Topics in Inequalities – Theorems and Techniques

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Contents

Introduction

Inequalities are useful in all fields of Mathematics. The aim of this *problem-oriented* book is to present elementary techniques in the theory of inequalities. The readers will meet classical theorems including *Schur's inequality*, *Muirhead's theorem*, *the Cauchy-Schwarz inequality*, *the Power Mean inequality*, *the AM-GM inequality*, and *Hölder's theorem*. I would greatly appreciate hearing about comments and corrections from my readers. You can send email to me at **ultrametric@gmail.com**

To Students

My target readers are challenging high schools students and undergraduate students. The given techniques in this book are just the tip of the inequalities iceberg. Young students should find their own methods to attack various problems. A great Hungarian Mathematician Paul Erdös was fond of saying that *God has a transfinite book with all the theorems and their best proofs*. I strongly encourage readers to send me their own creative solutions of the problems in this book. **Have fun!**

Acknowledgement

I'm indebted to ***Orlando Döhring*** and ***Darij Grinberg*** for providing me with TeX files including collections of interesting inequalities. I'd like to thank ***Marian Muresan*** for his excellent collection of problems. I'm also pleased that ***Cao Minh Quang*** sent me various vietnam problems and nice proofs of Nesbitt's inequality. I owe great debts to ***Stanley Rabinowitz*** who kindly sent me his paper *On The Computer Solution of Symmetric Homogeneous Triangle Inequalities*.

Resources on the Web

1. *MathLinks*, http://www.mathlinks.ro
2. *Art of Problem Solving*, http://www.artofproblemsolving.com
3. *MathPro Press*, http://www.mathpropress.com
4. K. S. Kedlaya, $A < B$, http://www.unl.edu/amc/a-activities/a4-for-students/s-index.html
5. T. J. Mildorf, *Olympiad Inequalities*, http://web.mit.edu/tmildorf/www

1 Geometric Inequalities

It gives me the same pleasure when someone else proves a good theorem as when I do it myself. E. Landau

1 Ravi Substitution

Many inequalities are simplified by some suitable substitutions. We begin with a classical inequality in triangle geometry. What is the first[1] *nontrivial* geometric inequality ? In 1746, Chapple showed that

Theorem 1.1. ***(Chapple 1746, Euler 1765)*** *Let R and r denote the radii of the circumcircle and incircle of the triangle ABC. Then, we have $R \geq 2r$ and the equality holds if and only if ABC is equilateral.*

Proof. Let $BC = a$, $CA = b$, $AB = c$, $s = \frac{a+b+c}{2}$ and $S = [ABC]$.[2] Recall the well-known identities : $S = \frac{abc}{4R}$, $S = rs$, $S^2 = s(s-a)(s-b)(s-c)$. Hence, $R \geq 2r$ is equivalent to $\frac{abc}{4S} \geq 2\frac{S}{s}$ or $abc \geq 8\frac{S^2}{s}$ or $abc \geq 8(s-a)(s-b)(s-c)$. We need to prove the following. □

[1] The first geometric inequality is the Triangle Inequality : $AB + BC \geq AC$

[2] In this book, $[P]$ stands for the area of the polygon P.

Theorem 1.2. ***([AP], A. Padoa)*** *Let a, b, c be the lengths of a triangle. Then, we have*

$$abc \geq 8(s-a)(s-b)(s-c) \text{ or } abc \geq (b+c-a)(c+a-b)(a+b-c)$$

and the equality holds if and only if $a = b = c$.

Proof. We use the *Ravi* Substitution : Since a, b, c are the lengths of a triangle, there are positive reals x, y, z such that $a = y+z$, $b = z+x$, $c = x+y$. (Why?) Then, the inequality is $(y+z)(z+x)(x+y) \geq 8xyz$ for x, y, $z > 0$. However, we get

$$(y+z)(z+x)(x+y)-8xyz = x(y-z)^2+y(z-x)^2+z(x-y)^2 \geq 0.$$

□

Exercise 1. *Let ABC be a right triangle. Show that*

$$R \geq (1+\sqrt{2})r.$$

When does the equality hold ?

It's natural to ask that the inequality in the theorem 2 holds for arbitrary positive reals a, b, c? Yes ! It's possible to prove the inequality without the additional condition that a, b, c are the lengths of a triangle :

Theorem 1.3. *Let x, y, $z > 0$. Then, we have $xyz \geq (y+z-x)(z+x-y)(x+y-z)$. The equality holds if and only if $x = y = z$.*

Proof. Since the inequality is symmetric in the variables, without loss of generality, we may assume that $x \geq y \geq z$. Then, we have $x+y > z$ and $z+x > y$. If $y+z > x$, then x, y, z are the lengths of the sides of a triangle. In this case, by the theorem 2, we get the result. Now, we may assume that $y+z \leq x$. Then, $xyz > 0 \geq (y+z-x)(z+x-y)(x+y-z)$. □

The inequality in the theorem 2 holds when some of x, y, z are zeros :

Theorem 1.4. *Let x, y, $z \geq 0$. Then, we have $xyz \geq (y+z-x)(z+x-y)(x+y-z)$.*

Proof. Since $x,y,z \geq 0$, we can find *positive* sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ for which

$$\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y,\ \lim_{n\to\infty} z_n = z.$$

Applying the theorem 2 yields

$$x_n y_n z_n \geq (y_n+z_n-x_n)(z_n+x_n-y_n)(x_n+y_n-z_n).$$

Now, taking the limits to both sides, we get the result.

□

Clearly, the equality holds when $x = y = z$. However, $xyz = (y+z-x)(z+x-y)(x+y-z)$ and x, y, $z \geq 0$ does not guarantee that $x = y = z$. In fact, for $x,y,z \geq 0$, the equality $xyz = (y+z-x)(z+x-y)(x+y-z)$ is equivalent to

$$x = y = z \text{ or } x = y, z = 0 \text{ or } y = z, x = 0 \text{ or } z = x, y = 0.$$

It's straightforward to verify the equality

$$xyz-(y+z-x)(z+x-y)(x+y-z) = x(x-y)(x-z)+y(y-z)(y-x)+z(z-x)(z-y).$$

Hence, the theorem 4 is a particular case of Schur's inequality.

Problem 1. ***(IMO 2000/2, Proposed by Titu Andreescu)*** *Let a,b,c be positive numbers such that $abc=1$. Prove that*

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\leq 1.$$

First Solution. Since $abc=1$, we make the substitution $a=\frac{x}{y}$, $b=\frac{y}{z}$, $c=\frac{z}{x}$ for $x,\ y,\ z>0$.[3] We rewrite the given inequality in the terms of $x,\ y,\ z$:

$$\left(\frac{x}{y}-1+\frac{z}{y}\right)\left(\frac{y}{z}-1+\frac{x}{z}\right)\left(\frac{z}{x}-1+\frac{y}{x}\right)\leq 1 \Leftrightarrow xyz\geq(y+z-x)(z+x-y)(x+y-z).$$

$\square$

The *Ravi* Substitution is useful for inequalities for the lengths a, b, c of a triangle. After the *Ravi* Substitution, we can remove the condition that they are the lengths of the sides of a triangle.

Problem 2. ***(IMO 1983/6)*** *Let a, b, c be the lengths of the sides of a triangle. Prove that*

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

First Solution. After setting $a=y+z$, $b=z+x$, $c=x+y$ for $x,y,z>0$, it becomes

$$x^3z+y^3x+z^3y\geq x^2yz+xy^2z+xyz^2 \ \text{ or } \ \frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq x+y+z,$$

which follows from the Cauchy-Schwarz inequality

$$(y+z+x)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)\geq(x+y+z)^2.$$

$\square$

Exercise 2. *Let a, b, c be the lengths of a triangle. Show that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<2.$$

Exercise 3. ***(Darij Grinberg)*** *Let a, b, c be the lengths of a triangle. Show the inequalities*

$$a^3+b^3+c^3+3abc-2b^2a-2c^2b-2a^2c\geq 0,$$

and

$$3a^2b+3b^2c+3c^2a-3abc-2b^2a-2c^2b-2a^2c\geq 0.$$

We now discuss Weitzenböck's inequality and related inequalities.

Problem 3. ***(IMO 1961/2, Weitzenböck's inequality)*** *Let a, b, c be the lengths of a triangle with area S. Show that*

$$a^2+b^2+c^2\geq 4\sqrt{3}S.$$

Solution. Write $a=y+z$, $b=z+x$, $c=x+y$ for $x,y,z>0$. It's equivalent to

$$((y+z)^2+(z+x)^2+(x+y)^2)^2\geq 48(x+y+z)xyz,$$

which can be obtained as following :

$$((y+z)^2+(z+x)^2+(x+y)^2)^2\geq 16(yz+zx+xy)^2\geq 16\cdot 3(xy\cdot yz+yz\cdot zx+xy\cdot yz).$$

Here, we used the well-known inequalities $p^2+q^2\geq 2pq$ and $(p+q+r)^2\geq 3(pq+qr+rp)$. $\square$

[3] For example, take $x=1$, $y=\frac{1}{a}$, $z=\frac{1}{ab}$.

Theorem 1.5. ***(Hadwiger-Finsler inequality)*** *For any triangle ABC with sides a, b, c and area F, the following inequality holds.*

$$2ab+2bc+2ca-(a^2+b^2+c^2)\geq 4\sqrt{3}F.$$

First Proof. After the substitution $a=y+z$, $b=z+x$, $c=x+y$, where $x,y,z>0$, it becomes

$$xy+yz+zx\geq\sqrt{3xyz(x+y+z)},$$

which follows from the identity

$$(xy+yz+zx)^2-3xyz(x+y+z)=\frac{(xy-yz)^2+(yz-zx)^2+(zx-xy)^2}{2}.$$

□

Second Proof. We give a convexity proof. There are many ways to deduce the following identity:

$$\frac{2ab+2bc+2ca-(a^2+b^2+c^2)}{4F}=\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}.$$

Since $\tan x$ is convex on $\left(0,\frac{\pi}{2}\right)$, Jensen's inequality shows that

$$\frac{2ab+2bc+2ca-(a^2+b^2+c^2)}{4F}\geq 3\tan\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)=\sqrt{3}.$$

□

Tsintsifas proved a simultaneous generalization of Weitzenböck's inequality and Nesbitt's inequality.

Theorem 1.6. ***(Tsintsifas)*** *Let p,q,r be positive real numbers and let a,b,c denote the sides of a triangle with area F. Then, we have*

$$\frac{p}{q+r}a^2+\frac{q}{r+p}b^2+\frac{r}{p+q}c^2\geq 2\sqrt{3}F.$$

Proof. (V. Pambuccian) By Hadwiger-Finsler inequality, it suffices to show that

$$\frac{p}{q+r}a^2+\frac{q}{r+p}b^2+\frac{r}{p+q}c^2\geq\frac{1}{2}(a+b+c)^2-(a^2+b^2+c^2)$$

or

$$\left(\frac{p+q+r}{q+r}\right)a^2+\left(\frac{p+q+r}{r+p}\right)b^2+\left(\frac{p+q+r}{p+q}\right)c^2\geq\frac{1}{2}(a+b+c)^2$$

or

$$((q+r)+(r+p)+(p+q))\left(\frac{1}{q+r}a^2+\frac{1}{r+p}b^2+\frac{1}{p+q}c^2\right)\geq(a+b+c)^2.$$

However, this is a straightforward consequence of the Cauchy-Schwarz inequality. □

Theorem 1.7. ***(Neuberg-Pedoe inequality)*** *Let a_1,b_1,c_1 denote the sides of the triangle $A_1B_1C_1$ with area F_1. Let a_2,b_2,c_2 denote the sides of the triangle $A_2B_2C_2$ with area F_2. Then, we have*

$${a_1}^2({b_2}^2+{c_2}^2-{a_2}^2)+{b_1}^2({c_2}^2+{a_2}^2-{b_2}^2)+{c_1}^2({a_2}^2+{b_2}^2-{c_2}^2)\geq 16F_1F_2.$$

Notice that it's a generalization of Weitzenböck's inequality.(Why?) In **[GC]**, G. Chang proved Neuberg-Pedoe inequality by using complex numbers. For very interesting geometric observations and proofs of Neuberg-Pedoe inequality, see **[DP]** or **[GI, pp.92-93]**. Here, we offer three algebraic proofs.

Lemma 1.1.

$$a_1{}^2(a_2{}^2+b_2{}^2-c_2{}^2)+b_1{}^2(b_2{}^2+c_2{}^2-a_2{}^2)+c_1{}^2(c_2{}^2+a_2{}^2-b_2{}^2)>0.$$

Proof. Observe that it's equivalent to

$$(a_1{}^2+b_1{}^2+c_1{}^2)(a_2{}^2+b_2{}^2+c_2{}^2)>2(a_1{}^2a_2{}^2+b_1{}^2b_2{}^2+c_1{}^2c_2{}^2).$$

From Heron's formula, we find that, for $i=1,2$,

$$16F_i^2=(a_i{}^2+b_i{}^2+c_i{}^2)^2-2(a_i{}^4+b_i{}^4+c_i{}^4)>0 \quad\text{or}\quad a_i{}^2+b_i{}^2+c_i{}^2>\sqrt{2(a_i{}^4+b_i{}^4+c_i{}^4)}\,.$$

The Cauchy-Schwarz inequality implies that

$$\begin{aligned}(a_1{}^2+b_1{}^2+c_1{}^2)(a_2{}^2+b_2{}^2+c_2{}^2) &> 2\sqrt{(a_1{}^4+b_1{}^4+c_1{}^4)(a_2{}^4+b_2{}^4+c_2{}^4)}\\ &\geq 2(a_1{}^2a_2{}^2+b_1{}^2b_2{}^2+c_1{}^2c_2{}^2).\end{aligned}$$

□

First Proof. **([LC1], Carlitz)** By the lemma, we obtain

$$L=a_1{}^2(b_2{}^2+c_2{}^2-a_2{}^2)+b_1{}^2(c_2{}^2+a_2{}^2-b_2{}^2)+c_1{}^2(a_2{}^2+b_2{}^2-c_2{}^2)>0,$$

Hence, we need to show that

$$L^2-(16F_1{}^2)(16F_2{}^2)\geq 0.$$

One may easily check the following identity

$$L^2-(16F_1{}^2)(16F_2{}^2)=-4(UV+VW+WU),$$

where

$$U=b_1{}^2c_2{}^2-b_2{}^2c_1{}^2,\ V=c_1{}^2a_2{}^2-c_2{}^2a_1{}^2 \ \text{and}\ W=a_1{}^2b_2{}^2-a_2{}^2b_1{}^2.$$

Using the identity

$$a_1{}^2U+b_1{}^2V+c_1{}^2W=0 \ \text{or}\ W=-\frac{a_1{}^2}{c_1{}^2}U-\frac{b_1{}^2}{c_1{}^2}V,$$

one may also deduce that

$$UV+VW+WU=-\frac{a_1{}^2}{c_1{}^2}\left(U-\frac{c_1{}^2-a_1{}^2-b_1{}^2}{2a_1{}^2}V\right)^2-\frac{4a_1{}^2b_1{}^2-(c_1{}^2-a_1{}^2-b_1{}^2)^2}{4a_1{}^2c_1{}^2}V^2.$$

It follows that

$$UV+VW+WU=-\frac{a_1{}^2}{c_1{}^2}\left(U-\frac{c_1{}^2-a_1{}^2-b_1{}^2}{2a_1{}^2}V\right)^2-\frac{16F_1{}^2}{4a_1{}^2c_1{}^2}V^2\leq 0.$$

□

Carlitz also observed that the Neuberg-Pedoe inequality can be deduced from Aczél's inequality.

Theorem 1.8. ***(Aczél's inequality)*** *Let $a_1,\cdots,a_n,b_1,\cdots,b_n$ be positive real numbers satisfying*

$$a_1{}^2\geq a_2{}^2+\cdots+a_n{}^2 \ \textit{and}\ b_1{}^2\geq b_2{}^2+\cdots+b_n{}^2.$$

Then, the following inequality holds.

$$a_1b_1-(a_2b_2+\cdots+a_nb_n)\geq\sqrt{\left(a_1{}^2-(a_2{}^2+\cdots+a_n{}^2)\right)\left(b_1{}^2-\left(b_2{}^2+\cdots+b_n{}^2\right)\right)}$$

Proof. **([AI])** The Cauchy-Schwarz inequality shows that

$$a_1b_1 \geq \sqrt{(a_2{}^2+\cdots+a_n{}^2)(b_2{}^2+\cdots+b_n{}^2)} \geq a_2b_2+\cdots+a_nb_n.$$

Then, the above inequality is equivalent to

$$(a_1b_1-(a_2b_2+\cdots+a_nb_n))^2 \geq \left(a_1{}^2-\left(a_2{}^2+\cdots+a_n{}^2\right)\right)\left(b_1{}^2-\left(b_2{}^2+\cdots+b_n{}^2\right)\right).$$

In case $a_1{}^2-(a_2{}^2+\cdots+a_n{}^2)=0$, it's trivial. Hence, we now assume that $a_1{}^2-(a_2{}^2+\cdots+a_n{}^2)>0$. The main trick is to think of the following quadratic polynomial

$$P(x)=(a_1x-b_1)^2-\sum_{i=2}^{n}(a_ix-b_i)^2=\left(a_1{}^2-\sum_{i=2}^{n}a_i{}^2\right)x^2+2\left(a_1b_1-\sum_{i=2}^{n}a_ib_i\right)x+\left(b_1{}^2-\sum_{i=2}^{n}b_i{}^2\right).$$

Since $P(\frac{b_1}{a_1})=-\sum_{i=2}^{n}\left(a_i\left(\frac{b_1}{a_1}\right)-b_i\right)^2\leq 0$ and since the coefficient of x^2 in the quadratic polynomial P is positive, P should have at least one real root. Therefore, P has nonnegative discriminant. It follows that

$$\left(2\left(a_1b_1-\sum_{i=2}^{n}a_ib_i\right)\right)^2-4\left(a_1{}^2-\sum_{i=2}^{n}a_i{}^2\right)\left(b_1{}^2-\sum_{i=2}^{n}b_i{}^2\right)\geq 0.$$

□

Second Proof of Neuberg-Pedoe inequality. **([LC2], Carlitz)** We rewrite it in terms of a_1, b_1, c_1, a_2, b_2, c_2:

$$(a_1{}^2+b_1{}^2+c_1{}^2)(a_2{}^2+b_2{}^2+c_2{}^2)-2(a_1{}^2a_2{}^2+b_1{}^2b_2{}^2+c_1{}^2c_2{}^2)$$

$$\geq\sqrt{\left(\left(a_1{}^2+b_1{}^2+c_1{}^2\right)^2-2(a_1{}^4+b_1{}^4+c_1{}^4)\right)\left(\left(a_2{}^2+b_2{}^2+c_2{}^2\right)^2-2(a_2{}^4+b_2{}^4+c_2{}^4)\right)}.$$

We employ the following substitutions

$$x_1=a_1{}^2+b_1{}^2+c_1{}^2, x_2=\sqrt{2}a_1{}^2, x_3=\sqrt{2}b_1{}^2, x_4=\sqrt{2}c_1{}^2,$$

$$y_1=a_2{}^2+b_2{}^2+c_2{}^2, y_2=\sqrt{2}a_2{}^2, y_3=\sqrt{2}b_2{}^2, y_4=\sqrt{2}c_2{}^2.$$

As in the proof of the lemma 5, we have

$$x_1{}^2>x_2{}^2+y_3{}^2+x_4{}^2 \text{ and } y_1{}^2>y_2{}^2+y_3{}^2+y_4{}^2.$$

We now apply Aczél's inequality to get the inequality

$$x_1y_1-x_2y_2-x_3y_3-x_4y_4\geq\sqrt{(x_1{}^2-(x_2{}^2+y_3{}^2+x_4{}^2))(y_1{}^2-(y_2{}^2+y_3{}^2+y_4{}^2))}.$$

□

We close this section with a very simple proof by a former student in KMO[4] summer program.

Third Proof. Toss two triangles $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ on $\mathbb{R}^2$:

$$A_1(0,p_1),\ B_1(p_2,0),\ C_1(p_3,0),\ A_2(0,q_1),\ B_2(q_2,0),\text{ and } C_2(q_3,0).$$

[4]Korean Mathematical Olympiads

It therefore follows from the inequality $x^2+y^2 \geq 2|xy|$ that

$$\begin{aligned}
& {a_1}^2({b_2}^2+{c_2}^2-{a_2}^2)+{b_1}^2({c_2}^2+{a_2}^2-{b_2}^2)+{c_1}^2({a_2}^2+{b_2}^2-{c_2}^2) \\
= \ & (p_3-p_2)^2(2{q_1}^2+2q_1q_2)+({p_1}^2+{p_3}^2)(2{q_2}^2-2q_2q_3)+({p_1}^2+{p_2}^2)(2{q_3}^2-2q_2q_3) \\
= \ & 2(p_3-p_2)^2{q_1}^2+2(q_3-q_2)^2{p_1}^2+2(p_3q_2-p_2q_3)^2 \\
\geq \ & 2((p_3-p_2)q_1)^2+2((q_3-q_2)p_1)^2 \\
\geq \ & 4|(p_3-p_2)q_1|\cdot|(q_3-q_2)p_1| \\
= \ & 16F_1F_2\ .
\end{aligned}$$

□

2 Trigonometric Methods

In this section, we employ trigonometric methods to attack geometric inequalities.

Theorem 1.9. ***(Erdös-Mordell Theorem)*** *If from a point P inside a given triangle ABC perpendiculars PH_1, PH_2, PH_3 are drawn to its sides, then $PA+PB+PC \geq 2(PH_1+PH_2+PH_3)$.*

This was conjectured by Paul Erdös in 1935, and first proved by Mordell in the same year. Several proofs of this inequality have been given, using Ptolemy's theorem by André Avez, angular computations with similar triangles by Leon Bankoff, area inequality by V. Komornik, or using trigonometry by Mordell and Barrow.

Proof. (**[MB], Mordell**) We transform it to a trigonometric inequality. Let $h_1=PH_1$, $h_2=PH_2$ and $h_3=PH_3$. Apply the Since Law and the Cosine Law to obtain

$$\begin{aligned}
PA\sin A=\overline{H_2H_3} &= \sqrt{{h_2}^2+{h_3}^2-2h_2h_3\cos(\pi-A)}, \\
PB\sin B=\overline{H_3H_1} &= \sqrt{{h_3}^2+{h_1}^2-2h_3h_1\cos(\pi-B)}, \\
PC\sin C=\overline{H_1H_2} &= \sqrt{{h_1}^2+{h_2}^2-2h_1h_2\cos(\pi-C)}.
\end{aligned}$$

So, we need to prove that

$$\sum_{\text{cyclic}}\frac{1}{\sin A}\sqrt{{h_2}^2+{h_3}^2-2h_2h_3\cos(\pi-A)} \geq 2(h_1+h_2+h_3).$$

The main trouble is that the left hand side has too *heavy* terms with square root expressions. Our strategy is to find a lower bound without square roots. To this end, we express the terms inside the square root as **the sum of two squares**.

$$\begin{aligned}
\overline{H_2H_3}^2 &= {h_2}^2+{h_3}^2-2h_2h_3\cos(\pi-A) \\
&= {h_2}^2+{h_3}^2-2h_2h_3\cos(B+C) \\
&= {h_2}^2+{h_3}^2-2h_2h_3(\cos B\cos C-\sin B\sin C).
\end{aligned}$$

Using $\cos^2 B+\sin^2 B=1$ and $\cos^2 C+\sin^2 C=1$, one finds that

$$\overline{H_2H_3}^2=(h_2\sin C+h_3\sin B)^2+(h_2\cos C-h_3\cos B)^2\ .$$

Since $(h_2\cos C - h_3\cos B)^2$ is clearly nonnegative, we get $\overline{H_2H_3} \geq h_2\sin C + h_3\sin B$. It follows that

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{\sqrt{{h_2}^2 + {h_3}^2 - 2h_2h_3\cos(\pi - A)}}{\sin A} &\geq \sum_{\text{cyclic}} \frac{h_2\sin C + h_3\sin B}{\sin A} \\
&= \sum_{\text{cyclic}} \left(\frac{\sin B}{\sin C} + \frac{\sin C}{\sin B}\right) h_1 \\
&\geq \sum_{\text{cyclic}} 2\sqrt{\frac{\sin B}{\sin C}\cdot\frac{\sin C}{\sin B}} h_1 \\
&= 2h_1 + 2h_2 + 2h_3.
\end{aligned}$$

□

We use the same techniques to attack the following geometric inequality.

Problem 4. ***(IMO Short-list 2005)*** *In an acute triangle ABC, let D, E, F, P, Q, R be the feet of perpendiculars from A, B, C, A, B, C to BC, CA, AB, EF, FD, DE, respectively. Prove that*

$$p(ABC)p(PQR) \geq p(DEF)^2,$$

where $p(T)$ denotes the perimeter of triangle T.

Solution. Let's ***euler***[5] this problem. Let ρ be the circumradius of the triangle ABC. It is easy to show that $BC = 2\rho\sin A$ and $EF = 2\rho\sin A\cos A$. Since $DQ = 2\rho\cdot\sin C\cos B\cos A$, $DR = 2\rho\cdot\sin B\cos C\cos A$, and $\angle FDE = \pi - 2A$, the Cosine Law gives us

$$\begin{aligned}
QR^2 &= DQ^2 + DR^2 - 2DQ\cdot DR\cos(\pi - 2A) \\
&= 4\rho^2\cos^2 A\left[(\sin C\cos B)^2 + (\sin B\cos C)^2 + 2\sin C\cos B\sin B\cos C\cos(2A)\right]
\end{aligned}$$

or

$$QR = 2\rho\cos A\sqrt{f(A,B,C)},$$

where

$$f(A,B,C) = (\sin C\cos B)^2 + (\sin B\cos C)^2 + 2\sin C\cos B\sin B\cos C\cos(2A).$$

So, what we need to attack is the following inequality:

$$\left(\sum_{\text{cyclic}} 2\rho\sin A\right)\left(\sum_{\text{cyclic}} 2\rho\cos A\sqrt{f(A,B,C)}\right) \geq \left(\sum_{\text{cyclic}} 2\rho\sin A\cos A\right)^2$$

or

$$\left(\sum_{\text{cyclic}} \sin A\right)\left(\sum_{\text{cyclic}} \cos A\sqrt{f(A,B,C)}\right) \geq \left(\sum_{\text{cyclic}} \sin A\cos A\right)^2.$$

Our job is now to find a reasonable lower bound of $\sqrt{f(A,B,C)}$. Once again, we express $f(A,B,C)$ as **the sum of two squares**. We observe that

$$\begin{aligned}
f(A,B,C) &= (\sin C\cos B)^2 + (\sin B\cos C)^2 + 2\sin C\cos B\sin B\cos C\cos(2A) \\
&= (\sin C\cos B + \sin B\cos C)^2 + 2\sin C\cos B\sin B\cos C\,[-1 + \cos(2A)] \\
&= \sin^2(C+B) - 2\sin C\cos B\sin B\cos C\cdot 2\sin^2 A \\
&= \sin^2 A\,[1 - 4\sin B\sin C\cos B\cos C].
\end{aligned}$$

[5]**euler** v. (in Mathematics) transform the problems in triangle geometry to trigonometric ones

So, we shall express $1-4\sin B\sin C\cos B\cos C$ as the sum of two squares. The trick is to replace 1 with $\left(\sin^2 B+\cos^2 B\right)\left(\sin^2 C+\cos^2 C\right)$. Indeed, we get

$$\begin{aligned}1-4\sin B\sin C\cos B\cos C &= \left(\sin^2 B+\cos^2 B\right)\left(\sin^2 C+\cos^2 C\right)-4\sin B\sin C\cos B\cos C\\ &= (\sin B\cos C-\sin C\cos B)^2+(\cos B\cos C-\sin B\sin C)^2\\ &= \sin^2(B-C)+\cos^2(B+C)\\ &= \sin^2(B-C)+\cos^2 A.\end{aligned}$$

It therefore follows that

$$f(A,B,C)=\sin^2 A\left[\sin^2(B-C)+\cos^2 A\right]\geq \sin^2 A\cos^2 A$$

so that

$$\sum_{\text{cyclic}}\cos A\sqrt{f(A,B,C)}\geq \sum_{\text{cyclic}}\sin A\cos^2 A.$$

So, we can complete the proof if we establish that

$$\left(\sum_{\text{cyclic}}\sin A\right)\left(\sum_{\text{cyclic}}\sin A\cos^2 A\right)\geq\left(\sum_{\text{cyclic}}\sin A\cos A\right)^2.$$

Indeed, one sees that it's a direct consequence of the Cauchy-Schwarz inequality

$$(p+q+r)(x+y+z)\geq(\sqrt{px}+\sqrt{qy}+\sqrt{rz})^2,$$

where p,q,r,x,y and z are positive real numbers. $\square$

Alternatively, one may obtain another lower bound of $f(A,B,C)$:

$$\begin{aligned}f(A,B,C) &= (\sin C\cos B)^2+(\sin B\cos C)^2+2\sin C\cos B\sin B\cos C\cos(2A)\\ &= (\sin C\cos B-\sin B\cos C)^2+2\sin C\cos B\sin B\cos C\,[1+\cos(2A)]\\ &= \sin^2(B-C)+2\frac{\sin(2B)}{2}\cdot\frac{\sin(2C)}{2}\cdot 2\cos^2 A\\ &\geq \cos^2 A\sin(2B)\sin(2C).\end{aligned}$$

Then, we can use this to offer a lower bound of the perimeter of triangle PQR:

$$p(PQR)=\sum_{\text{cyclic}}2\rho\cos A\sqrt{f(A,B,C)}\geq\sum_{\text{cyclic}}2\rho\cos^2 A\sqrt{\sin 2B\sin 2C}$$

So, one may consider the following inequality:

$$p(ABC)\sum_{\text{cyclic}}2\rho\cos^2 A\sqrt{\sin 2B\sin 2C}\geq p(DEF)^2$$

or

$$\left(2\rho\sum_{\text{cyclic}}\sin A\right)\left(\sum_{\text{cyclic}}2\rho\cos^2 A\sqrt{\sin 2B\sin 2C}\right)\geq\left(2\rho\sum_{\text{cyclic}}\sin A\cos A\right)^2.$$

or

$$\left(\sum_{\text{cyclic}}\sin A\right)\left(\sum_{\text{cyclic}}\cos^2 A\sqrt{\sin 2B\sin 2C}\right)\geq\left(\sum_{\text{cyclic}}\sin A\cos A\right)^2.$$

However, it turned out that this doesn't hold. Try to disprove this!

Problem 5. *Let I be the incenter of the triangle ABC with $BC = a$, $CA = b$ and $AB = c$. Prove that, for all points X,*

$$aXA^2 + bXB^2 + cXC^2 \geq abc.$$

Proof. This geometric inequality follows from the following geometric identity:

$$aXA^2 + bXB^2 + cXC^2 = (a+b+c)XI^2 + abc.^{6}$$

There are many ways to establish this identity. To ***euler*** this, we toss the picture on the cartesian plane so that $A(c\cos B, c\sin B)$, $B(0,0)$ and $C(a,0)$. Letting r be the inradius of ABC and $s = \frac{a+b+c}{2}$, we get $I(s-b, r)$. It's well-known that

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s}.$$

Set $X(p,q)$. On the one hand, we obtain

$$\begin{aligned}
& aXA^2 + bXB^2 + cXC^2 \\
= \; & a\left[(p-c\cos B)^2 + (q-c\sin B)^2\right] + b\left(p^2+q^2\right) + c\left[(p-a)^2+q^2\right] \\
= \; & (a+b+c)p^2 - 2acp(1+\cos B) + (a+b+c)q^2 - 2acq\sin B + ac^2 + a^2c \\
= \; & 2sp^2 - 2acp\left(1 + \frac{a^2+c^2-b^2}{2ac}\right) + 2sq^2 - 2acq\frac{[\triangle ABC]}{\frac{1}{2}ac} + ac^2 + a^2c \\
= \; & 2sp^2 - p(a+c+b)(a+c-b) + 2sq^2 - 4q[\triangle ABC] + ac^2 + a^2c \\
= \; & 2sp^2 - p(2s)(2s-2b) + 2sq^2 - 4qsr + ac^2 + a^2c \\
= \; & 2sp^2 - 4s(s-b)p + 2sq^2 - 4rsq + ac^2 + a^2c.
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
& (a+b+c)XI^2 + abc \\
= \; & 2s\left[(p-(s-b))^2 + (q-r)^2\right] \\
= \; & 2s\left[p^2 - 2(s-b)p + (s-b)^2 + q^2 - 2qr + r^2\right] \\
= \; & 2sp^2 - 4s(s-b)p + 2s(s-b)^2 + 2sq^2 - 4rsq + 2sr^2 + abc.
\end{aligned}$$

It follows that

$$\begin{aligned}
& aXA^2 + bXB^2 + cXC^2 - (a+b+c)XI^2 - abc. \\
= \; & ac^2 + a^2c - 2s(s-b)^2 - 2sr^2 - abc \\
= \; & ac(a+c) - 2s(s-b)^2 - 2(s-a)(s-b)(s-c) - abc \\
= \; & ac(a+c-b) - 2s(s-b)^2 - 2(s-a)(s-b)(s-c) \\
= \; & 2ac(s-b) - 2s(s-b)^2 - 2(s-a)(s-b)(s-c) \\
= \; & 2(s-b)\left[ac - s(s-b) - 2(s-a)(s-c)\right].
\end{aligned}$$

However, we compute $ac - s(s-b) - 2(s-a)(s-c) = -2s^2 + (a+b+c)s = 0$. □

Problem 6. ***(IMO 2001/1)*** *Let ABC be an acute-angled triangle with O as its circumcenter. Let P on line BC be the foot of the altitude from A. Assume that $\angle BCA \geq \angle ABC + 30^\circ$. Prove that $\angle CAB + \angle COP < 90^\circ$.*

[6] **IMO Short-list 1988**

Proof. The angle inequality $\angle CAB + \angle COP < 90^\circ$ can be written as $\angle COP < \angle PCO$. This can be shown if we establish the length inequality $OP > PC$. Since the power of P with respect to the circumcircle of ABC is $OP^2 = R^2 - BP \cdot PC$, where R is the circumradius of the triangle ABC, it becomes $R^2 - BP \cdot PC > PC^2$ or $R^2 > BC \cdot PC$. We ***euler*** this. It's an easy job to get $BC = 2R\sin A$ and $PC = 2R\sin B\cos C$. Hence, we show the inequality $R^2 > 2R\sin A \cdot 2R\sin B\cos C$ or $\sin A\sin B\cos C < \frac{1}{4}$. Since $\sin A < 1$, it suffices to show that $\sin A\sin B\cos C < \frac{1}{4}$. Finally, we use the angle condition $\angle C \geq \angle B + 30^\circ$ to obtain the trigonometric inequality

$$\sin B\cos C = \frac{\sin(B+C) - \sin(C-B)}{2} \leq \frac{1 - \sin(C-B)}{2} \leq \frac{1 - \sin 30^\circ}{2} = \frac{1}{4}.$$

□

We close this section with Barrows' inequality stronger than Erdös-Mordell Theorem. We need the following trigonometric inequality:

Proposition 1.1. *Let $x, y, z, \theta_1, \theta_2, \theta_3$ be real numbers with $\theta_1 + \theta_2 + \theta_3 = \pi$. Then,*

$$x^2 + y^2 + z^2 \geq 2(yz\cos\theta_1 + zx\cos\theta_2 + xy\cos\theta_3).$$

Proof. Using $\theta_3 = \pi - (\theta_1 + \theta_2)$, it's an easy job to check the following identity

$$x^2 + y^2 + z^2 - 2(yz\cos\theta_1 + zx\cos\theta_2 + xy\cos\theta_3) = (z - (x\cos\theta_2 + y\cos\theta_1))^2 + (x\sin\theta_2 - y\sin\theta_1)^2.$$

□

Corollary 1.1. *Let p, q, and r be positive real numbers. Let θ_1, θ_2, and θ_3 be real numbers satisfying $\theta_1 + \theta_2 + \theta_3 = \pi$. Then, the following inequality holds.*

$$p\cos\theta_1 + q\cos\theta_2 + r\cos\theta_3 \leq \frac{1}{2}\left(\frac{qr}{p} + \frac{rp}{q} + \frac{pq}{r}\right).$$

Proof. Take $(x,y,z) = \left(\sqrt{\frac{qr}{p}}, \sqrt{\frac{rp}{q}}, \sqrt{\frac{pq}{r}}\right)$ and apply the above proposition. □

Theorem 1.10. ***(Barrow's Inequality)*** *Let P be an interior point of a triangle ABC and let U, V, W be the points where the bisectors of angles BPC, CPA, APB cut the sides BC,CA,AB respectively. Prove that $PA + PB + PC \geq 2(PU + PV + PW)$.*

Proof. (**[MB] and [AK]**) Let $d_1 = PA$, $d_2 = PB$, $d_3 = PC$, $l_1 = PU$, $l_2 = PV$, $l_3 = PW$, $2\theta_1 = \angle BPC$, $2\theta_2 = \angle CPA$, and $2\theta_3 = \angle APB$. We need to show that $d_1 + d_2 + d_3 \geq 2(l_1 + l_2 + l_3)$. It's easy to deduce the following identities

$$l_1 = \frac{2d_2d_3}{d_2 + d_3}\cos\theta_1,\ l_2 = \frac{2d_3d_1}{d_3 + d_1}\cos\theta_2,\ \text{and}\ l_3 = \frac{2d_1d_2}{d_1 + d_2}\cos\theta_3,$$

By the AM-GM inequality and the above corollary, this means that

$$l_1 + l_2 + l_3 \leq \sqrt{d_2d_3}\cos\theta_1 + \sqrt{d_3d_1}\cos\theta_2 + \sqrt{d_1d_2}\cos\theta_3 \leq \frac{1}{2}(d_1 + d_2 + d_3).$$

□

As another application of the above trigonometric proposition, we establish the following inequality

Corollary 1.2. *(**[AK], Abi-Khuzam**) Let $x_1,\cdots,x_4$ be positive real numbers. Let $\theta_1,\cdots,\theta_4$ be real numbers such that $\theta_1+\cdots+\theta_4=\pi$. Then,*

$$x_1\cos\theta_1+x_2\cos\theta_2+x_3\cos\theta_3+x_4\cos\theta_4\le\sqrt{\frac{(x_1x_2+x_3x_4)(x_1x_3+x_2x_4)(x_1x_4+x_2x_3)}{x_1x_2x_3x_4}}.$$

Proof. Let $p=\frac{{x_1}^2+{x_2}^2}{2x_1x_2}+\frac{{x_3}^2+{x_4}^2}{2x_3x_4}$ $q=\frac{x_1x_2+x_3x_4}{2}$ and $\lambda=\sqrt{\frac{p}{q}}$. In the view of $\theta_1+\theta_2+(\theta_3+\theta_4)=\pi$ and $\theta_3+\theta_4+(\theta_1+\theta_2)=\pi$, the proposition implies that

$$x_1\cos\theta_1+x_2\cos\theta_2+\lambda\cos(\theta_3+\theta_4)\le p\lambda=\sqrt{pq},$$

and

$$x_3\cos\theta_3+x_4\cos\theta_4+\lambda\cos(\theta_1+\theta_2)\le\frac{q}{\lambda}=\sqrt{pq}.$$

Since $\cos(\theta_3+\theta_4)+\cos(\theta_1+\theta_2)=0$, adding these two above inequalities yields

$$x_1\cos\theta_1+x_2\cos\theta_2+x_3\cos\theta_3+x_4\cos\theta_4\le2\sqrt{pq}=\sqrt{\frac{(x_1x_2+x_3x_4)(x_1x_3+x_2x_4)(x_1x_4+x_2x_3)}{x_1x_2x_3x_4}}.$$

□

3 Applications of Complex Numbers

In this section, we discuss some applications of complex numbers to geometric inequality. Every complex number corresponds to a unique point in the complex plane. The standard symbol for the set of all complex numbers is $\mathbb{C}$, and we also refer to the complex plane as $\mathbb{C}$. The main tool is applications of the following fundamental inequality.

Theorem 1.11. *If $z_1,\cdots,z_n\in\mathbb{C}$, then $|z_1|+\cdots+|z_n|\ge|z_1+\cdots+z_n|$.*

Proof. Induction on n. □

Theorem 1.12. *(**Ptolemy's Inequality**) For any points A,B,C,D in the plane, we have*

$$\overline{AB}\cdot\overline{CD}+\overline{BC}\cdot\overline{DA}\ge\overline{AC}\cdot\overline{BD}.$$

Proof. Let a, b, c and 0 be complex numbers that correspond to A,B,C,D in the complex plane. It becomes

$$|a-b|\cdot|c|+|b-c|\cdot|a|\ge|a-c|\cdot|b|.$$

Applying the Triangle Inequality to the identity $(a-b)c+(b-c)a=(a-c)b$, we get the result. □

Problem 7. *(**[TD]**) Let P be an arbitrary point in the plane of a triangle ABC with the centroid G. Show the following inequalities*

(1) $\overline{BC}\cdot\overline{PB}\cdot\overline{PC}+\overline{AB}\cdot\overline{PA}\cdot\overline{PB}+\overline{CA}\cdot\overline{PC}\cdot\overline{PA}\ge\overline{BC}\cdot\overline{CA}\cdot\overline{AB}$ and
(2) $\overline{PA}^3\cdot\overline{BC}+\overline{PB}^3\cdot\overline{CA}+\overline{PC}^3\cdot\overline{AB}\ge3\overline{PG}\cdot\overline{BC}\cdot\overline{CA}\cdot\overline{AB}$.

Solution. We only check the first inequality. Regard A,B,C,P as complex numbers and assume that P corresponds to 0. We're required to prove that

$$|(B-C)BC|+|(A-B)AB|+|(C-A)CA|\ge|(B-C)(C-A)(A-B)|.$$

It remains to apply the Triangle Inequality to the identity

$$(B-C)BC+(A-B)AB+(C-A)CA=-(B-C)(C-A)(A-B).$$

□

Problem 8. ***(IMO Short-list 2002)*** *Let ABC be a triangle for which there exists an interior point F such that $\angle AFB = \angle BFC = \angle CFA$. Let the lines BF and CF meet the sides AC and AB at D and E, respectively. Prove that $\overline{AB} + \overline{AC} \geq 4\overline{DE}$.*

Solution. Let $\overline{AF} = x, \overline{BF} = y, \overline{CF} = z$ and let $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. We can toss the pictures on $\mathbb{C}$ so that the points $F, A, B, C, D,$ and E are represented by the complex numbers $0, x, y\omega, z\omega^2, d,$ and e. It's an easy exercise to establish that $\overline{DF} = \frac{xz}{x+z}$ and $\overline{EF} = \frac{xy}{x+y}$. This means that $d = -\frac{xz}{x+z}\omega$ and $e = -\frac{xy}{x+y}\omega$. We're now required to prove that

$$|x - y\omega| + |z\omega^2 - x| \geq 4\left|\frac{-zx}{z+x}\omega + \frac{xy}{x+y}\omega^2\right|.$$

Since $|\omega| = 1$ and $\omega^3 = 1$, we have $|z\omega^2 - x| = |\omega(z\omega^2 - x)| = |z - x\omega|$. Therefore, we need to prove

$$|x - y\omega| + |z - x\omega| \geq \left|\frac{4zx}{z+x} - \frac{4xy}{x+y}\omega\right|.$$

More strongly, we establish that $|(x - y\omega) + (z - x\omega)| \geq \left|\frac{4zx}{z+x} - \frac{4xy}{x+y}\omega\right|$ or $|p - q\omega| \geq |r - s\omega|$, where $p = z + x$, $q = y + x$, $r = \frac{4zx}{z+x}$ and $s = \frac{4xy}{x+y}$. It's clear that $p \geq r > 0$ and $q \geq s > 0$. It follows that

$$\begin{aligned} |p - q\omega|^2 - |r - s\omega|^2 &= (p - q\omega)\overline{(p - q\omega)} - (r - s\omega)\overline{(r - s\omega)} \\ &= (p^2 - r^2) + (pq - rs) + (q^2 - s^2) \geq 0. \end{aligned}$$

It's easy to check that the equality holds if and only if $\triangle ABC$ is equilateral. □

2 Four Basic Techniques

Differentiate! Shiing-shen Chern

1 Trigonometric Substitutions

If you are faced with an integral that contains square root expressions such as

$$\int \sqrt{1 - x^2}\,dx, \ \int \sqrt{1 + y^2}\,dy, \ \int \sqrt{z^2 - 1}\,dz$$

then trigonometric substitutions such as $x = \sin t$, $y = \tan t$, $z = \sec t$ are very useful. We will learn that making a suitable *trigonometric* substitution simplifies the given inequality.

Problem 9. ***(APMO 2004/5)*** *Prove that, for all positive real numbers a, b, c,*

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq 9(ab + bc + ca).$$

First Solution. Choose $A, B, C \in \left(0, \frac{\pi}{2}\right)$ with $a = \sqrt{2}\tan A$, $b = \sqrt{2}\tan B$, and $c = \sqrt{2}\tan C$. Using the well-known trigonometric identity $1 + \tan^2\theta = \frac{1}{cos^2\theta}$, one may rewrite it as

$$\frac{4}{9} \geq \cos A\cos B\cos C\left(\cos A\sin B\sin C + \sin A\cos B\sin C + \sin A\sin B\cos C\right).$$

One may easily check the following trigonometric identity

$$\cos(A + B + C) = \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\cos B\sin C - \sin A\sin B\cos C.$$

Then, the above trigonometric inequality takes the form

$$\frac{4}{9} \geq \cos A \cos B \cos C \left(\cos A \cos B \cos C - \cos(A+B+C)\right).$$

Let $\theta = \frac{A+B+C}{3}$. Applying the AM-GM inequality and Jesen's inequality, we have

$$\cos A \cos B \cos C \leq \left(\frac{\cos A + \cos B + \cos C}{3}\right)^3 \leq \cos^3 \theta.$$

We now need to show that

$$\frac{4}{9} \geq \cos^3 \theta (\cos^3 \theta - \cos 3\theta).$$

Using the trigonometric identity

$$\cos 3\theta = 4\cos^3 \theta - 3\cos \theta \text{ or } \cos^3 \theta - \cos 3\theta = 3\cos \theta - 3\cos^3 \theta,$$

it becomes

$$\frac{4}{27} \geq \cos^4 \theta \left(1 - \cos^2 \theta\right),$$

which follows from the AM-GM inequality

$$\left(\frac{\cos^2 \theta}{2} \cdot \frac{\cos^2 \theta}{2} \cdot \left(1 - \cos^2 \theta\right)\right)^{\frac{1}{3}} \leq \frac{1}{3}\left(\frac{\cos^2 \theta}{2} + \frac{\cos^2 \theta}{2} + \left(1 - \cos^2 \theta\right)\right) = \frac{1}{3}.$$

One find that the equality holds if and only if $\tan A = \tan B = \tan C = \frac{1}{\sqrt{2}}$ if and only if $a = b = c = 1$. ☐

Problem 10. ***(Latvia 2002)*** *Let a, b, c, d be the positive real numbers such that*

$$\frac{1}{1+a^4} + \frac{1}{1+b^4} + \frac{1}{1+c^4} + \frac{1}{1+d^4} = 1.$$

Prove that $abcd \geq 3$.

First Solution. We can write $a^2 = \tan A$, $b^2 = \tan B$, $c^2 = \tan C$, $d^2 = \tan D$, where $A, B, C, D \in \left(0, \frac{\pi}{2}\right)$. Then, the algebraic identity becomes the following trigonometric identity :

$$\cos^2 A + \cos^2 B + \cos^2 C + \cos^2 D = 1.$$

Applying the AM-GM inequality, we obtain

$$\sin^2 A = 1 - \cos^2 A = \cos^2 B + \cos^2 C + \cos^2 D \geq 3\left(\cos B \cos C \cos D\right)^{\frac{2}{3}}.$$

Similarly, we obtain

$$\sin^2 B \geq 3\left(\cos C \cos D \cos A\right)^{\frac{2}{3}}, \sin^2 C \geq 3\left(\cos D \cos A \cos B\right)^{\frac{2}{3}}, \text{ and } \sin^2 D \geq 3\left(\cos A \cos B \cos C\right)^{\frac{2}{3}}.$$

Multiplying these four inequalities, we get the result! ☐

Problem 11. ***(Korea 1998)*** *Let x, y, z be the positive reals with $x+y+z = xyz$. Show that*

$$\frac{1}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+y^2}} + \frac{1}{\sqrt{1+z^2}} \leq \frac{3}{2}.$$

Since the function f is not concave on $\mathbb{R}^+$, we cannot apply Jensen's inequality to the function $f(t) = \frac{1}{\sqrt{1+t^2}}$. However, the function $f(\tan \theta)$ is concave on $\left(0, \frac{\pi}{2}\right)$!

First Solution. We can write $x = \tan A$, $y = \tan B$, $z = \tan C$, where $A, B, C \in \left(0, \frac{\pi}{2}\right)$. Using the fact that $1 + \tan^2\theta = \left(\frac{1}{\cos\theta}\right)^2$, we rewrite it in the terms of A, B, C :

$$\cos A + \cos B + \cos C \leq \frac{3}{2}.$$

It follows from $\tan(\pi - C) = -z = \frac{x+y}{1-xy} = \tan(A+B)$ and from $\pi - C, A+B \in (0,\pi)$ that $\pi - C = A+B$ or $A+B+C = \pi$. Hence, it suffices to show the following. □

Theorem 2.1. *In any acute triangle ABC, we have* $\cos A + \cos B + \cos C \leq \frac{3}{2}$.

Proof. Since $\cos x$ is concave on $\left(0, \frac{\pi}{2}\right)$, it's a direct consequence of Jensen's inequality. □

We note that the function $\cos x$ is not concave on $(0, \pi)$. In fact, it's convex on $\left(\frac{\pi}{2}, \pi\right)$. One may think that the inequality $\cos A + \cos B + \cos C \leq \frac{3}{2}$ doesn't hold for any triangles. However, it's known that it holds for all triangles.

Theorem 2.2. *In any triangle ABC, we have* $\cos A + \cos B + \cos C \leq \frac{3}{2}$.

First Proof. It follows from $\pi - C = A + B$ that $\cos C = -\cos(A+B) = -\cos A\cos B + \sin A \sin B$ or

$$3 - 2(\cos A + \cos B + \cos C) = (\sin A - \sin B)^2 + (\cos A + \cos B - 1)^2 \geq 0.$$

□

Second Proof. Let $BC = a$, $CA = b$, $AB = c$. Use the Cosine Law to rewrite the given inequality in the terms of a, b, c :

$$\frac{b^2+c^2-a^2}{2bc} + \frac{c^2+a^2-b^2}{2ca} + \frac{a^2+b^2-c^2}{2ab} \leq \frac{3}{2}.$$

Clearing denominators, this becomes

$$3abc \geq a(b^2+c^2-a^2) + b(c^2+a^2-b^2) + c(a^2+b^2-c^2),$$

which is equivalent to $abc \geq (b+c-a)(c+a-b)(a+b-c)$ in the theorem 2. □

In the first chapter, we found that the *geometric* inequality $R \geq 2r$ is equivalent to the *algebraic* inequality $abc \geq (b+c-a)(c+a-b)(a+b-c)$. We now find that, in the proof of the above theorem, $abc \geq (b+c-a)(c+a-b)(a+b-c)$ is equivalent to the *trigonometric* inequality $\cos A + \cos B + \cos C \leq \frac{3}{2}$. One may ask that

> In any triangles ABC, is there a *natural* relation between $\cos A + \cos B + \cos C$ and $\frac{R}{r}$, where R and r are the radii of the circumcircle and incircle of ABC ?

Theorem 2.3. *Let R and r denote the radii of the circumcircle and incircle of the triangle ABC. Then, we have* $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$.

Proof. Use the identity $a(b^2+c^2-a^2) + b(c^2+a^2-b^2) + c(a^2+b^2-c^2) = 2abc + (b+c-a)(c+a-b)(a+b-c)$. We leave the details for the readers. □

Exercise 4. *(a) Let p, q, r be the positive real numbers such that $p^2+q^2+r^2+2pqr = 1$. Show that there exists an acute triangle ABC such that $p = \cos A$, $q = \cos B$, $r = \cos C$.*
(b) Let $p, q, r \geq 0$ with $p^2+q^2+r^2+2pqr = 1$. Show that there are $A, B, C \in \left[0, \frac{\pi}{2}\right]$ with $p = \cos A$, $q = \cos B$, $r = \cos C$, and $A + B + C = \pi$.

Problem 12. *(USA 2001) Let $a, b,$ and c be nonnegative real numbers such that $a^2+b^2+c^2+abc = 4$. Prove that $0 \leq ab+bc+ca-abc \leq 2$.*

Solution. Notice that $a,b,c>1$ implies that $a^2+b^2+c^2+abc>4$. If $a\le 1$, then we have $ab+bc+ca-abc\ge(1-a)bc\ge 0$. We now prove that $ab+bc+ca-abc\le 2$. Letting $a=2p$, $b=2q$, $c=2r$, we get $p^2+q^2+r^2+2pqr=1$. By the above exercise, we can write

$$a=2\cos A,\ b=2\cos B,\ c=2\cos C \text{ for some } A,B,C\in\left[0,\frac{\pi}{2}\right] \text{ with } A+B+C=\pi.$$

We are required to prove

$$\cos A\cos B+\cos B\cos C+\cos C\cos A-2\cos A\cos B\cos C\le\frac{1}{2}.$$

One may assume that $A\ge\frac{\pi}{3}$ or $1-2\cos A\ge 0$. Note that

$$\begin{aligned}&\cos A\cos B+\cos B\cos C+\cos C\cos A-2\cos A\cos B\cos C\\ =\ &\cos A(\cos B+\cos C)+\cos B\cos C(1-2\cos A).\end{aligned}$$

We apply Jensen's inequality to deduce $\cos B+\cos C\le\frac{3}{2}-\cos A$. Note that $2\cos B\cos C=\cos(B-C)+\cos(B+C)\le 1-\cos A$. These imply that

$$\cos A(\cos B+\cos C)+\cos B\cos C(1-2\cos A)\le\cos A\left(\frac{3}{2}-\cos A\right)+\left(\frac{1-\cos A}{2}\right)(1-2\cos A).$$

However, it's easy to verify that $\cos A\left(\frac{3}{2}-\cos A\right)+\left(\frac{1-\cos A}{2}\right)(1-2\cos A)=\frac{1}{2}$. $\square$

2 Algebraic Substitutions

We know that some inequalities in triangle geometry can be treated by the *Ravi* substitution and *trigonometric* substitutions. We can also transform the given inequalities into easier ones through some clever *algebraic* substitutions.

Problem 13. ***(IMO 2001/2)*** *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\ge 1.$$

First Solution. To remove the square roots, we make the following substitution :

$$x=\frac{a}{\sqrt{a^2+8bc}},\ y=\frac{b}{\sqrt{b^2+8ca}},\ z=\frac{c}{\sqrt{c^2+8ab}}.$$

Clearly, $x,y,z\in(0,1)$. Our aim is to show that $x+y+z\ge 1$. We notice that

$$\frac{a^2}{8bc}=\frac{x^2}{1-x^2},\ \frac{b^2}{8ac}=\frac{y^2}{1-y^2},\ \frac{c^2}{8ab}=\frac{z^2}{1-z^2}\implies\frac{1}{512}=\left(\frac{x^2}{1-x^2}\right)\left(\frac{y^2}{1-y^2}\right)\left(\frac{z^2}{1-z^2}\right).$$

Hence, we need to show that

$$x+y+z\ge 1, \text{ where } 0<x,y,z<1 \text{ and } (1-x^2)(1-y^2)(1-z^2)=512(xyz)^2.$$

However, $1>x+y+z$ implies that, by the AM-GM inequality,

$$\begin{aligned}(1-x^2)(1-y^2)(1-z^2)&>((x+y+z)^2-x^2)((x+y+z)^2-y^2)((x+y+z)^2-z^2)\\ &=(x+x+y+z)(y+z)\end{aligned}$$

$(x+y+y+z)(z+x)(x+y+z+z)(x+y)\ge 4(x^2yz)^{\frac{1}{4}}\cdot 2(yz)^{\frac{1}{2}}\cdot 4(y^2zx)^{\frac{1}{4}}\cdot 2(zx)^{\frac{1}{2}}\cdot 4(z^2xy)^{\frac{1}{4}}\cdot 2(xy)^{\frac{1}{2}}$
$=512(xyz)^2$. This is a contradiction ! $\square$

Problem 14. ***(IMO 1995/2)*** *Let a,b,c be positive numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

First Solution. After the substitution $a=\frac{1}{x}$, $b=\frac{1}{y}$, $c=\frac{1}{z}$, we get $xyz = 1$. The inequality takes the form

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\geq\frac{3}{2}.$$

It follows from the Cauchy-Schwarz inequality that

$$[(y+z)+(z+x)+(x+y)]\left(\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\right)\geq(x+y+z)^2$$

so that, by the AM-GM inequality,

$$\frac{x^2}{y+z}+\frac{y^2}{z+x}+\frac{z^2}{x+y}\geq\frac{x+y+z}{2}\geq\frac{3(xyz)^{\frac{1}{3}}}{2}=\frac{3}{2}.$$

□

(**Korea 1998**) Let x, y, z be the positive reals with $x+y+z = xyz$. Show that

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}}\leq\frac{3}{2}.$$

Second Solution. The starting point is letting $a=\frac{1}{x}$, $b=\frac{1}{y}$, $c=\frac{1}{z}$. We find that $a+b+c = abc$ is equivalent to $1 = xy+yz+zx$. The inequality becomes

$$\frac{x}{\sqrt{x^2+1}}+\frac{y}{\sqrt{y^2+1}}+\frac{z}{\sqrt{z^2+1}}\leq\frac{3}{2}$$

or

$$\frac{x}{\sqrt{x^2+xy+yz+zx}}+\frac{y}{\sqrt{y^2+xy+yz+zx}}+\frac{z}{\sqrt{z^2+xy+yz+zx}}\leq\frac{3}{2}$$

or

$$\frac{x}{\sqrt{(x+y)(x+z)}}+\frac{y}{\sqrt{(y+z)(y+x)}}+\frac{z}{\sqrt{(z+x)(z+y)}}\leq\frac{3}{2}.$$

By the AM-GM inequality, we have

$$\frac{x}{\sqrt{(x+y)(x+z)}}=\frac{x\sqrt{(x+y)(x+z)}}{(x+y)(x+z)}\leq\frac{1}{2}\frac{x[(x+y)+(x+z)]}{(x+y)(x+z)}=\frac{1}{2}\left(\frac{x}{x+z}+\frac{x}{x+z}\right).$$

In a like manner, we obtain

$$\frac{y}{\sqrt{(y+z)(y+x)}}\leq\frac{1}{2}\left(\frac{y}{y+z}+\frac{y}{y+x}\right)\ \text{ and }\ \frac{z}{\sqrt{(z+x)(z+y)}}\leq\frac{1}{2}\left(\frac{z}{z+x}+\frac{z}{z+y}\right).$$

Adding these three yields the required result. □

We now prove a classical theorem in various ways.

Theorem 2.4. ***(Nesbitt, 1903)*** *For all positive real numbers a,b,c, we have*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Proof 1. *After the substitution $x = b+c$, $y = c+a$, $z = a+b$, it becomes*

$$\sum_{\text{cyclic}} \frac{y+z-x}{2x} \geq \frac{3}{2} \quad or \quad \sum_{\text{cyclic}} \frac{y+z}{x} \geq 6,$$

which follows from the AM-GM inequality as following:

$$\sum_{\text{cyclic}} \frac{y+z}{x} = \frac{y}{x}+\frac{z}{x}+\frac{z}{y}+\frac{x}{y}+\frac{x}{z}+\frac{y}{z} \geq 6\left(\frac{y}{x}\cdot\frac{z}{x}\cdot\frac{z}{y}\cdot\frac{x}{y}\cdot\frac{x}{z}\cdot\frac{y}{z}\right)^{\frac{1}{6}} = 6.$$

Proof 2. *We make the substitution*

$$x = \frac{a}{b+c},\ y = \frac{b}{c+a},\ z = \frac{c}{a+b}.$$

It follows that

$$\sum_{\text{cyclic}} f(x) = \sum_{\text{cyclic}} \frac{a}{a+b+c} = 1, \quad where \quad f(t) = \frac{t}{1+t}.$$

Since f is concave on $(0,\infty)$, Jensen's inequality shows that

$$f\left(\frac{1}{2}\right) = \frac{1}{3} = \frac{1}{3}\sum_{\text{cyclic}} f(x) \leq f\left(\frac{x+y+z}{3}\right) \quad or \quad f\left(\frac{1}{2}\right) \leq f\left(\frac{x+y+z}{3}\right).$$

Since f is monotone increasing, this implies that

$$\frac{1}{2} \leq \frac{x+y+z}{3} \quad or \quad \sum_{\text{cyclic}} \frac{a}{b+c} = x+y+z \geq \frac{3}{2}.$$

Proof 3. *As in the previous proof, it suffices to show that*

$$T \geq \frac{1}{2}, \quad where \quad T = \frac{x+y+z}{3} \quad and \quad \sum_{\text{cyclic}} \frac{x}{1+x} = 1.$$

One can easily check that the condition

$$\sum_{\text{cyclic}} \frac{x}{1+x} = 1$$

becomes $1 = 2xyz + xy + yz + zx$. By the AM-GM inequality, we have

$$1 = 2xyz + xy + yz + zx \leq 2T^3 + 3T^2 \ \Rightarrow\ 2T^3 + 3T^2 - 1 \geq 0 \ \Rightarrow\ (2T-1)(T+1)^2 \geq 0 \ \Rightarrow\ T \geq \frac{1}{2}.$$

(**IMO 2000/2**) Let a, b, c be positive numbers such that $abc = 1$. Prove that

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leq 1.$$

Second Solution. (**[IV], Ilan Vardi**) Since $abc = 1$, we may assume that $a \geq 1 \geq b$. [7] It follows that

$$1-\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) = \left(c+\frac{1}{c}-2\right)\left(a+\frac{1}{b}-1\right)+\frac{(a-1)(1-b)}{a}. \text{ [8]}$$

□

[7] Why? Note that the inequality is not symmetric in the three variables. Check it!

[8] For a verification of the identity, see [IV].

Third Solution. As in the first solution, after the substitution $a = \frac{x}{y}$, $b = \frac{y}{z}$, $c = \frac{z}{x}$ for $x, y, z > 0$, we can rewrite it as $xyz \geq (y+z-x)(z+x-y)(x+y-z)$. Without loss of generality, we can assume that $z \geq y \geq x$. Set $y - x = p$ and $z - x = q$ with $p, q \geq 0$. It's straightforward to verify that

$$xyz - (y+z-x)(z+x-y)(x+y-z) = (p^2 - pq + q^2)x + (p^3 + q^3 - p^2q - pq^2).$$

Since $p^2 - pq + q^2 \geq (p-q)^2 \geq 0$ and $p^3 + q^3 - p^2q - pq^2 = (p-q)^2(p+q) \geq 0$, we get the result. $\square$

Fourth Solution. (**From the IMO 2000 Short List**) Using the condition $abc = 1$, it's straightforward to verify the equalities

$$2 = \frac{1}{a}\left(a - 1 + \frac{1}{b}\right) + c\left(b - 1 + \frac{1}{c}\right),$$
$$2 = \frac{1}{b}\left(b - 1 + \frac{1}{c}\right) + a\left(c - 1 + \frac{1}{a}\right),$$
$$2 = \frac{1}{c}\left(c - 1 + \frac{1}{a}\right) + b\left(a - 1 + \frac{1}{c}\right).$$

In particular, they show that at most one of the numbers $u = a - 1 + \frac{1}{b}$, $v = b - 1 + \frac{1}{c}$, $w = c - 1 + \frac{1}{a}$ is negative. If there is such a number, we have

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) = uvw < 0 < 1.$$

And if $u, v, w \geq 0$, the AM-GM inequality yields

$$2 = \frac{1}{a}u + cv \geq 2\sqrt{\frac{c}{a}uv},\ \ 2 = \frac{1}{b}v + aw \geq 2\sqrt{\frac{a}{b}vw},\ \ 2 = \frac{1}{c}w + aw \geq 2\sqrt{\frac{b}{c}wu}.$$

Thus, $uv \leq \frac{a}{c}$, $\quad vw \leq \frac{b}{a}$, $\quad wu \leq \frac{c}{b}$, so $(uvw)^2 \leq \frac{a}{c} \cdot \frac{b}{a} \cdot \frac{c}{b} = 1$. Since $u, v, w \geq 0$, this completes the proof. $\square$

Problem 15. *Let a, b, c be positive real numbers satisfying $a + b + c = 1$. Show that*

$$\frac{a}{a+bc} + \frac{b}{b+ca} + \frac{\sqrt{abc}}{c+ab} \leq 1 + \frac{3\sqrt{3}}{4}.$$

Solution. We want to establish that

$$\frac{1}{1+\frac{bc}{a}} + \frac{1}{1+\frac{ca}{b}} + \frac{\sqrt{\frac{ab}{c}}}{1+\frac{ab}{c}} \leq 1 + \frac{3\sqrt{3}}{4}.$$

Set $x = \sqrt{\frac{bc}{a}}, y = \sqrt{\frac{ca}{b}}, z = \sqrt{\frac{ab}{c}}$. We need to prove that

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{z}{1+z^2} \leq 1 + \frac{3\sqrt{3}}{4},$$

where $x, y, z > 0$ and $xy + yz + zx = 1$. It's not hard to show that there exists $A, B, C \in (0, \pi)$ with

$$x = \tan\frac{A}{2}, y = \tan\frac{B}{2}, z = \tan\frac{C}{2},\ \text{and}\ A + B + C = \pi.$$

The inequality becomes

$$\frac{1}{1+\left(\tan\frac{A}{2}\right)^2} + \frac{1}{1+\left(\tan\frac{B}{2}\right)^2} + \frac{\tan\frac{C}{2}}{1+\left(\tan\frac{C}{2}\right)^2} \leq 1 + \frac{3\sqrt{3}}{4}$$

or

$$1+\frac{1}{2}(\cos A+\cos B+\sin C)\leq 1+\frac{3\sqrt{3}}{4}$$

or

$$\cos A+\cos B+\sin C\leq \frac{3\sqrt{3}}{2}.$$

$\square$

Note that $\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$. Since $\left|\frac{A-B}{2}\right|<\frac{\pi}{2}$, this means that

$$\cos A+\cos B\leq 2\cos\left(\frac{A+B}{2}\right)=2\cos\left(\frac{\pi-C}{2}\right).$$

It will be enough to show that

$$2\cos\left(\frac{\pi-C}{2}\right)+\sin C\leq \frac{3\sqrt{3}}{2},$$

where $C\in(0,\pi)$. This is a one-variable inequality.[9] It's left as an exercise for the reader.

Here, we give another solution of the problem 10.

(**Latvia 2002**) Let a, b, c, d be the positive real numbers such that

$$\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1.$$

Prove that $abcd\geq 3$.

Second Solution. (given by Jeong Soo Sim at the KMO Weekend Program 2007) We need to prove the inequality $a^4b^4c^4d^4\geq 81$. After making the substitution

$$A=\frac{1}{1+a^4},\ B=\frac{1}{1+b^4},\ C=\frac{1}{1+c^4},\ D=\frac{1}{1+d^4},$$

we obtain

$$a^4=\frac{1-A}{A},\ b^4=\frac{1-B}{B},\ c^4=\frac{1-C}{C},\ d^4=\frac{1-D}{D}.$$

The constraint becomes $A+B+C+D=1$ and the inequality can be written as

$$\frac{1-A}{A}\cdot\frac{1-B}{B}\cdot\frac{1-C}{C}\cdot\frac{1-D}{D}\geq 81.$$

or

$$\frac{B+C+D}{A}\cdot\frac{C+D+A}{B}\cdot\frac{D+A+B}{C}\cdot\frac{A+B+C}{D}\geq 81.$$

or

$$(B+C+D)(C+D+A)(D+A+B)(A+B+C)\geq 81ABCD.$$

However, this is an immediate consequence of the AM-GM inequality:

$$(B+C+D)(C+D+A)(D+A+B)(A+B+C)\geq 3(BCD)^{\frac{1}{3}}\cdot 3(CDA)^{\frac{1}{3}}\cdot 3(DAB)^{\frac{1}{3}}\cdot 3(ABC)^{\frac{1}{3}}.$$

$\square$

[9] *Differentiate!* Shiing-shen Chern

Problem 16. ***(Iran 1998)*** *Prove that, for all $x,y,z>1$ such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$,*

$$\sqrt{x+y+z}\geq\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

First Solution. We begin with the algebraic substitution $a=\sqrt{x-1}$, $b=\sqrt{y-1}$, $c=\sqrt{z-1}$. Then, the condition becomes

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}=2 \;\Leftrightarrow\; a^2b^2+b^2c^2+c^2a^2+2a^2b^2c^2=1$$

and the inequality is equivalent to

$$\sqrt{a^2+b^2+c^2+3}\geq a+b+c \;\Leftrightarrow\; ab+bc+ca\leq\frac{3}{2}.$$

Let $p=bc$, $q=ca$, $r=ab$. Our job is to prove that $p+q+r\leq\frac{3}{2}$ where $p^2+q^2+r^2+2pqr=1$. By the exercise 7, we can make the trigonometric substitution

$$p=\cos A,\ q=\cos B,\ r=\cos C\ \text{ for some } A,B,C\in\left(0,\frac{\pi}{2}\right)\text{ with } A+B+C=\pi.$$

What we need to show is now that $\cos A+\cos B+\cos C\leq\frac{3}{2}$. It follows from Jensen's inequality. □

Problem 17. ***(Belarus 1998)*** *Prove that, for all $a,b,c>0$,*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{a+b}{b+c}+\frac{b+c}{c+a}+1.$$

Solution. After writing $x=\frac{a}{b}$ and $y=\frac{c}{b}$, we get

$$\frac{c}{a}=\frac{y}{x},\ \frac{a+b}{b+c}=\frac{x+1}{1+y},\ \frac{b+c}{c+a}=\frac{1+y}{y+x}.$$

One may rewrite the inequality as

$$x^3y^2+x^2+x+y^3+y^2\geq x^2y+2xy+2xy^2.$$

Apply the AM-GM inequality to obtain

$$\frac{x^3y^2+x}{2}\geq x^2y,\ \ \frac{x^3y^2+x+y^3+y^3}{2}\geq 2xy^2,\ \ x^2+y^2\geq 2xy.$$

Adding these three inequalities, we get the result. The equality holds if and only if $x=y=1$ or $a=b=c$. □

Problem 18. ***(IMO Short-list 2001)*** *Let $x_1,\cdots,x_n$ be arbitrary real numbers. Prove the inequality.*

$$\frac{x_1}{1+{x_1}^2}+\frac{x_2}{1+{x_1}^2+{x_2}^2}+\cdots+\frac{x_n}{1+{x_1}^2+\cdots+{x_n}^2}<\sqrt{n}.$$

First Solution. We only consider the case when $x_1,\cdots,x_n$ are all nonnegative real numbers.(Why?)[10] Let $x_0=1$. After the substitution $y_i={x_0}^2+\cdots+{x_i}^2$ for all $i=0,\cdots,n$, we obtain $x_i=\sqrt{y_i-y_{i-1}}$. We need to prove the following inequality

$$\sum_{i=0}^{n}\frac{\sqrt{y_i-y_{i-1}}}{y_i}<\sqrt{n}.$$

---

[10] $\frac{x_1}{1+{x_1}^2}+\frac{x_2}{1+{x_1}^2+{x_2}^2}+\cdots+\frac{x_n}{1+{x_1}^2+\cdots+{x_n}^2}\leq\frac{|x_1|}{1+{x_1}^2}+\frac{|x_2|}{1+{x_1}^2+{x_2}^2}+\cdots+\frac{|x_n|}{1+{x_1}^2+\cdots+{x_n}^2}$.

Since $y_i \geq y_{i-1}$ for all $i = 1, \cdots, n$, we have an upper bound of the left hand side:

$$\sum_{i=0}^{n} \frac{\sqrt{y_i - y_{i-1}}}{y_i} \leq \sum_{i=0}^{n} \frac{\sqrt{y_i - y_{i-1}}}{\sqrt{y_i y_{i-1}}} = \sum_{i=0}^{n} \sqrt{\frac{1}{y_{i-1}} - \frac{1}{y_i}}$$

We now apply the Cauchy-Schwarz inequality to give an upper bound of the last term:

$$\sum_{i=0}^{n} \sqrt{\frac{1}{y_{i-1}} - \frac{1}{y_i}} \leq \sqrt{n \sum_{i=0}^{n} \left(\frac{1}{y_{i-1}} - \frac{1}{y_i}\right)} = \sqrt{n \left(\frac{1}{y_0} - \frac{1}{y_n}\right)}.$$

Since $y_0 = 1$ and $y_n > 0$, this yields the desired upper bound $\sqrt{n}$.

□

Second Solution. We may assume that $x_1, \cdots, x_n$ are all nonnegative real numbers. Let $x_0 = 0$. We make the following *algebraic* substitution

$$t_i = \frac{x_i}{\sqrt{{x_0}^2 + \cdots + {x_i}^2}},\ c_i = \frac{1}{\sqrt{1 + {t_i}^2}} \text{ and } s_i = \frac{t_i}{\sqrt{1 + {t_i}^2}}$$

for all $i = 0, \cdots, n$. It's an easy exercise to show that $\frac{x_i}{{x_0}^2 + \cdots + {x_i}^2} = c_0 \cdots c_i s_i$. Since $s_i = \sqrt{1 - {c_i}^2}$, the desired inequality becomes

$$c_0 c_1 \sqrt{1 - {c_1}^2} + c_0 c_1 c_2 \sqrt{1 - {c_2}^2} + \cdots + c_0 c_1 \cdots c_n \sqrt{1 - {c_n}^2} < \sqrt{n}.$$

Since $0 < c_i \leq 1$ for all $i = 1, \cdots, n$, we have

$$\sum_{i=1}^{n} c_0 \cdots c_i \sqrt{1 - {c_i}^2} \leq \sum_{i=1}^{n} c_0 \cdots c_{i-1} \sqrt{1 - {c_i}^2} = \sum_{i=1}^{n} \sqrt{(c_0 \cdots c_{i-1})^2 - (c_0 \cdots c_{i-1} c_i)^2}.$$

Since $c_0 = 1$, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{i=1}^{n} \sqrt{(c_0 \cdots c_{i-1})^2 - (c_0 \cdots c_{i-1} c_i)^2} \leq \sqrt{n \sum_{i=1}^{n} [(c_0 \cdots c_{i-1})^2 - (c_0 \cdots c_{i-1} c_i)^2]} = \sqrt{n [1 - (c_0 \cdots c_n)^2]}.$$

□

3 Increasing Function Theorem

Theorem 2.5. ***(Increasing Function Theorem)*** *Let $f : (a,b) \longrightarrow \mathbb{R}$ be a differentiable function. If $f'(x) \geq 0$ for all $x \in (a,b)$, then f is monotone increasing on (a,b). If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing on (a,b).*

Proof. We first consider the case when $f'(x) > 0$ for all $x \in (a,b)$. Let $a < x_1 < x_2 < b$. We want to show that $f(x_1) < f(x_2)$. Applying the Mean Value Theorem, we find some $c \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Since $f'(c) > 0$, this equation means that $f(x_2) - f(x_1) > 0$. In case when $f'(x) \geq 0$ for all $x \in (a,b)$, we can also apply the Mean Value Theorem to get the result. □

Problem 19. ***(Ireland 2000)*** *Let $x, y \geq 0$ with $x + y = 2$. Prove that $x^2 y^2 (x^2 + y^2) \leq 2$.*

First Solution. After homogenizing it, we need to prove

$$2\left(\frac{x+y}{2}\right)^6 \geq x^2 y^2 (x^2 + y^2) \text{ or } (x+y)^6 \geq 32 x^2 y^2 (x^2 + y^2).$$

(Now, forget the constraint $x+y=2$!) In case $xy=0$, it clearly holds. We now assume that $xy\neq 0$. Because of the homogeneity of the inequality, this means that we may normalize to $xy=1$. Then, it becomes

$$\left(x+\frac{1}{x}\right)^6\geq 32\left(x^2+\frac{1}{x^2}\right)\text{ or } p^3\geq 32(p-2).$$

where $p=\left(x+\frac{1}{x}\right)^2\geq 4$. Our job is now to minimize $F(p)=p^3-32(p-2)$ on $[4,\infty)$. Since $F'(p)=3p^2-32\geq 0$, where $p\geq\sqrt{\frac{32}{3}}$, F is (monotone) increasing on $[4,\infty)$. So, $F(p)\geq F(4)=0$ for all $p\geq 4$. ☐

Second Solution. As in the first solution, we prove that $(x+y)^6\geq 32(x^2+y^2)(xy)^2$ for all $x,y\geq 0$. In case $x=y=0$, it's clear. Now, if $x^2+y^2>0$, then we may normalize to $x^2+y^2=2$. Setting $p=xy$, we have $0\leq p\leq\frac{x^2+y^2}{2}=1$ and $(x+y)^2=x^2+y^2+2xy=2+2p$. It now becomes

$$(2+2p)^3\geq 64p^2\text{ or } p^3-5p^2+3p+1\geq 0.$$

We want to minimize $F(p)=p^3-5p^2+3p+1$ on $[0,1]$. We compute $F'(p)=3\left(p-\frac{1}{3}\right)(p-3)$. We find that F is monotone increasing on $[0,\frac{1}{3}]$ and monotone decreasing on $[\frac{1}{3},1]$. Since $F(0)=1$ and $F(1)=0$, we conclude that $F(p)\geq F(1)=0$ for all $p\in[0,1]$. ☐

Third Solution. We show that $(x+y)^6\geq 32(x^2+y^2)(xy)^2$ where $x\geq y\geq 0$. We make the substitution $u=x+y$ and $v=x-y$. Then, we have $u\geq v\geq 0$. It becomes

$$u^6\geq 32\left(\frac{u^2+v^2}{2}\right)\left(\frac{u^2-v^2}{4}\right)^2\text{ or } u^6\geq(u^2+v^2)(u^2-v^2)^2.$$

Note that $u^4\geq u^4-v^4\geq 0$ and that $u^2\geq u^2-v^2\geq 0$. So, $u^6\geq(u^4-v^4)(u^2-v^2)=(u^2+v^2)(u^2-v^2)^2$. ☐

Problem 20. ***(IMO 1984/1)*** *Let x,y,z be nonnegative real numbers such that $x+y+z=1$. Prove that $0\leq xy+yz+zx-2xyz\leq\frac{7}{27}$.*

First Solution. Let $f(x,y,z)=xy+yz+zx-2xyz$. We may assume that $0\leq x\leq y\leq z\leq 1$. Since $x+y+z=1$, this implies that $x\leq\frac{1}{3}$. It follows that $f(x,y,z)=(1-3x)yz+xyz+zx+xy\geq 0$. Applying the AM-GM inequality, we obtain $yz\leq\left(\frac{y+z}{2}\right)^2=\left(\frac{1-x}{2}\right)^2$. Since $1-2x\geq 0$, this implies that

$$f(x,y,z)=x(y+z)+yz(1-2x)\leq x(1-x)+\left(\frac{1-x}{2}\right)^2(1-2x)=\frac{-2x^3+x^2+1}{4}.$$

Our job is now to maximize a one-variable function $F(x)=\frac{1}{4}(-2x^3+x^2+1)$, where $x\in\left[0,\frac{1}{3}\right]$. Since $F'(x)=\frac{3}{2}x\left(\frac{1}{3}-x\right)\geq 0$ on $\left[0,\frac{1}{3}\right]$, we conclude that $F(x)\leq F(\frac{1}{3})=\frac{7}{27}$ for all $x\in\left[0,\frac{1}{3}\right]$. ☐

(**IMO 2000/2**) Let a,b,c be positive numbers such that $abc=1$. Prove that

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\leq 1.$$

Fifth Solution. (based on work by an IMO 2000 contestant from Japan) Since $abc=1$, at least one of a, b, c is greater than or equal to 1. Say $b\geq 1$. Putting $c=\frac{1}{ab}$, it becomes

$$\left(a-1+\frac{1}{b}\right)(b-1+ab)\left(\frac{1}{ab}-1+\frac{1}{a}\right)\leq 1$$

or

$$a^3b^3 - a^2b^3 - ab^3 - a^2b^2 + 3ab^2 - ab + b^3 - b^2 - b + 1 \geq 0.$$

Setting $x = ab$, it becomes $f_b(x) \geq 0$, where

$$f_b(t) = t^3 + b^3 - b^2t - bt^2 + 3bt - t^2 - b^2 - t - b + 1.$$

Fix a positive number $b \geq 1$. We need to show that $F(t) := f_b(t) \geq 0$ for all $t \geq 0$. It follows from $b \geq 1$ that the cubic polynomial $F'(t) = 3t^2 - 2(b+1)t - (b^2 - 3b + 1)$ has two real roots

$$\frac{b+1-\sqrt{4b^2-7b+4}}{3} \text{ and } \lambda = \frac{b+1+\sqrt{4b^2-7b+4}}{3}.$$

Since F has a local minimum at $t = \lambda$, we find that $F(t) \geq Min\,\{F(0),\, F(\lambda)\}$ for all $t \geq 0$. We have to prove that $F(0) \geq 0$ and $F(\lambda) \geq 0$. We have $F(0) = b^3 - b^2 - b + 1 = (b-1)^2(b+1) \geq 0$. It remains to show that $F(\lambda) \geq 0$. Notice that λ is a root of $F^{/}(t)$. After long division, we get

$$F(t) = F'(t)\left(\frac{1}{3}t - \frac{b+1}{9}\right) + \frac{1}{9}\left((-8b^2 + 14b - 8)t + 8b^3 - 7b^2 - 7b + 8\right).$$

Putting $t = \lambda$, we have

$$F(\lambda) = \frac{1}{9}\left((-8b^2 + 14b - 8)\lambda + 8b^3 - 7b^2 - 7b + 8\right).$$

Thus, our job is now to establish that, for all $b \geq 0$,

$$(-8b^2 + 14b - 8)\left(\frac{b+1+\sqrt{4b^2-7b+4}}{3}\right) + 8b^3 - 7b^2 - 7b + 8 \geq 0,$$

which is equivalent to

$$16b^3 - 15b^2 - 15b + 16 \geq (8b^2 - 14b + 8)\sqrt{4b^2 - 7b + 4}\,.$$

Since both $16b^3 - 15b^2 - 15b + 16$ and $8b^2 - 14b + 8$ are positive,[11] it's equivalent to

$$(16b^3 - 15b^2 - 15b + 16)^2 \geq (8b^2 - 14b + 8)^2(4b^2 - 7b + 4)$$

or

$$864b^5 - 3375b^4 + 5022b^3 - 3375b^2 + 864b \geq 0 \text{ or } 864b^4 - 3375b^3 + 5022b^2 - 3375b + 864 \geq 0.$$

Let $G(x) = 864x^4 - 3375x^3 + 5022x^2 - 3375x + 864$. We prove that $G(x) \geq 0$ for all $x \in \mathbb{R}$. We find that

$$G'(x) = 3456x^3 - 10125x^2 + 10044x - 3375 = (x-1)(3456x^2 - 6669x + 3375).$$

Since $3456x^2 - 6669x + 3375 > 0$ for all $x \in \mathbb{R}$, we find that $G(x)$ and $x - 1$ have the same sign. It follows that G is monotone decreasing on $(-\infty, 1]$ and monotone increasing on $[1, \infty)$. We conclude that G has the global minimum at $x = 1$. Hence, $G(x) \geq G(1) = 0$ for all $x \in \mathbb{R}$. □

[11] It's easy to check that $16b^3 - 15b^2 - 15b + 16 = 16(b^3 - b^2 - b + 1) + b^2 + b > 16(b^2 - 1)(b - 1) \geq 0$ and $8b^2 - 14b + 8 = 8(b-1)^2 + 2b > 0$.

4 Establishing New Bounds

We first give two alternative ways to prove Nesbitt's inequality.

(**Nesbitt**) For all positive real numbers a,b,c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Proof 4. *From* $\left(\frac{a}{b+c}-\frac{1}{2}\right)^2\geq 0$, *we deduce that*

$$\frac{a}{b+c}\geq\frac{1}{4}\cdot\frac{\frac{8a}{b+c}-1}{\frac{a}{b+c}+1}=\frac{8a-b-c}{4(a+b+c)}.$$

It follows that

$$\sum_{\text{cyclic}}\frac{a}{b+c}\geq\sum_{\text{cyclic}}\frac{8a-b-c}{4(a+b+c)}=\frac{3}{2}.$$

Proof 5. *We claim that*

$$\frac{a}{b+c}\geq\frac{3a^{\frac{3}{2}}}{2\left(a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}\right)}\ \text{ or }\ 2\left(a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}\right)\geq 3a^{\frac{1}{2}}(b+c).$$

The AM-GM inequality gives $a^{\frac{3}{2}}+b^{\frac{3}{2}}+b^{\frac{3}{2}}\geq 3a^{\frac{1}{2}}b$ *and* $a^{\frac{3}{2}}+c^{\frac{3}{2}}+c^{\frac{3}{2}}\geq 3a^{\frac{1}{2}}c$ *. Adding these two inequalities yields* $2\left(a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}\right)\geq 3a^{\frac{1}{2}}(b+c)$, *as desired. Therefore, we have*

$$\sum_{\text{cyclic}}\frac{a}{b+c}\geq\frac{3}{2}\sum_{\text{cyclic}}\frac{a^{\frac{3}{2}}}{a^{\frac{3}{2}}+b^{\frac{3}{2}}+c^{\frac{3}{2}}}=\frac{3}{2}.$$

Some cyclic inequalities can be proved by finding new bounds. Suppose that we want to establish that

$$\sum_{\text{cyclic}}F(x,y,z)\geq C.$$

If a function G satisfies

(1) $F(x,y,z)\geq G(x,y,z)$ for all $x,y,z>0$, and
(2) $\sum_{\text{cyclic}}G(x,y,z)=C$ for all $x,y,z>0$,

then, we deduce that

$$\sum_{\text{cyclic}}F(x,y,z)\geq\sum_{\text{cyclic}}G(x,y,z)=C.$$

For example, if a function F satisfies

$$F(x,y,z)\geq\frac{x}{x+y+z}$$

for all $x,y,z>0$, then, taking the cyclic sum yields

$$\sum_{\text{cyclic}}F(x,y,z)\geq 1.$$

As we saw in the above two proofs of Nesbitt's inequality, there are various lower bounds.

Problem 21. *Let a, b, c be the lengths of a triangle. Show that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<2.$$

Proof. We don't employ the Ravi substitution. It follows from the triangle inequality that

$$\sum_{\text{cyclic}}\frac{a}{b+c}<\sum_{\text{cyclic}}\frac{a}{\frac{1}{2}(a+b+c)}=2.$$

□

One day, I tried finding a new lower bound of $(x+y+z)^2$ where $x,y,z>0$. There are well-known lower bounds such as $3(xy+yz+zx)$ and $9(xyz)^{\frac{2}{3}}$. But I wanted to find quite different one. I tried breaking the symmetry of the three variables x,y,z. Note that

$$(x+y+z)^2=x^2+y^2+z^2+xy+xy+yz+yz+zx+zx.$$

I applied the AM-GM inequality to the right hand side except the term x^2 :

$$y^2+z^2+xy+xy+yz+yz+zx+zx\geq 8x^{\frac{1}{2}}y^{\frac{3}{4}}z^{\frac{3}{4}}\ .$$

It follows that

$$(x+y+z)^2\geq x^2+8x^{\frac{1}{2}}y^{\frac{3}{4}}z^{\frac{3}{4}}=x^{\frac{1}{2}}\left(x^{\frac{3}{2}}+8y^{\frac{3}{4}}z^{\frac{3}{4}}\right).$$

(**IMO 2001/2**) Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Second Solution. We find that the above inequality also gives another lower bound of $x+y+z$, that is,

$$x+y+z\geq\sqrt{x^{\frac{1}{2}}\left(x^{\frac{3}{2}}+8y^{\frac{3}{4}}z^{\frac{3}{4}}\right)}.$$

It follows that

$$\sum_{\text{cyclic}}\frac{x^{\frac{3}{4}}}{\sqrt{x^{\frac{3}{2}}+8y^{\frac{3}{4}}z^{\frac{3}{4}}}}\geq\sum_{\text{cyclic}}\frac{x}{x+y+z}=1.$$

After the substitution $x=a^{\frac{4}{3}}, y=b^{\frac{4}{3}}$, and $z=c^{\frac{4}{3}}$, it now becomes

$$\sum_{\text{cyclic}}\frac{a}{\sqrt{a^2+8bc}}\geq 1.$$

□

Problem 22. *(**IMO 2005/3**) Let x, y, and z be positive numbers such that $xyz\geq 1$. Prove that*

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}\geq 0.$$

First Solution. It's equivalent to the following inequality

$$\left(\frac{x^2-x^5}{x^5+y^2+z^2}+1\right)+\left(\frac{y^2-y^5}{y^5+z^2+x^2}+1\right)+\left(\frac{z^2-z^5}{z^5+x^2+y^2}+1\right)\leq 3$$

or

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{x^2+y^2+z^2}{y^5+z^2+x^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2}\leq 3.$$

With the Cauchy-Schwarz inequality and the fact that $xyz \geq 1$, we have

$$(x^5+y^2+z^2)(yz+y^2+z^2)\geq (x^2+y^2+z^2)^2 \text{ or } \frac{x^2+y^2+z^2}{x^5+y^2+z^2}\leq \frac{yz+y^2+z^2}{x^2+y^2+z^2}.$$

Taking the cyclic sum and $x^2+y^2+z^2 \geq xy+yz+zx$ give us

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{x^2+y^2+z^2}{y^5+z^2+x^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2}\leq 2+\frac{xy+yz+zx}{x^2+y^2+z^2}\leq 3.$$

$\square$

Second Solution. The main idea is to think of 1 as follows :

$$\frac{x^5}{x^5+y^2+z^2}+\frac{y^5}{y^5+z^2+x^2}+\frac{z^5}{z^5+x^2+y^2}\geq 1\geq \frac{x^2}{x^5+y^2+z^2}+\frac{y^2}{y^5+z^2+x^2}+\frac{z^2}{z^5+x^2+y^2}.$$

We first show the left-hand. It follows from $y^4+z^4 \geq y^3z+yz^3 = yz(y^2+z^2)$ that

$$x(y^4+z^4)\geq xyz(y^2+z^2)\geq y^2+z^2 \text{ or } \frac{x^5}{x^5+y^2+z^2}\geq \frac{x^5}{x^5+xy^4+xz^4}=\frac{x^4}{x^4+y^4+z^4}.$$

Taking the cyclic sum, we have the required inequality. It remains to show the right-hand.

[First Way] As in the first solution, the Cauchy-Schwarz inequality and $xyz \geq 1$ imply that

$$(x^5+y^2+z^2)(yz+y^2+z^2)\geq (x^2+y^2+z^2)^2 \text{ or } \frac{x^2(yz+y^2+z^2)}{(x^2+y^2+z^2)^2}\geq \frac{x^2}{x^5+y^2+z^2}.$$

Taking the cyclic sum, we have

$$\sum_{\text{cyclic}}\frac{x^2(yz+y^2+z^2)}{(x^2+y^2+z^2)^2}\geq \sum_{\text{cyclic}}\frac{x^2}{x^5+y^2+z^2}.$$

Our job is now to establish the following homogeneous inequality

$$1\geq \sum_{\text{cyclic}}\frac{x^2(yz+y^2+z^2)}{(x^2+y^2+z^2)^2}\Leftrightarrow (x^2+y^2+z^2)^2\geq 2\sum_{\text{cyclic}}x^2y^2+\sum_{\text{cyclic}}x^2yz \Leftrightarrow \sum_{\text{cyclic}}x^4\geq \sum_{\text{cyclic}}x^2yz.$$

However, by the AM-GM inequality, we obtain

$$\sum_{\text{cyclic}}x^4=\sum_{\text{cyclic}}\frac{x^4+y^4}{2}\geq \sum_{\text{cyclic}}x^2y^2=\sum_{\text{cyclic}}x^2\left(\frac{y^2+z^2}{2}\right)\geq \sum_{\text{cyclic}}x^2yz.$$

[Second Way] We claim that

$$\frac{2x^4+y^4+z^4+4x^2y^2+4x^2z^2}{4(x^2+y^2+z^2)^2}\geq \frac{x^2}{x^5+y^2+z^2}.$$

We do this by proving

$$\frac{2x^4+y^4+z^4+4x^2y^2+4x^2z^2}{4(x^2+y^2+z^2)^2}\geq \frac{x^2yz}{x^4+y^3z+yz^3}$$

because $xyz \geq 1$ implies that

$$\frac{x^2yz}{x^4+y^3z+yz^3} = \frac{x^2}{\frac{x^5}{xyz}+y^2+z^2} \geq \frac{x^2}{x^5+y^2+z^2}.$$

Hence, we need to show the homogeneous inequality

$$(2x^4+y^4+z^4+4x^2y^2+4x^2z^2)(x^4+y^3z+yz^3) \geq 4x^2yz(x^2+y^2+z^2)^2.$$

However, this is a straightforward consequence of the AM-GM inequality.

$$\begin{aligned}
& (2x^4+y^4+z^4+4x^2y^2+4x^2z^2)(x^4+y^3z+yz^3) - 4x^2yz(x^2+y^2+z^2)^2 \\
= \; & (x^8+x^4y^4+x^6y^2+x^6y^2+y^7z+y^3z^5)+(x^8+x^4z^4+x^6z^2+x^6z^2+yz^7+y^5z^3) \\
& +2(x^6y^2+x^6z^2)-6x^4y^3z-6x^4yz^3-2x^6yz \\
\geq \; & 6\sqrt[6]{x^8\cdot x^4y^4\cdot x^6y^2\cdot x^6y^2\cdot y^7z\cdot y^3z^5}+6\sqrt[6]{x^8\cdot x^4z^4\cdot x^6z^2\cdot x^6z^2\cdot yz^7\cdot y^5z^3} \\
& +2\sqrt{x^6y^2\cdot x^6z^2}-6x^4y^3z-6x^4yz^3-2x^6yz \\
= \; & 0.
\end{aligned}$$

Taking the cyclic sum, we obtain

$$1 = \sum_{\text{cyclic}} \frac{2x^4+y^4+z^4+4x^2y^2+4x^2z^2}{4(x^2+y^2+z^2)^2} \geq \sum_{\text{cyclic}} \frac{x^2}{x^5+y^2+z^2}.$$

□

Third Solution. (by an IMO 2005 contestant Iurie Boreico[12] from Moldova) We establish that

$$\frac{x^5-x^2}{x^5+y^2+z^2} \geq \frac{x^5-x^2}{x^3(x^2+y^2+z^2)}.$$

It follows immediately from the identity

$$\frac{x^5-x^2}{x^5+y^2+z^2} - \frac{x^5-x^2}{x^3(x^2+y^2+z^2)} = \frac{(x^3-1)^2x^2(y^2+z^2)}{x^3(x^2+y^2+z^2)(x^5+y^2+z^2)}.$$

Taking the cyclic sum and using $xyz \geq 1$, we have

$$\sum_{\text{cyclic}} \frac{x^5-x^2}{x^5+y^2+z^2} \geq \frac{1}{x^5+y^2+z^2}\sum_{\text{cyclic}}\left(x^2-\frac{1}{x}\right) \geq \frac{1}{x^5+y^2+z^2}\sum_{\text{cyclic}}\left(x^2-yz\right) \geq 0.$$

□

Here is a brilliant solution of

Problem 23. ***(KMO Weekend Program 2007)*** *Prove that, for all $a,b,c,x,y,z > 0$,*

$$\frac{ax}{a+x}+\frac{by}{b+y}+\frac{cz}{c+z} \leq \frac{(a+b+c)(x+y+z)}{a+b+c+x+y+z}.$$

Solution. (by Sanghoon) We need the following lemma:

Lemma. For all $p,q,\omega_1,\omega_2 > 0$, we have

$$\frac{pq}{p+q} \leq \frac{{\omega_1}^2p+{\omega_2}^2q}{(\omega_1+\omega_2)^2}.$$

[12] He received the special prize for this solution.

Proof of lemma. It's equivalent to

$$(p+q)\left(\omega_1{}^2 p+\omega_2{}^2 q\right)-(\omega_1+\omega_2)^2\,pq\ge 0$$

or

$$(\omega_1 p-\omega_2 q)^2\ge 0.$$

Taking $(p,q,\omega_1,\omega_2)=(a,x,x+y+z,a+b+c)$ in the lemma, we get

$$\frac{ax}{a+x}\le\frac{(x+y+z)^2a+(a+b+c)^2x}{(x+y+z+a+b+c)^2}.$$

Similarly, we obtain

$$\frac{by}{b+y}\le\frac{(x+y+z)^2b+(a+b+c)^2y}{(x+y+z+a+b+c)^2}$$

and

$$\frac{cz}{c+z}\le\frac{(x+y+z)^2c+(a+b+c)^2z}{(x+y+z+a+b+c)^2}.$$

Adding the above three inequalities, we get

$$\frac{ax}{a+x}+\frac{by}{b+y}+\frac{cz}{c+z}\le\frac{(x+y+z)^2(a+b+c)+(a+b+c)^2(x+y+z)}{(x+y+z+a+b+c)^2}.$$

or

$$\frac{ax}{a+x}+\frac{by}{b+y}+\frac{cz}{c+z}\le\frac{(a+b+c)(x+y+z)}{a+b+c+x+y+z}.$$

$\square$

Exercise 5. ***(USAMO Summer Program 2002)*** *Let a, b, c be positive real numbers. Prove that*

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}}+\left(\frac{2b}{c+a}\right)^{\frac{2}{3}}+\left(\frac{2c}{a+b}\right)^{\frac{2}{3}}\ge 3.$$

(Hint. [TJM]) Establish the inequality $\left(\frac{2a}{b+c}\right)^{\frac{2}{3}}\ge 3\left(\frac{a}{a+b+c}\right)$.

Exercise 6. ***(APMO 2005)*** $(abc=8,\ a,b,c>0)$

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}}+\frac{b^2}{\sqrt{(1+b^3)(1+c^3)}}+\frac{c^2}{\sqrt{(1+c^3)(1+a^3)}}\ge\frac{4}{3}$$

(Hint.) Use the inequality $\frac{1}{\sqrt{1+x^3}}\ge\frac{2}{2+x^2}$ to give a lower bound of the left hand side.

3 Homogenizations and Normalizations

Every Mathematician Has Only a Few Tricks. *A long time ago an older and well-known number theorist made some disparaging remarks about Paul Erdös′s work. You admire Erdos′s contributions to mathematics as much as I do, and I felt annoyed when the older mathematician flatly and definitively stated that all of Erdos′s work could be reduced to a few tricks which Erdös repeatedly relied on in his proofs. What the number theorist did not realize is that other mathematicians, even the very best, also rely on a few tricks which they use over and over. Take Hilbert. The second volume of Hilbert′s collected papers contains Hilbert′s papers in invariant theory. I have made a point of reading some of these papers with care. It is sad to note that some of Hilbert′s beautiful results have been completely forgotten. But on reading the proofs of Hilbert′s striking and deep theorems in invariant theory, it was surprising to verify that Hilbert′s proofs relied on the same few tricks.* ***Even Hilbert had only a few tricks!*** Gian-Carlo Rota, *Ten Lessons I Wish I Had Been Taught*, Notices of the AMS, January 1997

1 Homogenizations

Many inequality problems come with constraints such as $ab = 1$, $xyz = 1$, $x+y+z = 1$. A non-homogeneous *symmetric* inequality can be transformed into a homogeneous one. Then we apply two powerful theorems : Shur's inequality and Muirhead's theorem. We begin with a simple example.

Problem 24. ***(Hungary 1996)*** *Let a and b be positive real numbers with $a+b = 1$. Prove that*

$$\frac{a^2}{a+1}+\frac{b^2}{b+1} \geq \frac{1}{3}.$$

Solution. Using the condition $a+b = 1$, we can reduce the given inequality to homogeneous one, i. e.,

$$\frac{1}{3} \leq \frac{a^2}{(a+b)(a+(a+b))}+\frac{b^2}{(a+b)(b+(a+b))} \text{ or } a^2b+ab^2 \leq a^3+b^3,$$

which follows from $(a^3+b^3)-(a^2b+ab^2) = (a-b)^2(a+b) \geq 0$. The equality holds if and only if $a = b = \frac{1}{2}$. □

The above inequality $a^2b+ab^2 \leq a^3+b^3$ can be generalized as following :

Theorem 3.1. *Let a_1,a_2,b_1,b_2 be positive real numbers such that $a_1+a_2 = b_1+b_2$ and $\max(a_1,a_2) \geq \max(b_1,b_2)$. Let x and y be nonnegative real numbers. Then, we have $x^{a_1}y^{a_2}+x^{a_2}y^{a_1} \geq x^{b_1}y^{b_2}+x^{b_2}y^{b_1}$.*

Proof. Without loss of generality, we can assume that $a_1 \geq a_2, b_1 \geq b_2, a_1 \geq b_1$. If x or y is zero, then it clearly holds. So, we assume that both x and y are nonzero. It follows from $a_1+a_2 = b_1+b_2$ that $a_1-a_2 = (b_1-a_2)+(b_2-a_2)$. It's easy to check

$$\begin{aligned} x^{a_1}y^{a_2}+x^{a_2}y^{a_1}-x^{b_1}y^{b_2}-x^{b_2}y^{b_1} &= x^{a_2}y^{a_2}\left(x^{a_1-a_2}+y^{a_1-a_2}-x^{b_1-a_2}y^{b_2-a_2}-x^{b_2-a_2}y^{b_1-a_2}\right) \\ &= x^{a_2}y^{a_2}\left(x^{b_1-a_2}-y^{b_1-a_2}\right)\left(x^{b_2-a_2}-y^{b_2-a_2}\right) \\ &= \frac{1}{x^{a_2}y^{a_2}}\left(x^{b_1}-y^{b_1}\right)\left(x^{b_2}-y^{b_2}\right) \geq 0. \end{aligned}$$

□

Remark 3.1. *When does the equality hold in the theorem 8?*

We now introduce two summation notations $\sum_{\text{cyclic}}$ and $\sum_{\text{sym}}$. Let $P(x,y,z)$ be a three variables function of x, y, z. Let us define :

$$\sum_{\text{cyclic}} P(x,y,z) = P(x,y,z)+P(y,z,x)+P(z,x,y),$$

$$\sum_{\text{sym}} P(x,y,z) = P(x,y,z)+P(x,z,y)+P(y,x,z)+P(y,z,x)+P(z,x,y)+P(z,y,x).$$

For example, we know that

$$\sum_{\text{cyclic}} x^3y = x^3y+y^3z+z^3x, \quad \sum_{\text{sym}} x^3 = 2(x^3+y^3+z^3)$$

$$\sum_{\text{sym}} x^2y = x^2y+x^2z+y^2z+y^2x+z^2x+z^2y, \quad \sum_{\text{sym}} xyz = 6xyz.$$

Problem 25. ***(IMO 1984/1)*** *Let x,y,z be nonnegative real numbers such that $x+y+z = 1$. Prove that $0 \leq xy+yz+zx-2xyz \leq \frac{7}{27}$.*

Second Solution. Using the condition $x+y+z=1$, we reduce the given inequality to homogeneous one, i. e.,

$$0 \leq (xy+yz+zx)(x+y+z)-2xyz \leq \frac{7}{27}(x+y+z)^3.$$

The left hand side inequality is trivial because it's equivalent to

$$0 \leq xyz + \sum_{\rm sym} x^2 y.$$

The right hand side inequality simplifies to

$$7\sum_{\rm cyclic} x^3 + 15xyz - 6\sum_{\rm sym} x^2 y \geq 0.$$

In the view of

$$7\sum_{\rm cyclic} x^3 + 15xyz - 6\sum_{\rm sym} x^2 y = \left(2\sum_{\rm cyclic} x^3 - \sum_{\rm sym} x^2 y\right) + 5\left(3xyz + \sum_{\rm cyclic} x^3 - \sum_{\rm sym} x^2 y\right),$$

it's enough to show that

$$2\sum_{\rm cyclic} x^3 \geq \sum_{\rm sym} x^2 y \text{ and } 3xyz + \sum_{\rm cyclic} x^3 \geq \sum_{\rm sym} x^2 y.$$

We note that

$$2\sum_{\rm cyclic} x^3 - \sum_{\rm sym} x^2 y = \sum_{\rm cyclic} (x^3+y^3) - \sum_{\rm cyclic} (x^2y+xy^2) = \sum_{\rm cyclic} (x^3+y^3-x^2y-xy^2) \geq 0.$$

The second inequality can be rewritten as

$$\sum_{\rm cyclic} x(x-y)(x-z) \geq 0,$$

which is a particular case of Schur's theorem in the next section. ☐

After homogenizing, sometimes we can find the *right* approach to see the inequalities:

(**Iran 1998**) Prove that, for all $x,y,z>1$ such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$,

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

Second Solution. After the algebraic substitution $a=\frac{1}{x}$, $b=\frac{1}{y}$, $c=\frac{1}{z}$, we are required to prove that

$$\sqrt{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \geq \sqrt{\frac{1-a}{a}}+\sqrt{\frac{1-b}{b}}+\sqrt{\frac{1-c}{c}},$$

where $a,b,c \in (0,1)$ and $a+b+c=2$. Using the constraint $a+b+c=2$, we obtain a homogeneous inequality

$$\sqrt{\frac{1}{2}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \geq \sqrt{\frac{\frac{a+b+c}{2}-a}{a}}+\sqrt{\frac{\frac{a+b+c}{2}-b}{b}}+\sqrt{\frac{\frac{a+b+c}{2}-c}{c}}$$

or

$$\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \geq \sqrt{\frac{b+c-a}{a}}+\sqrt{\frac{c+a-b}{b}}+\sqrt{\frac{a+b-c}{c}},$$

which immediately follows from the Cauchy-Schwarz inequality

$$\begin{aligned}
&\sqrt{[(b+c-a)+(c+a-b)+(a+b-c)]\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\\
\geq\ &\sqrt{\frac{b+c-a}{a}}+\sqrt{\frac{c+a-b}{b}}+\sqrt{\frac{a+b-c}{c}}.
\end{aligned}$$

□

2 Schur's Inequality and Muirhead's Theorem

Theorem 3.2. *(**Schur**) Let x,y,z be nonnegative real numbers. For any $r>0$, we have*

$$\sum_{\text{cyclic}} x^r(x-y)(x-z)\geq 0.$$

Proof. Since the inequality is symmetric in the three variables, we may assume without loss of generality that $x\geq y\geq z$. Then the given inequality may be rewritten as

$$(x-y)[x^r(x-z)-y^r(y-z)]+z^r(x-z)(y-z)\geq 0,$$

and every term on the left-hand side is clearly nonnegative. □

Remark 3.2. *When does the equality hold in Schur's Inequality?*

Exercise 7. *Disprove the following proposition: For all $a,b,c,d\geq 0$ and $r>0$, we have*

$$\begin{aligned}
a^r(a-b)(a-c)(a-d)+b^r(b-c)(b-d)(b-a)+&\\
c^r(c-a)(c-c)(a-d)+d^r(d-a)(d-b)(d-c)\quad&\geq\quad 0.
\end{aligned}$$

The following special case of Schur's inequality is useful :

$$\sum_{\text{cyclic}} x(x-y)(x-z)\geq 0 \;\Leftrightarrow\; 3xyz+\sum_{\text{cyclic}} x^3\geq\sum_{\text{sym}} x^2y \;\Leftrightarrow\; \sum_{\text{sym}} xyz+\sum_{\text{sym}} x^3\geq 2\sum_{\text{sym}} x^2y.$$

Corollary 3.1. *Let x,y,z be nonnegative real numbers. Then, we have*

$$3xyz+x^3+y^3+z^3\geq 2\left((xy)^{\frac{3}{2}}+(yz)^{\frac{3}{2}}+(zx)^{\frac{3}{2}}\right).$$

Proof. By Schur's inequality and the AM-GM inequality, we have

$$3xyz+\sum_{\text{cyclic}} x^3\geq\sum_{\text{cyclic}} x^2y+xy^2\geq\sum_{\text{cyclic}} 2(xy)^{\frac{3}{2}}.$$

□

We now use Schur's inequality to give an alternative solution of

(APMO 2004/5) Prove that, for all positive real numbers a,b,c,

$$(a^2+2)(b^2+2)(c^2+2)\geq 9(ab+bc+ca).$$

Second Solution. After expanding, it becomes

$$8+(abc)^2+2\sum_{\text{cyclic}} a^2b^2+4\sum_{\text{cyclic}} a^2 \geq 9\sum_{\text{cyclic}} ab.$$

From the inequality $(ab-1)^2+(bc-1)^2+(ca-1)^2 \geq 0$, we obtain

$$6+2\sum_{\text{cyclic}} a^2b^2 \geq 4\sum_{\text{cyclic}} ab.$$

Hence, it will be enough to show that

$$2+(abc)^2+4\sum_{\text{cyclic}} a^2 \geq 5\sum_{\text{cyclic}} ab.$$

Since $3(a^2+b^2+c^2) \geq 3(ab+bc+ca)$, it will be enough to show that

$$2+(abc)^2+\sum_{\text{cyclic}} a^2 \geq 2\sum_{\text{cyclic}} ab,$$

which is a particular case of the following result for $t=1$. $\square$

Corollary 3.2. *Let $t \in (0,3]$. For all $a,b,c \geq 0$, we have*

$$(3-t)+t(abc)^{\frac{2}{t}}+\sum_{\text{cyclic}} a^2 \geq 2\sum_{\text{cyclic}} ab.$$

In particular, we obtain non-homogeneous inequalities

$$\frac{5}{2}+\frac{1}{2}(abc)^4+a^2+b^2+c^2 \geq 2(ab+bc+ca),$$

$$2+(abc)^2+a^2+b^2+c^2 \geq 2(ab+bc+ca),$$

$$1+2abc+a^2+b^2+c^2 \geq 2(ab+bc+ca).$$

Proof. After setting $x=a^{\frac{2}{3}}, y=b^{\frac{2}{3}}, z=c^{\frac{2}{3}}$, it becomes

$$3-t+t(xyz)^{\frac{3}{t}}+\sum_{\text{cyclic}} x^3 \geq 2\sum_{\text{cyclic}} (xy)^{\frac{3}{2}}.$$

By the corollary 1, it will be enough to show that

$$3-t+t(xyz)^{\frac{3}{t}} \geq 3xyz,$$

which is a straightforward consequence of the weighted AM-GM inequality :

$$\frac{3-t}{3}\cdot 1+\frac{t}{3}(xyz)^{\frac{3}{t}} \geq 1^{\frac{3-t}{3}}\left((xyz)^{\frac{3}{t}}\right)^{\frac{t}{3}}=3xyz.$$

One may check that the equality holds if and only if $a=b=c=1$. $\square$

(IMO 2000/2) Let a,b,c be positive numbers such that $abc=1$. Prove that

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leq 1.$$

Second Solution. It is equivalent to the following homogeneous inequality[13] :

$$\left(a-(abc)^{1/3}+\frac{(abc)^{2/3}}{b}\right)\left(b-(abc)^{1/3}+\frac{(abc)^{2/3}}{c}\right)\left(c-(abc)^{1/3}+\frac{(abc)^{2/3}}{a}\right)\leq abc.$$

After the substitution $a=x^3,b=y^3,c=z^3$ with $x,y,z>0$, it becomes

$$\left(x^3-xyz+\frac{(xyz)^2}{y^3}\right)\left(y^3-xyz+\frac{(xyz)^2}{z^3}\right)\left(z^3-xyz+\frac{(xyz)^2}{x^3}\right)\leq x^3y^3z^3,$$

which simplifies to

$$\left(x^2y-y^2z+z^2x\right)\left(y^2z-z^2x+x^2y\right)\left(z^2x-x^2y+y^2z\right)\leq x^3y^3z^3$$

or

$$3x^3y^3z^3+\sum_{\text{cyclic}}x^6y^3\geq\sum_{\text{cyclic}}x^4y^4z+\sum_{\text{cyclic}}x^5y^2z^2$$

or

$$3(x^2y)(y^2z)(z^2x)+\sum_{\text{cyclic}}(x^2y)^3\geq\sum_{\text{sym}}(x^2y)^2(y^2z)$$

which is a special case of Schur's inequality. □

Here is another inequality problem with the constraint $abc=1$.

Problem 26. ***(Tournament of Towns 1997)*** *Let a,b,c be positive numbers such that $abc=1$. Prove that*

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\leq 1.$$

Solution. We can rewrite the given inequality as following :

$$\frac{1}{a+b+(abc)^{1/3}}+\frac{1}{b+c+(abc)^{1/3}}+\frac{1}{c+a+(abc)^{1/3}}\leq\frac{1}{(abc)^{1/3}}.$$

We make the substitution $a=x^3,b=y^3,c=z^3$ with $x,y,z>0$. Then, it becomes

$$\frac{1}{x^3+y^3+xyz}+\frac{1}{y^3+z^3+xyz}+\frac{1}{z^3+x^3+xyz}\leq\frac{1}{xyz}$$

which is equivalent to

$$xyz\sum_{\text{cyclic}}(x^3+y^3+xyz)(y^3+z^3+xyz)\leq(x^3+y^3+xyz)(y^3+z^3+xyz)(z^3+x^3+xyz)$$

or

$$\sum_{\text{sym}}x^6y^3\geq\sum_{\text{sym}}x^5y^2z^2\quad !$$

We apply the theorem 9 to obtain

$$\begin{aligned}\sum_{\text{sym}}x^6y^3 &= \sum_{\text{cyclic}}x^6y^3+y^6x^3\\ &\geq \sum_{\text{cyclic}}x^5y^4+y^5x^4\\ &= \sum_{\text{cyclic}}x^5(y^4+z^4)\\ &\geq \sum_{\text{cyclic}}x^5(y^2z^2+y^2z^2)\\ &= \sum_{\text{sym}}x^5y^2z^2.\end{aligned}$$

□

[13] For an alternative homogenization, see the problem 1 in the chapter 2.

Exercise 8. ***([TZ], pp.142)*** *Prove that for any acute triangle ABC,*

$$\cot^3 A+\cot^3 B+\cot^3 C+6\cot A\cot B\cot C \geq \cot A+\cot B+\cot C.$$

Exercise 9. ***(Korea 1998)*** *Let I be the incenter of a triangle ABC. Prove that*

$$IA^2+IB^2+IC^2 \geq \frac{BC^2+CA^2+AB^2}{3}.$$

Exercise 10. ***([IN], pp.103)*** *Let a,b,c be the lengths of a triangle. Prove that*

$$a^2b+a^2c+b^2c+b^2a+c^2a+c^2b > a^3+b^3+c^3+2abc.$$

Exercise 11. ***(Surányi's inequality))*** *Show that, for all $x_1,\cdots,x_n \geq 0$,*

$$(n-1)\left({x_1}^n+\cdots {x_n}^n\right)+nx_1\cdots x_n \geq (x_1+\cdots x_n)\left({x_1}^{n-1}+\cdots {x_n}^{n-1}\right).$$

Theorem 3.3. ***(Muirhead)*** *Let a_1,a_2,a_3,b_1,b_2,b_3 be real numbers such that*

$$a_1 \geq a_2 \geq a_3 \geq 0, b_1 \geq b_2 \geq b_3 \geq 0, a_1 \geq b_1, a_1+a_2 \geq b_1+b_2, a_1+a_2+a_3 = b_1+b_2+b_3.$$

Let x,y,z be positive real numbers. Then, we have $\sum_{\text{sym}} x^{a_1}y^{a_2}z^{a_3} \geq \sum_{\text{sym}} x^{b_1}y^{b_2}z^{b_3}$.

Proof. Case 1. $b_1 \geq a_2$: It follows from $a_1 \geq a_1+a_2-b_1$ and from $a_1 \geq b_1$ that $a_1 \geq max(a_1+a_2-b_1,b_1)$ so that $max(a_1,a_2) = a_1 \geq max(a_1+a_2-b_1,b_1)$. From $a_1+a_2-b_1 \geq b_1+a_3-b_1 = a_3$ and $a_1+a_2-b_1 \geq b_2 \geq b_3$, we have $max(a_1+a_2-b_1,a_3) \geq max(b_2,b_3)$. Apply the theorem 8 twice to obtain

$$\begin{aligned}
\sum_{\text{sym}} x^{a_1}y^{a_2}z^{a_3} &= \sum_{\text{cyclic}} z^{a_3}(x^{a_1}y^{a_2}+x^{a_2}y^{a_1}) \\
&\geq \sum_{\text{cyclic}} z^{a_3}(x^{a_1+a_2-b_1}y^{b_1}+x^{b_1}y^{a_1+a_2-b_1}) \\
&= \sum_{\text{cyclic}} x^{b_1}(y^{a_1+a_2-b_1}z^{a_3}+y^{a_3}z^{a_1+a_2-b_1}) \\
&\geq \sum_{\text{cyclic}} x^{b_1}(y^{b_2}z^{b_3}+y^{b_3}z^{b_2}) \\
&= \sum_{\text{sym}} x^{b_1}y^{b_2}z^{b_3}.
\end{aligned}$$

Case 2. $b_1 \leq a_2$: It follows from $3b_1 \geq b_1+b_2+b_3 = a_1+a_2+a_3 \geq b_1+a_2+a_3$ that $b_1 \geq a_2+a_3-b_1$ and that $a_1 \geq a_2 \geq b_1 \geq a_2+a_3-b_1$. Therefore, we have $max(a_2,a_3) \geq max(b_1,a_2+a_3-b_1)$ and $max(a_1,a_2+a_3-b_1) \geq max(b_2,b_3)$. Apply the theorem 8 twice to obtain

$$\begin{aligned}
\sum_{\text{sym}} x^{a_1}y^{a_2}z^{a_3} &= \sum_{\text{cyclic}} x^{a_1}(y^{a_2}z^{a_3}+y^{a_3}z^{a_2}) \\
&\geq \sum_{\text{cyclic}} x^{a_1}(y^{b_1}z^{a_2+a_3-b_1}+y^{a_2+a_3-b_1}z^{b_1}) \\
&= \sum_{\text{cyclic}} y^{b_1}(x^{a_1}z^{a_2+a_3-b_1}+x^{a_2+a_3-b_1}z^{a_1}) \\
&\geq \sum_{\text{cyclic}} y^{b_1}(x^{b_2}z^{b_3}+x^{b_3}z^{b_2}) \\
&= \sum_{\text{sym}} x^{b_1}y^{b_2}z^{b_3}.
\end{aligned}$$

□

Remark 3.3. *The equality holds if and only if $x=y=z$. However, if we allow $x=0$ or $y=0$ or $z=0$, then one may easily check that the equality holds when $a_1,a_2,a_3>0$ and $b_1,b_2,b_3>0$ if and only if*

$$x=y=z \text{ or } x=y,\ z=0 \text{ or } y=z,\ x=0 \text{ or } z=x,\ y=0.$$

We can use Muirhead's theorem to prove Nesbitt's inequality.

(**Nesbitt**) For all positive real numbers a,b,c, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

Proof 6. *Clearing the denominators of the inequality, it becomes*

$$2\sum_{\text{cyclic}}a(a+b)(a+c)\geq 3(a+b)(b+c)(c+a) \text{ or } \sum_{\text{sym}}a^3\geq\sum_{\text{sym}}a^2b.$$

(IMO 1995) Let a,b,c be positive numbers such that $abc=1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

Second Solution. It's equivalent to

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2(abc)^{4/3}}.$$

Set $a=x^3,b=y^3,c=z^3$ with $x,y,z>0$. Then, it becomes $\sum_{\text{cyclic}}\frac{1}{x^9(y^3+z^3)}\geq\frac{3}{2x^4y^4z^4}$. Clearing denominators, this becomes

$$\sum_{\text{sym}}x^{12}y^{12}+2\sum_{\text{sym}}x^{12}y^9z^3+\sum_{\text{sym}}x^9y^9z^6\geq 3\sum_{\text{sym}}x^{11}y^8z^5+6x^8y^8z^8$$

or

$$\left(\sum_{\text{sym}}x^{12}y^{12}-\sum_{\text{sym}}x^{11}y^8z^5\right)+2\left(\sum_{\text{sym}}x^{12}y^9z^3-\sum_{\text{sym}}x^{11}y^8z^5\right)+\left(\sum_{\text{sym}}x^9y^9z^6-\sum_{\text{sym}}x^8y^8z^8\right)\geq 0,$$

and every term on the left hand side is nonnegative by Muirhead's theorem. $\square$

Problem 27. ***(Iran 1996)*** *Let x,y,z be positive real numbers. Prove that*

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq\frac{9}{4}.$$

Proof. It's equivalent to

$$4\sum_{\text{sym}}x^5y+2\sum_{\text{cyclic}}x^4yz+6x^2y^2z^2-\sum_{\text{sym}}x^4y^2-6\sum_{\text{cyclic}}x^3y^3-2\sum_{\text{sym}}x^3y^2z\geq 0.$$

We rewrite this as following

$$\left(\sum_{\text{sym}}x^5y-\sum_{\text{sym}}x^4y^2\right)+3\left(\sum_{\text{sym}}x^5y-\sum_{\text{sym}}x^3y^3\right)+2xyz\left(3xyz+\sum_{\text{cyclic}}x^3-\sum_{\text{sym}}x^2y\right)\geq 0.$$

By Muirhead's theorem and Schur's inequality, it's a sum of three nonnegative terms. $\square$

Problem 28. *Let x, y, z be nonnegative real numbers with $xy + yz + zx = 1$. Prove that*

$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \geq \frac{5}{2}.$$

Proof. Using $xy + yz + zx = 1$, we homogenize the given inequality as following :

$$(xy+yz+zx)\left(\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x}\right)^2 \geq \left(\frac{5}{2}\right)^2$$

or

$$4\sum_{\text{sym}} x^5y + \sum_{\text{sym}} x^4yz + 14\sum_{\text{sym}} x^3y^2z + 38x^2y^2z^2 \geq \sum_{\text{sym}} x^4y^2 + 3\sum_{\text{sym}} x^3y^3$$

or

$$\left(\sum_{\text{sym}} x^5y - \sum_{\text{sym}} x^4y^2\right) + 3\left(\sum_{\text{sym}} x^5y - \sum_{\text{sym}} x^3y^3\right) + xyz\left(\sum_{\text{sym}} x^3 + 14\sum_{\text{sym}} x^2y + 38xyz\right) \geq 0.$$

By Muirhead's theorem, we get the result. In the above inequality, without the condition $xy + yz + zx = 1$, the equality holds if and only if $x = y, z = 0$ *or* $y = z, x = 0$ *or* $z = x, y = 0$. Since $xy + yz + zx = 1$, the equality occurs when $(x, y, z) = (1, 1, 0), (1, 0, 1), (0, 1, 1)$. □

3 Normalizations

In the previous sections, we transformed non-homogeneous inequalities into homogeneous ones. On the other hand, homogeneous inequalities also can be normalized in *various* ways. We offer two alternative solutions of the problem 8 by normalizations :

(IMO 2001/2) Let a, b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

Third Solution. We make the substitution $x = \frac{a}{a+b+c}$, $y = \frac{b}{a+b+c}$, $z = \frac{c}{a+b+c}$.[14] The problem is

$$xf(x^2+8yz) + yf(y^2+8zx) + zf(z^2+8xy) \geq 1,$$

where $f(t) = \frac{1}{\sqrt{t}}$. Since f is convex on $\mathbb{R}^+$ and $x + y + z = 1$, we apply (the weighted) Jensen's inequality to obtain

$$xf(x^2+8yz) + yf(y^2+8zx) + zf(z^2+8xy) \geq f(x(x^2+8yz) + y(y^2+8zx) + z(z^2+8xy)).$$

Note that $f(1) = 1$. Since the function f is strictly decreasing, it suffices to show that

$$1 \geq x(x^2+8yz) + y(y^2+8zx) + z(z^2+8xy).$$

Using $x + y + z = 1$, we homogenize it as $(x+y+z)^3 \geq x(x^2+8yz) + y(y^2+8zx) + z(z^2+8xy)$. However, this is easily seen from

$$(x+y+z)^3 - x(x^2+8yz) - y(y^2+8zx) - z(z^2+8xy) = 3[x(y-z)^2 + y(z-x)^2 + z(x-y)^2] \geq 0.$$

□

[14] Dividing by $a + b + c$ gives the equivalent inequality $\sum_{\text{cyclic}} \frac{\frac{a}{a+b+c}}{\sqrt{\frac{a^2}{(a+b+c)^2} + \frac{8bc}{(a+b+c)^2}}} \geq 1$.

In the above solution, we normalized to $x+y+z=1$. We now prove it by normalizing to $xyz=1$.

Fourth Solution. We make the *substitution* $x=\frac{bc}{a^2}$, $y=\frac{ca}{b^2}$, $z=\frac{ab}{c^2}$. Then, we get $xyz=1$ and the inequality becomes

$$\frac{1}{\sqrt{1+8x}}+\frac{1}{\sqrt{1+8y}}+\frac{1}{\sqrt{1+8z}}\geq 1$$

which is equivalent to

$$\sum_{\text{cyclic}}\sqrt{(1+8x)(1+8y)}\geq\sqrt{(1+8x)(1+8y)(1+8z)}.$$

After squaring both sides, it's equivalent to

$$8(x+y+z)+2\sqrt{(1+8x)(1+8y)(1+8z)}\sum_{\text{cyclic}}\sqrt{1+8x}\geq 510.$$

Recall that $xyz=1$. The AM-GM inequality gives us $x+y+z\geq 3$,

$$(1+8x)(1+8y)(1+8z)\geq 9x^{\frac{8}{9}}\cdot 9y^{\frac{8}{9}}\cdot 9z^{\frac{8}{9}}=729 \text{ and } \sum_{\text{cyclic}}\sqrt{1+8x}\geq\sum_{\text{cyclic}}\sqrt{9x^{\frac{8}{9}}}\geq 9(xyz)^{\frac{4}{27}}=9.$$

Using these three inequalities, we get the result. □

(IMO 1983/6) Let a, b, c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

Second Solution. After setting $a=y+z$, $b=z+x$, $c=x+y$ for $x,y,z>0$, it becomes

$$x^3z+y^3x+z^3y\geq x^2yz+xy^2z+xyz^2 \text{ or } \frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq x+y+z.$$

Since it's homogeneous, we can restrict our attention to the case $x+y+z=1$. Then, it becomes

$$yf\left(\frac{x}{y}\right)+zf\left(\frac{y}{z}\right)+xf\left(\frac{z}{x}\right)\geq 1,$$

where $f(t)=t^2$. Since f is convex on $\mathbb{R}$, we apply (the weighted) Jensen's inequality to obtain

$$yf\left(\frac{x}{y}\right)+zf\left(\frac{y}{z}\right)+xf\left(\frac{z}{x}\right)\geq f\left(y\cdot\frac{x}{y}+z\cdot\frac{y}{z}+x\cdot\frac{z}{x}\right)=f(1)=1.$$

□

Problem 29. ***(KMO Winter Program Test 2001)*** *Prove that, for all $a,b,c>0$,*

$$\sqrt{(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)}\geq abc+\sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)}$$

First Solution. Dividing by abc, it becomes

$$\sqrt{\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)\left(\frac{c}{a}+\frac{a}{b}+\frac{b}{c}\right)}\geq abc+\sqrt[3]{\left(\frac{a^2}{bc}+1\right)\left(\frac{b^2}{ca}+1\right)\left(\frac{c^2}{ab}+1\right)}.$$

After the substitution $x=\frac{a}{b}$, $y=\frac{b}{c}$, $z=\frac{c}{a}$, we obtain the constraint $xyz=1$. It takes the form

$$\sqrt{(x+y+z)(xy+yz+zx)}\geq 1+\sqrt[3]{\left(\frac{x}{z}+1\right)\left(\frac{y}{x}+1\right)\left(\frac{z}{y}+1\right)}.$$

From the constraint $xyz = 1$, we find two identities

$$\left(\frac{x}{z}+1\right)\left(\frac{y}{x}+1\right)\left(\frac{z}{y}+1\right) = \left(\frac{x+z}{z}\right)\left(\frac{y+x}{x}\right)\left(\frac{z+y}{y}\right) = (z+x)(x+y)(y+z),$$

$$(x+y+z)(xy+yz+zx) = (x+y)(y+z)(z+x)+xyz = (x+y)(y+z)(z+x)+1.$$

Letting $p = \sqrt[3]{(x+y)(y+z)(z+x)}$, the inequality now becomes $\sqrt{p^3+1} \geq 1+p$. Applying the AM-GM inequality, we have $p \geq \sqrt[3]{2\sqrt{xy}\cdot 2\sqrt{yz}\cdot 2\sqrt{zx}} = 2$. It follows that $(p^3+1)-(1+p)^2 = p(p+1)(p-2) \geq 0$. □

Problem 30. ***(IMO 1999/2)*** *Let n be an integer with $n \geq 2$.*

(a) Determine the least constant C such that the inequality

$$\sum_{1\leq i<j\leq n} x_i x_j (x_i^2+x_j^2) \leq C \left(\sum_{1\leq i\leq n} x_i\right)^4$$

holds for all real numbers $x_1, \cdots, x_n \geq 0$.
(b) For this constant C, determine when equality holds.

First Solution. (**Marcin E. Kuczma**[15]) For $x_1 = \cdots = x_n = 0$, it holds for any $C \geq 0$. Hence, we consider the case when $x_1+\cdots+x_n > 0$. Since the inequality is homogeneous, we may normalize to $x_1+\cdots+x_n = 1$. We denote

$$F(x_1,\cdots,x_n) = \sum_{1\leq i<j\leq n} x_i x_j (x_i^2+x_j^2).$$

From the assumption $x_1+\cdots+x_n = 1$, we have

$$F(x_1,\cdots,x_n) = \sum_{1\leq i<j\leq n} {x_i}^3 x_j + \sum_{1\leq i<j\leq n} x_i {x_j}^3 = \sum_{1\leq i\leq n} {x_i}^3 \sum_{j\neq i} x_i = \sum_{1\leq i\leq n} {x_i}^3(1-x_i)$$

$$= \sum_{i=1}^{n} x_i({x_i}^2-{x_i}^3).$$

We claim that $C = \frac{1}{8}$. It suffices to show that

$$F(x_1,\cdots,x_n) \leq \frac{1}{8} = F\left(\frac{1}{2},\frac{1}{2},0,\cdots,0\right).$$

Lemma 3.1. $0 \leq x \leq y \leq \frac{1}{2}$ *implies* $x^2-x^3 \leq y^2-y^3$.

Proof. Since $x+y \leq 1$, we get $x+y \geq (x+y)^2 \geq x^2+xy+y^2$. Since $y-x \geq 0$, this implies that $y^2-x^2 \geq y^3-x^3$ or $y^2-y^3 \geq x^2-x^3$, as desired. □

Case 1. $\frac{1}{2} \geq x_1 \geq x_2 \geq \cdots \geq x_n$

$$\sum_{i=1}^{n} x_i({x_i}^2-{x_i}^3) \leq \sum_{i=1}^{n} x_i\left(\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^3\right) = \frac{1}{8}\sum_{i=1}^{n} x_i = \frac{1}{8}.$$

[15] I slightly modified his solution in [Au99].

Case 2. *$x_1 \geq \frac{1}{2} \geq x_2 \geq \cdots \geq x_n$ Let $x_1 = x$ and $y = 1-x = x_2+\cdots+x_n$. Since $y \geq x_2, \cdots, x_n$,*

$$F(x_1,\cdots,x_n) = x^3y + \sum_{i=2}^{n} x_i({x_i}^2 - {x_i}^3) \leq x^3y + \sum_{i=2}^{n} x_i(y^2-y^3) = x^3y + y(y^2-y^3).$$

Since $x^3y + y(y^2-y^3) = x^3y + y^3(1-y) = xy(x^2+y^2)$, it remains to show that

$$xy(x^2+y^2) \leq \frac{1}{8}.$$

Using $x+y=1$, we homogenize the above inequality as following.

$$xy(x^2+y^2) \leq \frac{1}{8}(x+y)^4.$$

However, we immediately find that $(x+y)^4 - 8xy(x^2+y^2) = (x-y)^4 \geq 0$.

$\square$

Exercise 12. ***(IMO unused 1991)*** *Let n be a given integer with $n \geq 2$. Find the maximum value of*

$$\sum_{1\leq i<j\leq n} x_i x_j (x_i + x_j),$$

where $x_1,\cdots,x_n \geq 0$ and $x_1+\cdots+x_n = 1$.

We close this section with another proofs of Nesbitt's inequality.

(**Nesbitt**) For all positive real numbers a,b,c, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Proof 7. *We may normalize to $a+b+c=1$. Note that $0 < a,b,c < 1$. The problem is now to prove*

$$\sum_{\text{cyclic}} \frac{a}{b+c} = \sum_{\text{cyclic}} f(a) \geq \frac{3}{2}, \text{ where } f(x) = \frac{x}{1-x}.$$

Since f is convex on $(0,1)$, Jensen's inequality shows that

$$\frac{1}{3}\sum_{\text{cyclic}} f(a) \geq f\left(\frac{a+b+c}{3}\right) = f\left(\frac{1}{3}\right) = \frac{1}{2} \text{ or } \sum_{\text{cyclic}} f(a) \geq \frac{3}{2}.$$

Proof 8. *(Cao Minh Quang) Assume that $a+b+c=1$. Note that $ab+bc+ca \leq \frac{1}{3}(a+b+c)^2 = \frac{1}{3}$. More strongly, we establish that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq 3 - \frac{9}{2}(ab+bc+ca)$$

or

$$\left(\frac{a}{b+c} + \frac{9a(b+c)}{4}\right) + \left(\frac{b}{c+a} + \frac{9b(c+a)}{4}\right) + \left(\frac{c}{a+b} + \frac{9c(a+b)}{4}\right) \geq 3.$$

The AM-GM inequality shows that

$$\sum_{\text{cyclic}} \frac{a}{b+c} + \frac{9a(b+c)}{4} \geq \sum_{\text{cyclic}} 2\sqrt{\frac{a}{b+c} \cdot \frac{9a(b+c)}{4}} = \sum_{\text{cyclic}} 3a = 3.$$

Proof 9. *We now break the symmetry by a suitable normalization. Since the inequality is symmetric in the three variables, we may assume that $a \geq b \geq c$. After the substitution $x = \frac{a}{c}, y = \frac{b}{c}$, we have $x \geq y \geq 1$. It becomes*

$$\frac{\frac{a}{c}}{\frac{b}{c}+1} + \frac{\frac{b}{c}}{\frac{a}{c}+1} + \frac{1}{\frac{a}{c}+\frac{b}{c}} \geq \frac{3}{2} \quad \text{or} \quad \frac{x}{y+1} + \frac{y}{x+1} \geq \frac{3}{2} - \frac{1}{x+y}.$$

We apply the AM-GM inequality to obtain

$$\frac{x+1}{y+1} + \frac{y+1}{x+1} \geq 2 \quad \text{or} \quad \frac{x}{y+1} + \frac{y}{x+1} \geq 2 - \frac{1}{y+1} + \frac{1}{x+1}.$$

It's enough to show that

$$2 - \frac{1}{y+1} + \frac{1}{x+1} \geq \frac{3}{2} - \frac{1}{x+y} \quad \Leftrightarrow \quad \frac{1}{2} - \frac{1}{y+1} \geq \frac{1}{x+1} - \frac{1}{x+y} \quad \Leftrightarrow \quad \frac{y-1}{2(1+y)} \geq \frac{y-1}{(x+1)(x+y)}.$$

However, the last inequality clearly holds for $x \geq y \geq 1$.

Proof 10. *As in the previous proof, we may normalize to $c = 1$ with the assumption $a \geq b \geq 1$. We prove*

$$\frac{a}{b+1} + \frac{b}{a+1} + \frac{1}{a+b} \geq \frac{3}{2}.$$

Let $A = a+b$ and $B = ab$. It becomes

$$\frac{a^2+b^2+a+b}{(a+1)(b+1)} + \frac{1}{a+b} \geq \frac{3}{2} \quad \text{or} \quad \frac{A^2-2B+A}{A+B+1} + \frac{1}{A} \geq \frac{3}{2} \quad \text{or} \quad 2A^3 - A^2 - A + 2 \geq B(7A-2).$$

Since $7A - 2 > 2(a+b-1) > 0$ and $A^2 = (a+b)^2 \geq 4ab = 4B$, it's enough to show that

$$4(2A^3 - A^2 - A + 2) \geq A^2(7A-2) \quad \Leftrightarrow \quad A^3 - 2A^2 - 4A + 8 \geq 0.$$

However, it's easy to check that $A^3 - 2A^2 - 4A + 8 = (A-2)^2(A+2) \geq 0$.

4 Cauchy-Schwarz Inequality and Hölder's Inequality

We begin with the following famous theorem:

Theorem 3.4. ***(The Cauchy-Schwarz inequality)*** *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be real numbers. Then,*

$$({a_1}^2 + \cdots + {a_n}^2)({b_1}^2 + \cdots + {b_n}^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2.$$

Proof. Let $A = \sqrt{{a_1}^2 + \cdots + {a_n}^2}$ and $B = \sqrt{{b_1}^2 + \cdots + {b_n}^2}$. In the case when $A = 0$, we get $a_1 = \cdots = a_n = 0$. Thus, the given inequality clearly holds. So, we may assume that $A, B > 0$. We may normalize to

$$1 = {a_1}^2 + \cdots + {a_n}^2 = {b_1}^2 + \cdots + {b_n}^2.$$

Hence, we need to to show that

$$|a_1 b_1 + \cdots + a_n b_n| \leq 1.$$

We now apply the AM-GM inequality to deduce

$$|x_1 y_1 + \cdots + x_n y_n| \leq |x_1 y_1| + \cdots + |x_n y_n| \leq \frac{{x_1}^2 + {y_1}^2}{2} + \cdots + \frac{{x_n}^2 + {y_n}^2}{2} = 1.$$

$\square$

Exercise 13. *Prove the Lagrange identity :*

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \sum_{1\le i<j\le n} (a_i b_j - a_j b_i)^2 .$$

Exercise 14. ***(Darij Grinberg)*** *Suppose that $0 < a_1 \le \cdots \le a_n$ and $0 < b_1 \le \cdots \le b_n$ be real numbers. Show that*

$$\frac{1}{4}\left(\sum_{k=1}^{n} a_k\right)^2 \left(\sum_{k=1}^{n} b_k\right)^2 > \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \left(\sum_{k=1}^{n} a_k b_k\right)^2$$

Exercise 15. ***([PF], S. S. Wagner)*** *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be real numbers. Suppose that $x \in [0,1]$. Show that*

$$\left(\sum_{i=1}^{n} a_i^2 + 2x\sum_{i<j} a_i a_j\right)\left(\sum_{i=1}^{n} b_i^2 + 2x\sum_{i<j} b_i b_j\right) \ge \left(\sum_{i=1}^{n} a_i b_i + x\sum_{i\le j} a_i b_j\right)^2 .$$

Exercise 16. *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers. Show that*

$$\sqrt{(a_1+\cdots+a_n)(b_1+\cdots+b_n)} \ge \sqrt{a_1 b_1} + \cdots + \sqrt{a_n b_n}.$$

Exercise 17. *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers. Show that*

$$\frac{a_1^2}{b_1} + \cdots + \frac{a_n^2}{b_n} \ge \frac{(a_1+\cdots+a_n)^2}{b_1+\cdots+b_n}.$$

Exercise 18. *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers. Show that*

$$\frac{a_1}{b_1^2} + \cdots + \frac{a_n}{b_n^2} \ge \frac{1}{a_1+\cdots+a_n}\left(\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n}\right)^2 .$$

Exercise 19. *Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be positive real numbers. Show that*

$$\frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n} \ge \frac{(a_1+\cdots+a_n)^2}{a_1 b_1+\cdots+a_n b_n}.$$

As an application of the Cauchy-Schwarz inequality, we give a different solution of the following problem.

(**Iran 1998**) Prove that, for all $x, y, z > 1$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$,

$$\sqrt{x+y+z} \ge \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Third Solution. We note that $\frac{x-1}{x} + \frac{y-1}{y} + \frac{z-1}{z} = 1$. Apply the Cauchy-Schwarz inequality to deduce

$$\sqrt{x+y+z} = \sqrt{(x+y+z)\left(\frac{x-1}{x} + \frac{y-1}{y} + \frac{z-1}{z}\right)} \ge \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

□

We now apply the Cauchy-Schwarz inequality to prove Nesbitt's inequality.

(**Nesbitt**) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2}.$$

Proof 11. *Applying the Cauchy-Schwarz inequality, we have*

$$((b+c)+(c+a)+(a+b))\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\geq 3^2.$$

It follows that

$$\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b}\geq\frac{9}{2}\quad or\quad 3+\sum_{\text{cyclic}}\frac{a}{b+c}\geq\frac{9}{2}.$$

Proof 12. *The Cauchy-Schwarz inequality yields*

$$\sum_{\text{cyclic}}\frac{a}{b+c}\sum_{\text{cyclic}}a(b+c)\geq\left(\sum_{\text{cyclic}}a\right)^2\quad or\quad \sum_{\text{cyclic}}\frac{a}{b+c}\geq\frac{(a+b+c)^2}{2(ab+bc+ca)}\geq\frac{3}{2}.$$

Problem 31. *(**Gazeta Matematicã**) Prove that, for all $a,b,c>0$,*

$$\begin{aligned}&\sqrt{a^4+a^2b^2+b^4}+\sqrt{b^4+b^2c^2+c^4}+\sqrt{c^4+c^2a^2+a^4}\\ \geq\quad & a\sqrt{2a^2+bc}+b\sqrt{2b^2+ca}+c\sqrt{2c^2+ab}.\end{aligned}$$

Solution. We obtain the chain of equalities and inequalities

$$\begin{aligned}\sum_{\text{cyclic}}\sqrt{a^4+a^2b^2+b^4} &= \sum_{\text{cyclic}}\sqrt{\left(a^4+\frac{a^2b^2}{2}\right)+\left(b^4+\frac{a^2b^2}{2}\right)}\\ &\geq \frac{1}{\sqrt{2}}\sum_{\text{cyclic}}\left(\sqrt{a^4+\frac{a^2b^2}{2}}+\sqrt{b^4+\frac{a^2b^2}{2}}\right) && \text{(Cauchy-Schwarz)}\\ &= \frac{1}{\sqrt{2}}\sum_{\text{cyclic}}\left(\sqrt{a^4+\frac{a^2b^2}{2}}+\sqrt{a^4+\frac{a^2c^2}{2}}\right)\\ &\geq \sqrt{2}\sum_{\text{cyclic}}\sqrt[4]{\left(a^4+\frac{a^2b^2}{2}\right)\left(a^4+\frac{a^2c^2}{2}\right)} && \text{(AM-GM)}\\ &\geq \sqrt{2}\sum_{\text{cyclic}}\sqrt{a^4+\frac{a^2bc}{2}} && \text{(Cauchy-Schwarz)}\\ &= \sum_{\text{cyclic}}\sqrt{2a^4+a^2bc}\,.\end{aligned}$$

□

Here is an ingenious solution of

(**KMO Winter Program Test 2001**) Prove that, for all $a,b,c>0$,

$$\sqrt{(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)}\geq abc+\sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)}$$

Second Solution. (based on work by a winter program participant) We obtain

$$\begin{aligned}
& \sqrt{(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)} \\
= \; & \frac{1}{2}\sqrt{[b(a^2+bc)+c(b^2+ca)+a(c^2+ab)][c(a^2+bc)+a(b^2+ca)+b(c^2+ab)]} \\
\geq \; & \frac{1}{2}\left(\sqrt{bc}(a^2+bc)+\sqrt{ca}(b^2+ca)+\sqrt{ab}(c^2+ab)\right) && \text{(Cauchy-Schwarz)} \\
\geq \; & \frac{3}{2}\sqrt[3]{\sqrt{bc}(a^2+bc)\cdot\sqrt{ca}(b^2+ca)\cdot\sqrt{ab}(c^2+ab)} && \text{(AM-GM)} \\
= \; & \frac{1}{2}\sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)}+\sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)} \\
\geq \; & \frac{1}{2}\sqrt[3]{2\sqrt{a^3\cdot abc}\cdot 2\sqrt{b^3\cdot abc}\cdot 2\sqrt{c^3\cdot abc}}+\sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)} && \text{(AM-GM)} \\
= \; & abc+\sqrt[3]{(a^3+abc)(b^3+abc)(c^3+abc)}.
\end{aligned}$$

□

Problem 32. ***(Andrei Ciupan)*** *Let a, b, c be positive real numbers such that*

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\geq 1.$$

Show that $a+b+c\geq ab+bc+ca$.

First Solution. (by Andrei Ciupan) By applying the Cauchy-Schwarz inequality, we obtain

$$(a+b+1)(a+b+c^2)\geq(a+b+c)^2$$

or

$$\frac{1}{a+b+1}\leq\frac{c^2+a+b}{(a+b+c)^2}.$$

Now by summing cyclically, we obtain

$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\leq\frac{a^2+b^2+c^2+2(a+b+c)}{(a+b+c)^2}$$

But from the condition, we can see that

$$a^2+b^2+c^2+2(a+b+c)\geq(a+b+c)^2,$$

and therefore

$$a+b+c\geq ab+bc+ca.$$

We see that the equality occurs if and only if $a=b=c=1$. □

Second Solution. (by Cezar Lupu) We first observe that

$$2\geq\sum_{\text{cyclic}}\left(1-\frac{1}{a+b+1}\right)=\sum_{\text{cyclic}}\frac{a+b}{a+b+1}=\sum_{\text{cyclic}}\frac{(a+b)^2}{(a+b)^2+a+b}.$$

Apply the Cauchy-Schwarz inequality to get

$$2\geq\sum_{\text{cyclic}}\frac{(a+b)^2}{(a+b)^2+a+b}\geq\frac{(\sum a+b)^2}{\sum(a+b)^2+a+b}=\frac{4\sum a^2+8\sum ab}{2\sum a^2+2\sum ab+2\sum a}.$$

or

$$a+b+c\geq ab+bc+ca.$$

□

We now illustrate normalization techniques to establish classical theorems. Using the same idea in the proof of the Cauchy-Schwarz inequality, we find a natural generalization :

Theorem 3.5. *Let $a_{ij}(i,j=1,\cdots,n)$ be positive real numbers. Then, we have*

$$({a_{11}}^n+\cdots+{a_{1n}}^n)\cdots({a_{n1}}^n+\cdots+{a_{nn}}^n)\geq(a_{11}a_{21}\cdots a_{n1}+\cdots+a_{1n}a_{2n}\cdots a_{nn})^n.$$

Proof. Since the inequality is homogeneous, as in the proof of the theorem 11, we can normalize to

$$({a_{i1}}^n+\cdots+{a_{in}}^n)^{\frac{1}{n}}=1\ \text{ or }\ {a_{i1}}^n+\cdots+{a_{in}}^n=1\ \ (i=1,\cdots,n).$$

Then, the inequality takes the form $a_{11}a_{21}\cdots a_{n1}+\cdots+a_{1n}a_{2n}\cdots a_{nn}\leq 1$ or $\sum_{i=1}^n a_{i1}\cdots a_{in}\leq 1$. Hence, it suffices to show that, for all $i=1,\cdots,n$,

$$a_{i1}\cdots a_{in}\leq\frac{1}{n},\ \text{ where }\ {a_{i1}}^n+\cdots+{a_{in}}^n=1.$$

To finish the proof, it remains to show the following *homogeneous* inequality : ☐

Theorem 3.6. ***(AM-GM inequality)*** *Let $a_1,\cdots,a_n$ be positive real numbers. Then, we have*

$$\frac{a_1+\cdots+a_n}{n}\geq\sqrt[n]{a_1\cdots a_n}.$$

Proof. Since it's homogeneous, we may rescale $a_1,\cdots,a_n$ so that $a_1\cdots a_n=1$. [16] We want to show that

$$a_1\cdots a_n=1\implies a_1+\cdots+a_n\geq n.$$

The proof is by induction on n. If $n=1$, it's trivial. If $n=2$, then we get $a_1+a_2-2=a_1+a_2-2\sqrt{a_1a_2}=(\sqrt{a_1}-\sqrt{a_2})^2\geq 0$. Now, we assume that it holds for some positive integer $n\geq 2$. And let a_1, $\cdots$, a_{n+1} be positive numbers such that $a_1\cdots a_na_{n+1}$=1. We may assume that $a_1\geq 1\geq a_2$. (Why?) It follows that $a_1a_2+1-a_1-a_2=(a_1-1)(a_2-1)\leq 0$ so that $a_1a_2+1\leq a_1+a_2$. Since $(a_1a_2)a_3\cdots a_n=1$, by the induction hypothesis, we have $a_1a_2+a_3+\cdots+a_{n+1}\geq n$. Hence, $a_1+a_2-1+a_3+\cdots+a_{n+1}\geq n$. ☐

The following simple observation is not tricky :

Let $a,b>0$ and $m,n\in\mathbf{N}$. Take $x_1=\cdots=x_m=a$ and $x_{m+1}=\cdots=x_{x_{m+n}}=b$. Applying the AM-GM inequality to $x_1,\cdots,x_{m+n}>0$, we obtain

$$\frac{ma+nb}{m+n}\geq(a^mb^n)^{\frac{1}{m+n}}\ \text{ or }\ \frac{m}{m+n}a+\frac{n}{m+n}b\geq a^{\frac{m}{m+n}}b^{\frac{n}{m+n}}.$$

Hence, for all positive *rationals* ω_1 and ω_2 with $\omega_1+\omega_2=1$, we get

$$\omega_1\,a+\omega_2\,b\geq a^{\,\omega_1}b^{\,\omega_2}.$$

We immediately have

Theorem 3.7. *Let ω_1, $\omega_2>0$ with $\omega_1+\omega_2=1$. For all $x,y>0$, we have*

$$\omega_1\,x+\omega_2\,y\geq x^{\,\omega_1}y^{\,\omega_2}.$$

[16] Set $x_i=\frac{a_i}{(a_1\cdots a_n)^{\frac{1}{n}}}$ $(i=1,\cdots,n)$. Then, we get $x_1\cdots x_n=1$ and it becomes $x_1+\cdots+x_n\geq n$.

Proof. We can choose a positive *rational* sequence $a_1, a_2, a_3, \cdots$ such that

$$\lim_{n\to\infty} a_n = \omega_1.$$

And letting $b_i = 1 - a_i$, we get

$$\lim_{n\to\infty} b_n = \omega_2.$$

From the previous observation, we have

$$a_n\, x + b_n\, y \geq x^{a_n} y^{b_n}$$

By taking the limits to both sides, we get the result. □

Modifying slightly the above arguments, we see that the AM-GM inequality implies that

Theorem 3.8. ***(Weighted AM-GM inequality)*** *Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n > 0$, we have*

$$\omega_1\, x_1 + \cdots + \omega_n\, x_n \geq x_1^{\ \omega_1} \cdots x_n^{\ \omega_n}.$$

Alternatively, we find that it is a straightforward consequence of the concavity of $\ln x$. Indeed, the weighted Jensen's inequality says that $\ln(\omega_1\, x_1 + \cdots + \omega_n\, x_n) \geq \omega_1 \ln(x_1) + \cdots + \omega_n \ln(x_n) = \ln(x_1^{\ \omega_1} \cdots x_n^{\ \omega_n})$.

Recall that the AM-GM inequality is used to deduce the theorem 18, which is a generalization of the Cauchy-Schwarz inequality. Since we now get the *weighted* version of the AM-GM inequality, we establish *weighted* version of the Cauchy-Schwarz inequality.

Theorem 3.9. ***(Hölder)*** *Let x_{ij} $(i = 1, \cdots, m, j = 1, \cdots n)$ be positive real numbers. Suppose that $\omega_1, \cdots, \omega_n$ are positive real numbers satisfying $\omega_1 + \cdots + \omega_n = 1$. Then, we have*

$$\prod_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right)^{\omega_j} \geq \sum_{i=1}^{m} \left(\prod_{j=1}^{n} x_{ij}^{\ \omega_j} \right).$$

Proof. Because of the homogeneity of the inequality, as in the proof of the theorem 12, we may rescale $x_{1j}, \cdots, x_{mj}$ so that $x_{1j} + \cdots + x_{mj} = 1$ for each $j \in \{1, \cdots, n\}$. Then, we need to show that

$$\prod_{j=1}^{n} 1^{\omega_j} \geq \sum_{i=1}^{m} \prod_{j=1}^{n} x_{ij}^{\ \omega_j} \quad \text{or} \quad 1 \geq \sum_{i=1}^{m} \prod_{j=1}^{n} x_{ij}^{\ \omega_j}.$$

The weighted AM-GM inequality provides that

$$\sum_{j=1}^{n} \omega_j x_{ij} \geq \prod_{j=1}^{n} x_{ij}^{\ \omega_j} \;\; (i \in \{1, \cdots, m\}) \implies \sum_{i=1}^{m} \sum_{j=1}^{n} \omega_j x_{ij} \geq \sum_{i=1}^{m} \prod_{j=1}^{n} x_{ij}^{\ \omega_j}.$$

However, we immediately have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \omega_j x_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} \omega_j x_{ij} = \sum_{j=1}^{n} \omega_j \left(\sum_{i=1}^{m} x_{ij} \right) = \sum_{j=1}^{n} \omega_j = 1.$$

□

4 Convexity

Any good idea can be stated in fifty words or less. S. M. Ulam

1 Jensen's Inequality

In the previous chapter, we deduced the weighted AM-GM inequality from the AM-GM inequality. We use the same idea to study the following functional inequalities.

Proposition 4.1. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a **continuous** function. Then, the followings are equivalent.*

(1) For all $n \in \mathbb{N}$, the following inequality holds.

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n)$$

for all $x_1, \cdots, x_n \in [a,b]$ and $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$.
(2) For all $n \in \mathbb{N}$, the following inequality holds.

$$r_1 f(x_1) + \cdots + r_n f(x_n) \geq f(r_1\, x_1 + \cdots + r_n\, x_n)$$

for all $x_1, \cdots, x_n \in [a,b]$ and $r_1, \cdots, r_n \in \mathbb{Q}^+$ with $r_1 + \cdots + r_n = 1$.
(3) For all $N \in \mathbb{N}$, the following inequality holds.

$$\frac{f(y_1) + \cdots + f(y_N)}{N} \geq f\left(\frac{y_1 + \cdots + y_N}{N}\right)$$

for all $y_1, \cdots, y_N \in [a,b]$.
(4) For all $k \in \{0, 1, 2, \cdots\}$, the following inequality holds.

$$\frac{f(y_1) + \cdots + f(y_{2^k})}{2^k} \geq f\left(\frac{y_1 + \cdots + y_{2^k}}{2^k}\right)$$

for all $y_1, \cdots, y_{2^k} \in [a,b]$.
(5) We have $\frac{1}{2} f(x) + \frac{1}{2} f(y) \geq f\left(\frac{x+y}{2}\right)$ for all $x, y \in [a,b]$.
(6) We have $\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$ for all $x, y \in [a,b]$ and $\lambda \in (0,1)$.

Proof. $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ is obvious.

$(2) \Rightarrow (1)$ **:** Let $x_1, \cdots, x_n \in [a,b]$ and $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. One may see that there exist positive rational sequences $\{r_k(1)\}_{k\in\mathbb{N}}, \cdots, \{r_k(n)\}_{k\in\mathbb{N}}$ satisfying

$$\lim_{k\to\infty} r_k(j) = w_j \;\; (1 \leq j \leq n) \;\; \text{and} \;\; r_k(1) + \cdots + r_k(n) = 1 \;\; \text{for all} \;\; k \in \mathbb{N}.$$

By the hypothesis in (2), we obtain $r_k(1) f(x_1) + \cdots + r_k(n) f(x_n) \geq f(r_k(1)\, x_1 + \cdots + r_k(n)\, x_n)$. Since f is continuous, taking $k \to \infty$ to both sides yields the inequality

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1\, x_1 + \cdots + \omega_n\, x_n).$$

$(3) \Rightarrow (2)$ **:** Let $x_1, \cdots, x_n \in [a,b]$ and $r_1, \cdots, r_n \in \mathbb{Q}^+$ with $r_1 + \cdots + r_n = 1$. We can find a positive integer $N \in \mathbb{N}$ so that $Nr_1, \cdots, Nr_n \in \mathbb{N}$. For each $i \in \{1, \cdots, n\}$, we can write $r_i = \frac{p_i}{N}$, where $p_i \in \mathbb{N}$. It follows from $r_1 + \cdots + r_n = 1$ that $N = p_1 + \cdots + p_n$. Then, (3) implies that

$$\begin{aligned}
& r_1 f(x_1) + \cdots + r_n f(x_n) \\
= \;& \frac{\overbrace{f(x_1) + \cdots + f(x_1)}^{p_1 \text{ terms}} + \cdots + \overbrace{f(x_n) + \cdots + f(x_n)}^{p_n \text{ terms}}}{N} \\
\geq \;& f\left(\frac{\overbrace{x_1 + \cdots + x_1}^{p_1 \text{ terms}} + \cdots + \overbrace{x_n + \cdots + x_n}^{p_n \text{ terms}}}{N}\right) \\
= \;& f(r_1\, x_1 + \cdots + r_n\, x_n).
\end{aligned}$$

(4) $\Rightarrow$ (3) **:** Let $y_1,\cdots,y_N \in [a,b]$. Take a large $k \in \mathbb{N}$ so that $2^k > N$. Let $a = \frac{y_1+\cdots+y_N}{N}$. Then, (4) implies that

$$
\begin{aligned}
& \frac{f(y_1)+\cdots+f(y_N)+(2^k-n)f(a)}{2^k} \\
= \quad & \frac{f(y_1)+\cdots+f(y_N)+\overbrace{f(a)+\cdots+f(a)}^{(2^k-N)\text{ terms}}}{2^k} \\
\geq \quad & f\left(\frac{y_1+\cdots+y_N+\overbrace{a+\cdots+a}^{(2^k-N)\text{ terms}}}{2^k}\right) \\
= \quad & f(a)
\end{aligned}
$$

so that

$$f(y_1)+\cdots+f(y_N) \geq Nf(a) = Nf\left(\frac{y_1+\cdots+y_N}{N}\right).$$

(5) $\Rightarrow$ (4) **:** We use induction on k. In case $k=0,1,2$, it clearly holds. Suppose that (4) holds for some $k \geq 2$. Let $y_1,\cdots,y_{2^{k+1}} \in [a,b]$. By the induction hypothesis, we obtain

$$
\begin{aligned}
& f(y_1)+\cdots+f(y_{2^k})+f(y_{2^k+1})+\cdots+f(y_{2^{k+1}}) \\
\geq \quad & 2^k f\left(\frac{y_1+\cdots+y_{2^k}}{2^k}\right)+2^k f\left(\frac{y_{2^k+1}+\cdots+y_{2^{k+1}}}{2^k}\right) \\
= \quad & 2^{k+1}\frac{f\left(\frac{y_1+\cdots+y_{2^k}}{2^k}\right)+f\left(\frac{y_{2^k+1}+\cdots+y_{2^{k+1}}}{2^k}\right)}{2} \\
\geq \quad & 2^{k+1} f\left(\frac{\frac{y_1+\cdots+y_{2^k}}{2^k}+\frac{y_{2^k+1}+\cdots+y_{2^{k+1}}}{2^k}}{2}\right) \\
= \quad & 2^{k+1} f\left(\frac{y_1+\cdots+y_{2^{k+1}}}{2^{k+1}}\right).
\end{aligned}
$$

Hence, (4) holds for $k+1$. This completes the induction.

So far, we've established that (1), (2), (3), (4), (5) are all equivalent. Since (1) $\Rightarrow$ (6) $\Rightarrow$ (5) is obvious, this completes the proof. $\square$

Definition 4.1. *A real valued function f is said to be convex on $[a,b]$ if*

$$\lambda f(x)+(1-\lambda)f(y) \geq f(\lambda x+(1-\lambda)y)$$

for all $x,y \in [a,b]$ and $\lambda \in (0,1)$.

The above proposition says that

Corollary 4.1. *(**Jensen's inequality**) Let $f:[a,b] \longrightarrow \mathbb{R}$ be a continuous convex function. For all $x_1,\cdots,x_n \in [a,b]$, we have*

$$\frac{f(x_1)+\cdots+f(x_n)}{n} \geq f\left(\frac{x_1+\cdots+x_n}{n}\right).$$

Corollary 4.2. *(**Weighted Jensen's inequality**) Let $f:[a,b] \longrightarrow \mathbb{R}$ be a continuous convex function. Let $\omega_1,\cdots,\omega_n > 0$ with $\omega_1+\cdots+\omega_n = 1$. For all $x_1,\cdots,x_n \in [a,b]$, we have*

$$\omega_1 f(x_1)+\cdots+\omega_n f(x_n) \geq f(\omega_1 x_1+\cdots+\omega_n x_n).$$

In fact, we can almost drop the continuity of f. As an exercise, show that every convex function on $[a,b]$ is continuous on (a,b). So, every convex function on $\mathbb{R}$ is continuous on $\mathbb{R}$. By the proposition again, we get

Corollary 4.3. ***(Convexity Criterion I)*** *Let $f:[a,b]\longrightarrow\mathbb{R}$ be a continuous function. Suppose that*

$$\frac{f(x)+f(y)}{2}\geq f\left(\frac{x+y}{2}\right)$$

for all $x,y\in[a,b]$. Then, f is a convex function on $[a,b]$.

Exercise 20. ***(Convexity Criterion II)*** *Let $f:[a,b]\longrightarrow\mathbb{R}$ be a continuous function which are differentiable twice in (a,b). Show that the followings are equivalent.*

(1) $f''(x)\geq 0$ for all $x\in(a,b)$.
(2) f is convex on (a,b).

When we deduce $(5)\Rightarrow(4)\Rightarrow(3)\Rightarrow(2)$ in the proposition, we didn't use the continuity of f :

Corollary 4.4. *Let $f:[a,b]\longrightarrow\mathbb{R}$ be a function. Suppose that*

$$\frac{f(x)+f(y)}{2}\geq f\left(\frac{x+y}{2}\right)$$

for all $x,y\in[a,b]$. Then, we have

$$r_1 f(x_1)+\cdots+r_n f(x_n)\geq f(r_1\,x_1+\cdots+r_n\,x_n)$$

for all $x_1,\cdots,x_n\in[a,b]$ and $r_1,\cdots,r_n\in\mathbb{Q}^+$ with $r_1+\cdots+r_n=1$.

We close this section by presenting an well-known inductive proof of the weighted Jensen's inequality. It turns out that we can completely drop the continuity of f.

Second Proof. It clearly holds for $n=1,2$. We now assume that it holds for some $n\in\mathbb{N}$. Let $x_1,\cdots,x_n,x_{n+1}\in[a,b]$ and $\omega_1,\cdots,\omega_{n+1}>0$ with $\omega_1+\cdots+\omega_{n+1}=1$. Since $\frac{\omega_1}{1-\omega_{n+1}}+\cdots+\frac{\omega_n}{1-\omega_{n+1}}=1$, it follows from the induction hypothesis that

$$\begin{aligned}
& \omega_1 f(x_1)+\cdots+\omega_{n+1} f(x_{n+1})\\
= & (1-\omega_{n+1})\left(\frac{\omega_1}{1-\omega_{n+1}}f(x_1)+\cdots+\frac{\omega_n}{1-\omega_{n+1}}f(x_n)\right)+\omega_{n+1}f(x_{n+1})\\
\geq & (1-\omega_{n+1})f\left(\frac{\omega_1}{1-\omega_{n+1}}x_1+\cdots+\frac{\omega_n}{1-\omega_{n+1}}x_n\right)+\omega_{n+1}f(x_{n+1})\\
\geq & f\left((1-\omega_{n+1})\left[\frac{\omega_1}{1-\omega_{n+1}}x_1+\cdots+\frac{\omega_n}{1-\omega_{n+1}}x_n\right]+\omega_{n+1}x_{n+1}\right)\\
= & f(\omega_1x_1+\cdots+\omega_{n+1}x_{n+1}).
\end{aligned}$$

□

2 Power Means

Convexity is one of the most important concepts in analysis. Jensen's inequality is the most powerful tool in theory of inequalities. In this section, we shall establish the Power Mean inequality by applying Jensen's inequality in two ways. We begin with two simple lemmas.

Lemma 4.1. *Let a, b, and c be positive real numbers. Let us define a function* $f:\mathbb{R}\longrightarrow\mathbb{R}$ *by*

$$f(x)=\ln\left(\frac{a^x+b^x+c^x}{3}\right),$$

where $x\in\mathbb{R}$*. Then, we obtain* $f'(0)=\ln(abc)^{\frac{1}{3}}$*.*

Proof. We compute $f'(x)=\frac{a^x\ln a+b^x\ln b+c^x\ln c}{a^x+b^x+c^x}$. Then, $f'(0)=\frac{\ln a+\ln b+\ln c}{3}=\ln(abc)^{\frac{1}{3}}$. □

Lemma 4.2. *Let* $f:\mathbb{R}\longrightarrow\mathbb{R}$ *be a continuous function. Suppose that* f *is monotone increasing on* $(0,\infty)$ *and monotone increasing on* $(-\infty,0)$*. Then,* f *is monotone increasing on* $\mathbb{R}$*.*

Proof. We first show that f is monotone increasing on $[0,\infty)$. By the hypothesis, it remains to show that $f(x)\geq f(0)$ for all $x>0$. For all $\varepsilon\in(0,x)$, we have $f(x)\geq f(\varepsilon)$. Since f is continuous at 0, we obtain

$$f(x)\geq\lim_{\varepsilon\to0^+}f(\varepsilon)=f(0).$$

Similarly, we find that f is monotone increasing on $(-\infty,0]$. We now show that f is monotone increasing on $\mathbb{R}$. Let x and y be real numbers with $x>y$. We want to show that $f(x)\geq f(y)$. In case $0\notin(x,y)$, we get the result by the hypothesis. In case $x\geq0\geq y$, it follows that $f(x)\geq f(0)\geq f(y)$. □

Theorem 4.1. ***(Power Mean inequality for three variables*** *) Let a, b, and c be positive real numbers. We define a function* $M_{(a,b,c)}:\mathbb{R}\longrightarrow\mathbb{R}$ *by*

$$M_{(a,b,c)}(0)=\sqrt[3]{abc},\ M_{(a,b,c)}(r)=\left(\frac{a^r+b^r+c^r}{3}\right)^{\frac{1}{r}}\quad(r\neq0).$$

Then, $M_{(a,b,c)}$ *is a monotone increasing continuous function.*

First Proof. Write $M(r)=M_{(a,b,c)}(r)$. We first establish that M is continuous. Since M is continuous at r for all $r\neq0$, it's enough to show that

$$\lim_{r\to0}M(r)=\sqrt[3]{abc}.$$

Let $f(x)=\ln\left(\frac{a^x+b^x+c^x}{3}\right)$, where $x\in\mathbb{R}$. Since $f(0)=0$, the lemma 2 implies that

$$\lim_{r\to0}\frac{f(r)}{r}=\lim_{r\to0}\frac{f(r)-f(0)}{r-0}=f'(0)=\ln\sqrt[3]{abc}.$$

Since e^x is a continuous function, this means that

$$\lim_{r\to0}M(r)=\lim_{r\to0}e^{\frac{f(r)}{r}}=e^{\ln\sqrt[3]{abc}}=\sqrt[3]{abc}.$$

Now, we show that M is monotone increasing. By the lemma 3, it will be enough to establish that M is monotone increasing on $(0,\infty)$ and monotone increasing on $(-\infty,0)$. We first show that M is monotone increasing on $(0,\infty)$. Let $x\geq y>0$. We want to show that

$$\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}\geq\left(\frac{a^y+b^y+c^y}{3}\right)^{\frac{1}{y}}.$$

After the substitution $u=a^y$, $v=a^y$, $w=a^z$, it becomes

$$\left(\frac{u^{\frac{x}{y}}+v^{\frac{x}{y}}+w^{\frac{x}{y}}}{3}\right)^{\frac{1}{x}}\geq\left(\frac{u+v+w}{3}\right)^{\frac{1}{y}}.$$

Since it is homogeneous, we may normalize to $u+v+w=3$. We are now required to show that

$$\frac{G(u)+G(v)+G(w)}{3} \geq 1,$$

where $G(t)=t^{\frac{x}{y}}$, where $t>0$. Since $\frac{x}{y} \geq 1$, we find that G is convex. Jensen's inequality shows that

$$\frac{G(u)+G(v)+G(w)}{3} \geq G\left(\frac{u+v+w}{3}\right) = G(1) = 1.$$

Similarly, we may deduce that M is monotone increasing on $(-\infty, 0)$. $\square$

We've learned that the convexity of $f(x)=x^{\lambda}$ $(\lambda \geq 1)$ implies the monotonicity of the power means. Now, we shall show that the convexity of $x\ln x$ also implies the power mean inequality.

Second Proof of the Monotonicity. Write $f(x)=M_{(a,b,c)}(x)$. We use the increasing function theorem. By the lemma 3, it's enough to show that $f'(x) \geq 0$ for all $x \neq 0$. Let $x \in \mathbb{R}-\{0\}$. We compute

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}(\ln f(x)) = -\frac{1}{x^2}\ln\left(\frac{a^x+b^x+c^x}{3}\right) + \frac{1}{x}\frac{\frac{1}{3}(a^x\ln a+b^x\ln b+c^x\ln c)}{\frac{1}{3}(a^x+b^x+c^x)}$$

or

$$\frac{x^2 f'(x)}{f(x)} = -\ln\left(\frac{a^x+b^x+c^x}{3}\right) + \frac{a^x\ln a^x+b^x\ln b^x+c^x\ln c^x}{a^x+b^x+c^x}.$$

To establish $f'(x) \geq 0$, we now need to establish that

$$a^x\ln a^x+b^x\ln b^x+c^x\ln c^x \geq (a^x+b^x+c^x)\ln\left(\frac{a^x+b^x+c^x}{3}\right).$$

Let us introduce a function $f:(0,\infty) \longrightarrow \mathbf{R}$ by $f(t)=t\ln t$, where $t>0$. After the substitution $p=a^x$, $q=a^y$, $r=a^z$, it becomes

$$f(p)+f(q)+f(r) \geq 3f\left(\frac{p+q+r}{3}\right).$$

Since f is convex on $(0,\infty)$, it follows immediately from Jensen's inequality. $\square$

As a corollary, we obtain the RMS-AM-GM-HM inequality for three variables.

Corollary 4.5. *For all positive real numbers a, b, and c, we have*

$$\sqrt{\frac{a^2+b^2+c^2}{3}} \geq \frac{a+b+c}{3} \geq \sqrt[3]{abc} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$

Proof. The Power Mean inequality states that $M_{(a,b,c)}(2) \geq M_{(a,b,c)}(1) \geq M_{(a,b,c)}(0) \geq M_{(a,b,c)}(-1)$. $\square$

Using the convexity of $x\ln x$ or the convexity of x^{λ} $(\lambda \geq 1)$, we can also establish the monotonicity of the power means for n positive real numbers.

Theorem 4.2. ***(Power Mean inequality)*** *Let $x_1,\cdots,x_n>0$. The power mean of order r is defined by*

$$M_{(x_1,\cdots,x_n)}(0) = \sqrt[n]{x_1\cdots x_n},\ M_{(x_1,\cdots,x_n)}(r) = \left(\frac{x_1^{\,r}+\cdots+x_n^{\,r}}{n}\right)^{\frac{1}{r}} \quad (r \neq 0).$$

Then, $M_{(x_1,\cdots,x_n)}:\mathbb{R} \longrightarrow \mathbb{R}$ is continuous and monotone increasing.

We conclude that

Corollary 4.6. ***(Geometric Mean as a Limit)*** *Let* $x_1, \cdots, x_n > 0$*. Then,*

$$\sqrt[n]{x_1 \cdots x_n} = \lim_{r \to 0} \left(\frac{{x_1}^r + \cdots + {x_n}^r}{n} \right)^{\frac{1}{r}}.$$

Theorem 4.3. ***(RMS-AM-GM-HM inequality)*** *For all* $x_1, \cdots, x_n > 0$*, we have*

$$\sqrt{\frac{{x_1}^2 + \cdots + {x_n}^2}{n}} \geq \frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}.$$

3 Majorization Inequality

We say that a vector $\mathrm{x} = (x_1, \cdots, x_n)$ *majorizes* another vector $\mathrm{y} = (y_1, \cdots, y_n)$ if

(1) $x_1 \geq \cdots \geq x_n$, $y_1 \geq \cdots \geq y_n$,
(2) $x_1 + \cdots + x_k \geq y_1 + \cdots + y_k$ for all $1 \leq k \leq n-1$,
(3) $x_1 + \cdots + x_n = y_1 + \cdots + y_n$.

Theorem 4.4. ***(Majorization Inequality)*** *Let* $f : [a,b] \longrightarrow \mathbb{R}$ *be a convex function. Suppose that* $(x_1, \cdots, x_n)$ *majorizes* $(y_1, \cdots, y_n)$*, where* $x_1, \cdots, x_n, y_1, \cdots, y_n \in [a,b]$*. Then, we obtain*

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).$$

For example, we can minimize $\cos A + \cos B + \cos C$, where ABC is an acute triangle. Recall that $-\cos x$ is convex on $\left(0, \frac{\pi}{2}\right)$. Since $\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$ majorize (A, B, C), the majorization inequality implies that

$$\cos A + \cos B + \cos C \geq \cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) + \cos 0 = 1.$$

Also, in a triangle ABC, the convexity of $\tan^2\left(\frac{x}{4}\right)$ on $[0, \pi]$ and the majorization inequality show that

$$21 - 12\sqrt{3} = 3\tan^2\left(\frac{\pi}{12}\right) \leq \tan^2\left(\frac{A}{4}\right) + \tan^2\left(\frac{B}{4}\right) + \tan^2\left(\frac{C}{4}\right) \leq \tan^2\left(\frac{\pi}{4}\right) + \tan^2 0 + \tan^2 0 = 1.$$

(IMO 1999/2) Let n be an integer with $n \geq 2$.

Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{1 \leq i \leq n} x_i \right)^4$$

holds for all real numbers $x_1, \cdots, x_n \geq 0$.

Second Solution. **(Kin Y. Li[17])** As in the first solution, after normalizing $x_1 + \cdots + x_n = 1$, we maximize

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) = \sum_{i=1}^{n} f(x_i),$$

where $f(x) = x^3 - x^4$ is a convex function on $[0, \frac{1}{2}]$. Since the inequality is symmetric, we can restrict our attention to the case $x_1 \geq x_2 \geq \cdots \geq x_n$. If $\frac{1}{2} \geq x_1$, then we see that $\left(\frac{1}{2}, \frac{1}{2}, 0, \cdots 0\right)$ majorizes $(x_1, \cdots, x_n)$. Hence, the convexity of f on $[0, \frac{1}{2}]$ and the Majorization inequality show that

$$\sum_{i=1}^{n} f(x_i) \leq f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) + f(0) + \cdots + f(0) = \frac{1}{8}.$$

[17] I slightly modified his solution in [KYL].

We now consider the case when $\frac{1}{2} \geq x_1$. Write $x_1 = \frac{1}{2} - \varepsilon$ for some $\varepsilon \in \left[0, \frac{1}{2}\right]$. We find that $(1-x_1, 0, \cdots 0)$ majorizes $(x_2, \cdots, x_n)$. By the Majorization inequality, we find that

$$\sum_{i=2}^{n} f(x_i) \leq f(1-x_1) + f(0) + \cdots + f(0) = f(1-x_1)$$

so that

$$\begin{aligned}\sum_{i=1}^{n} f(x_i) &\leq f(x_1) + f(1-x_1) = x_1(1-x_1)[{x_1}^2 + (1-x_1)^2] \\ &= \left(\frac{1}{4} - \varepsilon^2\right)\left(\frac{1}{2} + 2\varepsilon^2\right) = 2\left(\frac{1}{16} - \varepsilon^4\right) \leq \frac{1}{8}.\end{aligned}$$

$\square$

4 Supporting Line Inequality

There is a simple way to find new bounds for given differentiable functions. We begin to show that every supporting lines are tangent lines in the following sense.

Proposition 4.2. ***(Characterization of Supporting Lines)*** *Let f be a real valued function. Let $m, n \in \mathbb{R}$. Suppose that*

(1) $f(\alpha) = m\alpha + n$ for some $\alpha \in \mathbb{R}$,
(2) $f(x) \geq mx + n$ for all x in some interval $(\varepsilon_1, \varepsilon_2)$ including α, and
(3) f is differentiable at α.

Then, the supporting line $y = mx + n$ of f is the tangent line of f at $x = \alpha$.

Proof. Let us define a function $F : (\varepsilon_1, \varepsilon_2) \longrightarrow \mathbb{R}$ by $F(x) = f(x) - mx - n$ for all $x \in (\varepsilon_1, \varepsilon_2)$. Then, F is differentiable at α and we obtain $F'(\alpha) = f'(\alpha) - m$. By the assumption (1) and (2), we see that F has a local minimum at α. So, the first derivative theorem for local extreme values implies that $0 = F'(\alpha) = f'(\alpha) - m$ so that $m = f'(\alpha)$ and that $n = f(\alpha) - m\alpha = f(\alpha) - f'(\alpha)\alpha$. It follows that $y = mx + n = f'(\alpha)(x - \alpha) + f(\alpha)$. $\square$

(**Nesbitt, 1903**) For all positive real numbers a, b, c, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Proof 13. *We may normalize to $a+b+c = 1$. Note that $0 < a, b, c < 1$. The problem is now to prove*

$$\sum_{\text{cyclic}} f(a) \geq \frac{3}{2} \Leftrightarrow \frac{f(a)+f(b)+f(c)}{3} \geq f\left(\frac{1}{3}\right), \text{ where } f(x) = \frac{x}{1-x}.$$

The equation of the tangent line of f at $x = \frac{1}{3}$ is given by $y = \frac{9x-1}{4}$. We claim that $f(x) \geq \frac{9x-1}{4}$ for all $x \in (0,1)$. It follows from the identity

$$f(x) - \frac{9x-1}{4} = \frac{(3x-1)^2}{4(1-x)}.$$

Now, we conclude that

$$\sum_{\text{cyclic}} \frac{a}{1-a} \geq \sum_{\text{cyclic}} \frac{9a-1}{4} = \frac{9}{4}\sum_{\text{cyclic}} a - \frac{3}{4} = \frac{3}{2}.$$

The above argument can be generalized. If a function f has a supporting line at some point on the graph of f, then f satisfies Jensen's inequality in the following sense.

Theorem 4.5. *(**Supporting Line Inequality**) Let $f : [a,b] \longrightarrow \mathbb{R}$ be a function. Suppose that $\alpha \in [a,b]$ and $m \in \mathbb{R}$ satisfy*

$$f(x) \geq m(x-\alpha) + f(\alpha)$$

for all $x \in [a,b]$. Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. Then, the following inequality holds

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\alpha)$$

for all $x_1, \cdots, x_n \in [a,b]$ such that $\alpha = \omega_1 x_1 + \cdots + \omega_n x_n$. In particular, we obtain

$$\frac{f(x_1) + \cdots + f(x_n)}{n} \geq f\left(\frac{s}{n}\right),$$

where $x_1, \cdots, x_n \in [a,b]$ with $x_1 + \cdots + x_n = s$ for some $s \in [na, nb]$.

Proof. It follows that $\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq \omega_1[m(x_1 - \alpha) + f(\alpha)] + \cdots + \omega_1[m(x_n - \alpha) + f(\alpha)] = f(\alpha)$. □

We can apply the supporting line inequality to deduce Jensen's inequality for differentiable functions.

Lemma 4.3. *Let $f : (a,b) \longrightarrow \mathbb{R}$ be a convex function which is differentiable twice on (a,b). Let $y = l_\alpha(x)$ be the tangent line at $\alpha \in (a,b)$. Then, $f(x) \geq l_\alpha(x)$ for all $x \in (a,b)$.*

Proof. Let $\alpha \in (a,b)$. We want to show that the tangent line $y = l_\alpha(x) = f'(\alpha)(x-\alpha) + f(\alpha)$ is the supporting line of f at $x = \alpha$ such that $f(x) \geq l_\alpha(x)$ for all $x \in (a,b)$. However, by Taylor's theorem, we can find a θ_x between α and x such that

$$f(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\theta_x)}{2}(x-\alpha)^2 \geq f(\alpha) + f'(\alpha)(x-\alpha).$$

□

(**Weighted Jensen's inequality**) Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous convex function which is differentiable twice on (a,b). Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n \in [a,b]$,

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq f(\omega_1 x_1 + \cdots + \omega_n x_n).$$

Third Proof. By the continuity of f, we may assume that $x_1, \cdots, x_n \in (a,b)$. Now, let $\mu = \omega_1 x_1 + \cdots + \omega_n x_n$. Then, $\mu \in (a,b)$. By the above lemma, f has the tangent line $y = l_\mu(x) = f'(\mu)(x-\mu) + f(\mu)$ at $x = \mu$ satisfying $f(x) \geq l_\mu(x)$ for all $x \in (a,b)$. Hence, the supporting line inequality shows that

$$\omega_1 f(x_1) + \cdots + \omega_n f(x_n) \geq \omega_1 f(\mu) + \cdots + \omega_n f(\mu) = f(\mu) = f(\omega_1 x_1 + \cdots + \omega_n x_n).$$

□

We note that the cosine function is concave on $\left[0, \frac{\pi}{2}\right]$ and convex on $\left[\frac{\pi}{2}, \pi\right]$. Non-convex functions can be locally convex and have supporting lines at some points. This means that the supporting line inequality is a powerful tool because we can also produce Jensen-type inequalities for non-convex functions.

(**Theorem 6**) In any triangle ABC, we have $\cos A + \cos B + \cos C \leq \frac{3}{2}$.

Third Proof. Let $f(x) = -\cos x$. Our goal is to establish a three-variables inequality

$$\frac{f(A)+f(B)+f(C)}{3} \geq f\left(\frac{\pi}{3}\right),$$

where $A,B,C \in (0,\pi)$ with $A+B+C=\pi$. We compute $f'(x)=\sin x$. The equation of the tangent line of f at $x=\frac{\pi}{3}$ is given by $y=\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)-\frac{1}{2}$. To apply the supporting line inequality, we need to show that

$$-\cos x \geq \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)-\frac{1}{2}$$

for all $x \in (0,\pi)$. This is a one-variable inequality! We omit the proof. □

Problem 33. ***(Japan 1997)*** *Let a, b, and c be positive real numbers. Prove that*

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}.$$

Proof. Because of the homogeneity of the inequality, we may normalize to $a+b+c=1$. It takes the form

$$\begin{aligned} & \frac{(1-2a)^2}{(1-a)^2+a^2}+\frac{(1-2b)^2}{(1-b)^2+b^2}+\frac{(1-2c)^2}{(1-c)^2+c^2} \geq \frac{3}{5} \\ \Leftrightarrow \quad & \frac{1}{2a^2-2a+1}+\frac{1}{2b^2-2b+1}+\frac{1}{2c^2-2c+1} \leq \frac{27}{5}. \end{aligned}$$

We find that the equation of the tangent line of $f(x)=\frac{1}{2x^2-2x+1}$ at $x=\frac{1}{3}$ is given by $y=\frac{54}{25}x+\frac{27}{25}$ and that

$$f(x)-\left(\frac{54}{25}x+\frac{27}{25}\right)=-\frac{2(3x-1)^2(6x+1)}{25(2x^2-2x+1)} \leq 0.$$

for all $x>0$. It follows that

$$\sum_{\text{cyclic}} f(a) \leq \sum_{\text{cyclic}} \frac{54}{25}a+\frac{27}{25}=\frac{27}{5}.$$

□

5 Problems, Problems, Problems

Each problem that I solved became a rule, which served afterwards to solve other problems. Rene Descartes

1 Multivariable Inequalities

M 1. ***(IMO short-list 2003)*** *Let $(x_1,x_2,\cdots,x_n)$ and $(y_1,y_2,\cdots,y_n)$ be two sequences of positive real numbers. Suppose that $(z_1,z_2,\cdots,z_n)$ is a sequence of positive real numbers such that*

$${z_{i+j}}^2 \geq x_i y_j$$

for all $1 \leq i,j \leq n$. Let $M=max\{z_2,\cdots,z_{2n}\}$. Prove that

$$\left(\frac{M+z_2+\cdots+z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1+\cdots+x_n}{n}\right)\left(\frac{y_1+\cdots+y_n}{n}\right).$$

M 2. ***(Bosnia and Herzegovina 2002)*** *Let $a_1,\cdots,a_n,b_1,\cdots,b_n,c_1,\cdots,c_n$ be positive real numbers. Prove the following inequality :*

$$\left(\sum_{i=1}^{n} {a_i}^3\right)\left(\sum_{i=1}^{n} {b_i}^3\right)\left(\sum_{i=1}^{n} {c_i}^3\right) \geq \left(\sum_{i=1}^{n} a_ib_ic_i\right)^3.$$

M 3. ***(C[18]2113, Marcin E. Kuczma)*** *Prove that inequality*

$$\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \geq \sum_{i=1}^{n} (a_i+b_i) \sum_{i=1}^{n} \frac{a_ib_i}{a_i+b_i}$$

for any positive real numbers $a_1,\cdots,a_n,b_1,\cdots,b_n$

M 4. ***(Yogoslavia 1998)*** *Let $n>1$ be a positive integer and $a_1,\cdots,a_n,b_1,\cdots,b_n$ be positive real numbers. Prove the following inequality.*

$$\left(\sum_{i\neq j} a_ib_j\right)^2 \geq \sum_{i\neq j} a_ia_j \sum_{i\neq j} b_ib_j.$$

M 5. ***(C2176, Sefket Arslanagic)*** *Prove that*

$$((a_1+b_1)\cdots(a_n+b_n))^{\frac{1}{n}} \geq (a_1\cdots a_n)^{\frac{1}{n}} + (b_1\cdots b_n)^{\frac{1}{n}}$$

where $a_1,\cdots,a_n,b_1,\cdots,b_n>0$

M 6. ***(Korea 2001)*** *Let $x_1,\cdots,x_n$ and $y_1,\cdots,y_n$ be real numbers satisfying*

$${x_1}^2+\cdots+{x_n}^2 = {y_1}^2+\cdots+{y_n}^2 = 1$$

Show that

$$2\left|1-\sum_{i=1}^{n} x_iy_i\right| \geq (x_1y_2-x_2y_1)^2$$

and determine when equality holds.

M 7. ***(Singapore 2001)*** *Let $a_1,\cdots,a_n,b_1,\cdots,b_n$ be real numbers between 1001 and 2002 inclusive. Suppose that*

$$\sum_{i=1}^{n} {a_i}^2 = \sum_{i=1}^{n} {b_i}^2.$$

Prove that

$$\sum_{i=1}^{n} \frac{{a_i}^3}{b_i} \leq \frac{17}{10}\sum_{i=1}^{n} {a_i}^2.$$

Determine when equality holds.

M 8. ***(Abel's inequality)*** *Let $a_1,\cdots,a_N,x_1,\cdots,x_N$ be real numbers with $x_n \geq x_{n+1} > 0$ for all n. Show that*

$$|a_1x_1+\cdots+a_Nx_N| \leq Ax_1$$

where

$$A = max\{|a_1|,|a_1+a_2|,\cdots,|a_1+\cdots+a_N|\}.$$

[18] CRUX with MAYHEM

M 9. ***(China 1992)*** *For every integer $n \geq 2$ find the smallest positive number $\lambda = \lambda(n)$ such that if*

$$0 \leq a_1, \cdots, a_n \leq \frac{1}{2},\ \ b_1, \cdots, b_n > 0,\ \ a_1 + \cdots + a_n = b_1 + \cdots + b_n = 1$$

then

$$b_1 \cdots b_n \leq \lambda(a_1 b_1 + \cdots + a_n b_n).$$

M 10. ***(C2551, Panos E. Tsaoussoglou)*** *Suppose that $a_1, \cdots, a_n$ are positive real numbers. Let $e_{j,k} = n-1$ if $j = k$ and $e_{j,k} = n-2$ otherwise. Let $d_{j,k} = 0$ if $j = k$ and $d_{j,k} = 1$ otherwise. Prove that*

$$\sum_{j=1}^{n} \prod_{k=1}^{n} e_{j,k} {a_k}^2 \geq \prod_{j=1}^{n} \left(\sum_{k=1}^{n} d_{j,k} a_k \right)^2$$

M 11. ***(C2627, Walther Janous)*** *Let $x_1, \cdots, x_n (n \geq 2)$ be positive real numbers and let $x_1 + \cdots + x_n$. Let $a_1, \cdots, a_n$ be non-negative real numbers. Determine the optimum constant C(n) such that*

$$\sum_{j=1}^{n} \frac{a_j(s_n - x_j)}{x_j} \geq C(n) \left(\prod_{j=1}^{n} a_j \right)^{\frac{1}{n}}.$$

M 12. ***(Hungary-Israel Binational Mathematical Competition 2000)*** *Suppose that k and l are two given positive integers and $a_{ij}(1 \leq i \leq k, 1 \leq j \leq l)$ are given positive numbers. Prove that if $q \geq p > 0$, then*

$$\left(\sum_{j=1}^{l} \left(\sum_{i=1}^{k} {a_{ij}}^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^{k} \left(\sum_{j=1}^{l} {a_{ij}}^q \right)^{\frac{p}{q}} \right)^{\frac{1}{p}}.$$

M 13. ***(Kantorovich inequality)*** *Suppose $x_1 < \cdots < x_n$ are given positive numbers. Let $\lambda_1, \cdots, \lambda_n \geq 0$ and $\lambda_1 + \cdots + \lambda_n = 1$. Prove that*

$$\left(\sum_{i=1}^{n} \lambda_i x_i \right) \left(\sum_{i=1}^{n} \frac{\lambda_i}{x_i} \right) \leq \frac{A^2}{G^2},$$

where $A = \frac{x_1 + x_n}{2}$ and $G = \sqrt{x_1 x_n}$.

M 14. ***(Czech-Slovak-Polish Match 2001)*** *Let $n \geq 2$ be an integer. Show that*

$$({a_1}^3 + 1)({a_2}^3 + 1) \cdots ({a_n}^3 + 1) \geq ({a_1}^2 a_2 + 1)({a_2}^2 a_3 + 1) \cdots ({a_n}^2 a_1 + 1)$$

for all nonnegative reals $a_1, \cdots, a_n$.

M 15. ***(C1868, De-jun Zhao)*** *Let $n \geq 3$, $a_1 > a_2 > \cdots > a_n > 0$, and $p > q > 0$. Show that*

$${a_1}^p {a_2}^q + {a_2}^p {a_3}^q + \cdots + {a_{n-1}}^p {a_n}^q + {a_n}^p {a_1}^q \geq {a_1}^q {a_2}^p + {a_2}^q {a_3}^p + \cdots + {a_{n-1}}^q {a_n}^p + {a_n}^q {a_1}^p$$

M 16. ***(Baltic Way 1996)*** *For which positive real numbers a, b does the inequality*

$$x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n + x_n x_1 \geq {x_1}^a {x_2}^b {x_3}^a + {x_2}^a {x_3}^b {x_4}^a + \cdots + {x_n}^a {x_1}^b {x_2}^a$$

holds for all integers $n > 2$ and positive real numbers $x_1, \cdots, x_n$.

M 17. ***(IMO short List 2000)*** *Let $x_1, x_2, \cdots, x_n$ be arbitrary real numbers. Prove the inequality*

$$\frac{x_1}{1 + {x_1}^2} + \frac{x_2}{1 + {x_1}^2 + {x_2}^2} + \cdots + \frac{x_n}{1 + {x_1}^2 + \cdots + {x_n}^2} < \sqrt{n}.$$

M 18. ***(MM[19]1479, Donald E. Knuth)*** *Let M_n be the maximum value of the quantity*

$$\frac{x_n}{(1+x_1+\cdots+x_n)^2}+\frac{x_2}{(1+x_2+\cdots+x_n)^2}+\cdots+\frac{x_1}{(1+x_n)^2}$$

over all nonnegative real numbers $(x_1,\cdots,x_n)$. At what point(s) does the maximum occur ? Express M_n in terms of M_{n-1}, and find $\lim_{n\to\infty} M_n$.

M 19. ***(IMO 1971)*** *Prove the following assertion is true for $n=3$ and $n=5$ and false for every other natural number $n>2$: if $a_1,\cdots,a_n$ are arbitrary real numbers, then*

$$\sum_{i=1}^{n}\prod_{i\neq j}(a_i-a_j)\geq 0.$$

M 20. ***(IMO 2003)*** *Let $x_1\leq x_2\leq\cdots\leq x_n$ be real numbers.*
(a) Prove that

$$\left(\sum_{1\leq i,j\leq n}|x_i-x_j|\right)^2\leq\frac{2(n^2-1)}{3}\sum_{1\leq i,j\leq n}(x_i-x_j)^2.$$

(b) Show that the equality holds if and only if $x_1,x_2,\cdots,x_n$ is an arithmetic sequence.

M 21. ***(Bulgaria 1995)*** *Let $n\geq 2$ and $0\leq x_1,\cdots,x_n\leq 1$. Show that*

$$(x_1+x_2+\cdots+x_n)-(x_1x_2+x_2x_3+\cdots+x_nx_1)\leq\left[\frac{n}{2}\right],$$

and determine when there is equality.

M 22. ***(MM1407, M. S. Klamkin)*** *Determine the maximum value of the sum*

$${x_1}^p+{x_2}^p+\cdots+{x_n}^p-{x_1}^q{x_2}^r-{x_2}^q{x_3}^r-\cdots{x_n}^q{x_1}^r,$$

where p,q,r are given numbers with $p\geq q\geq r\geq 0$ and $0\leq x_i\leq 1$ for all i.

M 23. ***(IMO Short List 1998)*** *Let $a_1,a_2,\cdots,a_n$ be positive real numbers such that*

$$a_1+a_2+\cdots+a_n<1.$$

Prove that

$$\frac{a_1a_2\cdots a_n(1-(a_1+a_2+\cdots+a_n))}{(a_1+a_2+\cdots+a_n)(1-a_1)(1-a_2)\cdots(1-a_n)}\leq\frac{1}{n^{n+1}}.$$

M 24. ***(IMO Short List 1998)*** *Let $r_1,r_2,\cdots,r_n$ be real numbers greater than or equal to 1. Prove that*

$$\frac{1}{r_1+1}+\cdots+\frac{1}{r_n+1}\geq\frac{n}{(r_1\cdots r_n)^{\frac{1}{n}}+1}.$$

M 25. ***(Baltic Way 1991)*** *Prove that, for any real numbers $a_1,\cdots,a_n$,*

$$\sum_{1\leq i,j\leq n}\frac{a_ia_j}{i+j-1}\geq 0.$$

M 26. ***(India 1995)*** *Let $x_1,x_2,\cdots,x_n$ be positive real numbers whose sum is 1. Prove that*

$$\frac{x_1}{1-x_1}+\cdots+\frac{x_n}{1-x_n}\geq\sqrt{\frac{n}{n-1}}.$$

[19]Mathematics Magazine

M 27. ***(Turkey 1997)*** *Given an integer $n \geq 2$, Find the minimal value of*

$$\frac{{x_1}^5}{x_2+x_3+\cdots+x_n}+\frac{{x_2}^5}{x_3+\cdots+x_n+x_1}+\cdots\frac{{x_n}^5}{x_1+x_3+\cdots+x_{n-1}}$$

for positive real numbers $x_1,\cdots,x_n$ subject to the condition ${x_1}^2+\cdots+{x_n}^2=1$.

M 28. ***(China 1996)*** *Suppose $n \in \mathbf{N}$, $x_0=0$, $x_1,\cdots,x_n>0$, and $x_1+\cdots+x_n=1$. Prove that*

$$1 \leq \sum_{i=1}^{n} \frac{x_i}{\sqrt{1+x_0+\cdots+x_{i-1}}\sqrt{x_i+\cdots+x_n}} < \frac{\pi}{2}$$

M 29. ***(Vietnam 1998)*** *Let $x_1,\cdots,x_n$ be positive real numbers satisfying*

$$\frac{1}{x_1+1998}+\cdots+\frac{1}{x_n+1998}=\frac{1}{1998}.$$

Prove that

$$\frac{(x_1\cdots x_n)^{\frac{1}{n}}}{n-1} \geq 1998$$

M 30. ***(C2768 Mohammed Aassila)*** *Let $x_1,\cdots,x_n$ be n positive real numbers. Prove that*

$$\frac{x_1}{\sqrt{x_1x_2+{x_2}^2}}+\frac{x_2}{\sqrt{x_2x_3+{x_3}^2}}+\cdots+\frac{x_n}{\sqrt{x_nx_1+{x_1}^2}} \geq \frac{n}{\sqrt{2}}$$

M 31. ***(C2842, George Tsintsifas)*** *Let $x_1,\cdots,x_n$ be positive real numbers. Prove that*

$$(a)\ \frac{{x_1}^n+\cdots+{x_n}^n}{nx_1\cdots x_n}+\frac{n(x_1\cdots x_n)^{\frac{1}{n}}}{x_1+\cdots+x_n} \geq 2,$$

$$(b)\ \frac{{x_1}^n+\cdots+{x_n}^n}{x_1\cdots x_n}+\frac{(x_1\cdots x_n)^{\frac{1}{n}}}{x_1+\cdots+x_n} \geq 1.$$

M 32. ***(C2423, Walther Janous)*** *Let $x_1,\cdots,x_n (n \geq 2)$ be positive real numbers such that $x_1+\cdots+x_n=1$. Prove that*

$$\left(1+\frac{1}{x_1}\right)\cdots\left(1+\frac{1}{x_n}\right) \geq \left(\frac{n-x_1}{1-x_1}\right)\cdots\left(\frac{n-x_n}{1-x_n}\right)$$

Determine the cases of equality.

M 33. ***(C1851, Walther Janous)*** *Let $x_1,\cdots,x_n (n \geq 2)$ be positive real numbers such that*

$${x_1}^2+\cdots+{x_n}^2=1.$$

Prove that

$$\frac{2\sqrt{n}-1}{5\sqrt{n}-1} \leq \sum_{i=1}^{n}\frac{2+x_i}{5+x_i} \leq \frac{2\sqrt{n}+1}{5\sqrt{n}+1}.$$

M 34. ***(C1429, D. S. Mitirinovic, J. E. Pecaric)*** *Show that*

$$\sum_{i=1}^{n}\frac{x_i}{{x_i}^2+x_{i+1}x_{i+2}} \leq n-1$$

where $x_1,\cdots,x_n$ are $n \geq 3$ positive real numbers. Of course, $x_{n+1}=x_1, x_{n+2}=x_2$. [20]

[20] Original version is to show that $sup\sum_{i=1}^{n}\frac{x_i}{{x_i}^2+x_{i+1}x_{i+2}}=n-1$.

M 35. ***(Belarus 1998 S. Sobolevski)*** *Let $a_1 \leq a_2 \leq \cdots \leq a_n$ be positive real numbers. Prove the inequalities*

$$(a)\quad \frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}} \geq \frac{a_1}{a_n}\cdot\frac{a_1+\cdots+a_n}{n},$$

$$(b)\quad \frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}} \geq \frac{2k}{1+k^2}\cdot\frac{a_1+\cdots+a_n}{n},$$

where $k=\frac{a_n}{a_1}$.

M 36. ***(Hong Kong 2000)*** *Let $a_1 \leq a_2 \leq \cdots \leq a_n$ be n real numbers such that*

$$a_1+a_2+\cdots+a_n=0.$$

Show that

$${a_1}^2+{a_2}^2+\cdots+{a_n}^2+na_1a_n \leq 0.$$

M 37. ***(Poland 2001)*** *Let $n \geq 2$ be an integer. Show that*

$$\sum_{i=1}^{n} {x_i}^i + \binom{n}{2} \geq \sum_{i=1}^{n} ix_i$$

for all nonnegative reals $x_1,\cdots,x_n$.

M 38. ***(Korea 1997)*** *Let $a_1,\cdots,a_n$ be positive numbers, and define*

$$A=\frac{a_1+\cdots+a_n}{n}, G=(a_1\cdots{}_n)^{\frac{1}{n}}, H=\frac{n}{\frac{1}{a_1}+\cdots+\frac{1}{a_n}}$$

(a) If n is even, show that

$$\frac{A}{H} \leq -1+2\left(\frac{A}{G}\right)^n.$$

(b) If n is odd, show that

$$\frac{A}{H} \leq -\frac{n-2}{n}+\frac{2(n-1)}{n}\left(\frac{A}{G}\right)^n.$$

M 39. ***(Romania 1996)*** *Let $x_1,\cdots,x_n,x_{n+1}$ be positive reals such that*

$$x_{n+1}=x_1+\cdots+x_n.$$

Prove that

$$\sum_{i=1}^{n}\sqrt{x_i(x_{n+1}-x_i)} \leq \sqrt{x_{n+1}(x_{n+1}-x_i)}$$

M 40. ***(C2730, Peter Y. Woo)*** *Let $AM(x_1,\cdots,x_n)$ and $GM(x_1,\cdots,x_n)$ denote the arithmetic mean and the geometric mean of the positive real numbers $x_1,\cdots,x_n$ respectively. Given positive real numbers $a_1,\cdots,a_n,b_1,\cdots,b_n$, (a) prove that*

$$GM(a_1+b_1,\cdots,a_n+b_n) \geq GM(a_1,\cdots,a_n)+GM(b_1,\cdots,b_n).$$

For each real number $t \geq 0$, define

$$f(t)=GM(t+b_1,t+b_2,\cdots,t+b_n)-t$$

(b) Prove that f is a monotonic increasing function, and that

$$\lim_{t\to\infty} f(t)=AM(b_1,\cdots,b_n)$$

M 41. ***(C1578, O. Johnson, C. S. Goodlad)*** *For each fixed positive real number a_n, maximize*

$$\frac{a_1a_2\cdots a_n}{(1+a_1)(a_1+a_2)(a_2+a_3)\cdots(a_{n-1}+a_n)}$$

over all positive real numbers $a_1,\cdots,a_{n-1}$.

M 42. ***(C1630, Isao Ashiba)*** *Maximize*

$$a_1a_2+a_3a_4+\cdots+a_{2n-1}a_{2n}$$

over all permutations $a_1,\cdots,a_{2n}$ of the set $\{1,2,\cdots,2n\}$

M 43. ***(C1662, M. S. Klamkin)*** *Prove that*

$$\frac{{x_1}^{2r+1}}{s-x_1}+\frac{{x_2}^{2r+1}}{s-x_2}+\cdots\frac{{x_n}^{2r+1}}{s-x_n}\geq\frac{4^r}{(n-1)n^{2r-1}}\left(x_1x_2+x_2x_3+\cdots+x_nx_1\right)^r$$

where $n>3$, $r\geq\frac{1}{2}$, $x_i\geq 0$ for all i, and $s=x_1+\cdots+x_n$. Also, Find some values of n and r such that the inequality is sharp.

M 44. ***(C1674, M. S. Klamkin)*** *Given positive real numbers r,s and an integer $n>\frac{r}{s}$, find positive real numbers $x_1,\cdots,x_n$ so as to minimize*

$$\left(\frac{1}{{x_1}^r}+\frac{1}{{x_2}^r}+\cdots+\frac{1}{{x_n}^r}\right)(1+x_1)^s(1+x_2)^s\cdots(1+x_n)^s.$$

M 45. ***(C1691, Walther Janous)*** *Let $n\geq 2$. Determine the best upper bound of*

$$\frac{x_1}{x_2x_3\cdots x_n+1}+\frac{x_2}{x_1x_3\cdots x_n+1}+\cdots+\frac{x_n}{x_1x_2\cdots x_{n-1}+1}$$

over all $x_1,\cdots,x_n\in[0,1]$.

M 46. ***(C1892, Marcin E. Kuczma)*** *Let $n\geq 4$ be an integer. Find the exact upper and lower bounds for the cyclic sum*

$$\sum_{i=1}^{n}\frac{x_i}{x_{i-1}+x_i+x_{i+1}}$$

over all n-tuples of nonnegative numbers $x_1,\cdots,x_n$ such that $x_{i-1}+x_i+x_{i+1}>0$ for all i. Of course, $x_{n+1}=x_1$, $x_0=x_n$. Characterize all cases in which either one of these bounds is attained.

M 47. ***(C1953, M. S. Klamkin)*** *Determine a necessary and sucient condition on real constants $r_1,\cdots,r_n$ such that*

$${x_1}^2+{x_2}^2+\cdot+{x_n}^2\geq(r_1x_1+r_2x_2+\cdots+r_nx_n)^2$$

holds for all real numbers $x_1,\cdots,x_n$.

M 48. ***(C2018, Marcin E. Kuczma)*** *How many permutations $(x_1,\cdots,x_n)$ of $\{1,2,\cdots,n\}$ are there such that the cyclic sum*

$$|x_1-x_2|+|x_2-x_3|+\cdots+|x_{n-1}-x_n|+|x_n-x_1|$$

is (a) a minimum, (b) a maximum ?

M 49. ***(C2214, Walther Janous)*** *Let $n\geq 2$ be a natural number. Show that there exists a constant $C=C(n)$ such that for all $x_1,\cdots,x_n\geq 0$ we have*

$$\sum_{i=1}^{n}\sqrt{x_i}\leq\sqrt{\prod_{i=1}^{n}(x_i+C)}$$

Determine the minimum $C(n)$ for some values of n. (For example, $C(2)=1$.)

M 50. ***(C2615, M. S. Klamkin)*** *Suppose that $x_1,\cdots,x_n$ are non-negative numbers such that*

$$\sum {x_i}^2 \sum (x_i x_{i+1})^2 = \frac{n(n+1)}{2}$$

where e the sums here and subsequently are symmetric over the subscripts $\{1,\cdots,n\}$. (a) Determine the maximum of $\sum x_i$. (b) Prove or disprove that the minimum of $\sum x_i$ is $\sqrt{\frac{n(n+1)}{2}}$.

M 51. ***(Turkey 1996)*** *Given real numbers $0 = x_1 < x_2 < \cdots < x_{2n}, x_{2n+1} = 1$ with $x_{i+1} - x_i \le h$ for $1 \le i \le n$, show that*

$$\frac{1-h}{2} < \sum_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) < \frac{1+h}{2}.$$

M 52. ***(Poland 2002)*** *Prove that for every integer $n \ge 3$ and every sequence of positive numbers $x_1,\cdots,x_n$ at least one of the two inequalities is satsified :*

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1}+x_{i+2}} \ge \frac{n}{2}, \quad \sum_{i=1}^{n} \frac{x_i}{x_{i-1}+x_{i-2}} \ge \frac{n}{2}.$$

Here, $x_{n+1} = x_1, x_{n+2} = x_2, x_0 = x_n, x_{-1} = x_{n-1}$.

M 53. ***(China 1997)*** *Let $x_1,\cdots,x_{1997}$ be real numbers satisfying the following conditions:*

$$-\frac{1}{\sqrt{3}} \le x_1,\cdots,x_{1997} \le \sqrt{3}, x_1 + \cdots + x_{1997} = -318\sqrt{3}$$

Determine the maximum value of ${x_1}^{12} + \cdots + {x_{1997}}^{12}$.

M 54. ***(C2673, George Baloglou)*** *Let $n > 1$ be an integer. (a) Show that*

$$(1 + a_1 \cdots a_n)^n \ge a_1 \cdots a_n (1 + {a_1}^{n-2}) \cdots (1 + {a_1}^{n-2})$$

for all $a_1,\cdots,a_n \in [1,\infty)$ if and only if $n \ge 4$.

(b) Show that

$$\frac{1}{a_1(1+{a_2}^{n-2})} + \frac{1}{a_2(1+{a_3}^{n-2})} + \cdots + \frac{1}{a_n(1+{a_1}^{n-2})} \ge \frac{n}{1+a_1\cdots a_n}$$

for all $a_1,\cdots,a_n > 0$ if and only if $n \le 3$.

(c) Show that

$$\frac{1}{a_1(1+{a_1}^{n-2})} + \frac{1}{a_2(1+{a_2}^{n-2})} + \cdots + \frac{1}{a_n(1+{a_n}^{n-2})} \ge \frac{n}{1+a_1\cdots a_n}$$

for all $a_1,\cdots,a_n > 0$ if and only if $n \le 8$.

M 55. ***(C2557, Gord Sinnamon,Hans Heinig)*** *(a) Show that for all positive sequences $\{x_i\}$*

$$\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} x_i \le 2\sum_{k=1}^{n}\left(\sum_{j=1}^{k} x_j\right)^2 \frac{1}{x_k}.$$

(b) Does the above inequality remain true without the factor 2? (c) What is the minimum constant c that can replace the factor 2 in the above inequality?

M 56. ***(C1472, Walther Janous)*** *For each integer $n \ge 2$, Find the largest constant C_n such that*

$$C_n \sum_{i=1}^{n} |a_i| \le \sum_{1\le i<j\le n} |a_i - a_j|$$

for all real numbers $a_1,\cdots,a_n$ satisfying $\sum_{i=1}^{n} a_i = 0$.

M 57. ***(China 2002)*** *Given $c \in \left(\frac{1}{2}, 1\right)$. Find the smallest constant M such that, for any integer $n \geq 2$ and real numbers $1 < a_1 \leq a_2 \leq \cdots \leq a_n$, if*

$$\frac{1}{n}\sum_{k=1}^{n} ka_k \leq c\sum_{k=1}^{n} a_k,$$

then

$$\sum_{k=1}^{n} a_k \leq M\sum_{k=1}^{m} ka_k,$$

where m is the largest integer not greater than cn.

M 58. ***(Serbia 1998)*** *Let $x_1, x_2, \cdots, x_n$ be positive numbers such that*

$$x_1 + x_2 + \cdots + x_n = 1.$$

Prove the inequality

$$\frac{a^{x_1 - x_2}}{x_1 + x_2} + \frac{a^{x_2 - x_3}}{x_2 + x_3} + \cdots \frac{a^{x_n - x_1}}{x_n + x_1} \geq \frac{n^2}{2},$$

holds true for every positive real number a. Determine also when the equality holds.

M 59. ***(MM1488, Heinz-Jurgen Seiffert)*** *Let n be a positive integer. Show that if $0 < x_1 \leq x_2 \leq x_n$, then*

$$\prod_{i=1}^{n}(1 + x_i)\left(\sum_{j=0}^{n}\prod_{k=1}^{j}\frac{1}{x_k}\right) \geq 2^n(n+1)$$

with equality if and only if $x_1 = \cdots = x_n = 1$.

M 60. ***(Leningrad Mathematical Olympiads 1968)*** *Let $a_1, a_2, \cdots, a_p$ be real numbers. Let $M = \max S$ and $m = \min S$. Show that*

$$(p-1)(M-m) \leq \sum_{1 \leq i,j \leq n} |a_i - a_j| \leq \frac{p^2}{4}(M-m)$$

M 61. ***(Leningrad Mathematical Olympiads 1973)*** *Establish the following inequality*

$$\sum_{i=0}^{8} 2^i \cos\left(\frac{\pi}{2^{i+2}}\right)\left(1 - \cos\left(\frac{\pi}{2^{i+2}}\right)\right) < \frac{1}{2}.$$

M 62. ***(Leningrad Mathematical Olympiads 2000)*** *Show that, for all $0 < x_1 \leq x_2 \leq \ldots \leq x_n$,*

$$\frac{x_1 x_2}{x_3} + \frac{x_2 x_3}{x_4} + \cdots + \frac{x_{n_1} x_1}{x_2} + \frac{x_n x_1}{x_2} \geq \sum_{i=1}^{n} x_i$$

M 63. ***(Mongolia 1996)*** *Show that, for all $0 < a_1 \leq a_2 \leq \ldots \leq a_n$,*

$$\left(\frac{a_1 + a_2}{2}\right)\left(\frac{a_2 + a_3}{2}\right)\cdots\left(\frac{a_n + a_1}{2}\right) \leq \left(\frac{a_1 + a_2 + a_3}{3}\right)\left(\frac{a_2 + a_3 + a_4}{3}\right)\cdots\left(\frac{a_n + a_1 + a_2}{3}\right).$$

2 Problems for Putnam Seminar

P 1. $\boxed{\textit{Putnam 04A6}}$ *Suppose that $f(x,y)$ is a continuous real-valued function on the unit square $0 \leq x \leq 1, 0 \leq y \leq 1$. Show that*

$$\int_0^1 \left(\int_0^1 f(x,y)dx\right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)dy\right)^2 dx$$
$$\leq \left(\int_0^1 \int_0^1 f(x,y)dx\,dy\right)^2 + \int_0^1 \int_0^1 (f(x,y))^2 \,dx\,dy.$$

P 2. $\boxed{\textbf{Putnam 04B2}}$ *Let m and n be positive integers. Show that*

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

P 3. $\boxed{\textbf{Putnam 03A2}}$ *Let $a_1,a_2,\dots,a_n$ and $b_1,b_2,\dots,b_n$ be nonnegative real numbers. Show that*

$$(a_1a_2\cdots a_n)^{1/n}+(b_1b_2\cdots b_n)^{1/n} \leq [(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)]^{1/n}.$$

P 4. $\boxed{\textbf{Putnam 03A3}}$ *Find the minimum value of*

$$|\sin x+\cos x+\tan x+\cot x+\sec x+\csc x|$$

for real numbers x.

P 5. $\boxed{\textbf{Putnam 03A4}}$ *Suppose that a,b,c,A,B,C are real numbers, $a\neq 0$ and $A\neq 0$, such that*

$$|ax^2+bx+c| \leq |Ax^2+Bx+C|$$

for all real numbers x. Show that

$$|b^2-4ac| \leq |B^2-4AC|.$$

P 6. $\boxed{\textbf{Putnam 03B6}}$ *Let $f(x)$ be a continuous real-valued function defined on the interval $[0,1]$. Show that*

$$\int_0^1\int_0^1 |f(x)+f(y)|\,dx\,dy \geq \int_0^1 |f(x)|\,dx.$$

P 7. $\boxed{\textbf{Putnam 02B3}}$ *Show that, for all integers $n>1$,*

$$\frac{1}{2ne} < \frac{1}{e}-\left(1-\frac{1}{n}\right)^n < \frac{1}{ne}.$$

P 8. $\boxed{\textbf{Putnam 01A6}}$ *Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?*

P 9. $\boxed{\textbf{Putnam 99A5}}$ *Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then*

$$|p(0)| \leq C\int_{-1}^1 |p(x)|\,dx.$$

P 10. $\boxed{\textbf{Putnam 99B4}}$ *Let f be a real function with a continuous third derivative such that $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$ are positive for all x. Suppose that $f'''(x)\leq f(x)$ for all x. Show that $f'(x)<2f(x)$ for all x.*

P 11. $\boxed{\textbf{Putnam 98B4}}$ *Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1+x+x^2)^m$. Prove that for all integers $k\geq 0$,*

$$0 \leq \sum_{i=0}^{\lfloor \frac{2k}{3}\rfloor} (-1)^i a_{k-i,i} \leq 1.$$

P 12. $\boxed{\textbf{Putnam 98B1}}$ *Find the minimum value of*

$$\frac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)}$$

for $x>0$.

P 13. $\boxed{\textbf{\textit{Putnam 96B2}}}$ *Show that for every positive integer n,*

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1\cdot 3\cdot 5\cdots(2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

P 14. $\boxed{\textbf{\textit{Putnam 96B3}}}$ *Given that* $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$*, find, with proof, the largest possible value, as a function of n (with* $n \geq 2$*), of*

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.$$

P 15. $\boxed{\textbf{\textit{Putnam 91B6}}}$ *Let a and b be positive numbers. Find the largest number c, in terms of a and b, such that*

$$a^x b^{1-x} \leq a\frac{\sinh ux}{\sinh u} + b\frac{\sinh u(1-x)}{\sinh u}$$

for all u with $0 < |u| \leq c$ *and for all x,* $0 < x < 1$.

P 16. ***(CMJ[21]416, Joanne Harris)*** *For what real values of c is*

$$\frac{e^x + e^{-x}}{2} \leq e^{cx^2}.$$

for all real x?

P 17. ***(CMJ420, Edward T. H. Wang)*** *It is known [Daniel I. A. Cohen, Basic Techniques of Combinatorial Theory, p.56] and easy to show that* $2^n < \binom{2n}{n} < 2^{2n}$ *for all integers* $n > 1$*. Prove that the stronger inequalities*

$$\frac{2^{2n-1}}{\sqrt{n}} < \binom{2n}{n} < \frac{2^{2n}}{\sqrt{n}}$$

hold for all $n \geq 4$.

P 18. ***(CMJ379, Mohammad K. Azarian)*** *Let x be any real number. Prove that*

$$(1-\cos x)\left|\sum_{k=1}^{n}\sin(kx)\right|\left|\sum_{k=1}^{n}\cos(kx)\right| \leq 2.$$

P 19. ***(CMJ392 Robert Jones)*** *Prove that*

$$\left(1+\frac{1}{x^2}\right)\left(x\sin\frac{1}{x}\right) > 1 \ \textit{for}\ \ x \geq \frac{1}{\sqrt{5}}.$$

P 20. ***(CMJ431 R. S. Luthar)*** *Let* $0 < \phi < \theta < \frac{\pi}{2}$*. Prove that*

$$[(1+\tan^2\phi)(1+\sin^2\phi)]^{\csc^2\phi} < [(1+\tan^2\theta)(1+\sin^2\theta)]^{\csc^2\theta}.$$

P 21. ***(CMJ451, Mohammad K. Azarian)*** *Prove that*

$$\pi^{\sec^2\alpha}\cos^2\alpha + \pi^{\csc^2\alpha}\sin^2\alpha \geq \pi^2,$$

provided $0 < \alpha < \frac{\pi}{2}$.

P 22. ***(CMJ446, Norman Schaumberger)*** *If x, y, and z are the radian measures of the angles in a (non-degenerate) triangle, prove that*

$$\pi\sin\frac{3}{\pi} \geq x\sin\frac{1}{x} + y\sin\frac{1}{y} + z\sin\frac{1}{z}.$$

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P 23. ***(CMJ461, Alex Necochea)*** *Let $0 < x < \frac{\pi}{2}$ and $0 < y < 1$. Prove that*

$$x - \arcsin y \le \frac{\sqrt{1-y^2} - \cos x}{y},$$

with equality holding if and only if $y = \sin x$.

P 24. ***(CMJ485 Norman Schaumberger)*** *Prove that*

(1) if $a \ge b > 1$ or $1 > a \ge b > 0$, then $a^{b^b} b^{a^a} \ge a^{b^a} b^{a^b}$; and

(2) if $a > 1 > b > 0$, then $a^{b^b} b^{a^a} \le a^{b^a} b^{a^b}$.

P 25. ***(CMJ524 Norman Schaumberger)*** *Let a, b, and c be positive real numbers. Show that*

$$a^a b^b c^c \ge \left(\frac{a+b}{2}\right)^a \left(\frac{b+c}{2}\right)^b \left(\frac{c+a}{2}\right)^c \ge b^a c^b a^c.$$

P 26. ***(CMJ567 H.-J. Seiffert)*** *Show that for all ditinct positive real numbers x and y,*

$$\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 < \frac{x-y}{2\sinh\frac{x-y}{x+y}} < \frac{x+y}{2}.$$

P 27. ***(CMJ572, George Baloglou and Robert Underwood)*** *Prove or disprove that for $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, $\cosh\theta \le \frac{1}{\sqrt{1-\tan^2\theta}}$.*

P 28. ***(CMJ603, Juan-Bosco Romero Marquez)*** *Let a and b be distinct positive real numbers and let n be a positive integer. Prove that*

$$\frac{a+b}{2} \le \sqrt[n]{\frac{b^{n+1}-a^{n+1}}{(n+1)(b-a)}} \le \sqrt[n]{\frac{a^n+b^n}{2}}.$$

P 29. ***(MM[22]904, Norman Schaumberger)*** *For $x > 2$, prove that*

$$\ln\left(\frac{x}{x-1}\right) \le \sum_{j=0}^{\infty} \frac{1}{x^{2j}} \le \ln\left(\frac{x-1}{x-2}\right).$$

P 30. ***(MM1590, Constantin P. Niculescu)*** *For given a, $0 < a < \frac{\pi}{2}$, determine the minimum value of $\alpha \ge 0$ and the maximum value of $\beta \ge 0$ for which*

$$\left(\frac{x}{a}\right)^\alpha \le \frac{\sin x}{\sin a} \le \left(\frac{x}{a}\right)^\beta.$$

(This generalize the well-known inequality due to Jordan, which asserts that $\frac{2x}{\pi} \le \sin x \le 1$ on $[0, \frac{\pi}{2}]$.)

P 31. ***(MM1597, Constantin P. Niculescu)*** *For every $x, y \in \left(0, \sqrt{\frac{\pi}{2}}\right)$ with $x \ne y$, prove that*

$$\left(\ln\frac{1-\sin xy}{1+\sin xy}\right)^2 \ge \ln\frac{1-\sin x^2}{1+\sin x^2}\ln\frac{1-\sin y^2}{1+\sin y^2}.$$

P 32. ***(MM1599, Ice B. Risteski)*** *Given $\alpha > \beta > 0$ and $f(x) = x^\alpha(1-x)^\beta$. If $0 < a < b < 1$ and $f(a) = f(b)$, show that $f'(\alpha) < -f'(\beta)$.*

P 33. ***(MM Q197, Norman Schaumberger)*** *Prove that if $b > a > 0$, then $\left(\frac{a}{b}\right)^a \ge \frac{e^a}{e^b} \ge \left(\frac{a}{b}\right)^b$.*

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P 34. ***(MM1618, Michael Golomb)*** *Prove that* $0 < x < \pi$,

$$x\frac{\pi - x}{\pi + x} < \sin x < \left(3 - \frac{x}{\pi}\right) x\frac{\pi - x}{\pi + x}.$$

P 35. ***(MM1634, Constantin P. Niculescu)*** *Find the smallest constant* $k > 0$ *such that*

$$\frac{ab}{a+b+2c} + \frac{bc}{b+c+2a} + \frac{ca}{c+a+2b} \le k(a+b+c)$$

for every $a, b, c > 0$.

P 36. ***(MM1233, Robert E. Shafer)*** *Prove that if* $x > -1$ *and* $x \neq 0$, *then*

$$\frac{x^2}{1 + x + \frac{x^2}{2} - \frac{\frac{x^4}{120}}{1+x+\frac{31}{252}x^2}} < [\ln(1+x)]^2 < \frac{x^2}{1 + x + \frac{x^2}{2} - \frac{\frac{x^4}{240}}{1+x+\frac{1}{20}x^2}}.$$

P 37. ***(MM1236, Mihaly Bencze)*** *Let the functions* f *and* g *be defined by*

$$f(x) = \frac{\pi^2 x}{2\pi^2 + 8x^2} \text{ and } g(x) = \frac{8x}{4\pi^2 + \pi x^2}$$

for all real x. Prove that if A, B, and C are the angles of an acuted-angle triangle, and R is its circumradius then

$$f(A) + f(B) + f(C) < \frac{a+b+c}{4R} < g(A) + g(B) + g(C).$$

P 38. ***(MM1245, Fouad Nakhli)*** *For each number x in open interval* $(1, e)$ *it is easy to show that there is a unique number y in* (e, ∞) *such that* $\frac{\ln y}{y} = \frac{\ln x}{x}$. *For such an x and y, show that* $x + y > x\ln y + y\ln x$.

P 39. ***(MM Q725, S. Kung)*** *Show that* $(\sin x)y \le \sin(xy)$, *where* $0 < x < \pi$ *and* $0 < y < 1$.

P 40. ***(MM Q771, Norman Schaumberger)*** *Show that if* $0 < \theta < \frac{\pi}{2}$, *then* $\sin 2\theta \ge (\tan\theta)^{\cos 2\theta}$.

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