دانشگاه صنعتی قم

Impact Mechanics

Plasticity Theory

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- Constitutive material models
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- Elementary theory of plastic bending
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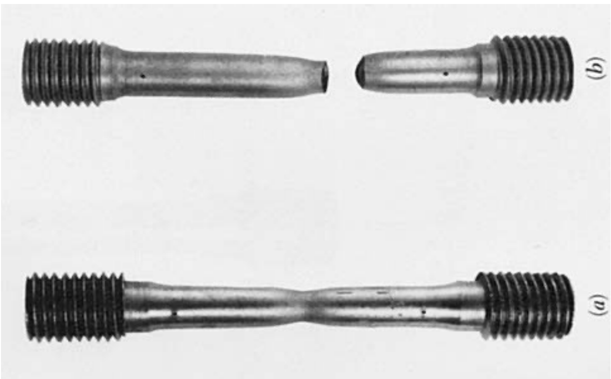
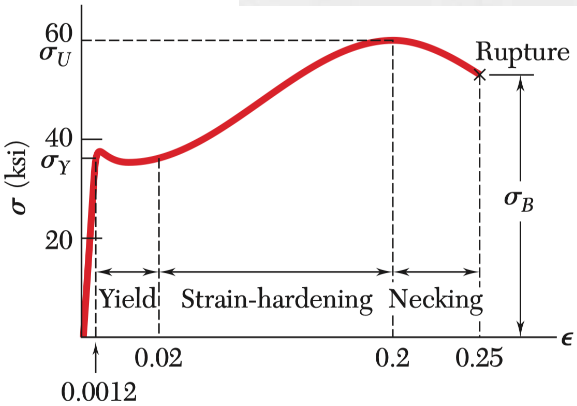
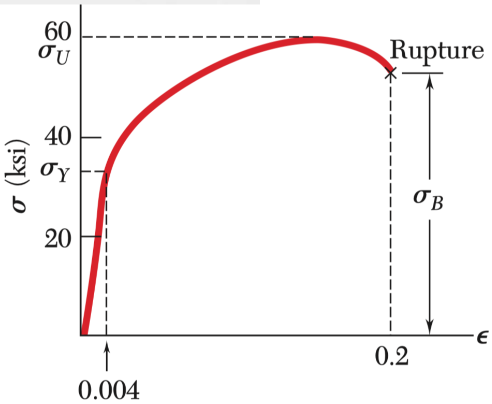
Tensile Stress, Strain and Strain-rate

Tension Test





(*a*) Low-carbon steel



(*b*) Aluminum alloy

Volume Change

- A unit cube of material is subject to the principal stress σ_1.
- Volume of the cube:

$$V = (1 + e_1)(1 + e_2)(1 + e_3)$$

- Change in volume:

$$\Delta V = e_1 + e_2 + e_3 = \sigma_1(1 - 2\nu)/E$$

- For $\nu = 1/3$, and $\sigma_1 = Y$, $\Delta V = Y/3E$.
  - Thus for typical elastic stress conditions, changes in volume are certainly very small—less than 0.001
- Measurements show that in the tensile test, the more plastic does the specimen become the more nearly does ν approach 0.5.

Volume Change

- So, for an incompressible material,

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$$

- For a solid cylinder:

$$\frac{\pi}{4} \cdot D_0^2 L_0 = \frac{\pi}{4} D^2 L$$

$$2 \ln D/D_0 + \ln L/L_0 = 0$$

$$2\varepsilon_\theta + \varepsilon_l = 0$$

- Annulus cylinder:

$$\pi D_0 \cdot (dD_0) h_0 = \pi D \cdot (dD) \cdot h$$

$$\ln D/D_0 + \ln(dD)/(dD_0) + \ln h/h_0 = 0$$

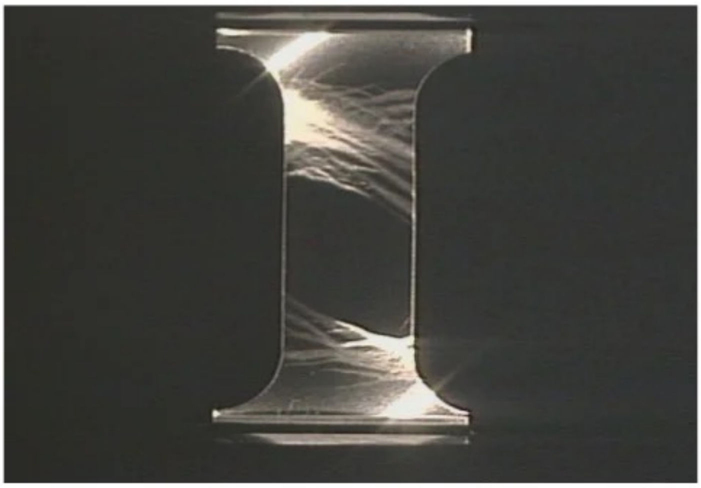
$$\varepsilon_\theta + \varepsilon_r + \varepsilon_z = 0$$

$$\varepsilon_\theta = \varepsilon_r$$

Upper Yield Point: Luder's Bands

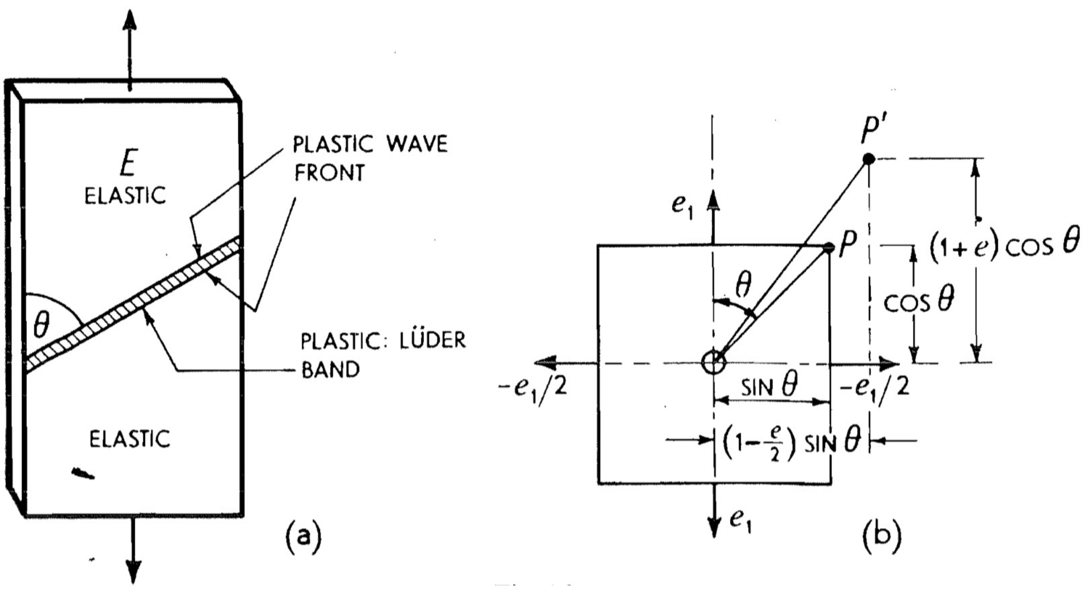
- If annealed mild steel strip is strained just to the upper yield point, then across particular sections, yielding will occur and Luder's bands appear.
- These are visible as grey-black bands inclined at a particular angle to the direction of the tensile stress.
- The material inside a Luder's band is plastic and that on either side of it, elastic.
- The plastic flow is constrained by the two elastic regions and does not allow plastic strain in the direction of the length of the band.
- Thus the direction along which bands lie is that of zero extension in the plane of the strip.

Luder's Bands



Luder's Bands

Luder's Bands



Luder's Bands

- If the principal strain in the direction of the tensile stress is e, that $-e/2$ is the strain in each of the transverse directions.
- Thus, if an element of the strip, OP, is of unit length, then in straining P moves to P', and the condition for zero extension is OP' = OP, so that we have,

$$(1+e)^2 \cdot \cos^2\theta + (1-e/2)^2 \cdot \sin^2\theta = 1$$
$$\tan^2\theta = 2 \text{ or } \theta = 54^\circ 44'$$

  - Terms in e^2 are neglected.
- The Luder bands thus form at , $\sim \pm\ 55°$ to the specimen axis.

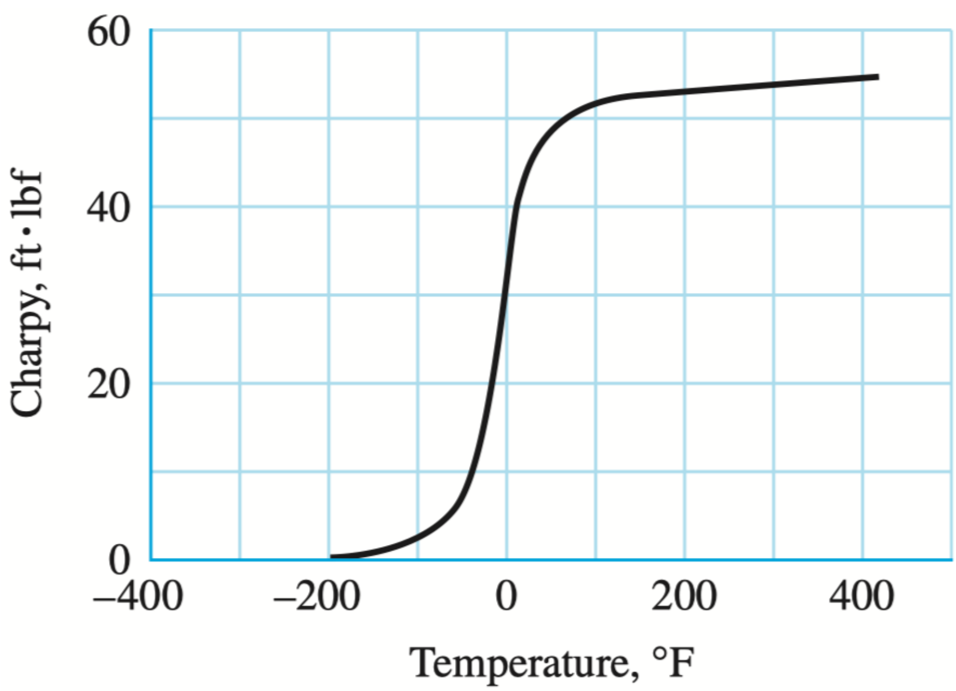
Luder's Bands

- Material inside the bands undergoes a finite amount of strain of about 0.01 whilst neighboring material remains elastic.
- The mixture of elastic and plastic zones alters with increase in extension until eventually the whole specimen becomes plastic.
- As more and more Luder bands form, each to have a strain of about 0.01, the total measured extension of the strip increases.
- On either side of an individual Luder's band is a very thin wave-front of thickness s which advances at an extremely low speed v, and as material is encompassed by it, an increment of strain Δe is undergone.

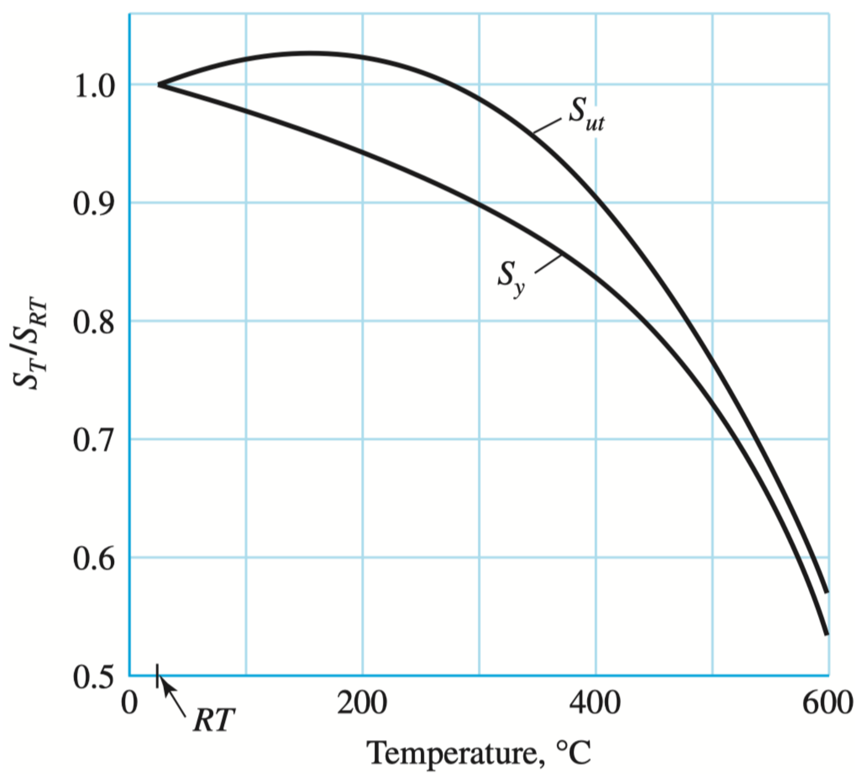
Luder's Bands

- Effectively there is a strain discontinuity at the wave-front or at the head of this plastic wave.
- The strain rate $\dot{e}$ is given by $\dot{e} = \dfrac{\Delta e}{s/v}$
- If $v = 1cm/s$, $\Delta e = 0.01$ and $s = 0.01\ cm$ (i.e. a magnitude typically a grain size in diameter) then the strain rate $\dot{e} = 1\ /s$ which is relatively high for a standard tensile test.
- The difference in magnitude between the upper and lower yield point stress is sensitive to the rate of loading and depends on moving dislocations from 'pinning' nitrogen atoms.

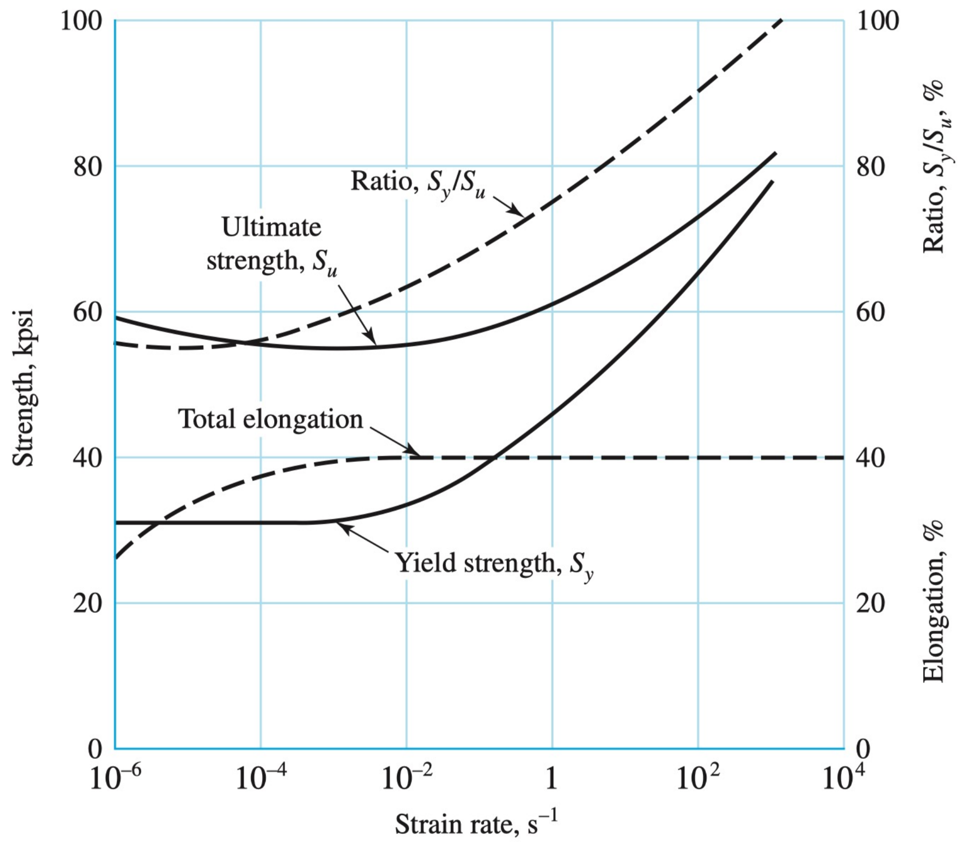
Temperature Effect



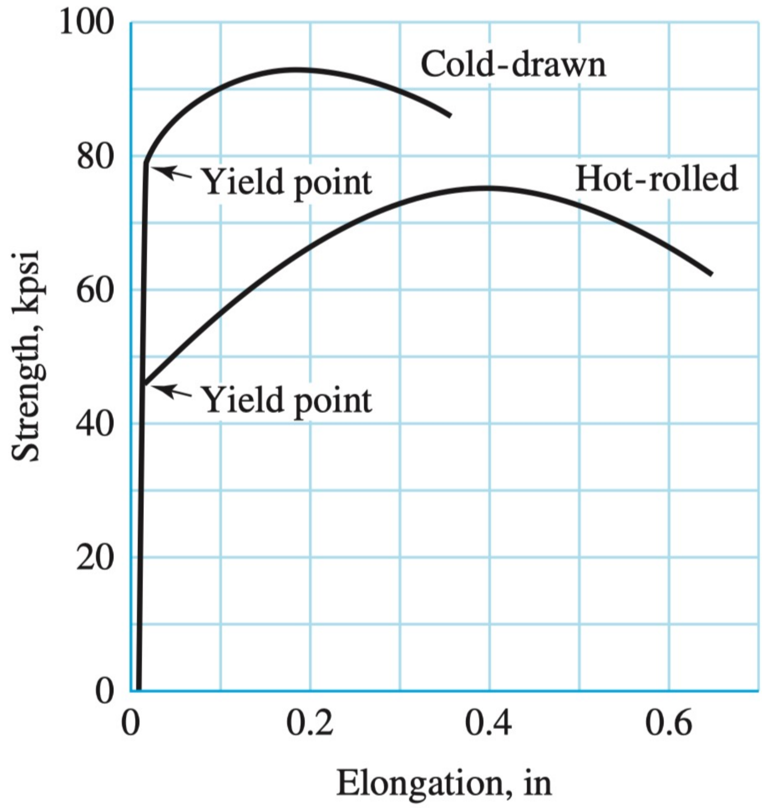
Temperature Effect



Strain Rate Effect



Cold/hot Working



Recrystallisation

- Above a somewhat indefinite temperature called the recrystallisation temperature T_r, a metal may be worked and after a certain small strain, cease to harden with increasing strain.
- A balance would have been established between the tendency to strain-harden and the tendency to soften due to thermal activation.
  - These two competing tendencies are first seriously concurrent over a narrow range of temperature near T_r; below this range the material perceptibly hardens during straining, but above it the yield stress is constant, though highly strain-rate dependent.
- The softening processes are recovery, recrystallisation and grain growth, and they are all thermally activated.

Recrystallisation

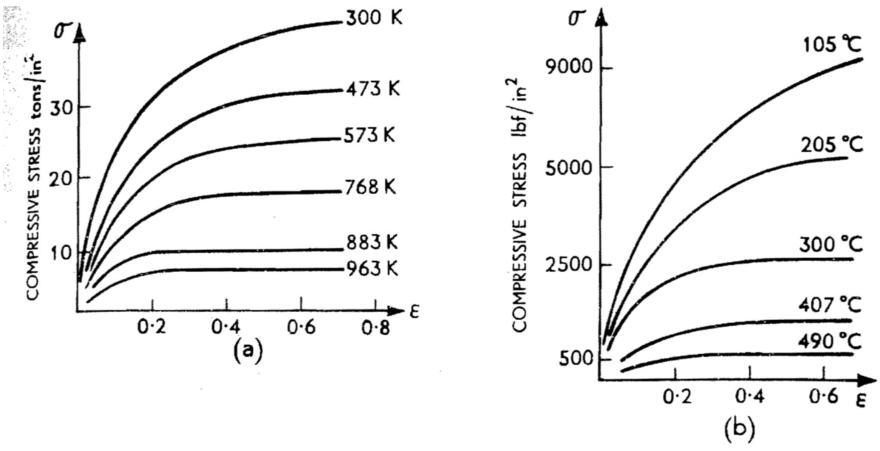
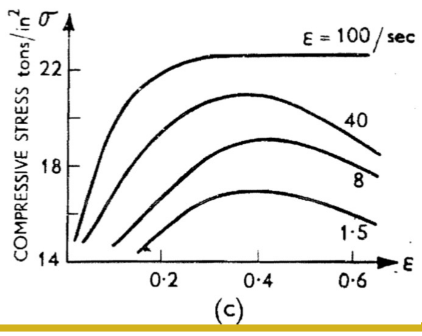


Fig. 4.4 Quasi-static (compression) stress–strain curves at various temperatures and $\dot{\varepsilon} \simeq 10^{-3}$/sec,
(a) copper, and
(b) super pure aluminium.
(c) Showing the compressive stress–strain curve for EN 52 steel at 1000 °C at various rates of strain.



Recrystallisation

- The homologous temperature, T_H:

$$T_H = \frac{\text{Testing temperature } T(K)}{\text{Melting point temperature } T_M(K)}$$

- The recrystallisation temperature is usually found to be between 0.4 and 0.55 but is dependent on the strain and strain-rate.
- Hardness testing is fundamentally related to the stress-strain properties of a material, and thus indentation tests may be conducted for the purpose of illuminating material behavior generally.

Recrystallisation

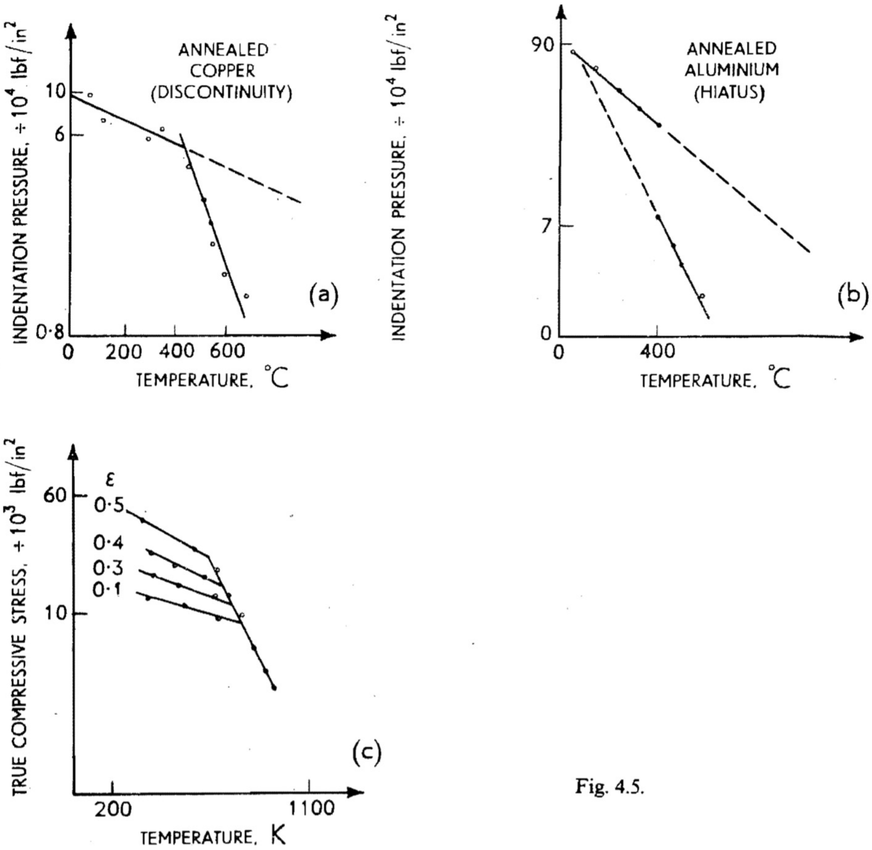


Fig. 4.5.

Recrystallisation

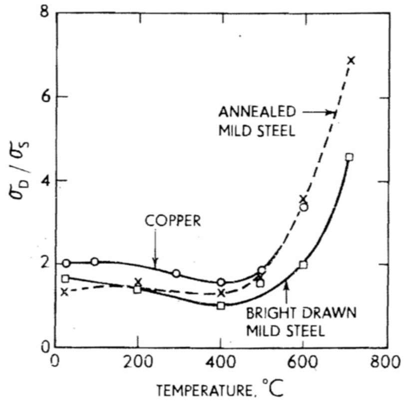
- It is observed that in (a) there is a discontinuity of slope, or as in (b) a hiatus at about the recrystallisation temperature.
- The pivotal role thus played by the recrystallisation temperature is clearly evident.
- A distinction between cold and hot working can be usefully based on the recrystallisation temperature, T_r.
- Cold working occurring if the temperature at which it starts is less than T_r.
- If the temperature of working exceeds T_r; it is hot working.

Dynamic vs. Static Flow Stress

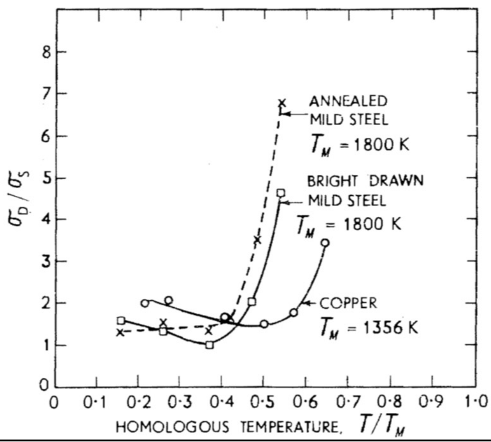
- It is very useful to consider how the ratio of the dynamic, σ_D, to the static, σ_S, simple compressive flow stress of a metal at a specified strain varies with temperature.
- Static flow stress is referred to the strain rate $0.001/s$–a typical slow speed compression test.
- Dynamic flow stress is associated with an impact test, e.g. that due to a falling hammer, which is likely to be about $100/s$.
- Thus σ_D/σ_S refers to a strain-rate ratio of about 10^5.

Dynamic vs. Static Flow Stress

- When strains imposed are large, say between 0.05 and 0.5,
  - (i) strain-rate effects below the recrystallisation temperature are not pronounced and approximately, 1<σ_D/σ_S<2;
  - (ii) above T_r, σ_D/σ_S is very sensitive to strain rate; when T_H~0.6 or 0.7. For many metals the ratio is 10, more or less.



(a)



(b)

Yield Stress

- From the work of Hopkinson in 1905 to that by Taylor, Campbell and others in the 1950's, the conclusion which seems to emerge fairly clearly from quasi-static and impact tension tests (strain rates 10^{-3} to 10^{2} or 10^{3} /sec) is, that the ratio of dynamic to static yield stress in mild steel is in excess of unity, is usually about two and sometimes as high as three.
- The time to yield is the shorter the higher the load; the maximum delay period at $25°C$ is of order $1\ sec$; for a delay period of only $10^{-6}\ sec$ the yield stress may have to be 2-5 times as great as its normal value.

Constitutive Material Models

Constitutive Relations

- Ludwik (1909) proposed a semi-logarithmic dependence of tensile yield strength on strain-rate,

$$\sigma = \sigma_1 + \sigma_0 \ln \dot{\varepsilon} / \dot{\varepsilon_0}$$

- Manjoine and Nadai (1940) found that tensile strength at elevated temperature varied nearly linearly with the logarithm of strain-rate.
- Alder and Phillips (1954) in work on copper at up to 600 °C, on aluminum at up to 500°C and on steel at up to 930 to 1200°C, were best able to summarize their results in the strain-rate range 1 to $40/\sec$ by

$$\sigma = \sigma_0 \cdot \dot{\varepsilon}^n$$

Constitutive Relations

- Macgregor and Fisher (1946) introduced the idea of a "velocity modified temperature", T_m, so that,

$$T_m = T(1 - m \ln \frac{\dot{\varepsilon}}{\dot{\varepsilon}_0})$$

  - This form gives the expected qualitative result that an increase in strain-rate is equivalent to a decrease in temperature.
- Inouye has used the expression,

$$\sigma = \sigma_0 \varepsilon^n \cdot \dot{\varepsilon}^m \exp(A/Tk)$$

- Malvern (1965) introduced an equation of the forms,

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + D\left(\frac{\sigma}{\sigma_0} - 1\right)^p$$

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + A\left[\exp\left(\frac{\sigma}{\sigma_0} - 1\right)^q - 1\right]$$

Constitutive Relations

- Ripperberger (1965) has maintained and shown that his experimental data in plastic wave propagation work can be best summarized by adapting Malvern's equation to,

$$\dot{\varepsilon} = \frac{1}{\tau}\left[\frac{\sigma-\sigma_0(\varepsilon)}{\sigma_0(\varepsilon)}\right]^m$$

  - Where τ is the relaxation time of the material, $\sigma_0(\varepsilon)$ the static stress at strain ε, and thus $\sigma - \sigma_0(\varepsilon)$ is the 'over stress', or the 'excess stress' to which the material is subject at the given strain-rate.
  - For Armco iron (99.85% Fe) $\tau \sim 1.2\ ms$ and $m \sim 1.18$.

Constitutive Relations

- Bell claims to have demonstrated the "wide and remarkable generality" of the following stress-strain function,

$$\sigma = \left(\frac{2}{3}\right)^{r/2} \cdot \mu(0) \cdot B_0\left(1 - \frac{T}{T_M}\right) \cdot e^{1/2}$$

  - where $\mu(0)$ is the zero-point isotropic linear elastic shear modulus; B_0 is a dimensionless universal constant having the value B_0 = 0.0280 and r is an integral index, 1, 2 ... etc.
  - This equation is said to be applicable over the entire temperature scale from 4 °K to within 20° of the melting point of crystalline solids and is known experimentally to apply for strain rates $\dot{e} = 10^{-9}/s$ to $\dot{e} = 10^{4}/s$

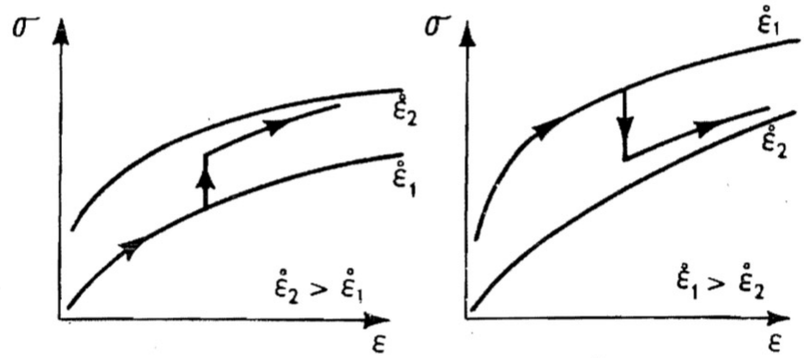
Constitutive Relations

- Bell determines the stress-strain function from,

$$\sigma = \int_0^e \rho_0 \cdot c_p^2(e) \cdot de$$

  - where ρ_0 denotes density and $c_p(e)$ is the plastic wave speed.

- Tests carried out in which, after a given amount of strain ε at a rate of $\dot{\varepsilon}_1$, there is a sudden change in rate of strain to $\dot{\varepsilon}_2$, show the flow stress to be immediately changed also.



Strain-rate vs. Recrystallisation

- Alder and Phillips have shown that the logarithm of compressive stress for up to 50% compression in the range 1 to 40/sec varies with logarithmic strain-rate at specific temperatures, for aluminum, copper and mild steel.

$$\sigma = \sigma_0 \cdot \dot{\varepsilon}^n$$

- The experimental data for compressive strains up to 0.5, shows the exponent n to vary with homologous temperature, thus,
  - For $T_H < 0.55, n \cong 0.055$
  - For $T_H > 0.55, n \cong 0.43$

Power-law Isotropic Plasticity

- This model provides the dynamic yield strength σ_y of the material (Hallquist, 2005) as:

$$\sigma_y = k(\varepsilon_{yp} + \bar{\varepsilon}_p)^n \left[1 + \left(\frac{\dot{\varepsilon}}{C}\right)^{\left(\frac{1}{P}\right)}\right]$$

  - where ε_{yp} is the elastic strain to yield; k is a constitutive coefficient; n is the hardening index, $\overline{\varepsilon_p}$ is the effective (logarithmic) plastic strain; $\dot{\varepsilon}$ is the strain rate; and C and P are the Cowper–Symonds strain rate parameters.
  - This model can be used for thick, and soft metals and alloy steel targets impacted by projectiles at ordnance velocities where thermal softening is small and both elastic and plastic strains and strain rate effects are considerable.

Johnson–Cook Material Model

- The Johnson-Cook (J-C) model was introduced in 1983 and was primarily intended for computational work.
- The yield stress is given by the following expression:

$$\sigma_y = \left[A + B\bar{\varepsilon}_p^n\right]\left[1 + C\ln\left(\dot{\varepsilon}^*\right)\right]\left[1 - \left(T^*\right)^m\right]$$

  - where

$$\dot{\varepsilon}^* = \frac{\dot{\bar{\varepsilon}}_p}{\dot{\varepsilon}_0}; \quad \text{and} \quad T^* = \frac{T - T_r}{T_m - T_r}$$

  - in which A is the initial yield stress, B is the strain hardening coefficient and n is the strain hardening exponent, $\bar{\varepsilon}_p$ is the effective plastic strain, $\dot{\bar{\varepsilon}}_p$ is the effective plastic strain rate, $\dot{\varepsilon}_0$ is the user defined reference strain rate (normally taken as $1.0\ s^{-1}$) and C is the strain rate coefficient.

Determination of Parameters

- The first step in this process is to determine the constants in the first set of brackets.
- At room temperature and for the strain rate of interest, $\dot{\varepsilon}^* = 1.0$, the J-C material model can be written as:

$$\sigma_y = \left[A + B\bar{\varepsilon}_p^n\right]$$

- The flow stress at zero plastic strain i.e., $A = \sigma_0 = \sigma_y$ can now be obtained from an experimental data.
- The quantity $\sigma_y - \sigma_0$ is plotted against plastic strain $\bar{\varepsilon}_p$ on a log-log plot, and applying least squares fit of the experimental data to the power law equation provides the other two material constants B and n.

Determination of Parameters

- In the second step, the J-C's strain rate parameter C is determined from data of σ_y and $\dot{\bar{\varepsilon}}_p$.
- The constitutive equation at a constant temperature and for a constant strain, can be written as:

$$\sigma_y = \sigma_s[1 + C\ln(\dot{\varepsilon}^*)] \quad \text{or} \quad \frac{\sigma_y}{\sigma_c} - 1 = C\ln(\dot{\varepsilon}^*)$$

  - where σ_s is the stress at a given strain rate, $\dot{\varepsilon}^* = 1.0$.
  - For a constant strain, the value of σ_s can be calculated, and $[\sigma_y/\sigma_s$-1] can be plotted against $\dot{\bar{\varepsilon}}_p$ on a semi-log plot.
  - A least squares fit to the data gives us the value of J-C's strain rate parameter C.

Determination of Parameters

- In the third step, the J-C's thermal parameter m is determined from the stress-temperature response of the material.
- At constant strain rate, the constitutive equation can be written as:

$$\sigma_y = \sigma_t[1 - (T^*)^m] \quad \text{or} \quad \frac{\sigma_y}{\sigma_t} = [1 - (T^*)^m]$$

  - where σ_t is the stress at room temperature.
- For a constant strain and constant strain rate, the value of σ_t can be calculated.
  - The quantity (σ_y/σ_t) is plotted against T^*. The thermal parameter m is determined after applying least squares fit of the plotted data.

J-C Parameters

Material	*A* (MPa)	*B* (MPa)	*n*	*C*	*m*	Source
Copper	90	292	0.310	0.0250	1.09	Johnson and Cook (1983)
Weldox 460E steel	490	807	0.730	0.0120	0.94	Borvik *et al*. (1999)
Weldox 700E steel	819	308	0.640	0.0098	1.00	Borvik *et al*. (2009)
Lead	24	300	1.000	0.1000	1.00	
Brass	206	505	0.420	0.0100	1.68	
Steel AISI 1006	350	275	0.360	0.0220	1.00	Steinberg *et al*. (1980)
Steel AISI 4340	792	510	0.260	0.0140	1.03	
Catridge brass	112	505	0.420	0.0090	1.68	
Tungsten	730	562	0.075	0.0290	0.15	Batra and Wilson (1998)
Fe-Ni-W alloy	150	546	0.208	0.0840	0.20	
Alloy steel 100Cr6	2033	895	0.300	0.0095	1.03	Bilici (2007)
AA 7075-T651	520	477	0.520	0.0010	1.00	Borvik *et al*. (2010)
AA 7010	547	602	0.650	0.0022	1.30	Panov (2006)
AA 2024	369	684	0.730	0.0083	1.70	

Zerilli-Armstrong Material Model

- The Zerilli-Armstrong (Z-A) constitutive model was proposed in 1987 and is based on thermally activated dislocation mechanics.
- They proposed two microstructurally based constitutive equations that show a very good match with experimental results.
- Two models were developed, one for face-centered cubic (FCC) materials and another for body-centered cubic (BCC) materials.
- This material model treats the FCC and BCC materials differently, because the strain rate and temperature sensitivities are totally different for these two class of materials.

Zerilli-Armstrong Material Model

- These constitutive material models are given by,
- BCC:

$$\sigma_y = C_0 + C_1 \exp\left[-C_3 T + C_4 T \ln\left(\dot{\bar{\varepsilon}}_p\right)\right] + C_5 \bar{\varepsilon}_p^n$$

- FCC:

$$\sigma_y = C_0 + C_2 \sqrt{\bar{\varepsilon}_p} \exp\left[-C_3 T + C_4 T \ln\left(\dot{\bar{\varepsilon}}_p\right)\right]$$

  - where C_0, C_1, C_2, C_3 and C_4 are fit parameters, C_5 is the strain hardening coefficient and n is the strain-hardening exponent.
- The primary difference between the two material models provided respectively for BCC and FCC metals is that the plastic strain is uncoupled from strain rate and temperature in BCC metals unlike that in FCC metals.

Modified Zerilli-Armstrong

- The modified material model enhances the thermal softening behavior of materials.
- This modified Zerilli-Amstrong material model (Hallquist, 2005) is a strain rate and temperature sensitive model and is given by,
- BCC:

$$\sigma_y = C_0 + C_2 \exp\left[-C_3 T + C_4 T \ln\left(\dot{\varepsilon}^*\right)\right] + \left[C_5 \bar{\varepsilon}_p^n + C_6\right]\left[B_1 + B_2 T + B_3 T^2\right]$$

- FCC:

$$\sigma_y = C_0 + \left\{C_2 \sqrt{\bar{\varepsilon}_p} \exp\left[-C_3 T + C_4 T \ln\left(\dot{\varepsilon}^*\right)\right] + C_5\right\} \times \left[B_1 + B_2 T + B_3 T^2\right]$$

Modified Zerilli-Armstrong

- Materials which are sensitive for changes in temperature and strain rate are better described by the Zerilli-Armstrong model since the temperature and strain rate terms are in the exponent.
- Due to the coupled dependency of temperature and strain rate, determination of the fit parameters is more difficult in comparison to the Johnson and Cook model.

Combined J-C and Z-A

- This material model combines the yield and strain hardening portion of the J-C model with temperature and strain rate portion of Z-A model.
- According to Holmquist and Johnson (1991), the flow stress can be expressed as:

$$\sigma_y = \left(A + B\bar{\varepsilon}_p^n\right)\exp\left[-C_3T + C_4T\ln\left(\dot{\bar{\varepsilon}}_p\right)\right]$$

  - This combination has been proven to be accurate for most metals and obtaining the constitutive parameters is relatively easy compared to Z-A material model.
  - However, its implementation in hydrocodes is limited and its application is not common other than few special cases.

Steinberg-Guinan Material Model

- A constitutive model for metals applicable at high-strain rates ($10^5\ s^{-1}$) was originally proposed by Steinberg and Guinan in 1978 and later enhanced by Steinberg, Cochran and Guinan in 1980.
- Experiments proved that at high pressure ($> 5\ GPa$), the rate dependency becomes insignificant.
- The Steinberg-Guinan model for the flow stress is written as:

$$\sigma_y = [A + B(\bar{\varepsilon}_p + \varepsilon_0)]^n\left[1 + H_1\frac{p}{J^{1/3}} - H_2(T - 300)\right]$$

  - $J = V/V_0$ is the volume ratio; A is the initial yield stress; B and n are work-hardening parameters; ε_0 is the initial equivalent plastic strain, normally zero; and H_1 and H_2 are model parameters.

Steinberg-Guinan Material Model

- This material model is used for lead (Richards et al, 1999).
- It is also used for copper in the software of the CTH computer code developed at Sandia National Laboratory (Zerilli and Armstrong, 1987).

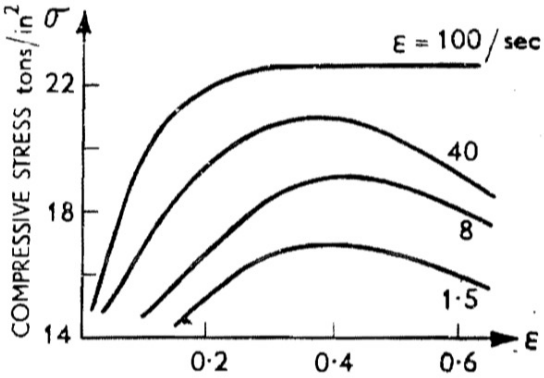
Material	*A* (MPa)	*B* (MPa)	*n*	H_1 (10^{-12}/Pa)	H_2 (10^{-3}/K)	Source
Lead	8	880	0.52	116	1.16	Steinberg *et al.* (1980)

Strain-softening

- During the working of a metal, work is done on the specimen which manifests itself as a temperature increase.
- Over a substantial range of temperature, any temperature increase, and hence softening tendency due to working seems not to be too influentive.
- However it is evident that a point will be reached when the contribution of the latter to the softening tendency will be decisive.
- It may be expected that a region of behavior will exist in which the effects of the softening rate will predominate over the effects of the hardening rate.

Strain-softening

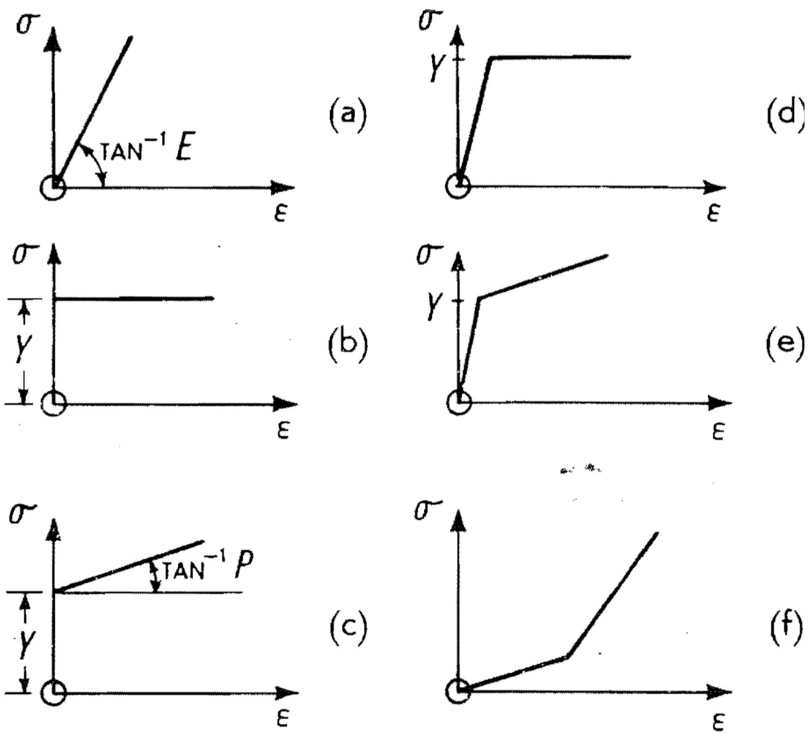
- Experiments using a cam plastometer show that after sufficient high constant rate of strain, at a sufficiently elevated temperature, the flow stress of metal may decrease with increasing strain.



- As the temperatures move closer to the melting temperature, resistance to deformation also becomes increasingly dependent on the inertia of the metal.

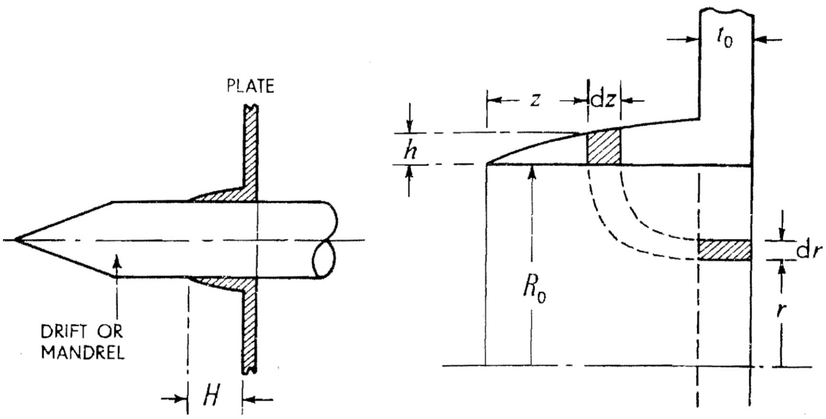
Strain-rate Independent Behavior

- a. Linear Elastic
- b. rigid-perfectly plastic
- c. rigid-linear hardening
- d. elastic-perfectly plastic
- e. elastic-linear hardening
- f. bi-linear locking



Perforation of a Thin Plate

Perforation



- Each element of the lip attains its final position by the simple process of 'hoop' stretching and rotation from its initial position in the plate.
  - All bending and shear force effects are neglected.

Perforation

- Each element in the plate at initial radius r, is conceived as being subject to circumferential stress only—as in a tensile test—whilst being stretched to final radius R_0, and the element thickness changes from t_0 to h and the radial length from dr to dz.
- For the volume of the element to remain constant,

$$2\pi r \cdot dr \cdot t_0 = 2\pi R_0 \cdot dz \cdot h \qquad \frac{R_0}{r} \cdot \frac{dz}{dr} \cdot \frac{h}{t_0} = 1$$

$$\varepsilon_r + \varepsilon_\theta + \varepsilon_l = 0$$

- Since only hoop stress prevails, then the thinning strains in the two transverse directions are equal, i.e. $\varepsilon_r = \varepsilon_l$

$$\varepsilon_\theta = -2\varepsilon_r \quad \text{or} \quad \frac{R_0}{r} = \left(\frac{dr}{dz}\right)^2$$

Perforation

- Thus,

$$\int_0^H \sqrt{R_0} \cdot dz = \int_0^{R_0} \sqrt{r} \cdot dr$$

$$H = \frac{2}{3} R_0$$

- If the drift creates a lip starting from a hole in the plate of radius R_1,

$$z = \frac{2}{3} R_0 \left[\left(\frac{r}{R_0}\right)^{3/2} - \left(\frac{R_1}{R_0}\right)^{3/2} \right]$$

- Thus that the lip height is

$$H = \frac{2}{3} R_0 \left[1 - \left(\frac{R_1}{R_0}\right)^{3/2} \right] \qquad h = t_0 \left[\frac{3}{2} \cdot \frac{z}{R_0} + \left(\frac{R_1}{R_0}\right)^{3/2} \right]^{1/3}$$

Perforation

- An assumption on which this analysis is based is that the plate material is sufficiently ductile for the mode of deformation assumed to be possible.
- We assume that plate material initially located along the drift axis undergoes infinite hoop strain, or if perforation starts from a hole of radius R_1, that the maximum hoop strain is $\ln R_0/R_1$.
- Punch-displacement/load diagrams for a perforation are nearly triangular in shape.
- There is considerable (experimental) evidence of bending even some distance away from the lip and out into the plate.

Perforation

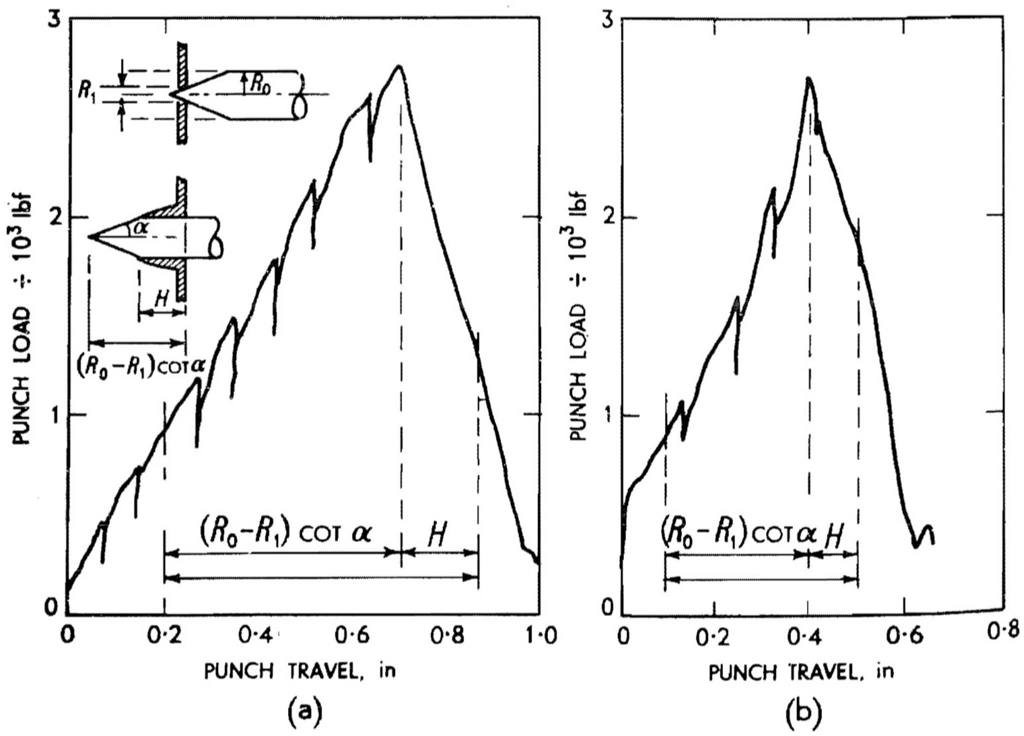


Fig. 4.10 Punch load-displacement diagrams for $\frac{1}{8}$ in thick copper specimens using $R_0 = \frac{1}{4}$ in radius punch with a nose semi-angle of $22\frac{1}{2}°$, (a) initial hole radius $R_1 = \frac{1}{32}$ in, (b) initial hole radius $R_1 = \frac{1}{8}$ in.

Empirical Equations

- Many authors have given equations relating uniaxial stress and strain for a strain-hardening material:

$$\sigma = B\varepsilon^n$$
$$\sigma = Y + P\varepsilon$$
$$\sigma = B(C + \varepsilon)^n$$
$$\sigma = Y_0 \tanh^{-1}(E\varepsilon/Y_0)$$

- Empirical equations can only be made to describe the real stress-strain curve of a material over a limited range of strain.
- B is typically of the order of $10^5\ lb/in^2$ $(10^8\ N/m^2)$ and n is about 0.25; $n{\sim}0.5$ for stainless steel and this is unusually large.

Tensile Instability

- At the maximum load when a prismatic bar is subject to an axial force F, $dF = 0$, so that because $\sigma = \frac{F(1+e)}{A_0}$,
  - Note that for incompressible plastic deformation, $Al = A_0 l_0$, thus $\sigma = F/A = F/(A_0 l_0/l) = F/A_0 \times l/l_0 = \sigma_0(1+e)$

$$d\left(\frac{A_0\sigma}{1+e}\right) = 0 \qquad \frac{d\sigma}{1+e} - \frac{\sigma \cdot de}{(1+e)^2} = 0$$

  - since $\varepsilon = \ln l/l_0 = \ln(1+e) \rightarrow \quad d\varepsilon = de/(1+e)$,

$$\frac{d\sigma}{de} = \frac{\sigma}{1+e} = \sigma_0 \qquad \frac{d\sigma}{d\varepsilon} = \sigma$$

- If the first empirical equation is given, then,

$$\frac{d\sigma}{d\varepsilon} = Bn \cdot \varepsilon^{n-1} = \sigma = B\varepsilon^n$$

Tensile Instability

- Thus, at the maximum load, the strain is,

$$\varepsilon = n \quad \text{or} \quad e = [\exp(n) - 1]$$

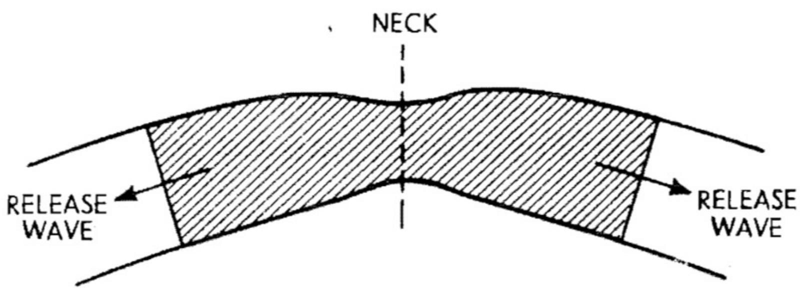
- In physical terms, as extension of a bar proceeds, its capacity to carry load is reduced in as far as its cross-section reduces.
- However, the act of stretching the bar strengthens or hardens it, so that up the maximum load an equilibrium is maintained. Two competing factors are:
  - Geometrical—the reducing area, a feature which depends on the specimen shape and how it is load,
  - Metallurgical—the multiplication of interacting dislocations.

Tensile Instability

- At maximum load, a further slight extension causes a reduction in area and hence a reduced capacity to carry load.
- Thus, equilibrium is not maintained and depending upon the manner of loading or unloading the specimen, a concentrated or localized straining occurs, soon followed by fracture.
- These maximum loads are often referred to as loads causing tensile instability.
- The argument just presented applies qualitatively to the radial expansion of spheres and cylinders, and to sheet metal in equal biaxial tension.

Tensile Instability

- When a thin ring is expanded with a sufficiently high radial speed, a neck will form somewhere around the circumference, so that the hoop force there begins to fall and in the un-necked portion of the ring the stress begins to decrease.
- This reduction is communicated circumferentially in opposite directions from the neck at the elastic wave speed:

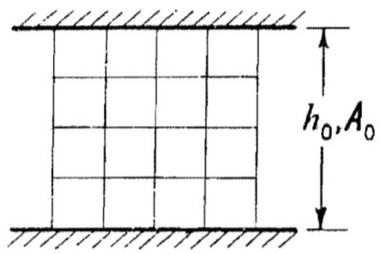


Tensile Instability

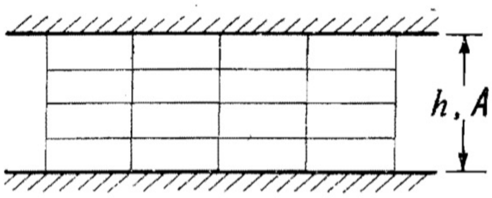
- It is possible, if the radial speed is large enough for a second (or third, etc.) neck to form in the un-necked portion of the ring, because the release wave from the first neck has not yet passed through it.
  - From the second neck, new elastic release waves are propagated.
- Reductions in area in necks, impart both circumferential speed and acceleration to the material at each side of the neck and thus give rise to inertia forces.
  - Inertia thus provides a stabilizing influence and hence in the high speed expansion of a ring, there is reason to expect multiple necks and stable straining to a higher degree than in the corresponding quasi-static process.

Homogeneous Compression

- A cylindrical block whose height is less than about twice its diameter if plastically compressed between rigid, parallel and substantially frictionless plates is a case of homogeneous compression.



- When the height of the block is reduced to h, the cross-sectional area is A_0h_0/h and the force to cause further deformation is A_0h_0Y/h; this compressive force thus increases hyperbolically.



Homogeneous Compression

- If the current height of the block is h and it is compressed a further amount $-dh$, then the axial compressive logarithmic strain increment $d\varepsilon_c$ is,

$$d\varepsilon_c = -dh/h$$

- Thus,

$$\varepsilon_c = \int_{h_0}^{h} -\frac{dh}{h} = \ln\frac{h_0}{h}$$

- To compress the block an amount $-dh$, an amount of plastic work dW must be performed:

$$dW = F \cdot (-dh) = \frac{A_0 h_0}{h} \cdot Y \cdot (-dh)$$

$$W = A_0 h_0 Y \int_{h_0}^{h} -\frac{dh}{h} = A_0 h_0 Y \ln\frac{h_0}{h}$$

Homogeneous Compression

- The work done per unit volume of the material is

$$w = \frac{W}{A_0 h_0} = Y \ln\frac{h_0}{h} \quad \text{or} \quad w = Y \cdot \varepsilon_c$$

- It follows that the alteration in shape of the block is achieved with maximum efficiency.
- Homogeneous straining, or shape-changing, is the most efficient method of securing deformation and is the criterion by which all other methods of securing the same final shape are judged.
- Homogeneous deformation is an idea of great value in the study of metal deformation processes.

Homogeneous Compression

- A bar of non-hardening material of length l_0 homogeneously stretched in tension to length l, i.e. on which a logarithmic strain $\varepsilon_l = \ln l/l_0$ is imposed, has an amount of work done on it per unit volume of

$$w = Y \ln l/l_0 = Y\varepsilon_l$$

- Similarly, for a thin ring of non-hardening material given a simple principal hoop strain ε_θ, the plastic energy required is $w = Y.\varepsilon_\theta$ per unit volume.

Temperature Rise

- The work done in frictionlessly and homogeneously compressing a cylindrical block from height h_0 to h, when its true stress-strain curve is $\sigma = B\varepsilon^n$, is

$$W = \int_{h_0}^{h} -A\sigma \cdot dh = \int_{h_0}^{h} -A_0 h_0 \sigma \cdot \frac{dh}{h}$$

- So, $$W = -\int_0^{\varepsilon} A_0 h_0 \cdot \sigma \cdot d\varepsilon$$

$$= -\int_0^{\varepsilon} A_0 h_0 \cdot B\varepsilon^n \cdot d\varepsilon$$

$$w = \frac{W}{A_0 h_0} = B\left[\frac{\varepsilon^{n+1}}{n+1}\right]_0^{\varepsilon}$$

$$w = B \cdot \frac{[\ln h_0/h]^{n+1}}{n+1}$$

Temperature Rise

- Experimentally it is found that about 90% of the energy supplied to effect plastic compression reappears as heat and causes a rise in temperature of the material.
- If there is no change in the material properties of the metal with increase in temperature, then,

$$\rho c \cdot \Delta\theta \cdot J = w \quad \rightarrow \quad \Delta\theta = \frac{w}{J\rho c}$$

  - J is the mechanical equivalent of heat, c is the specific heat.
  - A bar of mild steel, which at an ultimate strength of about 692 MPa had a strain of 0.3 would, if adiabatically stretched, undergo a uniform temperature rise of about 57 °C.

Penetration: Energy Calculations

- Each element of circumferential length $2\pi r$ and cross-sectional area $t_0 . dr$, is subject to hoop strain $\varepsilon_\theta = \ln R_0/r$
- So, if the plate is of non-hardening material, the total work done is,

$$W = \int_0^{R_0} 2\pi t_0 Y r \ln R_0/r \cdot dr = \tfrac{1}{2}\pi R_0^2 t_0 Y$$

- If the material of the plate is linear strain-hardening,

$$\int_0^w dW = \int_0^R (2\pi r t_0) \cdot \sigma \cdot \ln R_0/r \cdot dr$$

$$W = 2\pi t_0 \int_0^{R_0} r\left(Y + P \ln \tfrac{R_0}{r}\right) \ln \tfrac{R_0}{r} \cdot dr$$

$$= \tfrac{1}{2}\pi R_0^2 t_0 (Y_0 + P)$$

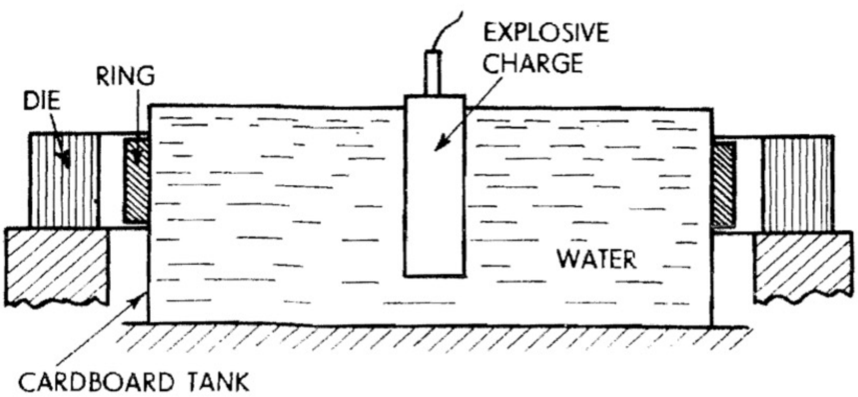
Penetration: Energy Calculations

- If, instead of a drift we consider a projectile, and assume that all dynamical effects may be neglected, we may estimate the change in speed of the projectile in order to achieve penetration by equating the kinetic energy loss of the projectile to the plastic work done in perforating the plate.
- Thus, if the projectile weight is W, if its initial speed is v_0 and its speed on emerging from the plate is v, then

$$\frac{1}{2} \cdot \frac{W}{g}\left(v_0^2 - v^2\right) = \frac{1}{2}\pi \cdot R_0^2 t_0 \cdot Y$$

Explosive Forming

- By detonating a charge of explosive in a tank of water, a radially moving pressure wave is generated which, with various reflections, acts on a given circular ring causing each element of it to acquire a radial speed v_0.



Explosive Forming

- The magnitude of v_0 in order that the ring be expanded from diameter D_0 to D is easily estimated by assuming that all the plastic work which is required to be done on the ring is acquired initially as kinetic energy.

$$\int_0^{\varepsilon_0} B\varepsilon^n \cdot d\varepsilon = B\varepsilon_0^{n+1}/(n+1)$$

  - Where $\varepsilon_0 = \ln D/D_0$.

- The kinetic energy acquired per unit volume is $\rho v_0^2/2$. Thus,

$$v_0 = \left\{ \frac{2B(\ln D/D_0)^{n+1}}{\rho(n+1)} \right\}^{1/2}$$

Explosive Forming

- If $D_0 = 36\ in$ and $D = 42\ in$, then for a ring of killed (deoxidized) steel for which $B = 91000\ \varepsilon^{0.2}$, and $\rho g = 0.283\ lb/in^2$, it is found that $v_0 \cong 500\ ft/s$.

- The maximum engineering hoop strain rate occurs at the beginning of the process and is

$$\dot{e}_\theta = v_0/(D_0/2) = 500/1\tfrac{1}{2} = 333/s$$

- The ring would contract in length as well as thin, the strains in the radial and axial directions being equal to about one half the final hoop strain.

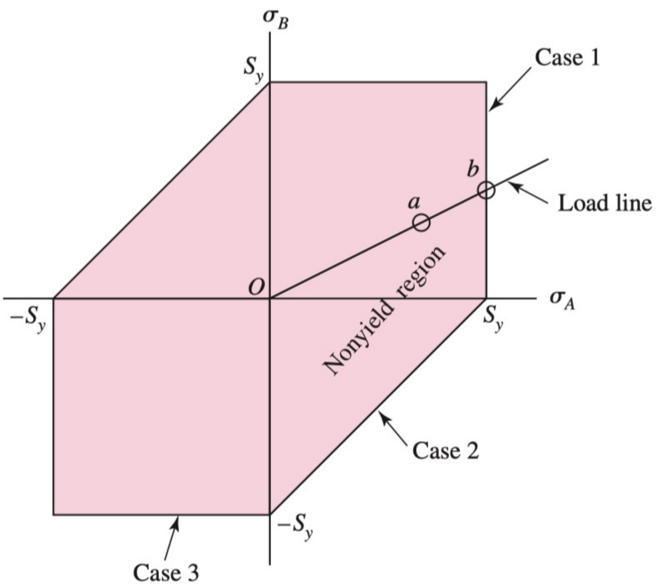
Yield Criteria

Tresca Criterion

- If each of the principal stresses is known, then when the greatest difference between any pair of them reaches a specific quantity, yield occurs.

$$\sigma_1 > \sigma_2 > \sigma_3$$

$$\sigma_1 - \sigma_3 = Y$$



von-Mises Criterion

- This condition for plastic yield is,

$$J_2 = k^2$$

$$\frac{1}{2}\sigma'_{ij}\sigma'_{ij} = k^2$$

$$\sigma'^2_x + \sigma'^2_y + \sigma'^2_z + 2\left(\tau^2_{xy} + \tau^2_{yz} + \tau^2_{zx}\right) = 2k^2$$

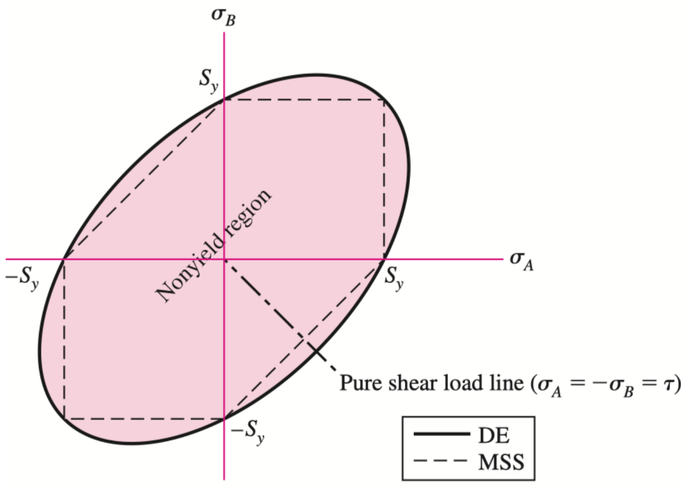
$$\begin{bmatrix} (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 \\ +6\left(\tau^2_{xy} + \tau^2_{yz} + \tau^2_{zx}\right) \end{bmatrix} = 6k^2$$

$$\sigma'_{ij} = \sigma_{ij} - \bar{\sigma}\delta_{ij} \qquad k = \frac{1}{\sqrt{3}}Y$$

von-Mises Criterion

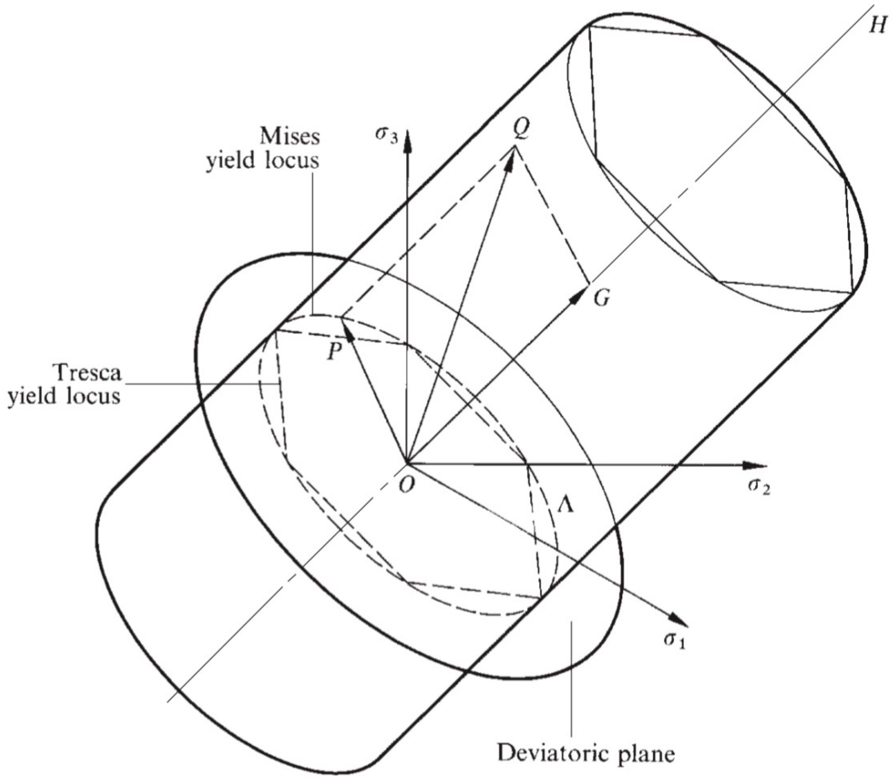
$$\sigma_e = \left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right]^{1/2}$$

$$= \frac{1}{\sqrt{2}}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau^2_{xy} + \tau^2_{yz} + \tau^2_{zx}\right)\right]^{1/2}$$



von-Mises Criterion

- The Mises criterion when plotted appears as a right circular cylinder whose axis, is equally inclined to all three coordinate axes and whose radius is $Y\sqrt{2/3}$.

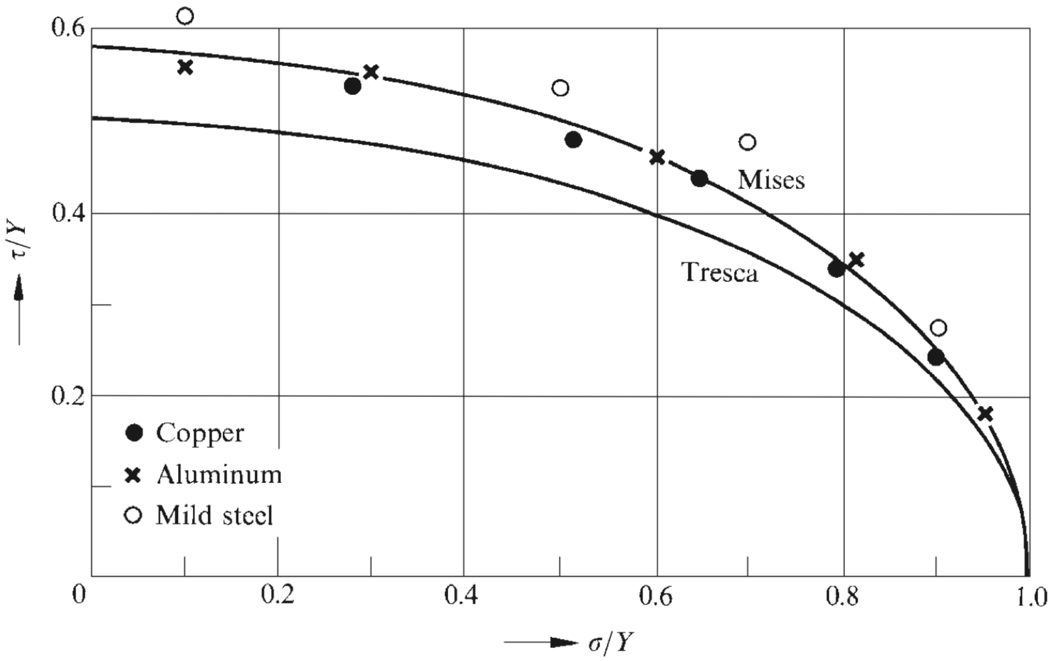


Yield Criteria

- If a thin-wall tube of mean radius a and thickness t, is loaded by an axial force P and a torque T, the mean axial stress $\sigma = P/(2\pi at)$ and the mean shear stress $\tau = T/(2\pi a^2 t)$
- The principal stresses are, $2\sigma_1 = \sigma + (\sigma^2 + 4\tau^2)^{1/2}$, $2\sigma_3 = \sigma - (\sigma^2 + 4\tau^2)^{1/2}$, and $\sigma_2 = 0$.
- Tresca: $\sigma^2 + 4\tau^2 = Y^2$ or $a^2P^2 + 4T^2 = (2\pi a^2 t)^2 \cdot Y^2$
- Mises: $\sigma^2 + 3\tau^2 = Y^2$ or $a^2P^2 + 3T^2 = (2\pi a^2 t)^2 \cdot Y^2$

Yield Criteria

- Tresca and Mises ellipses on the (σ, τ) plane together with the experimental results of Taylor and Quinney.



The Plastic Potential

- The equation of the yield surface, may be written in the form,

$$g(J_2, J_3) = \text{const.}$$

  - For von-Mises, $g = J_2$, and for Tresca, $g = \sigma_1 - \sigma_3$.

- The plastic strain increment, regarded as a vector in a nine-dimensional space, is directed along the outward normal to the yield surface at the considered stress point.

$$d\varepsilon_{ij}^p = n_{ij} d\lambda, \quad n_{ij} d\sigma_{ij} = 0, \quad n_{ij} = \frac{\partial g}{\partial \sigma'_{ij}}$$

  - where $d\lambda$ is a positive scalar representing the magnitude of the plastic strain increment vector.
  - The condition $n_{ij} d\sigma_{ij} = 0$ implies that the stress point must remain on the yield surface during an increment of plastic strain.

Levy-Mises

- In elasticity the strain is only dependent on state, in plasticity it is dependent on stress history.
  - The final or total strain undergone by a body which has been plastic is found by summing the increments of strain in accordance with change, or history, of stress development.
- When circumstances are such that the elastic components of strain are negligible by comparison with plastic components, it is usual to employ the Levy-Mises equation,

$$\frac{d\varepsilon_1}{\sigma'_1} = \frac{d\varepsilon_2}{\sigma'_2} = \frac{d\varepsilon_3}{\sigma'_3} = d\lambda$$

  - σ'_i are principal stress deviators, $d\lambda$ is just a constant of proportionality which may change as σ_i change.

Levy-Mises

- Instead of the previous equation, we may write,

$$d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = \sigma'_1 : \sigma'_2 : \sigma'_3$$

  - This form is intimately related to the Mises' yield criterion in conjunction with which it should be used.
- The corresponding form for the Tresca yield criterion is

$$d\varepsilon_1 : d\varepsilon_2 : d\varepsilon_3 = 1 : 0 : -1$$

Plane-strain Bending

- Consider a plate bent by end couples which render it plastic through its entire thickness.
- Strain in the O2-direction will be zero.
- So, $\mathrm{d}\varepsilon_2 = 0$. since $\mathrm{d}\varepsilon_1 + \mathrm{d}\varepsilon_2 + \mathrm{d}\varepsilon_3 = 0$, then $\mathrm{d}\varepsilon_1 + \mathrm{d}\varepsilon_3 = 0$.
- Using the Levy-Mises equation, we have $\mathrm{d}\varepsilon_1 : \mathrm{d}\varepsilon_3 = \sigma_1' : \sigma_3'$
- So, $\sigma_1' + \sigma_3' = 0$, $\sigma_1' = \sigma_1 - \bar{\sigma} = (2\sigma_1 - \sigma_2)/3$ and $\sigma_3' = \sigma_3 - \bar{\sigma} = -(\sigma_1 + \sigma_2)/3$ with $\bar{\sigma} = \frac{1}{3}(\sigma_1 + \sigma_2 + 0)$,
- Gives $\sigma_2 = \sigma_1/2$.
- Substituting in the Mises yield criterion,

$$\sigma_1 = 2Y\sqrt{3}/3$$

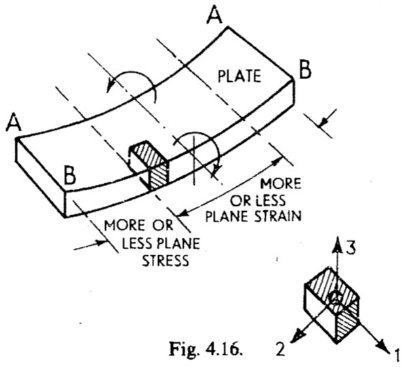
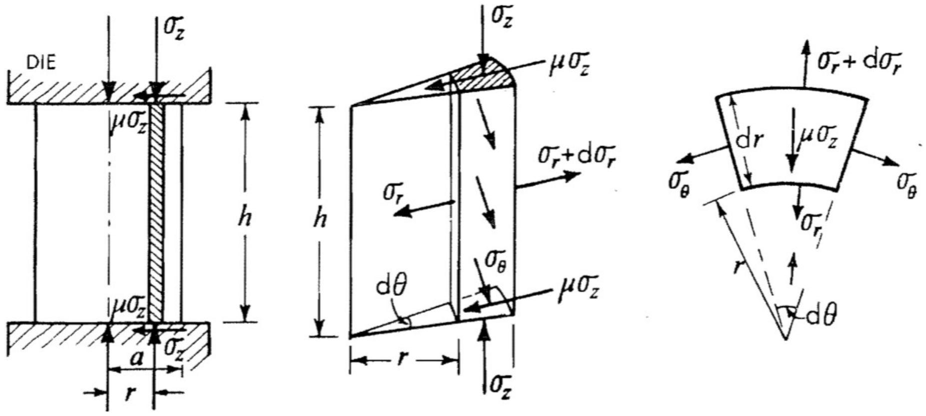


Fig. 4.16.

Compression of Cylinders and Spherical Shells

Compression of a Short Cylinder

- We consider a short circular cylinder of rigid-perfectly plastic material whose height to radius ratio, h/a is less than about 3, compressed between rigid parallel dies at a relatively constant slow speed.
- Equation of equilibrium: $\frac{\sigma_r - \sigma_\theta}{r} + \frac{\partial \sigma_r}{\partial r} = -\frac{2\mu\sigma_z}{h}$



Compression of a Short Cylinder

- Referring to no volume change of a cylinder (slide 6), we have $\mathrm{d}\varepsilon_\theta = \mathrm{d}\varepsilon_r$. Using Levy-Mises equation, $\sigma_\theta = \sigma_r$.
- Thus, $d\sigma_r/dr = -2\mu/h \cdot \sigma_z$.
- Using Tresca's yield criterion, $\sigma_r - \sigma_z = Y \quad \rightarrow \quad d\sigma_r = d\sigma_z$.
- So, $\ln \sigma_z = -2\mu r/h + c$.
- At $r = a, \sigma_r = 0$, and $\sigma_z = -Y$. So the above becomes,

$$p = Y \exp(2\mu(a-r)/h)$$

- If μ is small,

$$p \simeq Y\left\{1 + \frac{2\mu(a-r)}{h}\right\}$$

Compression of a Short Cylinder

- The total force over a compression die is P,

$$P \simeq \pi a^2 Y\left\{1 + \frac{2}{3}\mu \cdot \frac{a}{h}\right\}$$

- This pressure distribution is known as a friction-hill.
  - If $a/h = 3$ and $\mu = 0.12$, then the load to effect compression of the cylinder is increased by 24% because of end interfacial friction.

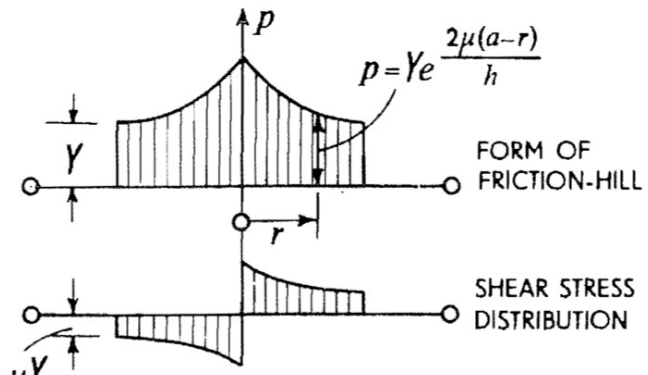


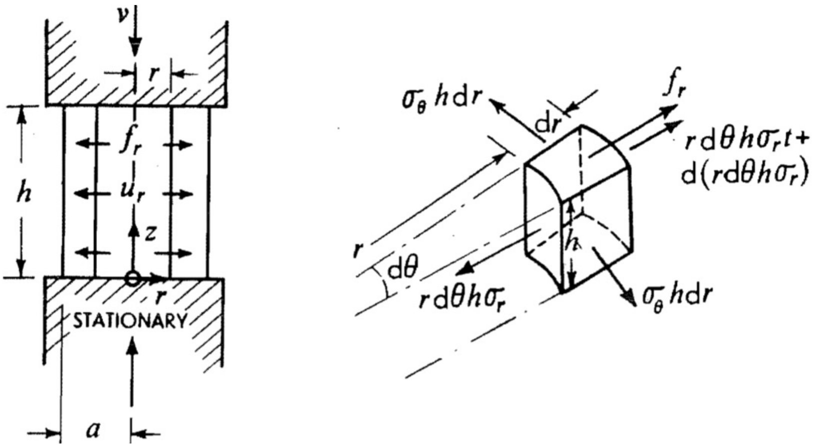
Fig. 4.18.

Fast Compression of a Cylinder

- When a short, uniform circular cylinder is rapidly compressed between rigid parallel plates or dies, inertia stresses and forces are generated which modify those ordinarily required to bring about plastic yield and compression in the quasi-static process.
  - In the range of about 100 to 1000 ft/sec (30 to 300 m/s).
  - For speeds in excess of about 1000 ft/sec (30 m/s), stress wave effects will assume importance.
  - It is possible for any elastic or plastic stress waves initiated to travel up and down any cylindrical block many times during the course of compression.

Fast Compression of a Cylinder

- First, the load ‘felt’ by each of the upper and lower dies will be different; the upper die is subject to a load in excess of the quasi-static load in the early part of the compressive process whilst it is accelerating material.
  - The load will be reduced in the last stages of the compression since the inertia so acts as to assist the compression sought.



Fast Compression of a Cylinder

- The equation for constancy of volume is,

$$v \cdot \pi r^2 = 2\pi r h \cdot u_r \qquad u_r = \frac{r}{2h} \cdot v$$

  - Where v is the speed of die, u_r, f_r are the radial speed and acceleration of cylinder.

- Also,

$$f_r = \frac{du_r}{dt} = \frac{\partial r}{\partial t} \cdot \frac{v}{2h} + \frac{r}{2h}\frac{\partial v}{\partial t} - \frac{rv}{2h^2}\frac{\partial h}{\partial t}$$

$$= \frac{u_r v}{2h} + \frac{r}{2h} \cdot \dot{v} + \frac{rv^2}{2h^2}$$

- So,

$$f_r = \frac{rv^2}{4h^2} + \frac{r\dot{v}}{2h} + \frac{rv^2}{2h^2}$$

$$= \frac{3rv^2}{4h^2} + \frac{r\dot{v}}{2h}$$

Fast Compression of a Cylinder

- Thus the radial acceleration f_r is just $3rv^2/4h^2$ for a constant speed upper die.
- The equation of radial motion of an element is,

$$d(rd\theta \cdot h\sigma_r) - \sigma_\theta h dr d\theta = rd\theta \cdot hdr\rho f_r$$

- Which reduces to

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = \rho f_r$$

- Assuming $\sigma_r = \sigma_\theta$ and recalling that $f_r = 3rv^2/4h^2$,

$$\frac{\partial \sigma_r}{\partial r} = \frac{3\rho v^2}{4h^2} r \quad \rightarrow \quad \sigma_r = \frac{3\rho v^2}{4h^2} \cdot \frac{r^2}{2} + c$$

Fast Compression of a Cylinder

- Noting that $\sigma_r = 0$ when $r = a$,

$$\sigma_r = \frac{3\rho v^2}{8h^2} \cdot \left(r^2 - a^2\right)$$

- In the absence of acceleration parallel to the axis of the cylinder,

$$\sigma_r - \sigma_z = Y$$

- For the compressive stress on the dies p,

$$p = -\sigma_z = Y - \sigma_r = Y + \frac{3\rho v^2}{8h^2} \cdot \left(a^2 - r^2\right)$$

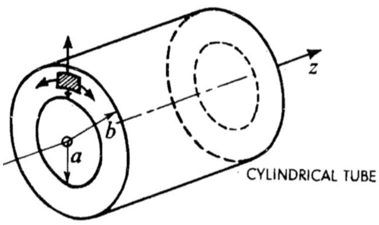
- The load on the dies is P and,

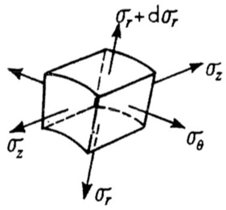
$$P = \pi a^2 Y \cdot \left[1 + \frac{3}{16} \cdot \frac{\rho v^2}{Y} \cdot \left(\frac{a}{h}\right)^2\right]$$

  - The second term on the right hand side, for steel, and with $a/h = 1/2$ and using $Y = 246\ MPa$ would only contribute 1% of the total load if the velocity was 100 ft/sec.

Thick-walled Cylinders

- For a thick-wall uniform cylinder of inner radius a and outer radius b, subjected to an internal pressure $-p_i$ and an external pressure $-p_o$, under conditions of plane strain,

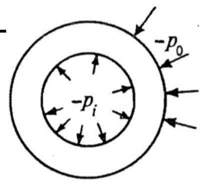


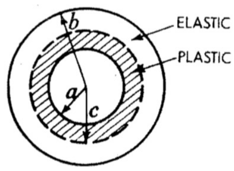


$$\sigma_r = \frac{a^2 b^2 (p_0 - p_i)}{b^2 - a^2} \cdot \frac{1}{r^2} + \frac{p_i a^2 - p_0 b^2}{b^2 - a^2}$$

$$\sigma_\theta = -\frac{a^2 b^2 (p_0 - p_i)}{b^2 - a^2} \cdot \frac{1}{r^2} + \frac{p_i a^2 - p_0 b^2}{b^2 - a^2}$$

$$\sigma_z = \frac{p_i a^2 - p_0 b^2}{b^2 - a^2} \cdot 2v$$





Thick-walled Cylinders

- For $p_0 = 0$, the greatest principal stress difference is $\sigma_\theta - \sigma_r$,

$$\sigma_\theta - \sigma_r = \frac{p_i}{m^2 - 1} \cdot \frac{2b^2}{r^2}$$

- where $m = b/a$. Thus, plastic yield first occurs at the bore if we assume the Tresca yield criterion to apply,

$$\frac{p_i^*}{Y} = \frac{m^2 - 1}{2m^2}$$

- If the Mises' criterion is used, plastic yield still occurs first at the bore but now,

$$\frac{p_i^*}{Y} = \frac{m^2 - 1}{2m^2} \cdot \frac{1}{\left(\frac{3}{4} + \frac{(1-2v)^2}{4m^4}\right)^{1/2}}$$

Thick-walled Cylinders

- When the applied pressure exceeds p_i^*, a cylindrically symmetric zone of yielded material extends outwards from the bore, part way through the cylinder wall.
- Now the radial equilibrium equation for an element of the cylinder wall is,

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

- and if the Tresca criterion applies,

$$\frac{d\sigma_r}{dr} = \frac{Y}{r}$$

$$\sigma_r = Y \ln r + c$$

Thick-walled Cylinders

- At $r = a,\ \sigma_r = -p_i$ and if the zone of plastic yield extends out to radius c, the radial stress is σ_c,

$$\sigma_c = Y \ln(c/a) - p_i$$

- For $r > c$, the cylinder is elastic but at $r = c$, i.e. the elastic-plastic interface it is just on the point of yield so that recalling $p_i^*/Y = (m^2 - 1)/2m^2$,

$$-\frac{\sigma_c}{Y} = \frac{b^2 - c^2}{2b^2}$$

- Eliminating σ_c,

$$\frac{p_i}{Y} = \ln\frac{c}{a} + \frac{1}{2}\left(1 - \frac{c^2}{b^2}\right)$$

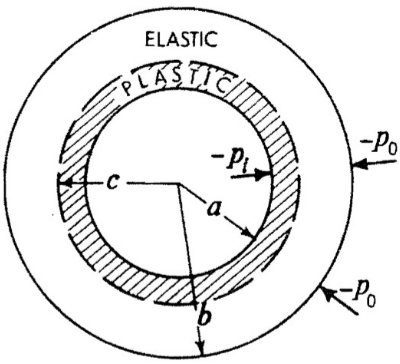
- When $c = b$,

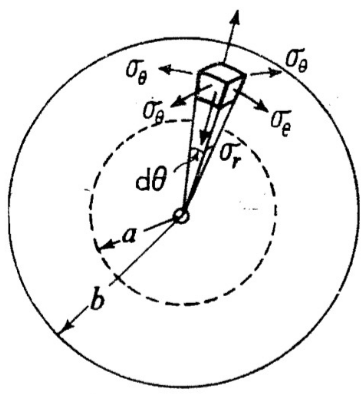
$$\frac{p_i^{**}}{Y} = \ln\frac{b}{a}$$

Thick Spherical Shells

- For a spherical shell of inner radius a and outer radius b, subjected to an internal pressure $-p_i$ and an external pressure $-p_o$,

$$\sigma_r = -\frac{(p_i - p_0)(b^3 - r^3)a^3}{r^3(a^3 - b^3)} \qquad \sigma_\theta = -\frac{(p_i - p_0)(2r^3 + b^3)a^3}{2r^3(b^3 - a^3)}$$





Thick Spherical Shells

- For $p_0 = 0$, the principal stress difference is,

$$\sigma_\theta - \sigma_r = -\frac{3p_i a^3 b^3}{2r^3(b^3 - a^3)}$$

  - and this is greatest when $r = a$.

- Thus, plastic yielding commences at the inner surface of a spherical shell at pressure p_i^*, and for either a Tresca or a Mises criterion, is given by,

$$\sigma_\theta - \sigma_r = Y = -\frac{3p_i^* b^3}{2(b^3 - a^3)} \quad \text{or} \quad \frac{p_i^*}{Y} = \frac{2(m^3 - 1)}{3m^3}$$

  - where $m = b/a$.
  - When the applied pressure exceeds $-p_i^*$, a symmetric zone of plastically-yielded material surrounds the inner surface out to radius c, which locates the elastic-plastic interface.

Thick Spherical Shells

- Now, for the plastic spherical shell, the equation of radial equilibrium is,
$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}x} + \frac{2(\sigma_r - \sigma_\theta)}{r} = 0$$
- Putting $\sigma_\theta - \sigma_r = Y \rightarrow d\sigma_r = 2Y.dr/r \rightarrow \sigma_r = 2Y \ln r + c$.
- At $r = a, \ \sigma_r = -p_i \rightarrow \ \sigma_r = -p_i + 2Y \ln r/a$.
- At $r = c$, the outer elastic shell is on the point of yielding and hence the radial pressure, σ_c,
$$Y = \frac{-3\sigma_c b^3}{2(b^3 - c^3)}$$
- But, $\sigma_c = -p_i + 2Y \ln c/a$.
- And hence eliminating $\sigma_c, \ p_i/Y = \ln(c/a) + \frac{1}{3}(1 - c^3/b^3)$.

Thick Spherical Shells

- When $c = b$, the shell is wholly plastic and the internal pressure is given by,
$$\frac{p_i^{**}}{2Y} = \ln\frac{b}{a}$$
- The radial expansion Δb^{**} when pressure p_i^{**} is reached is obtained by realizing that the outermost element of the shell is stressed just to the point of yielding.
- We have at $r = b$, $Ee_\theta = \sigma_\theta - \nu(\sigma_\theta + \sigma_r)$, but $\sigma_r = 0$, so,
$$\Delta b^{**} = \frac{b\,\sigma_\theta(1-\nu)}{E} = \frac{b\,Y(1-\nu)}{E}$$
- Similarly, the radial expansion Δa^* at the inner surface for pressure p_i^* is,
$$E \cdot \frac{\Delta a^*}{a} = \sigma_\theta - \nu(\sigma_\theta - p_i^*)$$

Thick Spherical Shells

- Or,

$$\Delta a^* = \frac{aY}{3E}\left[(1+\nu) + \frac{2}{m^3}(1-2\nu)\right]$$

- If the material of the sphere is incompressible,

$$\Delta a^* = \frac{aY}{2E} \quad \text{and} \quad \Delta b^{**} = \frac{bY}{2E}$$

- Since $b^3 - a^3 = b^{*3} - a^{*3} = b^{**3} - a^{**3}$,

$$b^* = b\frac{\left[m^3 - 1 + \left(1 + \frac{Y}{2E}\right)^3\right]^{1/3}}{m} \cong b\left(1 + \frac{1}{m^3}\cdot\frac{Y}{2E}\right)$$

$$a^{**} = a\left[m^3\left(1 + \frac{Y}{2E}\right)^3 - m^3 + 1\right]^{1/3} \cong a\left(1 + \frac{3Y}{2E}m^3\right)^{1/3}$$

Expansion of Cylinders and Spherical Shells

Deformation

- The calculations took no account of geometry changes or the expansion of the shell during the application of the pressure.
- It was assumed that at the moment the whole shell became plastic that the internal and external radii had exactly the same values as they did when there was no pressure at all applied.
- When the pressure p^{**} is reached,

$$a^{**} \cong a\left(1+\frac{2E}{3Y}m^3\right)^{1/3} \quad \text{and} \quad b^{**} \cong b\left(1+\frac{Y}{2E}\right)$$

  - Typically, $Y/E = 0.001$, so that for $m = 10$, $m^{**} = 7.37$ which is very significantly different from the initial ratio.

Deformation

- Thus if account is taken of the radial displacements undergone by the spherical shell wall in the act of straining and becoming fully plastic, then for the elastic-perfectly plastic incompressible material,

$$\frac{b^{**}}{a^{**}} = \frac{b\left(1+\frac{Y}{2E}\right)}{a\left(1+m^3\cdot\frac{3Y}{2E}\right)^{1/3}} \simeq \left[\frac{m^3}{1+m^3\cdot\frac{3Y}{2E}}\right]^{1/3}$$

$$\frac{p^{**}}{Y} = 2\ln\frac{b^{**}}{a^{**}} = \frac{2}{3}\ln\frac{m^3}{\left(1+\frac{3m^3Y}{2E}\right)}$$

- Clearly, for large values of m, $p^{**}/Y \to \frac{2}{3}\ln(2E/3Y)$.

Deformation

TABLE 4.1 ASSUMING, $Y/E = \frac{1}{1000}$

$m = b/a$	1·5	2	4	6	10	∞
b^{**}/a^{**}	1·5	1·99	3·87	5·45	7·37	—
$p^{**}/2Y$ (using equation (4.93ii))	0·40	0·69	1·35	1·70	1·99	2·17
$p^{**}/2Y$ (using equation (4.93i))	0·41	0·69	1·39	1·79	2·30	—

Linear Hardening

- The analysis* of the 'infinite radius ratio' spherical shell (i.e. spherical cavity expanding in an infinite medium) possessing a linear strain-hardening law of the form, $\sigma = Y + P.\varepsilon$, leads to a pressure for p^{**}, to bring the whole shell to a plastic state, of

$$p^{**} = \frac{2}{3} Y \ln \frac{2E}{3Y} + \frac{2\pi^2}{27} P$$

- If $E/Y = 0.001, P/Y = 1/3 \ \rightarrow p^{**}/Y = 4.58.$

Hill, R. Mathematical Theory of Plasticity, Oxford University Press, 1950.

Work to Expand

- The work required to plastically expand a spherical shell of incompressible non-hardening material from internal initial radius a_0 to some current radius a by quasi-static internal pressure, is given by

$$W = \int_{a_0}^{a} 4\pi a^2 \cdot p_i \cdot da$$

- Neglecting the work required to be done in order to bring the shell up to full plastic yield, (recall $p_i^{**}/2Y = \ln b/a$),

$$W = \int_{a_0}^{a} 4\pi a^2 \cdot \frac{2Y}{3} \ln \frac{b^3}{a^3} \cdot da$$

$$= \frac{8\pi Y}{3} \int_{a_0}^{a} a^2 \ln\left(1 + \frac{b_0^3 - a_0^3}{a^3}\right) da$$

- since $(b_0^3 - a_0^3) = b^3 - a^3$, where b_0 in the initial external radius.

Radial Speed of Inner Surface

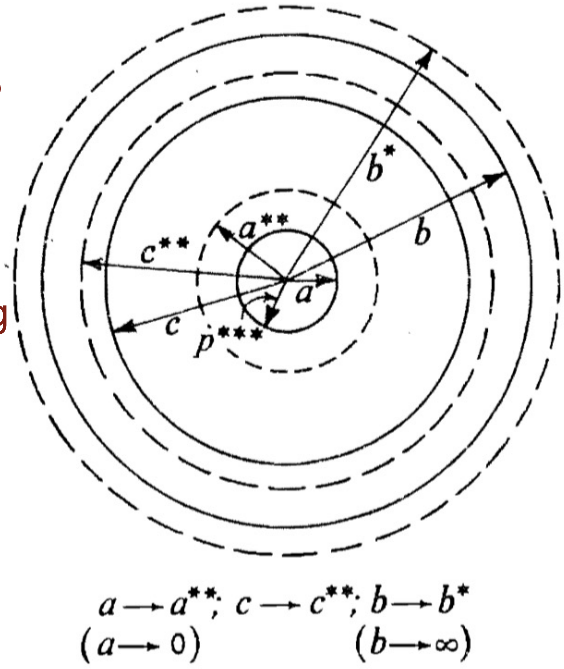
- A spherical shell when packed with explosive and detonated, causes the shell to rapidly expand, the inner surface acquiring some initial radial speed v_0.
- If all stress wave effects are neglected, then the speed to achieve a specified expansion to radius a, or b which is considerably in excess of b^{**}, may be estimated for the incompressible material as follows.
- The speed of an element of the shell at radius r is $a_0^2 v_0/r^2$ and hence, the total kinetic energy of the shell wall is,

$$\int_{a_0}^{b_0} \frac{1}{2} \cdot 4\pi r^2 \rho \cdot \left(\frac{a_0^2 v_0}{r^2}\right)^2 \cdot dr = 2\pi\rho a_0^3 (b_0 - a_0) v_0^2 / b_0$$

- By equating the work to expand to the above, v_0 is found.

Infinitely Small Cavity

- Application of internal pressure, p^{***}, over the inner surface of radius a creates,
  - (i) a spherically symmetric, fully plastic shell of radius a^{**}, out to radius c^{**}, and
  - (ii) from c^{**} to the outer radius b^*, an outer shell which is entirely elastic, apart from being in a state of plastic yield on its inner surface of radius c^{**}.



Infinitely Small Cavity

- The spherical shell in its expanded state may thus be considered as consisting of two concentric spherical shells,
  - (i) $$\frac{p^{***}}{Y} = \frac{2}{3}\ln\left(\frac{c^{**}}{a^{**}}\right)^3 + \frac{\sigma_{c^{**}}}{Y}$$
  - $\sigma_{c^{**}}$ denotes the radial pressure at radius c^{**}.
  - (ii) from slide 98: $$\frac{\sigma_{c^{**}}}{Y} = \frac{2}{3}, \quad \text{since} \quad \frac{b^*}{c^{**}} \to \infty$$
- Hence, $$\frac{p^{***}}{Y} = \frac{2}{3}\left(1 + \ln\frac{2E}{3Y}\right)$$
  - The elastic-plastic interface extends to c^{**} and $c^{**}/\,a^{**} \to \left(\frac{2E}{3Y}\right)^{1/3}$

Quasi-static Approach

- Let a pressure p', which is a function of time and radius be applied to the inside surface of a cavity such that initially, i.e. at $t = 0$, the cavity radius is infinitely small.
- When the current cavity radius is a and its radial speed $\dot{a}$, the kinetic energy E of the whole shell is

$$E = \int_a^\infty \frac{1}{2}\left(4\pi r^2 dr \rho\right) \cdot \dot{r}^2$$

- The incompressibility condition gives $4\pi a^2 . \dot{a} = 4\pi r^2 . \dot{r}$,

$$E = \int_a^\infty 2\pi\rho r^2 \left(\frac{a^2 \cdot \dot{a}}{r^2}\right)^2 dr$$

$$= 2\pi\rho a^4 \cdot \dot{a}^2 \left[-\frac{1}{r}\right]_a^\infty = 2\pi\rho a^3 \cdot \dot{a}^2$$

Quasi-static Approach

- At time t, pressure p' is doing work at a rate $p' . 4\pi a^2 . \dot{a}$

$$p' .4\pi a^2 \cdot \dot{a} = p^{***} \cdot 4\pi a^2 \cdot \dot{a} + \dot{E}$$

- Since $\dot{E} = 2\pi\rho(3a^2 . \dot{a}^3 + 2a^3 . \dot{a} . \ddot{a})$,

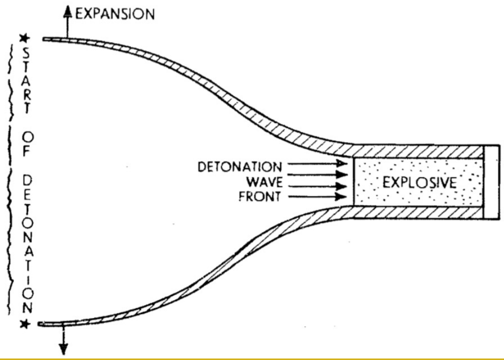
$$p' = p^{***} + \rho\left(\frac{3}{2}\dot{a}^2 + a \cdot \ddot{a}\right)$$

- If when $\ddot{a} = 0, \dot{a} = V_0$,

$$p'(t) = \frac{2}{3}Y\left[1 + \ln\frac{2E}{3Y}\right] + \frac{3}{2}\rho V_0^2$$

Bombs

- Bombs are usually constructed from suitably corrugated iron or steel casings packed with high explosive.
  - At detonation, the high pressure of the explosion products drives the wall of the bomb outwards at high speed such that multi-fractures are initiated, and as expansion proceeds further, complete fragmentation occurs.
  - The onset of the latter is detected from the appearance of smoke when high speed cine-films are taken of bomb explosions.

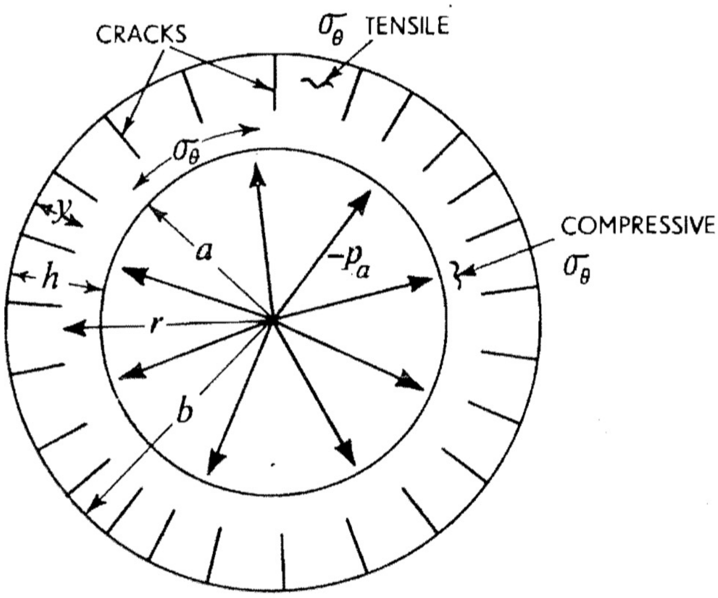


Bombs

- Taylor in a paper in 1948 presented a simple and elegant analysis that, with the help of a few assumptions, accounted for the radius of fracture at complete fragmentation in the explosion of a thin tubular bomb.
- This analysis makes use of the fact that when very high pressure acts on the inside of the casing, compressive hoop stresses exist over an inner region of the casing, and the depth of this region is governed predominantly by the magnitude of this applied pressure.
- Taylor's assumption is that radial cracks are initiated only in the region of tensile stress and cannot propagate through and into the compressive zone.

Bombs

- Taylor concluded that complete fragmentation takes place only when the compressive region completely disappears, hence allowing cracks to propagate right up to the inside surface.

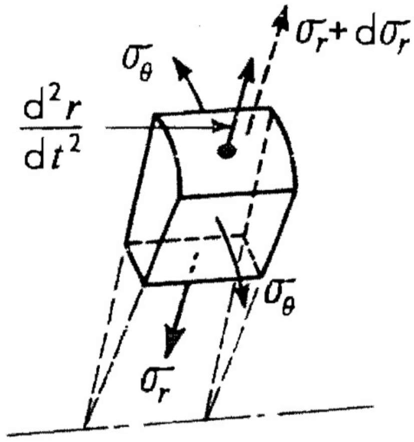


Bombs

- His analysis predicts that the depth, y, to which the tensile hoop stress region extends at any time during the expansion process, measured from the outside surface, is approximately Yh/p_a.
  - where h is the current cylinder wall thickness, Y is the uniaxial yield stress of the material and p_a is the current internal radial pressure.
- Thus, if the condition for complete fragmentation is given by $y = h$, this occurs when $p_a = Y$.
- Hence if p_a can be calculated as a function of radius a, the fracture radius may be deduced.

Bombs

- The approach below is essentially the same as that of Taylor; it is however more complete and pertains more generally to thick-walled cylinders.
- The idealized equation of radial motion of an element is



$$\frac{\sigma_r - \sigma_\theta}{r} + \frac{\partial \sigma_r}{\partial r} = \rho \frac{d^2 r}{dt^2}$$

- No volume change:

$$r^2 - a^2 = r_0^2 - a_0^2$$

  - The symbol r denotes the current radial distance of the element from the center line of the bomb, r_0 is its original location and a_0 and a are its original and the current internal radius; b_0 is the original external radius of the bomb.

Bombs

- Hence

$$r\frac{\partial r}{\partial t} - a\frac{\partial a}{\partial t} = 0$$

- Differentiating again, but writing $\partial a/\partial t = v_a$

$$\frac{d^2 r}{dt^2} = \frac{v_a^2}{r} + \frac{a}{r}\frac{\partial v_a}{\partial t} - \frac{a^2}{r^3} v_a^2$$

- Using the Tresca yield criterion as,

$$\sigma_\theta - \sigma_r = Y$$

- Substituting

$$\frac{\partial \sigma_r}{\partial r} = \frac{Y}{r} + \rho\left(\frac{v_a^2}{r} + \frac{a}{r}\frac{\partial v_a}{\partial t} - \frac{a^2 v_a^2}{r^3}\right)$$

Bombs

- After integration gives,

$$\sigma_r = \left[Y + \rho\left(v_a^2 + a\frac{\partial v_a}{\partial t}\right)\right] \ln r + \frac{1}{2}\rho\frac{a^2 v_a^2}{r^2} + C$$

- At $r = a, \;\; \sigma_r = -p_a,$

$$-p_a = \left[Y + \rho\left(v_a^2 + a\frac{\partial v_a}{\partial t}\right)\right] \ln a + \frac{1}{2}\rho v_a^2 + C$$

- Eliminating C,

$$\sigma_r + p_a = \left[Y + \rho\left(v_a^2 + a\frac{\partial v_a}{\partial t}\right)\right] \ln \frac{r}{a} + \frac{1}{2}\rho v_a^2\left(\frac{a^2}{r^2} - 1\right)$$

- So,

$$\sigma_\theta = Y - p_a + \left[Y + \rho\left(v_a^2 + a\frac{\partial v_a}{\partial t}\right)\right] \ln \frac{r}{a} + \frac{1}{2}\rho v_a^2\left(\frac{a^2}{r^2} - 1\right)$$

Bombs

- Two different regions of stress distribution may be distinguished within the tube wall:
- the boundary between them being defined by $\sigma_\theta = 0$, i.e. at the depth $y = b - r$.
- Using this definition in the previous equation, the value of y is given by,

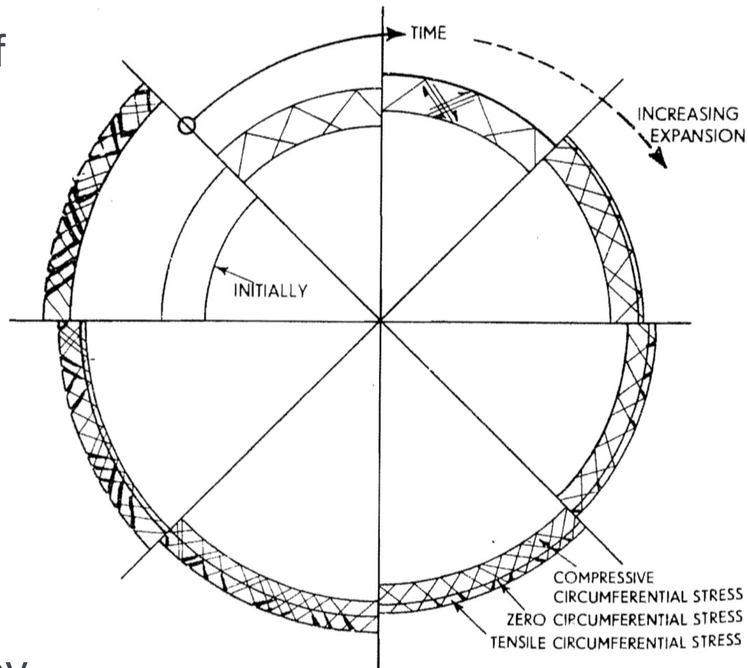


Fig. 4.26.

$$p_a - Y = \left[Y + \rho\left(v_a^2 + a\frac{\partial v_a}{\partial t}\right)\right] \ln\left(\frac{b-y}{a}\right) + \frac{1}{2}\rho v_a^2\left[\frac{a^2}{(b-y)^2} - 1\right]$$

Bombs

- The region of compressive stress i.e. $\sigma_\theta < 0$ and $\sigma_r < 0$, will just disappear when $y = b - a = h$. Then, $p_a = Y$.
  - This is the same condition as that originally obtained by Taylor.
- However, to determine when complete fracture takes place, an expression for the pressure exerted by the gaseous explosion products on the inner wall of the casing as a function of internal wall radius is required.
- This expression can be approximated reasonably well by assuming an isentropic expansion of the explosion products. Thus,

$$p_a = p_0 \left(\frac{a}{a_0} \right)^{-2\gamma}$$

  - where p_0 is the effective detonation pressure acting on the inside surface when $a = a_0$ and γ is the adiabatic exponent.

Bombs

- Typically, for TNT, γ has an initial value of about 3.4 when the pressure p_0 is about $15.8\times10^9\ Pa$ and, when the density is reduced to about one quarter of its initial value, γ is about 1.9.
  - For a moderate expansion to fracture, the fixed value $\gamma = 3$ is a satisfactory approximation.
- The criterion for complete fracture of the casing when $p_a = Y$,

$$\frac{a}{a_0} = \left(\frac{p_0}{Y} \right)^{1/2\gamma} \cong \left(\frac{p_0}{Y} \right)^{1/6}$$

  - Note that the higher the yield stress, Y, of the casing material, the smaller the resistance to crack propagation as measured by the expansion radius at complete fracture.

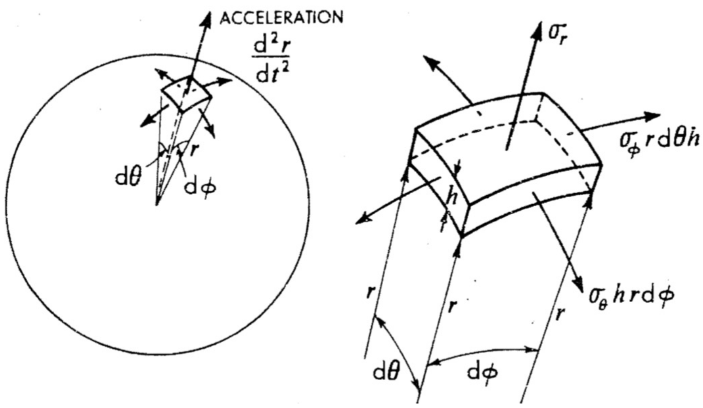
Bombs

- The procedure for analyzing the fragmentation of spherical bombs may be carried out in a similar manner to that used above for cylindrical shells.
- The casing radius of fracture is now given by,

$$\frac{a}{a_0} = \left(\frac{p_0}{Y}\right)^{1/3\gamma}$$

Initial Speed for Expansion

- The permanent plastic straining of a spherical or cylindrical shell of initial radius r_0 of strain-hardening material by imparting a radial initial velocity, v_0, to give a radius of r, is considered in order to find the radius to which the (thin) sphere or cylinder is expanded.
- For an element of the spherical shell, the equation of radial motion is,



$$2\sigma_\theta hrd\theta \cdot d\phi + \rho hr^2 d\theta \cdot d\phi \cdot \frac{d^2r}{dt^2} = 0 \quad \rightarrow \frac{d^2r}{dt^2} + \frac{2\sigma_\theta}{\rho r} = 0$$

Initial Speed for Expansion

- Throughout the expansion process $\sigma_\theta = \sigma_\varphi$ and $\sigma_r = 0$, so the von-Mises stress is $\bar{\sigma} = \sigma_\theta$ and since in this circumstance of uniform expansion it is permissible to use total strains for which, $\varepsilon_\theta = \varepsilon_\varphi = -\varepsilon_t/2$, so $\bar{\varepsilon} = 2\varepsilon_\theta$.
- Adapting strain-hardening model, $\bar{\sigma} = B(C + \bar{\varepsilon})^n$ $\Rightarrow \sigma_\theta = B(C + 2\varepsilon_\theta)^n$. Hence noting $d^2r/dt^2 = v.dv/dr$,

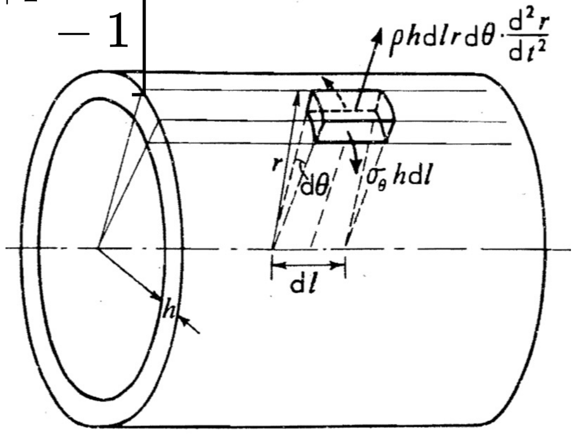
$$\int_{v_0}^{0} v \cdot dv = -\frac{2B}{\rho} \int_{r_0}^{r} \frac{(C+2\varepsilon_\theta)^n}{r} \cdot dr$$

- Thus, $$v_0^2 = \frac{2BC^{n+1}}{\rho(n+1)}\left[\left(1 + \frac{2}{C}\ln\frac{r}{r_0}\right)^{n+1} - 1\right]$$
  - Where $\varepsilon_\theta = \ln r/r_0$

Initial Speed for Expansion

- For the element of the cylindrical shell shown, the equation of radial motion is, $$\frac{d^2r}{dt^2} + \frac{\sigma_\theta}{\rho r} = 0$$
- Proceeding as previously, it is easily shown that for plane strain expansion,

$$v_0^2 = \frac{2BC^{n+1}}{\rho(n+1)}\left[\left(1 + \frac{1}{C}\ln\frac{r}{r_0}\right)^{n+1} - 1\right]$$



Initial Speed for Expansion

- Expressions for v_0 to cause the shells to reach the tensile instability strain may now easily be arrived at.
- These strains are those prevailing when some slowly applied hydrostatic pressure reaches its greatest value.
- For the spherical shell at instability, the hoop strain, $\ln r/r_0$ is equal to $(2n-3C)/6$. The initial velocity v_0^* required to produce strain equal to the instability strain is

$$v_0^{*2} = \frac{2BC^{n+1}}{\rho(n+1)}\left[\left(\frac{2n}{3C}\right)^{n+1} - 1\right]$$

- The initial velocity v_0^{**} required to produce strains equal to the static instability strain in a long cylindrical shell is

$$v_0^{**2} = \frac{2BC^{n+1}}{\rho(n+1)}\left[\left(\frac{n}{C\sqrt{3}}\right)^{n+1} - 1\right]$$

Initial Speed for Expansion

- Some representative values of v_0^* and v_0^{**} given in this Table, can be arrived at for stainless steel, half-hard copper and half-hard brass by selecting suitable values of B, C and n.

Material	Density (Slugs/ft^3)	$B \div 10^6$ (lb/ft^2)	C	n	ft/sec v_0^* sphere	ft/sec v_0^{**} cylinder	Instability hoop strain, ε_θ sphere	Instability hoop strain, ε_θ cylinder
Copper ($\frac{1}{2}$-hard)	17·1	0·895	0·114	0·3	227	186	0·043	0·051
Brass ($\frac{1}{2}$-hard)	16·6	1·58	0·127	0·48	414	357	0·177	0·130
Stainless steel	15·1	3·20	0·016	0·50	734	658	0·240	0·236

Internal Blast Loading

- Consider a thin spherical shell of radius a subjected to an internal blast load which evokes only an elastic response, or vibration of the shell.
- If $p(t)$ denotes the variation of pulse pressure with time, t, and u, is the (small) radial displacement of the shell of thickness h,

$$\frac{d^2 u_r}{dt^2} + \frac{2\sigma_\theta}{\rho a} = \frac{p(t)}{\rho h}$$

- Using the elastic stress-strain equation $\sigma_\theta = E e_\theta (1 - \nu)$, where $e_\theta = u_r / a$,

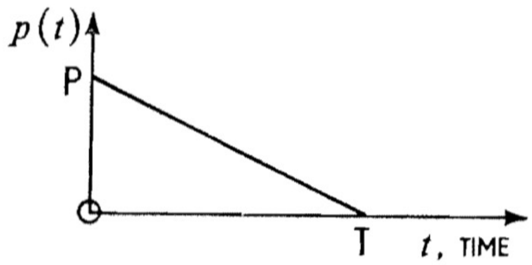
$$\rho \frac{d^2 u_r}{dt^2} + \frac{2E}{(1-v)} \cdot \frac{u_r}{a^2} = \frac{p(t)}{h}$$

Internal Blast Loading

- The simple fundamental radial mode of vibration found by letting $p(t) = 0$ shows the shell to have a period of vibration of $\pi a [2\rho(1-\nu)/E]^{1/2}$.
- Suppose the blast pulse is defined by $p(t) = P(l - t/T)$ for $0 < t < T$ and that $p(t) = 0$ for $t > T$.

$$u_r = \frac{P}{\omega^2 \rho h_0}\left[1 - \frac{t}{T} - \cos\omega t + \frac{\sin\omega t}{\omega T}\right], \quad 0 < t \leq T$$

$$u_r = A\cos\omega(t-T) + B\sin\omega(t-T), \qquad t \geq T$$



$$\omega^2 = \frac{2E}{\rho a^2 (1-\nu)}$$

$$A = \frac{P}{\omega^2 \rho h_0}\left[\frac{\sin\omega T}{\omega T} - \cos\omega T\right]$$

$$B = \frac{P}{\omega^2 \rho h_0}\left[\sin\omega T + \frac{\cos\omega T}{\omega T} - \frac{1}{\omega T}\right]$$

Internal Blast Loading

- With the constants $\nu = 0.3$, $\rho g = 2.21 \times 10^3 N/m^2$, $E = 82.5\ GPa$ and $h/a = 0.0435$, the computed value of the frequency ω is 2680 Hz.
- The maximum displacement $(A^2 + B^2)^{1/2}$ is 1.475×10^{-3} in, and $e_{\theta_{max}} = 1.23 \times 10^{-4}$.
- Experimental data:

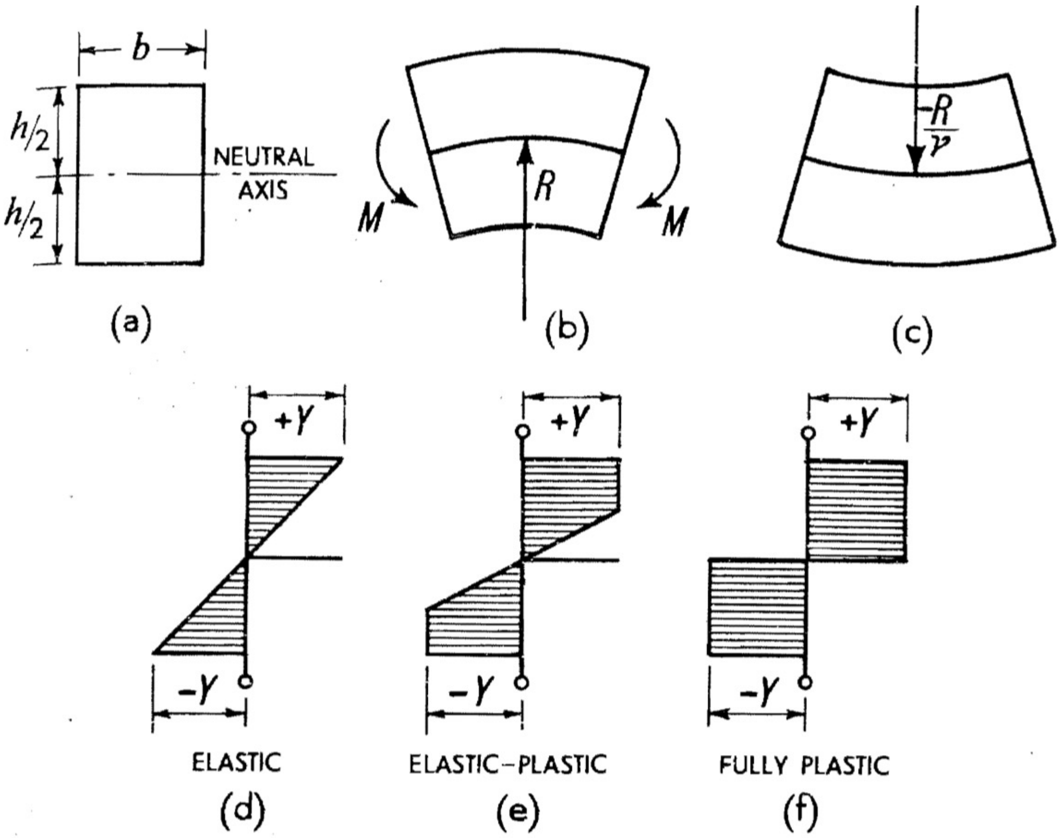
Loading			Response	
P lbf/in^2	Impulse lbf/in^2 ms	T ms	Vibration frequency, cycles/sec	Strain amplitude $e_\theta \cdot 10^4$
528	10·5	0·040	2190	1·68
467	7·9	0·034	2260	1·87

Elementary Theory of Plastic Bending

Plastic Bending

- A straight beam of rectangular cross-section $b \times h$, of elastic-perfectly plastic material, when subjected to an elastic bending moment M, is bent into the arc of a circle of radius R.
- The stress distribution across the section is linear and elastic for $M \leq M_E = bh^2Y/6$; when $M = M_E$ the normal stress on a section in the extreme fibers is just the yield stress.
- For $M_E \leq M \leq M_p = bh^2Y/4$, the stress distribution is elastic-plastic.
- In the limit, the whole section becomes plastic, $M = M_p = [bh/2.Y].h/2 = bh^2Y/4$. M_p is called the full plastic bending moment for the section.

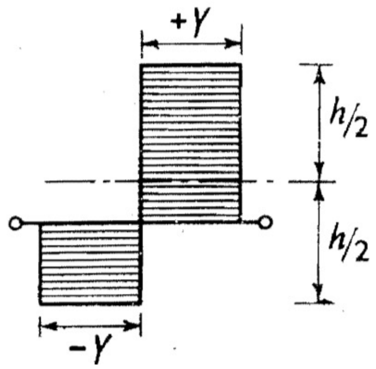
Plastic Bending



Plastic Bending

- The ratio M_p/M_E is called the shape factor for the section and for the rectangle it is 1.5. For a circular section it is $16/3\pi$.
- The stress distribution across the section of a fully plastic rectangular beam which is subject to an axial force, F, and a bending moment, M, is shown.
- The force, F, obviously affects the distribution of normal stress in a section and it may easily be verified that, if $F_p = bhY$, then

$$\left(\frac{M}{M_p}\right) + \left(\frac{F}{F_p}\right)^2 = 1$$

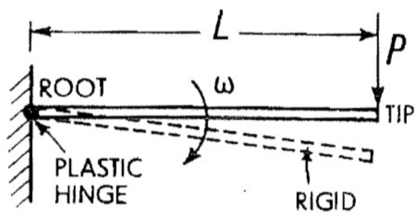


Plastic Hinges in Beams

- The maximum bending moment occurs at the built-in end and when plastic collapse occurs, the tendency is for the bean to rotate about its root as a rigid body.
- The whole section at the root is plastic an the moment prevailing is M_p.
- At the instant collapse occurs, work will be done by load P moving downwards with speed $L\omega$, where ω is the angular speed of the beam about the root;

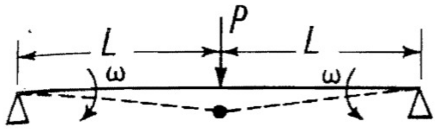
$$PL\omega = M_p\omega, \quad \text{or} \quad P = M_p/L$$

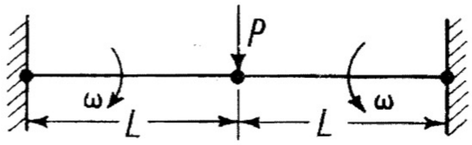
- The fully plastic section at the root of the beam is called a plastic hinge.



Plastic Hinges in Beams

- For a simply supported beam, $P = 2M_p/L$. For a beam clamped at both ends, four hinges arise, and $P = 4M_p/L$.





- If a mass M moving with speed v_0 impinges on the end of a rigid-perfectly plastic cantilever of mass m, then it may be supposed that the end deflection Δ may simply be estimated by equating the plastic work done in the hinge at the root, to the initial kinetic energy of M.

$$\frac{1}{2}Mv_0^2 = M_p \cdot \frac{\Delta}{L} \quad \rightarrow \quad \Delta = \frac{Mv_0^2 L}{2M_p}$$

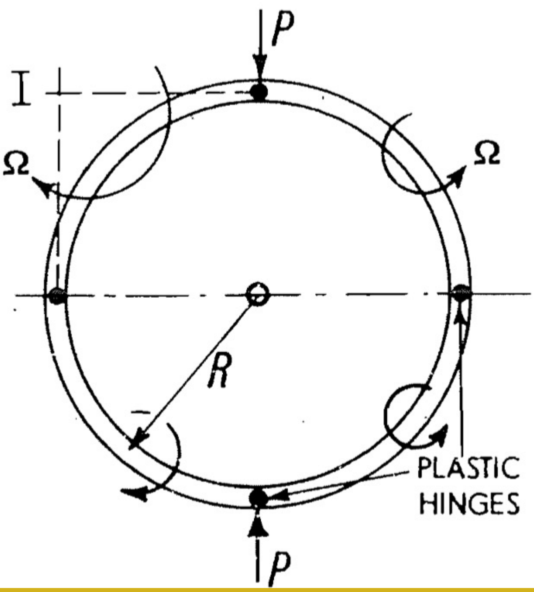
Plastic Hinges in Beams

- If a cantilever rotates as a rigid bar about its root, the tip having a downward speed of u_0, its total kinetic energy is $\frac{1}{2}I\omega^2$ where I is the moment of inertia of the cantilever about the root, i.e. $bL^3h\rho/3$ and angular velocity $\omega = u_0/L$.
- The end deflection Δ for deflections which are not too large is given by,

$$\tfrac{1}{2}I\omega^2 = M_p \cdot \tfrac{\Delta}{L}, \qquad \Delta = bL^2h\rho u_0^2/6M_p$$

Plastic Hinges in Circular Rings

- A circular ring which is neither too thin nor too flexible, of mean radius R when subjected to diametral loads P can only collapse plastically when four hinges have been formed to permit it to behave as a mechanism.



Plastic Hinges in Circular Rings

- It is easy to see that, if the center of the ring, O, remains stationary, then at collapse the four rigid portions of the ring between the hinges rotate with angular speed, Ω, about instantaneous center, I, and the forces P, move towards the center of the ring with speed $\mathrm{R}.\Omega$.
- The work input rate is therefore $2\mathrm{P}.\mathrm{R}.\Omega$ and the plastic work dissipation rate is $8\mathrm{M_p}\Omega$. Thus,

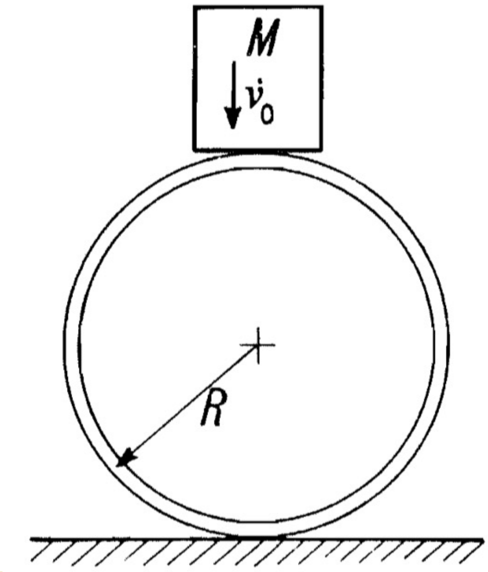
$$2P \cdot R\Omega = 8M_p \cdot \Omega \quad \text{and hence} \quad P = 4M_p/R$$

Plastic Hinges in Circular Rings

- When a large mass, M, impinges with low speed, v_0, on a supported, circular ring of diameter D and small mass, then if all inertia effects in the ring can be neglected, and all the kinetic energy in the mass is used up in enforcing plastic deformation on the ring, the diametral compression ΔD, provided $\Delta D/D$ is small, is very approximately given by,

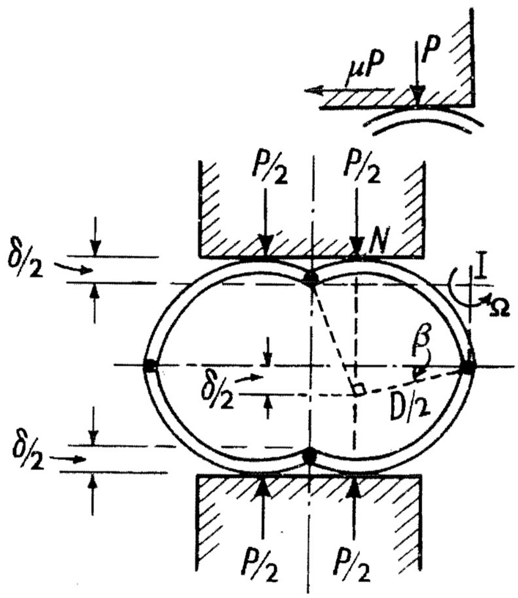
$$\frac{4M_p}{D/2}\cdot\Delta D=\frac{1}{2}Mv_0^2$$

$$\Delta D=\frac{Mv_0^2 D}{16M_p}$$



Plastic Hinges in Circular Rings

- Experiments show that in order to allow for stress wave transmission and other forms of energy loss, it is useful to assume that about 85% of the kinetic energy available is used in doing plastic work.
- When continued quasi-static crushing of a ring occurs between rigid, parallel surfaces, it is evident that the crushing force increases with reduction in vertical diameter of the ring.



Plastic Hinges in Circular Rings

- The original four plastic hinges are maintained but the point(s) of application of the compressing force(s) moves away the center line, splitting into two equal components, $P/2$.

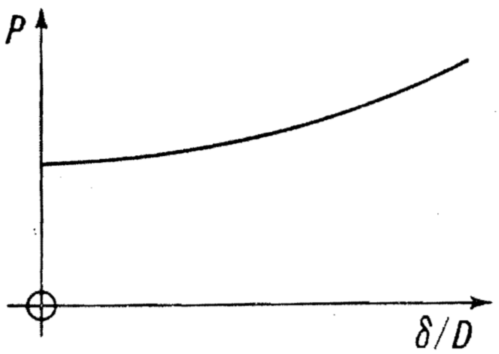
- Applying the same work approach,

$$\frac{P}{2}.NI.\Omega = 2M_p.\Omega, \quad \rightarrow \quad P = \frac{4M_p}{NI}$$

- If b denotes the total ring deflection, $NI^2 = (D^2 - \delta^2)/4$ and thus,

$$\frac{P}{4M_p/R} = \frac{1}{[1-(\delta/D)^2]^{1/2}}$$

Plastic Hinges in Circular Rings



- The form of this $P/(\delta/D)$ curve is shown. This solution holds for $0 \leq \beta \leq \pi/4$.

- A frictional force of $\mu P/2$ would modify to,

$$\frac{P}{2} \cdot NI \cdot \Omega + \mu \frac{P}{2} \cdot MI \cdot \Omega = 2M_p \cdot \Omega$$

$$\frac{P}{4M_p/R} = \frac{1}{[1-(\delta/D)^2]^{1/2} + \mu(1-\cos\beta)}$$

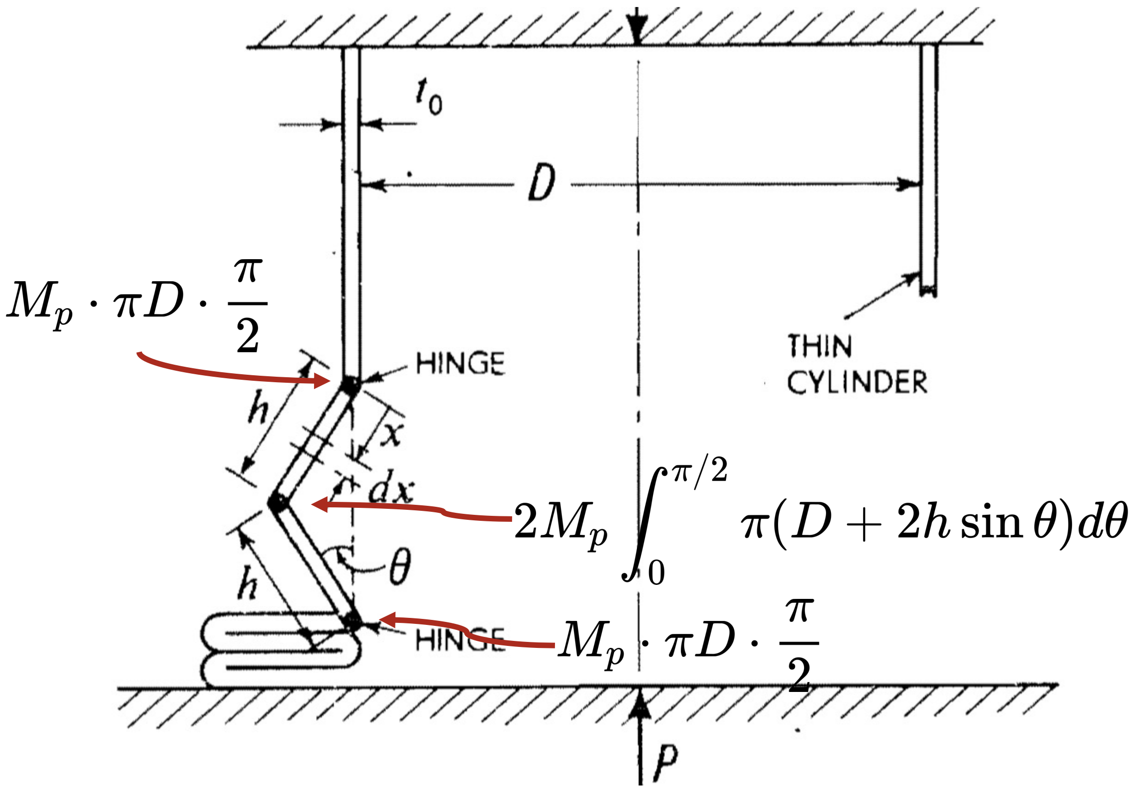
Axial Crushing of Thin Tubes

Crushing a Thin Cylinder

- It is arranged that the dropped component falls on to a thin, uniform cylinder that will buckle lengthwise; it is necessary to be able to estimate the energy absorbing capacity of the cylinder.
  - (i) the energy dissipated due to plastic bending, W_B, in the four circular hinges and
  - (ii) the energy dissipated in stretching, W_S, under uniform tensile yield hoop stress in the metal between the hinges.
- Assuming the material of the cylinder is rigid-perfectly plastic,

$$\begin{aligned} W_B &= 2M_p \cdot \pi D \cdot \frac{\pi}{2} + 2M_p \int_0^{\pi/2} \pi(D + 2h\sin\theta)d\theta \\ &= M_p\pi^2 D + 2M_p\pi\{D\pi/2 + 2h\} \\ &= 2M_p\pi(\pi D + 2h) \end{aligned}$$

Crushing a Thin Cylinder



Crushing a Thin Cylinder

- Using the Mises criterion, $M_p = 2Yt_0^2/4\sqrt{3}$ and

$$W_s = 2\int_0^x Y \cdot \pi D t_0 dx \cdot \ln(D + 2x\sin\theta)/D$$

$$\simeq 2Y \int_0^h \pi D t_0 \tfrac{2x}{D} \cdot dx, \text{ when } \theta \text{ is } 90^\circ$$

$$= 2Y\pi t_0 h^2$$

- The energy for plastic dissipation is supplied by the axial compressive force, P, and is $P.2h$. Thus,

$$P \cdot 2h = 2\left(\frac{2}{\sqrt{3}} \cdot \frac{Yt_0^2}{4}\right)\pi(\pi D + 2h) + 2Y\pi t_0 h^2$$

$$\frac{P}{Y} = \frac{\pi t_0^2}{\sqrt{3}}\left(\frac{\pi D}{2h} + 1\right) + \pi h t_0$$

Crushing a Thin Cylinder

- The value of h is now determined by minimizing (P/Y) with respect to h and thus,

$$\frac{d}{dh}\left(\frac{P}{Y}\right) = \frac{\pi t_0^2}{\sqrt{3}}\left(-\frac{\pi D}{2h^2}\right) + \pi t_0 = 0$$

$$h = \left(\frac{\pi}{2\sqrt{3}}\right)^{1/2} (Dt_0)^{1/2} \simeq 0 \cdot 95 \cdot (Dt_0)^{1/2}$$

- Substituting h, $P/Y \simeq 6t_0(Dt_0)^{1/2} + 1.8\, t_0^2$
- If the buckling convolutions had been entirely internal, rather than wholly external as treated above,

$$P/Y \simeq 6t_0(Dt_0)^{1/2} - 1.8\, t_0^2$$

Crushing a Thin Cylinder

- Denoting the crushed length by x, then if a steady force P applies during the process and assuming all the initial kinetic energy is dissipated in plastic deformation, we have,

$$Px = \frac{1}{2}Mv^2 \quad \text{or} \quad x = \frac{Mv^2}{12Yt_0\sqrt{Dt_0}}$$

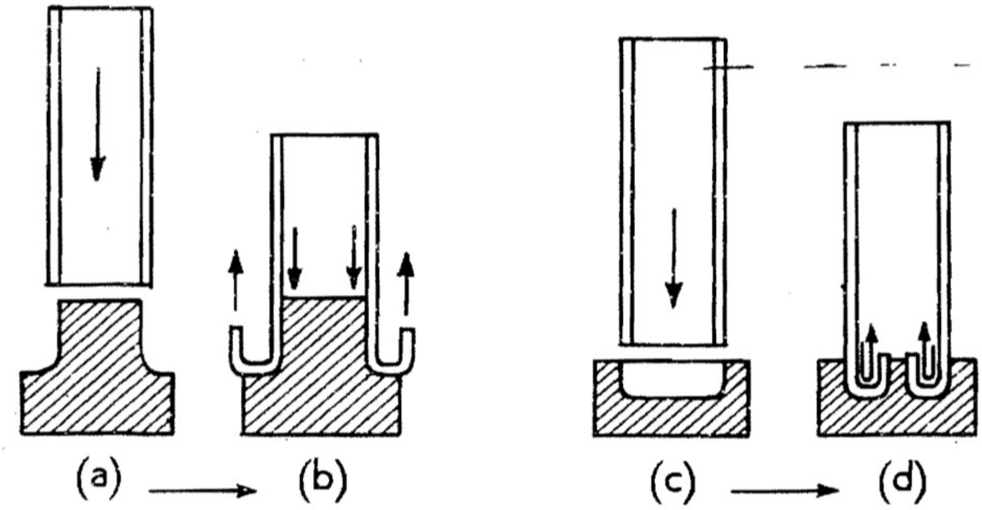
  - where M is the mass of the structure.
- Putting $M = \rho\pi D t_0 L$ where L is the structure length,

$$\frac{x}{L} = \frac{\pi}{12} \cdot \sqrt{\frac{D}{t_0}} \cdot \left(\frac{\rho v^2}{Y}\right)$$

  - This equation could be useful in connection with vehicle design.

Inversion Tubes

- A simple expendable energy absorber which possesses a rectangular force displacement characteristic and a high energy absorption capacity is the 'Inver' tube device.



Inversion Tubes

- Of course inversion is not the only mode of deformation possible for an inver-tube device; alternative modes are
  - (i) column or Euler buckling
  - (ii) concertinaing or axisymmetric buckling,
  - (iii) diamond buckling,
  - (iv) tearing of the tube,
  - (v) brittle fracture, or shattering, and
  - (vi) uniform compression.

Inversion Tubes

- An approximate analytical expression for the steady compressive load necessary to maintain an inside-out inversion is easily arrived at by assuming that the tube material is perfectly plastic, that no tube length or thickness changes occur during bending, that the energy dissipation consists solely in
  - (i) bending and unbending the tube and
  - (ii) in increasing its radius.
- If the radius of inversion is c and the tube thickness is t_0, the work done per unit time in bending a straight element of tube, W_B, at the entrance to the bending zone is,

$$W_B \simeq 2\int_0^{t_0/2} (\pi D \cdot dy \cdot u) Y \cdot \left(\frac{y\,\theta}{c\,\theta}\right) = \frac{\pi D\, t_0^2\, u}{4\,c}\, Y$$

Inversion Tubes

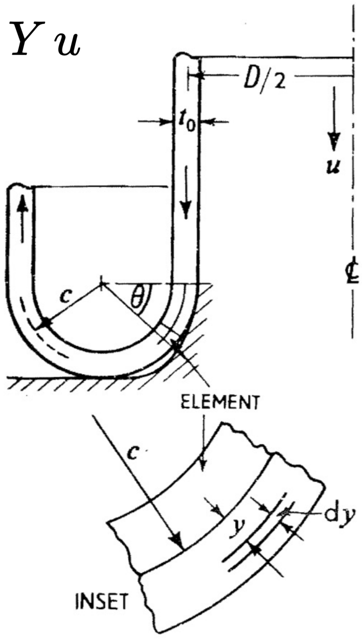
- The amount of work done per unit time in extending tube elements W_E is,

$$W_E = \pi D\, t_0\, u\, Y \ln\left(1 + \frac{2c}{D/2}\right) \cong 4\pi\, t_0\, c\, Y\, u$$

- The rate at which work is done by the compressing force P,

$$P.u = 2W_B + W_E = \pi t_0 Y \left\{\frac{D t_0}{2c} + 4c\right\} u$$

  - The work done in unbending at exit from the curved zone is assumed equal to that done in bending at the entrance to it.
  - c is found by supposing it to acquire a value which makes F a minimum.



$$\frac{D t_0}{2c^2} = 4 \quad \text{or} \quad c = \sqrt{\frac{D t_0}{8}}$$

Inversion Tubes

- So,

$$P = \pi t_0 Y (8Dt_0)^{1/2}$$

- The energy, E, dissipated by plastically inverting the tube, per unit weight, w, of the tube, is,

$$E = \frac{PL}{\pi D t_0 L w} = \frac{Y}{w}\sqrt{\frac{8t_0}{D}}$$

- The mean strain ε_m imparted to each element of the tube following the above equation is the work done on the tube per unit time, divided by the volume deformed times Y.

$$\varepsilon_m = \frac{P \cdot L}{Y \cdot \pi D t_0 L} = \sqrt{\frac{8t_0}{D}}$$

Inversion Tubes

- For a material described by $\sigma = B\varepsilon^n$,

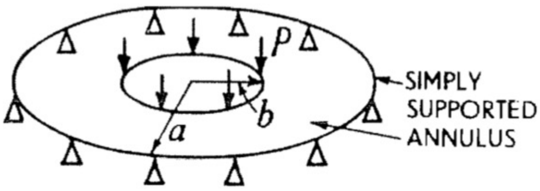
$$\bar{Y} \cdot \varepsilon_m = \int_0^{\varepsilon_m} \sigma \, d\varepsilon = \frac{B\,\varepsilon_m^{n+1}}{n+1} \quad \rightarrow \quad \bar{Y} = \frac{B\,\varepsilon_m^n}{n+1}$$

- Tubes with low strain hardening rates buckle rather than invert.
- The axisymmetric buckling load P_B for a tube whose reduced modulus is E_k is approximately $P_B = 4\pi t_0^2 E_k/3$.
- Thus, inversion would occur only if $P < P_B$ or $E_k/Y > 3(D/2t_0)^{1/2}$

Plastic Bending of Thin Flat Plates

Plastic Bending of Plates

- Annular plate position fixed (zero fixing moment) at its outer periphery and free at its inner boundary.



- The rate at which work is done by the externally applied load is $2\pi b. P. u_0$ and hence,

$$2\pi b. P. u_0 = (a - b).2\pi\omega. M_p \quad \rightarrow \quad P = \frac{M_p}{b}$$

- The total applied load is,

$$P_0 = 2\pi b. P = 2\pi. M_p$$

Plastic Bending of Plates

- If the annulus was subjected to a uniformly distributed pressure, p,

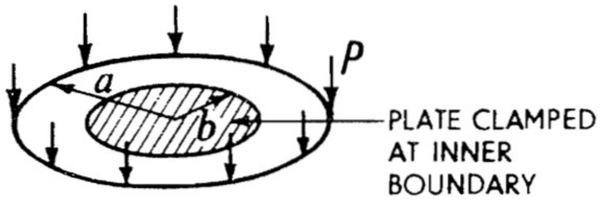
$$p = \frac{6M_p}{(a-b)(a+2b)}$$

- For the circular disc, $b = 0$, and,

$$p = \frac{6M_p}{a^2}$$

- Obviously, this situation is basically similar to that in which the initial condition prescribed for the plate is, that it is position-fixed at its inner radius, b. If it carries a uniform load P' at its outer edge, then $P' = P(b/a)$.

Plastic Bending of Plates



- If the annulus is line-loaded along its outer boundary,

$$P = \frac{M_p}{a-b} \qquad P = 2\pi \frac{a}{a-b} M_p$$

- If the annulus is loaded with a uniform transverse pressure, p, then the plastic collapse pressure is,

$$p = \frac{6\, M_p\, a}{(a-b)^2\,(2a+b)}$$