessary. For example, a choice should be made for how to generalize the two-agent correlation device to a three-agent system. The most straightforward way to do this, keeping the interpretation of the device as a publicly known signal, is to define the correlation device as $p_{\mu\nu\rho}$, which just like in the two-agent case precisely specifies the probability for each possible outcome. In terms of the response probabilities of the three agents, the renormalized correlations become

$$p^R_{\mu\nu\rho} = \sum_{\mu'\nu'\rho'} P^1_{\mu\leftarrow\mu'} P^2_{\nu\leftarrow\nu'} P^3_{\rho\leftarrow\rho'} p_{\mu\nu\rho}. \tag{4.1}$$

It turns out to be very useful to re-write the full correlation device into effective two-agent correlation devices, since the payoff only depends on the expressions

$$\begin{cases} p^{(1,2)}_{\mu\nu} = p_{\mu\nu C} + p_{\mu\nu D}, \\ p^{(1,3)}_{\mu\nu} = p_{\mu C\nu} + p_{\mu D\nu}, \\ p^{(2,3)}_{\mu\nu} = p_{C\mu\nu} + p_{D\mu\nu}, \end{cases} \tag{4.2}$$

which renormalize just like in the two-agent case:

$$p^{(i,j),R}_{\mu\nu} = \sum_{\mu'\nu'} P^i_{\mu\leftarrow\mu'} P^j_{\nu\leftarrow\nu'} p^{(i,j)}_{\mu\nu}. \tag{4.3}$$

The expected outcome of each agent is determined by the renormalized probabilities in combination with the payoff function. In terms of the full correlation device and payoff function, this becomes

$$\langle u^i \rangle = \sum_{\mu\nu\rho} U^i_{\mu\nu\rho} p^R_{\mu\nu\rho}. \tag{4.4}$$

In terms of the effective two-agent devices and the more familiar two-agent payoff matrices (which now are denoted with a superscript to specify which of the agents it concerns) this can also be written as

$$\langle u^i \rangle = \sum_j \sum_{\mu\nu} u^{(i,j)}_{\mu\nu} p^{(i,j),R}_{\mu\nu} \quad j \in \{1,2,3\} \setminus i. \tag{4.5}$$

This expression is very similar to what the expected payoff of a two-agent system; this similarity suggests the possibility to define slope coefficients

$$\langle u^i \rangle = \sum_j (C^{(i,j)}_C P^i_{FC} + C^{(i,j)}_D P^i_{FD} + C^{(i,j)}) \quad j \in \{1,2,3\} \setminus i. \tag{4.6}$$

By writing out the sum explicitly and grouping terms, it becomes clear that the slope coefficients for agent i now are the combination of the influences from each connected agent. As an example, the expected payoff of agent 1 can be written as

$$\langle u^1 \rangle = (C_C^{(1,2)} + C_C^{(1,3)})P_{FC}^1 + (C_D^{(1,2)} + C_D^{(1,3)})P_{FD}^1 + C^{(1,2)} + C^{(1,3)}, \tag{4.7}$$

which already gives some intuition about the equilibrium solutions. Since the slope coefficients are the sum of the contributions from each connection, we can deduce that the average of these contributions determines the full slope coefficient. How this translates in concrete results will become clear in the following sections, where the specific network structures are analyzed.

Before presenting the concrete analysis, a few more things are discussed. First of all, it is important to note that, by introducing the effective correlation in Equation 4.2, one degree of freedom is lost. Another way of stating that one degree of freedom is lost by rewriting the correlation device, is that the effective two-agent correlations are invariant under the transformation:

$$\begin{cases} p_{CCC} \to p_{CCC} + \eta \\ p_{CCD} \to p_{CCD} - \eta \\ p_{CDC} \to p_{CDC} - \eta \\ p_{DCC} \to p_{DCC} - \eta \\ p_{CDD} \to p_{CDD} + \eta \\ p_{DCD} \to p_{DCD} + \eta \\ p_{DDC} \to p_{DDC} + \eta \\ p_{DDD} \to p_{DDD} - \eta \end{cases}. \tag{4.8}$$

This is an interesting observation, which would be loosely comparable to a gauge invariance in physics. For any consistent choice of η, all agents are correlated to each other in exactly the same way. Also, this means that the expected payoff of the agents is equal for each choice. The fact which makes this not exactly comparable to gauge invariance is of course that the differences for different values of η are not entirely unobservable: the observable probability distribution does change.

The three-agent correlation device has 7 degrees of freedom, and therefore it is not possible to plot the whole phase space in one figure. Since some graphical presentation of the results is desirable, some convenient constraints were used to be able to plot a wide range of results anyway[1]. If we impose that for the three-agent correlation device the conditions are such that $p_{CD}^{(i,j)} = p_{DC}^{(i,j)}$, then all three of these effective two-agent correlation devices can be plotted in the same figure. In terms of the full correlation device, this leads to two extra constraints (on top of the requirement that the probabilities sum to 1). These can be formulated in multiple ways, but one choice

[1]These constraints basically are a logical continuation of the constraint which was for plotting in two-agent Battle of the Sexes, namely $p_{CD} = p_{DC}$.

is

$$\begin{cases} p_{CCD} &= p_{DCC} + p_{DDC} - p_{CDD} \\ p_{CDC} &= p_{DCC} + p_{DCD} - p_{CDD} \\ p_{DDD} &= 1 - p_{CCC} - p_{DCC} - p_{CDD} - p_{DCD} - p_{DDC} - p_{CCD} - p_{CDC} \end{cases} . \quad (4.9)$$

It can be directly verified that these constraints, when substituted in Equation 4.2, indeed make the effective correlation devices symmetric, and of the form:

$$\begin{aligned} p_{\mu\nu}^{(1,2)} &= \begin{bmatrix} p_{CC} & p_{CD} \\ p_{CD} & p_{DD} \end{bmatrix}, \\ p_{\mu\nu}^{(1,3)} &= \begin{bmatrix} p_{CC} - \delta & p_{CD} + \delta \\ p_{CD} + \delta & p_{DD} - \delta \end{bmatrix}, \\ p_{\mu\nu}^{(2,3)} &= \begin{bmatrix} p_{CC} - \epsilon & p_{CD} + \epsilon \\ p_{CD} + \epsilon & p_{DD} - \epsilon \end{bmatrix}. \end{aligned} \quad (4.10)$$

In terms of the full correlation device values, the different variables in these expressions are

$$\begin{cases} p_{CC} &= p_{CCC} + p_{DCC} + p_{DDC} - p_{CDD} \\ p_{CD} &= p_{DCC} + p_{DCD} \\ p_{DD} &= 1 - p_{CCC} - 3p_{DCC} - 2p_{DCD} - p_{DDC} + p_{CDD} \\ \delta &= p_{DDC} - p_{DCD} \\ \epsilon &= p_{DDC} - p_{CDD} \end{cases} . \quad (4.11)$$

In this form, all three effective correlation devices can be plotted in the same plane as three different points, being shifted diagonally with respect to each other.

4.2 Network structure: D-C-D

The first concrete network structure which is worked out in this chapter is shown in Figure 4.2. The corresponding payoff matrix is shown in Table 4.1.

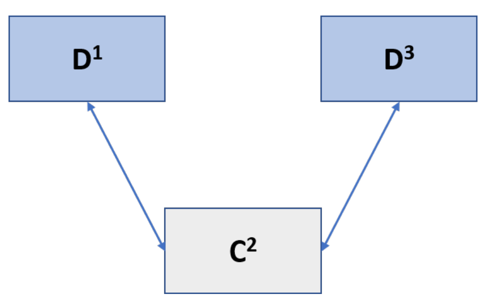


Figure 4.2: Network structure D-C-D.

Table 4.1: Pay-off table *D*-*C*-*D*.

3		C		D	
2		C	D	C	D
1	C	(S,2,S)	(0,0,0)	(S,1,0)	(0,1,S)
	D	(0,1,S)	(1,S,0)	(0,0,0)	(1,2S,1)

First of all, the slope coefficients, in the form of Equation 4.6, should be determined for each agent. For agents 1 and 3, denoted as $i \in \{1,3\}$, this gives

$$\begin{cases} C_C^{(i,2)}(P_{FC}^2, P_{FD}^2) & = p_{CC}^{(i,2)}((S+1)P_{FC}^2 - 1) - p_{CD}^{(i,2)}((S+1)P_{FD}^2 - S) \\ C_D^{(i,2)}(P_{FC}^2, P_{FD}^2) & = p_{DD}^{(i,2)}((S+1)P_{FD}^2 - S) - p_{DC}^{(i,2)}((S+1)P_{FC}^2 - 1) \end{cases}, \tag{4.12}$$

whereas agent 2 now has two contributions to the slope coefficient:

$$\begin{cases} C_C^{(2,i)}(P_{FC}^i, P_{FD}^i) & = p_{CC}^{(2,i)}((S+1)P_{FC}^i - S) - p_{DC}^{(2,i)}((S+1)P_{FD}^i - 1) \\ C_D^{(2,i)}(P_{FC}^i, P_{FD}^i) & = p_{DD}^{(2,i)}((S+1)P_{FD}^i - 1) - p_{CD}^{(2,i)}((S+1)P_{FC}^i - S) \end{cases}. \tag{4.13}$$

In order to determine the Nash equilibria, it is useful to see that, given a certain strategy from agent 2, the situation for agents 1 and 3 is the same as in the two-agent case. Therefore, for each particular response strategy from agent 2, we already know for what kind of correlation devices agent 1 and 3 are optimized given their strategy. So this part of the calculations is straightforward. Agent 2, who has two connections, is what makes this network truly different than the two-agent cases.

Slope analysis

In this section, the various equilibria are worked out in detail. As was previously mentioned, not all possible combinations of slope coefficients are considered. The pure equilibria and, if it exists in the network, the mixed equilibrium are calculated. Of the 'class 4' equilibria, one example of the many possible slope conditions is considered.

Class 1: pure equilibria

There are once again two pure equilibria in this network: one where each agent always plays C, and one were each agent always plays D. The expected payoffs of the 'always C' equilibrium are

$$\begin{aligned} \langle u^1 \rangle &= S, \\ \langle u^2 \rangle &= 2, \\ \langle u^3 \rangle &= S, \end{aligned} \tag{4.14}$$

and the expected payoffs of the 'always D' equilibrium are

$$\begin{aligned} \langle u^1 \rangle &= 1, \\ \langle u^2 \rangle &= 2S, \\ \langle u^3 \rangle &= 1. \end{aligned} \tag{4.15}$$

Class 2: mixed equilibrium

There is a mixed equilibrium in this system, where each agent plays such that the others are indifferent between the outcomes. This is achieved by the response strategies

$$\begin{aligned} P_{FC}^{1*} + P_{FC}^{3*} &= \frac{2S}{S+1}, \\ P_{FD}^{1*} + P_{FD}^{3*} &= \frac{2}{S+1}, \\ P_{FC}^{2*} &= \frac{1}{S+1}, \\ P_{FD}^{2*} &= \frac{S}{S+1}. \end{aligned} \tag{4.16}$$

The expected payoff for each agent is

$$\begin{aligned} \langle u^1 \rangle &= \frac{2S}{(S+1)^2}, \\ \langle u^2 \rangle &= \frac{4S}{(S+1)^2}, \\ \langle u^3 \rangle &= \frac{2S}{(S+1)^2}. \end{aligned} \tag{4.17}$$

Although this is of course mathematically possible to achieve, it is a complicating factor that, in order to reach the mixed equilibrium, agent 1 and agent 3 need to coordinate to make agent 2 indifferent. This might not be something which is likely to happen spontaneously, since the agents cannot communicate.

Class 3: stable correlated equilibria

From the discussion of two-agent Battle of the Sexes, we know that, if agent 2 plays strategy 'always follow', the optimal strategy for agent 1 and agent 3 depends on the determinant of the correlation device. If this is positive, the agents want to keep the positive correlations as much as possible by always following as well. If the determinant is negative, the agents want to flip the sign of the (negative) correlations by never following the device. If agent 2 plays 'never follows', the opposite is true. We therefore know that there are once again four response strategies which result in Nash equilibria. These are considered in this section in order.

The first case to consider is 'everyone always follows the device', which in the notation $(P_{FC}^1, P_{FD}^1, P_{FC}^2, P_{FD}^2, P_{FC}^3, P_{FD}^3)$ is denoted by $(1,1,1,1,1,1)$. The equilibrium conditions for this response strategy are:

$$\begin{cases} C_C^{(1,2)}(1,1) > 0 \\ C_D^{(1,2)}(1,1) > 0 \end{cases} \quad \begin{cases} C_C^{(2,1)}(1,1) + C_C^{(2,3)}(1,1) > 0 \\ C_D^{(2,1)}(1,1) + C_D^{(2,3)}(1,1) > 0 \end{cases} \quad \begin{cases} C_C^{(3,2)}(1,1) > 0 \\ C_D^{(3,2)}(1,1) > 0 \end{cases} \tag{4.18}$$

In terms of the effective correlation devices as defined by Equation 4.2, agents 1 and 3 are optimized if

$$\begin{aligned} p_{CC}^{(i,2)} &> \frac{p_{CD}^{(i,2)}}{S}, \\ p_{DD}^{(i,2)} &> S p_{DC}^{(i,2)}. \end{aligned} \tag{4.19}$$

Now that we know the equilibrium conditions of agents 1 and 3, the next step is to find the equilibrium conditions for agent 2 given the strategy of the others. The optimal strategy is, as is clear from Equation 4.18, a result of the combined contributions of the other agents:

$$\begin{aligned} (p_{CC}^{(2,1)} + p_{CC}^{(2,3)}) &> S(p_{DC}^{(2,1)} + p_{DC}^{(2,3)}), \\ (p_{DD}^{(2,1)} + p_{DD}^{(2,3)}) &> \frac{p_{CD}^{(2,1)} + p_{CD}^{(2,3)}}{S}. \end{aligned} \tag{4.20}$$

With these constraints, we know for a general correlation device whether this response strategy is an equilibrium or not. However, these results cannot be visualized yet. In order to be able to make graphs, the constraints as given by Equation 4.10 are now used. With these constraints on the device, the constraints for agent 2 become

$$\begin{aligned} 2(p_{CC} - S p_{DC}) &> \delta \cdot (s+1), \\ 2(S p_{DD} - p_{DC}) &> \delta \cdot (s+1). \end{aligned} \tag{4.21}$$

A graphical interpretation of these constraints is as follows: agent 2 is in equilibrium if the *midpoint* of the line joining the values of $p_{\mu\nu}^{(2,1)}$ and $p_{\mu\nu}^{(2,3)}$ is within the area where he, when playing against only one agent, would be in equilibrium. This reflects the intuitive notion that only the combination of what agents 1 and 3 play matters and that their influence is equal. In Figure 4.3, some examples are shown to illustrate when a situation is an equilibrium given these response strategies. In each of the sub-figures, three different coordinates are plotted, with arrows connecting them. The point marks the values of $p_{\mu\nu}^{(1,2)}$, the small circle marks the values of $p_{\mu\nu}^{(2,3)}$ and the big circle marks the values of $p_{\mu\nu}^{(1,3)}$. If $p_{\mu\nu}^{(1,2)}$ and $p_{\mu\nu}^{(2,3)}$ are within the red area, agents 1 and 3 are in equilibrium with the response strategy 'always follow'. For agent 2, the midpoint between the values of $p_{\mu\nu}^{(1,2)}$ and $p_{\mu\nu}^{(2,3)}$ should be within the blue area for the equilibrium conditions to hold. The notation $(p_{CCC}, p_{CCD}, p_{CDC}, p_{DCC}, p_{CDD}, p_{DCD}, p_{DDC}, p_{DDD})$ is used from now to specify the specific correlation device values used in the figures. In Figure 4.3a, which is generated with correlation device values $(0.2, 0.08, 0., 0.04, 0.06, 0.02, 0.1, 0.5)$, agents 1 and 3 are optimized since both $p_{\mu\nu}^{(1,2)}$ and $p_{\mu\nu}^{(2,3)}$ are in the red area. Agent 2 is optimized as well, since the midpoint between $p_{\mu\nu}^{(1,2)}$ and $p_{\mu\nu}^{(2,3)}$ is within the blue area. Hence, it is a Nash equilibrium. In Figure 4.3b, which is generated with the values $(0.2, 0.08, 0., 0.04, 0.06, 0.02, 0.1, 0.5)$, agent 3 is not optimized. Hence the strategy is not a Nash equilibrium for this correlation device. In Figure 4.3c, which is generated with the values $(0.5, 0.1, 0., 0.05, 0.05, 0, 0.1, 0.2)$, all agents are optimized again, which means that this is an equilibrium. In Figure 4.3d, which is generated with the values $(0.42, 0.12, 0.08, 0.02, 0, 0.06, 0.1, 0.2)$, agent 2 is not optimized.

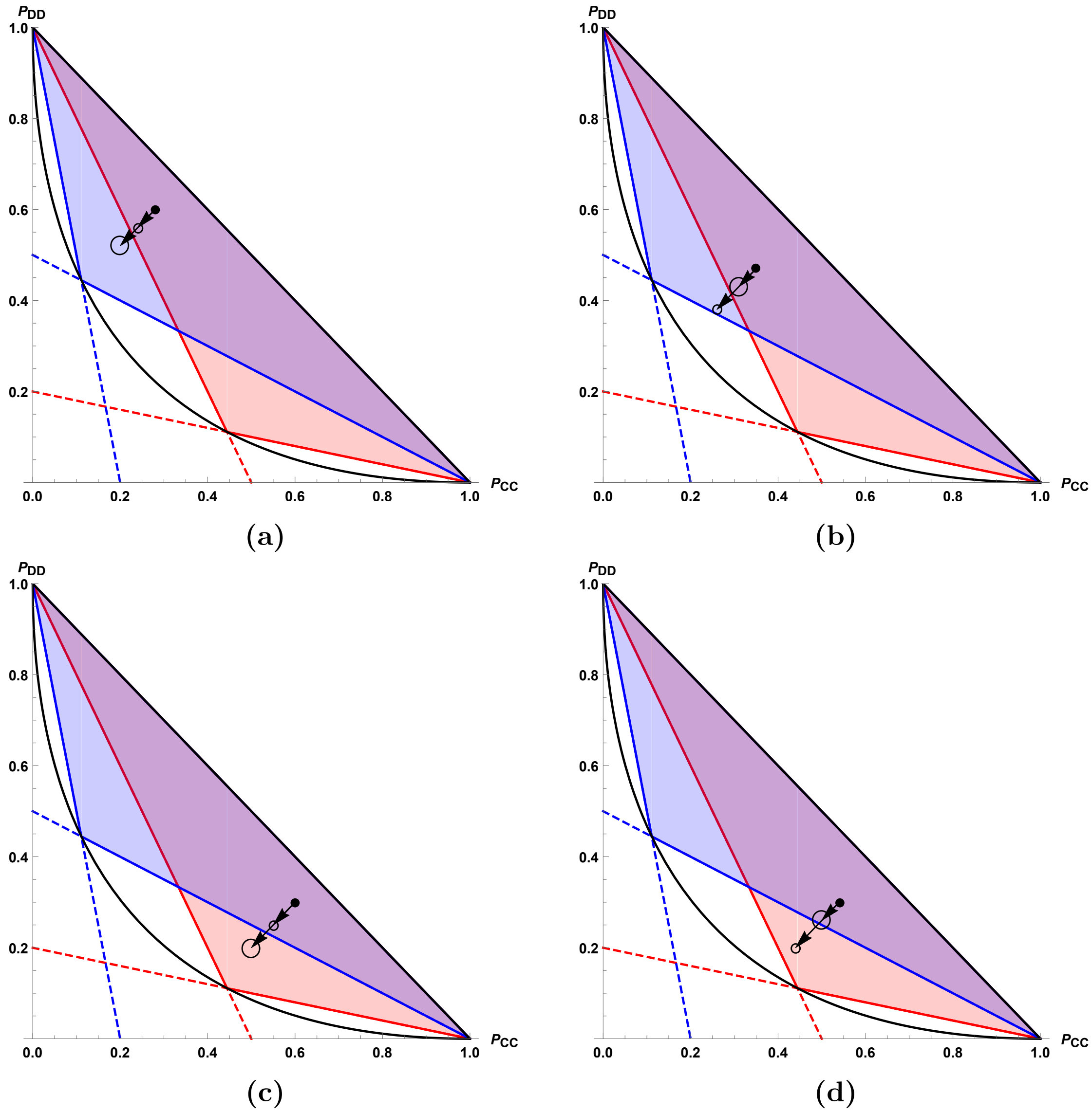


Figure 4.3: In this figure, four situations which illustrate the equilibrium conditions for strategy $(1, 1, 1, 1, 1, 1)$ are shown, for $S = .5$. The dot shows the coordinates of $p_{\mu\nu}^{(1,2)}$, the large circle shows the coordinates of $p_{\mu\nu}^{(1,3)}$ and the small circle shows the coordinates of $p_{\mu\nu}^{(2,3)}$. The blue region shows the equilibrium constraints for 'D-type' agents, whereas the red region shows the equilibrium constraints for the 'C-type' agents. When there are multiple influences on an agent, the midpoint between the two relevant effective correlation devices should be within the equilibrium area for the agent to be optimized with the strategy 'always follow'. Which of these sub-figures are equilibrium situations depends on the network structure; see the text for the concrete analysis.

Now, lets consider what happens when agent 2 never follows the device. The only way for agents 1 and 3 to optimize with a positively correlating device is by never following as well. This gives the slope conditions:

$$\begin{cases} C_C^{(1,2)}(0,0) < 0 \\ C_D^{(1,2)}(0,0) < 0 \end{cases} \qquad \begin{cases} C_C^{(2,1)}(0,0) + C_C^{(2,3)}(0,0) < 0 \\ C_D^{(2,1)}(0,0) + C_D^{(2,3)}(0,0) < 0 \end{cases} \qquad \begin{cases} C_C^{(3,2)}(0,0) < 0 \\ C_D^{(3,2)}(0,0) < 0 \end{cases} . \tag{4.22}$$

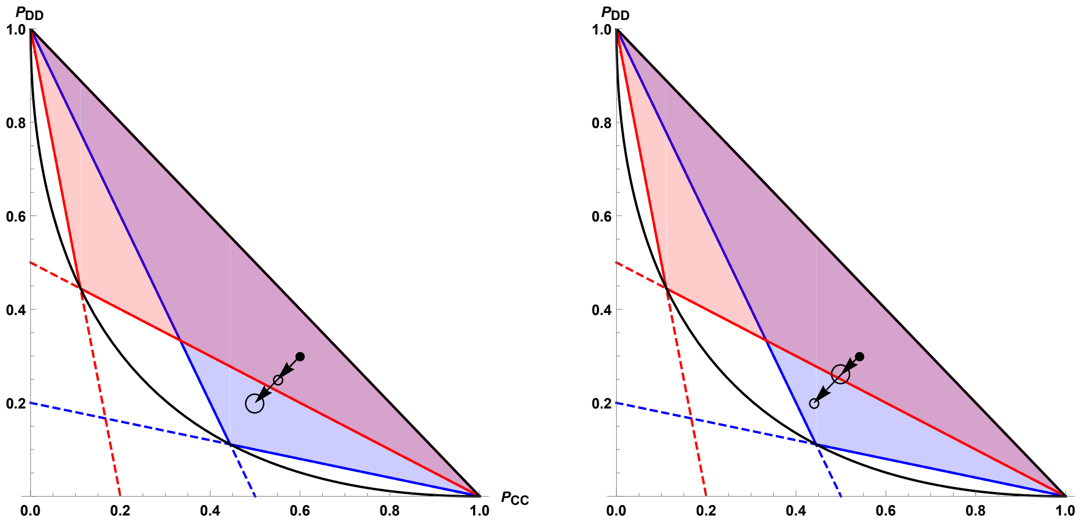
Under these conditions, agents 1 and 3 are optimized if

$$\begin{aligned} p_{CC}^{(i,2)} &> S p_{CD}^{(i,2)}, \\ p_{DD}^{(i,2)} &> \frac{p_{DC}^{(i,2)}}{S}, \end{aligned} \tag{4.23}$$

while agent 2 is optimized if

$$\begin{aligned} (p_{CC}^{(2,1)} + p_{CC}^{(2,3)}) &> \frac{p_{DC}^{(2,1)} + p_{DC}^{(2,3)}}{S}, \\ (p_{DD}^{(2,1)} + p_{DD}^{(2,3)}) &> S(p_{CD}^{(2,1)} + p_{CD}^{(2,3)}). \end{aligned} \tag{4.24}$$

In Figure 4.4, two example are given of concrete situations. Again, the assumption that the correlation devices are symmetric has been used here, in order to reduce the degrees of freedom to an amount which can be plotted. Figure 4.4a, which was generated with values $(0.5, 0.1, 0., 0.05, 0.05, 0, 0.1, 0.2)$, is an equilibrium since all equilibrium conditions are met. Figure 4.4b, which was generated with values $(0.42, 0.12, 0.08, 0.02, 0, 0.06, 0.1,\ 0.2)$, is not an equilibrium, since agent 3 is not optimized.



(a) **(b)**

Figure 4.4: In this figure, two situations which illustrate the equilibrium conditions for strategy $(0, 0, 0, 0, 0, 0)$ are shown, for $S = .5$. The dot shows the coordinates of $p_{\mu\nu}^{(1,2)}$, the large circle shows the coordinates of $p_{\mu\nu}^{(1,3)}$ and the small circle shows the coordinates of $p_{\mu\nu}^{(2,3)}$. The blue region shows the equilibrium constraints for 'D-type' agents, whereas the red region shows the equilibrium constraints for the 'C-type' agents. When there are multiple influences on an agent, the midpoint between the two relevant effective correlation devices should be within the equilibrium area for the agent to be optimized with the strategy 'never follow'. Which of these sub-figures are equilibrium situations depends on the network structure; see the text for the concrete analysis.

Class 4: unstable correlated equilibria

As was already mentioned in the introduction, working out all the different combinations of slope conditions which result in the 'class 4' Nash equilibria, such as defined in Table 3.3, would be too much work to write out. Also, it is not very instructive. However, it is worth working out one example to show how these behave in a three-agent system. For this example it is once again assumed that the correlations induced by the device are positive.

Consider that agent 2 plays strategy 'always follow C' and 'sometimes follow D', which can be described as $(1, P_{FD}^{2*})^2$. The slope coefficients for agents 1 and 3 now are

$$\begin{cases} C_C^{(i,2)}(1, P_{FD}^{2*}) & = S p_{CC}^{(i,2)} - p_{CD}^{(i,2)}((S+1)P_{FD}^{2*} - S) \\ C_D^{(i,2)}(1, P_{FD}^{2*}) & = p_{DD}^{(i,2)}((S+1)P_{FD}^{2*} - S) - S p_{DC}^{(i,2)} \end{cases}. \tag{4.25}$$

It can be expected from our two-agent results that for agents 1 and 3 both $(1, P_{FD}^{1*})$ and $(P_{FC}^{1*}, 1)$ could give equilibrium solutions for some range in the correlation device phase space. This gives the following candidate strategy profiles to consider:

1. $(1, P_{FD}^{1*}, 1, P_{FD}^{2*}, 1, P_{FD}^{3*})$

2. $(1, P_{FD}^{1*}, 1, P_{FD}^{2*}, P_{FC}^{3*}, 1)$

3. $(P_{FC}^{1*}, 1, 1, P_{FD}^{2*}, 1, P_{FD}^{3*})$

4. $(P_{FC}^{1*}, 1, 1, P_{FD}^{2*}, P_{FC}^{3*}, 1)$

These are the options when requiring positive correlations from the device. When allowing any correlation device, the number of cases to consider becomes a lot larger, which illustrates how quickly the pool of these kinds of equilibria increases with increasing network complexity. Since we are not interested in working out all here, but just want to show an example of the kind of results which you get, $(1, P_{FD}^{1*}, 1, P_{FD}^{2*}, 1, P_{FD}^{3*})$ is the only example strategy which is worked out explicitly. With this response strategy, the slope coefficients for agent 2 become the sum of the two contributions

$$\begin{cases} C_C^{(2,i)}(P_{FC}^i, P_{FD}^i) & = p_{CC}^{(2,i)} - p_{DC}^{(2,i)}((S+1)P_{FD}^{i*} - 1) \\ C_D^{(2,i)}(P_{FC}^i, P_{FD}^i) & = p_{DD}^{(2,i)}((S+1)P_{FD}^{i*} - 1) - p_{CD}^{(2,i)} \end{cases}. \tag{4.26}$$

The slope conditions now are:

$$\begin{cases} C_C^{(1,2)}(1, P_{FD}^{2*}) > 0 \\ C_D^{(1,2)}(1, P_{FD}^{2*}) = 0 \end{cases} \quad \begin{cases} C_C^{(2,1)}(1, P_{FD}^{1*}) + C_C^{(2,3)}(1, P_{FD}^{3*}) > 0 \\ C_D^{(2,1)}(1, P_{FD}^{1*}) + C_D^{(2,3)}(1, P_{FD}^{3*}) = 0 \end{cases} \quad \begin{cases} C_C^{(3,2)}(1, P_{FD}^{2*}) > 0 \\ C_D^{(3,2)}(1, P_{FD}^{2*}) = 0 \end{cases}. \tag{4.27}$$

It follows from the equality constraints of agents 1 and 3 that

$$\begin{aligned} P_{FD}^{2*} &= \frac{S}{S+1}\left(1 + \frac{p_{DC}^{(1,2)}}{p_{DD}^{(1,2)}}\right), \\ P_{FD}^{2*} &= \frac{S}{S+1}\left(1 + \frac{p_{DC}^{(3,2)}}{p_{DD}^{(3,2)}}\right). \end{aligned} \tag{4.28}$$

This is a very remarkable difference compared to the two-agent results, since we now get an extra constraint on the correlation device:

$$\frac{p_{DC}^{(1,2)}}{p_{DD}^{(1,2)}} = \frac{p_{DC}^{(3,2)}}{p_{DD}^{(3,2)}} \to \frac{p_{DCC} + p_{DCD}}{p_{DDC} + p_{DDD}} = \frac{p_{CDC} + p_{DDC}}{p_{DDD} + p_{CDD}}. \tag{4.29}$$

The equality constraint for agent 2 gives one equation with two unknowns:

$$p_{DD}^{(2,1)}((S+1)P_{FD}^{1*} - 1) - p_{CD}^{(2,1)} + p_{DD}^{(2,3)}((S+1)P_{FD}^{3*} - 1) - p_{CD}^{(2,3)} = 0. \tag{4.30}$$

This degree of freedom is a result from the fact that agents 1 and 3 together can make agent 2 indifferent, and there is no unique way to do this. However, the most logical solution of the range of possibilities is

$$\begin{aligned} P_{FD}^{1*} &= \frac{1}{S+1}\left(1 + \frac{p_{CD}^{(2,1)}}{p_{DD}^{(2,1)}}\right), \\ P_{FD}^{3*} &= \frac{1}{S+1}\left(1 + \frac{p_{CD}^{(3,2)}}{p_{DD}^{(3,2)}}\right). \end{aligned} \tag{4.31}$$

From solving the inequalities of the slope coefficients, we get the constraints that the effective correlations are positive:

$$\begin{aligned} p_{CC}^{(1,2)}p_{DD}^{(1,2)} &> p_{CD}^{(1,2)}p_{DC}^{(1,2)}, \\ p_{CC}^{(1,3)}p_{DD}^{(1,3)} &> p_{CD}^{(1,3)}p_{DC}^{(1,3)}, \\ p_{CC}^{(2,3)}p_{DD}^{(2,3)} &> p_{CD}^{(2,3)}p_{DC}^{(2,3)}. \end{aligned} \tag{4.32}$$

Requiring that $0 < P_{FD}^{i*} < 1$, we get a new set of constraints on the correlation device:

$$\begin{aligned} p_{DD}^{(1,2)} &< Sp_{DC}^{(1,2)}, \\ p_{DD}^{(3,2)} &< Sp_{DC}^{(3,2)}, \\ p_{DD}^{(2,1)} &< \frac{p_{CD}^{(2,1)}}{S}, \\ p_{DD}^{(2,3)} &< \frac{p_{CD}^{(2,3)}}{S}. \end{aligned} \tag{4.33}$$

In conclusion, it can be observed that theoretically these constraints can definitely be obeyed. For example, if $p_{\mu\nu}^{(2,1)} = p_{\mu\nu}^{(2,3)}$, the constraints are almost identical to the two-agent case. However, the equality constraint of Equation 4.29 is a significant difference when arguing about the likelihood that this equilibrium will be realized in realistic situations. The equality means that very strict conditions have to be obeyed in order for this equilibrium to be realizable. Therefore, there are relatively few correlation devices, which have to be precisely tuned, which could result in this kind of equilibrium.

4.3 Network structure: C-C-D

The network which is considered now is shown in Figure 4.5 and has a payoff matrix as shown in Table 4.2. The most important difference between this network and the one from the previous discussion is that there is no symmetry between agents 1 and 3: they are no longer interchangeable without affecting the payoff matrix. However, in determining the Nash equilibria this turns out to have no significant impact.

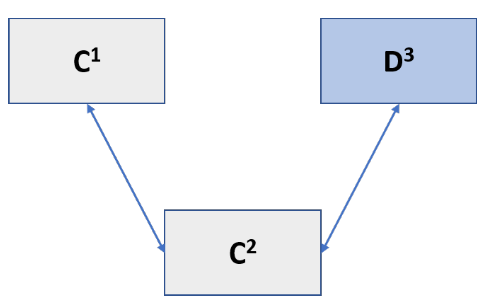


Figure 4.5: Network structure C-C-D.

Table 4.2: Pay-off table C-C-D.

3		C		D	
2		C	D	C	D
1	C	(1,2,S)	(0,0,0)	(1,1,0)	(0,S,1)
	D	(0,1,S)	(S,S,0)	(0,0,0)	(S,2S,1)

The slope coefficients of agent 1, in the form of Equation 4.6, are

$$\begin{cases} C_C^{(1,2)}(P_{FC}^2, P_{FD}^2) & = p_{CC}^{(1,2)}((S+1)P_{FC}^2 - S) - p_{CD}^{(1,2)}((S+1)P_{FD}^2 - 1) \\ C_D^{(1,2)}(P_{FC}^2, P_{FD}^2) & = p_{DD}^{(1,2)}((S+1)P_{FD}^2 - 1) - p_{DC}^{(1,2)}((S+1)P_{FC}^2 - S) \end{cases}, \tag{4.34}$$

for agent 3, they are

$$\begin{cases} C_C^{(3,2)}(P_{FC}^2, P_{FD}^2) & = p_{CC}^{(3,2)}((S+1)P_{FC}^2 - 1) - p_{CD}^{(3,2)}((S+1)P_{FD}^2 - S) \\ C_D^{(3,2)}(P_{FC}^2, P_{FD}^2) & = p_{DD}^{(3,2)}((S+1)P_{FD}^2 - S) - p_{DC}^{(3,2)}((S+1)P_{FC}^2 - 1) \end{cases}, \tag{4.35}$$

and agent 2 again has two contributions to the slope coefficient:

$$\begin{cases} C_C^{(2,i)}(P_{FC}^i, P_{FD}^i) & = p_{CC}^{(2,i)}((S+1)P_{FC}^i - S) - p_{DC}^{(2,i)}((S+1)P_{FD}^i - 1) \\ C_D^{(2,i)}(P_{FC}^i, P_{FD}^i) & = p_{DD}^{(2,i)}((S+1)P_{FD}^i - 1) - p_{CD}^{(2,i)}((S+1)P_{FC}^i - S) \end{cases}. \tag{4.36}$$

Just like in the previous network, it is convenient to start with assuming a strategy for agent 2. Whether or not agent 1 and 3 are optimized given this strategy and a particular correlation device follows from the same analysis as two-agent Battle of the Sexes (for agent 3) and two-agent pure coordination (for agent 1). With this insight, we know immediately that if the correlation device gives positive correlations, the only optimal strategy if agent 2 plays 'always follow' is for agent 1 and agent 3 to do the same. Also, if agent 2 plays 'never follow', this is what agent 1 and agent 3 should do as well.

Slope analysis

In this section, the stable equilibria are worked out for positively correlating devices. Also, the mixed equilibrium is discussed. The 'class 4' equilibria are not considered.

Class 1: pure equilibria

There are two pure equilibria in this network: one where each agent always plays C, and one were each agent always plays D. The expected payoffs of the 'always C' equilibrium are

$$\begin{aligned} \langle u^1 \rangle &= 1, \\ \langle u^2 \rangle &= 2, \\ \langle u^3 \rangle &= S, \end{aligned} \tag{4.37}$$

whereas the expected payoffs of the 'always D' equilibrium are

$$\begin{aligned}\langle u^1\rangle &= S,\\ \langle u^2\rangle &= 2S,\\ \langle u^3\rangle &= 1.\end{aligned} \tag{4.38}$$

Class 2: mixed equilibrium

In this Network, there is no mixed equilibrium. This is because it is impossible for agent 2 to make both agent 1 and agent 3 indifferent between their outcomes. In order to make agent 1 indifferent, the response strategies would have to be

$$\begin{aligned}P^{2*}_{FC} &= \frac{S}{S+1},\\ P^{2*}_{FD} &= \frac{1}{S+1}.\end{aligned} \tag{4.39}$$

However, agent 3 is indifferent with the choice

$$\begin{aligned}P^{2*}_{FC} &= \frac{1}{S+1},\\ P^{2*}_{FD} &= \frac{S}{S+1}.\end{aligned} \tag{4.40}$$

Since agent 2 cannot do both at the same time, the conclusion is that the mixed equilibrium does not exist.

Class 3: stable pure equilibria

The first case to consider is 'everyone always follows the device', which is denoted by $(1,1,1,1,1,1)$. The equilibrium conditions for this response strategy are

$$\begin{cases}C^{(1,2)}_C(1,1) > 0\\ C^{(1,2)}_D(1,1) > 0\end{cases} \quad \begin{cases}C^{(2,1)}_C(1,1) + C^{(2,3)}_C(1,1) > 0\\ C^{(2,1)}_D(1,1) + C^{(2,3)}_D(1,1) > 0\end{cases} \quad \begin{cases}C^{(3,2)}_C(1,1) > 0\\ C^{(3,2)}_D(1,1) > 0\end{cases}. \tag{4.41}$$

Under these conditions, agent 1 is optimized if

$$\begin{aligned}p^{(1,2)}_{CC} &> Sp^{(1,2)}_{CD},\\ p^{(1,2)}_{DD} &> \frac{p^{(1,2)}_{DC}}{S},\end{aligned} \tag{4.42}$$

agent 3 is optimized if

$$\begin{aligned}p^{(3,2)}_{CC} &> \frac{p^{(3,2)}_{CD}}{S},\\ p^{(3,2)}_{DD} &> Sp^{(3,2)}_{DC},\end{aligned} \tag{4.43}$$

and agent 2 is optimized if

$$\begin{aligned}(p_{CC}^{(2,1)} + p_{CC}^{(2,3)}) &> S(p_{DC}^{(2,1)} + p_{DC}^{(2,3)}),\\ (p_{DD}^{(2,1)} + p_{DD}^{(2,3)}) &> \frac{p_{CD}^{(2,1)} + p_{CD}^{(2,3)}}{S}.\end{aligned} \tag{4.44}$$

A graphical interpretation of these results is very similar to the previously discussed network. New figures with examples could be made, but 4.3 can just as well be re-used, with a new interpretation of the lines. In particular, agent 1 now has a different equilibrium area. The same graphical representation is used, where a point marks the values of $p_{\mu\nu}^{(1,2)}$, a large circle the values of $p_{\mu\nu}^{(1,3)}$ and a small circle the values of $p_{\mu\nu}^{(2,3)}$. Using the same example as before, the blue area in Figure 4.3 can now can be interpreted as the region in which $p_{\mu\nu}^{(1,2)}$ should be such that agent 1 is optimized, as well as the region in which the midpoint between $p_{\mu\nu}^{(1,2)}$ and $p_{\mu\nu}^{(2,3)}$ should be to optimize agent 2. The red area indicates the region in which $p_{\mu\nu}^{(2,3)}$ should be for agent 3 to be optimized. Therefore, in this network structure, Figure 4.3a is an equilibrium, Figure 4.3b is not an equilibrium (agent 3 isn't optimized), Figure 4.3c is an equilibrium as well and Figure 4.3d is not an equilibrium (agent 2 is not optimized).

If agent 2 never follows the device, then (assuming a positive correlation device) agents 1 and 3 should also never follow the device. This gives slope conditions

$$\begin{cases} C_C^{(1,2)}(0,0) < 0 \\ C_D^{(1,2)}(0,0) < 0 \end{cases} \quad \begin{cases} C_C^{(2,1)}(0,0) + C_C^{(2,3)}(0,0) < 0 \\ C_D^{(2,1)}(0,0) + C_D^{(2,3)}(0,0) < 0 \end{cases} \quad \begin{cases} C_C^{(3,2)}(0,0) < 0 \\ C_D^{(3,2)}(0,0) < 0 \end{cases}. \tag{4.45}$$

Under these conditions, agent 1 is optimized if

$$\begin{aligned}p_{CC}^{(1,2)} &> \frac{p_{CD}^{(1,2)}}{S},\\ p_{DD}^{(1,2)} &> S p_{DC}^{(1,2)},\end{aligned} \tag{4.46}$$

agent 3 is optimized if

$$\begin{aligned}p_{CC}^{(3,2)} &> S p_{CD}^{(3,2)},\\ p_{DD}^{(3,2)} &> \frac{p_{DC}^{(3,2)}}{S},\end{aligned} \tag{4.47}$$

and agent 2 is optimized if

$$\begin{aligned}(p_{CC}^{(2,1)} + p_{CC}^{(2,3)}) &> \frac{p_{DC}^{(2,1)} + p_{DC}^{(2,3)}}{S},\\ (p_{DD}^{(2,1)} + p_{DD}^{(2,3)}) &> S(p_{CD}^{(2,1)} + p_{CD}^{(2,3)}).\end{aligned} \tag{4.48}$$

We can again use the examples of Figure 4.4 with the new interpretation of the coloured regions (blue for agents 1 and 2, red for agent 3). It can be concluded that Figure 4.4a is an equilibrium, whereas Figure 4.4b now is not an equilibrium (agent 3 is not optimized).

4.4 Network structure: -C-D-C-

In this network, which is as shown in Figure 4.6 and Table 4.3, all agents are connected. This has the implication that, unlike in the two previously considered cases, none of the agents has the payoff matrix of a two-agent network; everyone has multiple influences.

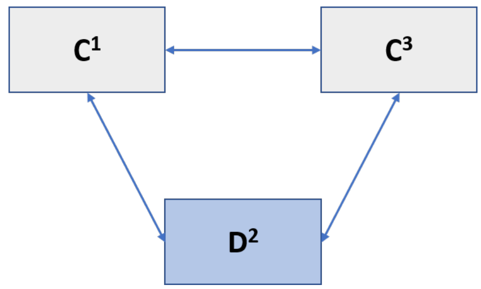


Figure 4.6: Network structure -C-D-C-.

Table 4.3: Pay-off table -C-D-C-.

3		C		D	
2		C	D	C	D
1	C	(2,2S,2)	(1,0,1)	(1,S,0)	(0,1,S)
	D	(0,S,1)	(S,1,0)	(S,0,S)	(2S,2,2S)

In this system, the slope coefficients for agents 1 are

$$\begin{cases} C_C^{(1,i)}(P_{FC}^i, P_{FD}^i) &= p_{CC}^{(1,i)}((S+1)P_{FC}^i - S) - p_{CD}^{(1,i)}((S+1)P_{FD}^i - 1) \\ C_D^{(1,i)}(P_{FC}^i, P_{FD}^i) &= p_{DD}^{(1,i)}((S+1)P_{FD}^i - 1) - p_{DC}^{(1,i)}((S+1)P_{FC}^i - S) \end{cases}, \tag{4.49}$$

the coefficients for agent 3 are

$$\begin{cases} C_C^{(3,i)}(P_{FC}^i, P_{FD}^i) &= p_{CC}^{(3,i)}((S+1)P_{FC}^i - S) - p_{CD}^{(3,i)}((S+1)P_{FD}^i - 1) \\ C_D^{(3,i)}(P_{FC}^i, P_{FD}^i) &= p_{DD}^{(3,i)}((S+1)P_{FD}^i - 1) - p_{DC}^{(3,i)}((S+1)P_{FC}^i - S) \end{cases}, \tag{4.50}$$

and agent 2, who now prefers outcome C, has slope coefficients

$$\begin{cases} C_C^{(2,i)}(P_{FC}^i, P_{FD}^i) &= p_{CC}^{(2,i)}((S+1)P_{FC}^i - 1) - p_{DC}^{(2,i)}((S+1)P_{FD}^i - S) \\ C_D^{(2,i)}(P_{FC}^i, P_{FD}^i) &= p_{DD}^{(2,i)}((S+1)P_{FD}^i - S) - p_{CD}^{(2,i)}((S+1)P_{FC}^i - 1) \end{cases}. \tag{4.51}$$

It now is not as easy to show what the optimal responses for agents 1 and 3 are if agent 2 plays a certain strategy, since they also interact with each other (and hence agent 2 does not solely determine whether or not agents 1 and 3 are optimal with a certain strategy).

Slope analysis

In this section, the stable equilibria are worked out for positively correlating devices. Also, the mixed equilibrium is considered.

Class 1: pure equilibria

The two pure equilibria in this network again are one where each agent always plays C, and one were each agent always plays D. The expected payoffs of the 'always C' equilibrium are

$$\begin{aligned}\langle u^1\rangle &= 2,\\ \langle u^2\rangle &= 2S,\\ \langle u^3\rangle &= 2,\end{aligned} \tag{4.52}$$

whereas the expected payoffs of the 'always D' equilibrium are

$$\begin{aligned}\langle u^1\rangle &= 2S,\\ \langle u^2\rangle &= 2,\\ \langle u^3\rangle &= 2S.\end{aligned} \tag{4.53}$$

Class 2: Mixed equilibrium

The mixed equilibrium in this system can not be realized for a general three-agent correlation device. Under the conditions that $p_{\mu\nu}^{(1,2)} = p_{\mu\nu}^{(1,3)} = p_{\mu\nu}^{(2,3)}$, it is possible to create a mixed equilibrium, with the equilibrium solution

$$\begin{aligned}P_{FC}^1 + P_{FC}^3 &= \frac{2}{S+1},\\ P_{FC}^2 + P_{FC}^3 &= \frac{2S}{S+1},\\ P_{FC}^1 + P_{FC}^2 &= \frac{2S}{S+1}.\end{aligned} \tag{4.54}$$

Again, achieving this this clearly requires coordination between the agents, since it is not possible to make the other indifferent by yourself. The unique solution is

$$\begin{aligned}P_{FC}^{1*} &= \frac{1}{S+1},\\ P_{FC}^{2*} &= \frac{2S-1}{S+1},\\ P_{FC}^{3*} &= \frac{1}{S+1}.\end{aligned} \tag{4.55}$$

It can be seen from this equation that for $S < .5$ the response probability for agent 2 becomes negative. So even in the special case where the correlation device allows for a mixed equilibrium, it is not realizable for a general payoff matrix.

Class 3: stable correlated equilibria

The first step of the analysis is to find the possible optimal responses of agents 1 and 3 if agent 2 always follows the instructions of the device. This gives the following set of equations:

$$\begin{cases} C_C^{(1,2)}(1,1) + C_C^{(1,3)}(1,1) > 0 \\ C_D^{(1,2)}(1,1) + C_D^{(1,3)}(1,1) > 0 \\ C_C^{(2,1)}(1,1) + C_C^{(2,3)}(1,1) > 0 \\ C_D^{(2,1)}(1,1) + C_D^{(2,3)}(1,1) > 0 \\ C_C^{(3,1)}(1,1) + C_C^{(3,2)}(1,1) > 0 \\ C_D^{(3,1)}(1,1) + C_D^{(3,2)}(1,1) > 0 \end{cases} . \tag{4.56}$$

The only equilibrium solution for positively correlating devices is again that both agent 1 and agent 3 always follow the instructions[2]. This now results in the following set of constraints for agents 1 and 3:

$$\begin{aligned} (p_{CC}^{(1,2)} + p_{CC}^{(1,3)}) &> S(p_{CD}^{(1,2)} + p_{CD}^{(1,3)}), \\ (p_{DD}^{(1,2)} + p_{DD}^{(1,3)}) &> \frac{p_{DC}^{(1,2)} + p_{DC}^{(1,3)}}{S}, \\ (p_{CC}^{(3,1)} + p_{CC}^{(3,2)}) &> S(p_{CD}^{(3,1)} + p_{CD}^{(3,2)}), \\ (p_{DD}^{(3,1)} + p_{DD}^{(3,2)}) &> \frac{p_{DC}^{(3,1)} + p_{DC}^{(3,2)}}{S}. \end{aligned} \tag{4.57}$$

The conditions for agent 2 are

$$\begin{aligned} (p_{CC}^{(2,1)} + p_{CC}^{(2,3)}) &> \frac{p_{DC}^{(2,1)} + p_{DC}^{(2,3)}}{S}, \\ (p_{DD}^{(2,1)} + p_{DD}^{(2,3)}) &> S(p_{CD}^{(2,1)} + p_{CD}^{(2,3)}). \end{aligned} \tag{4.58}$$

Using the same kind of graphical interpretation as before, it now follows that the red area in Figure 4.3 corresponds to the conditions for agent 2, whereas the blue area corresponds to the conditions for agents 1 and 3. However, the main difference now is that all agents have two contributions to their outcome. This has the effect that all three agents are optimal if the midpoint connecting the values of the two effective devices which influence them lies within the area. This means that in Figure 4.3a all agents are optimized, and hence it is an equilibrium. In Figure 4.3b, however, agent 2 is not in equilibrium. In Figure 4.3c everyone except agent 3 is optimized, which means that this is not an equilibrium. Figure 4.3d is not an equilibrium, since both agent 3 is not optimized.

[2] In this situation, agents 1 and 3 also influence each other, which makes it slightly less straightforward to make this conclusion. However, explicit calculation shows that this is the case.

If agent 2 chooses to never follow the device it can be shown that, even though agents 1 and 3 influence each other, the only equilibrium response is to always follow the device as well. This again is assuming a correlation device which positively correlates each link. The slope conditions in this case are

$$\begin{cases} C_C^{(1,2)}(0,0) + C_C^{(1,3)}(0,0) < 0 \\ C_D^{(1,2)}(0,0) + C_D^{(1,3)}(0,0) < 0 \\ C_C^{(2,1)}(0,0) + C_C^{(2,3)}(0,0) < 0 \\ C_D^{(2,1)}(0,0) + C_D^{(2,3)}(0,0) < 0 \\ C_C^{(3,1)}(0,0) + C_C^{(3,2)}(0,0) < 0 \\ C_D^{(3,1)}(0,0) + C_D^{(3,2)}(0,0) < 0 \end{cases}, \tag{4.59}$$

which results in the following set of constraints:

$$\begin{aligned}
(p_{CC}^{(1,2)} + p_{CC}^{(1,3)}) &> \frac{p_{CD}^{(1,2)} + p_{CD}^{(1,3)}}{S}, \\
(p_{DD}^{(1,2)} + p_{DD}^{(1,2)}) &> S(p_{DC}^{(1,2)} + p_{DC}^{(1,3)}), \\
(p_{CC}^{(3,1)} + p_{CC}^{(3,2)}) &> \frac{p_{CD}^{(3,1)} + p_{CD}^{(3,1)}}{S}, \\
(p_{DD}^{(3,1)} + p_{DD}^{(3,2)}) &> S(p_{DC}^{(3,1)} + p_{DC}^{(3,1)}), \\
(p_{CC}^{(2,1)} + p_{CC}^{(2,3)}) &> S(p_{DC}^{(2,1)} + p_{DC}^{(2,3)}), \\
(p_{DD}^{(2,1)} + p_{DD}^{(2,3)}) &> \frac{p_{CD}^{(2,1)} + p_{CD}^{(2,3)}}{S}.
\end{aligned} \tag{4.60}$$

In Figure 4.4, these constraints should be interpreted as follows: the blue area corresponds to the conditions for agents 1 and 3, whereas the red area stands for the conditions for agent 2. The two influences on each agent should on average be in this area for equilibrium conditions to be satisfied. Therefore, Figure 4.4a is an equilibrium, whereas Figure 4.4b is not an equilibrium (agent 2 is not optimized).

4.5 Discussion of results

The discussion of the results for the three-agent system in a way is more challenging than for the two-agent systems, since it is difficult to think of any way to visualize the outcomes other than the figures such as included earlier in this chapter. It is not possible to plot the entire phase space, which rules out the option of showing the results as a mapping. Also, it is impossible to, on a plane, show where the correlated equilibria are better than the worst pure equilibrium. By lack of these options, it was chosen to include a small section which reflects on the results in words.

First of all, it was found that the mixed equilibrium is not realizable in all three-agent networks. As soon as one agent has to make two other indifferent, the mixed equilibrium is only possible whenever those two agents have an equal payoff matrix. The 'class 4' equilibria, which were only in one example discussed, are only possible with quite strict constraints on the initial correlation device and sometimes as well the payoff matrix. The situation which was worked out in this chapter has a quite symmetric network structure; for less symmetric networks the constraints become even stricter. If one would work out the equilibria for even more complex networks, with more than three agents, it can therefore be expected that the mixed equilibria usually do not exist (they only can be realized if there happen to be convenient symmetries in the network). This is, besides the instability of these equilibria, another motivation to argue that one can safely ignore these equilibrium solutions when describing observable behaviour.

The analysis of the 'class 3' equilibria turns out to be not that much more complicated than in the two-agent case. The slope coefficients define equivalent area's in the phase space as in the two-agent case, with the important difference that averaging over all contributions now determines whether a situation is in equilibrium or not. Without the symmetric constraints on the correlation device, this averaging between contributions still works. However, in that case the midpoint of the two points in the full phase space should be calculated. The only pure equilibria in three-agent networks are, just like in two-agent coordination games, the two cases where everyone plays the same outcome.

Towards bigger networks

Even though in this thesis networks with four or more agents are not yet precisely studied, the choice was made to share some thoughts and insights here about how these results could be generalized to more complex systems. First of all, if $p_{\mu_1 .... \mu_N}$ denotes the correlation device for an N agent network, the effective two-agent interactions can be defined by summing over all outcomes which do not affect that particular link:

$$p^{(1,2)}_{\mu\nu} = p_{\mu\nu C....C} + p_{\mu\nu C....D} + ... + p_{\mu\nu D....D}. \tag{4.61}$$

By doing this, similar plots as Figure 4.3 can be made, with the difference that there are now more effective correlation devices which need to be shown in the figure. For example, if one agent has three links, the midpoint of those three links determines whether that situation is an equilibrium or not. With this knowledge, it should be possible to analyze the 'class 3' equilibria in a similar way as in three-agent systems. However, already starting from a four-agent network, it can be expected that the analysis becomes more complicated for another reason: there can be more than two pure Nash equilibria. For example, the network structure shown in Figure 4.7 has three pure equilibria. The two familiar ones, where everyone plays either C or D, and one completely different one, where everyone plays his or her preferred option. This is the most simple example of the phenomenon that 'subsystems' can form which are correlated internally, but anti-correlated with each other. If you have N neighbours, of which n play your preferred option with probability 1, whereas the rest plays the other option with probability 1, it is optimal for you to play your preferred option if

$$\frac{n}{N} \geq \frac{S}{S+1}. \tag{4.62}$$

Otherwise it is optimal to play your least preferred option. It would be interesting to analyze the correlated equilibria for networks in which this leads to non-trivial outcomes.

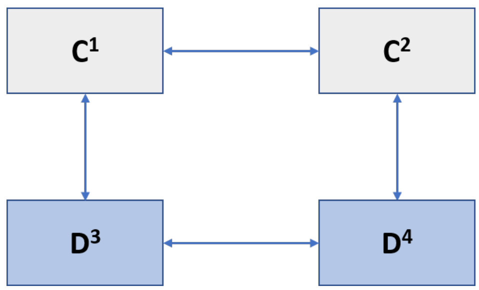


Figure 4.7: A four-agent network structure.

Chapter 5

The Ising model analogy

In this chapter, an exact mapping is made between the game-theoretical results and the Ising model from physics. The motivation for making such an analogy is that, as was proposed by Correia and Stoof [6], there is a striking analogy between the Ising model and Battle of the Sexes-like networks. By establishing a mapping to the Ising model, a better understanding of the equilibrium conditions within the response strategy of correlated games can possibly be obtained. Since the Ising model is exactly solvable for many one- and two-dimensional networks, the mapping could also be useful for obtaining analytical results in systems with many agents. The first implementation of this analogy for Battle of the Sexes was done by Peters [14]. In this chapter, his results are generalized and expanded upon. The two-agent results are presented in a way which enables the mapping for any renormalized outcomes. The general mapping for a three-agent system is stated as well. In order to make this mapping work, three body interactions are needed. The first part of this chapter is a general introduction to the Ising model, thereafter the results are discussed.

5.1 The Ising model

The Ising model is a simple quantum mechanical model to describe magnetic interactions between interacting particles, which assumes that only the z-components of the particle spins interact[1] [8]. The strength of this interaction is determined by the direction of the spin s_i of each particle and the interaction strength J_{ij} between the particles. We will assume that the particles are spin $\frac{1}{2}$ here (meaning that the value of the spin can either be $+\frac{1}{2}$ or $-\frac{1}{2}$), since this choice gives each particle two different states. Also, particles can couple to an external magnetic field B. How strongly they couple to the field is determined by the gyro-magnetic ratio γ_i, which is a property of the particle.

[1]The introduction to this model given here necessarily assumes some knowledge about physics, and is rather compact. For an extensive introduction, see for example [3].

The probability to find a particle in a certain state is determined by its Hamiltonian

$$p(s) = \frac{1}{Z} e^{-\beta H(s)}, \tag{5.1}$$

where s denotes a certain state and $H(s)$ is the value of the Hamiltonian in that state. The dynamics of a system are completely determined by its Hamiltonian. The Ising Hamiltonian in its usual form, which is consistent with the description above, is defined as

$$H(s) = -\frac{1}{2} \sum_{i,j} J_{ij} s_i s_j - B \sum_i \gamma_i s_i. \tag{5.2}$$

In this equation, the factor one half in front compensates for double counting of pairs, the parameter β is related to the inverse of the temperature of the system and Z is the partition function, which is the normalization factor for the probabilities:

$$Z = \sum_{s_i} e^{-\beta H(s_i)}. \tag{5.3}$$

With this information, the probability of each possible outcome of the system can be determined.

There are also other variations of the Ising model, in which the interactions are slightly different. One such variant includes three-body interactions. Including these gives a slightly different Hamiltonian, which is of the form

$$H(s) = -\frac{1}{2} \sum_{i,j} J_{ij} s_i s_j - B \sum_i \gamma_i s_i - A s_1 s_2 s_3. \tag{5.4}$$

Correspondence to game theory

The following discussion clarifies how the correspondence between the Ising model and game-theoretical networks precisely works. First of all, note that in the (spin $\frac{1}{2}$) Ising model each particle has two possible states: spin up (which can be denoted as $\uparrow$) and spin down (which can be denoted as $\downarrow$). In the game theory problem, each agent has to choose between two options, which are C and D. If we now identify C with $\uparrow$ and D with $\downarrow$, we can start to see how the analogy works. By equating the equilibrium strategies from Battle of the Sexes to Equation 5.1, the values for the parameters B, J and γ_i such that the Ising model gives identical outcomes can be found. This can be done on the renormalized level as well as on the non-renozmalized level:

$$p^R_{\mu\nu} = \frac{1}{Z} e^{-H^R_{\mu\nu}} \quad \text{or} \quad p_{\mu\nu} = \frac{1}{Z} e^{-H_{\mu\nu}}. \tag{5.5}$$

By working out both of these mappings, and analyzing how the renormalization affects the Ising parameters, new insights in the game theoretical problem are obtained. Since parameter β does not influence the outcomes (it is just a constant factor which stays in front of the results), it is conveniently chosen that the value is 1. Also, because of the structure of the problem, it can safely be assumed that J_{ij} has the fixed value J between each of the agents that is interacting.

5.2 Mapping for two agents

In this section, the analogy between two-agent Battle of the Sexes and the two-particle Ising model is made. Since for the analogy with the Battle of the Sexes game it can be assumed that the interaction between particles is symmetric, the general expression of the Hamiltonian in Equation 5.2 becomes

$$H_{\mu\nu} = \begin{bmatrix} -J - B(\gamma_1 + \gamma_2) & J - B(\gamma_1 - \gamma_2) \\ J + B(\gamma_1 - \gamma_2) & -J + B(\gamma_1 + \gamma_2) \end{bmatrix}. \tag{5.6}$$

It is possible to create any energy landscape by tuning these parameters, and therefore there is a unique mapping from any set of probabilities $p_{\mu\nu}$ to the corresponding Ising parameters. These are given in terms of the probabilities as

$$\begin{aligned} \gamma_1 B &= \frac{1}{4}\log(\frac{p_{CC}p_{CD}}{p_{DD}p_{DC}}), \\ \gamma_2 B &= \frac{1}{4}\log(\frac{p_{CC}p_{DC}}{p_{DD}p_{CD}}), \\ J &= \frac{1}{4}\log(\frac{p_{CC}p_{DD}}{p_{CD}p_{DC}}). \end{aligned} \tag{5.7}$$

Similarly, the renormalized Ising parameters are of course defined as

$$\begin{aligned} \gamma_1 B^R &= \frac{1}{4}\log(\frac{p^R_{CC}p^R_{CD}}{p^R_{DD}p^R_{DC}}), \\ \gamma_2 B^R &= \frac{1}{4}\log(\frac{p^R_{CC}p^R_{DC}}{p^R_{DD}p^R_{CD}}), \\ J^R &= \frac{1}{4}\log(\frac{p^R_{CC}p^R_{DD}}{p^R_{CD}p^R_{DC}}). \end{aligned} \tag{5.8}$$

These relations are general, from which it follows that any probability distribution from Battle of the Sexes can be mapped to a unique corresponding Ising model. These general relations are in agreement with the results of Kevin Peters [14], who made the mapping for some of the equilibria. The advantage of these results is that any mapping can be made.

Analysis per Nash equilibrium

The different Nash Equilibria in two-agent Battle of the Sexes as well as the Pure Coordination game are mapped to the corresponding Ising models. Since we are mainly interested in the correlated equilibria, these are the ones discussed here.

The 'class 3' equilibria seem to result in identical functions on the p_{CC}-p_{DD} plane, as was already noticed by Peters [14]. However, it turns out that this is not the case for the other equilibria, as is clear from the following discussion. First of all, why do these equilibria result in the same function? This can be explained by the observation that in these equilibria, the values of the correlation device are 'swapped' in the process of renormalization, but not changed in magnitude. To illustrate this, consider response strategy $(0,0,1,1)$. In this case, $p^R_{CC} = p_{DC}$, $p^R_{CD} = p_{DD}$, $p^R_{DC} = p_{CC}$ and $p^R_{DD} = p_{CD}$. When substituting these values in the expression for J in Equation 5.8, the following expression is obtained:

$$J^R = \frac{1}{4}\log(\frac{p_{DC}p_{CD}}{p_{DD}p_{CC}}). \tag{5.9}$$

For strategy $(1,1,0,0)$, exactly the same equation would be obtained. Response strategies $(0,0,0,0)$ and $(1,1,1,1)$ would both result in the expression:

$$J^R = \frac{1}{4}\log(\frac{p_{CC}p_{DD}}{p_{CD}p_{DC}}). \tag{5.10}$$

It is clear that these are equal up to a minus sign. Because the region in the correlation device phase space where these $(1,1,0,0)$ and $(0,0,1,1)$ are optimal has the constraint that $p_{CC}p_{DD} < p_{CD}p_{DC}$, it can be concluded that in all these expressions the interactions are positive; this is what we would expect in a coordination game.

Similar analysis shows why, with appropriate choice of γ_i, the function for B is governed by the same function for all four response probabilities. For example for response strategy $(1,1,0,0)$ we see that

$$\gamma_1 B^R = \frac{1}{4}\log(\frac{p_{CD}p_{CC}}{p_{DC}p_{DD}}), \tag{5.11}$$

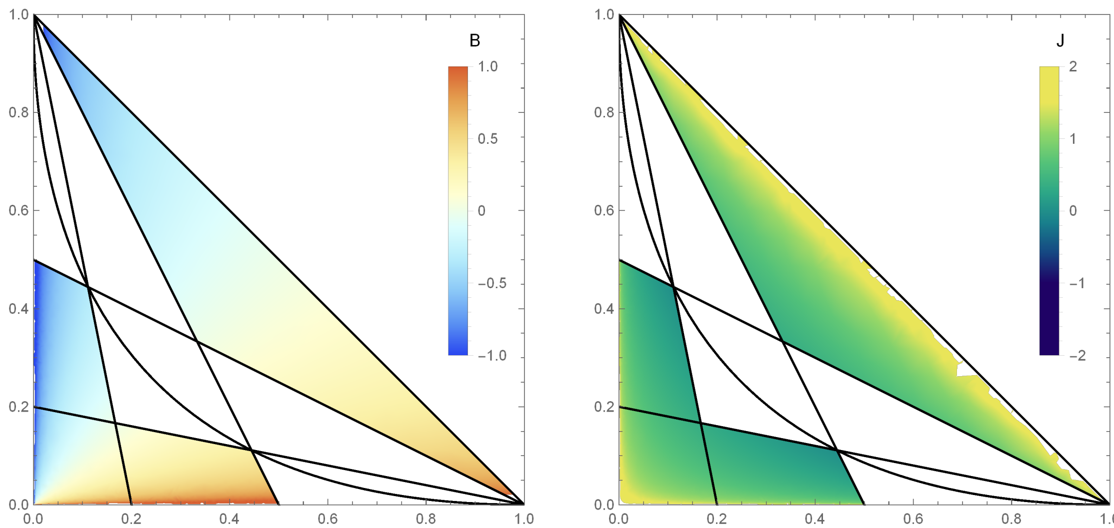
which is exactly equal to the function for response strategy $(1,1,1,1)$. However, now

$$\gamma_2 B^R = \frac{1}{4}\log(\frac{p_{CD}p_{DD}}{p_{DC}p_{CC}}), \tag{5.12}$$

which differs by a minus sign from the results for $(1,1,1,1)$. This minus sign can be interpreted as a different coupling to the magnetic field. With this logic, it was found that the following set of γ_i give consistent results:

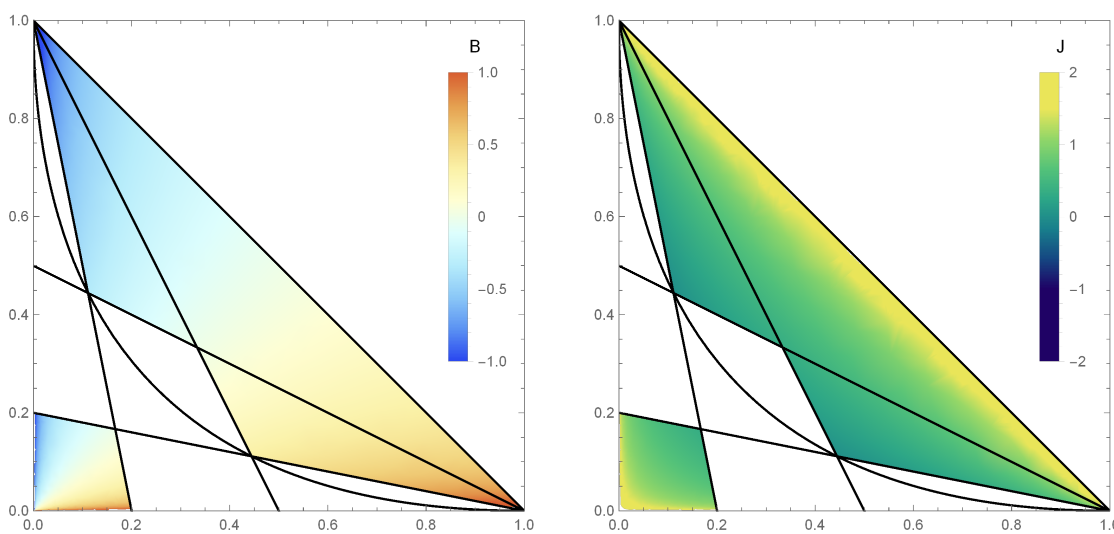
$$\begin{aligned}
(1,1,1,1) \rightarrow &\quad \gamma_1 = 1 \quad \wedge \quad \gamma_2 = 1,\\
(0,0,0,0) \rightarrow &\quad \gamma_1 = -1 \quad \wedge \quad \gamma_2 = -1,\\
(1,1,0,0) \rightarrow &\quad \gamma_1 = 1 \quad \wedge \quad \gamma_2 = -1,\\
(0,0,1,1) \rightarrow &\quad \gamma_1 = -1 \quad \wedge \quad \gamma_2 = 1.
\end{aligned} \tag{5.13}$$

With these values for the coupling, the equations for B^R and J^R are plotted in Figure 5.1 for Battle of the Sexes, and in Figure 5.2 for the Pure Coordination game.



(a) **(b)**

Figure 5.1: In this figure, the Ising parameters for the 'class 3' equilibria in Battle of the Sexes are shown, for $S = .5$. In **a):** the magnetic field strength is shown and in **b):** the interaction strength is shown. The colours indicate the strength of the Ising parameters, with the absolute values shown in the legend. In the different equilibria, the coupling to the magnetic field is different.



(a) **(b)**

Figure 5.2: In this figure, the Ising parameters for the 'class 3' equilibria in the Pure Coordination game are shown, for $S = .5$. In **a):** the magnetic field strength is shown and in **b):** the interaction strength is shown. The colours indicate the strength of the Ising parameters, with the absolute values shown in the legend. In the different equilibria, the coupling to the magnetic field is different.

These expressions and figures look very logical and simple. However, when considering the 'class 4' Nash equilibria it turns out that that the results are more complicated. This is because unlike in the 'class 3' equilibria, where the initial correlation device indices are 'shuffled' by the renormalization, in these cases the values change completely. The values of J^R are shown for six examples of equilibrium regions in Figure 5.3. In Figure 5.3a, the results for $(1, P_{FD}^{1*}, 1, P_{FD}^{2*})$ are shown. It can be seen that the values of J^R are in general lower than for the previously considered cases, which reflects that the expected outcome of these unstable equilibria are sub-optimal. In Fig 5.3b, the results for $(P_{FC}^{1*}, 1, P_{FC}^{2*}, 1)$ are shown, which are mirrored in the diagonal compared to Figure 5.3a. Where the domains meet, the values exactly match up with the results of $(1, P_{FD}^{1*}, P_{FC}^{2*}, 1)$, which are shown in Figure 5.3c. The remaining three examples in the figure are $(P_{FC}^{1*}, 0, P_{FC}^{2*}, 1)$, which is shown in Figure 5.6d, $(P_{FC}^{1*}, 1, P_{FC}^{2*}, 0)$, which is shown in Figure 5.3e and $(P_{FC}^{1*}, 1, 0, P_{FD}^{2*})$, which is shown in Figure 5.3f.

Similarly, plots can be made for the magnetic field. However, no convenient values for γ_i were found. Therefore the results, which are shown for the same regions as before in Figures 5.4 and 5.5, are shown in terms of $\gamma_i B^R$. The main difference compared to the 'class 3' results, is that now the renormalized values are completely different from the correlation device values. Therefore a coupling strength of ± 1 cannot be the case. The coupling is therefore some non-trivial expression, which also seems to be a function of the coordinates in the correlation device phase space. It might be possible that there is a convenient way to do make sense of this after all, but this was not found.

This entire discussion was mainly focused on Battle of the Sexes, but of course similar reasoning is valid for the Pure Coordination game. The choice was made not to reproduce every graph made for Battle of the Sexes, since this would not add much to the understanding.

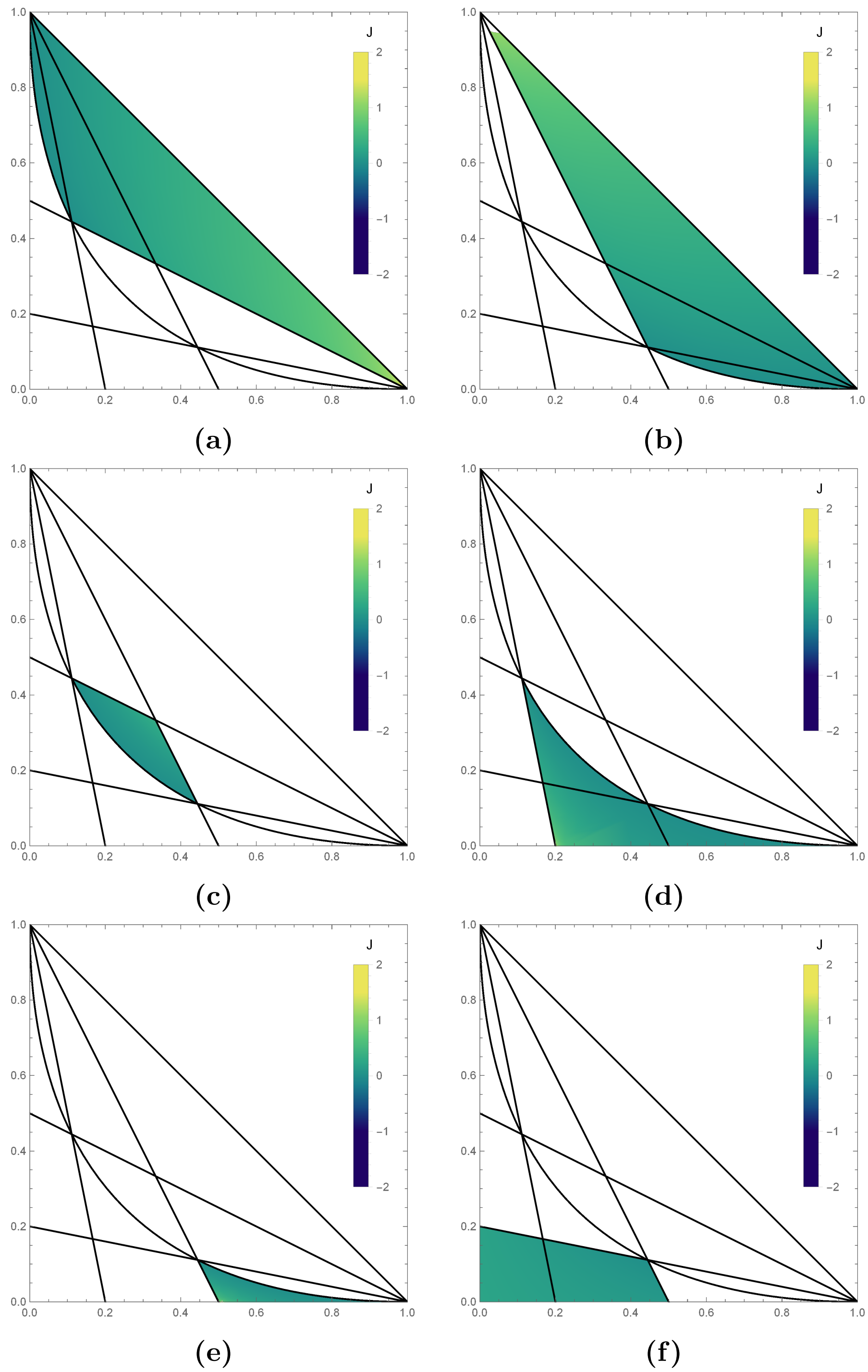


Figure 5.3: The Ising parameter value J^R in the 'class 4' equilibrium regions for Battle of the Sexes, for $S = .5$. The colour shows the value of this parameter. **a):** $(1, P^{1*}_{FD}, 1, P^{2*}_{FD})$, **b):** $(P^{1*}_{FC}, 1, P^{2*}_{FC}, 1)$, **c):** $(1, P^{1*}_{FD}, P^{2*}_{FC}, 1)$, **d):** $(P^{1*}_{FC}, 0, P^{2*}_{FC}, 1)$, **e):** $(P^{1*}_{FC}, 1, P^{2*}_{FC}, 0)$ and **f):** $(P^{1*}_{FC}, 1, 0, P^{2*}_{FD})$.

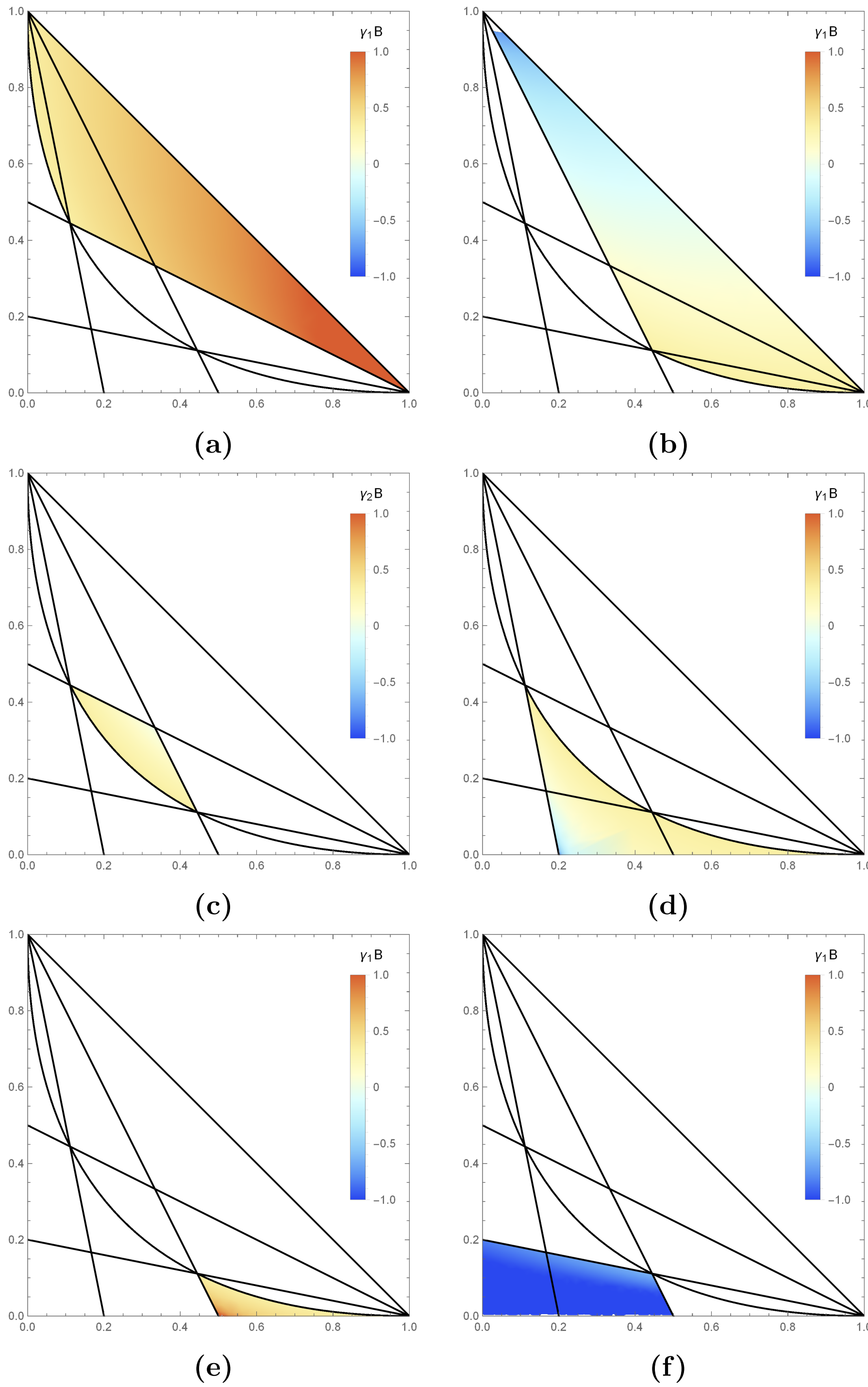


Figure 5.4: The Ising parameter value $\gamma_1 B^R$ in the 'class 4' equilibrium regions for Battle of the Sexes, for $S = .5$. The colour shows the value of this parameter. **a):** $(1, P^{1*}_{FD}, 1, P^{2*}_{FD})$, **b):** $(P^{1*}_{FC}, 1, P^{2*}_{FC}, 1)$, **c):** $(1, P^{1*}_{FD}, P^{2*}_{FC}, 1)$, **d):** $(P^{1*}_{FC}, 0, P^{2*}_{FC}, 1)$, **e):** $(P^{1*}_{FC}, 1, P^{2*}_{FC}, 0)$ and **f):** $(P^{1*}_{FC}, 1, 0, P^{2*}_{FD})$.

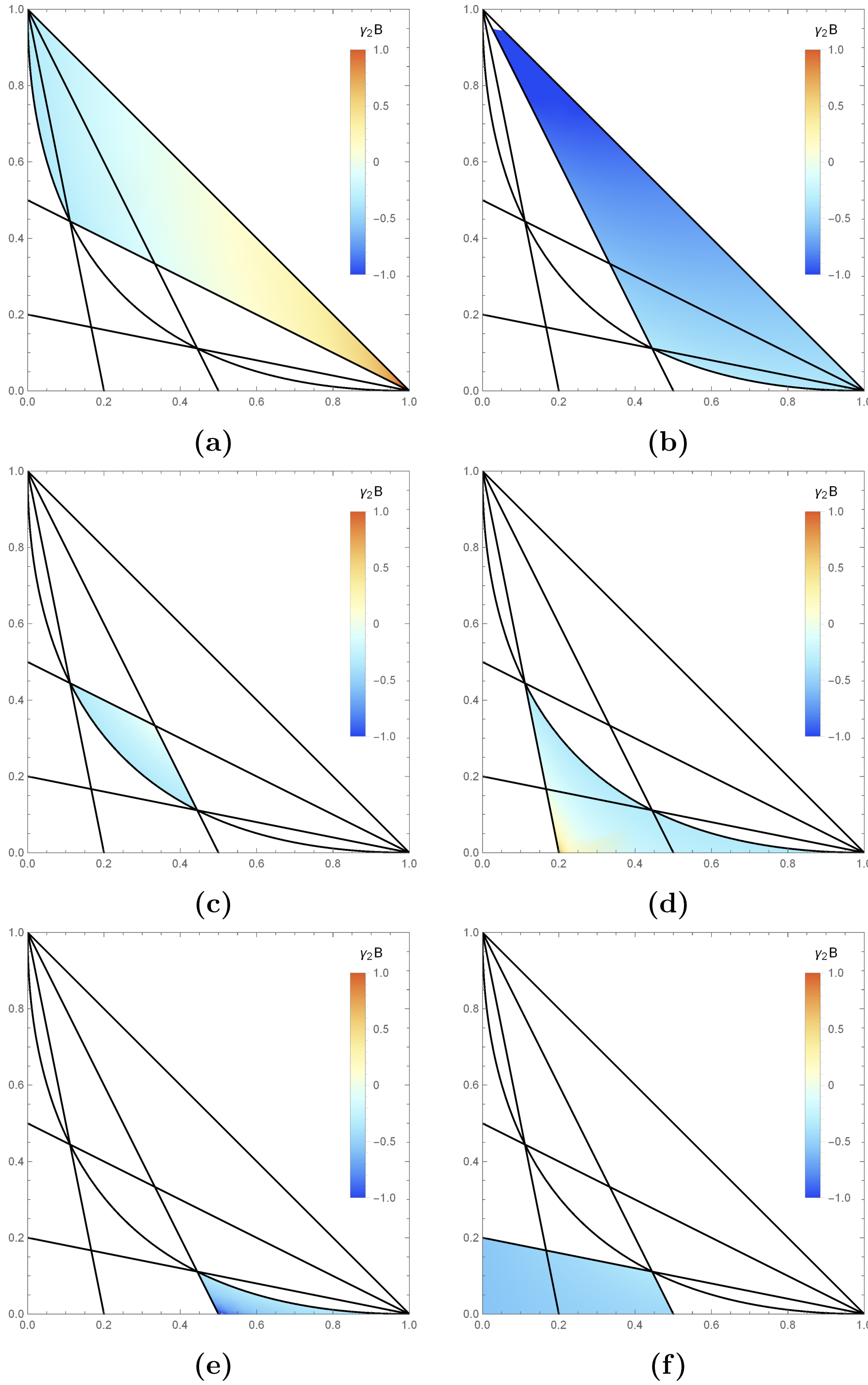


Figure 5.5: The Ising parameter value $\gamma_2 B^R$ in the 'class 4' equilibrium regions for Battle of the Sexes, for $S = .5$. The colour shows the value of this parameter. **a):** $(1, P^{1*}_{FD}, 1, P^{2*}_{FD})$, **b):** $(P^{1*}_{FC}, 1, P^{2*}_{FC}, 1)$, **c):** $(1, P^{1*}_{FD}, P^{2*}_{FC}, 1)$, **d):** $(P^{1*}_{FC}, 0, P^{2*}_{FC}, 1)$, **e):** $(P^{1*}_{FC}, 1, P^{2*}_{FC}, 0)$ and **f):** $(P^{1*}_{FC}, 1, 0, P^{2*}_{FD})$.

5.3 Mapping for three agents

In this section, the corresponding Ising models for all three-agent networks are discussed. This already is somewhat less straightforward than the two-agent case, since there are multiple possible network structures where the different links may or may not be realized. However, the first step once more is to establish a general relation between any three-agent probability distribution and the corresponding Ising model. Seven degrees of freedom are required to make this work, which are obtained with one degree of freedom ($\gamma_i B$) per agent to couple to the magnetic field, one interaction term $J_{i,j}$ between each pair of agents and a three-body interaction term A. Introducing the three-body interactions makes the Hamiltonian deviate from the form introduced in Equation 5.2:

$$H(s) = -\frac{1}{2}\sum_{i,j} J_{ij}s_i s_j - B\sum_i \gamma_i s_i - A s_1 s_2 s_3. \tag{5.14}$$

The goal is now to find a general solution to the relation

$$p_{\mu\nu\rho} = \frac{1}{Z}e^{-H_{\mu\nu\rho}}, \tag{5.15}$$

in which the Hamiltonian, written out explicitly, consists of the components

$$\begin{aligned}
H_{CCC} &= -J_{12} - J_{23} - J_{13} - \gamma_1 B - \gamma_2 B - \gamma_3 B - A,\\
H_{CCD} &= -J_{12} + J_{23} + J_{13} - \gamma_1 B - \gamma_2 B + \gamma_3 B + A,\\
H_{CDC} &= J_{12} + J_{23} - J_{13} - \gamma_1 B + \gamma_2 B - \gamma_3 B + A,\\
H_{DCC} &= J_{12} - J_{23} + J_{13} + \gamma_1 B - \gamma_2 B - \gamma_3 B + A,\\
H_{CDD} &= J_{12} - J_{23} + J_{13} - \gamma_1 B + \gamma_2 B + \gamma_3 B - A,\\
H_{DCD} &= J_{12} + J_{23} - J_{13} + \gamma_1 B - \gamma_2 B + \gamma_3 B - A,\\
H_{DDC} &= -J_{12} + J_{23} + J_{13} + \gamma_1 B + \gamma_2 B - \gamma_3 B - A,\\
H_{DDD} &= -J_{12} - J_{23} - J_{13} + \gamma_1 B + \gamma_2 B + \gamma_3 B + A.
\end{aligned} \tag{5.16}$$

The general solution of Equation 5.15 in terms of the probabilities was found to be

$$
\begin{aligned}
\gamma_1 B &= \frac{1}{8}\log(\frac{p_{CCC}p_{CDC}}{p_{DDD}p_{DCD}}\frac{p_{CCD}p_{CDD}}{p_{DDC}p_{CDD}}),\\
\gamma_2 B &= \frac{1}{8}\log(\frac{p_{CCC}p_{DCD}}{p_{DDD}p_{CDC}}\frac{p_{CCD}p_{DCC}}{p_{DDC}p_{CDD}}),\\
\gamma_3 B &= \frac{1}{8}\log(\frac{p_{CCC}p_{CDC}}{p_{DDD}p_{DCD}}\frac{p_{DDC}p_{CDD}}{p_{CCD}p_{DCC}}),\\
J_{12} &= \frac{1}{8}\log(\frac{p_{DDD}p_{CCC}}{p_{CDC}p_{DCD}}\frac{p_{CCD}p_{DDC}}{p_{DCC}p_{CDD}}),\\
J_{23} &= \frac{1}{8}\log(\frac{p_{DDD}p_{CCC}}{p_{CDC}p_{DCD}}\frac{p_{DCC}p_{CDD}}{p_{CCD}p_{DDC}}),\\
J_{13} &= \frac{1}{8}\log(\frac{p_{CCC}p_{DDD}}{p_{DDC}p_{CCD}}\frac{p_{CDC}p_{DCD}}{p_{DCC}p_{CDD}}),\\
A &= \frac{1}{8}\log(\frac{p_{CCC}p_{DCD}}{p_{DCD}p_{CDC}}\frac{p_{DDC}p_{CDD}}{p_{CCD}p_{DCC}}).
\end{aligned}
\tag{5.17}
$$

The mapping for the renormalized results of course has exactly the same form.

General analysis

In this section, some general analysis of the previously found results is provided. First of all, it is important to note that the mapping is independent of the specific network structure of the game-theoretical model. Therefore, there could be situations in which there is no link in the network between two agents in the BoS case, but that there is an interaction between those agents in the Ising case. Also, considering the three-agent BoS results, it becomes clear that there are cases where the same Ising model corresponds to a Nash equilibrium in one network structure, whereas it is not a Nash equilibrium in some other network structure. As a concrete example, consider Figure 4.3. From the equilibrium analysis it became clear that the examples in the subfigures are equilibrium situations for some network structures, but adding or removing a link between the agents changes that. Still, since the renormalized outcomes are indistinguishable, they all map to the same Ising model. These complications could be seen as a sign that the optimization process in the Ising model is quite different from the optimization process in game theory; it is always possible to define a mapping which gives the same results, but the underlying systems which produce these results are so different that they are hardly comparable anymore. Still, the Ising model might be useful to gain new insight in the system. To better clarify what exactly happens in this mapping and renormalization process, some analysis is included in the rest of this section.

In the mapping $H_{\mu\nu\rho} \to H^R_{\mu\nu\rho}$, the initial parameters renormalize to observable values. It is therefore interesting to investigate what happens to a Hamiltonian which starts within a certain sub-space of the possible values upon renormalization. Does it stay in that subspace? Or does the renormalization process mix it up with the entire phase space? To investigate this, the following examples of sub-spaces for the correlation device are worked out:

1. There are no initial correlations (all J's and A are zero).
2. There are initial correlations between one pair (two J's and A are zero).
3. There are no three-body interactions (A equals zero).

When all J's and A are zero, there are no initial correlations in the system, and hence $p_{\mu\nu\rho} = p^1_\mu p^2_\nu p^3_\rho$. In this equation, p^i_μ corresponds to the probability that agent i plays μ. To clarify why this must be the case in the Ising model language, observe that

$$p^i_\mu = \frac{e^{\gamma_i B}}{e^{\gamma_i B} + e^{-\gamma_i B}}. \tag{5.18}$$

It is easy to show that multiplying these gives the full Hamiltonian of the system where all J's and A are zero:

$$p^1_\mu p^2_\nu p^3_\rho = \frac{e^{-H_{\mu\nu\rho}}}{Z}. \tag{5.19}$$

Similarly, when substituting $p_{\mu\nu\rho} = p^1_\mu p^2_\nu p^3_\rho$ in Equation 5.17 it directly follows that all J's and A must be zero.

So what happens when we renormalize these values?

$$p^R_{\mu\nu\rho} = \sum_{\mu'\nu'\rho'} P_{\mu\leftarrow\mu'} P_{\mu\leftarrow\nu'} P_{\mu\leftarrow\rho'} p_\mu p_\nu p_\rho. \tag{5.20}$$

By explicit calculation, it follows that, for any possible renormalization, the resulting expression is uncorrelated as well:

$$p^R_{\mu\nu\rho} = p^R_\mu p^R_\nu p^R_\rho. \tag{5.21}$$

The expressions in these equation are defined as

$$p^R_\mu = \sum_{\mu'} P_{\mu\leftarrow\mu'} p_{\mu'}. \tag{5.22}$$

From this result, it indeed immediately follows that the renormalized J's and B's are zero by substituting the expressions in the renormalized form of Equation 5.17. Although none of this analysis is for the Ising model is particularly new, it is very useful to have proven that the process of renormalization cannot introduce correlations in the system.

The case where there is one source of correlations in the correlation device (either J_{12}, J_{23} or J_{13}) without three-body interactions is discussed now. For the analysis the situation where J_{12} is non-zero is chosen, but exactly the same reasoning applies when there are correlations on one of the other links. In this situation, the correlation device does not correlate the third agent to the other two, which manifests itself in the product separability

$$p_{\mu\nu\rho} = p_{\mu\nu} p_\rho. \tag{5.23}$$

With some tedious calculations it be shown that renormalizing this cannot create correlations between the agent three and the other two:

$$p^R_{\mu\nu\rho} = \sum_{\mu'\nu'\rho'} P_{\mu\leftarrow\mu'} P_{\mu\leftarrow\nu'} P_{\mu\leftarrow\rho'} p_{\mu\nu} p_\rho = p^R_{\mu\nu} p^R_\rho. \tag{5.24}$$

From this, a simple substitution in Equation 5.17 shows that the renormalization process does not introduce a non-zero J_{23}, J_{13} or A, and therefore the systems stays in the subspace where the only correlations are between agent 1 and agent 2.

As a last example, let's consider the situations where there are correlations between each of the agents. It turns out that this becomes a very complicated system; even if the correlation device does not have three body interactions in its Ising representation, these interactions are in general present in the renormalized system. This seems to be mainly due to the fact that the optimization process is different in the Ising model compared to the game theory model, which makes the mapping somewhat arbitrary. The result is that even a simple situation in game theory can map into a difficult Ising model. As an example of how the response probabilities 'turn on' the Ising parameters, even if they were not present initially, Figure 5.6 is included. In this figure, the values of the three J's and A are shown as the response probability P^1_{FC} is varied, while the other response probabilities are all equal to 1. The correlation device was designed such that $B_1 = B_2 = B_3 = J_{12} = J_{13} = 1$, $J_{23} = A = 0$. As can be seen, when P^1_{FC} is smaller than 1, the parameters which were first zero do gain a finite value. This illustrates that the system does not remain in a subspace by the process of renormalization. Also, these figures nicely illustrate that the process of renormalization cannot increase the correlations in the system. This property becomes clear indirectly because the interaction strengths J_{12} and J_{13} only decrease by 'turning off' P^1_{FC}.

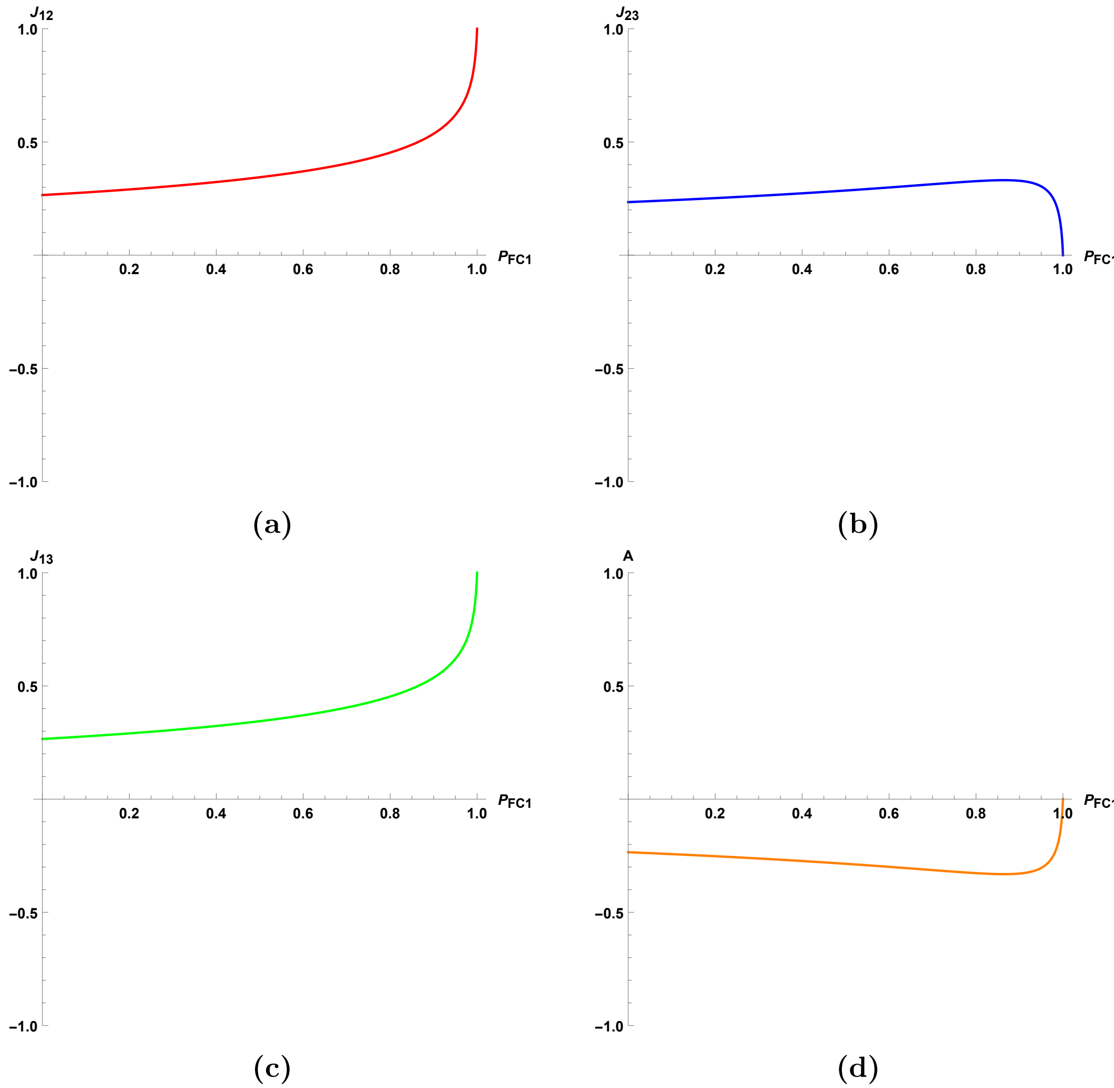


Figure 5.6: These sub-figures show the values of **a)**: J^R_{12}, **b)**: J^R_{23}, **c)**: J^R_{12} and **d)**: A as a function of P^1_{FC}, when all other response probabilities equal 1. These are the results for $B_1 = B_2 = B_3 = J_{12} = J_{13} = 1$, $J_{23} = A = 0$; in terms of the correlation device this implies that $(p_{CCC}, p_{CCD}, p_{CDC}, p_{DCC}, p_{CDD}, p_{DCD}, p_{DDC}, p_{DDD}) \approx (0.955, 0.0175, 0.0175, 0.002, 0.000, 0.002, 0.002, 0.002)$.

Chapter 6

Conclusion

The results of this thesis contribute to a better understanding of correlated equilibria in response strategy as well as the correspondence between the Ising model and coordination games. For two-agent Battle of the Sexes as well as the Pure Coordination game, the equilibrium solutions have been mapped out for the entire correlation device phase space. This has lead to some interesting results. First of all, the observation that there are four different equilibrium mappings to each possible outcome is important for understanding the underlying logic of the game. If we consider the correlation device to be 'real', then for each possible correlation device the equilibrium strategies are important. If we merely want to be able to describe all possible observable outcome, this degeneracy of the outcomes suggests that it is possible to focus attention to part of the correlation device space without loss of generality. Also, the analysis of the stability of the different Nash equilibria shows that not all equilibrium solutions are stable under infinitesimal deviations. Particularly in iterative variants of the game, it follows that the unstable ones are most likely not observed. These insights resulted in a classification of the different equilibrium strategies based on whether they are stable or not and whether or not they are accessible from any correlation device. These insights proved to be useful for the analysis of three-agent systems, since the entire phase space here already is enormous. When focusing on that part of the phase space where the correlation device positively correlated each link, the stable Nash equilibria were mapped out in full. This analysis turned out to be very much alike the two-agent system, with the important difference that, if an agent is connected to two other agents, the equilibrium conditions are determined by averaging over the two contributions. This insight suggests a geometric representation from which it is convenient to visualize whether or not a strategy is an equilibrium solution given the correlation device. Of essence in the success of the analysis of this system was the fact that the full correlation device can in fact be rewritten in terms of effective correlation devices between each pair of agents. Of the 'class 4' correlated equilibria, which are shown to be unstable, one example was worked out in detail. From this analysis it is clear that quite strict conditions need to be obeyed for such equilibria to be possible. These conditions need to be met since, whenever an agent is connected to two or more others, it becomes complicated to make everyone indifferent between outcomes at the same time. This is another reason to suspect that these types of equilibria are not likely to be realized in real situations. A last contribution was the definition of a mapping from any two- or three-agent network to a unique Ising model. Although this general mapping gave some useful insights, it also made clear that this mapping is not as straightforward as one might have hoped. First of all, it is not possible to make a general mapping from games on networks to an Ising model with the same network structure; it is necessary to use a fully connected Ising model (with three-body interactions) to describe game-theoretical networks which are not fully connected. Also, the Ising model can't determine what is a Nash equilibrium, since the Hamiltonian does not contain the necessary information to do so. This manifests itself in the observation that different models in game theory, each with their own equilibrium conditions, map to the same Ising model. For the reasons stated above, it is to us unclear how the Ising model mapping in the form as used in this thesis can concretely be helpful in solving problems in game theory.

There are many interesting open questions as well as possible directions for future research. First of all, it would be very interesting to work out more complex structures within the restriction of positively correlating devices and stable equilibria. The insight that 'whenever an agent has two connections, the equilibrium conditions are determined by the average of the two contributions' can be generalized to more complex systems. With the techniques introduced in Chapter 4, it is possible to analyze as well as plot the result for systems with more agents without too much extra effort. This in turn could be used to test the validity of the theory against results from either simulations or experiments. Another interesting angle would be to define a dynamic theory to describe the path toward an equilibrium. Since agents base their expectations on previous experience, there is clearly a path dependence. One could aim to formulate a theory which, from an initial uncorrelated situation, shows how the agents, with limited information, in each round make the best decisions, and thereby eventually end up in one of the fully correlated states. We believe that the response strategy of correlated games would be a suitable starting point for such a dynamic theory, because for any possible situation with limited information, which is contained in the device, the best responses to the strategies of the other agents can be calculated. Besides these possibilities to directly build on the results of this thesis, there are also still research gaps in working out the two-agent response dynamics for payoff matrices which have not been considered yet. Concerning the Ising model analogy, it would be worth trying to understand better how exactly the mapping works; even though the exact mapping was defined and analyzed, there are still questions concerning what precisely is going on. It is certain however that a mapping to the Ising model by no means is a simple solution to obtain results for complicated models in game theory. It can for example be expected that a mapping for four agents has to include four-body interactions, which would complicate the matter even more. However, since the mapping for four agents was not yet established in this thesis, this is still an open question.

Bibliography

[1] C. Adami and A. Hintze. “Thermodynamics of evolutionary games”. In: *Physical Review E* 97.6 (2018).

[2] R. Aumann. “Subjectivity and correlations in randomized strategies”. In: *Journal of Mathematical Economics* 1.1 (1974), pp. 67–96.

[3] R. J. Baxter. *Exactly Solved Model in Statistical Mechanics*. Dover Publications, 1989.

[4] J. Broere et al. “An experimental study of network effects on coordination in asymmetric games”. In: *Scientific Reports* 9 (2019).

[5] J. Broere et al. “Network effects on coordination in asymmetric games”. In: *Scientific Reports* 7.1 (2017), pp. 1–9.

[6] A.D. Correia and H.T.C. Stoof. “Nash Equilibria in the Response Strategy of Correlated Games”. In: *Scientific Reports* 9.1 (2019).

[7] D. Fudenberg and J. Tirole. *Game Theory*. The MIT Press, 1991.

[8] E. Ising. “Beitrag zur Theorie des Ferromagnetismus”. In: *Zeitschrift für Physik* 31.1 (1925), pp. 253–258.

[9] E. Kohlherg and J. Mertens. “On the Strategic Stability of Equilibria”. In: *Game and Economic Theory: Selected Contributions in Honor of Robert J. Aumann* 54.5 (1995), p. 47.

[10] D.M. Kreps. *Game theory and economic modelling*. Oxford University Press, 1990.

[11] J. Morrow. *Game theory for political scientists*. Princeton University Press, 1994.

[12] J.F. Nash. “Equilibrium points in N person games”. In: *Proceedings of the National Academy of Sciences* 36.1 (1950), pp. 48–49.

[13] J. von Neumann. “Zur Theorie der Gesellschaftsspiele”. In: *Mathematische Annalen* 100.1 (1928), pp. 295–320.

[14] K.J.H. Peters and H.T.C. Stoof. “A Response Strategy for the Battle of the Sexes Game with Intrinsic Correlations Author :” in: *Master Thesis at Utrecht University.* (2018).

[15] E. Rasmusen. *Games and Information: An Introduction to Game Theory*. 2007.

[16] L. Samuelson. "Game theory in economics and beyond". In: *Voprosy Ekonomiki* 30.5 (2017), pp. 89–115.

[17] L. Samuelson. "Stochastic Stability in Games with Alternative Best Replies". In: *Elsevier* 64.1 (1994), pp. 35–65.

[18] J. Smith and G. Price. "The logic of animal conflict". In: *Nature* 246 (1973), pp. 15–18.