
MATHEMATICAL PRINCIPLES
OF
NATURAL PHILOSOPHY¹

Definitions

DEFINITION I

The quantity of matter is the measure of the same, arising from its density and bulk conjointly.²

THUS AIR of a double density, in a double space, is quadruple in quantity; in a triple space, sextuple in quantity. The same thing is to be understood of snow, and fine dust or powders, that are condensed by compression or liquefaction, and of all bodies that are by any causes whatever differently condensed. I have no regard in this place to a medium, if any such there is, that freely pervades the interstices between the parts of bodies. It is this quantity that I mean hereafter everywhere under the name of body or mass. And the same is known by the weight of each body, for it is proportional to the weight, as I have found by experiments on pendulums, very accurately made, which shall be shown hereafter.

DEFINITION II³

The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly.

The motion of the whole is the sum of the motions of all the parts; and therefore in a body double in quantity, with equal velocity, the motion is double; with twice the velocity, it is quadruple.

[¹ Appendix, Note 10.] [² Appendix, Note 11.] [³ Appendix, Note 12.]

DEFINITION III

The vis insita, or innate force of matter, is a power of resisting, by which every body, as much as in it lies, continues in its present state, whether it be of rest, or of moving uniformly forwards in a right line.

This force is always proportional to the body whose force it is and differs nothing from the inactivity of the mass, but in our manner of conceiving it. A body, from the inert nature of matter, is not without difficulty put out of its state of rest or motion. Upon which account, this *vis insita* may, by a most significant name, be called inertia (*vis inertiae*) or force of inactivity. But a body only exerts this force when another force, impressed upon it, endeavors to change its condition; and the exercise of this force may be considered as both resistance and impulse; it is resistance so far as the body, for maintaining its present state, opposes the force impressed; it is impulse so far as the body, by not easily giving way to the impressed force of another, endeavors to change the state of that other. Resistance is usually ascribed to bodies at rest, and impulse to those in motion; but motion and rest, as commonly conceived, are only relatively distinguished; nor are those bodies always truly at rest, which commonly are taken to be so.

DEFINITION IV

An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of uniform motion in a right line.

This force consists in the action only, and remains no longer in the body when the action is over. For a body maintains every new state it acquires, by its inertia only. But impressed forces are of different origins, as from percussion, from pressure, from centripetal force.

DEFINITION V

A centripetal force is that by which bodies are drawn or impelled, or any way tend, towards a point as to a centre.

Of this sort is gravity, by which bodies tend to the centre of the earth; magnetism, by which iron tends to the loadstone; and that force, whatever it is, by which the planets are continually drawn aside from the rectilinear motions, which otherwise they would pursue, and made to revolve in curvilinear orbits. A stone, whirled about in a sling, endeavors to recede from the

hand that turns it; and by that endeavor, distends the sling, and that with so much the greater force, as it is revolved with the greater velocity, and as soon as it is let go, flies away. That force which opposes itself to this endeavor, and by which the sling continually draws back the stone towards the hand, and retains it in its orbit, because it is directed to the hand as the centre of the orbit, I call the centripetal force. And the same thing is to be understood of all bodies, revolved in any orbits. They all endeavor to recede from the centres of their orbits; and were it not for the opposition of a contrary force which restrains them to, and detains them in their orbits, which I therefore call centripetal, would fly off in right lines, with an uniform motion. A projectile, if it was not for the force of gravity, would not deviate towards the earth, but would go off from it in a right line, and that with an uniform motion, if the resistance of the air was taken away. It is by its gravity that it is drawn aside continually from its rectilinear course, and made to deviate towards the earth, more or less, according to the force of its gravity, and the velocity of its motion. The less its gravity is, or the quantity of its matter, or the greater the velocity with which it is projected, the less will it deviate from a rectilinear course, and the farther it will go. If a leaden ball, projected from the top of a mountain by the force of gunpowder, with a given velocity, and in a direction parallel to the horizon, is carried in a curved line to the distance of two miles before it falls to the ground; the same, if the resistance of the air were taken away, with a double or decuple velocity, would fly twice or ten times as far. And by increasing the velocity, we may at pleasure increase the distance to which it might be projected, and diminish the curvature of the line which it might describe, till at last it should fall at the distance of 10, 30, or 90 degrees, or even might go quite round the whole earth before it falls; or lastly, so that it might never fall to the earth, but go forwards into the celestial spaces, and proceed in its motion *in infinitum*. And after the same manner that a projectile, by the force of gravity, may be made to revolve in an orbit, and go round the whole earth, the moon also, either by the force of gravity, if it is endued with gravity, or by any other force, that impels it towards the earth, may be continually drawn aside towards the earth, out of the rectilinear way which by its innate force it would pursue; and would be made to revolve in the orbit which it now describes; nor could the moon without some such force be retained in its

orbit. If this force was too small, it would not sufficiently turn the moon out of a rectilinear course; if it was too great, it would turn it too much, and draw down the moon from its orbit towards the earth. It is necessary that the force be of a just quantity, and it belongs to the mathematicians to find the force that may serve exactly to retain a body in a given orbit with a given velocity; and *vice versa*, to determine the curvilinear way into which a body projected from a given place, with a given velocity, may be made to deviate from its natural rectilinear way, by means of a given force.

The quantity of any centripetal force may be considered as of three kinds: absolute, accelerative, and motive.

DEFINITION VI

The absolute quantity of a centripetal force is the measure of the same, proportional to the efficacy of the cause that propagates it from the centre, through the spaces round about.

Thus the magnetic force is greater in one loadstone and less in another, according to their sizes and strength of intensity.

DEFINITION VII

The accelerative quantity of a centripetal force is the measure of the same, proportional to the velocity which it generates in a given time.

Thus the force of the same loadstone is greater at a less distance, and less at a greater: also the force of gravity is greater in valleys, less on tops of exceeding high mountains; and yet less (as shall hereafter be shown), at greater distances from the body of the earth; but at equal distances, it is the same everywhere; because (taking away, or allowing for, the resistance of the air), it equally accelerates all falling bodies, whether heavy or light, great or small.

DEFINITION VIII

The motive quantity of a centripetal force is the measure of the same, proportional to the motion which it generates in a given time.

Thus the weight is greater in a greater body, less in a less body; and, in the same body, it is greater near to the earth, and less at remoter distances. This sort of quantity is the centripetency, or propension of the whole body towards the centre, or, as I may say, its weight; and it is always known by

the quantity of an equal and contrary force just sufficient to hinder the descent of the body.

These quantities of forces, we may, for the sake of brevity, call by the names of motive, accelerative, and absolute forces; and, for the sake of distinction, consider them with respect to the bodies that tend to the centre, to the places of those bodies, and to the centre of force towards which they tend; that is to say, I refer the motive force to the body as an endeavor and propensity of the whole towards a centre, arising from the propensities of the several parts taken together; the accelerative force to the place of the body, as a certain power diffused from the centre to all places around to move the bodies that are in them; and the absolute force to the centre, as endued with some cause, without which those motive forces would not be propagated through the spaces round about; whether that cause be some central body (such as is the magnet in the centre of the magnetic force, or the earth in the centre of the gravitating force), or anything else that does not yet appear. For I here design only to give a mathematical notion of those forces, without considering their physical causes and seats.

Wherefore the accelerative force will stand in the same relation to the motive, as celerity does to motion. For the quantity of motion arises from the celerity multiplied by the quantity of matter; and the motive force arises from the accelerative force multiplied by the same quantity of matter. For the sum of the actions of the accelerative force, upon the several particles of the body, is the motive force of the whole. Hence it is, that near the surface of the earth, where the accelerative gravity, or force productive of gravity, in all bodies is the same, the motive gravity or the weight is as the body; but if we should ascend to higher regions, where the accelerative gravity is less, the weight would be equally diminished, and would always be as the product of the body, by the accelerative gravity. So in those regions, where the accelerative gravity is diminished into one-half, the weight of a body two or three times less, will be four or six times less.

I likewise call attractions and impulses, in the same sense, accelerative, and motive; and use the words attraction, impulse, or propensity of any sort towards a centre, promiscuously, and indifferently, one for another; considering those forces not physically, but mathematically: wherefore the reader is not to imagine that by those words I anywhere take upon me to

define the kind, or the manner of any action, the causes or the physical reason thereof, or that I attribute forces, in a true and physical sense, to certain centres (which are only mathematical points); when at any time I happen to speak of centres as attracting, or as endued with attractive powers.

SCHOLIUM¹

Hitherto I have laid down the definitions of such words as are less known, and explained the sense in which I would have them to be understood in the following discourse. I do not define time, space, place, and motion, as being well known to all. Only I must observe, that the common people conceive those quantities under no other notions but from the relation they bear to sensible objects. And thence arise certain prejudices, for the removing of which it will be convenient to distinguish them into absolute and relative, true and apparent, mathematical and common.

I. Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year.

II. Absolute space, in its own nature, without relation to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies; and which is commonly taken for immovable space; such is the dimension of a subterraneous, an aerial, or celestial space, determined by its position in respect of the earth. Absolute and relative space are the same in figure and magnitude; but they do not remain always numerically the same. For if the earth, for instance, moves, a space of our air, which relatively and in respect of the earth remains always the same, will at one time be one part of the absolute space into which the air passes; at another time it will be another part of the same, and so, absolutely understood, it will be continually changed.

III. Place is a part of space which a body takes up, and is according to the space, either absolute or relative. I say, a part of space; not the situation, nor the external surface of the body. For the places of equal solids are always

[¹ Appendix, Note 13.]

equal; but their surfaces, by reason of their dissimilar figures, are often unequal. Positions properly have no quantity, nor are they so much the places themselves, as the properties of places. The motion of the whole is the same with the sum of the motions of the parts; that is, the translation of the whole, out of its place, is the same thing with the sum of the translations of the parts out of their places; and therefore the place of the whole is the same as the sum of the places of the parts, and for that reason, it is internal, and in the whole body.

IV. Absolute motion is the translation of a body from one absolute place into another; and relative motion, the translation from one relative place into another. Thus in a ship under sail, the relative place of a body is that part of the ship which the body possesses; or that part of the cavity which the body fills, and which therefore moves together with the ship: and relative rest is the continuance of the body in the same part of the ship, or of its cavity. But real, absolute rest, is the continuance of the body in the same part of that immovable space, in which the ship itself, its cavity, and all that it contains, is moved. Wherefore, if the earth is really at rest, the body, which relatively rests in the ship, will really and absolutely move with the same velocity which the ship has on the earth. But if the earth also moves, the true and absolute motion of the body will arise, partly from the true motion of the earth, in immovable space, partly from the relative motion of the ship on the earth; and if the body moves also relatively in the ship, its true motion will arise, partly from the true motion of the earth, in immovable space, and partly from the relative motions as well of the ship on the earth, as of the body in the ship; and from these relative motions will arise the relative motion of the body on the earth. As if that part of the earth, where the ship is, was truly moved towards the east, with a velocity of 10010 parts; while the ship itself, with a fresh gale, and full sails, is carried towards the west, with a velocity expressed by 10 of those parts; but a sailor walks in the ship towards the east, with 1 part of the said velocity; then the sailor will be moved truly in immovable space towards the east, with a velocity of 10001 parts, and relatively on the earth towards the west, with a velocity of 9 of those parts.

Absolute time, in astronomy, is distinguished from relative, by the equation or correction of the apparent time. For the natural days are truly un-

equal, though they are commonly considered as equal, and used for a measure of time; astronomers correct this inequality that they may measure the celestial motions by a more accurate time. It may be, that there is no such thing as an equable motion, whereby time may be accurately measured. All motions may be accelerated and retarded, but the flowing of absolute time is not liable to any change. The duration or perseverance of the existence of things remains the same, whether the motions are swift or slow, or none at all: and therefore this duration ought to be distinguished from what are only sensible measures thereof; and from which we deduce it, by means of the astronomical equation. The necessity of this equation, for determining the times of a phenomenon, is evinced as well from the experiments of the pendulum clock, as by eclipses of the satellites of Jupiter.

As the order of the parts of time is immutable, so also is the order of the parts of space. Suppose those parts to be moved out of their places, and they will be moved (if the expression may be allowed) out of themselves. For times and spaces are, as it were, the places as well of themselves as of all other things. All things are placed in time as to order of succession; and in space as to order of situation. It is from their essence or nature that they are places; and that the primary places of things should be movable, is absurd. These are therefore the absolute places; and translations out of those places, are the only absolute motions.

But because the parts of space cannot be seen, or distinguished from one another by our senses, therefore in their stead we use sensible measures of them. For from the positions and distances of things from any body considered as immovable, we define all places; and then with respect to such places, we estimate all motions, considering bodies as transferred from some of those places into others. And so, instead of absolute places and motions, we use relative ones; and that without any inconvenience in common affairs; but in philosophical disquisitions, we ought to abstract from our senses, and consider things themselves, distinct from what are only sensible measures of them. For it may be that there is no body really at rest, to which the places and motions of others may be referred.

But we may distinguish rest and motion, absolute and relative, one from the other by their properties, causes, and effects. It is a property of rest, that bodies really at rest do rest in respect to one another. And therefore as it is

possible, that in the remote regions of the fixed stars, or perhaps far beyond them, there may be some body absolutely at rest; but impossible to know, from the position of bodies to one another in our regions, whether any of these do keep the same position to that remote body, it follows that absolute rest cannot be determined from the position of bodies in our regions.

It is a property of motion, that the parts, which retain given positions to their wholes, do partake of the motions of those wholes. For all the parts of revolving bodies endeavor to recede from the axis of motion; and the impetus of bodies moving forwards arises from the joint impetus of all the parts. Therefore, if surrounding bodies are moved, those that are relatively at rest within them will partake of their motion. Upon which account, the true and absolute motion of a body cannot be determined by the translation of it from those which only seem to rest; for the external bodies ought not only to appear at rest, but to be really at rest. For otherwise, all included bodies, besides their translation from near the surrounding ones, partake likewise of their true motions; and though that translation were not made, they would not be really at rest, but only seem to be so. For the surrounding bodies stand in the like relation to the surrounded as the exterior part of a whole does to the interior, or as the shell does to the kernel; but if the shell moves, the kernel will also move, as being part of the whole, without any removal from near the shell.

A property, near akin to the preceding, is this, that if a place is moved, whatever is placed therein moves along with it; and therefore a body, which is moved from a place in motion, partakes also of the motion of its place. Upon which account, all motions, from places in motion, are no other than parts of entire and absolute motions; and every entire motion is composed of the motion of the body out of its first place, and the motion of this place out of its place; and so on, until we come to some immovable place, as in the before-mentioned example of the sailor. Wherefore, entire and absolute motions can be no otherwise determined than by immovable places; and for that reason I did before refer those absolute motions to immovable places, but relative ones to movable places. Now no other places are immovable but those that, from infinity to infinity, do all retain the same given position one to another; and upon this account must ever remain unmoved; and do thereby constitute immovable space.

The causes by which true and relative motions are distinguished, one from the other, are the forces impressed upon bodies to generate motion. True motion is neither generated nor altered, but by some force impressed upon the body moved; but relative motion may be generated or altered without any force impressed upon the body. For it is sufficient only to impress some force on other bodies with which the former is compared, that by their giving way, that relation may be changed, in which the relative rest or motion of this other body did consist. Again, true motion suffers always some change from any force impressed upon the moving body; but relative motion does not necessarily undergo any change by such forces. For if the same forces are likewise impressed on those other bodies, with which the comparison is made, that the relative position may be preserved, then that condition will be preserved in which the relative motion consists. And therefore any relative motion may be changed when the true motion remains unaltered, and the relative may be preserved when the true suffers some change. Thus, true motion by no means consists in such relations.

The effects which distinguish absolute from relative motion are, the forces of receding from the axis of circular motion. For there are no such forces in a circular motion purely relative, but in a true and absolute circular motion, they are greater or less, according to the quantity of the motion. If a vessel, hung by a long cord, is so often turned about that the cord is strongly twisted, then filled with water, and held at rest together with the water; thereupon, by the sudden action of another force, it is whirled about the contrary way, and while the cord is untwisting itself, the vessel continues for some time in this motion; the surface of the water will at first be plain, as before the vessel began to move; but after that, the vessel, by gradually communicating its motion to the water, will make it begin sensibly to revolve, and recede by little and little from the middle, and ascend to the sides of the vessel, forming itself into a concave figure (as I have experienced), and the swifter the motion becomes, the higher will the water rise, till at last, performing its revolutions in the same times with the vessel, it becomes relatively at rest in it. This ascent of the water shows its endeavor to recede from the axis of its motion; and the true and absolute circular motion of the water, which is here directly contrary to the relative, becomes known, and may be measured by this endeavor. At first, when the relative

motion of the water in the vessel was greatest, it produced no endeavor to recede from the axis; the water showed no tendency to the circumference, nor any ascent towards the sides of the vessel, but remained of a plain surface, and therefore its true circular motion had not yet begun. But afterwards, when the relative motion of the water had decreased, the ascent thereof towards the sides of the vessel proved its endeavor to recede from the axis; and this endeavor showed the real circular motion of the water continually increasing, till it had acquired its greatest quantity, when the water rested relatively in the vessel. And therefore this endeavor does not depend upon any translation of the water in respect of the ambient bodies, nor can true circular motion be defined by such translation. There is only one real circular motion of any one revolving body, corresponding to only one power of endeavoring to recede from its axis of motion, as its proper and adequate effect; but relative motions, in one and the same body, are innumerable, according to the various relations it bears to external bodies, and, like other relations, are altogether destitute of any real effect, any otherwise than they may perhaps partake of that one only true motion. And therefore in their system who suppose that our heavens, revolving below the sphere of the fixed stars, carry the planets along with them; the several parts of those heavens, and the planets, which are indeed relatively at rest in their heavens, do yet really move. For they change their position one to another (which never happens to bodies truly at rest), and being carried together with their heavens, partake of their motions, and as parts of revolving wholes, endeavor to recede from the axis of their motions.

Wherefore relative quantities are not the quantities themselves, whose names they bear, but those sensible measures of them (either accurate or inaccurate), which are commonly used instead of the measured quantities themselves. And if the meaning of words is to be determined by their use, then by the names time, space, place, and motion, their [sensible] measures are properly to be understood; and the expression will be unusual, and purely mathematical, if the measured quantities themselves are meant. On this account, those violate the accuracy of language, which ought to be kept precise, who interpret these words for the measured quantities. Nor do those less defile the purity of mathematical and philosophical truths, who confound real quantities with their relations and sensible measures.

It is indeed a matter of great difficulty to discover, and effectually to distinguish, the true motions of particular bodies from the apparent; because the parts of that immovable space, in which those motions are performed, do by no means come under the observation of our senses. Yet the thing is not altogether desperate; for we have some arguments to guide us, partly from the apparent motions, which are the differences of the true motions; partly from the forces, which are the causes and effects of the true motions. For instance, if two globes, kept at a given distance one from the other by means of a cord that connects them, were revolved about their common centre of gravity, we might, from the tension of the cord, discover the endeavor of the globes to recede from the axis of their motion, and from thence we might compute the quantity of their circular motions. And then if any equal forces should be impressed at once on the alternate faces of the globes to augment or diminish their circular motions, from the increase or decrease of the tension of the cord, we might infer the increment or decrement of their motions; and thence would be found on what faces those forces ought to be impressed, that the motions of the globes might be most augmented; that is, we might discover their hindmost faces, or those which, in the circular motion, do follow. But the faces which follow being known, and consequently the opposite ones that precede, we should likewise know the determination of their motions. And thus we might find both the quantity and the determination of this circular motion, even in an immense vacuum, where there was nothing external or sensible with which the globes could be compared. But now, if in that space some remote bodies were placed that kept always a given position one to another, as the fixed stars do in our regions, we could not indeed determine from the relative translation of the globes among those bodies, whether the motion did belong to the globes or to the bodies. But if we observed the cord, and found that its tension was that very tension which the motions of the globes required, we might conclude the motion to be in the globes, and the bodies to be at rest; and then, lastly, from the translation of the globes among the bodies, we should find the determination of their motions. But how we are to obtain the true motions from their causes, effects, and apparent differences, and the converse, shall be explained more at large in the following treatise. For to this end it was that I composed it.

AXIOMS, OR LAWS OF MOTION¹

LAW I

Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

PROJECTILES continue in their motions, so far as they are not retarded by the resistance of the air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are continually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by the air. The greater bodies of the planets and comets, meeting with less resistance in freer spaces, preserve their motions both progressive and circular for a much longer time.

LAW II²

The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force be impressed altogether and at once, or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added to or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.

LAW III

To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also pressed by the

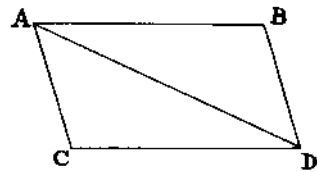
[¹ Appendix, Note 14.] [² Appendix, Note 15.]

stone. If a horse draws a stone tied to a rope, the horse (if I may so say) will be equally drawn back towards the stone; for the distended rope, by the same endeavor to relax or unbend itself, will draw the horse as much towards the stone as it does the stone towards the horse, and will obstruct the progress of the one as much as it advances that of the other. If a body impinge upon another, and by its force change the motion of the other, that body also (because of the equality of the mutual pressure) will undergo an equal change, in its own motion, towards the contrary part. The changes made by these actions are equal, not in the velocities but in the motions of bodies; that is to say, if the bodies are not hindered by any other impediments. For, because the motions are equally changed, the changes of the velocities made towards contrary parts are inversely proportional to the bodies. This law takes place also in attractions, as will be proved in the next Scholium.

COROLLARY I

A body, acted on by two forces simultaneously, will describe the diagonal of a parallelogram in the same time as it would describe the sides by those forces separately.

If a body in a given time, by the force M impressed apart in the place A, should with an uniform motion be carried from A to B, and by the force N impressed apart in the same place, should be carried from A to C, let the



parallelogram ABCD be completed, and, by both forces acting together, it will in the same time be carried in the diagonal from A to D. For since the force N acts in the direction of the line AC, parallel to BD, this force (by the second Law) will not at all alter the velocity generated by the other

force M, by which the body is carried towards the line BD. The body therefore will arrive at the line BD in the same time, whether the force N be impressed or not; and therefore at the end of that time it will be found somewhere in the line BD. By the same argument, at the end of the same time it will be found somewhere in the line CD. Therefore it will be found in the point D, where both lines meet. But it will move in a right line from A to D, by Law 1.

As these radii lie in the same right line they will be equipollent, and so remain in equilibrium; which is the well-known property of the balance, the lever, and the wheel. If either weight is greater than in this ratio, its force to move the wheel will be so much greater.

If the weight $p = P$, is partly suspended by the cord Np , partly sustained by the oblique plane pG ; draw pH , NH , the former perpendicular to the horizon, the latter to the plane pG ; and if the force of the weight p tending downwards is represented by the line pH , it may be resolved into the forces pN , HN . If there was any plane pQ , perpendicular to the cord pN , cutting the other plane pG in a line parallel to the horizon, and the weight p was supported only by those planes pQ , pG , it would press those planes perpendicularly with the forces pN , HN ; to wit, the plane pQ with the force pN , and the plane pG with the force HN . And therefore if the plane pQ was taken away, so that the weight might stretch the cord, because the cord, now sustaining the weight, supplied the place of the plane that was removed, it would be strained by the same force pN which pressed upon the plane before. Therefore, the

$$\text{tension of } pN : \text{tension of } PN = \text{line } pN : \text{line } pH.$$

Therefore, if

$$p : A = OK : OL = \text{line } pH : \text{line } pN,$$

then the weights p and A will have the same effect towards moving the wheel, and will therefore sustain each other; as anyone may find by experiment.

But the weight p pressing upon those two oblique planes, may be considered as a wedge between the two internal surfaces of a body split by it; and hence the forces of the wedge and the mallet may be determined: because the force with which the weight p presses the plane pQ is to the force with which the same, whether by its own gravity, or by the blow of a mallet, is impelled in the direction of the line pH towards both the planes, as

$$pN : pH;$$

and to the force with which it presses the other plane pG , as

$$pN : NH.$$

And thus the force of the screw may be deduced from a like resolution of forces; it being no other than a wedge impelled with the force of a lever. Therefore the use of this Corollary spreads far and wide, and by that diffusive extent the truth thereof is further confirmed. For on what has been said depends the whole doctrine of mechanics variously demonstrated by different authors. For from hence are easily deduced the forces of machines, which are compounded of wheels, pullics, levers, cords, and weights, ascending directly or obliquely, and other mechanical powers; as also the force of the tendons to move the bones of animals.

COROLLARY III

The quantity of motion, which is obtained by taking the sum of the motions directed towards the same parts, and the difference of those that are directed to contrary parts, suffers no change from the action of bodies among themselves.

For action and its opposite reaction are equal, by Law III, and therefore, by Law II, they produce in the motions equal changes towards opposite parts. Therefore if the motions are directed towards the same parts, whatever is added to the motion of the preceding body will be subtracted from the motion of that which follows; so that the sum will be the same as before. If the bodies meet, with contrary motions, there will be an equal deduction from the motions of both; and therefore the difference of the motions directed towards opposite parts will remain the same.

Thus, if a spherical body A is 3 times greater than the spherical body B, and has a velocity = 2, and B follows in the same direction with a velocity = 10, then the

$$\text{motion of A : motion of B} = 6 : 10.$$

Suppose, then, their motions to be of 6 parts and of 10 parts, and the sum will be 16 parts. Therefore, upon the meeting of the bodies, if A acquire 3, 4, or 5 parts of motion, B will lose as many; and therefore after reflection A will proceed with 9, 10, or 11 parts, and B with 7, 6, or 5 parts; the sum remaining always of 16 parts as before. If the body A acquire 9, 10, 11, or 12 parts of motion, and therefore after meeting proceed with 15, 16, 17, or

18 parts, the body B, losing so many parts as A has got, will either proceed with 1 part, having lost 9, or stop and remain at rest, as having lost its whole progressive motion of 10 parts; or it will go back with 1 part, having not only lost its whole motion, but (if I may so say) one part more; or it will go back with 2 parts, because a progressive motion of 12 parts is taken off. And so the sums of the conspiring motions,

$$15+1 \text{ or } 16+0,$$

and the differences of the contrary motions,

$$17-1 \text{ and } 18-2,$$

will always be equal to 16 parts, as they were before the meeting and reflection of the bodies. But the motions being known with which the bodies proceed after reflection, the velocity of either will be also known, by taking the velocity after to the velocity before reflection, as the motion after is to the motion before. As in the last case, where the

$$\begin{aligned} &\text{motion of A before reflection (6) : motion of A after (18)} \\ &= \text{velocity of A before (2) : velocity of A after (x)}; \end{aligned}$$

that is,

$$6 : 18 = 2 : x, x = 6.$$

But if the bodies are either not spherical, or, moving in different right lines, impinge obliquely one upon the other, and their motions after reflection are required, in those cases we are first to determine the position of the plane that touches the bodies in the point of impact, then the motion of each body (by Cor. 11) is to be resolved into two, one perpendicular to that plane, and the other parallel to it. This done, because the bodies act upon each other in the direction of a line perpendicular to this plane, the parallel motions are to be retained the same after reflection as before; and to the perpendicular motions we are to assign equal changes towards the contrary parts; in such manner that the sum of the conspiring and the difference of the contrary motions may remain the same as before. From such kind of reflections sometimes arise also the circular motions of bodies about their own centres. But these are cases which I do not consider in what follows; and it would be too tedious to demonstrate every particular case that relates to this subject.

COROLLARY IV

The common centre of gravity of two or more bodies does not alter its state of motion or rest by the actions of the bodies among themselves; and therefore the common centre of gravity of all bodies acting upon each other (excluding external actions and impediments) is either at rest, or moves uniformly in a right line.

For if two points proceed with an uniform motion in right lines, and their distance be divided in a given ratio, the dividing point will be either at rest, or proceed uniformly in a right line. This is demonstrated hereafter in Lem. xxiii and Corollary, when the points are moved in the same plane; and by a like way of arguing, it may be demonstrated when the points are not moved in the same plane. Therefore if any number of bodies move uniformly in right lines, the common centre of gravity of any two of them is either at rest, or proceeds uniformly in a right line; because the line which connects the centres of those two bodies so moving is divided at that common centre in a given ratio. In like manner the common centre of those two and that of a third body will be either at rest or moving uniformly in a right line; because at that centre the distance between the common centre of the two bodies, and the centre of this last, is divided in a given ratio. In like manner the common centre of these three, and of a fourth body, is either at rest, or moves uniformly in a right line; because the distance between the common centre of the three bodies, and the centre of the fourth, is there also divided in a given ratio, and so on *in infinitum*. Therefore, in a system of bodies where there is neither any mutual action among themselves, nor any foreign force impressed upon them from without, and which consequently move uniformly in right lines, the common centre of gravity of them all is either at rest or moves uniformly forwards in a right line.

Moreover, in a system of two bodies acting upon each other, since the distances between their centres and the common centre of gravity of both are reciprocally as the bodies, the relative motions of those bodies, whether of approaching to or of receding from that centre, will be equal among themselves. Therefore since the changes which happen to motions are equal and directed to contrary parts, the common centre of those bodies, by their mutual action between themselves, is neither accelerated nor retarded, nor

suffers any change as to its state of motion or rest. But in a system of several bodies, because the common centre of gravity of any two acting upon each other suffers no change in its state by that action; and much less the common centre of gravity of the others with which that action does not intervene; but the distance between those two centres is divided by the common centre of gravity of all the bodies into parts inversely proportional to the total sums of those bodies whose centres they are; and therefore while those two centres retain their state of motion or rest, the common centre of all does also retain its state: it is manifest that the common centre of all never suffers any change in the state of its motion or rest from the actions of any two bodies between themselves. But in such a system all the actions of the bodies among themselves either happen between two bodies, or are composed of actions interchanged between some two bodies; and therefore they do never produce any alteration in the common centre of all as to its state of motion or rest. Wherefore since that centre, when the bodies do not act one upon another, either is at rest or moves uniformly forwards in some right line, it will, notwithstanding the mutual actions of the bodies among themselves, always continue in its state, either of rest, or of proceeding uniformly in a right line, unless it is forced out of this state by the action of some power impressed from without upon the whole system. And therefore the same law takes place in a system consisting of many bodies as in one single body, with regard to their persevering in their state of motion or of rest. For the progressive motion, whether of one single body, or of a whole system of bodies, is always to be estimated from the motion of the centre of gravity.

COROLLARY V

The motions of bodies included in a given space are the same among themselves, whether that space is at rest, or moves uniformly forwards in a right line without any circular motion.

For the differences of the motions tending towards the same parts, and the sums of those that tend towards contrary parts, are, at first (by supposition), in both cases the same; and it is from those sums and differences that the collisions and impulses do arise with which the bodies impinge one upon another. Wherefore (by Law II), the effects of those collisions will be

equal in both cases; and therefore the mutual motions of the bodies among themselves in the one case will remain equal to the motions of the bodies among themselves in the other. A clear proof of this we have from the experiment of a ship; where all motions happen after the same manner, whether the ship is at rest, or is carried uniformly forwards in a right line.

COROLLARY VI

If bodies, moved in any manner among themselves, are urged in the direction of parallel lines by equal accelerative forces, they will all continue to move among themselves, after the same manner as if they had not been urged by those forces.

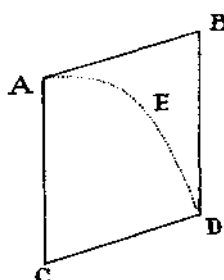
For these forces acting equally (with respect to the quantities of the bodies to be moved), and in the direction of parallel lines, will (by Law II) move all the bodies equally (as to velocity), and therefore will never produce any change in the positions or motions of the bodies among themselves.

SCHOLIUM¹

Hitherto I have laid down such principles as have been received by mathematicians, and are confirmed by abundance of experiments. By the first two Laws and the first two Corollaries, *Galileo* discovered that the descent of bodies varied as the square of the time (*in duplicata ratione temporis*) and that the motion of projectiles was in the curve of a parabola; experience agreeing with both, unless so far as these motions are a little retarded by the resistance of the air. When a body is falling, the uniform force of its gravity acting equally, impresses, in equal intervals of time, equal forces upon that body, and therefore generates equal velocities; and in the whole time impresses a whole force, and generates a whole velocity proportional to the time. And the spaces described in proportional times are as the product of the velocities and the times; that is, as the squares of the times. And when a body is thrown upwards, its uniform gravity impresses forces and reduces velocities proportional to the times; and the times of ascending to the greatest heights are as the velocities to be taken away, and those heights are as the product of the velocities and the times, or as the squares of the velocities. And if a body be projected in any direction, the motion arising from its projection is compounded with the motion arising from its gravity. Thus,

[¹ Appendix, Note 16.]

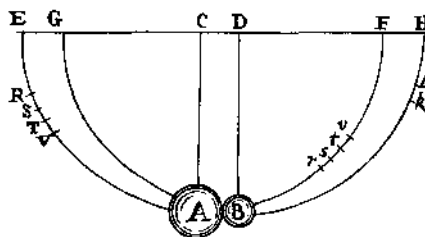
if the body *A* by its motion of projection alone could describe in a given time the right line *AB*, and with its motion of falling alone could describe in the same time the altitude *AC*; complete the parallelogram *ABCD*, and the body by that compounded motion will at the end of the time be found



in the place *D*; and the curved line *AED*, which that body describes, will be a parabola, to which the right line *AB* will be a tangent at *A*; and whose ordinate *BD* will be as the square of the line *AB*. On the same Laws and Corollaries depend those things which have been demonstrated concerning the times of the vibration of pendulums, and are confirmed by the daily experiments of pendulum clocks. By the same, together with Law III, Sir *Christopher Wren*, Dr. *Wallis*, and Mr. *Huygens*,

the greatest geometers of our times, did severally determine the rules of the impact and reflection of hard bodies, and about the same time communicated their discoveries to the *Royal Society*, exactly agreeing among themselves as to those rules. Dr. *Wallis*, indeed, was somewhat earlier in the publication; then followed Sir *Christopher Wren*, and, lastly, Mr. *Huygens*. But Sir *Christopher Wren* confirmed the truth of the thing before the *Royal Society* by the experiments on pendulums, which M. *Mariotte* soon after thought fit to explain in a treatise entirely upon that subject.¹ But to bring this experiment to an accurate agreement with the theory, we are to have due regard as well to the resistance of the air as to the elastic force of the concurring bodies.

Let the spherical bodies *A*, *B* be suspended by the parallel and equal strings *AC*, *BD*, from the centres *C*, *D*. About these centres, with those lengths as radii, describe the semicircles *EAF*, *GBH*, bisected respectively by the radii *CA*, *DB*. Bring the



body *A* to any point *R* of the arc *EAF*, and (withdrawing the body *B*) let it go from thence, and after one oscillation suppose it to return to the point

[¹ Appendix, Note 17.]

V: then RV will be the retardation arising from the resistance of the air. Of this RV let ST be a fourth part, situated in the middle, namely, so that

$$RS = TV,$$

and

$$RS : ST = 3 : 2,$$

then will ST represent very nearly the retardation during the descent from S to A. Restore the body B to its place: and, supposing the body A to be let fall from the point S, the velocity thereof in the place of reflection A, without sensible error, will be the same as if it had descended *in vacuo* from the point T. Upon which account this velocity may be represented by the chord of the arc TA. For it is a proposition well known to geometers, that the velocity of a pendulous body in the lowest point is as the chord of the arc which it has described in its descent. After reflection, suppose the body A comes to the place *s*, and the body B to the place *k*. Withdraw the body B, and find the place *v*, from which if the body A, being let go, should after one oscillation return to the place *r*, *st* may be a fourth part of *rv*, so placed in the middle thereof as to leave *rs* equal to *tv*, and let the chord of the arc *tA* represent the velocity which the body A had in the place A immediately after reflection. For *t* will be the true and correct place to which the body A should have ascended, if the resistance of the air had been taken off. In the same way we are to correct the place *k* to which the body B ascends, by finding the place *l* to which it should have ascended *in vacuo*. And thus everything may be subjected to experiment, in the same manner as if we were really placed *in vacuo*. These things being done, we are to take the product (if I may so say) of the body A, by the chord of the arc TA (which represents its velocity), that we may have its motion in the place A immediately before reflection; and then by the chord of the arc *tA*, that we may have its motion in the place A immediately after reflection. And so we are to take the product of the body B by the chord of the arc *Bl*, that we may have the motion of the same immediately after reflection. And in like manner, when two bodies are let go together from different places, we are to find the motion of each, as well before as after reflection; and then we may compare the motions between themselves, and collect the effects of the reflection. Thus trying the thing with pendulums of 10 feet, in unequal as well as equal bodies, and making the bodies to concur after a descent through large

spaces, as of 8, 12, or 16 feet, I found always, without an error of 3 inches, that when the bodies concurred together directly, equal changes towards the contrary parts were produced in their motions, and, of consequence, that the action and reaction were always equal. As if the body A impinged upon the body B at rest with 9 parts of motion, and losing 7, proceeded after reflection with 2, the body B was carried backwards with those 7 parts. If the bodies concurred with contrary motions, A with 12 parts of motion, and B with 6, then if A receded with 2, B receded with 8; namely, with a deduction of 14 parts of motion on each side. For from the motion of A subtracting 12 parts, nothing will remain; but subtracting 2 parts more, a motion will be generated of 2 parts towards the contrary way; and so, from the motion of the body B of 6 parts, subtracting 14 parts, a motion is generated of 8 parts towards the contrary way. But if the bodies were made both to move towards the same way, A, the swifter, with 14 parts of motion, B, the slower, with 5, and after reflection A went on with 5, B likewise went on with 14 parts; 9 parts being transferred from A to B. And so in other cases. By the meeting and collision of bodies, the quantity of motion, obtained from the sum of the motions directed towards the same way, or from the difference of those that were directed towards contrary ways, was never changed. For the error of an inch or two in measures may be easily ascribed to the difficulty of executing everything with accuracy. It was not easy to let go the two pendulums so exactly together that the bodies should impinge one upon the other in the lowermost place AB; nor to mark the places *s*, and *k*, to which the bodies ascended after impact. Nay, and some errors, too, might have happened from the unequal density of the parts of the pendulous bodies themselves, and from the irregularity of the texture proceeding from other causes.

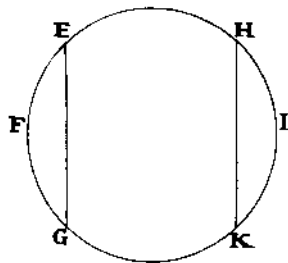
But to prevent an objection that may perhaps be alleged against the rule, for the proof of which this experiment was made, as if this rule did suppose that the bodies were either absolutely hard, or at least perfectly elastic (whereas no such bodies are to be found in Nature), I must add, that the experiments we have been describing, by no means depending upon that quality of hardness, do succeed as well in soft as in hard bodies. For if the rule is to be tried in bodies not perfectly hard, we are only to diminish the reflection in such a certain proportion as the quantity of the elastic force

requires. By the theory of *Wren* and *Huygens*, bodies absolutely hard return one from another with the same velocity with which they meet. But this may be affirmed with more certainty of bodies perfectly elastic. In bodies imperfectly elastic the velocity of the return is to be diminished together with the elastic force; because that force (except when the parts of bodies are bruised by their impact, or suffer some such extension as happens under the strokes of a hammer) is (as far as I can perceive) certain and determined, and makes the bodies to return one from the other with a relative velocity, which is in a given ratio to that relative velocity with which they met. This I tried in balls of wool, made up tightly, and strongly compressed. For, first, by letting go the pendulous bodies, and measuring their reflection, I determined the quantity of their elastic force; and then, according to this force, estimated the reflections that ought to happen in other cases of impact. And with this computation other experiments made afterwards did accordingly agree; the balls always receding one from the other with a relative velocity, which was to the relative velocity with which they met as about 5 to 9. Balls of steel returned with almost the same velocity; those of cork with a velocity something less; but in balls of glass the proportion was as about 15 to 16. And thus the third Law, so far as it regards percussions and reflections, is proved by a theory exactly agreeing with experience.

In attractions, I briefly demonstrate the thing after this manner. Suppose an obstacle is interposed to hinder the meeting of any two bodies A, B, attracting one the other: then if either body, as A, is more attracted towards the other body B, than that other body B is towards the first body A, the obstacle will be more strongly urged by the pressure of the body A than by the pressure of the body B, and therefore will not remain in equilibrium: but the stronger pressure will prevail, and will make the system of the two bodies, together with the obstacle, to move directly towards the parts on which B lies; and in free spaces, to go forwards *in infinitum* with a motion continually accelerated; which is absurd and contrary to the first Law. For, by the first Law, the system ought to continue in its state of rest, or of moving uniformly forwards in a right line; and therefore the bodies must equally press the obstacle, and be equally attracted one by the other. I made the experiment on the loadstone and iron. If these, placed apart in proper

vessels, are made to float by one another in standing water, neither of them will propel the other; but, by being equally attracted, they will sustain each other's pressure, and rest at last in an equilibrium.

So the gravitation between the earth and its parts is mutual. Let the earth FI be cut by any plane EG into two parts EGF and EGI, and their weights one towards the other will be mutually equal. For if by another plane HK, parallel to the former EG, the greater part EGI is cut into two parts EGKH and HKI, whereof HKI is equal to the part



EFG, first cut off, it is evident that the middle part EGKH will have no propension by its proper weight towards either side, but will hang as it were, and rest in an equilibrium between both. But the one extreme part HKI will with its whole weight bear upon and press the middle part towards the other extreme part EGF; and therefore the force with

which EGI, the sum of the parts HKI and EGKH, tends towards the third part EGF, is equal to the weight of the part HKI, that is, to the weight of the third part EGF. And therefore the weights of the two parts EGI and EGF, one towards the other, are equal, as I was to prove. And indeed if those weights were not equal, the whole earth floating in the nonresisting ether would give way to the greater weight, and, retiring from it, would be carried off *in infinitum*.

And as those bodies are equipollent in the impact and reflection, whose velocities are inversely as their innate forces, so in the use of mechanic instruments those agents are equipollent, and mutually sustain each the contrary pressure of the other, whose velocities, estimated according to the determination of the forces, are inversely as the forces.

So those weights are of equal force to move the arms of a balance, which during the play of the balance are inversely as their velocities upwards and downwards; that is, if the ascent or descent is direct, those weights are of equal force, which are inversely as the distances of the points at which they are suspended from the axis of the balance; but if they are turned aside by the interposition of oblique planes, or other obstacles, and made to ascend or descend obliquely, those bodies will be equipollent, which are inversely

as the heights of their ascent and descent taken according to the perpendicular; and that on account of the determination of gravity downwards.

And in like manner in the pulley, or in a combination of pulleys, the force of a hand drawing the rope directly, which is to the weight, whether ascending directly or obliquely, as the velocity of the perpendicular ascent of the weight to the velocity of the hand that draws the rope, will sustain the weight.

In clocks and such like instruments, made up from a combination of wheels, the contrary forces that promote and impede the motion of the wheels, if they are inversely as the velocities of the parts of the wheel on which they are impressed, will mutually sustain each other.

The force of the screw to press a body is to the force of the hand that turns the handles by which it is moved as the circular velocity of the handle in that part where it is impelled by the hand is to the progressive velocity of the screw towards the pressed body.

The forces by which the wedge presses or drives the two parts of the wood it cleaves are to the force of the mallet upon the wedge as the progress of the wedge in the direction of the force impressed upon it by the mallet is to the velocity with which the parts of the wood yield to the wedge, in the direction of lines perpendicular to the sides of the wedge. And the like account is to be given of all machines.

The power and use of machines consist only in this, that by diminishing the velocity we may augment the force, and the contrary; from whence, in all sorts of proper machines, we have the solution of this problem: *To move a given weight with a given power*, or with a given force to overcome any other given resistance. For if machines are so contrived that the velocities of the agent and resistant are inversely as their forces, the agent will just sustain the resistant, but with a greater disparity of velocity will overcome it. So that if the disparity of velocities is so great as to overcome all that resistance which commonly arises either from the friction of contiguous bodies as they slide by one another, or from the cohesion of continuous bodies that are to be separated, or from the weights of bodies to be raised, the excess of the force remaining, after all those resistances are overcome, will produce an acceleration of motion proportional thereto, as well in the parts of the machine as in the resisting body. But to treat of mechanics is

not my present business. I was aiming only to show by those examples the great extent and certainty of the third Law of Motion. For if we estimate the action of the agent from the product of its force and velocity, and likewise the reaction of the impediment from the product of the velocities of its several parts, and the forces of resistance arising from the friction, cohesion, weight, and acceleration of those parts, the action and reaction in the use of all sorts of machines will be found always equal to one another. And so far as the action is propagated by the intervening instruments, and at last impressed upon the resisting body, the ultimate action will be always contrary to the reaction.

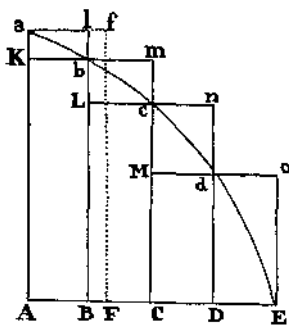
inscribed figure $AKbLcMdD$, the circumscribed figure $AalbmcndoE$, and curvilinear figure $AabcdE$, will have to one another, are ratios of equality.

For the difference of the inscribed and circumscribed figures is the sum of the parallelograms Kl , Lm , Mn , Do , that is (from the equality of all their bases), the rectangle under one of their bases Kb and the sum of their altitudes Aa , that is, the rectangle $ABla$. But this rectangle, because its breadth AB is supposed diminished *in infinitum*, becomes less than any given space. And therefore (by Lem. 1) the figures inscribed and circumscribed become ultimately equal one to the other; and much more will the intermediate curvilinear figure be ultimately equal to either. Q.E.D.

LEMMA III

The same ultimate ratios are also ratios of equality, when the breadths AB , BC , DC , &c., of the parallelograms are unequal, and are all diminished in infinitum.

For suppose AF equal to the greatest breadth, and complete the parallelogram $FAaf$. This parallelogram will be greater than the difference of the inscribed and circumscribed figures; but, because its breadth AF is diminished *in infinitum*, it will become less than any given rectangle. Q.E.D.



COR. I. Hence the ultimate sum of those evanescent parallelograms will in all parts coincide with the curvilinear figure.

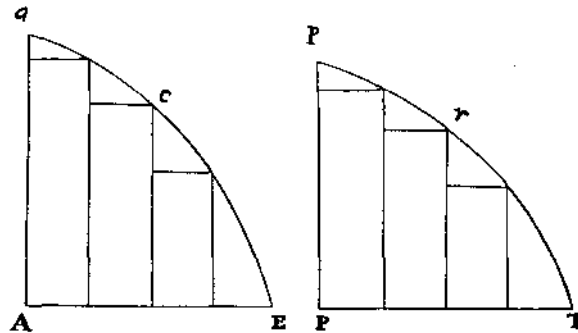
COR. II. Much more will the rectilinear figure comprehended under the chords of the evanescent arcs ab , bc , cd , &c., ultimately coincide with the curvilinear figure.

COR. III. And also the circumscribed rectilinear figure comprehended under the tangents of the same arcs.

COR. IV. And therefore these ultimate figures (as to their perimeters acE) are not rectilinear, but curvilinear limits of rectilinear figures.

LEMMA IV

If in two figures $AacE$, $PprT$, there are inscribed (as before) two series of parallelograms, an equal number in each series, and, their breadths being diminished in infinitum, if the ultimate ratios of the parallelograms in one figure to those in the other, each to each respectively, are the same: I say, that those two figures, $AacE$, $PprT$, are to each other in that same ratio.



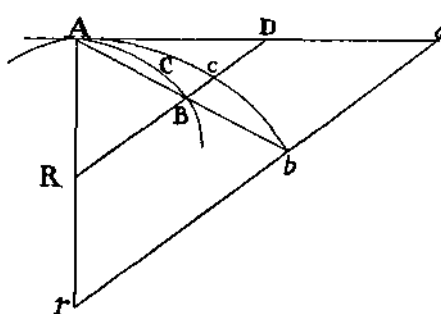
For as the parallelograms in the one are severally to the parallelograms in the other, so (by composition) is the sum of all in the one to the sum of all in the other; and so is the one figure to the other; because (by Lem. III) the former figure to the former sum, and the latter figure to the latter sum, are both in the ratio of equality. Q.E.D.

COR. Hence if two quantities of any kind are divided in any manner into an equal number of parts, and those parts, when their number is augmented, and their magnitude diminished *in infinitum*, have a given ratio to each other, the first to the first, the second to the second, and so on in order, all of them taken together will be to each other in that same given ratio. For if, in the figures of this Lemma, the parallelograms are taken to each other in the ratio of the parts, the sum of the parts will always be as the sum of the parallelograms; and therefore supposing the number of the parallelograms and parts to be augmented, and their magnitudes diminished *in infinitum*, those sums will be in the ultimate ratio of the parallelogram in the one figure to the correspondent parallelogram in the other; that is (by the supposition), in the ultimate ratio of any part of the one quantity to the correspondent part of the other.

LEMMA V

All homologous sides of similar figures, whether curvilinear or rectilinear, are proportional; and the areas are as the squares of the homologous sides.

LEMMA VI



If any arc ACB, given in position, is subtended by its chord AB, and in any point A, in the middle of the continued curvature, is touched by a right line AD, produced both ways; then if the points A and B approach one another and meet, I say, the angle BAD, contained between the chord and the tangent, will be diminished in infinitum, and ultimately will vanish.

For if that angle does not vanish, the arc ACB will contain with the tangent AD an angle equal to a rectilinear angle; and therefore the curvature at the point A will not be continued, which is against the supposition.

LEMMA VII

The same things being supposed, I say that the ultimate ratio of the arc, chord, and tangent, any one to any other, is the ratio of equality.

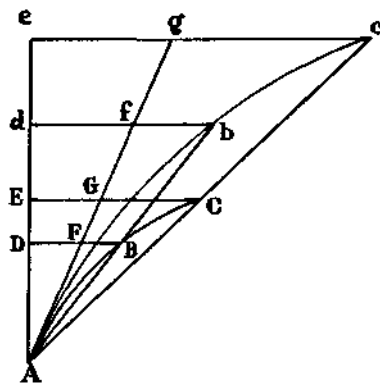
For while the point B approaches towards the point A, consider always AB and AD as produced to the remote points b and d ; and parallel to the secant BD draw bd ; and let the arc $Ac b$ be always similar to the arc ACB. Then, supposing the points A and B to coincide, the angle dAb will vanish, by the preceding Lemma; and therefore the right lines Ab , Ad (which are always finite), and the intermediate arc $Ac b$, will coincide, and become equal among themselves. Wherefore, the right lines AB, AD, and the intermediate arc ACB (which are always proportional to the former), will vanish, and ultimately acquire the ratio of equality. Q.E.D.

COR. I. Whence if through B we draw BF parallel to the tangent, always cutting any right line AF passing through A in F, this line BF will be

LEMMA IX

If a right line AE , and a curved line ABC , both given by position, cut each other in a given angle, A ; and to that right line, in another given angle, BD , CE are ordinately applied, meeting the curve in B , C ; and the points B and C together approach towards and meet in the point A : I say, that the areas of the triangles ABD , ACE , will ultimately be to each other as the squares of homologous sides.

For while the points B , C , approach towards the point A , suppose always AD to be produced to the remote points d and e , so as Ad , Ae may be proportional to AD , AE ; and the ordinates db , ec , to be drawn parallel to the



ordinates DB and EC , and meeting AB and AC produced in b and c . Let the curve Abc be similar to the curve ABC , and draw the right line Ag so as to touch both curves in A , and cut the ordinates DB , EC , db , ec , in F , G , f , g . Then, supposing the length Ae to remain the same, let the points B and C meet in the point A ; and the angle cAg vanishing, the curvilinear areas Abd , Ace will coincide with the rectilinear areas Afd , Age ; and

therefore (by Lem. v) will be one to the other in the duplicate ratio of the sides Ad , Ae . But the areas ABD , ACE are always proportional to these areas; and so the sides AD , AE are to these sides. And therefore the areas ABD , ACE are ultimately to each other as the squares of the sides AD , AE . Q.E.D.

LEMMA X

The spaces which a body describes by any finite force urging it, whether that force is determined and immutable, or is continually augmented or continually diminished, are in the very beginning of the motion to each other as the squares of the times.

Let the times be represented by the lines AD , AE , and the velocities generated in those times by the ordinates DB , EC . The spaces described with

these velocities will be as the areas ABD, ACE, described by those ordinates, that is, at the very beginning of the motion (by Lem. ix), in the duplicate ratio of the times AD, AE. Q.E.D.

COR. I. And hence one may easily infer, that the errors of bodies describing similar parts of similar figures in proportional times, the errors being generated by any equal forces similarly applied to the bodies, and measured by the distances of the bodies from those places of the similar figures, at which, without the action of those forces, the bodies would have arrived in those proportional times—are nearly as the squares of the times in which they are generated.

COR. II. But the errors that are generated by proportional forces, similarly applied to the bodies at similar parts of the similar figures, are as the product of the forces and the squares of the times.

COR. III. The same thing is to be understood of any spaces whatsoever described by bodies urged with different forces; all which, in the very beginning of the motion, are as the product of the forces and the squares of the times.

COR. IV. And therefore the forces are directly as the spaces described in the very beginning of the motion, and inversely as the squares of the times.

COR. V. And the squares of the times are directly as the spaces described, and inversely as the forces.

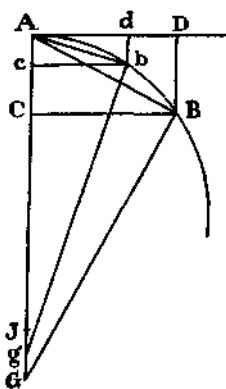
SCHOLIUM

If in comparing with each other indeterminate quantities of different sorts, any one is said to be directly or inversely as any other, the meaning is, that the former is augmented or diminished in the same ratio as the latter, or as its reciprocal. And if any one is said to be as any other two or more, directly or inversely, the meaning is, that the first is augmented or diminished in the ratio compounded of the ratios in which the others, or the reciprocals of the others, are augmented or diminished. Thus, if A is said to be as B directly, and C directly, and D inversely, the meaning is, that A is augmented or diminished in the same ratio as $B \cdot C \cdot \frac{1}{D}$, that is to say, that A and $\frac{BC}{D}$ are to each other in a given ratio.

LEMMA XI¹

The evanescent subtense of the angle of contact, in all curves which at the point of contact have a finite curvature, is ultimately as the square of the subtense of the conterminous arc.

CASE I. Let AB be that arc, AD its tangent, BD the subtense of the angle of contact perpendicular on the tangent, AB the subtense of the arc. Draw BG perpendicular to the subtense AB , and AG perpendicular to the tangent



AD , meeting in G ; then let the points D , B , and G approach to the points d , b , and g , and suppose J to be the ultimate intersection of the lines BG , AG , when the points D , B have come to A . It is evident that the distance GJ may be less than any assignable distance. But (from the nature of the circles passing through the points A , B , G , and through A , b , g),

$$AB^2 = AG \cdot BD, \text{ and}$$

$$Ab^2 = Ag \cdot bd.$$

But because GJ may be assumed of less length than any assignable, the ratio of AG to Ag may be such as to differ from unity by less than any assignable difference; and therefore the ratio of AB^2 to Ab^2 may be such as to differ from the ratio of BD to bd by less than any assignable difference. Therefore, by Lem. 1, ultimately,

$$AB^2 : Ab^2 = BD : bd.$$

Q.E.D.

CASE 2. Now let BD be inclined to AD in any given angle, and the ultimate ratio of BD to bd will always be the same as before, and therefore the same with the ratio of AB^2 to Ab^2 . Q.E.D.

CASE 3. And if we suppose the angle D not to be given, but that the right line BD converges to a given point, or is determined by any other condition whatever; nevertheless the angles D , d , being determined by the same law, will always draw nearer to equality, and approach nearer to each other than by any assigned difference, and therefore, by Lem. 1, will at last be equal; and therefore the lines BD , bd are in the same ratio to each other as before. Q.E.D.

[¹ Appendix, Note 17.]

And if DB be made successively as AD^2 , $AD^{3/2}$, $AD^{4/3}$, $AD^{5/4}$, $AD^{6/5}$, $AD^{7/6}$, &c., we shall have another infinite series of angles of contact, the first of which is of the same sort with those of circles, the second infinitely greater, and every succeeding one infinitely greater than the preceding. But between any two of these angles another series of intermediate angles of contact may be interposed, proceeding both ways *in infinitum*, wherein every succeeding angle shall be infinitely greater or infinitely less than the preceding. As if between the terms AD^2 and AD^3 there were interposed the series $AD^{13/6}$, $AD^{15/6}$, $AD^{17/6}$, $AD^{19/6}$, $AD^{21/6}$, $AD^{23/6}$, $AD^{25/6}$, $AD^{27/6}$, $AD^{29/6}$, $AD^{31/6}$, &c. And again, between any two angles of this series, a new series of intermediate angles may be interposed, differing from one another by infinite intervals. Nor is Nature confined to any bounds.

Those things which have been demonstrated of curved lines, and the surfaces which they comprehend, may be easily applied to the curved surfaces and contents of solids. These Lemmas are premised to avoid the tediousness of deducing involved demonstrations *ad absurdum*, according to the method of the ancient geometers. For demonstrations are shorter by the method of indivisibles; but because the hypothesis of indivisibles seems somewhat harsh, and therefore that method is reckoned less geometrical, I chose rather to reduce the demonstrations of the following Propositions to the first and last sums and ratios of nascent and evanescent quantities, that is, to the limits of those sums and ratios, and so to premise, as short as I could, the demonstrations of those limits. For hereby the same thing is performed as by the method of indivisibles; and now those principles being demonstrated, we may use them with greater safety. Therefore if hereafter I should happen to consider quantities as made up of particles, or should use little curved lines for right ones, I would not be understood to mean indivisibles, but evanescent divisible quantities; not the sums and ratios of determinate parts, but always the limits of sums and ratios; and that the force of such demonstrations always depends on the method laid down in the foregoing Lemmas.

Perhaps it may be objected, that there is no ultimate proportion of evanescent quantities; because the proportion, before the quantities have vanished, is not the ultimate, and when they are vanished, is none. But by the same argument it may be alleged that a body arriving at a certain place, and

there stopping, has no ultimate velocity; because the velocity, before the body comes to the place, is not its ultimate velocity; when it has arrived, there is none. But the answer is easy; for by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but at the very instant it arrives; that is, that velocity with which the body arrives at its last place, and with which the motion ceases. And in like manner, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities not before they vanish, nor afterwards, but with which they vanish. In like manner the first ratio of nascent quantities is that with which they begin to be. And the first or last sum is that with which they begin and cease to be (or to be augmented or diminished). There is a limit which the velocity at the end of the motion may attain, but not exceed. This is the ultimate velocity. And there is the like limit in all quantities and proportions that begin and cease to be. And since such limits are certain and definite, to determine the same is a problem strictly geometrical. But whatever is geometrical we may use in determining and demonstrating any other thing that is also geometrical.

It may also be objected, that if the ultimate ratios of evanescent quantities are given, their ultimate magnitudes will be also given: and so all quantities will consist of indivisibles, which is contrary to what *Euclid* has demonstrated concerning incommensurables, in the tenth Book of his *Elements*. But this objection is founded on a false supposition. For those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but limits towards which the ratios of quantities decreasing without limit do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *in infinitum*. This thing will appear more evident in quantities infinitely great. If two quantities, whose difference is given, be augmented *in infinitum*, the ultimate ratio of these quantities will be given, namely, the ratio of equality; but it does not from thence follow, that the ultimate or greatest quantities themselves, whose ratio that is, will be given. Therefore if in what follows, for the sake of being more easily understood, I should happen to mention quantities as least, or evanescent, or ultimate, you are not to suppose that quantities of any determinate magnitude are meant, but such as are conceived to be always diminished without end.

Draw cC parallel to BS , meeting BC in C ; and at the end of the second part of the time, the body (by Cor. 1 of the Laws) will be found in C , in the same plane with the triangle ASB . Join SC , and, because SB and Cc are parallel, the triangle SBC will be equal to the triangle Sbc , and therefore also to the triangle SAB . By the like argument, if the centripetal force acts successively in C , D , E , &c., and makes the body, in each single particle of time, to describe the right lines CD , DE , EF , &c., they will all lie in the same plane; and the triangle SCD will be equal to the triangle Sbc , and SDE to SCD , and SEF to SDE . And therefore, in equal times, equal areas are described in one immovable plane: and, by composition, any sums $SADS$, $SAFS$, of those areas, are to each other as the times in which they are described. Now let the number of those triangles be augmented, and their breadth diminished *in infinitum*; and (by Cor. iv, Lem. iii) their ultimate perimeter ADF will be a curved line: and therefore the centripetal force, by which the body is continually drawn back from the tangent of this curve, will act continually; and any described areas $SADS$, $SAFS$, which are always proportional to the times of description, will, in this case also, be proportional to those times. Q.E.D.

COR. I. The velocity of a body attracted towards an immovable centre, in spaces void of resistance, is inversely as the perpendicular let fall from that centre on the right line that touches the orbit. For the velocities in those places A , B , C , D , E , are as the bases AB , BC , CD , DE , EF , of equal triangles; and these bases are inversely as the perpendiculars let fall upon them.

COR. II. If the chords AB , BC of two arcs, successively described in equal times by the same body, in spaces void of resistance, are completed into a parallelogram $ABCV$, and the diagonal BV of this parallelogram, in the position which it ultimately acquires when those arcs are diminished *in infinitum*, is produced both ways, it will pass through the centre of force.

COR. III. If the chords AB , BC , and DE , EF , of arcs described in equal times, in spaces void of resistance, are completed into the parallelograms $ABCV$, $DEFZ$, the forces in B and E are one to the other in the ultimate ratio of the diagonals BV , EZ , when those arcs are diminished *in infinitum*. For the motions BC and EF of the body (by Cor. 1 of the Laws) are compounded of the motions Bc , BV , and Ef , EZ ; but BV and EZ , which are equal to Cc and Ff , in the demonstration of this Proposition, were generated

by the impulses of the centripetal force in B and E, and are therefore proportional to those impulses.

COR. IV. The forces by which bodies, in spaces void of resistance, are drawn back from rectilinear motions, and turned into curvilinear orbits, are to each other as the versed sines of arcs described in equal times; which versed sines tend to the centre of force, and bisect the chords when those arcs are diminished to infinity. For such versed sines are the halves of the diagonals mentioned in Cor. III.

COR. V. And therefore those forces are to the force of gravity as the said versed sines to the versed sines perpendicular to the horizon of those parabolic arcs which projectiles describe in the same time.

COR. VI. And the same things do all hold good (by Cor. v of the Laws) when the planes in which the bodies are moved, together with the centres of force which are placed in those planes, are not at rest, but move uniformly forwards in right lines.

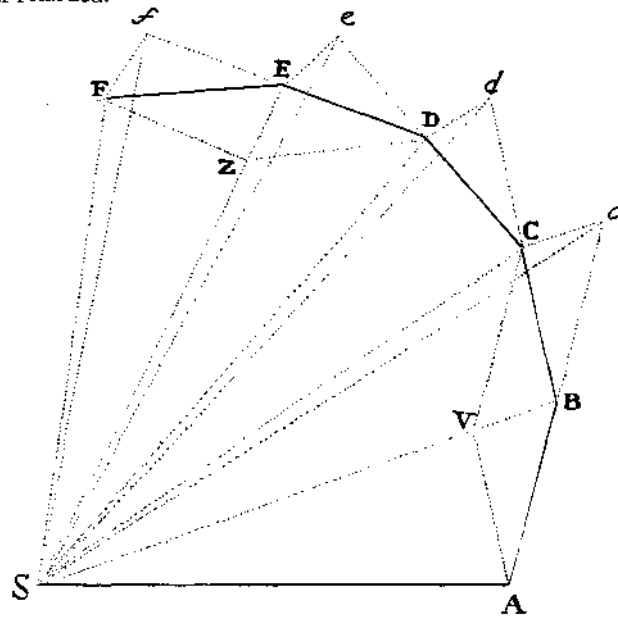
PROPOSITION II. THEOREM II

Every body that moves in any curved line described in a plane, and by a radius drawn to a point either immovable, or moving forwards with an uniform rectilinear motion, describes about that point areas proportional to the times, is urged by a centripetal force directed to that point.

CASE I. For every body that moves in a curved line is (by Law I) turned aside from its rectilinear course by the action of some force that impels it. And that force by which the body is turned off from its rectilinear course, and is made to describe, in equal times, the equal least triangles SAB, SBC, SCD, &c., about the immovable point S (by Prop. XI, Book I, *Elem. of Euclid*, and Law II), acts in the place B, according to the direction of a line parallel to cC , that is, in the direction of the line BS; and in the place C, according to the direction of a line parallel to dD , that is, in the direction of the line CS, &c.; and therefore acts always in the direction of lines tending to the immovable point S. Q.E.D.

CASE 2. And (by Cor. v of the Laws) it is indifferent whether the surface in which a body describes a curvilinear figure be at rest, or moves together with the body, the figure described, and its point S, uniformly forwards in a right line.

COR. I. In nonresisting spaces or mediums, if the areas are not proportional to the times, the forces are not directed to the point in which the radii meet, but deviate therefrom towards the part to which the motion is directed, if the description of the areas is accelerated, and away from that part, if retarded.



COR. II. And even in resisting mediums, if the description of the areas is accelerated, the directions of the forces deviate from the point in which the radii meet, towards the part to which the motion tends.

SCHOLIUM

A body may be urged by a centripetal force compounded of several forces; in which case the meaning of the Proposition is, that the force which results out of all tends to the point S. But if any force acts continually in the direction of lines perpendicular to the described surface, this force will make the body to deviate from the plane of its motion; but will neither augment nor diminish the area of the described surface, and is therefore to be neglected in the composition of forces.

PROPOSITION III. THEOREM III

Every body, that by a radius drawn to the centre of another body, howsoever moved, describes areas about that centre proportional to the times, is urged by a force compounded of the centripetal force tending to that other body, and of all the accelerative force by which that other body is impelled.

Let L represent the one, and T the other body; and (by Cor. vi of the Laws) if both bodies are urged in the direction of parallel lines, by a new force equal and contrary to that by which the second body T is urged, the first body L will go on to describe about the other body T the same areas as before: but the force by which that other body T was urged will be now destroyed by an equal and contrary force; and therefore (by Law 1) that other body T, now left to itself, will either rest, or move uniformly forwards in a right line: and the first body L, impelled by the difference of the forces, that is, by the force remaining, will go on to describe about the other body T areas proportional to the times. And therefore (by Theor. ii) the difference of the forces is directed to the other body T as its centre. Q.E.D.

COR. I. Hence if the one body L, by a radius drawn to the other body T, describes areas proportional to the times; and from the whole force, by which the first body L is urged (whether that force is simple, or, according to Cor. ii of the Laws, compounded out of several forces), we subtract (by the same Cor.) that whole accelerative force by which the other body is urged; the whole remaining force by which the first body is urged will tend to the other body T, as its centre.

COR. II. And, if these areas are proportional to the times nearly, the remaining force will tend to the other body T nearly.

COR. III. And *vice versa*, if the remaining force tends nearly to the other body T, those areas will be nearly proportional to the times.

COR. IV. If the body L, by a radius drawn to the other body T, describes areas, which, compared with the times, are very unequal; and that other body T be either at rest, or moves uniformly forwards in a right line: the action of the centripetal force tending to that other body T is either none at all, or it is mixed and compounded with very powerful actions of other forces: and the whole force compounded of them all, if they are many, is directed to another (immovable or movable) centre. The same thing

obtains, when the other body is moved by any motion whatsoever; provided that centripetal force is taken, which remains after subtracting that whole force acting upon that other body T.

SCHOLIUM

Since the equable description of areas indicates that there is a centre to which tends that force by which the body is most affected, and by which it is drawn back from its rectilinear motion, and retained in its orbit, why may we not be allowed, in the following discourse, to use the equable description of areas as an indication of a centre, about which all circular motion is performed in free spaces?

PROPOSITION IV. THEOREM IV¹

The centripetal forces of bodies, which by equable motions describe different circles, tend to the centres of the same circles; and are to each other as the squares of the arcs described in equal times divided respectively by the radii of the circles.

These forces tend to the centres of the circles (by Prop. II, and Cor. II, Prop. I), and are to one another as the versed sines of the least arcs described in equal times (by Cor. IV, Prop. I); that is, as the squares of the same arcs divided by the diameters of the circles (by Lem. VII); and therefore since those arcs are as arcs described in any equal times, and the diameters are as the radii, the forces will be as the squares of any arcs described in the same time divided by the radii of the circles. Q.E.D.

COR. I. Therefore, since those arcs are as the velocities of the bodies, the centripetal forces are as the squares of the velocities divided by the radii.

COR. II. And since the periodic times are as the radii divided by the velocities, the centripetal forces are as the radii divided by the square of the periodic times.

COR. III. Whence if the periodic times are equal, and the velocities therefore as the radii, the centripetal forces will be also as the radii; and conversely.

COR. IV. If the periodic times and the velocities are both as the square roots of the radii, the centripetal forces will be equal among themselves; and conversely.

[¹ Appendix. Note 15.]

COR. v. If the periodic times are as the radii, and therefore the velocities equal, the centripetal forces will be inversely as the radii; and conversely.

COR. vi. If the periodic times are as the $\frac{3}{2}$ th powers of the radii, and therefore the velocities inversely as the square roots of the radii, the centripetal forces will be inversely as the squares of the radii; and conversely.

COR. vii. And universally, if the periodic time is as any power R^n of the radius R , and therefore the velocity inversely as the power R^{n-1} of the radius, the centripetal force will be inversely as the power R^{2n-1} of the radius; and conversely.

COR. viii. The same things hold concerning the times, the velocities, and the forces by which bodies describe the similar parts of any similar figures that have their centres in a similar position with those figures; as appears by applying the demonstration of the preceding cases to those. And the application is easy, by only substituting the equable description of areas in the place of equable motion, and using the distances of the bodies from the centres instead of the radii.

COR. ix. From the same demonstration it likewise follows, that the arc which a body, uniformly revolving in a circle with a given centripetal force, describes in any time, is a mean proportional between the diameter of the circle, and the space which the same body falling by the same given force would describe in the same given time.

SCHOLIUM

The case of the sixth Corollary obtains in the celestial bodies (as Sir *Christopher Wren*, Dr. *Hooke*, and Dr. *Halley* have severally observed); and therefore in what follows, I intend to treat more at large of those things which relate to centripetal force decreasing as the squares of the distances from the centres.

Moreover, by means of the preceding Proposition and its Corollaries, we may discover the proportion of a centripetal force to any other known force, such as that of gravity. For if a body by means of its gravity revolves in a circle concentric to the earth, this gravity is the centripetal force of that body. But from the descent of heavy bodies, the time of one entire revolution, as well as the arc described in any given time, is given (by Cor. ix of this Prop.). And by such propositions, Mr. *Huygens*, in his excellent book

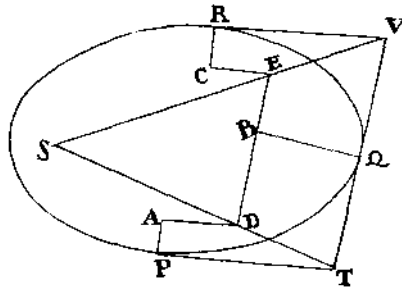
De horologio oscillatorio, has compared the force of gravity with the centrifugal forces of revolving bodies.

The preceding Proposition may be likewise demonstrated after this manner. In any circle suppose a polygon to be inscribed of any number of sides. And if a body, moved with a given velocity along the sides of the polygon, is reflected from the circle at the several angular points, the force, with which at every reflection it strikes the circle, will be as its velocity: and therefore the sum of the forces, in a given time, will be as the product of that velocity and the number of reflections; that is (if the species of the polygon be given), as the length described in that given time, and increased or diminished in the ratio of the same length to the radius of the circle; that is, as the square of that length divided by the radius; and therefore the polygon, by having its sides diminished *in infinitum*, coincides with the circle, as the square of the arc described in a given time divided by the radius. This is the centrifugal force, with which the body impels the circle; and to which the contrary force, wherewith the circle continually repels the body towards the centre, is equal.

PROPOSITION V. PROBLEM I

There being given, in any places, the velocity with which a body describes a given figure, by means of forces directed to some common centre: to find that centre.

Let the three right lines PT, TQV, VR touch the figure described in as many points, P, Q, R, and meet in T and V. On the tangents erect the perpendiculars PA, QB, RC, inversely proportional to the velocities of the body in the points P, Q, R, from which the perpendiculars were raised; that is, so that PA may be to QB as the velocity in Q to the velocity in P, and QB to RC as the velocity in R to the velocity in Q. Through the ends A, B, C of the perpendiculars draw



AD, DBE, EC, at right angles, meeting in D and E: and the right lines TD, VE produced, will meet in S, the centre required.

For the perpendiculars let fall from the centre S on the tangents PT, QT, are inversely as the velocities of the bodies in the points P and Q (by Cor. 1, Prop. 1), and therefore, by construction, directly as the perpendiculars AP, BQ; that is, as the perpendiculars let fall from the point D on the tangents. Whence it is easy to infer that the points S, D, T are in one right line. And by the like argument the points S, E, V are also in one right line; and therefore the centre S is in the point where the right lines TD, VE meet. Q.E.D.

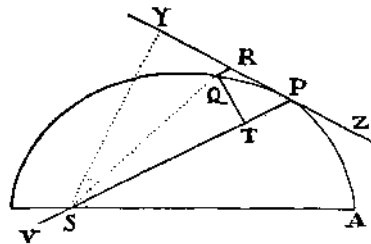
PROPOSITION VI. THEOREM V

In a space void of resistance, if a body revolves in any orbit about an immovable centre, and in the least time describes any arc just then nascent; and the versed sine of that arc is supposed to be drawn bisecting the chord, and produced passing through the centre of force: the centripetal force in the middle of the arc will be directly as the versed sine and inversely as the square of the time.

For the versed sine in a given time is as the force (by Cor. iv, Prop. 1); and augmenting the time in any ratio, because the arc will be augmented in the same ratio, the versed sine will be augmented in the square of that ratio (by Cor. ii and iii, Lem. xi), and therefore is as the force and the square of the time. Divide both sides by the square of the time, and the force will be directly as the versed sine, and inversely as the square of the time. Q.E.D.

And the same thing may also be easily demonstrated by Cor. iv, Lem. x.

COR. 1. If a body P revolving about the centre S describes a curved line APQ, which a right line ZPR touches in any point P; and from any other point Q of the curve, QR is drawn parallel to the distance SP, meeting the tangent in R; and QT is drawn perpendicular to the distance SP; the centripetal force will be inversely as the solid $\frac{SP^2 \cdot QT^2}{QR}$, if the solid be



taken of that magnitude which it ultimately acquires when the points P and Q coincide. For QR is equal to the versed sine of double the arc QP, whose middle is P: and double the triangle SQP, or $SP \cdot QT$ is proportional to the time in which that double arc is described; and therefore may be used to represent the time.

COR. II. By a like reasoning, the centripetal force is inversely as the solid $\frac{SY^2 \cdot QP^2}{QR}$; if SY is a perpendicular from the centre of force on PR, the tangent of the orbit. For the rectangles $SY \cdot QP$ and $SP \cdot QT$ are equal.

COR. III. If the orbit is either a circle, or touches or cuts a circle concentrically, that is, contains with a circle the least angle of contact or section, having the same curvature and the same radius of curvature at the point P; and if PV be a chord of this circle, drawn from the body through the centre of force; the centripetal force will be inversely as the solid $SY^2 \cdot PV$. For PV is $\frac{QP^2}{QR}$.

COR. IV. The same things being supposed, the centripetal force is as the square of the velocity directly, and that chord inversely. For the velocity is reciprocally as the perpendicular SY, by Cor. I, Prop. I.

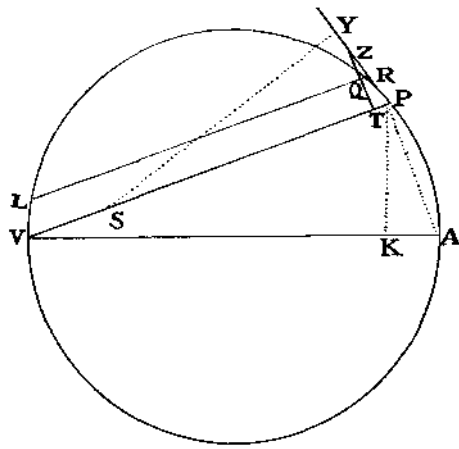
COR. V. Hence if any curvilinear figure APQ is given, and therein a point S is also given, to which a centripetal force is continually directed, that law of centripetal force may be found, by which the body P will be continually drawn back from a rectilinear course, and, being detained in the perimeter of that figure, will describe the same by a continual revolution. That is, we are to find, by computation, either the solid $\frac{SP^2 \cdot QT^2}{QR}$ or the solid $SY^2 \cdot PV$, inversely proportional to this force. Examples of this we shall give in the following Problems.

PROPOSITION VII. PROBLEM II

If a body revolves in the circumference of a circle, it is proposed to find the law of centripetal force directed to any given point.

Let VQPA be the circumference of the circle; S the given point to which as to a centre the force tends; P the body moving in the circumference; Q the next place into which it is to move; and PRZ the tangent of the circle

at the preceding place. Through the point S draw the chord PV , and the diameter VA of the circle; join AP , and draw QT perpendicular to SP , which produced, may meet the tangent PR in Z ; and lastly, through the



point Q , draw LR parallel to SP , meeting the circle in L , and the tangent PZ in R . And, because of the similar triangles ZQR , ZTP , VPA , we shall have

$$RP^2 : QT^2 = AV^2 : PV^2.$$

Since $RP^2 = RL \cdot QR$,

$$QT^2 = \frac{RL \cdot QR \cdot PV^2}{AV^2}.$$

Multiply those equals by $\frac{SP^2}{QR}$,

and the points P and Q coinciding, for RL write PV ; then we shall have

$$\frac{SP^2 \cdot PV^3}{AV^2} = \frac{SP^2 \cdot QT^2}{QR}.$$

And therefore (by Cor. 1 and v, Prop. vi) the centripetal force is inversely as $\frac{SP^2 \cdot PV^3}{AV^2}$; that is (because AV^2 is given), inversely as the product of SP^2 and PV^3 . Q.E.I.

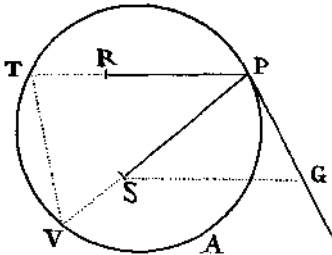
The same otherwise.

On the tangent PR produced let fall the perpendicular SY ; and (because of the similar triangles SYP , VPA) we shall have AV to PV as SP to SY , and therefore $\frac{SP \cdot PV}{AV} = SY$, and $\frac{SP^2 \cdot PV^3}{AV^2} = SY^2 \cdot PV$. And therefore (by

Cor. III and v, Prop. vi) the centripetal force is inversely as $\frac{SP^2 \cdot PV^3}{AV^2}$; that is (because AV is given), inversely as $SP^2 \cdot PV^3$. Q.E.I.

COR. 1. Hence if the given point S , to which the centripetal force always tends, is placed in the circumference of the circle, as at V , the centripetal force will be inversely as the fifth power of the altitude SP .

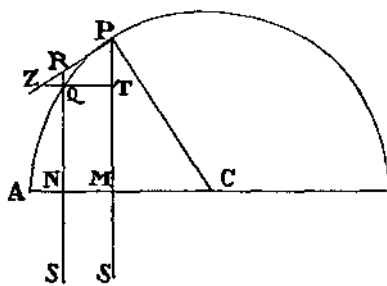
COR. II. The force by which the body P in the circle APTV revolves about the centre of force S is to the force by which the same body P may revolve in the same circle, and in the same periodic time, about any other centre of force R, as $RP^2 \cdot SP$ to the cube of the right line SG, which from the first centre of force S is drawn parallel to the distance PR of the body from the second centre of force R, meeting the tangent PG of the orbit in G. For by the construction



to the latter as $RP^2 \cdot PT^3$ to $SP^2 \cdot PV^2$; that is, as $SP \cdot RP^2$ to $\frac{SP^3 \cdot PV^2}{PT^3}$; or (because of the similar triangles PSG, TPV) to SG^3 .

COR. III. The force by which the body P in any orbit revolves about the centre of force S, is to the force by which the same body may revolve in the same orbit, and the same periodic time, about any other centre of force R, as the solid $SP \cdot RP^2$, contained under the distance of the body from the first centre of force S, and the square of its distance from the second centre of force R, to the cube of the right line SG, drawn from the first centre of the force S, parallel to the distance RP of the body from the second centre of force R, meeting the tangent PG of the orbit in G. For the force in this orbit at any point P is the same as in a circle of the same curvature.

PROPOSITION VIII. PROBLEM III



If a body moves in the semicircumference PQA; it is proposed to find the law of the centripetal force tending to a point S, so remote, that all the lines PS, RS drawn thereto, may be taken for parallels.

From C, the centre of the semicircle, let the semidiameter CA be drawn, cutting the parallels at right angles in M and N, and join CP. Because of the similar triangles CPM,

PZT, and RZQ, we shall have $CP^2 : PM^2 = PR^2 : QT^2$. From the nature of the circle, $PR^2 = QR(RN + QN) = QR \cdot 2PM$, when the points P and Q coincide. Therefore $CP^2 : PM^2 = QR \cdot 2PM : QT^2$; and $\frac{QT^2}{QR} = \frac{2PM^2}{CP^2}$, and $\frac{QT^2 \cdot SP^2}{QR} = \frac{2PM^2 \cdot SP^2}{CP^2}$. And therefore (by Cor. 1 and v, Prop. vi) the centripetal force is inversely as $\frac{2PM^2 \cdot SP^2}{CP^2}$; that is (neglecting the given ratio $\frac{2SP^2}{CP^2}$), inversely as PM^2 . Q. E. I.

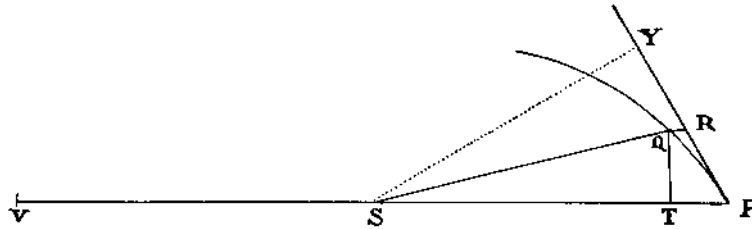
And the same thing is likewise easily inferred from the preceding Proposition.

SCHOLIUM

And by a like reasoning, a body will be moved in an ellipse, or even in an hyperbola, or parabola, by a centripetal force which is inversely as the cube of the ordinate directed to an infinitely remote centre of force.

PROPOSITION IX. PROBLEM IV

If a body revolves in a spiral PQS, cutting all the radii SP, SQ, &c., in a given angle; it is proposed to find the law of the centripetal force tending to the centre of that spiral.



Suppose the indefinitely small angle PSQ to be given; because, then, all the angles are given, the figure SPRQT will be given in kind. Therefore the ratio $\frac{QT}{QR}$ is also given, and $\frac{QT^2}{QR}$ is as QT, that is (because the figure is given in kind), as SP. But if the angle PSQ is any way changed, the right line QR, subtending the angle of contact QPR (by Lem. xi) will

be changed in the ratio of PR^2 or QT^2 . Therefore the ratio $\frac{QT^2}{QR}$ remains the same as before, that is, as SP . And $\frac{QT^2 \cdot SP^2}{QR}$ is as SP^3 , and therefore (by Cor. 1 and v, Prop. vi) the centripetal force is inversely as the cube of the distance SP . Q.E.I.

The same otherwise.

The perpendicular SY let fall upon the tangent, and the chord PV of the circle concentrically cutting the spiral, are in given ratios to the height SP ; and therefore SP^3 is as $SY^2 \cdot PV$, that is (by Cor. iii and v, Prop. vi) inversely as the centripetal force.

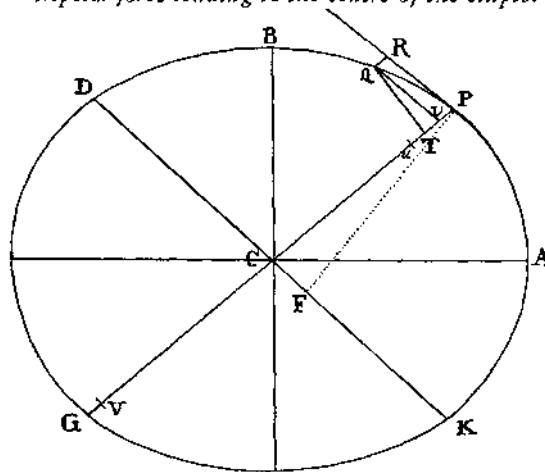
LEMMA XII

All parallelograms circumscribed about any conjugate diameters of a given ellipse or hyperbola are equal among themselves.

This is demonstrated by the writers on the conic sections.

PROPOSITION X. PROBLEM V

If a body revolves in an ellipse; it is proposed to find the law of the centripetal force tending to the centre of the ellipse.



Suppose CA, CB to be semiaxes of the ellipse; GP, DK , conjugate diameters; PF, QT , perpendiculars to those diameters; Qv , an ordinate to the

diameter GP; and if the parallelogram QvPR be completed, then (by the properties of the conic sections) $Pv \cdot vG : Qv^2 = PC^2 : CD^2$, and, because of the similar triangles QvT, PCF, $Qv^2 : QT^2 = PC^2 : PF^2$; and by eliminating Qv^2 , $vG : \frac{QT^2}{Pv} = PC^2 : \frac{CD^2 \cdot PF^2}{PC^2}$. Since QR = Pv, and (by Lem. XII)

$BC \cdot CA = CD \cdot PF$, and, when the points P and Q coincide, $2PC = vG$, we shall have, multiplying the extremes and means together,

$$\frac{QT^2 \cdot PC^2}{QR} = \frac{2BC^2 \cdot CA^2}{PC}.$$

Therefore (by Cor. v, Prop. vi), the centripetal force is inversely as $\frac{2BC^2 \cdot CA^2}{PC}$; that is (because $2BC^2 \cdot CA^2$ is given), inversely as $\frac{1}{PC}$; that is, directly as the distance PC. Q.E.I.

The same otherwise.

In the right line PG on the other side of the point T, take the point u so that Tu may be equal to Tv; then take uV, such that uV : vG = DC² : PC². Since, by the conic sections, $Qv^2 : Pv \cdot vG = DC^2 : PC^2$, we have $Qv^2 = Pv \cdot uV$. Add Pu · Pv to both sides, and the square of the chord of the arc PQ will be equal to the rectangle PV · Pv; and therefore a circle which touches the conic section in P, and passes through the point Q, will pass also through the point V. Now let the points P and Q meet, and the ratio of uV to vG, which is the same with the ratio of DC² to PC², will become the ratio of PV to PG, or PV to 2PC; and therefore PV will be equal to $\frac{2DC^2}{PC}$. And therefore the force by which the body P revolves in the ellipse will be inversely as $\frac{2DC^2}{PC} \cdot PF^2$ (by Cor. III, Prop. vi); that is (because $2DC^2 \cdot PF^2$ is given), directly as PC. Q.E.I.

COR. I. And therefore the force is as the distance of the body from the centre of the ellipse; and, *vice versa*, if the force is as the distance, the body will move in an ellipse whose centre coincides with the centre of force, or perhaps in a circle into which the ellipse may degenerate.

COR. II. And the periodic times of the revolutions made in all ellipses whatsoever about the same centre will be equal. For those times in similar

ellipses will be equal (by Cor. III and VIII, Prop. IV); but in ellipses that have their greater axis common, they are to each other as the whole areas of the ellipses directly, and the parts of the areas described in the same time inversely; that is, as the lesser axes directly, and the velocities of the bodies in their principal vertices inversely; that is, as those lesser axes directly, and the ordinates to the same point of the common axes inversely; and therefore (because of the equality of the direct and inverse ratios) in the ratio of equality, 1 : 1.

SCHOLIUM

If the ellipse, by having its centre removed to an infinite distance, degenerates into a parabola, the body will move in this parabola; and the force, now tending to a centre infinitely remote, will become constant. This is *Galileo's* theorem. And if the parabolic section of the cone (by changing the inclination of the cutting plane to the cone) degenerates into an hyperbola, the body will move in the perimeter of this hyperbola, having its centripetal force changed into a centrifugal force. And in like manner as in the circle, or in the ellipse, if the forces are directed to the centre of the figure placed in the abscissa, those forces by increasing or diminishing the ordinates in any given ratio, or even by changing the angle of the inclination of the ordinates to the abscissa, are always augmented or diminished in the ratio of the distances from the centre; provided the periodic times remain equal; so also in all figures whatsoever, if the ordinates are augmented or diminished in any given ratio, or their inclination is any way changed, the periodic time remaining the same, the forces directed to any centre placed in the abscissa are in the several ordinates augmented or diminished in the ratio of the distances from the centre.