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Physics 352 – Winter 2008

3 REFRACTION

When light travels from one medium to another, it may change direction. This phenomenon – familiar whenever we see the "bent" shape of a straw poking out of a glass of water – is known as **refraction**. (The light may also change its intensity at a boundary between media, which we'll discuss soon.) Refraction, like diffraction, is a consequence of the wave nature of light, and our analysis below applies also to waves in water, sound waves, etc.

3.1 Snell's Law

The basic setup for issues of refraction is shown in Figure 3.1: A ray of light crosses the boundary between two media, with indices of refraction n_1 and n_2, making angles θ_1 and θ_2 with respect to the normal in each medium, respectively. The question is: *How are θ_1 and θ_2 related?* The answer is crucial to the propagation of light, and the design of lenses and other optical elements.

The answer, as we'll derive, is that light obeys **Snell's Law:**

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}.$$

There are several ways to derive this result. We could directly examine the wave equations for electromagnetic fields and look for solutions consistent with the presence of a boundary between two media, but this would be both painful and un-illuminating. There are simpler ways of thinking about light propagation; we'll briefly mention one, and then more fully discuss another.

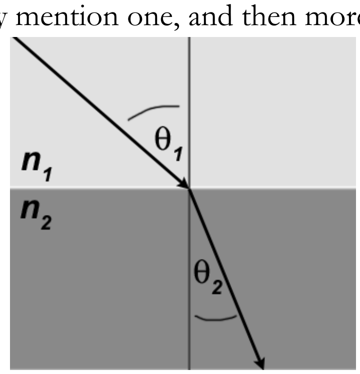

Figure 3.1. Refraction: The path taken by light traveling between two media bends at the interface; the relation between the angles θ_1 and θ_2 depends on the indices of refraction of the two materials, and is given by Snell's Law.

3.2 Huygen's Principle

Several centuries ago, Huygens came up with an interesting way of thinking about propagating waves, realizing that: **Each point in a propagating wave can be treated as a source**

of spherical waves. (This idea was later "fleshed out" by Fresnel and others.) There's nothing too shocking here – each point at a wave is, by definition, a point, and points generate spherical waves, as we noted in Section 1.1. The "envelope" of the wavelets generated by all the points defines the shape of the wavefront as the wave progresses. There are some conceptual difficulties inherent in Huygens idea which we won't discuss. Keep in mind that it's just a "picture" that we employ to save us from having to solve the full wave equation. Furthermore, it's often painful to use Huygen's principle to actually calculate anything quantitative. It does, however, provide a means for deriving **Snell's Law**, as described in last quarter's textbook (A.P. French, *Vibrations and Waves*, p. 270-274). We won't go into the derivation here, but will rather take a different, and more generally useful, approach.

3.3 Fermat's Principle

3.3.1 Minimal time paths and Snell's Law

Another "general" principle describing wave motion was put forth by Fermat. It is sometimes stated as "light travels from one point to another along the path that takes the minimal amount of time." This isn't quite correct – we'll fix it in a few paragraphs – but it's a good place to start. We'll also return to justifying Fermat's principle shortly. First, let's use it to derive Snell's Law.

Imagine you're on a beach, and someone in the ocean is drowning. You rush out to help, which requires both running on land and swimming in the water. You can run faster than you can swim. What path should you take? With a bit of thought, you'll realize that a straight line between you and the drowning person isn't the best idea – rather, you should reduce the length of the swim to minimize the overall time to your target. How much should you run and how much should you swim?

The same dilemma is encountered by our light beam, traveling from position A in a medium of index of refraction n_1 (your position on the beach, in the above analogy) to position B in a medium of n_2, (the swimmer's position, in the water) in Figure 3.3. The speed of light in medium 1 is $v_1 = c/n_1$ and in medium 2 is $v_2 = c/n_2$. Within each medium the light travels in a straight line – itself a consequence of Fermat's principle, as you can convince yourself later. There are many possible paths between A and B, as illustrated in the figure, that we can label based on the position x at which they cross the interface. One of these – let's call it the one that goes through position x_0 – minimizes the total travel time. What is this path? (In other words, what is x_0?)

The travel time in medium 1, t_1, is the distance traveled in medium 1 divided by the speed in medium 1: $t_1 = d_1 / v_1 = \frac{n_1}{c}\sqrt{y_1^2 + x^2}$; similarly the travel time in medium 2 is

$t_2 = d_2 / v_2 = \frac{n_2}{c}\sqrt{y_2^2 + (L-x)^2}$. (See Figure 3.3 for the geometry.) The total travel time is therefore

$$t = t_1 + t_2 = \frac{n_1}{c}\sqrt{y_1^2 + x^2} + \frac{n_2}{c}\sqrt{y_2^2 + (L-x)^2}.$$

To find the minimal time, we determine the x for which $\frac{dt}{dx} = 0$; call this x_0:

$$\frac{dt}{dx} = \frac{n_1}{c}\frac{x}{\sqrt{y_1^2 + x^2}} - \frac{n_2}{c}\frac{(L-x)}{\sqrt{y_2^2 + (L-x)^2}}$$

$$\left.\frac{dt}{dx}\right|_{x=x_0} = 0 \rightarrow n_1 \frac{x_0}{\sqrt{y_1^2 + x_0^2}} = n_2 \frac{(L-x_0)}{\sqrt{y_2^2 + (L-x_0)^2}}.$$

(To show that x_0 is a minimum we should also examine the second derivative of t, but as we'll see later it does not actually matter if x_0 is the site of a minimum or a maximum. Furthermore, we can intuit from the form of $t(x)$ that this extremum is, in fact, a minimum.)

Note from geometry that

$$\sin\theta_1 = \frac{x_0}{\sqrt{y_1^2 + x_0^2}}$$

$$\sin\theta_2 = \frac{(L-x_0)}{\sqrt{y_2^2 + (L-x_0)^2}}$$

Therefore the above condition becomes

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}.$$

We've shown that when the above condition is met, the travel time for light propagation is minimized. This is Snell's Law! (And a valuable tool for saving drowning swimmers.)

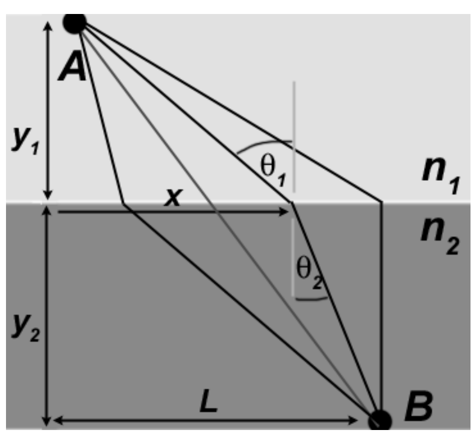

Figure 3.3. Light traveling from point A in medium 1 to point B in medium 2 can take infinitely many possible paths consisting of line segments in each medium, four of which are illustrated here. Each path crosses the interface at some position x, and hence makes some angles θ_1 and θ_2 with respect to the normal to the interface, as illustrated for one of the paths. Fermat's principle states that the **actual** path the light takes is that for which the total

travel time is minimal. (Actually, Fermat's principle states something slightly more subtle, as we'll discuss shortly.) Applying this principle, we can derive Snell's Law.

We can also use Fermat's principle to derive Snell's **Law of Reflection**, which states that the *reflected* ray makes the same angle with the interface as the incident ray.

3.3.2 Explaining Fermat's Principle

Now let's explain Fermat's Principle. Suppose light travels along many paths, all of which interfere with one another[1]. Paths for which the phase difference is near zero will constructively interfere. Consider the minimal time path. As we saw in our derivation of Snell's Law above, this is the path for which $\sum n_i d_i$ is minimal, where the sum runs over all the segments being considered (two segments in the above example) and d_i is the length of segment i. Recall from Section 1.2.1, though, that minimizing $\sum n_i d_i$ is the equivalent to minimizing the *phase* traversed by the wave along the path. Therefore, the minimal time path is also the path of minimal phase, and is also the path of minimal $\sum n_i d_i$ – all these statements are equivalent. This sum $\sum n_i d_i$ is more properly written as an integral, and is called the optical path length: $OPL = \int_A^B n(x)\, dx$.

Why should the path of minimal *OPL* be the path light takes? Let's call this path *P*. By construction, $\left.\frac{d(OPL)}{d"s"}\right|_P$ is zero, where "s" indicates any variable that characterizes the paths. Therefore nearby paths are similar in phase, and so constructively interfere. Consider a path for which $\frac{d(OPL)}{d"s"}$ is **not** zero – moving to a slightly different path, the *OPL* can change appreciably, perhaps higher in one direction, lower in another, etc., and so we would not expect constructive interference.

You may be thinking: the minimal OPL path isn't the only one for which we can guarantee constructive interference. What about the *maximal* OPL path? This too provides constructive interference. And so the proper formulation of Fermat's principle is that **light travels along paths of extremal optical path length.** Typically these are minimal OPL paths, but in various geometries they can be maximal OPL paths as well.

As we'll see in class, Fermat's principle provides a simple way to think about **mirages** on hot desert roads.

3.4 Total Internal Reflection

Let's look more carefully at Snell's Law. What if $n_1 > n_2$, and θ_1 is large, so that $\frac{n_1}{n_2}\sin\theta_1 > 1$? What θ_2 can satisfy Snell's Law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$? **None**, since $\sin$ is bounded by

[1] This can be seen as a restatement of Huygen's Principle.

± 1. This means that *there can be no wave transmitted to medium 2*; the light from medium 1 is totally reflected at the interface, a condition known as **total internal reflection**.

Fiber optics, which underpin much of modern communication, work because of total internal reflection. Consider a glass fiber ($n = 1.5$) surrounded by air ($n = 1.0$). We want the light to travel down the fiber and not leak out into the air (Figure 3.4). This is automatically taken care of by total internal reflection – note the higher index of refraction of the glass and the large incident angles of light traveling along the fiber (as long as the fiber is not severely bent). In a fiber optic cable, light can propagate for kilometers with losses of a fraction of a percent! Total internal reflection also has interesting applications in microscopy that we will discuss later.

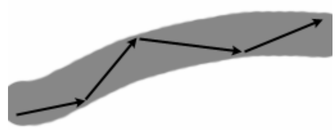

Figure 3.4. A fiber optic cable. Glass (gray) is surrounded by a material of lower index of refraction, e.g. air.