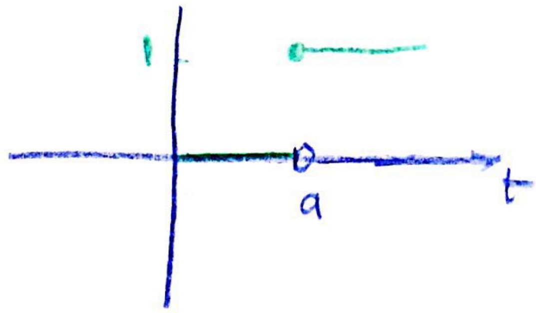
## P: A before 81

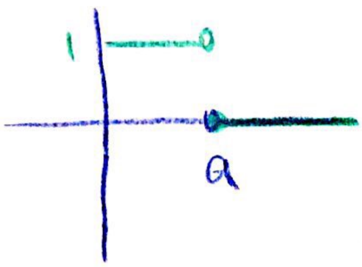
$$H(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases} \qquad a > 0$$

تابع پله‌ای واحد




Ex

$$1 - H(t-a) = \begin{cases} 1 & t < a \\ 0 & t \geq a \end{cases}$$




Ex $H(t-a) - H(t-b) = f(t) = ?$ $\quad a, b > 0 \quad a < b$

| t | | a | | b | |
|---|---|---|---|---|---|
| $H(t-a)$ | 0 | | 1 | | 1 |
| $H(t-b)$ | 0 | | 0 | | 1 |
| $f(t)$ | 0 | | 1 | | 0 |

$$f(t) = \begin{cases} 0 & t < a \\ 1 & a \leq t < b \\ 0 & t \geq b \end{cases}$$

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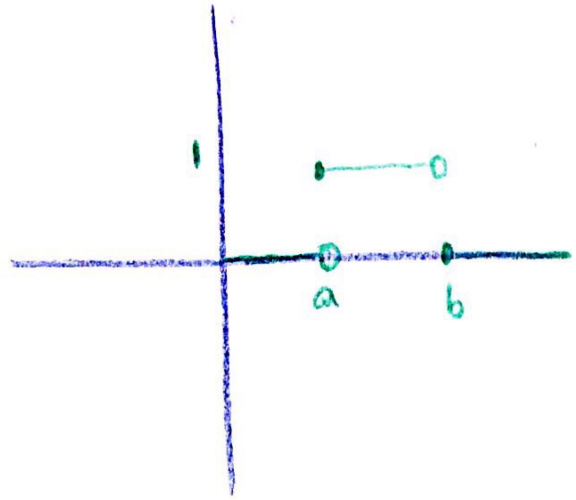


تبدیل لاپلاس تابع پله‌ای واحد

$$\mathcal{L}\{H(t-a)\} = \frac{e^{-as}}{s} \quad , \quad \mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\} = H(t-a)$$

زیرا

$$\mathcal{L}\{H(t-a)\} = \int_0^\infty e^{-st} H(t-a)\, dt$$

$$= \int_0^a e^{-st} \times 0\, dt + \int_a^\infty e^{-st}\, dt = \lim_{b \to \infty} \int_a^b e^{-st}\, dt$$

$$= \lim_{b \to \infty} \frac{-1}{s} e^{-st} \Big|_a^b = \lim_{b \to \infty} \frac{-1}{s} e^{-sb} + \frac{1}{s} e^{-as}$$

$$= 0 + \frac{1}{s} e^{-as} = \frac{1}{s} e^{-as}$$

| $f(t)$ | $F(s)$ |
|---|---|
| $H(t-a)$ | $\frac{e^{-as}}{s}$ |

Ex $\mathcal{L}\{H(t-1)\} = \frac{e^{-s}}{s}$.

Ex $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s}\right\} = H(t-\pi)$.

✱ قضیه دوم انتقال : فرض کنید $\mathcal{L}\{f(t)\} = F(s)$ در این صورت

$$\mathcal{L}\{H(t-a)f(t-a)\} = e^{-as}F(s).$$

یا

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = H(t-a)f(t-a).$$

Ex: تبدیل لاپلاس توابع زیر را بدست آورید.

(1) $f(t) = \begin{cases} 0 & 0 \le t \le \pi \\ \sin t & \pi \le t < 2\pi \\ 0 & t \geqslant 2\pi \end{cases}$

$\mathcal{L}\{f(t)\} = ?$

حل : می‌دانیم

$$H(t-\pi) - H(t-2\pi) = \begin{cases} 0 & 0 < t < \pi \\ 1 & \pi \le t < 2\pi \\ 0 & t \geqslant 2\pi \end{cases}$$

و

$f(t) = \sin t\,(H(t-\pi) - H(t-2\pi))$,

$\mathcal{L}\{f(t)\} = \mathcal{L}\{(\sin t)H(t-\pi)\} - \mathcal{L}\{(\sin t)H(t-2\pi)\}$

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$$\mathcal{L}\{f(t)\} = \mathcal{L}\{-\sin(t-\pi)\, H(t-\pi)\} - \mathcal{L}\{\sin(t-2\pi) H(t-2\pi)\}$$

$$= -e^{-\pi s}\frac{1}{s^2+1} - e^{-2\pi s}\frac{1}{s^2+1}$$

$$= \frac{-1}{s^2+1}\left(e^{-\pi s} + e^{-2\pi s}\right)$$

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② $g(t) = \begin{cases} t & 0 \le t < 1 \\ 2-t & 1 \le t < 2 \\ 1 & t \ge 2 \end{cases}$ , $\mathcal{L}\{g(t)\} = ?$

$$g(t) = t\begin{cases} 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases} + (2-t)\begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ 0 & t \ge 2 \end{cases}$$ :حل

$$+\begin{cases} 0 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \\ 1 & t \ge 2 \end{cases}$$

$$\Rightarrow g(t) = t\left(1 - H(t-1)\right) + (2-t)\left(H(t-1) - H(t-2)\right)$$
$$+ H(t-2)$$

$$\rightarrow g(t) = t - tH(t-1) + \cancel{2H(t-1)} - 2H(t-2) - \cancel{tH(t-1)} + tH(t-2)$$
$$+ H(t-2)$$

$$\rightarrow g(t) = t - 2(t-1)H(t-1) + (t-2)H(t-2) + H(t-2)$$

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$$\mathcal{L}\{g(t)\} = \frac{1}{s^2} - 2e^{-s}\frac{1}{s^2} + e^{-2s}\frac{1}{s^2} + \frac{e^{-2\pi}}{s}.$$

③ $h(t) = (t^2+1)H(t-2)$ ; $\mathcal{L}\{h(t)\} = ?$

حل:

$$h(t) = (t^2+1)\,H(t-2) = \left((t-2)^2 + 4t - 3\right)H(t-2)$$

$$\mathcal{L}\{h(t)\} = \mathcal{L}\{(t-2)^2 H(t-2)\} + \mathcal{L}\{(4t-8+5)H(t-2)\}$$

$$= \frac{2e^{-2s}}{s^3} + \mathcal{L}\{4(t-2)H(t-2)\} + 5\mathcal{L}\{H(t-2)\}$$

$$= \frac{2e^{-2s}}{s^3} + 4\frac{e^{-2s}}{s^2} + 5\frac{e^{-2s}}{s}.$$

مثال) لاپلاس معکوس توابع زیر را حساب کنید

① $\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\} \overset{\text{انتقال}}{=} f(t-1)\,H(t-1)$

$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} \overset{\text{تقسیم بر s}}{=} \mathcal{L}^{-1}\left\{\frac{\frac{1}{s+1}}{s}\right\}$ خاصیت

$$= \int_0^t e^{-x}\,dx = -e^{-x}\Big|_0^t = -e^{-t}+1 = 1-e^{-t}.$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\} = \left(1-e^{-(t-1)}\right)H(t-1)$$

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② $$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^3+s}\right\} = f(t-\pi)\, H(t-\pi)$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^3+s}\right\}$$ جواب ما

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3+s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{s^2+1}}{s}\right\}$$

لاپلاس انتگرال

$$= \int_0^t \sin x\, dx = -\cos x\Big|_0^t = -\cos t + 1$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^3+1}\right\} = \big(1-\cos(t-\pi)\big)\, H(t-\pi)$$

$$= (1+\cos t)\, H(t-\pi).$$

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حل برخی مسائل مقدار اولیه با کمک تبدیل لاپلاس

مسائل مقدار اولیه زیر را حل کنید

① $$y' - y = t - 1 \; ; \; y(0) = 0$$

حل: دقت کنید که $y$ تابعی بر حسب $t$ است، $\mathcal{L}\{y(t)\} := Y(s)$ در نظر می‌گیریم. برای حل معادله داده از طرفین آن تبدیل می‌گیریم داریم

$$\underbrace{\mathcal{L}\{y'\}}_{\text{لاپلاس مشتق}} - \mathcal{L}\{y\} = \mathcal{L}\{t\} - \mathcal{L}\{1\}$$

$$sY(s) - y(0) - Y(s) = \frac{1}{s^2} - \frac{1}{s}.$$

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$$(S-1)Y(S) = \frac{1}{S^2} - \frac{1}{S}$$

$$\Rightarrow Y(S) = \frac{1}{S^2(S-1)} - \frac{1}{S(S-1)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{S^2(S-1)} \right\} - \mathcal{L}^{-1}\left\{ \frac{1}{S(S-1)} \right\}$$

$$\Rightarrow y(t) \overset{*}{=} \cancel{e^t} - t - 1 - \cancel{e^t} + 1 = -t \Longrightarrow y(t) = -t.$$

$$\boxed{\begin{aligned} &\mathcal{L}^{-1}\left\{ \frac{\frac{1}{S-1}}{S} \right\} = \int_0^t e^x\,dx = e^x\Big|_0^t = e^t - 1 \\ &\mathcal{L}^{-1}\left\{ \frac{\frac{1}{S(S-1)}}{S} \right\} = \int_0^t (e^x - 1)\,dx = e^x - x\Big|_0^t = e^t - t - 1 \end{aligned}}$$

* روش انتگرال

② $y'' - y' - 6y = 0 \qquad y(0) = 1, \quad y'(0) = -1$

حل:

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = 0$$

$$S^2 Y(S) - Sy(0) - y'(0) - (SY(S) - y(0)) - 6Y(S) = 0$$

$$(S^2 - S - 6)Y(S) - S + 1 + 1 = 0$$

$$Y(S) = \frac{S-2}{S^2 - S - 6} \Longrightarrow y(t) = \mathcal{L}^{-1}\left\{ \frac{S-2}{(S-3)(S+2)} \right\}$$

ادامه در صفحه بعد

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$$\frac{S-2}{(S-3)(S+2)} = \frac{A}{S-3} + \frac{B}{S+2}$$

$$\Rightarrow S-2 = A(S+2) + B(S-3)$$

$$\Rightarrow S-2 = (A+B)S + 2A - 3B$$

$$\xrightarrow{3} \begin{cases} 1 = A+B \\ -2 = 2A-3B \end{cases} \qquad A = 1/5, \quad B = 4/5$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{S-2}{(S-3)(S+2)} \right\} \stackrel{*}{=} \mathcal{L}^{-1}\left\{ \frac{\frac{1}{5}}{S-3} \right\} + \mathcal{L}^{-1}\left\{ \frac{\frac{4}{5}}{S+2} \right\}$$

$$y(t) = \frac{1}{5} e^{3t} + \frac{4}{4} e^{-2t}$$

③ $y'' - 2y' + 2y = e^{-t}$ $\qquad y(0) = 0;\ y'(0) = 1$

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \frac{1}{S+1}$$

$$\to S^2 Y(S) - S y(0) - y'(0) - 2(S Y(S) - y(0)) + 2Y(S) = \frac{1}{S+1}$$

$$\to (S^2 - 2S + 2)\, Y(S) = \frac{1}{1+S} + 1$$

$$\to (S^2 - 2S + 2)\, Y(S) = \frac{1+S+1}{S+1}$$

$$\to Y(S) = \frac{S+2}{(S+1)(S^2-2S+2)} \to \text{ok.}$$

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$$y(t) = \mathcal{L}^{-1}\left\{ \frac{S+2}{(S+1)(S^2-2S+2)} \right\}$$

$$\frac{S+2}{(S+1)(S^2-2S+2)} = \frac{A}{S+1} + \frac{BS+C}{S^2-2S+2}$$

$$\rightarrow S+2 = A(S^2-2S+2) + (BS+C)(S+1)$$

$$\rightarrow S+2 = (A+B)S^2 + (-2A+B+C)S + 2A + C$$

$$\rightarrow \begin{cases} 0 = A+B \\ 1 = -2A+B+C \\ 2 = 2A+C \end{cases} \rightarrow \begin{cases} 1 = -3A+C \\ 2 = 2A+C \end{cases} \rightarrow A = 1/5,\ C = \frac{8}{5}$$

$$B = -\frac{1}{5}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{1/5}{S+1} \right\} + \mathcal{L}^{-1}\left\{ \frac{-\frac{1}{5}S + \frac{8}{5}}{(S-1)^2+1} \right\}$$

$$y(t) = \frac{1}{5}e^{-t} - \frac{1}{5}\mathcal{L}^{-1}\left\{ \frac{S-8}{(S-1)^2+1} \right\}$$

$$y(t) = \frac{1}{5}e^{-t} - \frac{1}{5}\mathcal{L}^{-1}\left\{ \frac{S-1-7}{(S-1)^2+1} \right\}$$

$$y(t) = \frac{1}{5}e^{-t} - \frac{1}{5}\mathcal{L}^{-1}\left\{ \frac{S-1}{(S-1)^2+1} \right\} + \frac{7}{5}\mathcal{L}^{-1}\left\{ \frac{1}{(S-1)^2+1} \right\}$$

$$y(t) = \frac{1}{5}e^{-t} - \frac{1}{5}e^{t}\cos t + \frac{7}{5}e^{t}\sin t.$$