Impact Mechanics

1D Elastic Stress Waves

Dr. Jaafar Ghanbari
Ghanbari@qut.ac.ir

Content

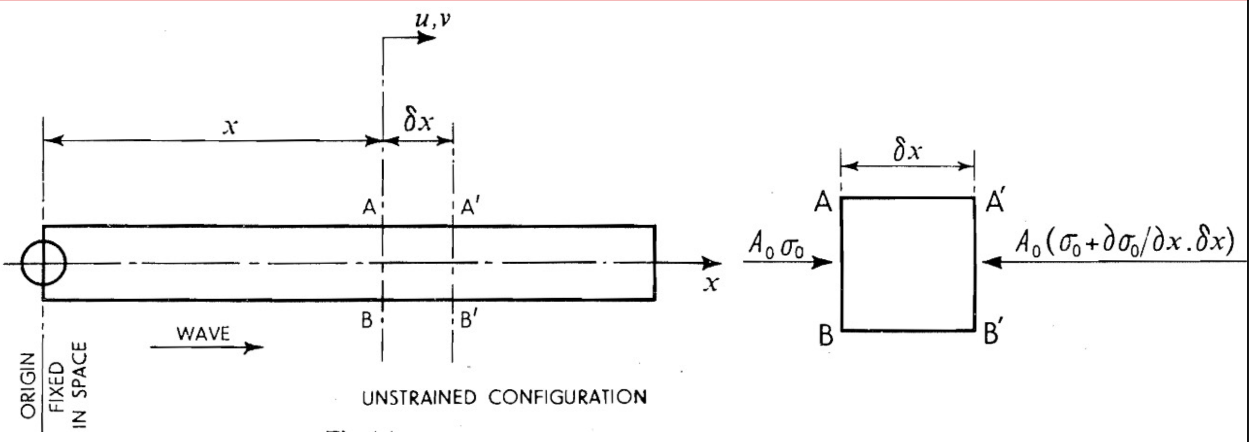
- Longitudinal waves
- Wave transmission along a uniform bar
- Coaxial collision of bars
- Reflection and superposition of waves
- Impact of bars and spheres
- Energy and momentum transmission
- Propagation of torsional waves

1D Elastic Wave

Definitions

- Stress wave:
  - A pulse transmitted through a body when different parts of it are not at equilibrium.
- Body waves:
  - Waves traveling through the mass of a body
- Surface waves:
  - Waves traveling over the surface of a body
- Wave types:
  - Longitudinal, torsional, bending

Compressive wave



- Equation of motion:

$$-\frac{\partial \sigma_0}{\partial x} . \delta x \cdot A_0 = A_0 \rho_0 \delta x \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial \sigma_0}{\partial x} = -\rho_0 \frac{\partial^2 u}{\partial t^2}$$

Compressive wave

- Strain: $\varepsilon_x = \frac{\partial u}{\partial x}$
- The Hooke's law is: $-\frac{\sigma_0}{\partial u / \partial x} = E$
- Differentiating the above: $\frac{\partial \sigma_0}{\partial x} = -E \frac{\partial^2 u}{\partial x^2}$
- Substituting into the equation of motion:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = E \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho_0} \cdot \frac{\partial^2 u}{\partial x^2} = c_L^2 \frac{\partial^2 u}{\partial x^2}$$

$$c_L = \sqrt{\frac{E}{\rho_0}}$$

- The wave equation.

Solution

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

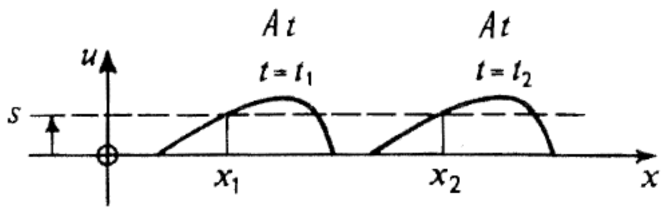
- Try: $u = f(x - ct) + F(x + ct)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 f''(x - ct) + c^2 F''(x + ct)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x - ct) + F''(x + ct)$$

- Hence, $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Wave speed



- Suppose, $u = f(x - ct)$
- If u=s when x=x1 and t=t1 and also u = s when t = t2 and x = x2

$$s = f(x_1 - ct_1) = f(x_2 - ct_2)$$

- Thus, $$x_1 - ct_1 = x_2 - ct_2$$
- and $$c = \frac{x_2 - x_1}{t_2 - t_1}$$

Wave speed

- c_L is the speed of elastic wave propagation along the fixed axis of the bar

$$c_L = \sqrt{E/\rho_0}$$

  - Note that the speed of propagation is independent of $\partial u/\partial t$, or the local velocity of the elements transmitting the wave.
  - c_L depends only on the elastic properties of the transmitting medium and its density.

- Similarly, for torsional waves,

$$c_T = \sqrt{G/\rho_0}$$

Wave speed

- Wave speed for common materials:

	Cast Iron	Carbon Steel	Brass	Copper	Lead	Alum-inium	Glass
E lbf/in^2	$16{\cdot}5 \,.\, 10^6$	$29{\cdot}5 \,.\, 10^6$	$13{\cdot}5 \,.\, 10^6$	$16{\cdot}5 \,.\, 10^6$	$2{\cdot}5 \,.\, 10^6$	$10 \,.\, 10^6$	$8 \,.\, 10^6$
ρ_0 = lb/in^3	0·26	0·28	0·30	0·32	0·41	0.·096	0·070
$c_L = \sqrt{E/\rho_0}$ ft/sec ($g \simeq 384$ in/sec/sec)	13 025	16 900	11 000	12 100	3 900	16 700	17 500
$c_T = \sqrt{G/\rho_0}$ ft/sec	8 100	10 600	6 700	7 500	2 300	10 200	10 700

Intensity of stress

- The stress propagated in the material:

$$\sigma_0 = -E\partial u/\partial x = -(E/c_L)\partial u/\partial t$$

- By $v_0 = \partial u/\partial t$, we have $\sigma_0 = Ev_0/c_L$. Or

$$\sigma_0 = \rho_0 c_L v_0$$

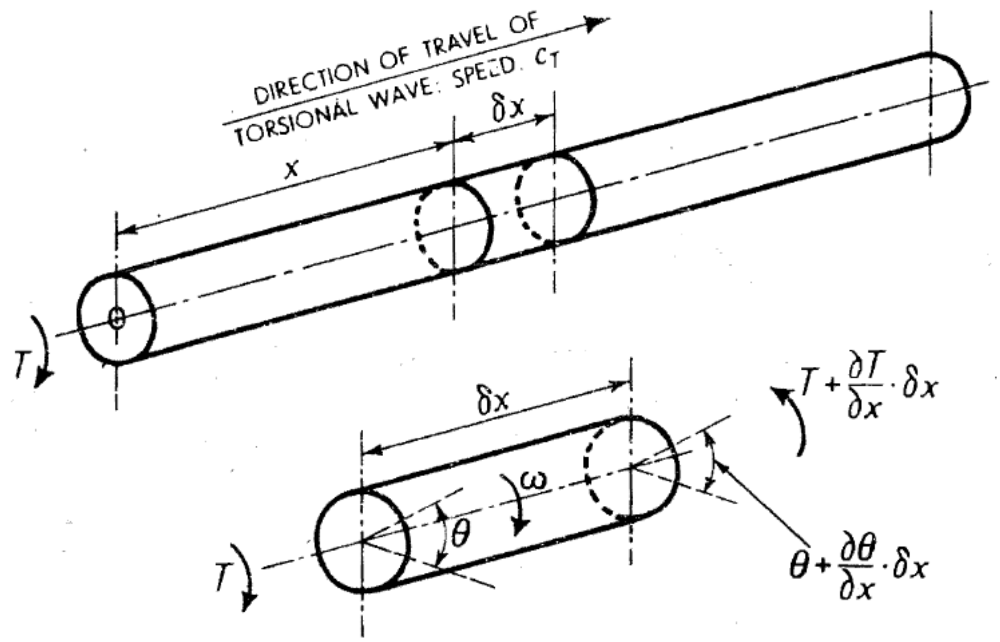
  - for steel if the stress is 16 $tonf/in^2$, the particle speed would be

$$v_0 = \frac{16\times 2240\text{in/sec}}{\sqrt{30\cdot 10^6\cdot 0\cdot 28/384}} \simeq 20\text{ft/sec}$$

  - For pure lead, at its yield stress of about 1 $tonf/in^2$, v0 is only about 4 ft/sec.

- The quantity $\rho_0 c_L$ is often referred to as the mechanical impedance of the bar.

Torsional wave



- Equation of motion: $\frac{\partial T}{\partial x} = JG\frac{\partial^2\theta}{\partial x^2} = I\frac{\partial^2\theta}{\partial t^2}$

Torsional wave

- Or

$$\frac{\partial^2\theta}{\partial t^2} = \frac{JG}{I}\frac{\partial^2\theta}{\partial x^2}$$

$$\frac{\partial^2\theta}{\partial t^2} = c_T^2\frac{\partial^2\theta}{\partial x^2}$$

- Where $c_T^2 = \frac{JG}{I}$
- is the torsional wave speed.
- For a circular cylinder:

$$c_T^2 = \frac{G{\cdot}\pi a^4/2}{\rho_0\pi a^4/2} = \frac{G}{\rho_0}$$

Torsional stress

- With rotation $\theta = g(x - c_T t),$

$$\frac{\partial\theta}{\partial x} = g'(x - c_T t)$$

$$\frac{\partial\theta}{\partial t} = -g' c_T(x - c_T t)$$

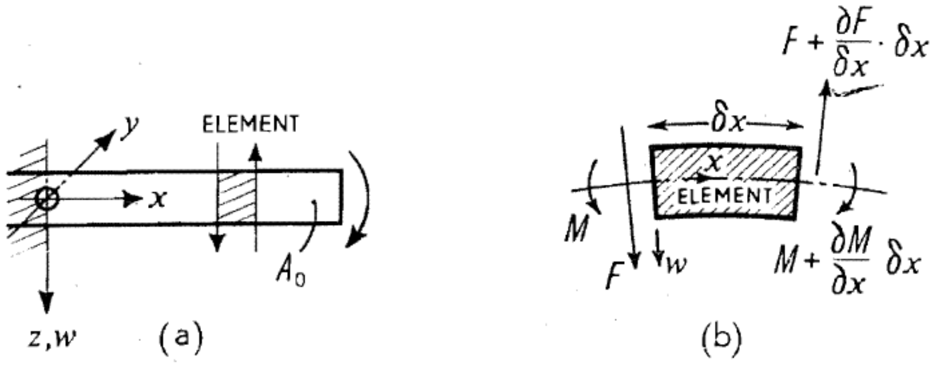
$$\frac{\partial\theta}{\partial x} = -\frac{1}{c_T}\frac{\partial\theta}{\partial t}$$

$$T = JG\frac{\partial\theta}{\partial x}$$

$$T = -\frac{JG}{c_T}\cdot\frac{\partial\theta}{\partial t} = J\sqrt{G\rho_0}\cdot\omega.$$

- Stress can be obtained with known torsional torque.

Flexural wave



- Equation of motion: $-(\rho_0 A_0 dx)\frac{\partial^2 w}{\partial t^2} = \frac{\partial F}{\partial x} \cdot dx$
- From beam theory, $EI\frac{\partial^3 w}{\partial x^3} = F$

$$\rho_0 A_0 \frac{\partial^2 w}{\partial t^2} = -EI\frac{\partial^4 w}{\partial x^4}$$

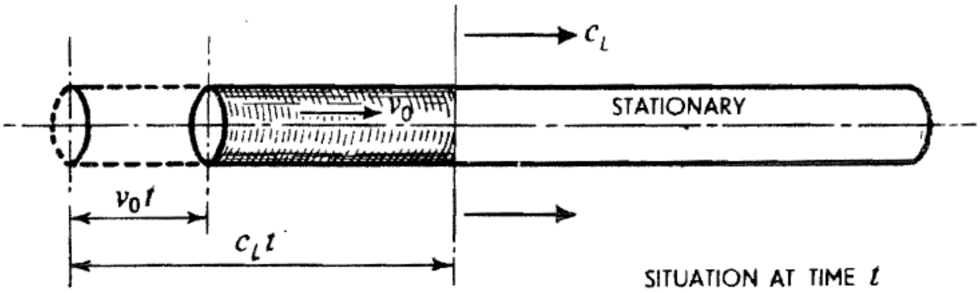
$$\frac{\partial^2 w}{\partial t^2} = -c^2 k^2 \frac{\partial^4 w}{\partial x^4}$$

Flexural wave

- k denotes the radius of gyration of the cross-section about an axis in the neutral surface
- If we try a solution of the form $w = f(x - ct)$ or $w = f(x + ct)$ the equation is found not to be satisfied.
- Thus flexural disturbances of arbitrary form are not propagated without dispersion.

Momentum approach

- Longitudinal wave



- After time t, the compressed zone length is $c_L t$.
- If the bar is originally stationary and the end face is caused to move with and maintains uniform speed v_0 , then the whole length will be moving with uniform speed v_0 at time t.
- Equating the change in momentum of this length $c_L t$ to the impulse, we have

$$(A_0 c_L t \rho_0) v_0 = (\sigma_0 A_0) \cdot t$$

Momentum approach

- Thus $\sigma_0 = \rho_0 c_L v_0$
- The strain is $v_0 t / c_L t$, so $\sigma_0 = E\, v_0 / c_L$
- Substituting, $E\, v_0 / c_L = \rho_0 c_L v_0$
- Hence, $c_L = \sqrt{E/\rho_0}$

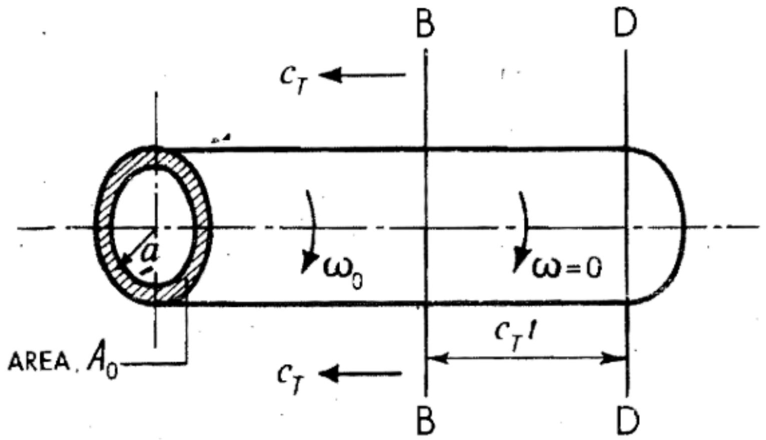
Energy

- The total energy acquired by the rod at time t is made up of
  - (a) kinetic energy $\frac{1}{2}A_0(c_Lt)\rho_0v_0^2$
  - (b) stored strain energy $A_0(c_Lt)\sigma_0^2/2E$

$$A_0c_Lt\cdot\frac{\sigma_0^2}{2E}=A_0(c_Lt)\cdot\frac{\rho_0^2c_L^2v_0^2}{2E}=\frac{1}{2}\cdot A_0(c_Lt)\rho_0v_0^2$$

  - and thus the total energy acquired by the bar at time t is composed equally of strain energy and kinetic energy.

Torsional wave



- A uniform thin hollow tube of radius a and cross-sectional area A0 is rotating at angular speed ω when at time t = 0 the end DD is suddenly brought to rest.
- a length c_Tt will have been brought to rest and the remainder of the tube will still be rotating at speed ω_0

Torsional wave

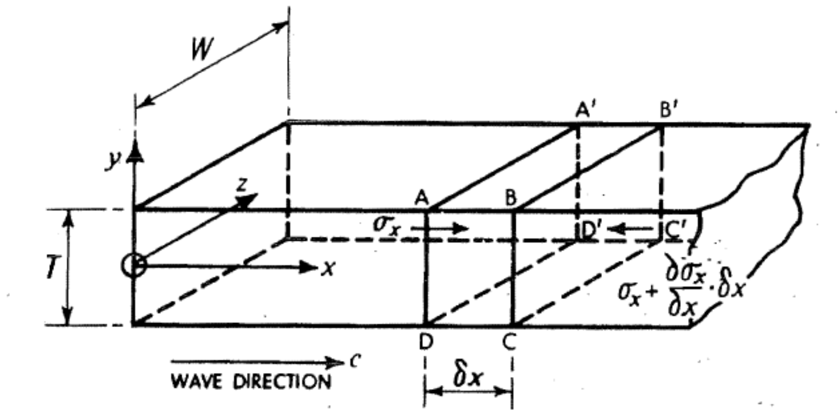
- The mean shear stress τ which prevails in length $c_T t$ is arrived at by equating the impulsive torque, $t.(A_0 \tau. a)$, to the loss in angular momentum of the tube in time t,

$$[(tc_T.A_0)\rho_0.a^2].w_0$$

- Thus $tA_0\tau a = t.c_T.A_0 a^2 \rho_0 \omega_0$
- The torsional strain is $\omega_0 ta/c_T t$
- Hence, $\tau = G.\phi = G \cdot \frac{\omega_0 ta}{c_T t}$
- Substituting, $G \cdot \frac{\omega_0 a}{c_T} = \rho_0 c_T a \omega_0$

$$c_T = \sqrt{G/\rho_0}$$

Plane strain condition



- Equation of motion:

$$-A_0 \frac{\partial \sigma_x}{\partial x} \cdot \delta x = A_0 \rho_0 \cdot \delta x \cdot \frac{\hat{c}^2 u}{\partial t^2}$$

$$\frac{\partial \sigma_x}{\partial x} = -\rho_0 \cdot \frac{\hat{c}^2 u}{\partial t^2}$$

Plane strain condition

- Plane strain condition:

$$e_z = (-\sigma_z - \nu\sigma_x)/E = 0,$$

- Thus $\sigma_z = -\nu\sigma_x$
- Also, $e_x = \frac{-\sigma_x - v\sigma_y - v\sigma_z}{E} = \frac{-\sigma_x - v \cdot 0 + v \cdot v\sigma_x}{E}$
- Hence

$$\frac{\partial u}{\partial x} = e_x = -\frac{(1 - v^2)}{E} \cdot \sigma_x$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{(1 - v^2)}{E} \cdot \frac{\partial \sigma_x}{\partial x}$$

Plane strain condition

- Hence

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho_0 (1 - v^2)} \cdot \frac{\partial^2 u}{\partial x^2}$$

- The longitudinal wave speed is

$$c_L' = \sqrt{\frac{E}{\rho_0 (1 - v^2)}}$$

$$\frac{c_L'}{c_L} = \frac{1}{\sqrt{1 - v^2}}$$

Transversely constrained

- Wave transmission along a uniform bar constrained to have zero transverse deformation.
- From symmetry: $e_y = e_z; \ \ \sigma_y = \sigma_z$
- So,

$$\begin{aligned} Ee_y &= \sigma_y - \nu(\sigma_y - \sigma_x) \\ &= \sigma_y(1-\nu) + \nu\sigma_x \end{aligned}$$

- For zero transverse strain, $\sigma_y = -\nu\sigma_x/(1-\nu)$
- With

$$\frac{\partial u}{\partial x} = e_x = \frac{-\sigma_x - 2v\sigma_y}{E}$$

Transversely constrained

$$\frac{\partial u}{\partial x} = -\sigma_x \frac{(1 - v - 2v^2)}{E(1-v)}$$

$$\frac{\partial \sigma_x}{\partial x} = -E\frac{(1-v)}{(1+v)(1-2v)} \cdot \frac{\partial^2 u}{\partial x^2}$$

- Substituting,

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho_0} \cdot \frac{(1-v)}{(1+v)(1-2v)} \cdot \frac{\partial^2 u}{\partial x^2}$$

Transversely constrained

$$c_L'' = \sqrt{\frac{E}{\rho_0} \cdot \frac{(1-v)}{(1+v)(1-2v)}}$$

$$\frac{c_L''}{c_L} = \sqrt{\frac{(1-v)}{(1+v)(1-2v)}}$$

$$c_L'' = \sqrt{(\lambda + 2G)/\rho_0}$$

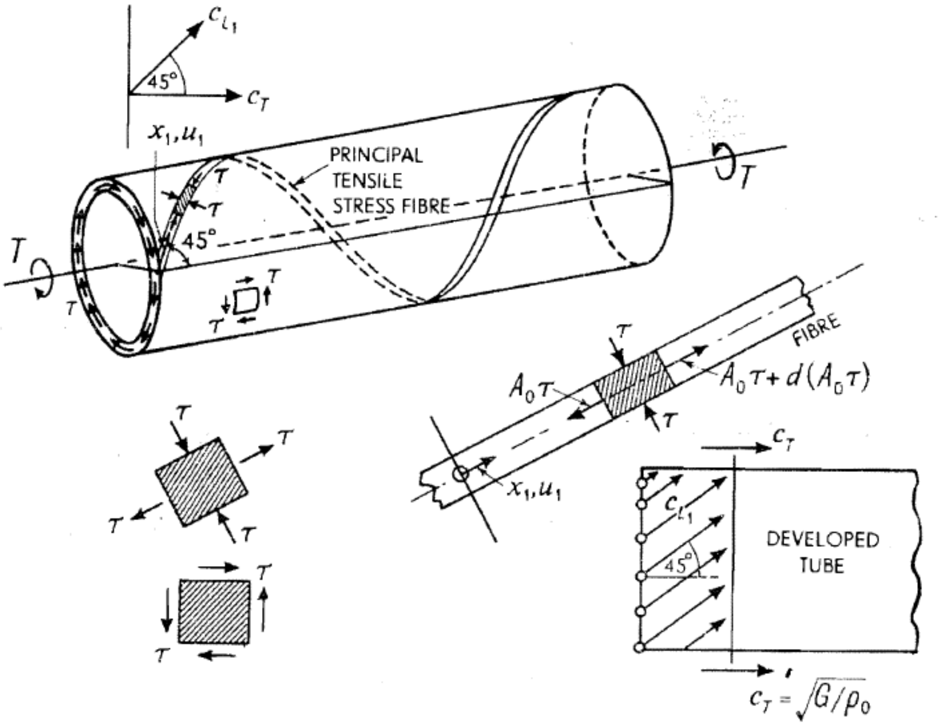
$$\lambda = vE/(1+v)(1-2v).$$

Different conditions

- Wave speed for 1D, plane strain, and transversely constrained conditions

v	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$
c_L'/c_L (from (1.20))	$\frac{4}{\sqrt{15}}$ $\simeq 1{\cdot}03$	$3\sqrt{2}/4$ $\simeq 1{\cdot}06$	$2\sqrt{3}/3$ $\simeq 1.15$
c_L''/c_L (from (1.24))	$\sqrt{1{\cdot}2}$ $\simeq 1.1$	$\sqrt{1{\cdot}5}$ $\simeq 1{\cdot}22$	∞

Torsional wave via longitudinal wave



Torsional wave via longitudinal wave

- If the transverse shear stress in the tube wall is τ, the principal stresses in a long uniform helical fiber at 45° to the tube axis are + τ and - τ.
- The fiber lying along a principal axis may be considered simply as a bar.
- The initiation of a torsional wave at the end of the tube can be identified with the propagation of a longitudinal wave along the fiber.
- Equation of motion:

$$\rho_0 A_0 dx_1 \cdot \frac{\partial^2 u_1}{\partial t^2} = A_0 \cdot d\tau$$

Torsional wave via longitudinal wave

- With

$$e_1 = \frac{\partial u_1}{\partial x_1} = \frac{\tau(1+v)}{E}$$

- Substituting into eq. of motion:

$$\frac{\partial^2 u_1}{\partial t^2} = \frac{E}{\rho_0(1+v)} \cdot \frac{\partial^2 u_1}{\partial x_1^2}$$

- The speed of longitudinal wave propagation:

$$c_{L_1} = \sqrt{\frac{E}{\rho_0(1+v)}}$$

Torsional wave via longitudinal wave

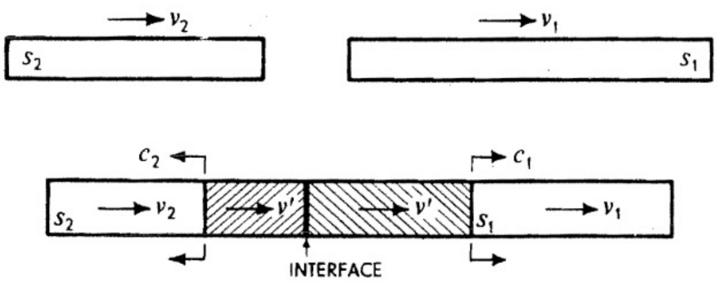
- The speed of the wave parallel to the axis of the tube which is just the torsional wave speed c_T is c_{L_1} cos 45^o

$$c_T = \frac{c_{L_1}}{\sqrt{2}} = \sqrt{\frac{E}{2(1+v)\rho_0}} = \sqrt{\frac{G}{\rho_0}}$$

Collision of bars

Collision of bars

- Before impact let the two square-ended bars which have different mechanical impedances ($\rho_0 . c$) possess speeds v_1 and v_2 , where $v_1 < v_2$
- After impact, compressive longitudinal waves will propagate from the impact interface into each bar.
- the impact interface and the material engulfed by each wave will all have the same speed v'
- The stress created in each bar is the same.



Collision of bars

- Earlier we derived the stress as $\sigma_0 = \rho_0 c_L v_0$ for stationary bar.
  - For a bar with initial speed v_1, the expression will be $\rho_0 c_L (v' - v_1)$, where v' is the new particle speed after the wave has travelled through it.
  - Thus, v_0 is properly to be understood as a change in particle speed due to the passage of a wave or the magnitude of the velocity discontinuity across a wave front.
  - The quantity $(v' - v_1)$ in respect of the initially translating bar and v_0 for the stationary bar are therefore identical quantities for the purpose of calculation.
  - Henceforth, when we use the expression $\sigma_0 = \rho_0 c_L v_0$ it must be remembered that v_0 refers to a change in particle speed.

Collision of bars

- Thus,

$$\sigma = \rho_2 c_2 (v_2 - v') = \rho_1 c_1 (v' - v_1)$$

$$v' = \frac{\rho_1 c_1 v_1 + \rho_2 c_2 v_2}{\rho_1 c_1 + \rho_2 c_2}$$

$$\sigma = \frac{v_2 - v_1}{\frac{1}{\rho_1 c_1} + \frac{1}{\rho_2 c_2}}$$

- If $\rho_1 c_1 = \rho_2 c_2$, and $v_1 = -v_2$ then $v' = 0$ and $\sigma = \rho_1 c_1 v_1$

Impact with water

- Consider a square-ended projectile of initial density ρ_0 impinging normally upon a plane sheet of water.
  - Let the elastic compressive stress generated in the cylinder, which is initially moving with speed v_0 , be σ_0
  - Then the particle speed in that part of the cylinder traversed by the stress wave is reduced to $v = v_0 - \sigma_0/(\rho_0 c_L)$
  - at the instant of impact, v is also the particle speed of the contiguous water surface
  - The compressive stress immediately created in the water σ_w:

$$\sigma_w = \rho_w c_w v = \rho_w c_w \left(v_0 - \frac{\sigma_0}{\rho_0 c_0} \right)$$

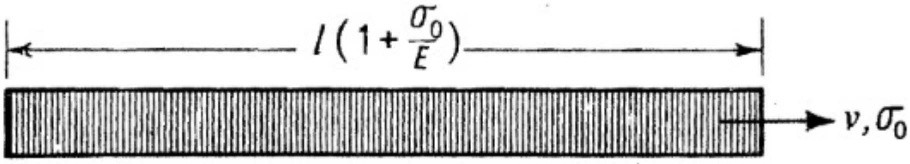
Impact with water

- The stress in the cylinder ($\sigma_w = \sigma_0$):

$$\sigma_0 = \frac{\rho_w c_w v_0}{1+\frac{\rho_w c_w}{\rho_0 c_0}} = \frac{v_0}{\frac{1}{\rho_w c_w}+\frac{1}{\rho_0 c_0}}$$

- a square-ended steel bullet moving at 2500 ft/sec. would give rise to an elastic stress σ_0 of $283000\ lbf/in^2$ $\approx 126\ tonf/in^2$, on hitting the water

Stressed material



- Let the right hand end of the stationary rod be given a speed of v which is maintained until the whole of the bar has the same speed.
- The length of the bar will have then become $l\left(1+\frac{\sigma_0}{E}\right)$
- Denote the wave speed through the space occupied by the bar in its final stressed state. by C_E:

$$\frac{l\left(1+\frac{\sigma_0}{E}\right)}{C_E} = \frac{l\frac{\sigma_0}{E}}{v}$$

Stressed material

- Hence,

$$c_E = \frac{v(1+\sigma_0/E)}{\sigma_0/E} = c_L + v$$

- Or

$$c_E = c_0 + v$$

  - c_0 is the wave speed in the unstressed bar
  - c_E is the wave speed in the stressed bar
  - v is the velocity of the particles

- If the bar had been put into compression instead of tension,

$$c'_E = c_0 - v$$

Stressed material

- Since an element initially of unit length when stressed by a tension wave becomes (1 + e),

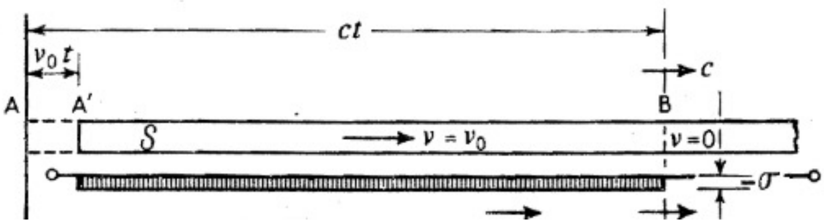
$$\frac{1}{c_0} = \frac{1+e}{c_E}$$

- Or

$$\frac{c_E}{c_0} = 1 + \frac{v}{c_0}$$

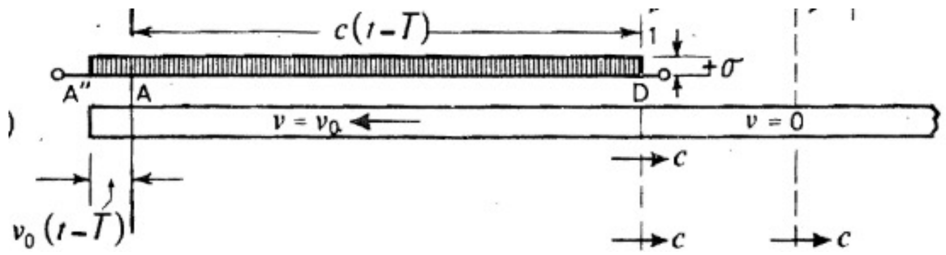
  - Since speed, v, for elastic behavior is of the order of 10 ft/sec and c_0 is of the order of 10000 ft/sec, then v/c_0 , or v/c_E are $\approx 1/1000$.
  - Thus for cases of elastic impact, for all practical purposes, we need not distinguish between c_0 and c_E.

Rectangular pulse



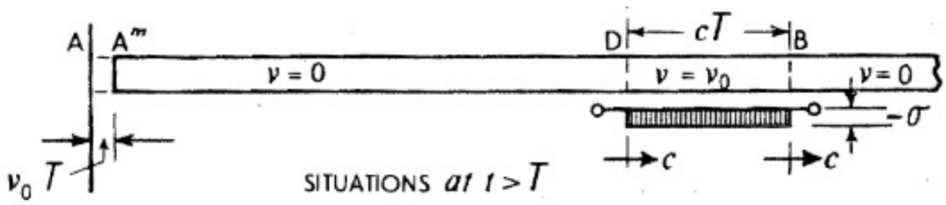
- Let the end A of an unstrained stationary bar S, be moved with constant speed v_0 to the right for a time t, so that A moves to A' and $AA' = v_0 t$.
- At the same time a compressive wave $\sigma = \rho_0 c_0 v_0$ is caused to travel along the bar from A to the right as far as B, with speed c_0, so that $AB = c_0 t$.

Rectangular pulse



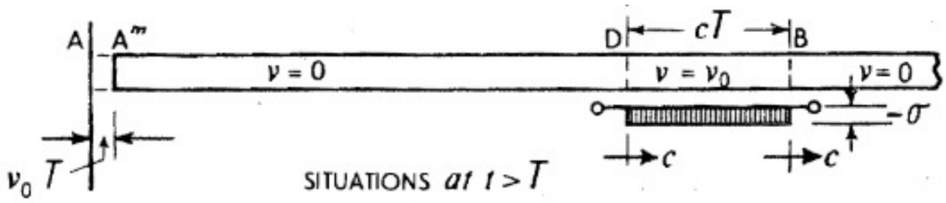
- Consider a bar identical with S the end A of which is caused to move to the left to A with speed v_0 but only commencing at a time t = T.
- The magnitude of the induced tensile stress is $\sigma = \rho_0 c_0 v_0$ and the wave travels to the right so that at time t > T, AD = c(t - T); also $AA" = v_0(\mathrm{t} - \mathrm{T})$.

Rectangular pulse



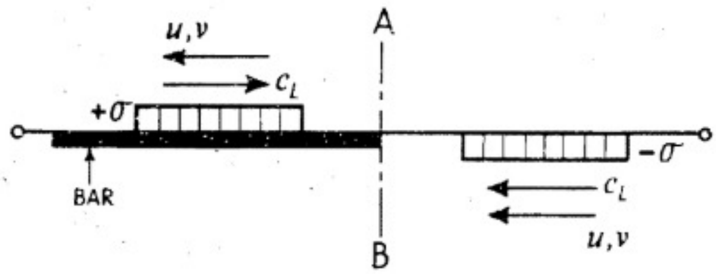
- Suppose now that the left hand end of unstrained stationary bar S, at $t = 0$ is moved to the right and that this is maintained for all t; also, suppose that time $t = T$, there is imposed on this the previous situation and to the end of the bar is added a speed to the left of v_0.
- Length $A'''\mathrm{D}$ is stationary and unstressed having undergone a rigid body movement to the right of AA''' or $v_0 T$
- DB which has an unstrained length $c_0 T$ is subject to compressive stress σ, is actually compressed amount $v_0 T$

Rectangular pulse



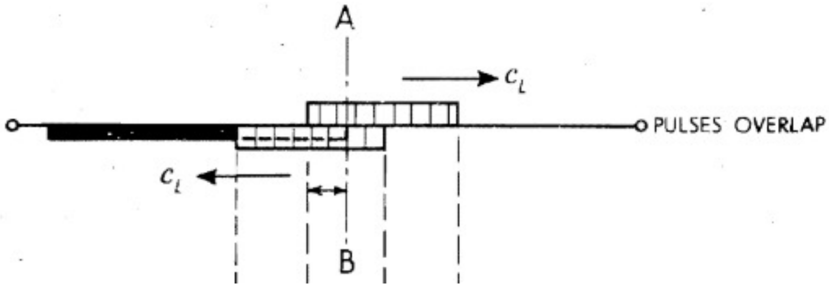
- The portion of bar to the right of B of course stationary and unstressed.
- A rectangular pulse of length cT can thus be considered in itself

Reflection



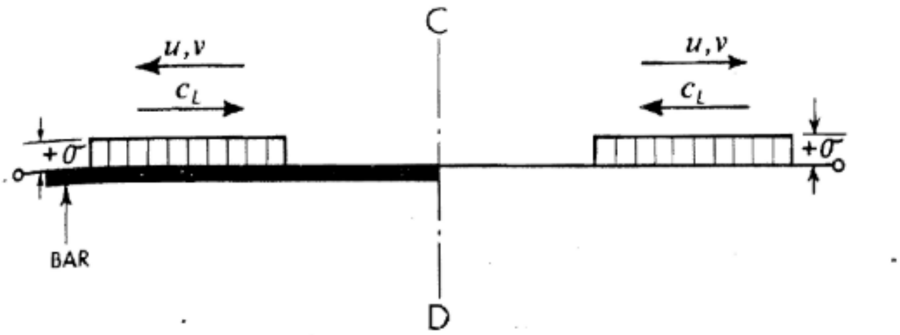
- Assume an elastic rectangular pulse of tension to be moving along a uniform rod in the positive x-direction.
  - Introduce a hypothetical rectangular compressive wave length and stress magnitude moving in the opposite direction.
  - At AB the two wave fronts first meet and after some time have moved through one another completely.

Reflection



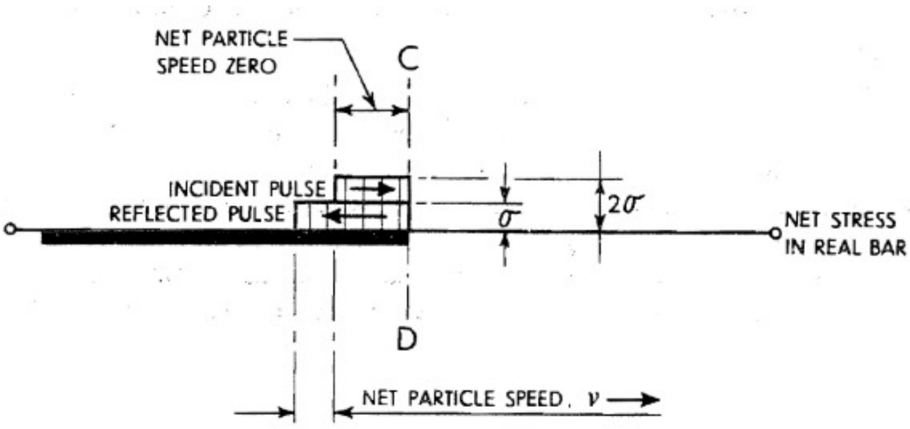
- At AB, the stresses in these waves annul one another so that the stress is zero: a free end.
- Hence, for a free-ended bar, a tensile wave is reflected as a compressive wave.
- In the portion of the bar where the two pulses overlap the total stress is zero and the particle speed is twice what it was when covered by the incident tensile wave alone.

Reflection



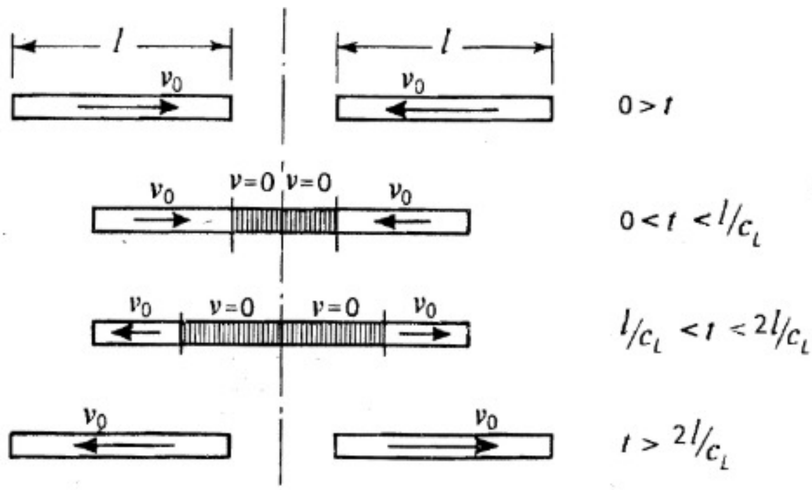
- Consider two identical tensile waves moving towards one another.
  - at CD, where the heads of the two pulse first meet, the stress is doubled and the particle speed is zero: fixed end.
  - Hence, an elastic wave reflected from a fixed-ended bar is entirely unchanged in shape or intensity.

Reflection



- It should now be evident that the net stress on, or the speed of particles in, a given plane are easily obtained by adding together the separate effects of the operative waves at that plane, e.g. the incident and reflected waves, provided, of course, that the waves are elastic.

Impact of identical bars



- Immediately after impact a compressive wave of intensity, $\rho_0 v_0 c_L$, moves into each bar from the common plane of impact.

Impact of identical bars

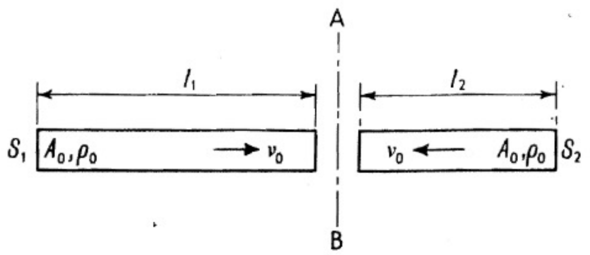
- Energies:

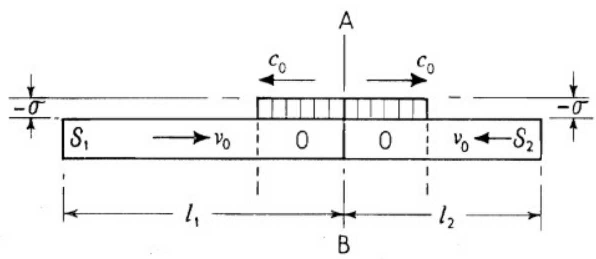
$$K = \frac{1}{2}(\text{ mass of bar }) \cdot v_0^2 = \frac{1}{2} \cdot A_0 l \cdot \rho_0 \cdot v_0^2$$

$$E = \text{ volume } \cdot \frac{\sigma^2}{2E} = \frac{A_0 l \cdot (\rho_0 v_0 c_L)^2}{2E} = \frac{A_0 l \rho_0 v_0^2}{2}$$

- Thus at $t = 2l/c_L$ the particles in the common plane of impact will move away from one another with equal but opposite speeds.
- The bars will thus rebound as unstressed bodies at a time $t = 2l/c_L$ after impact first took place.
- The coefficient of restitution $e = 1$ in this case.

Impact of bars





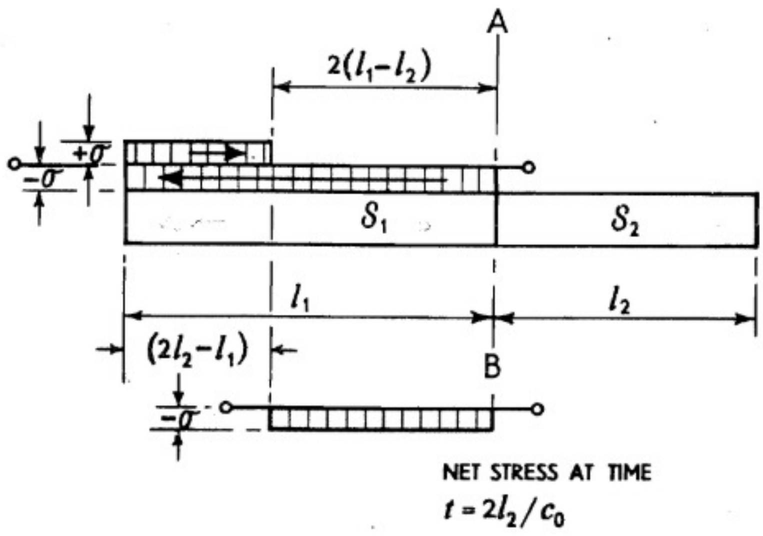
- A compressive stress $\sigma = \rho_0 c_0 v_0$ will have been propagated away from the common plane of impact.
- The compressive pulse is reflected as a wave of tension from the free end of each bar, so that for the period of time $\frac{l_2}{c_0} < t < \frac{2l_2}{c_0}$, S_2 will be being unloaded.
  - When $t = 2l_2/c_0$, S_2 is just completely stress free and the particle speed in it will everywhere have been exactly reversed.

Impact of bars

- at this instant an unloading wave travels into S_1 from S_2 so that the particles at the right hand end of S_1 move to the right with speed v_0.
- Contact ceases at $t = 2l_2/c_2$ when the wave reflected from the left band end of S_1 reaches the right hand end of S_2 and so cancels the speed there of v_0.

Impact of bars

- At this instant, i.e. $t = 2l_2/c_0$, let us ascertain how the energy is distributed in S_1 and S_2 if $2l_2 > l_1$:



Impact of bars

- Kinetic energy of S_2: $\frac{1}{2}A_0\rho_0 l_2 v_0^2$
- And S_1: $\frac{1}{2}A_0\rho_0(2l_2 - l_1\ )v_0^2$

- the elastic strain energy in S_2 is zero and that in S_1:

$$= \frac{1}{2}A_0[2(l_1 - l_2)]\frac{\sigma^2}{E}$$
$$= \frac{1}{2}A_0[2(l_1 - l_2)]v_0^2\frac{E\rho_0}{E},$$

- Because $\sigma = v_0\sqrt{E\rho_0}$

$$= \frac{1}{2}\rho_0 A_0 v_0^2(2l_2 - 2l_1)$$

Impact of bars

- Total energy of the bars:

$$\frac{1}{2}A_0\rho_0 l_2 v_0^2 + \frac{1}{2}A_0\rho_0 v_0^2(2l_2 - l_1) + \frac{1}{2}A_0\rho_0 v_0^2(2l_1 - 2l_2)$$
$$= \frac{1}{2}A_0\rho_0 v_0^2[l_2 + 2l_2 - l_1 + 2l_1 - 2l_2]$$
$$= \frac{1}{2}A_0\rho_0 v_0^2[l_1 + l_2]$$

- which is just the kinetic energy of l_1 and l_2 before impact took place.

Impact of bars

- If we consider the case of $l_1 = 2l_2$, then at $t = l_1/c_0$, the whole of S_1 will be compressed and stationary.
  - At $t = 1.5l_1/c_0$, S_1 will be completely unstrained.
  - The left hand half of S_1 will be moving to the left with speed v_0 and right hand half to the right with speed v_0.
  - At time $t = 2l_1/c_0$ each half of the bar will be in tension.
  - At $t = 2.5l_1/c_0$ the halves will be unloaded.
  - At $t = 3l_1/c_0$ they will again be entirely in compression

Impact of bars

- The coefficient of restitution e at the impact of S_1 and S_2, when $l_1 = 2l_2$, calculated by reference to the center of gravity of each, is

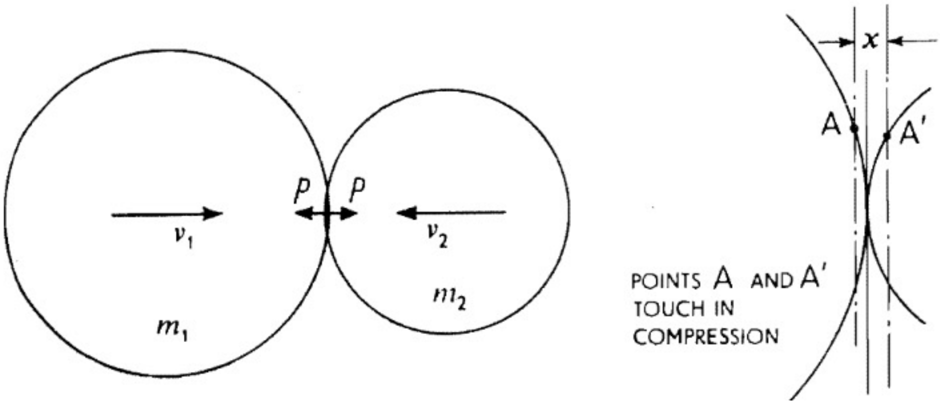
velocity of separation of bar = -e (velocity of approach of bars)

$$v_0 - (-v_0) = -e(0 - v_0)$$

$$e = \frac{1}{2}$$

Impact of spheres

Impact of spheres



- Equations of motion:

$$m_1 \frac{dv_1}{dt} = -P \text{ and } m_2 \frac{dv_2}{dt} = -P$$

$$\frac{dx}{dt} = v_1 + v_2 \text{ and } \frac{d^2x}{dt^2} = \frac{dv_1}{dt} + \frac{dv_2}{dt}$$

$$\frac{d^2x}{dt^2} = -P \cdot \frac{m_1+m_2}{m_1 m_2} = -P\mu$$

Impact of spheres

- The force-displacement relation for static conditions is given by ref. 1.4 as

$$P = kx^{3/2}$$

- Where

$$k = \frac{4}{3\pi\left[\frac{1-v_1^2}{\pi E_1}+\frac{1-v_2^2}{\pi E_2}\right]} \cdot \left(\frac{R_1 R_2}{R_1+R_2}\right)^{1/2}$$

- Substituting, $\ddot{x} = \frac{d^2x}{dt^2} = -k\mu x^{3/2}$
- Integrating, $\frac{1}{2}\left(\dot{x}^2 - v_0^2\right) = -\frac{2}{5}k\mu x^{5/2}$

Impact of spheres

- Where v_0 denotes the value of $(v_1 + v_2)$ when $t = 0$.
- Putting $\dot{x} = 0$, the maximum compression x_0 is,

$$x_0 = \left(\frac{5}{4}\frac{v_0^2}{k\mu}\right)^{2/5}$$

- We have

$$\dot{x} = \frac{dx}{dt} = v_0\left[1 - \frac{4}{5}\frac{k\mu x^{5/2}}{v_0^2}\right]^{1/2} = v_0\left[1 - \left(\frac{x}{x_0}\right)^{5/2}\right]^{1/2}$$

- Hence the time to maximum compression, T, is

$$T = \frac{x_0}{v_0}\int_0^1 \frac{dv}{\left[1-v^{5/2}\right]^{1/2}} \cong 1.47\frac{x_0}{v_0}$$

- The radius of the circle of contact, d, is given by

$$d = \left[3P\left(\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2}\right)\frac{R_1 R_2}{4(R_1+R_2)}\right]^{1/3}$$

Impact of spheres

- For a spherical body, or a spherical-nosed projectile, radius R_2 , impinging against a plane surface, the maximum compressive force is

$$P_{max} = kx_0^{3/2} = \frac{4 \cdot R_2^{1/5} \left(\frac{15}{16} \pi m_2 v_0^2 \right)^{3/5}}{3\pi \cdot \left(\frac{1-v_1^2}{\pi E_1} + \frac{1-v_2^2}{\pi E_2} \right)^{2/5}}$$

- and

$$d_{\max} = R_2^{2/5} \left[\frac{15\pi \left(\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2} \right) m_2 v_0^2}{16} \right]^{1/5}$$

Impact of spheres

- Also,

$$q_{\max} = \frac{3 \cdot P_{\max}}{2\pi \cdot d_{\max}^2}$$

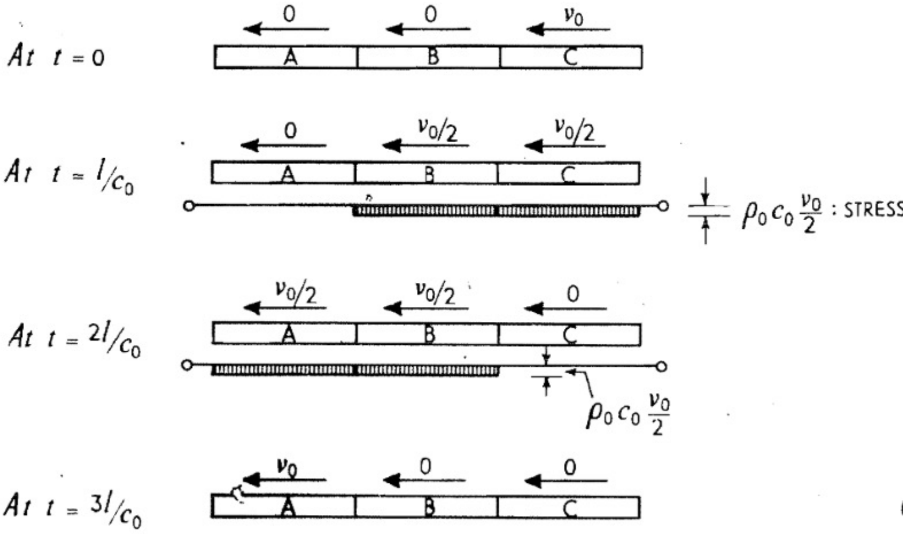
$$= \frac{E}{\pi(1-v^2)} \cdot \left(\frac{x_0}{R_2} \right)^{1/2}$$

- The maximum approach distance for identical spheres is given by

$$\frac{x_0}{R} = \left[\frac{5\sqrt{2}\pi\rho}{4} \cdot \frac{1-v^2}{E} \cdot v_0^2 \right]^{2/5}$$

Impact of multiple bars

Impact of three bars



- Two identical bars, A and B, of mass m, length l and cross-sectional area A_0 , are placed end-to-end touching one another, and one of them is hit on one end by a third identical bar, C, moving with speed v_0

Impact of three bars

- The initial momentum is mv_0 and the initial kinetic energy is ½mv_0^2
  - the momentum of B and C are each equal to $\frac{mv_0}{2}$
  - The strain energy in each is equal to

$$A_0 l \cdot [\rho_0 c_0 (v_0/2)]^2/2E, \text{ or } A_0 l. \frac{\rho_0^2 c_0^2 v_0^2/4}{2E} = mv_0^2/8$$

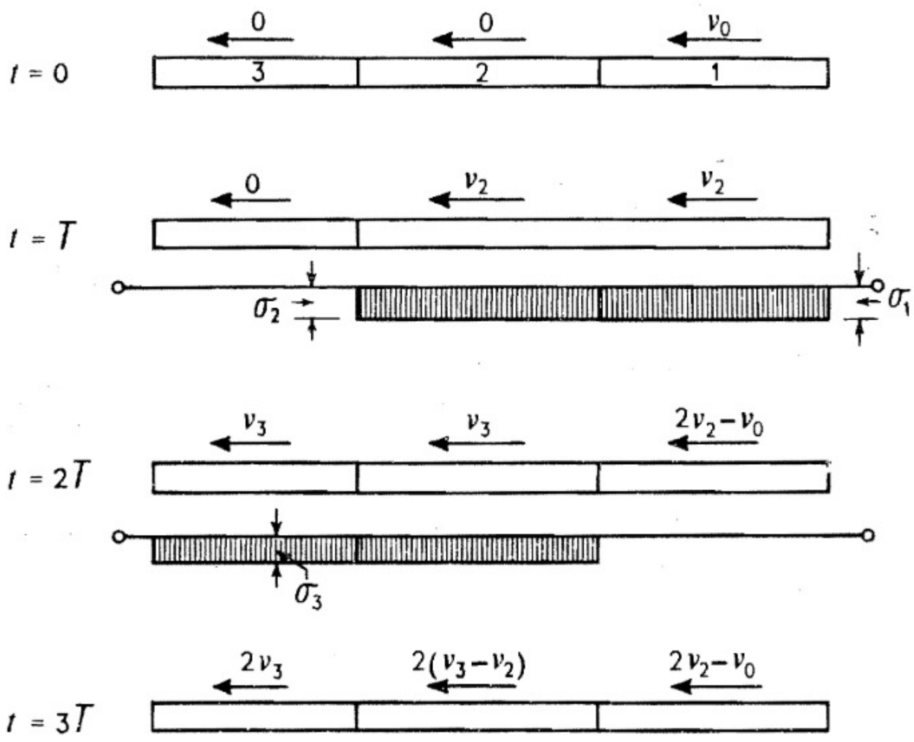
  - Thus, the total energy is

$$= 2 \times \tfrac{1}{2} m \tfrac{v_0^2}{4} + 2 \times m \tfrac{v_0^2}{8} = \tfrac{1}{2} m v_0^2 = \text{ original kinetic energy}$$

Impact of three bars

- After stress relief as between B and C at $t = 2l/c_0$ an unloading wave moves into B from its right hand end and an equally intense unloading wave moves into A from its left hand end.
- At time $t = 3l\ /c_0$, B will come to rest since the unloading tensile wave nullifies the compression in B and at the same time cancels the velocity, $v_0/2$, of particles in B.
- At $t = 3l/c_0$, B and C are stress free and stationary whilst A is stress free but moving to the left, every particle having the same speed v_0
  - Initial momentum and kinetic energy of C is wholly transferred to A.

Bars of unequal impedance



Bars of unequal impedance

- The mechanical impedance ($\rho_0 c_0$) of each of the three bars is different, but subject to the condition that $\frac{l_1}{c_1} = \frac{l_2}{c_2} = \frac{l_3}{c_3} = T$
  - The second and third bars are initially stationary and in contact, and that the first bar impinges collinearly on the second with an initial speed of v_0
  - For $0 < t < T$:

$$\sigma_1 = \rho_1 c_1 (v_0 - v_2)$$
$$\sigma_2 = \rho_2 c_2 \cdot v_2$$
$$\sigma_1 = \sigma_2.$$

  - Hence,

$$v_2 = \frac{\rho_1 c_1}{\rho_1 c_1 + \rho_2 c_2} \cdot v_0$$
$$\sigma_2 = \sigma_1 = \frac{\rho_1 c_1 \cdot \rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2} \cdot v_0$$

Bars of unequal impedance

- For $T < t < 2T$, the compressive wave in the first bar reflected from its free end after t = T; the first bar is completely stress free at t = 2T:

$$v_2 - (v_0 - v_2) = 2v_2 - v_0 = (\rho_1 c_1 - \rho_2 c_2)v_0/(\rho_1 c_1 + \rho_2 c_2)$$

$$\sigma_3 = \rho_3 c_3 v_3$$
$$\sigma_2' = \rho_2 c_2 (v_2 - v_3)$$
$$\sigma_3 = \sigma_2 + \sigma_2'$$
$$v_3 = \frac{2\rho_2 c_2}{\rho_2 c_2 + \rho_3 c_3} \cdot v_2$$
$$v_3 = \frac{2\rho_1 c_1 \cdot \rho_2 c_2}{(\rho_1 c_1 + \rho_2 c_2)(\rho_2 c_2 + \rho_3 c_3)} \cdot v_0,$$
$$\sigma_3 = \frac{2\rho_1 c_1 \cdot \rho_2 c_2 \cdot \rho_3 c_3}{(\rho_1 c_1 + \rho_2 c_2)(\rho_2 c_2 + \rho_3 c_3)} \cdot v_0$$
$$2v_2 - v_3 = \frac{2\rho_1 c_1 \cdot \rho_3 c_3}{(\rho_1 c_1 + \rho_2 c_2)(\rho_2 c_2 + \rho_3 c_3)} \cdot v_0$$

Bars of unequal impedance

- For $2T < t < 3T$

$$2v_3 - 2v_2 > 2v_2 - v_0$$
$$v_0 > 2(2v_2 - v_3)$$
$$\left(1 + \frac{\rho_1 c_1}{\rho_2 c_2}\right)\left(1 + \frac{\rho_3 c_3}{\rho_2 c_2}\right) > 4\frac{\rho_1 c_1}{\rho_2 c_2} \cdot \frac{\rho_3 c_3}{\rho_2 c_2}$$

- At $t = 3T$:
  - Kinetic energy of 1st bar: $E_1 = \frac{1}{2} A_0 l_1 \rho_1 \left[\frac{i_1 - i_2}{i_1 + i_2} \cdot v_0\right]^2$
    - Where $i_1 = \rho_1 c_1$ and $i_2 = \rho_2 c_2$
  - Its momentum: $M_1 = A_0 l_1 \rho_1 \frac{i_1 - i_2}{i_1 + i_2} v_0$
  - The 2nd bar will also be completely unloaded
    $v_3 - (2v_2 - v_3) = 2(v_3 - v_2) = 2i_1(i_2 - i_3)v_0/(i_1 + i_2)(i_2 + i_3)$

Bars of unequal impedance

- Its kinetic energy:

$$E_2 = \frac{1}{2} A_0 l_2 \rho_2 \left[\frac{2i_1(i_2 - i_3)}{(i_1+i_2)(i_2+i_3)} \cdot v_0 \right]^2$$

- And momentum:

$$M_2 = A l_2 \rho_2 \left[\frac{2i_1(i_2 - i_3)v_0}{(i_1+i_2)(i_2+i_3)} \right]$$

- The 3rd bar is stress-free but the linear speed of the whole bar is $2v_3$; thus its kinetic energy E_3, is

$$E_3 = \frac{1}{2} A_0 l_3 \rho_3 \left[\frac{4 \cdot i_1 i_2 v_0}{(i_1+i_2)(i_2+i_3)} \right]^2$$

- And its momentum:

$$M_3 = A l_3 \rho_3 \left[\frac{4 i_1 i_2 v_0}{(i_1+i_2)(i_2+i_3)} \right]$$

Bars of unequal impedance

- The total kinetic energy will be

$$= \frac{1}{2} A l_1 \rho_1 v_0^2 \left[\left(\frac{i_1 - i_2}{i_1 + i_2} \right)^2 + \left(\frac{l_2}{l_1} \cdot \frac{\rho_2}{\rho_1} \right) \cdot 4 \left(\frac{i_1 \cdot (i_2 - i_3)}{(i_1 + i_2)(i_2 + i_3)} \right)^2 \right.$$

$$\left. + \left(\frac{l_3}{l_1} \cdot \frac{\rho_3}{\rho_1} \right) \cdot 16 \cdot \left(\frac{i_1 i_2}{(i_1 + i_2)(i_2 + i_3)} \right)^2 \right]$$

$$= \frac{1}{2} m v_0^2 \left[(i_1 - i_2)^2 \cdot (i_2 + i_3)^2 + 4 \cdot \frac{i_2}{i_1} \cdot i_1^2 (i_2 - i_3)^2 \right.$$

$$\left. + 16 \cdot \frac{i_3}{i_1} \cdot i_1^2 \cdot i_2^2 \right] / (i_1 + i_2)^2 \cdot (i_2 + i_3)^2$$

$$= \frac{1}{2} m v_0^2 \frac{\left[(i_1 - i_2)^2 \cdot (i_2 + i_3)^2 + 4 i_1 i_2 (i_2 + i_3)^2 - 16 \cdot i_1 i_2^2 i_3 + 16 i_1 \cdot i_2^2 i_3 \right]}{(i_1 + i_2)^2 \cdot (i_2 + i_3)^2}$$

$= \frac{1}{2} m v_0^2 =$ the original kinetic energy of the first bar whose mass $m = A_0 l_1 \rho_1$

Bars of unequal impedance

- Total momentum:

$$= A_0 l_1 \rho_1 v_0 \left[\frac{i_1 - i_2}{i_1 + i_2} + \frac{l_2 \rho_2}{l_1 \rho_1} \cdot \frac{2 i_1 (i_2 - i_3)}{(i_1 + i_2)(i_2 + i_3)} + \frac{l_3 \rho_3}{l_1 \rho_1} \cdot \frac{4 i_1 i_2}{(i_1 + i_2)(i_2 + i_3)} \right]$$

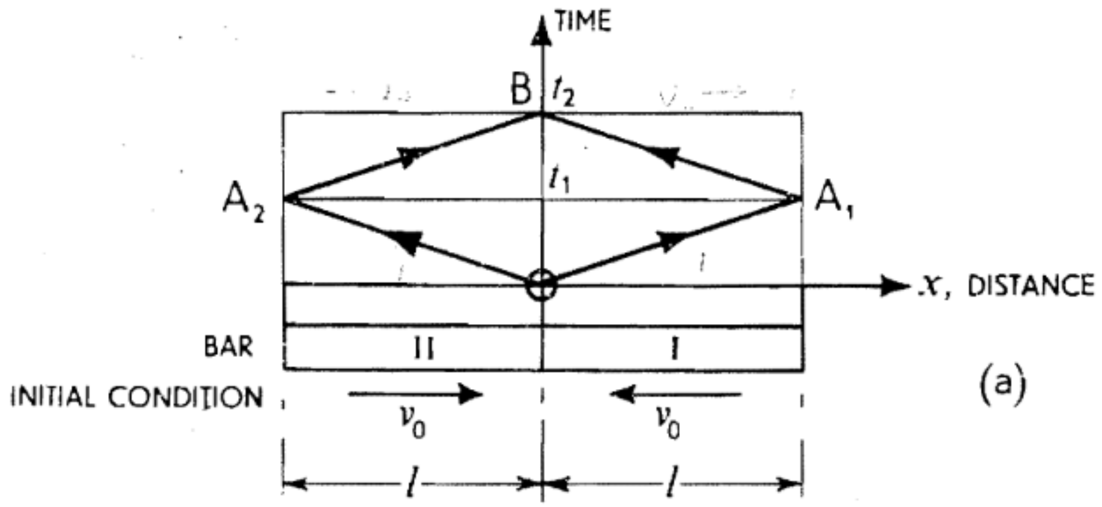
$$= m v_0 \left[\frac{(i_1 - i_2)(i_2 + i_3) + 2 i_2 (i_2 - i_3) + 4 i_2 i_3}{(i_1 + i_2)(i_2 + i_3)} \right]$$

$$= m v_0 \left[\frac{(i_2 + i_3)(i_1 + i_2) - 2 i_2 (i_2 + i_3) + 2 i_2 (i_2 + i_3)]}{(i_1 + i_2)(i_2 + i_3)} \right]$$

- $= m v_0 =$ the original momentum of the first bar.

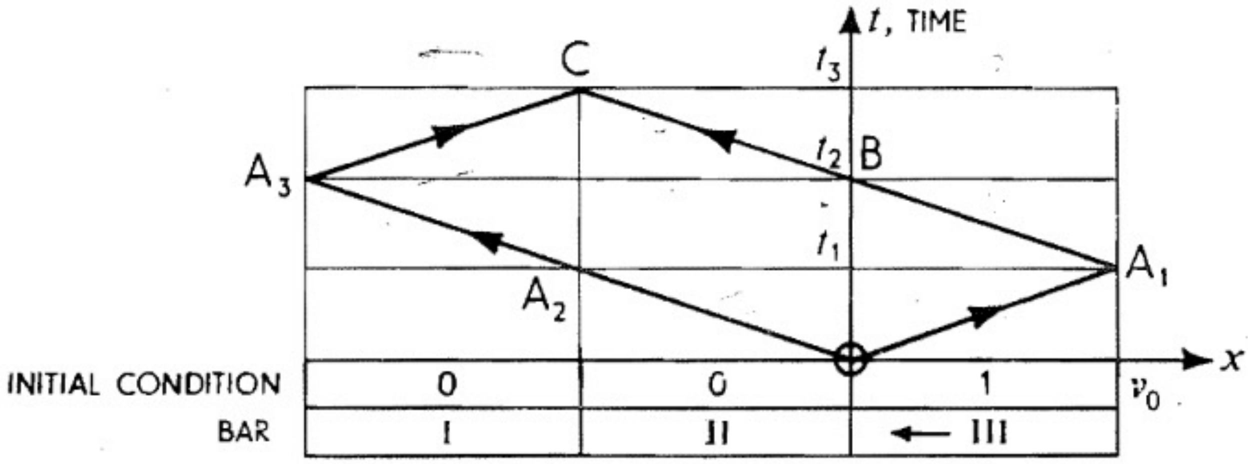
Space-time diagram

- Two identical bars:



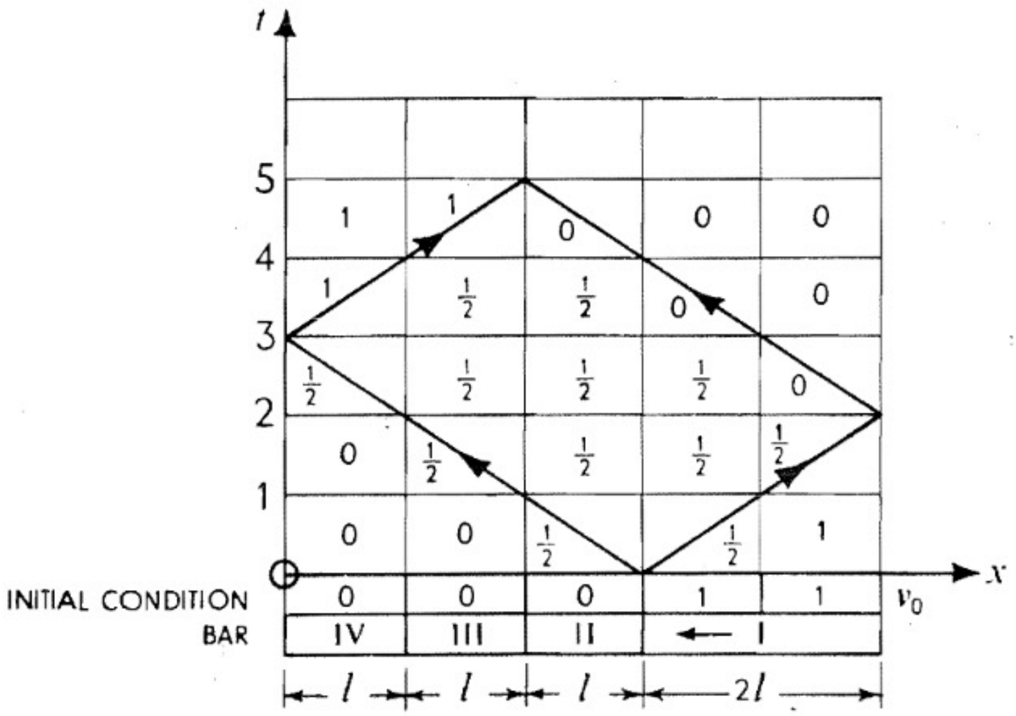
Space-time diagram

- Three identical bar:

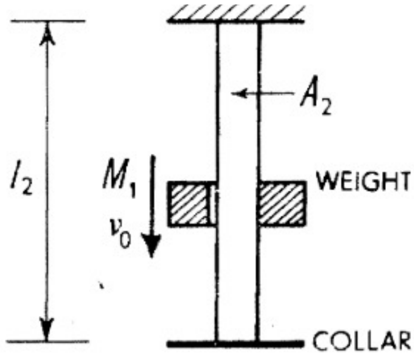


Space-time diagram

- Four bars:



Falling weight



- Assuming that all the kinetic energy of M_1 is absorbed as uniformly distributed strain energy in the rod:

$$A_2 l_2 \cdot \frac{\sigma^2}{2E_2} = \frac{1}{2} M_1 v_0^2$$

$$\sigma^2 = \frac{M_1 v_0^2}{M_2/\rho_2} \cdot E_2 = \frac{M_1}{M_2} \cdot v_0^2 \cdot E_2 \rho_2$$

Falling weight

- The dynamic stress at impact:

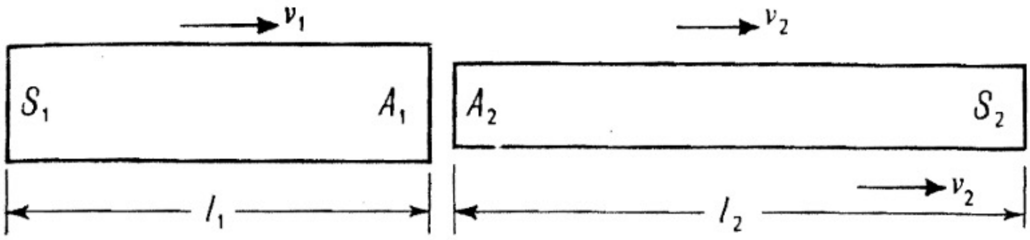
$$\sigma_0 = \rho_2 v_0 \sqrt{\frac{E_2}{\rho_2}} \text{ or } \sigma_0^2 = \rho_2 v_0^2 E_2$$

  - σ_0 is independent of M_1

$$\frac{\sigma}{\sigma_0} = \sqrt{\frac{M_1}{M_2}} = \sqrt{\frac{\text{Striker mass}}{\text{Struck rod mass}}}$$

  - No matter how small M_1 , for a given v_0 the stress σ_0 would be generated at impact.

Arbitrary bars



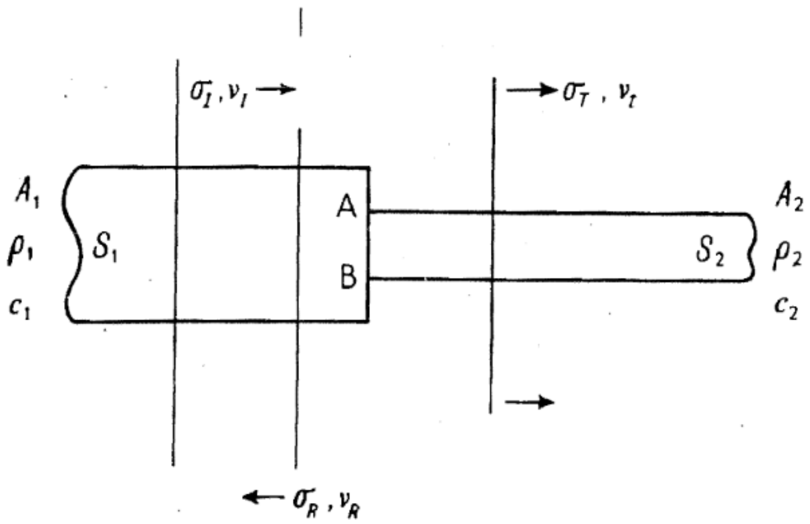
- The forces at the interface are equal: $A_1\sigma_1 = A_2\sigma_2$
- Thus $A_1\rho_1 c_1(v_1 - u_0) = A_2\rho_2 c_2(u_0 - v_2)$
- So
$$u_0 = \frac{v_2 + \frac{A_1\rho_1 c_1}{A_2\rho_2 c_2}\cdot v_1}{1+\frac{A_1\rho_1 c_1}{A_2\rho_2 c_2}}$$
- The stresses would be
$$\sigma_1 = \frac{\rho_1 c_1 v_1}{\left(1+\frac{A_1\rho_1 c_1}{A_2\rho_2 c_2}\right)}\cdot\left[1-\frac{v_2}{v_1}\right]$$
$$\sigma_2 = \frac{\rho_2 c_2 v_2}{\left(1+\frac{A_1\rho_1 c_1}{A_2\rho_2 c_2}\right)}\cdot\left(\frac{v_1}{v_2}-1\right)\cdot\frac{A_1\rho_1 c_1}{A_2\rho_2 c_2}$$

Arbitrary bars

- If $\rho_1 c_1 = \rho_2 c_2$, and assuming $\frac{A_1}{A_2} = \mu$
$$u_0 = \frac{v_2+\mu v_1}{1+\mu}$$
- For the case $u_0 = 0, l_1 = 2l_2, \rho_1 = \rho_2$, when S_2 is completely unloaded at time $t = 2l_2/c_2$, S_2 will be moving as a wholly unstressed bar having a translational speed of $(2u_0 - v_2)$.
- Coefficient of restitution:
$$2u_0 - v_2 = -e(v_2 - v_1)$$
$$e = \frac{-2\frac{(v_2+\mu v_1)}{(1+\mu)} + v_2}{v_2 - v_1} = \frac{\mu}{1+\mu}$$

Wave transmission in stepped and conical bars

Stepped shafts



- If A_2 was zero the wave would be wholly reflected whilst if S_1 and S_2 were identical the wave would be wholly transmitted.

Stepped shafts

- Equilibrium at the discontinuity:

$$A_1(\sigma_I + \sigma_R) = A_2\sigma_T$$

- Continuity of displacement:

$$v_I - v_R = v_T \quad \text{or} \quad \frac{\sigma_I}{\rho_1 c_1} - \frac{\sigma_R}{\rho_1 c_1} = \frac{\sigma_T}{\rho_2 c_2}$$

- Hence

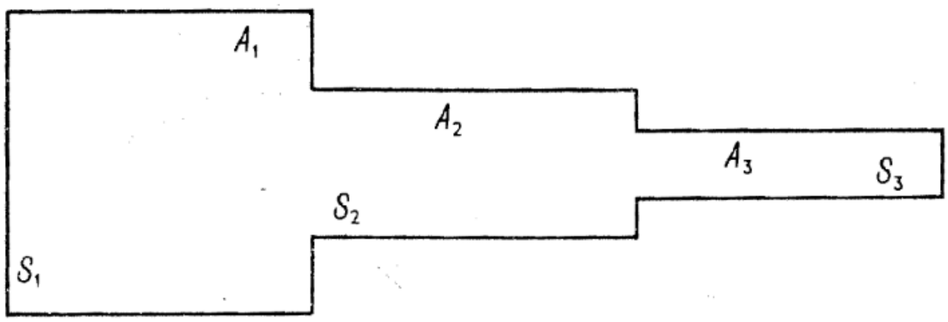
$$\sigma_T = \frac{2A_1\rho_2 c_2}{A_2\rho_2 c_2 + A_1\rho_1 c_1} \cdot \sigma_I$$
$$\sigma_R = \frac{A_2\rho_2 c_2 - A_1\rho_1 c_1}{A_2\rho_2 c_2 + A_1\rho_1 c_1} \cdot \sigma_I$$

- Note to the special cases!

Stepped shafts

- These results show that a small shaft on the end of larger one can act as a wave trap to a pulse or blow on the far end of the large shaft.
- For free end, the stress magnification factor is 2. (wave reflection).
- This intensification factor is not reduced by using a solid two-step shaft (at the end) but, on the contrary, is increased!

Stepped shafts



- Let $\frac{A_1}{A_3} = 4, \frac{A_1}{A_2} = \frac{A_2}{A_3} = 2$
- Transmitted stress in S_2: $\frac{\sigma_{T_2}}{\sigma_I} = \frac{2A_1}{A_1+A_2} = \frac{2\times2}{2+1} = \frac{4}{3}$
- S3: $\sigma_{T_3} = \frac{2A_2}{A_2+A_3} \cdot \sigma_{T_2} = \frac{2\times2}{2+1} \cdot \frac{4}{3}\sigma_I = 1 \cdot 78\sigma_I$
- If S_2 is omitted: $\sigma_{T_3} = \frac{2A_1}{A_1+A_3} \cdot \sigma_I = \frac{2\times4}{4+1} \cdot \sigma_I = 1 \cdot 60\sigma_I$

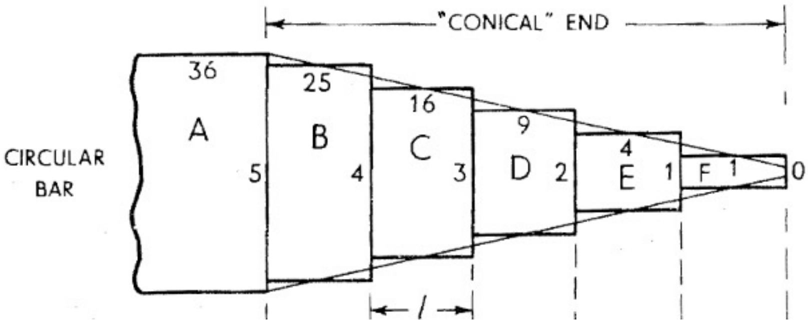
Stepped shafts

- For no wave to be reflected from the discontinuity in the bar we require $\sigma_R = 0$ and then $A_2\rho_2c_2 = A_1\rho_1c_1$ so that

$$\sigma_T = \sigma_I\sqrt{E_2\rho_2/E_1\rho_1}$$

- Ensuring that $A_2\rho_2c_2 = A_1\rho_1c_1$ is known as impedance matching.

Conical bar



- Assuming $\sigma_I = 1$, at $t = 0$ and at the interface 5:

$$\sigma_{T_5} = \frac{2\times 36}{25+36} \cdot 1 = 1.18$$

$$\sigma_{R_5} = \frac{25-36}{25+36} \cdot 1 = -0.18$$

Conical bar

- At $t = T$, $\sigma_{T\,5}$ reaches section 4:

$$\sigma_{T_4} = \frac{2\times 25}{16+25} \cdot 1.18 = 1.44$$

$$\sigma_{R_4} = \frac{16-25}{16+25} \cdot 1.18 = -0.26$$

- At $t = 2T$, $\sigma_{T\,4}$ will reach section 3:

$$\sigma_{T_3} = \frac{2\times 16}{9+16} \cdot 1.44 = 1.84$$

$$\sigma_{R_3} = \frac{9-16}{9+16} \cdot 1.44 = -0.403$$

- $\sigma_{R\,4}$ reaches section 5:

$$\sigma_{R_{4,5}} = \frac{36-25}{36+25} \cdot (-0.26) = -0.047$$

Conical bar

- At $t = 3T$, σ_{T3} reaches section 2:

$$\sigma_{T_2} = \frac{2\times 9}{4+9}\cdot 1.84 = 2.55$$

- $\sigma_{R4,5}$ will reach section 4:

$$\sigma_{R_{4,5,4}} = \frac{2\times 25}{25+16}\cdot(-0.047) = -0.0574$$

- σ_{R3} reaches section 4:

$$\sigma_{R_{3,4}} = \frac{25-16}{25+16}\cdot(-0.403) = -0.0885$$

- The total intensity of stress proceeding across section 4 is (-0.0574) + (-0.0885) = -0.146

Conical bar

- At $t = 4T$, at section 1:

$$\sigma_{T_1} = \frac{2\times 4}{1+4}\cdot 2.55 = 4.08$$

$$\sigma_{R_1} = \frac{1-4}{1+4}\cdot 2.55 = -1.53$$

- At section 3:

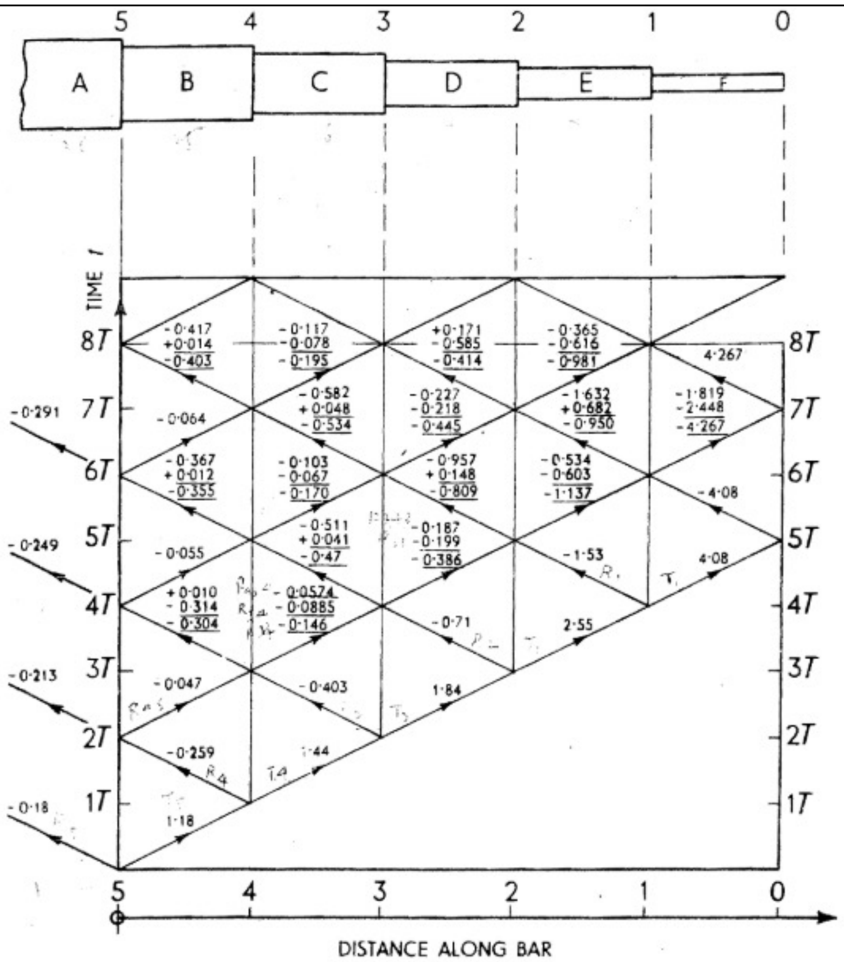
$$\sigma_{R_{2,3}} = \frac{16-9}{16+9}\cdot(-0.71) = -0.199$$

$$\sigma_{R_{3,4,3}} = \frac{2\times 16}{9+16}\cdot(-0.1461) = -0.187$$

- Thus the total intensity of stress two segment lengths behind the head of the successively transmitted initial unit pulse is (-0.199) + (-0.187) = -0.386

Conical bar

- It should be noted particularly that as the pulse travels towards the end of the bar, the stress intensity at the head increases, whilst a tail to the pulse is developed which has a smaller intensity but one of opposite sign.
- In a bar which possesses an end segment, F, of constant length, a high compressive stress will be attained and after reaching the end of the bar, it will be reflected as a tensile stress pulse of equal intensity.



Finite lateral restraint

- Consider a bar a finite length of which is wholly restrained against any lateral expansion.
- A rectangular compressive stress wave of intensity σ_0 is then reflected, σ_R, and transmitted, σ_T, at the boundary between the two regions:

$$\frac{\sigma_R}{\sigma_0} = \frac{c'/c-1}{c'/c+1} \text{ and } \frac{\sigma_T}{\sigma_0} = \frac{2(c'/c)}{c'/c+1},$$

- If the restraint applies over a finite length of bar, then on emerging from the restraint zone, reflected and transmitted waves will again be generated:

$$\frac{\sigma'_R}{\sigma_T} = \frac{c/c'-1}{c/c'+1} \quad \text{and thus} \quad \frac{\sigma'_R}{\sigma_0} = \frac{-2(c'/c)(c'/c-1)}{(c'/c+1)^2}$$

$$\frac{\sigma'_T}{\sigma_T} = \frac{2\cdot c/c'}{c/c'+1} \quad \text{and thus} \quad \frac{\sigma'_T}{\sigma_0} = \frac{2(c/c')}{c/c'+1}\cdot\frac{2(c'/c)}{c'/c+1} = \frac{4(c'/c)}{(c'/c+1)^2}$$