08.03 Euler's Method for Ordinary Differential Equations

After reading this chapter, you should be able to:

1. *Understand Euler's Method for solving ordinary differential equations and how to use it to solve problems.*

What is Euler's Method?

Euler's method is a numerical technique to solve ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0 \qquad (1)$$

So only first order ordinary differential equations can be solved by using Euler's method. In other sections, we will discuss how Euler's method is used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

How does one write a first order differential equation in the above form?

Example 1:

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x,y)=1.3e^{-x}-2y$$

Another example is given as follows

Example 2:

$$e^{y}\frac{dy}{dx}+x^2y^2=2Sin(3x),\ y(0)=5$$

is rewritten as

$$\frac{dy}{dx}=\frac{2Sin(3x)-x^2y^2}{e^{y}},\ y(0)=5$$

In this case

$$f(x,y)=\frac{2Sin(3x)-x^2y^2}{e^{y}}$$

At $x=0$, we are given the value of $y=y_0$. Let us call $x=0$ as x_0. Now since we know the slope of y with respect to x, that is, $f(x,y)$, then at $x=x_0$, the slope is $f(x_0,y_0)$. Both x_0 and y_0 are known as they from the initial condition $y(x_0)=y_0$.

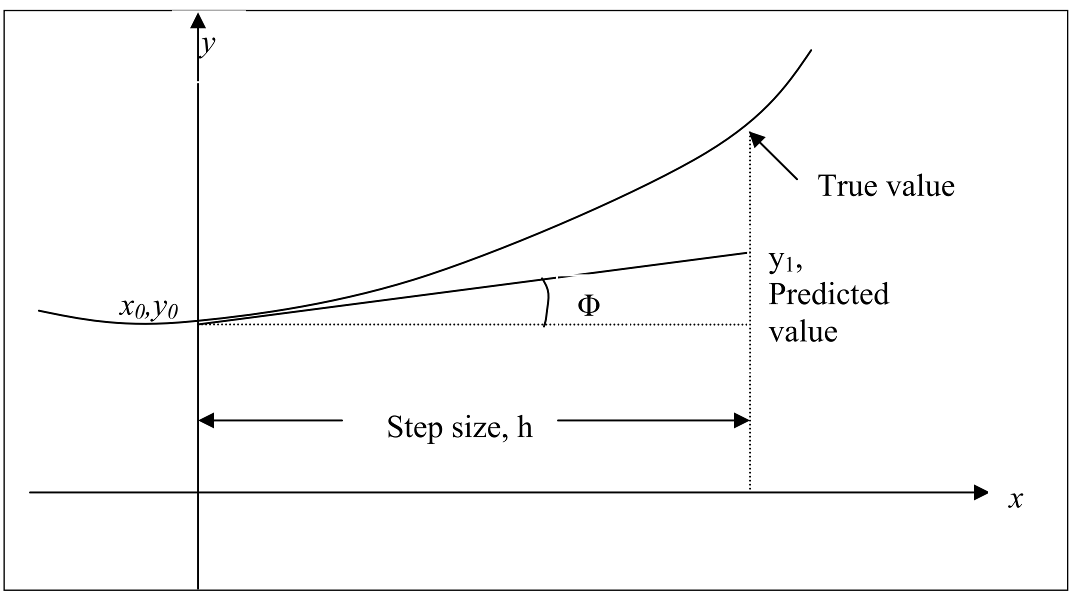

Figure 1. Graphical interpretation of the first step of Euler's method

So the slope at $x = x_0$ as shown in Figure 1 is

$$\text{Slope} = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

From here

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

Calling $x_1 - x_0$ as a step size h, we get

$$y_1 = y_0 + f(x_0, y_0)h. \qquad (2)$$

One can now use the value of y_1 (an approximate value of y at $x = x_1$) to calculate y_2, and that would be the predicted value at x_2,

$$y_2 = y_1 + f(x_1, y_1)h$$

$$x_2 = x_1 + h$$

Based on the above equations, if we now know the value of $y = y_i$ at x_i, then

$$y_{i+1} = y_i + f(x_i, y_i)h \qquad (3)$$

This formula is known as the Euler's method and is illustrated graphically in Figure 2. In some books, it is also called the Euler-Cauchy method.

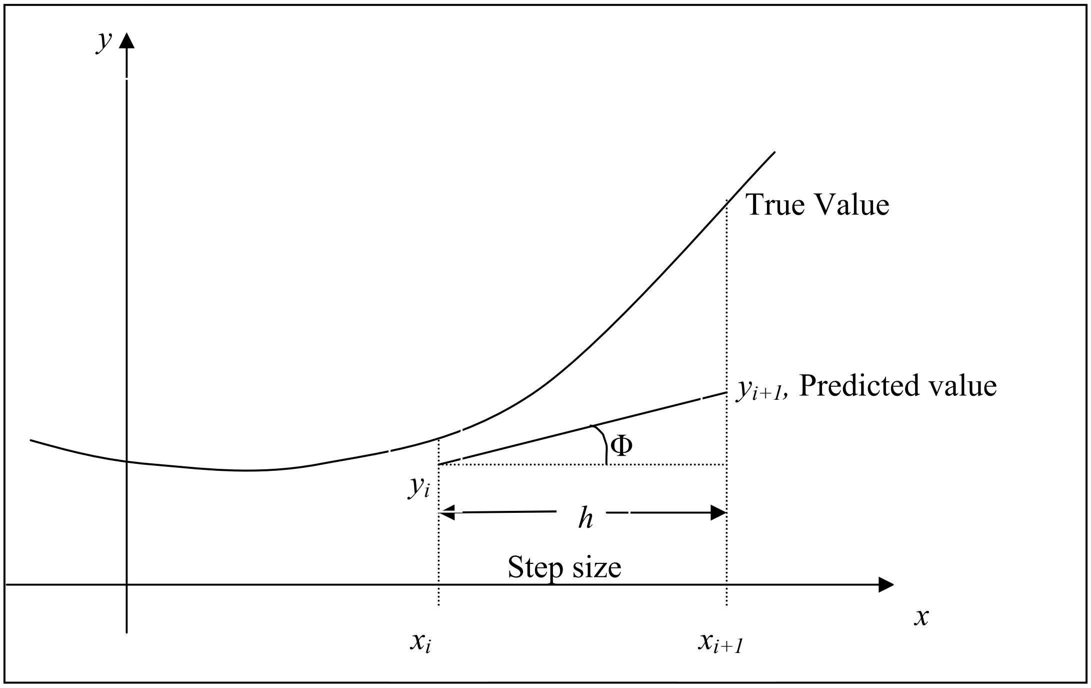

Figure 2. General graphical interpretation of Euler's method

Example 3:

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right), \theta(0) = 1200K$$

where θ is in K and t in seconds. Find the temperature at $t = 480$ seconds using Euler's method. Assume a step size of $h = 240$ seconds.

Solution

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

Per equation 3, the Euler's method reduces to

$$\theta_{i+1} = \theta_i + f(t_i, \theta_i)h$$

For $i = 0$, $t_0 = 0$, $\theta_0 = 1200$

$$\begin{aligned}\theta_1 &= \theta_0 + f(t_0, \theta_0)h \\ &= 1200 + f(0,1200)240 \\ &= 1200 + \left(-2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right)\right)240 \\ &= 1200 + (-4.5579)240 \\ &= 106.09K\end{aligned}$$

θ_1 is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 240 = 240$$

$$\theta_1 = \theta(240) \cong 106.09K$$

For $i = 1$, $t_1 = 240$, $\theta_1 = 106.09$

$$\begin{aligned}\theta_2 &= \theta_1 + f(t_1, \theta_1)h \\ &= 106.09 + f(240,106.09)240 \\ &= 106.09 + \left(-2.2067\times10^{-12}\left(106.09^4 - 81\times10^8\right)\right)240 \\ &= 106.09 + (0.017595)240 \\ &= 110.32K\end{aligned}$$

θ_2 is the approximate temperature at $t = t_2$

$$t = t_2 = t_1 + h = 240 + 240 \quad = 480$$

$$\theta_2 = \theta(480) \cong 110.32K$$

Figure 3 compares the exact solution with the numerical solution from Euler's method for the step size of *h=240*.

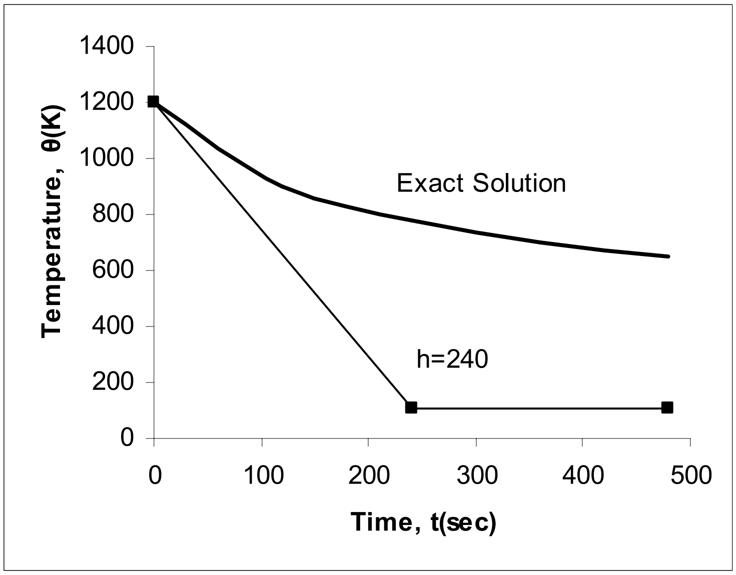

Figure 3. Comparing exact and Euler's method

The problem was solved again using a smaller step size. The results are given below in Table 1.

Table 1. Temperature at 480 seconds as a function of step size, h

Step size, h	$\theta(480)$	E_t	$\lvert\in_t\rvert\,\%$
480	-987.81	1635.4	252.54
240	110.32	537.26	82.964
120	546.77	100.80	15.566
60	614.97	32.607	5.0352
30	632.77	14.806	2.2864

Figure 4 shows how the temperature varies as a function of time for different step sizes

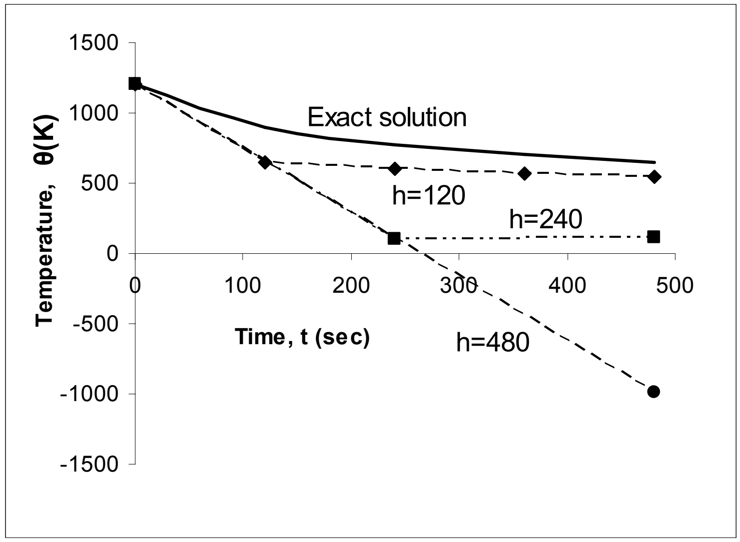

Figure 4. Comparison of Euler's method with exact solution for different step sizes

while the values of the calculated temperature at $t=480s$ as a function of step size are plotted in Figure 5.

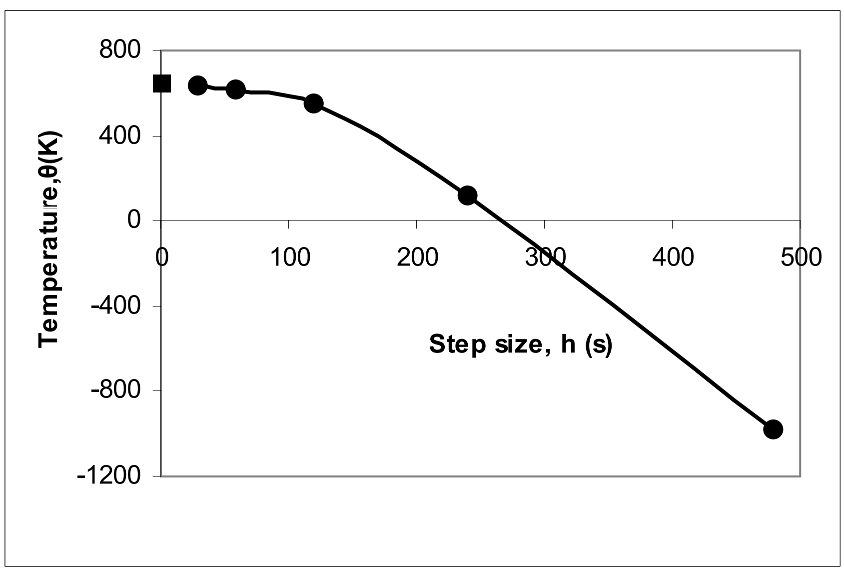

Figure 5. Effect of step size in Euler's method.

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta-300}{\theta+300}-1.8519\tan^{-1}\left(0.333\times10^{-2}\theta\right)=-0.22067\times10^{-3}t-2.9282 \qquad (4)$$

The solution to this nonlinear equation is

$$\theta = 647.57K$$

It can be seen that Euler's method has large errors. This can be illustrated using Taylor series.

$$y_{i+1}=y_i+\left.\frac{dy}{dx}\right|_{x_i,y_i}(x_{i+1}-x_i)+\frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i,y_i}(x_{i+1}-x_i)^2+\frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i,y_i}(x_{i+1}-x_i)^3+\ldots \qquad (5)$$

$$y_{i+1}=y_i+f(x_i,y_i)+\frac{1}{2!}f'(x_i,y_i)(x_{i+1}-x_i)^2+\frac{1}{3!}f''(x_i,y_i)(x_{i+1}-x_i)^3+\ldots \qquad (6)$$

As you can see the first two terms of the Taylor series

$$y_{i+1}=y_i+f(x_i,y_i)h$$

are the Euler's method.

The true error in the approximation is given by

$$E_t=\frac{f'(x_i,y_i)}{2!}h^2+\frac{f''(x_i,y_i)}{3!}h^3+\ldots \qquad (7)$$

The true error hence is approximately proportional to the square of the step size, that is, as the step size is halved, the true error gets approximately quartered. However from Table 1, we see that as the step size gets halved, the true error only gets approximately halved. This is because the true error being proportioned to the square of the step size is the local truncation error, that is, error from one point to the next. The global truncation error is however proportional only to the step size as the error keeps propagating from one point to another.

ORDINARY DIFFERENTIAL EQUATIONS	
Topic	Euler's Method for Ordinary Differential Equations
Summary	Textbook notes on Euler's method for solving Ordinary Differential Equations
Major	All Majors of Engineering
Authors	Autar Kaw
Last Revised	April 21, 2008
Web Site	http://numericalmethods.eng.usf.edu