Frobenius Solution to Coupled Differential Equations

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Equations

Here are your coupled differential equations:

$$\mathbf{eqns} = \{u''(s)\, s^2 + u'(s)\, s - \mu^2\, u(s) + \mu^2 \cot(\alpha)\, (s\, \nu\, w'(s) - w(s)) == 0,$$
$$\eta^2\, (s^2\, w''(s) - s\, w'(s)) - \cot(\alpha)\, (\nu\, u'(s)\, s^3 + \cot(\alpha)\, w(s)\, s^2 + u(s)) == 0\};$$

Numerical Example

As a particular example, for the parameters

$$\mathbf{params} = \left\{\alpha \to \frac{\pi}{4}, \nu \to \frac{1}{3}, \mu \to 3, \eta \to 1\right\};$$

here is the numerical solution of the coupled system for a particular set of initial (final) conditions over $0 < s \leq 1$.

$$\mathbf{nsoln} = \mathrm{First@NDSolve}[\mathrm{Join}[\mathrm{eqns}\ /.\ \mathrm{params}, \{u(1) == 4, u'(1) == 5, w(1) == 6, w'(1) == -1\}], \{u, w\}, \{s, 10^{-30}, 1\}]$$

$\{u \to \mathrm{InterpolatingFunction}[(\ 1. \times 10^{-30}\quad 1.\ ), <>], w \to \mathrm{InterpolatingFunction}[(\ 1. \times 10^{-30}\quad 1.\ ), <>]\}$

These solutions are singular at $s = 0$. For example, here are the values of $\{u(0.01), w(0.01)\}$.

$$\{u(0.01), w(0.01)\}\ /.\ \mathbf{nsoln}$$

$\{8.78284982 \times 10^6, 532482.9068\}$

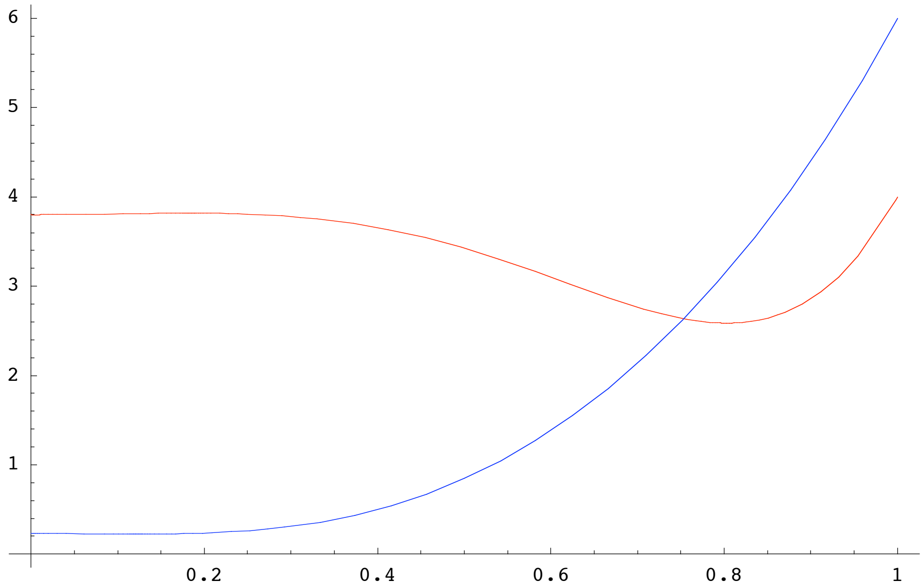
Later analysis shows that the maximal singulartity is of the form s^σ where, for this particular set of parameters, $\sigma \approx -3.18194$. Dividing the solutions by this singular term leads to finite behaviour at $s = 0$. Here are the initial values of $f(0)$ and $g(0)$.

initial = $s^{3.18194}$ {$u(s)$, $w(s)$} /. $s \to 10^{-30}$ /. nsoln

{3.800485798, 0.2304910197}

Here is a plot of $s^{\sigma}\ u(s)$ (—) and $s^{\sigma}\ w(s)$ (—).

nplot = Plot[Evaluate[$s^{3.18194}$ {$u(s)$, $w(s)$} /. nsoln], {s, 0, 1}, PlotStyle → {Red, Blue}]



- Graphics -

Separable Solution

A trivial separable solution is obtained if $\cot(\alpha) \to 0$,

DSolve[eqns /. cot(α) → 0, {$u(s)$, $w(s)$}, s]

$$\left\{\left\{u(s) \to c_1 \cosh(\mu \log(s)) + i\, c_2 \sinh(\mu \log(s)),\ w(s) \to \frac{c_3\, s^2}{2} + c_4\right\}\right\}$$

and the hyperbolic trig functions reduce to simple powers.

$u(s)$ /. First[%] // TrigToExp

$$\frac{1}{2}\, c_1\, s^{-\mu} - \frac{1}{2}\, i\, c_2\, s^{-\mu} + \frac{c_1\, s^{\mu}}{2} + \frac{1}{2}\, i\, c_2\, s^{\mu}$$

Frobenius Solution

Let's use the method of Frobenius, assuming that the series solutions are of the form $u(s) = s^{\sigma} \sum_{k=0}^{\infty} u_k\, s^k = s^{\sigma}\ f(s)$ and $w(s) = s^{\sigma} \sum_{k=0}^{\infty} w_k\, s^k = s^{\sigma}\ g(s)$. These (truncated) series expansions can implemented as follows:

u(s_) = Series[$s^{\sigma}\ f(s)$, {s, 0, 7}]

$$s^{\sigma}\left(f(0) + f'(0)\, s + \frac{1}{2}\, f''(0)\, s^2 + \frac{1}{6}\, f^{(3)}(0)\, s^3 + \frac{1}{24}\, f^{(4)}(0)\, s^4 + \frac{1}{120}\, f^{(5)}(0)\, s^5 + \frac{1}{720}\, f^{(6)}(0)\, s^6 + \frac{f^{(7)}(0)\, s^7}{5040} + O(s^8)\right)$$

w(s_) = Series[s^σ g(s), {s, 0, 7}]

$$s^\sigma\left(g(0)+g'(0)\,s+\frac{1}{2}\,g''(0)\,s^2+\frac{1}{6}\,g^{(3)}(0)\,s^3+\frac{1}{24}\,g^{(4)}(0)\,s^4+\frac{1}{120}\,g^{(5)}(0)\,s^5+\frac{1}{720}\,g^{(6)}(0)\,s^6+\frac{g^{(7)}(0)\,s^7}{5040}+O(s^8)\right)$$

■ Solving the first equation

Solving the first equation for the undetermined coefficients $g^{(n)}(0)$ yields

gsoln = Solve[ExpandAll[$\frac{\text{eqns[[1, 1]]}}{s^\sigma}$] == 0, Table[$g^{(n)}$(0), {n, 0, 7}]] // Simplify // First

$$\Big\{g(0)\to\frac{(\mu^2-\sigma^2)\,f(0)\tan(\alpha)}{\mu^2\,(\nu\,\sigma-1)},\; g'(0)\to\frac{(\mu^2-(\sigma+1)^2)\tan(\alpha)\,f'(0)}{\mu^2\,(\sigma\,\nu+\nu-1)},\; g''(0)\to\frac{(\mu^2-(\sigma+2)^2)\tan(\alpha)\,f''(0)}{\mu^2\,(\nu\,(\sigma+2)-1)},$$
$$g^{(3)}(0)\to\frac{(\mu^2-(\sigma+3)^2)\tan(\alpha)\,f^{(3)}(0)}{\mu^2\,(\nu\,(\sigma+3)-1)},\; g^{(4)}(0)\to\frac{(\mu^2-(\sigma+4)^2)\tan(\alpha)\,f^{(4)}(0)}{\mu^2\,(\nu\,(\sigma+4)-1)},\; g^{(5)}(0)\to\frac{(\mu^2-(\sigma+5)^2)\tan(\alpha)\,f^{(5)}(0)}{\mu^2\,(\nu\,(\sigma+5)-1)},$$
$$g^{(6)}(0)\to\frac{(\mu^2-(\sigma+6)^2)\tan(\alpha)\,f^{(6)}(0)}{\mu^2\,(\nu\,(\sigma+6)-1)},\; g^{(7)}(0)\to\frac{(\mu^2-(\sigma+7)^2)\tan(\alpha)\,f^{(7)}(0)}{\mu^2\,(\nu\,(\sigma+7)-1)}\Big\}$$

Clearly, in general one has that

$$\frac{\tan(\alpha)}{\mu^2}\,f^{(n)}(0)=\frac{(\nu\,(\sigma+n)-1)}{\mu^2-(\sigma+n)^2}\,g^{(n)}(0).$$

Partial fraction decomposition of the right-hand side,

Apart[$\frac{(\nu\,(\sigma+n)-1)}{\mu^2-(\sigma+n)^2}$]

$$\frac{-\mu\,\nu-1}{2\,\mu\,(n+\mu+\sigma)}+\frac{1-\mu\,\nu}{2\,\mu\,(n-\mu+\sigma)}$$

and using

$\frac{1-\mu\,\nu}{2\,\mu}\,x^{\mu-\sigma}\int x^{n+\sigma-\mu-1}\,dx-\frac{1+\mu\,\nu}{2\,\mu}\,x^{-\sigma-\mu}\int x^{n+\sigma+\mu-1}\,dx$ // Together // FullSimplify

$$\frac{x^n\,(1-\nu\,(n+\sigma))}{(n-\mu+\sigma)\,(n+\mu+\sigma)}$$

shows that

$$f(s)=\mu\cot(\alpha)\left(\frac{1-\mu\,\nu}{2}\,s^{\mu-\sigma}\int^s g(x)\,x^{\sigma-\mu-1}\,dx-\frac{1+\mu\,\nu}{2}\,s^{-\sigma-\mu}\int^s g(x)\,x^{\sigma+\mu-1}\,dx\right)$$

or equivalently

$$u(s)=\frac{1}{2}\,\mu\cot(\alpha)\left((1-\mu\,\nu)\,s^\mu\int^s w(x)\,x^{-\mu-1}\,dx-(1+\mu\,\nu)\,s^{-\mu}\int^s w(x)\,x^{\mu-1}\,dx\right)=$$
$$\frac{1}{2}\,\mu\cot(\alpha)\int^s\frac{w(x)}{x}\left((1-\mu\,\nu)\left(\frac{s}{x}\right)^\mu-(1+\mu\,\nu)\left(\frac{x}{s}\right)^\mu\right)dx$$

As a check, we substitute this solution back into the first equation.

$$\textbf{eqns}[\![1]\!]\ \textbf{/.}\ u \to \textbf{Function}\Big[s,\ \frac{1}{2}\,\mu\cot(\alpha)\int_1^s \frac{w(x)}{x}\Big((1-\mu\,\nu)\Big(\frac{s}{x}\Big)^{\mu} - (1+\mu\,\nu)\Big(\frac{x}{s}\Big)^{\mu}\Big)\,dx\Big]\ \textbf{/. gsoln // Simplify}$$

$s^{\sigma}\,O(s^8) = 0$

■ Solving the second equation

Substituting the values of $g^{(n)}(0)$ into the left-hand side of the second equation yields

$$\textbf{fseries = ExpandAll}\Big[\frac{\textbf{eqns}[\![2,1]\!]}{s^{\sigma}}\Big] == 0\ \textbf{/. gsoln;}$$

To solve the pair of differential equations, the coefficient of each power of s in this series solution must vanish identically. The simplest non-trivial solution arises when $f(0) \neq 0$.

Coefficient[First@fseries, *s*, 0] // Simplify

$$-\frac{f(0)\,((\sigma-2)\,\sigma\,(\sigma^2-\mu^2)\tan(\alpha)\,\eta^2 + \mu^2\,(\nu\,\sigma-1)\cot(\alpha))}{\mu^2\,(\nu\,\sigma-1)}$$

■ Indicial equation

Assuming that $\sigma\,\nu \neq 1$, one requires that

$$\textbf{soln[0]} = \frac{\textbf{Numerator[\%]}}{f(0)} = 0\ \textbf{// FullSimplify}$$

$(\sigma-2)\,\sigma\,(\sigma^2-\mu^2)\tan(\alpha)\,\eta^2 + \mu^2\,(\nu\,\sigma-1)\cot(\alpha) = 0$

The roots of this quartic in σ determine the Frobenius solution; however, in general, the solutions in terms of radicals are messy.

Assuming that $\sigma \neq 0, 2, \pm\mu$ (reasonable, since these are not, in general, solutions to the quartic), one can eliminate η as follows:

$$\textbf{soln[0] /.}\ \eta^2 \to \frac{\mu^2\,(\nu\,\sigma-1)\cot^2(\alpha)}{(\sigma-2)\,\sigma\,(\mu^2-\sigma^2)}\ \textbf{// Simplify}$$

True

With this implied relation for σ, one can determine the $f^{(n)}(0)$ as follows.

$$\textbf{fsoln = Solve}\Big[\textbf{fseries /.}\ \eta^2 \to \frac{\mu^2\,(\nu\,\sigma-1)\cot^2(\alpha)}{(\sigma-2)\,\sigma\,(\mu^2-\sigma^2)},\ \textbf{Table}[f^{(n)}(0),\{n,7\}]\Big]\ \textbf{// FullSimplify // First;}$$

Substituting the $f^{(n)}(0)$ and $g^{(n)}(0)$ back into series expansion, one obtains the solutions $u(s)$ and $w(s)$

_u_(s_) = _u_(_s_) /. fsoln // Simplify;

and

_w_(s_) = _w_(_s_) /. gsoln /. fsoln // Simplify;

■ Numerical example

For the numerical parameters above, the indicial equation becomes

First@soln[0] /. params // Factor

$(\sigma - 3)(\sigma^3 + \sigma^2 - 6\sigma + 3)$

Although exact solution is possible, numerical solution suffices.

nroots = NSolve[% == 0, σ]

$\{\{\sigma \to -3.181943336\}, \{\sigma \to 0.5935793454\}, \{\sigma \to 1.588363991\}, \{\sigma \to 3.\}\}$

Also, for these parameters, here are the Frobenius solutions, divided by $s^\sigma\ f(0)$.

Frobenius = $\mathbf{\dfrac{\{u(s),\ w(s)\}}{s^\sigma\ f(0)}}$ **/. params // Simplify**

$$\left\{1 - \frac{(\sigma-2)(\sigma+3)s^2}{6\sigma+16} + \frac{\sigma(\sigma^2+\sigma-6)^2 s^4}{8(9\sigma^3+66\sigma^2+154\sigma+112)} - \frac{\left(\sigma^2(\sigma^2+\sigma-6)^3\right)s^6}{48(27\sigma^5+378\sigma^4+2106\sigma^3+5792\sigma^2+7784\sigma+4032)} + O(s^8),\ \frac{1}{3}(-\sigma-3) + \frac{(\sigma-2)(\sigma+3)(\sigma+5)s^2}{6(3\sigma+8)} - \frac{\left(\sigma(\sigma+7)(\sigma^2+\sigma-6)^2\right)s^4}{24(9\sigma^3+66\sigma^2+154\sigma+112)} + \frac{\sigma^2(\sigma+9)(\sigma^2+\sigma-6)^3 s^6}{144(27\sigma^5+378\sigma^4+2106\sigma^3+5792\sigma^2+7784\sigma+4032)} + O(s^8)\right\}$$

Using the numerically computed initial value $f(0)$, and the most singular value of σ, we compare the numerical solution to the Frobenius solution.

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DisplayTogether[nplot,
  Plot[Evaluate[initial[[1]] Normal[Frobenius] /. First[nroots]], {s, 0, 1}, PlotStyle → {LightPink, LightBlue}]]
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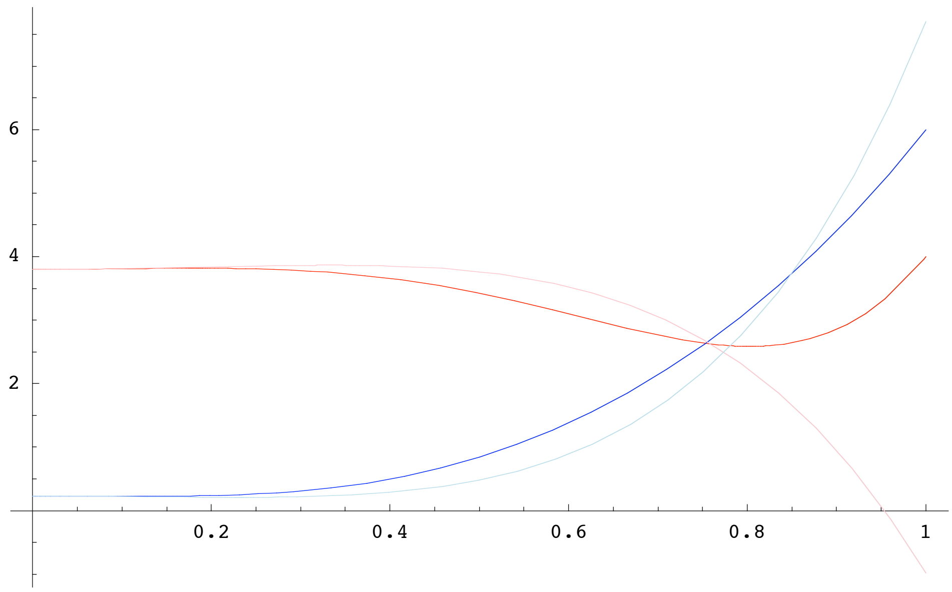

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The agreement is, as expected, very good for small s.