The 17th CSI International Symposium on
Computer Architecture & Digital Systems

Gaussian and EJ Networks
Some Efficient Interconnection Topologies for Parallel Systems

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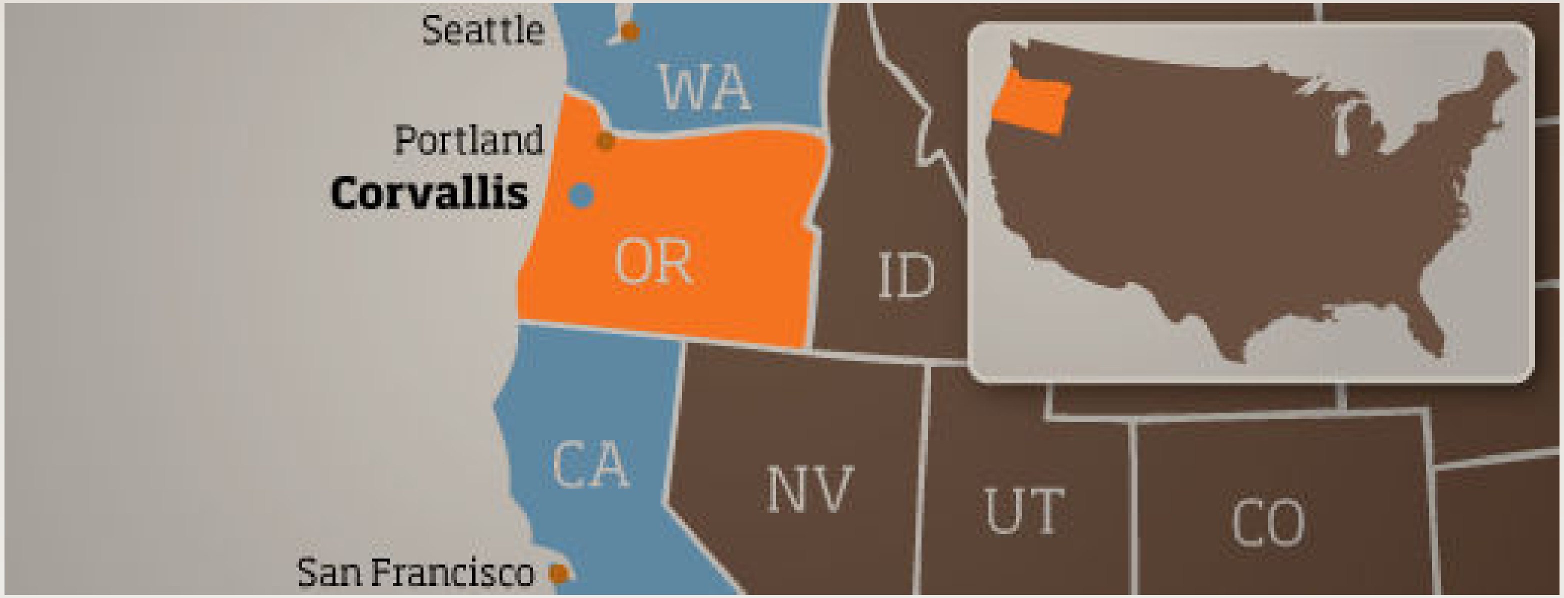
Location


Seattle
WA
Portland
Corvallis
OR
ID
CA
NV
UT
CO
San Francisco

Oregon State University, Corvallis

- # of students 26,393
- Students from all 50 States and over 100 countries
- Close to 2,500 international Students
- 200 undergrad degree programs
- More than 80 grad degree programs

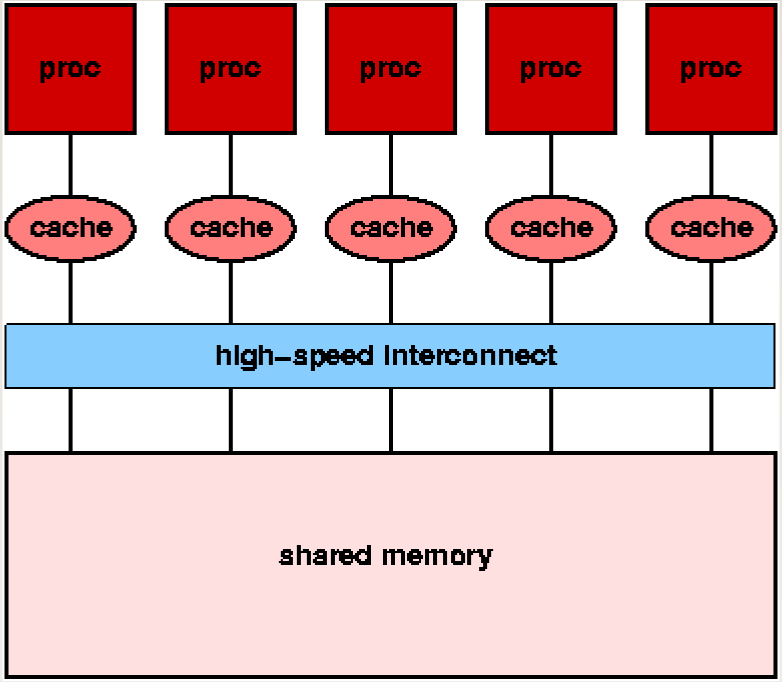
School of EECS (KEC)

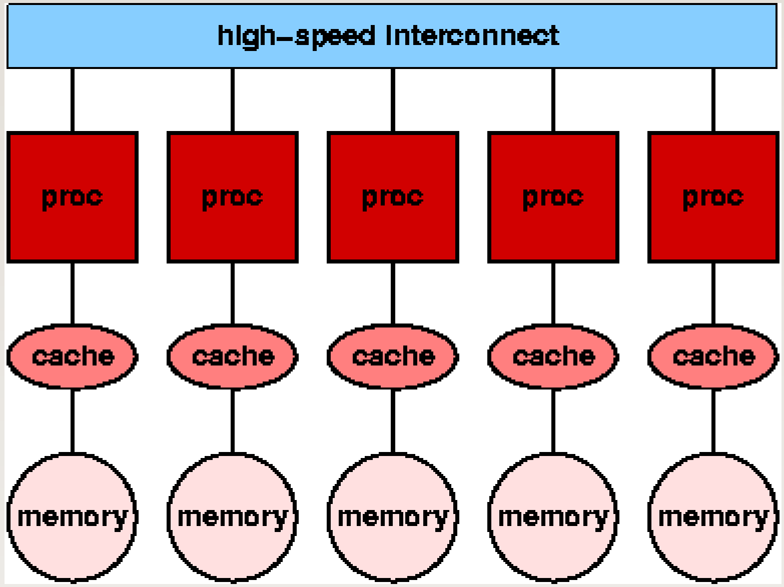
- 46 Faculty
- 2000 undergraduate students
- 180 PhD and 140 MS students
- 12 IEEE and ACM fellows
- Authored more than 50 text-books

Outline

- Introduction
- Gaussian Networks
  - Gaussian Integers
  - Mod Operation over $a + bi$
  - Interconnection Topology
  - Routing
  - Broadcasting
  - Hamiltonian Decomposition
- EJ Networks
  - EJ Integers
  - Mod Operation
  - Interconnection Topology
  - Routing
  - Hamiltonian Decomposition
- Summary

Parallel Computing





Shared-memory multiprocessors

Distributed-memory multi-computers

http://rtinfo.uits.iu.edu/hpa/mpi_tutorial/s1_basic_concepts.html

Introduction

- In the last decade, supercomputers with thousands of processors have been built.
- **Sequoia - BlueGene/Q**
  - **Manufacturer:** IBM
  - **Performance:** 17,173.2 TFlop/s (3rd in last June)

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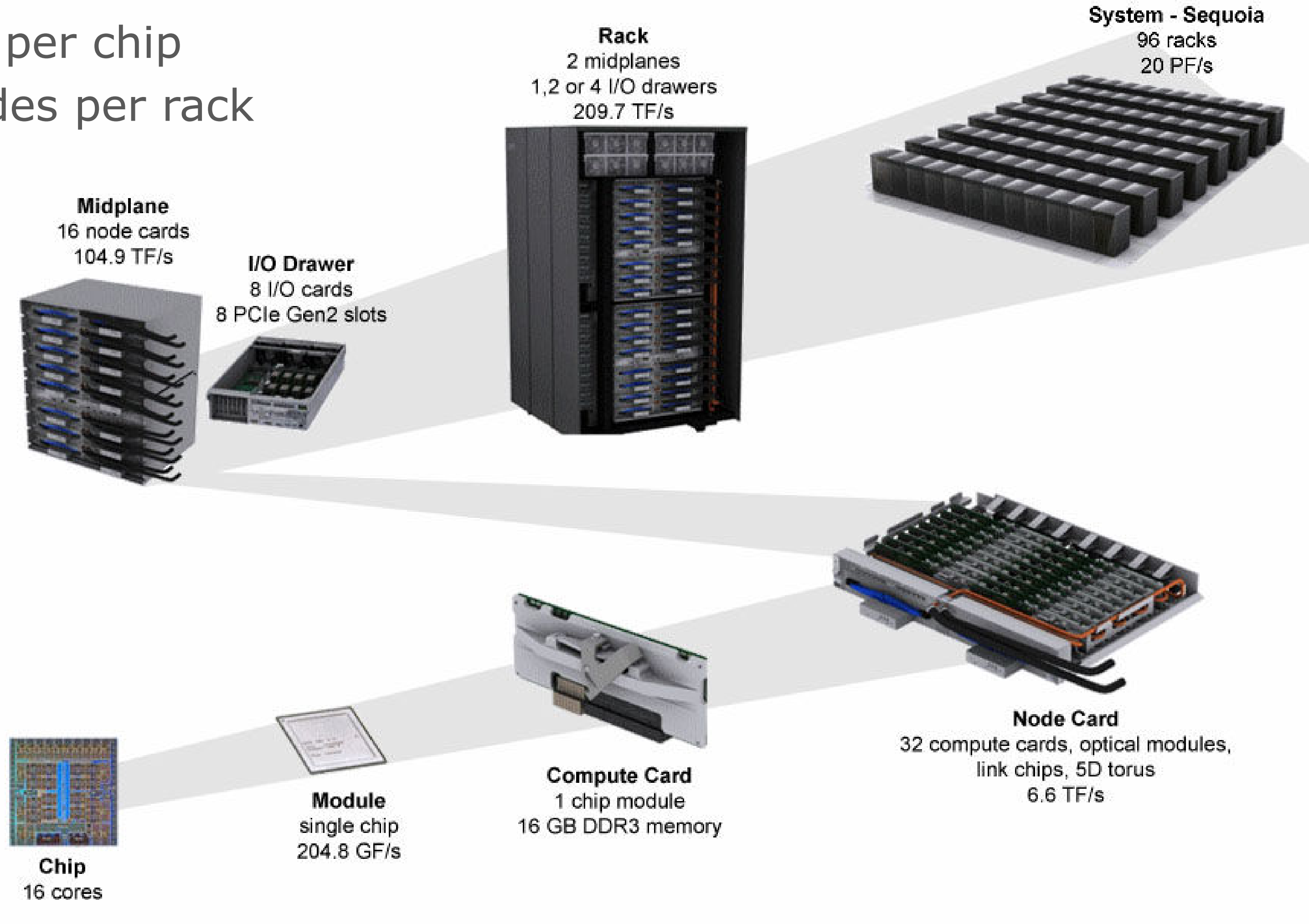
	NAME	SPECS	SITE	COUNTRY	CORES	RMAX PFLOP/S	POWER MW
1	Tianhe-2 (Milkyway-2)	NUDT, Intel Ivy Bridge (12C, 2.2 GHz) & Xeon Phi (57C, 1.1 GHz), Custom interconnect	NUDT	China	3,120,000	**33.9**	17.8
2	Titan	Cray XK7, Opteron 6274 (16C, 2.2 GHz) + Nvidia Kepler (14C, .732 GHz), Custom interconnect	DOE/SC/ORNL	USA	560,640	**17.6**	8.3
3	Sequoia	IBM BlueGene/Q, Power BQC (16C, 1.60 GHz), Custom interconnect	DOE/NNSA/LLNL	USA	1,572,864	**17.2**	7.9
4	K computer	Fujitsu SPARC64 VIIIfx (8C, 2.0GHz), Custom interconnect	RIKEN AICS	Japan	705,024	**10.5**	12.7
5	Mira	IBM BlueGene/Q, Power BQC (16C, 1.60 GHz), Custom interconnect	DOE/SC/ANL	USA	786,432	**8.16**	3.95

Introduction

- **Sequoia - BlueGene/Q**
  - **Cores:** 1.6 million
    - 16 cores per chip
    - 1024 nodes per rack
    - 96 racks



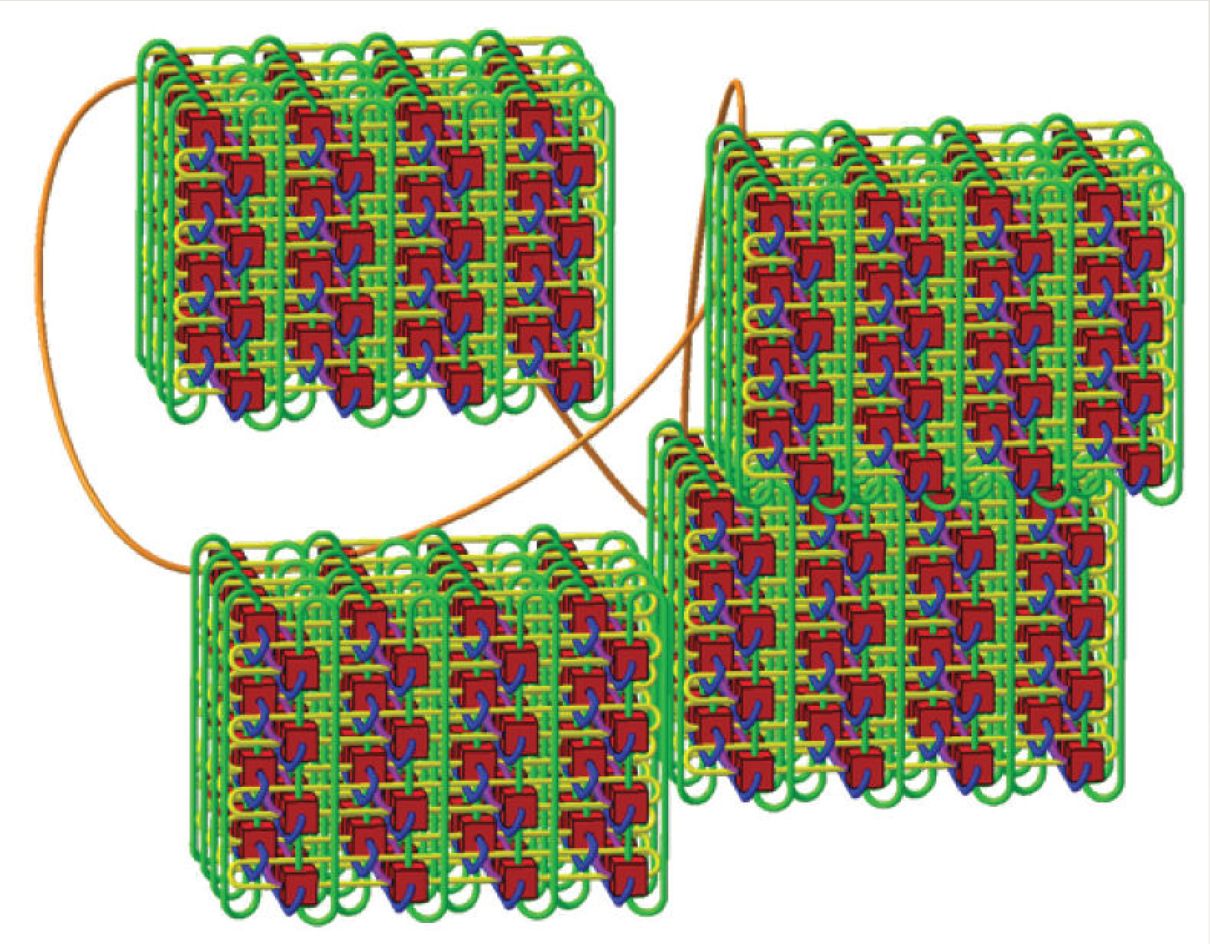
Introduction

- **Sequoia - BlueGene/Q**
  - **Cores:** 1.6 million

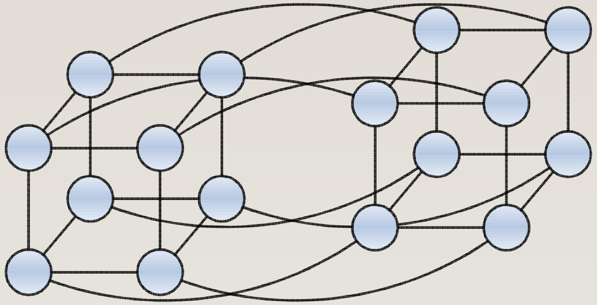
Sequoia's 96 racks during installation (Photo: Lawrence Livermore National Laboratory)
It covers an area of about 3,000 square feet (280 m^2).

Introduction

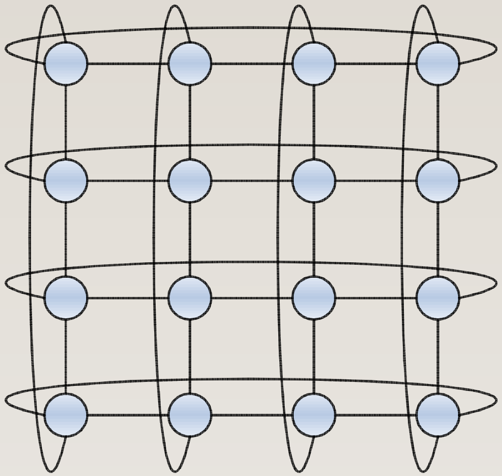
- **Sequoia - BlueGene/Q**
  - **Interconnect:** 5D Tours



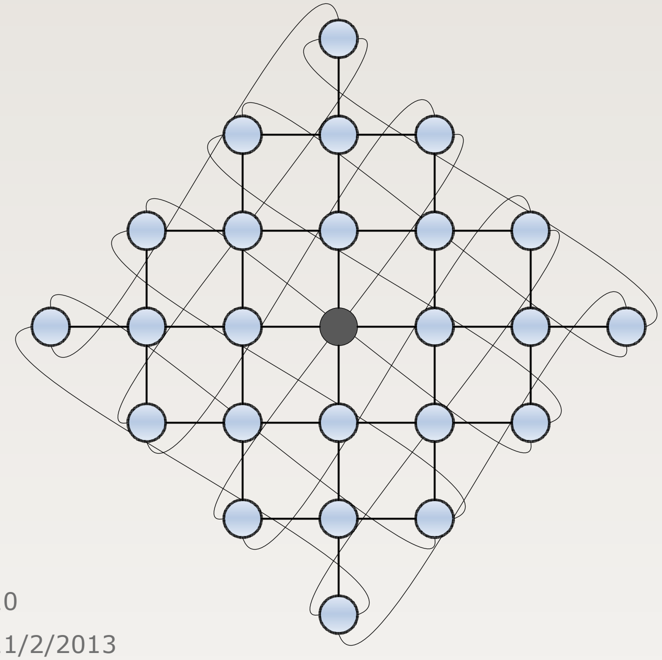
Topologies



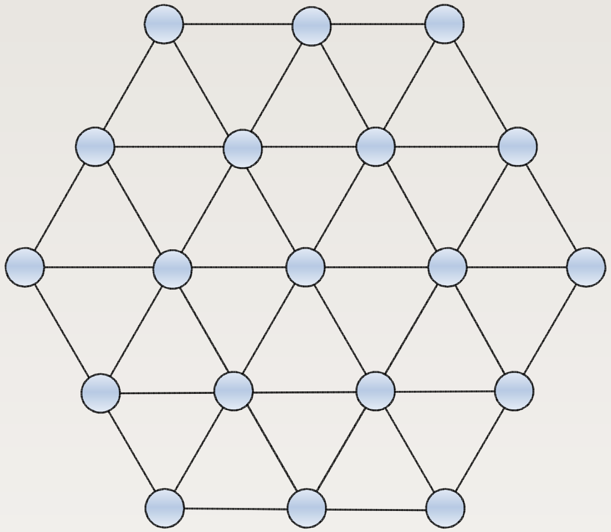
hypercube



torus



Gaussian



hexagonal mesh

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- Introduction
- Gaussian Networks
  - Gaussian Integers
  - Mod Operation over $a + bi$
  - Interconnection Topology
  - Routing
  - Broadcasting
  - Hamiltonian Decomposition
- EJ Networks
  - EJ Integers
  - Mod Operation
  - Interconnection Topology
  - Routing
  - Hamiltonian Decomposition
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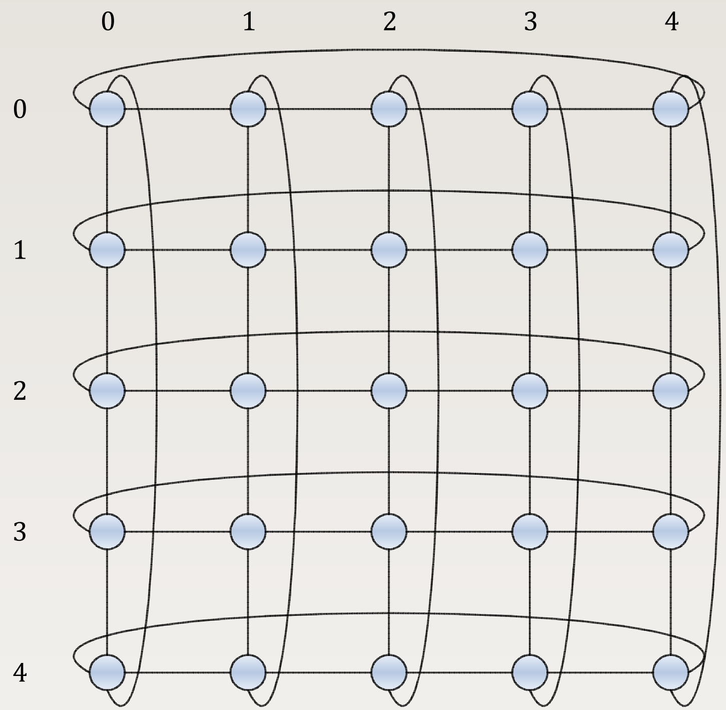
Torus Interconnection Networks

- Node addresses over $\mathbb{Z}_k \times \mathbb{Z}_k \times \cdots \times \mathbb{Z}_k$
- A node $A = (a_{n-1}a_{n-2}\ldots a_0)$ is adjacent to node $B = (b_{n-1}b_{n-2}\ldots b_0)$ if the address digits differ in one position by $\pm 1 \; mod \; k$
  - # of nodes = k^n
  - Degree of each node = $2n$
  - Diameter = $n\left\lfloor\frac{k}{2}\right\rfloor$
- Example: 5×5 Torus
  - # of nodes =5^2
  - Degree of each node = 4
  - Diameter 16



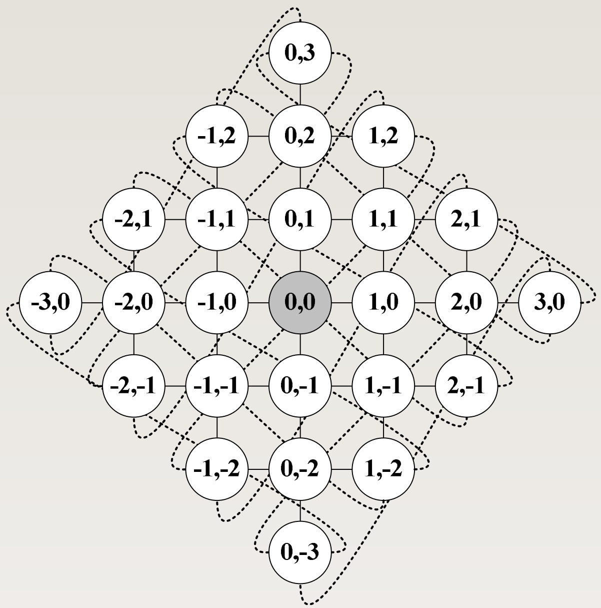
Gaussian Networks

- A generator $\alpha = a + bi \in \mathbb{Z}[i]$
- Node addresses over $\mathbb{Z}_\alpha \times \mathbb{Z}_\alpha \times \cdots \times \mathbb{Z}_\alpha$

- $A = (a_{n-1}a_{n-2}\ldots a_0)$, $B = (b_{n-1}b_{n-2}\ldots b_0)$ are adjacent
  - if the address digits differ in one position by $\pm 1, or \pm i \bmod \alpha$

  - $\#$ of nodes $(a^2 + b^2)^n$
  - Degree of each node $4n$
  - Diameter $\begin{cases} nb & if\, a^2 + b^2 \text{ is even} \\ n(b-1) & if\, a^2 + b^2 \text{ is odd} \end{cases}$

- Note if $(a = 0, b \neq 0)$ or $(a \neq 0, b = 0) \Rightarrow$ Torus network

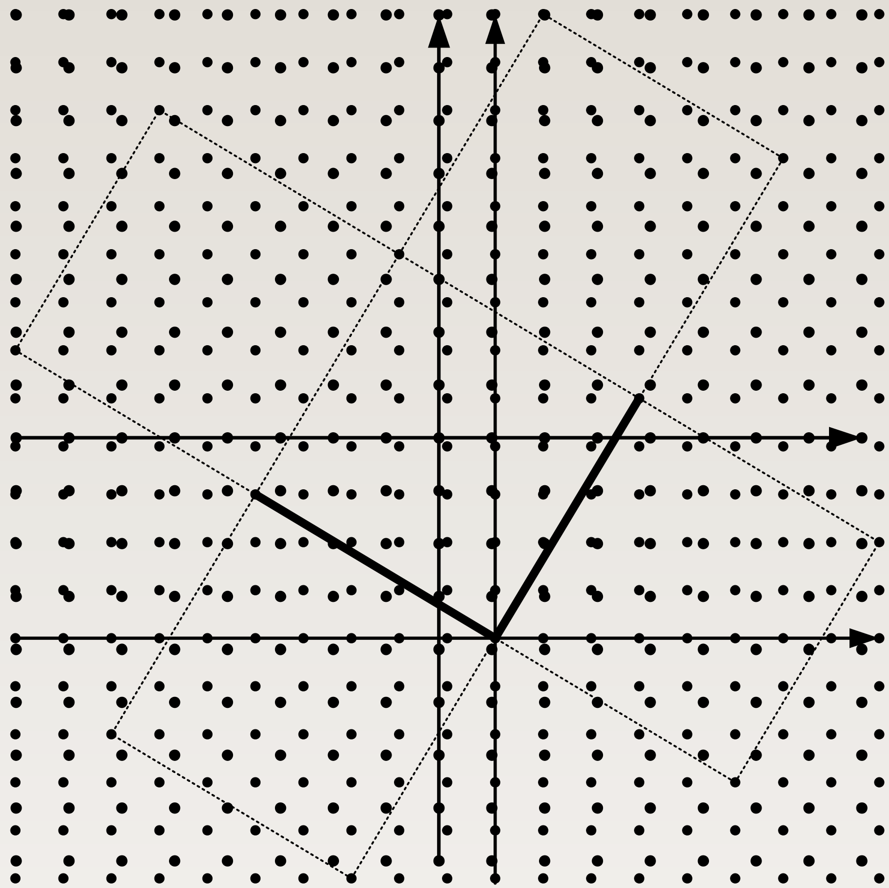


Advantages of Gaussian Networks

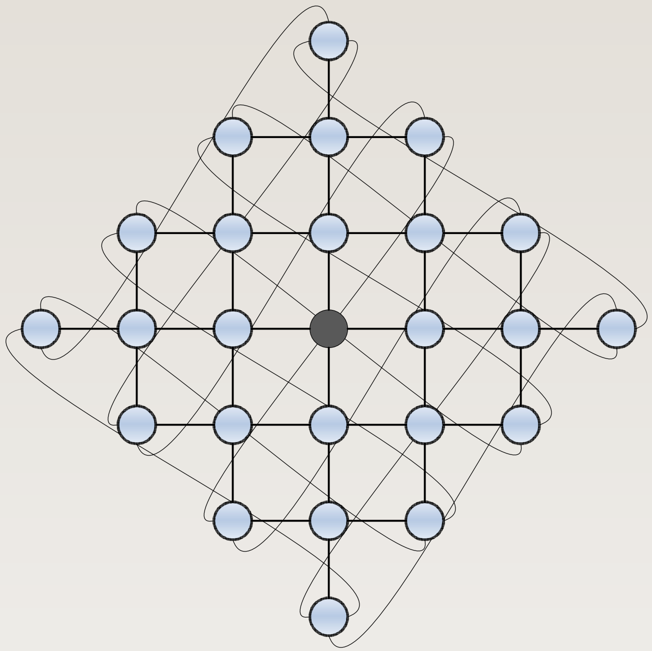
- Example 1: Let $\alpha = 10 + 10i$
  - One dimensional Gaussian network
    - # of nodes = 200
    - Degree = 4
    - Diameter = 10
- Note any 2D torus with 200 nodes will have diameter at least 14
- Example 2: Let $\alpha = 10 + 10i$
  - Two dimensional Gaussian network
    - # of nodes = 40,000
    - Degree = 8
    - Diameter = 20
- Note any 4D torus with 40,000 nodes will have diameter at least 28

What are Gaussian integers

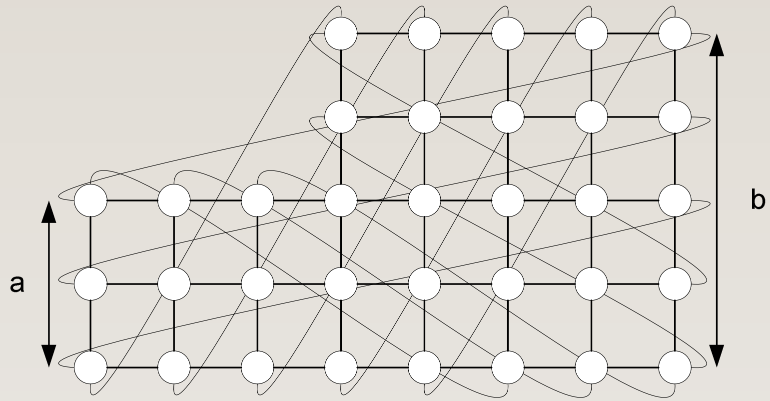
- A subset of complex numbers
  - $\mathbb{Z}[i] = \{x + yi | x, y \in \mathbb{Z}\}$
- A generator
  - $\alpha = a + bi \in \mathbb{Z}[i]$
- # of nodes:
  - $Norm(\alpha) = \mathcal{N}(\alpha) = a^2 + b^2$
- Example: $\alpha = 3+5i$



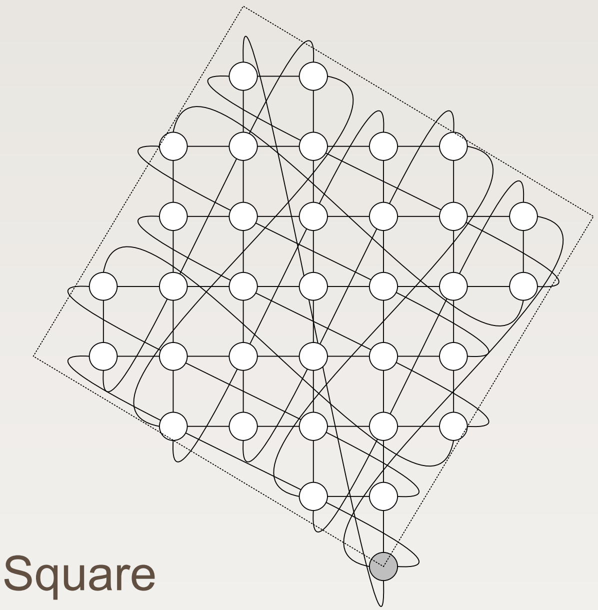
Different Representations



Distance-based

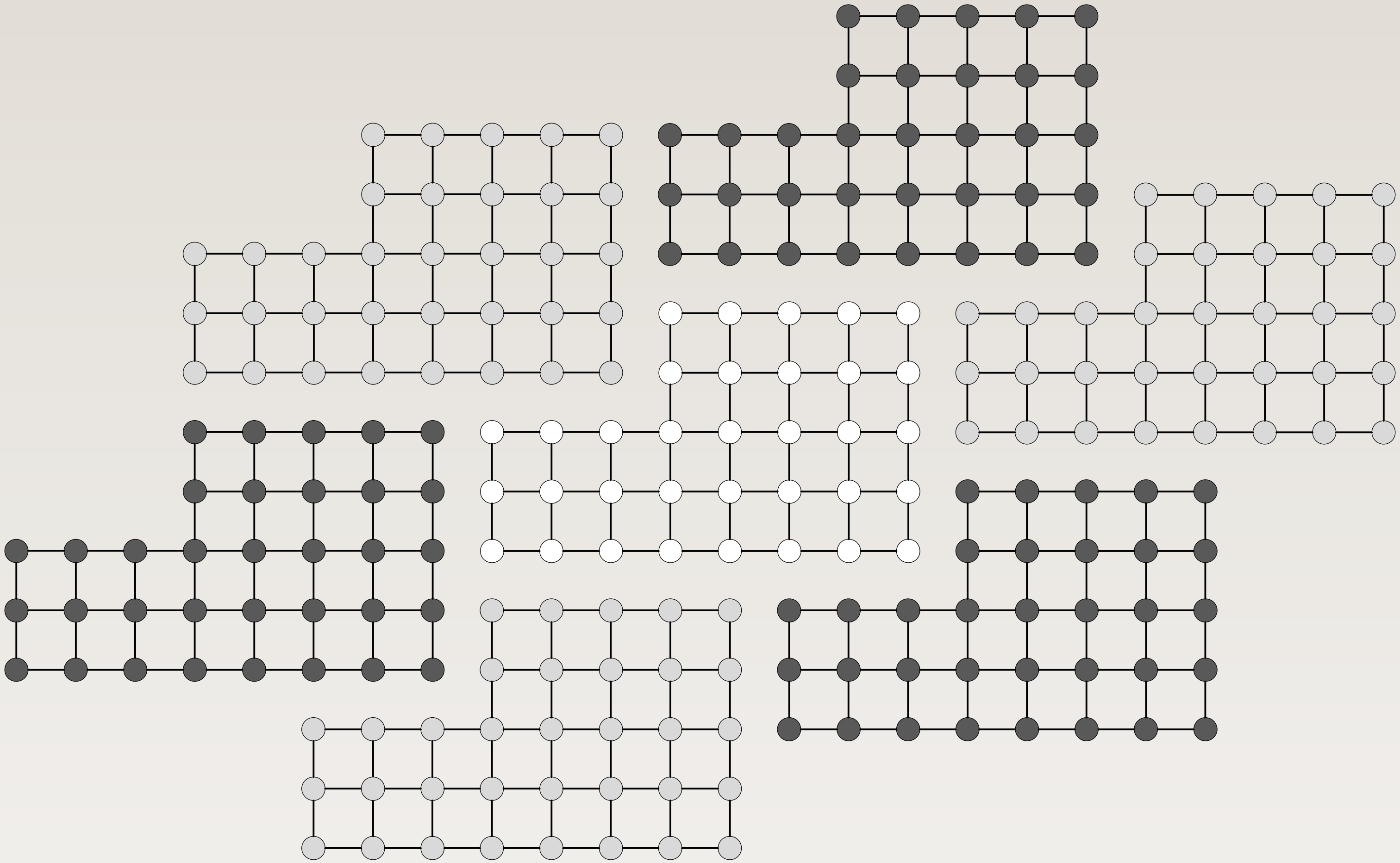


Utah

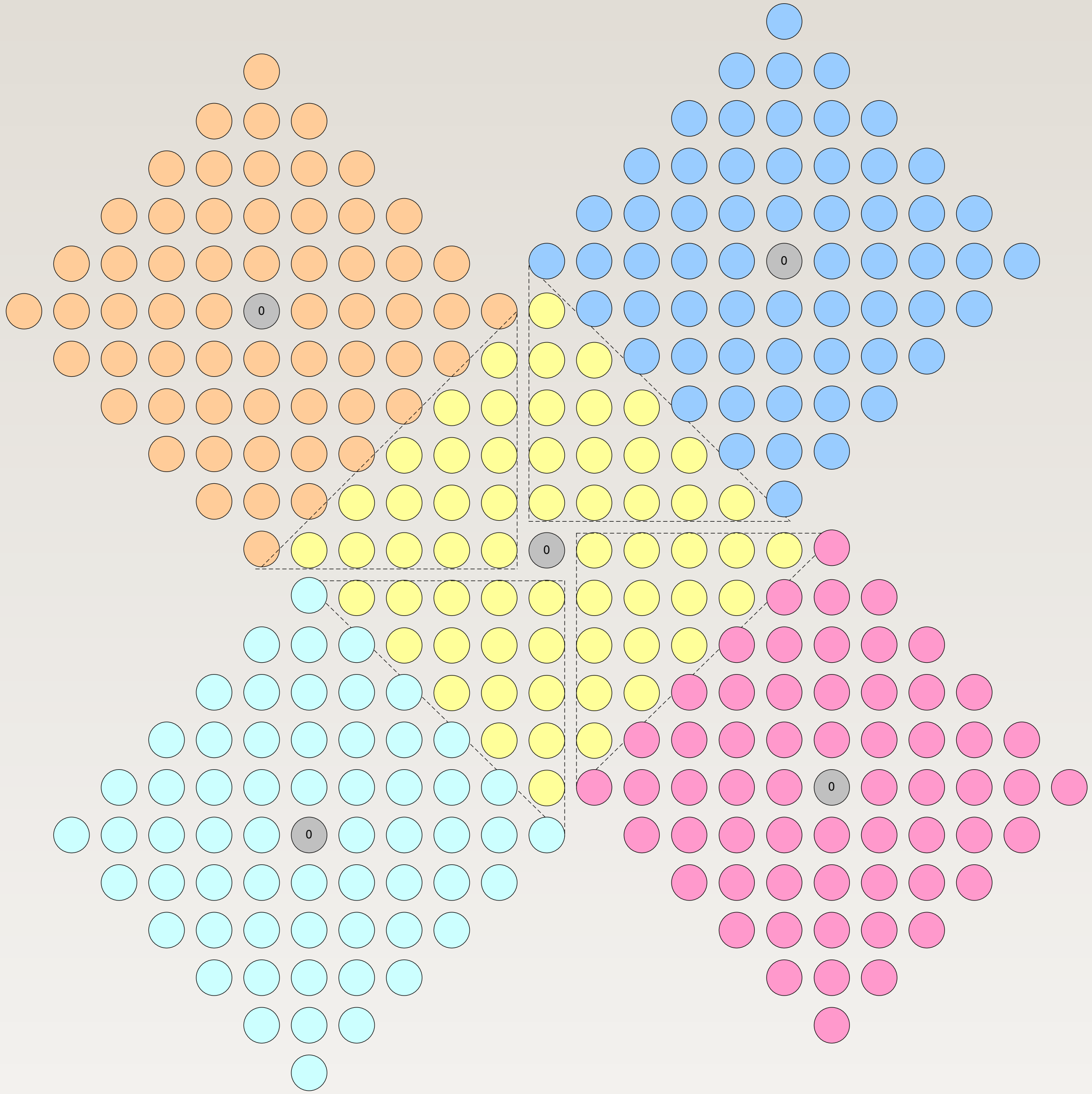


Square

Utah Representation Tiling

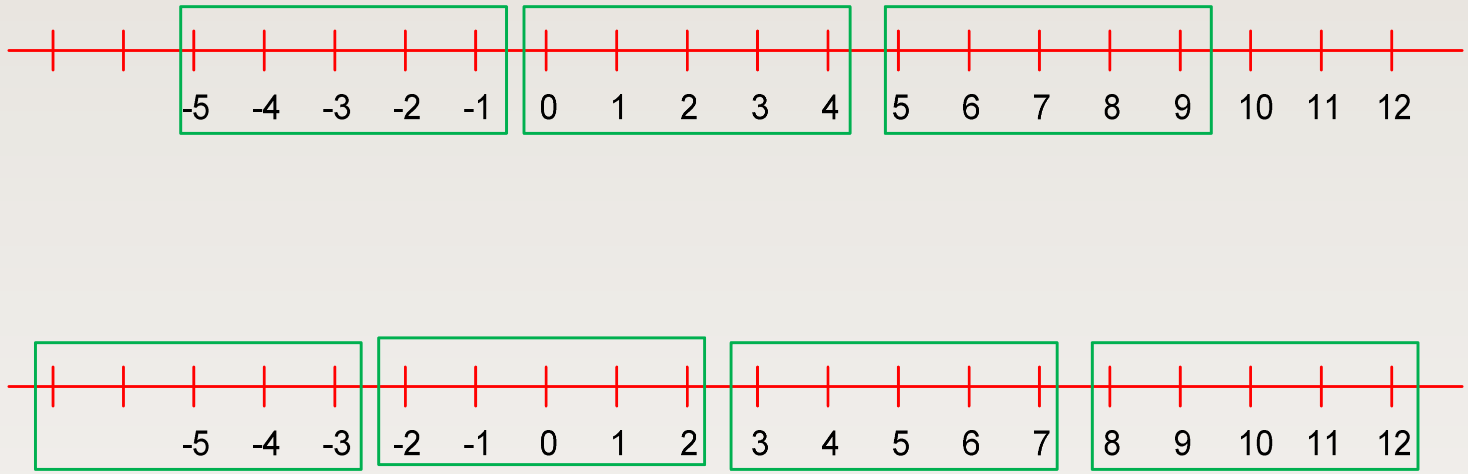


Distance-based Representation



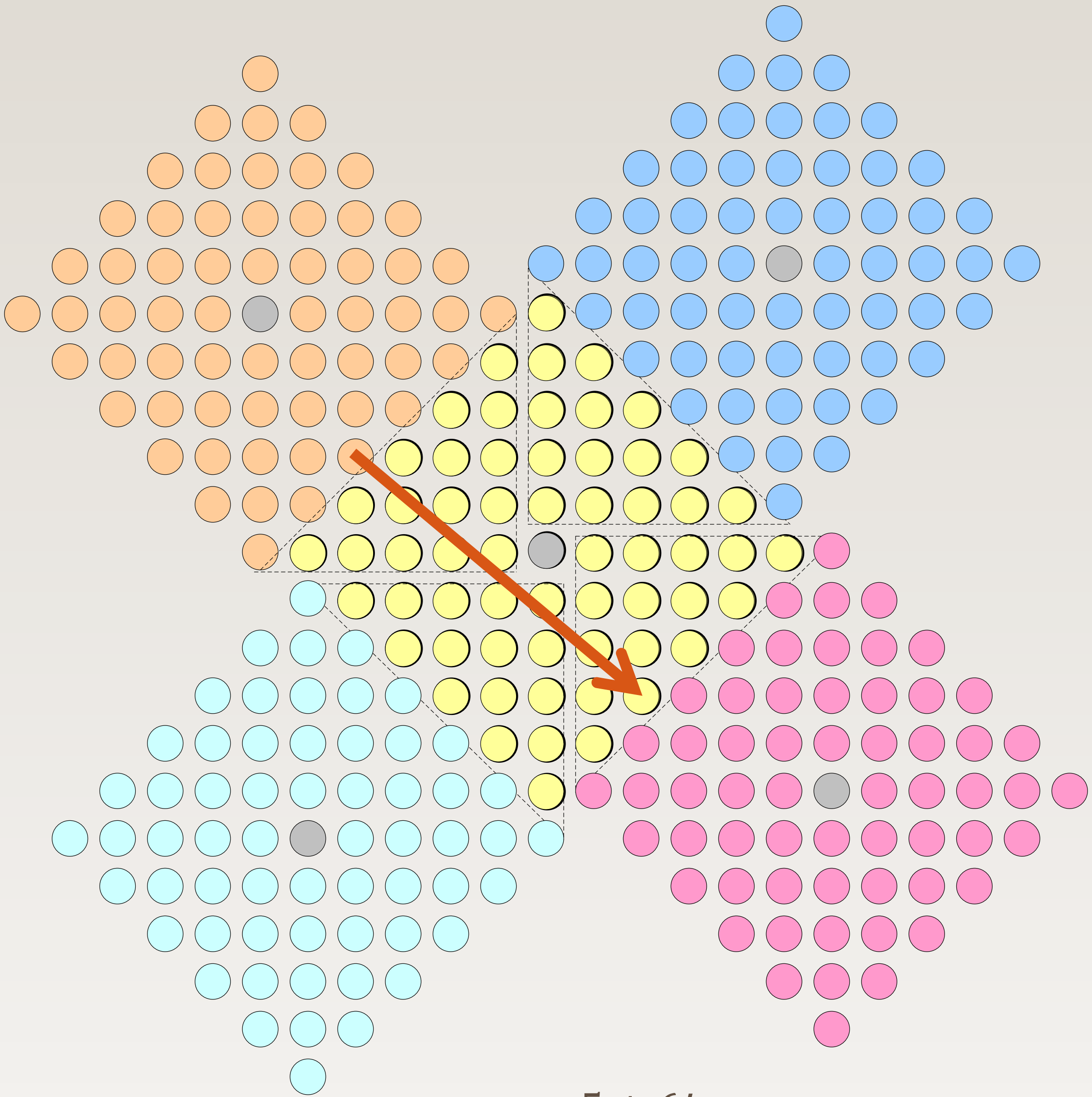
How to do mod α operation

- $\alpha = a + bi; a^2 + b^2$ elements
  - $Norm(\alpha) = \mathcal{N}(\alpha) = a^2 + b^2$
- How to do mod k, when k is integer
  - Example: mod 5



Mod Operation

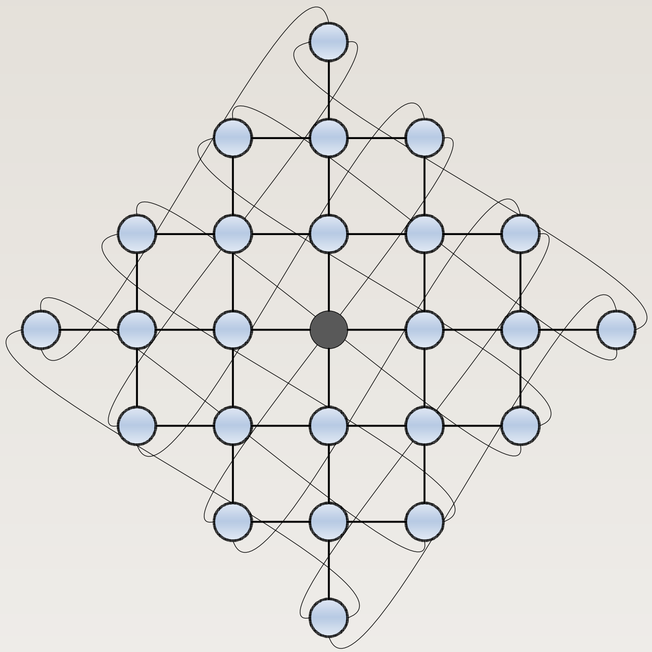
$$-4 + 2i \; mod \; 5 + 6i$$
$$= \; (-4 + 2i) - (-6 + 5i)$$
$$= \; 2 + 3i$$

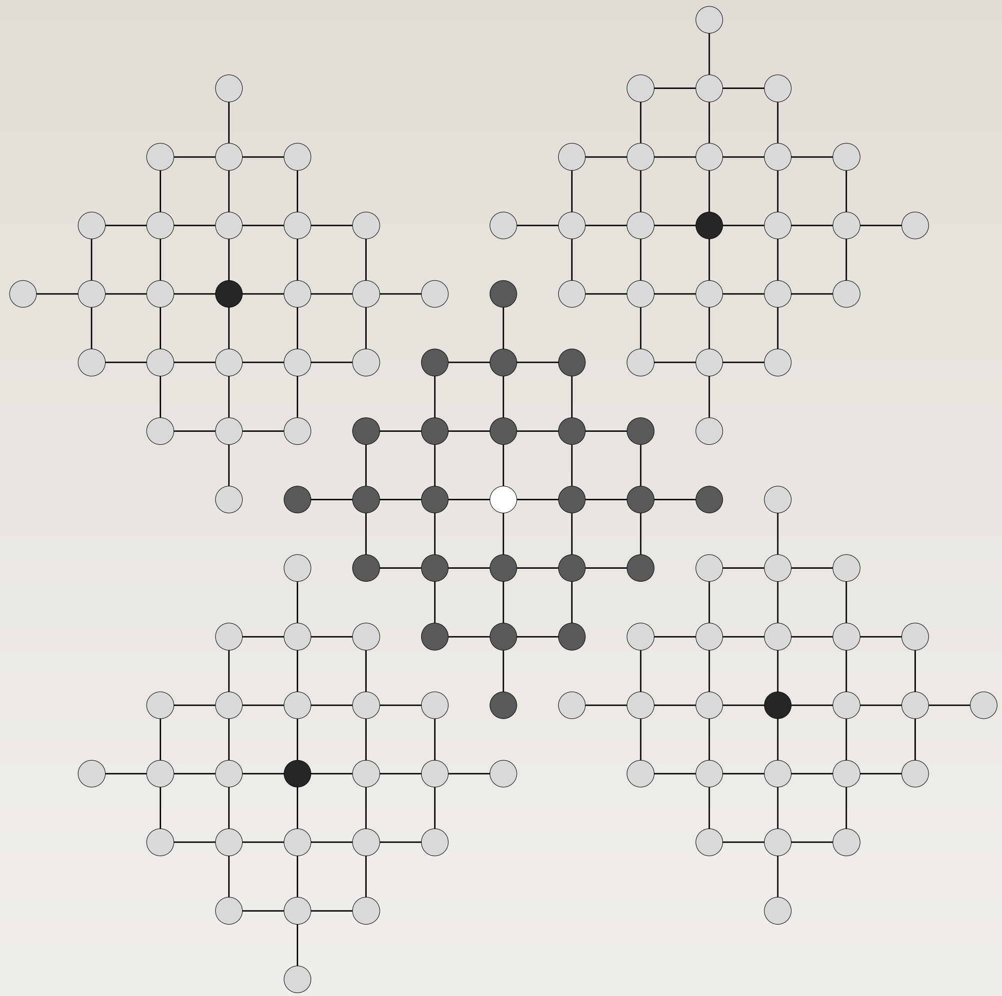


$\alpha = 5 + 6i$

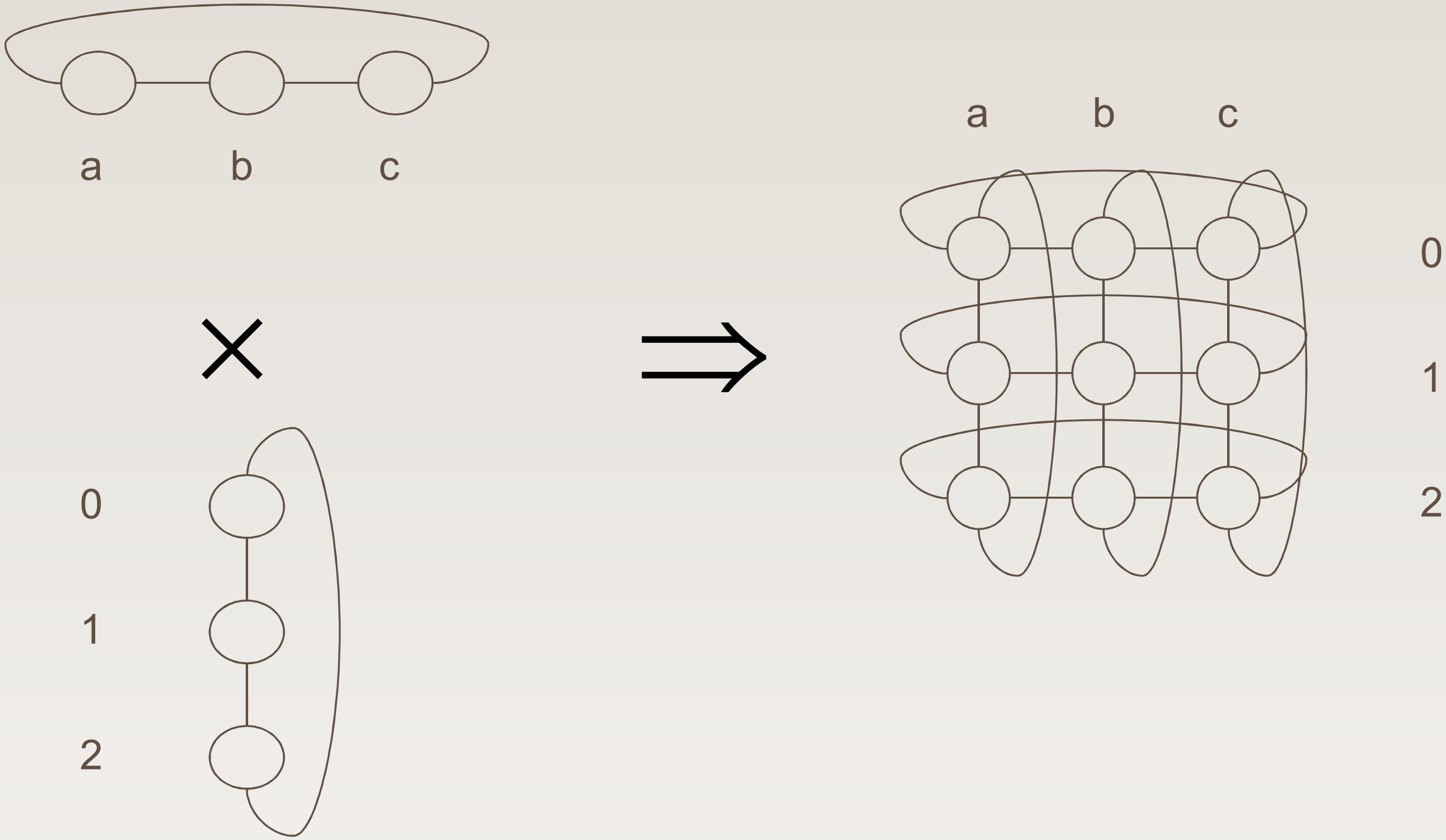
Gaussian Interconnection Network (1-D)

- $\alpha = 3 + 4i$





Cross Product of Graphs

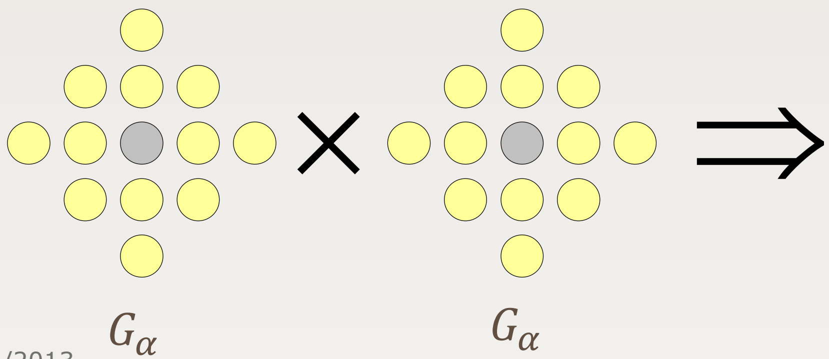


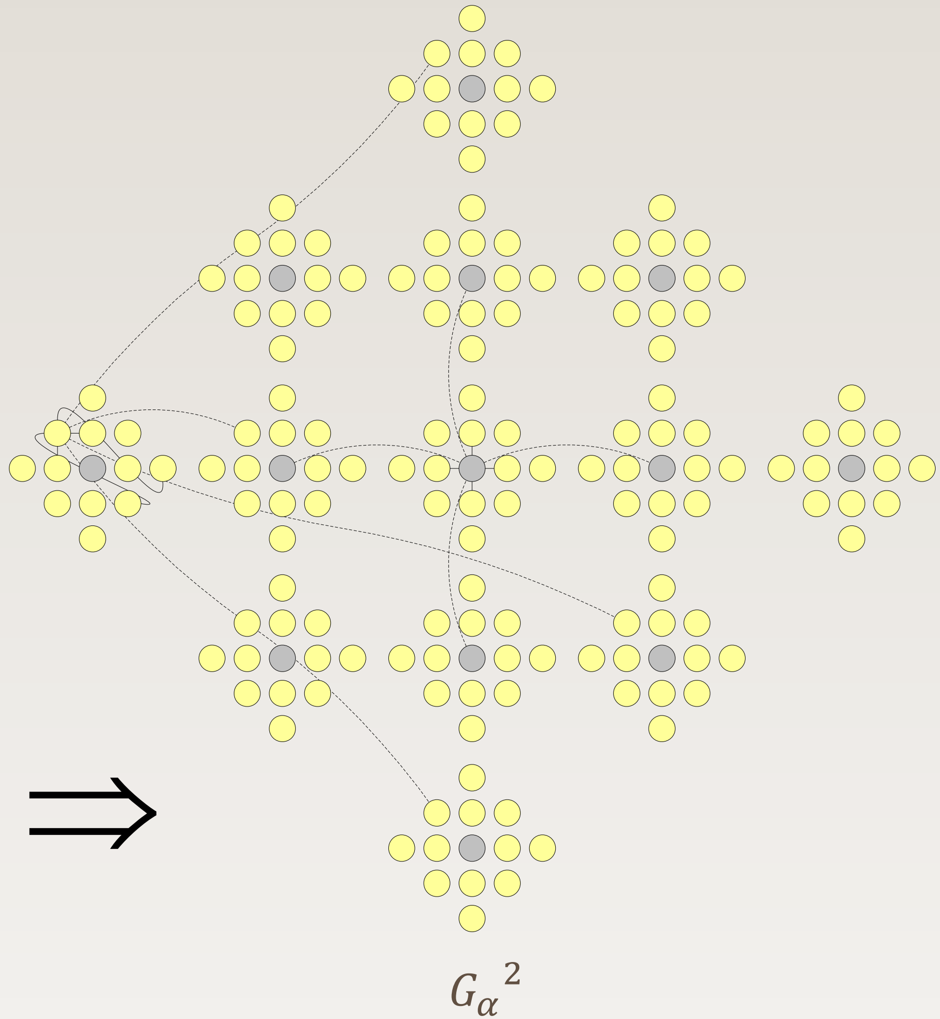
Example of 2D Gaussian Network

- Cross product of two Gaussian networks

- Let $\alpha = 2 + 3i$
- ${G_\alpha}^2 = G_\alpha \times G_\alpha$
  - \# of nodes = 169
  - Degree = 8
  - Diameter = 4

- The links shown for
  - (0,0)
  - (-2,-1+i)

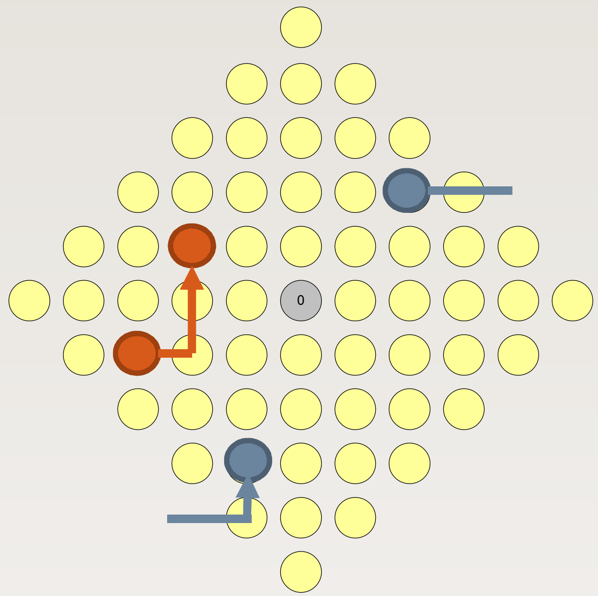




Routing in 1-D Gaussian Networks

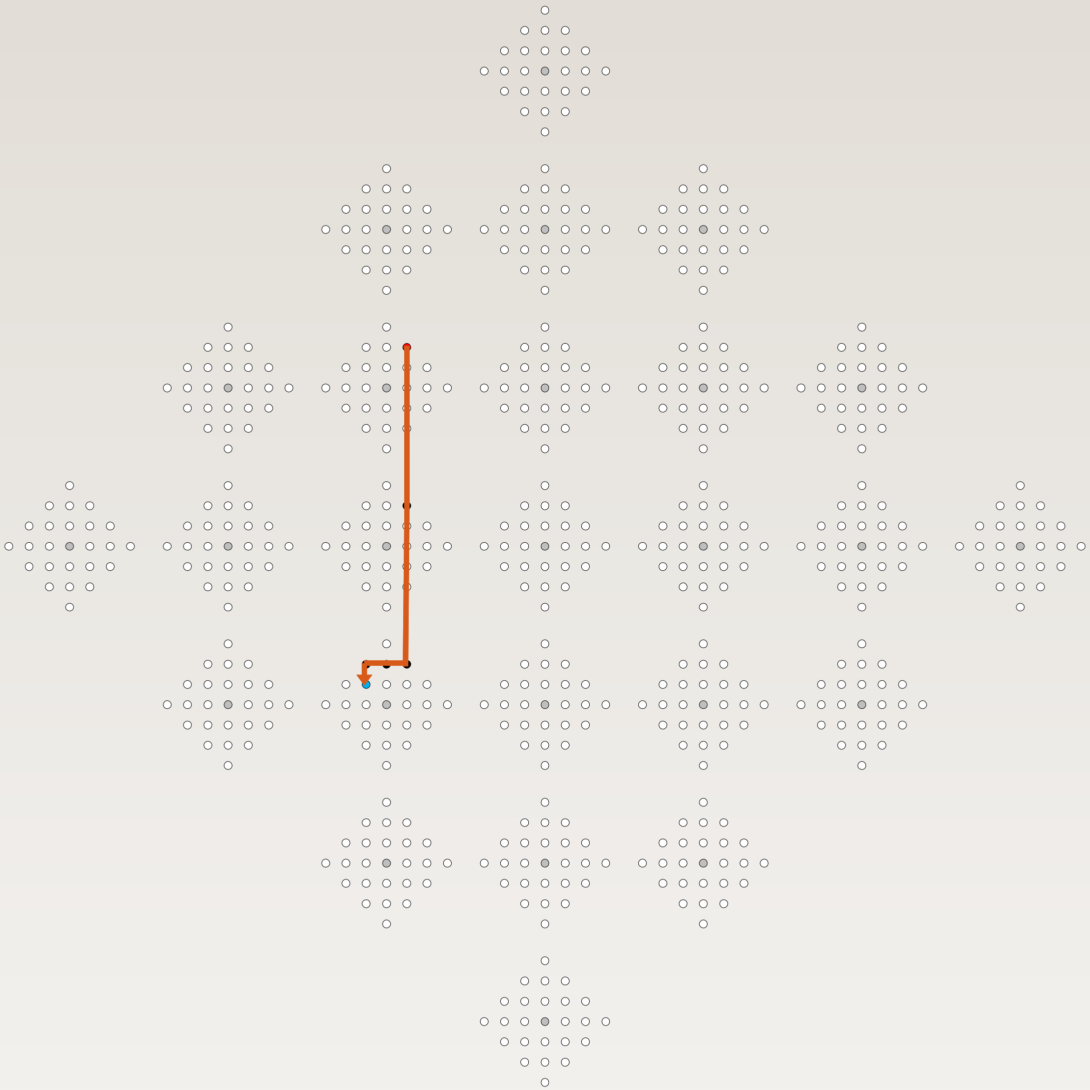
- $\alpha = 5 + 6i$
  - $S = -3 - i$
  - $D = -2 + i$
  - $D - s = 1 + 2i$

  - $S = 2 + 2i$
  - $D = -1 - 3i$
  - $D - s = -3 - 5i$
  - $-3 - 5i \; - i^2(5 + 6i) = 2 + i$



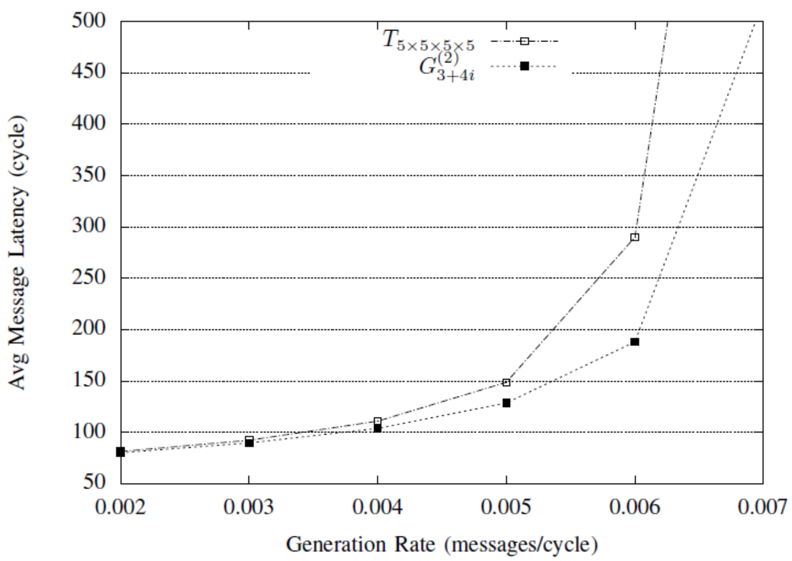
Routing in 2D Gaussian

- $S = (-1 + i, 1 + 2i)$
- $D = (-1 - i, -1 + i)$
- $D - S = (-2i, -2 - i)$

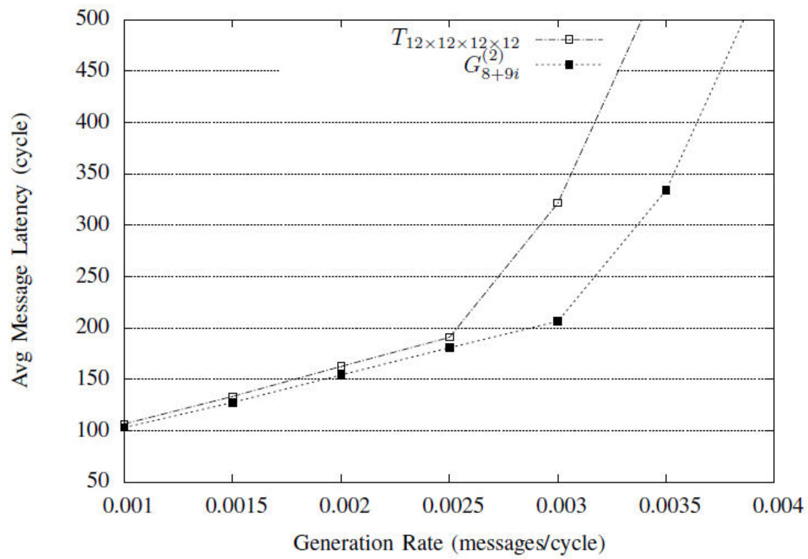


Simulation Results

- Dimension order routing
- Simulations are performed using XMulator



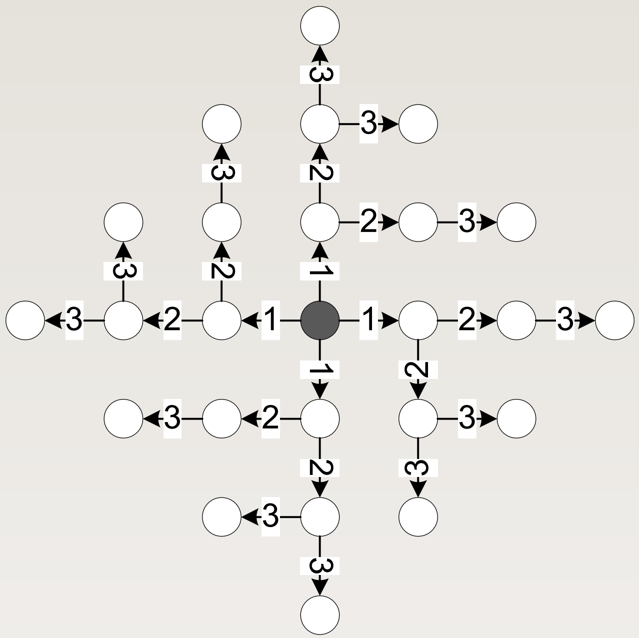
- 4D torus with 625 nodes
- 2D Gaussian with 625 nodes

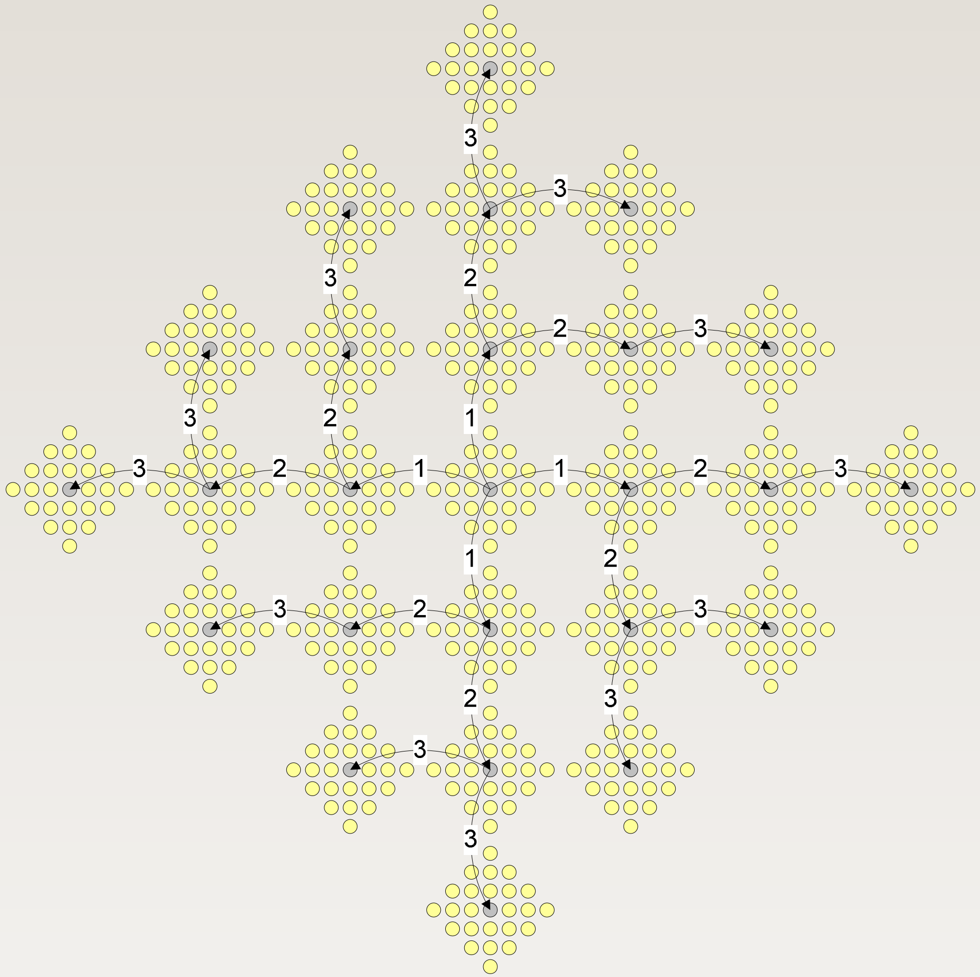


- 4D torus with 20736 nodes
- 2D Gaussian with 21025 nodes

Broadcasting

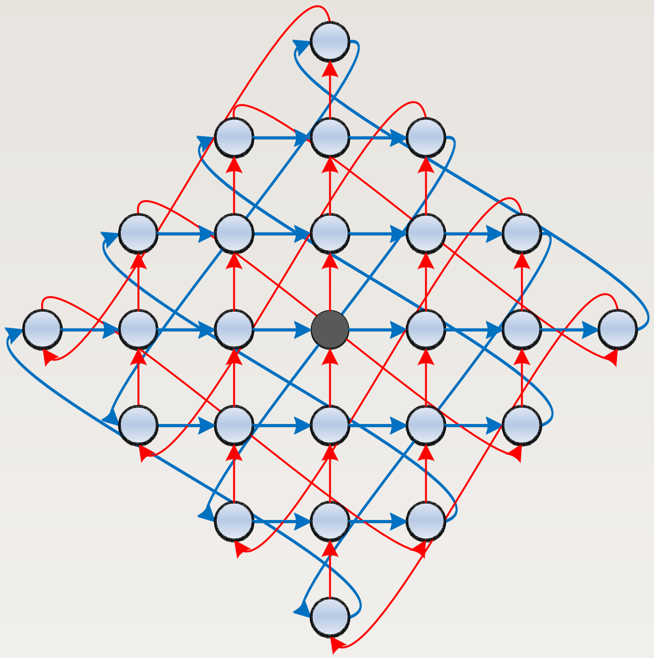
- Multiport I/O model



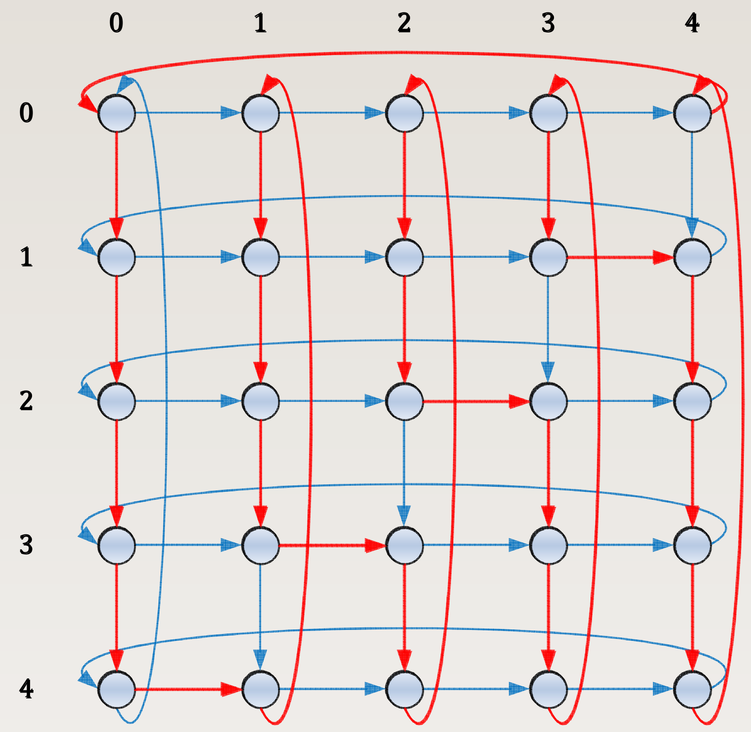


Hamiltonian Decomposition

- $\alpha = a + bi; N = a^2 + b^2; GCD(a, b) = 1$
- Example $\alpha = 3 + 4i$
  1. Start at 0, Go along $1,2,3, \ldots, N-1$
  2. Start at 0, Go along $i, 2i, 3i, \ldots, (N-1)i$



Hamiltonian Cycles in 2D Torus



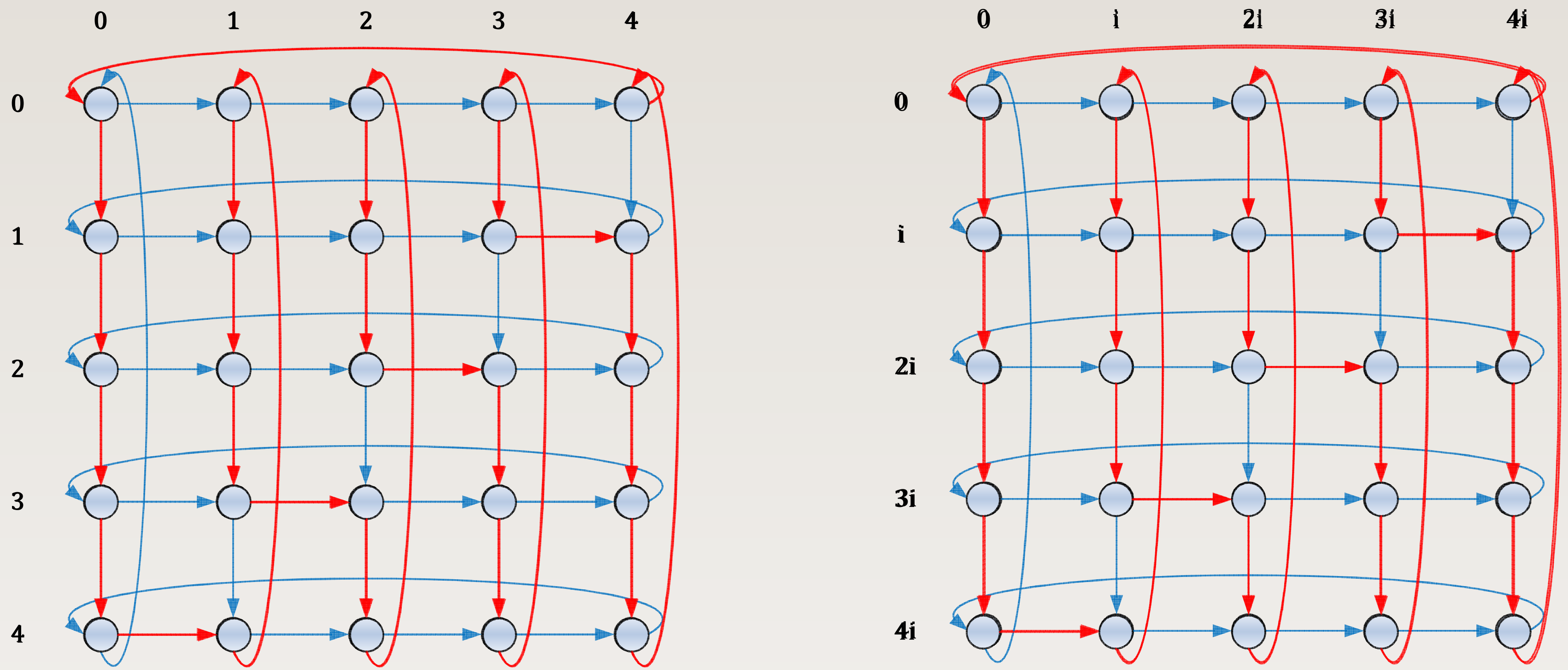
Multi Dimensional

- $\alpha = a + bi; N = a^2 + b^2$
- If $GCD(a, b) = 1$ then there is a 1-1 correspondence between $\mathbb{Z}_\alpha$ and $\mathbb{Z}_N$

$$
\begin{aligned}
G_\alpha^{(2)} &= G_\alpha \otimes G_\alpha \\
&= (H_1^N \oplus H_2^N) \otimes (H_3^N \oplus H_4^N) \\
&= (H_1^N \otimes H_3^N) \oplus (H_2^N \otimes H_4^N) \\
&= T_{N\times N}^1 \oplus T_{N\times N}^2 \\
&= (H_1^{N\times N} \oplus H_2^{N\times N}) \oplus (H_3^{N\times N} \oplus H_4^{N\times N})
\end{aligned}
$$

$$
\begin{aligned}
G_{1+2i}^{(2)} &= G_{1+2i} \otimes G_{1+2i} \\
&= (\langle 0, 1, 2, 3, 4\rangle \oplus \langle 0, i, 2i, 3i, 4i\rangle) \\
&\otimes (\langle 0, 1, 2, 3, 4\rangle \oplus \langle 0, i, 2i, 3i, 4i\rangle) \\
&= (\langle 0, 1, 2, 3, 4\rangle \otimes \langle 0, 1, 2, 3, 4\rangle) \\
&\oplus (\langle 0, i, 2i, 3i, 4i\rangle \otimes \langle 0, i, 2i, 3i, 4i\rangle) \\
&= (H_1' \oplus H_2') \oplus (H_3' \oplus H_4')
\end{aligned}
$$

Hamiltonian Cycles in 2D Gaussian



Gray Codes & Edge-disjoint Hamiltonian Cycles

Let $G_{N,1} = (a_0, a_1, \ldots, a_{N-1})$ with $a_i \in \mathbb{Z}_N$ and all a_i's distinct

Let $G_{N,1}^t$ be the cyclic t-shift of $G_{N,1}$. For example:

$$
\begin{aligned}
G_{N,1}^0 &= G_{N,1} = (a_0, a_1, \ldots, a_{N-1}) \\
G_{N,1}^1 &= (a_{N-1}, a_0, a_1, \ldots, a_{N-2}) \\
G_{N,1}^2 &= (a_{N-2}, a_{N-1}, a_0, a_1, \ldots) \\
&\vdots \\
G_{N,1}^j &= (a_{N-j}, a_{N-j+1}, \ldots, a_{N-1}, a_0, a_1, \ldots)
\end{aligned}
$$

Define $G_{N,2m}$ as follows:

$$
\begin{aligned}
G_{N,2m} &= G_{N,m} \otimes G_{N,m} \\
&= \{A_j G_{N,m}^j | A_j \text{ is the } j\text{-th word in the } G_{N,m}\} , \\
&\text{for } j = 0, 1, 2, \ldots, N-1\}
\end{aligned}
$$

$G_{N,n}$ forms a Gray code over $\mathbb{Z}_N^n$ for n a power of 2, i.e, $n = 2^r$

Example of Gray Codes

Let $\alpha = 1 + 2i$. *Then* $N = 5$ *and we have:*

$$G_{5,1} = G^0_{5,1} = (0, 1, 2, 3, 4)$$
$$G^1_{5,1} = (4, 0, 1, 2, 3)$$
$$G^2_{5,1} = (3, 4, 0, 1, 2)$$
$$G^3_{5,1} = (2, 3, 4, 0, 1)$$
$$G^4_{5,1} = (1, 2, 3, 4, 0)$$
$$G_{5,2} = G_{5,1} \otimes G_{5,1} \text{ as shown in Table 1}$$
$$G_{5,4} = G_{5,2} \otimes G_{5,2} \text{ as shown in Table 2}$$

Table 1: Gray Codes in $G^{(2)}_\alpha$ where $\alpha = 1 + 2i$.

$0G^0_{5,1}$	$1G^1_{5,1}$	$2G^2_{5,1}$	$3G^3_{5,1}$	$4G^4_{5,1}$
00	14	23	32	41
01	10	24	33	42
02	11	20	34	43
03	12	21	30	44
04	13	22	31	40

Table 2: Gray Codes in $G^{(4)}_\alpha = G_{5,2} \otimes G_{5,2}$ where $\alpha = 1 + 2i$.

$00\,G^0_{5,2}$	$01\,G^1_{5,2}$	...	$44\,G^{23}_{5,2}$	$40\,G^{24}_{5,2}$
00 00	01 40	...	44 02	40 01
00 01	01 00	...	44 03	40 02
00 02	01 01	...	44 04	40 03
00 03	01 02	...	44 14	40 04
00 04	01 03	...	44 10	40 14
00 14	01 04	...	44 11	40 10
00 10	01 14	...	44 12	40 11
00 11	01 10	...	44 13	40 12
00 12	01 11	...	44 23	40 13
00 13	01 12	...	44 24	40 23
00 23	01 13	...	44 20	40 24
00 24	01 23	...	44 21	40 20
00 20	01 24	...	44 22	40 21
00 21	01 20	...	44 32	40 22
00 22	01 21	...	44 33	40 32
00 32	01 22	...	44 34	40 33
00 33	01 32	...	44 30	40 34
00 34	01 33	...	44 31	40 30
00 30	01 34	...	44 41	40 31
00 31	01 30	...	44 42	40 41
00 41	01 31	...	44 43	40 42
00 42	01 41	...	44 44	40 43
00 43	01 42	...	44 40	40 44
00 44	01 43	...	44 00	40 40
00 40	01 44	...	44 01	40 00

Permutations of Gray Codes

All other edge-disjoint Hamiltonian cycles in G_α^n can be obtained by some form of permutation and multiplication over the digits of $G_{N,n}$.

Let $P_0(G_{N,2^3}) = P_{000}(G_{N,2^3}) = (e_7, e_6, e_5, e_4, e_3, e_2, e_1, e_0)$. Then

$$P_1(G_{N,2^3}) = P_{001}(G_{N,2^3}) = (\underline{e_6}, \underline{e_7}, \underline{e_4}, \underline{e_5}, \underline{e_2}, \underline{e_3}, \underline{e_0}, \underline{e_1})$$

$$P_2(G_{N,2^3}) = P_{010}(G_{N,2^3}) = (\underline{e_5, e_4}, \underline{e_7, e_6}, \underline{e_1, e_0}, \underline{e_3, e_2})$$

$$P_3(G_{N,2^3}) = P_{011}(G_{N,2^3}) = (\underline{\underline{e_4}, \underline{e_5}}, \underline{\underline{e_6}, \underline{e_7}}, \underline{\underline{e_0}, \underline{e_1}}, \underline{\underline{e_2}, \underline{e_3}})$$

$$P_4(G_{N,2^3}) = P_{100}(G_{N,2^3}) = (\underline{e_3, e_2, e_1, e_0}, \underline{e_7, e_6, e_5, e_4})$$

$$P_5(G_{N,2^3}) = P_{101}(G_{N,2^3}) = (\underline{\underline{e_2}, \underline{e_3}, \underline{e_0}, \underline{e_1}}, \underline{\underline{e_6}, \underline{e_7}, \underline{e_4}, \underline{e_5}})$$

$$P_6(G_{N,2^3}) = P_{110}(G_{N,2^3}) = (\underline{\underline{e_1, e_0}, \underline{e_3, e_2}}, \underline{\underline{e_5, e_4}, \underline{e_7, e_6}})$$

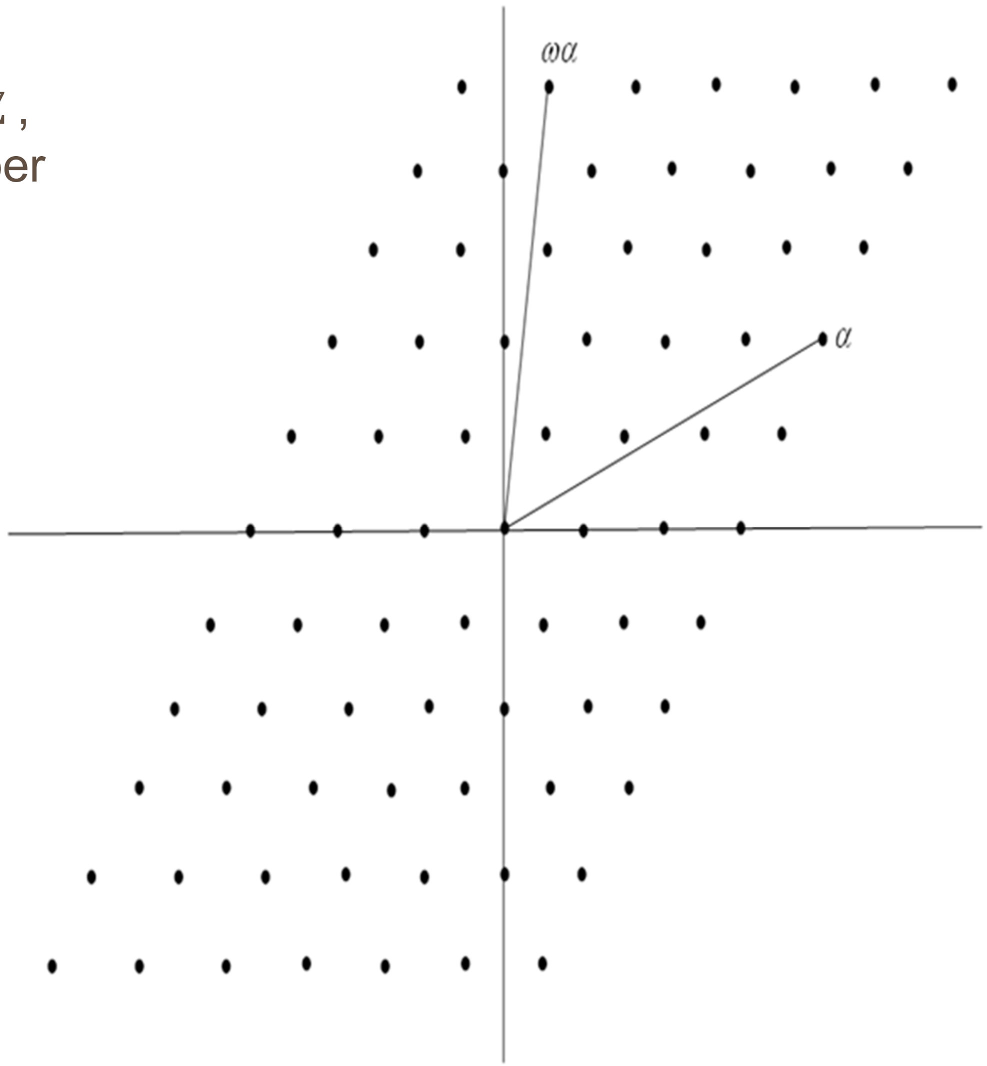
$$P_7(G_{N,2^3}) = P_{111}(G_{N,2^3}) = (\underline{\underline{\underline{e_0}, \underline{e_1}}, \underline{\underline{e_2}, \underline{e_3}}}, \underline{\underline{\underline{e_4}, \underline{e_5}}, \underline{\underline{e_6}, \underline{e_7}}})$$

Outline

- Introduction
- Gaussian Networks
  - Gaussian Integers
  - Mod Operation over $a + bi$
  - Interconnection Topology
  - Routing
  - Broadcasting
  - Hamiltonian Decomposition
- EJ Networks
  - EJ Integers
  - Mod Operation
  - Interconnection Topology
  - Routing
  - Hamiltonian Decomposition
- Summary

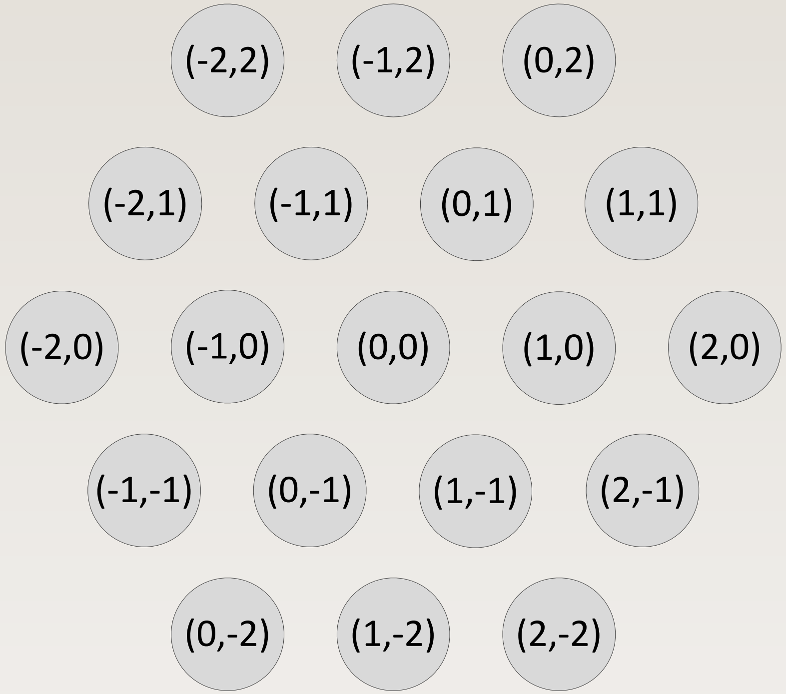
Eisenstein-Jacobi (EJ) numbers

- $(x, y) = x + y\omega; x, y \in \mathbb{Z}$, Eisenstein-Jacobi number
- where $\omega = \frac{1+i\sqrt{3}}{2}$

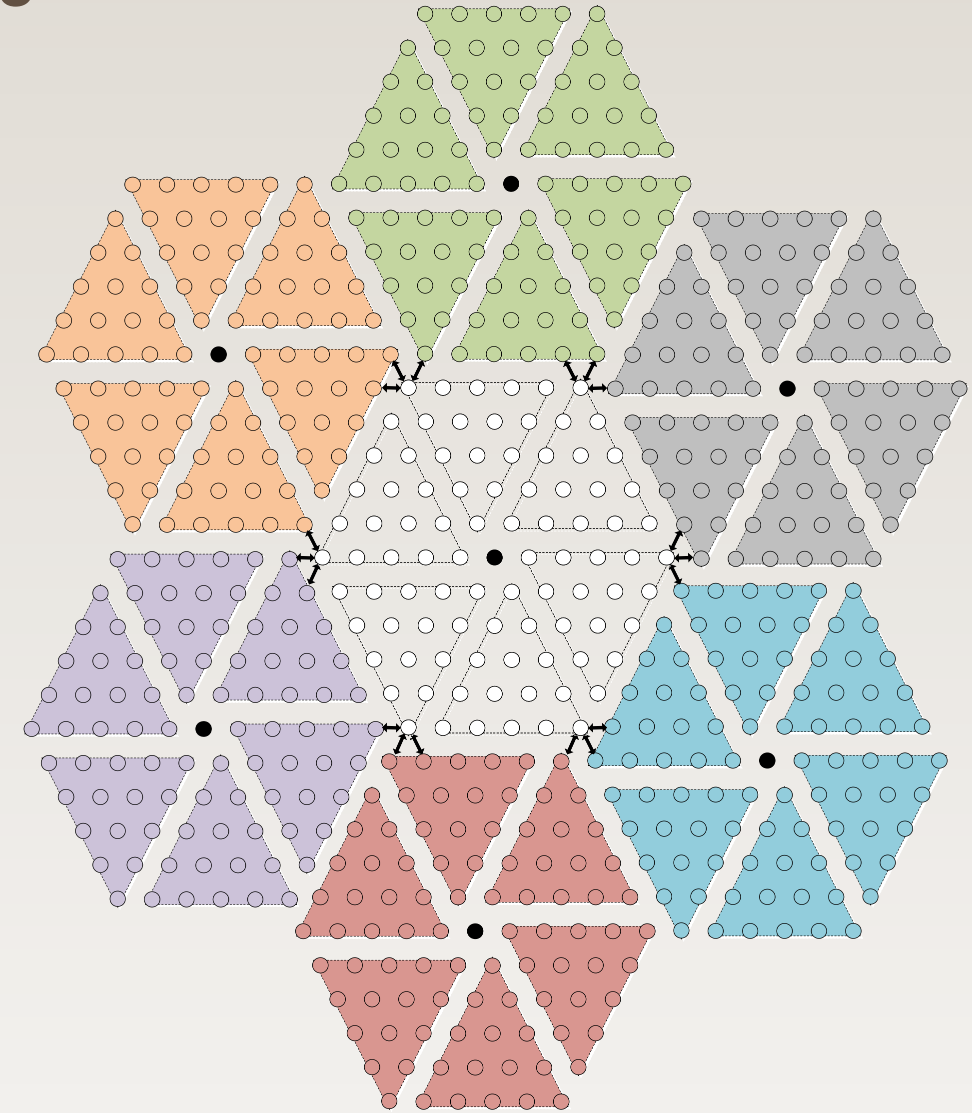


Eisenstein-Jacobi (EJ) Networks

- EJ network is generated by a fixed EJ number $\alpha = a + b\omega$
- $A = x_1 + y_1\omega$ and $B = x_2 + y_2\omega$ are adjacent
  - iff $A - B = \pm 1 \ or \pm \omega \ or \pm \omega^2 \ mod \ \alpha$
- The HARTS network $\mathcal{H}_n$ is the EJ network generated by α=n+(n−1)ω
  - No. of nodes $N = a^2 + ab + b^2$
  - Diameter $n - 1$

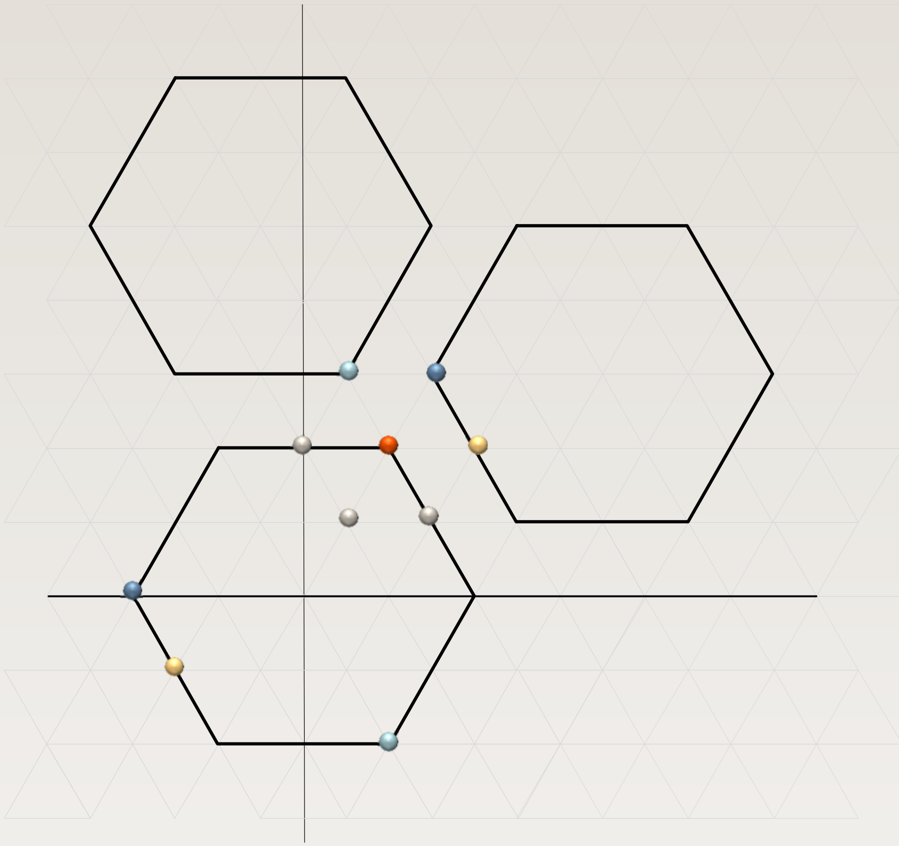


Tiling in EJ



Wraparound – Mod Operation

- $\mathcal{H}_3$ generated by $\alpha = 3 + 2\omega$
- Node $\langle 0,2 \rangle$ is adjacent to six nodes:
- $\langle 1,1 \rangle$, $\langle 0,1 \rangle$, $\langle -1,2 \rangle$,$\langle -1,3 \rangle$, $\langle 0,3 \rangle$, $\langle 1,2 \rangle$
- $\langle 2,-2 \rangle$, $\langle -2,0 \rangle$, $\langle -1,-1 \rangle$
- The hexagons are translated to be centered at the origin
- Modulo α operation

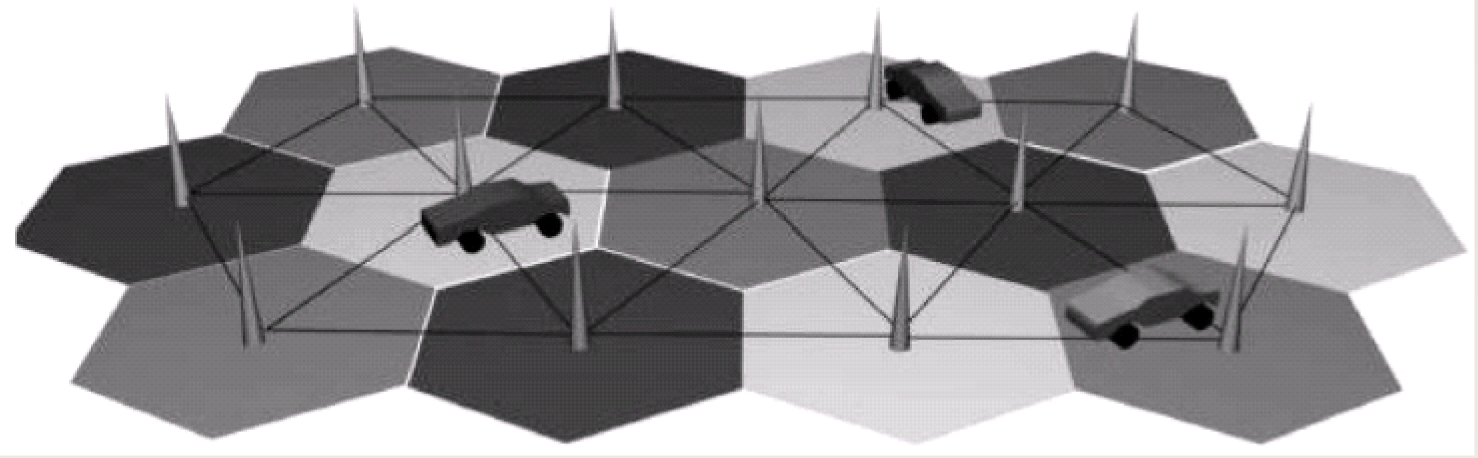


Advantage of EJ Number Representation

- First attempt (Chen, Shin, Kandlur: IEEE Trans. Computers, 1990)
  - 3 components
  - Each component gives the order of a node in its cycle
  - # of bits required: $3\lceil \log N \rceil$
- Second attempt (Garcia, Stojmenovic, Zhang: IEEE Trans. PDS, 2002)
  - 3 components
  - Each component gives the order of a node in 3 directions 120 degree apart
  - # of bits required: $3(\lceil \log t \rceil + 1)$ where t is diameter
- Our method
  - 2 components
  - Each component gives the order of a node in E and NE directions
  - # of bits required: $2(\lceil \log t \rceil + 1)$ where t is diameter
- For example t=3
  - First method needs 18 bits
  - Second method needs 9 bits
  - Our method needs 6 bits

EJ Networks Application

- Wireless Networks

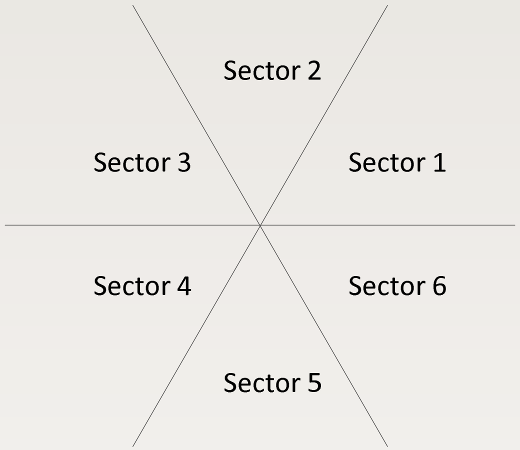


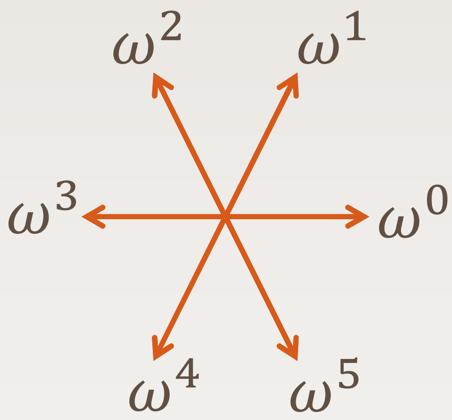
Shortest Path Routing

Note that: $\omega^2 = \omega - 1; \omega^3 = -1;\ \omega^4 = -\omega; \omega^5 = 1 - \omega; \omega^6 = 1;$

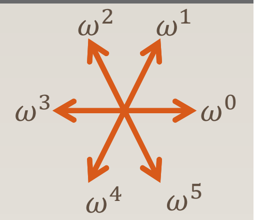
$$D - S = x + y\omega = a\omega^{j-1} + b\omega^j$$
$$x, y, a, \mathrm{b}\ \in \mathbb{Z}\,; \mathrm{a}, \mathrm{b} \geq 0$$

- The sign of x, y defines the sector j
- Shortest path: a nodes in ω^{j-1} and b nodes in ω^j direction





Routing Examples



$\mathcal{H}_5$ generated by $\alpha = 5 + 4\omega$

$S = 4 - \omega; D = 2\omega$

$D - S = -4 + 3\omega = \omega^3 + 3\omega^2$

Sector 3

- 1 nodes in ω^3, 3 nodes in ω^2

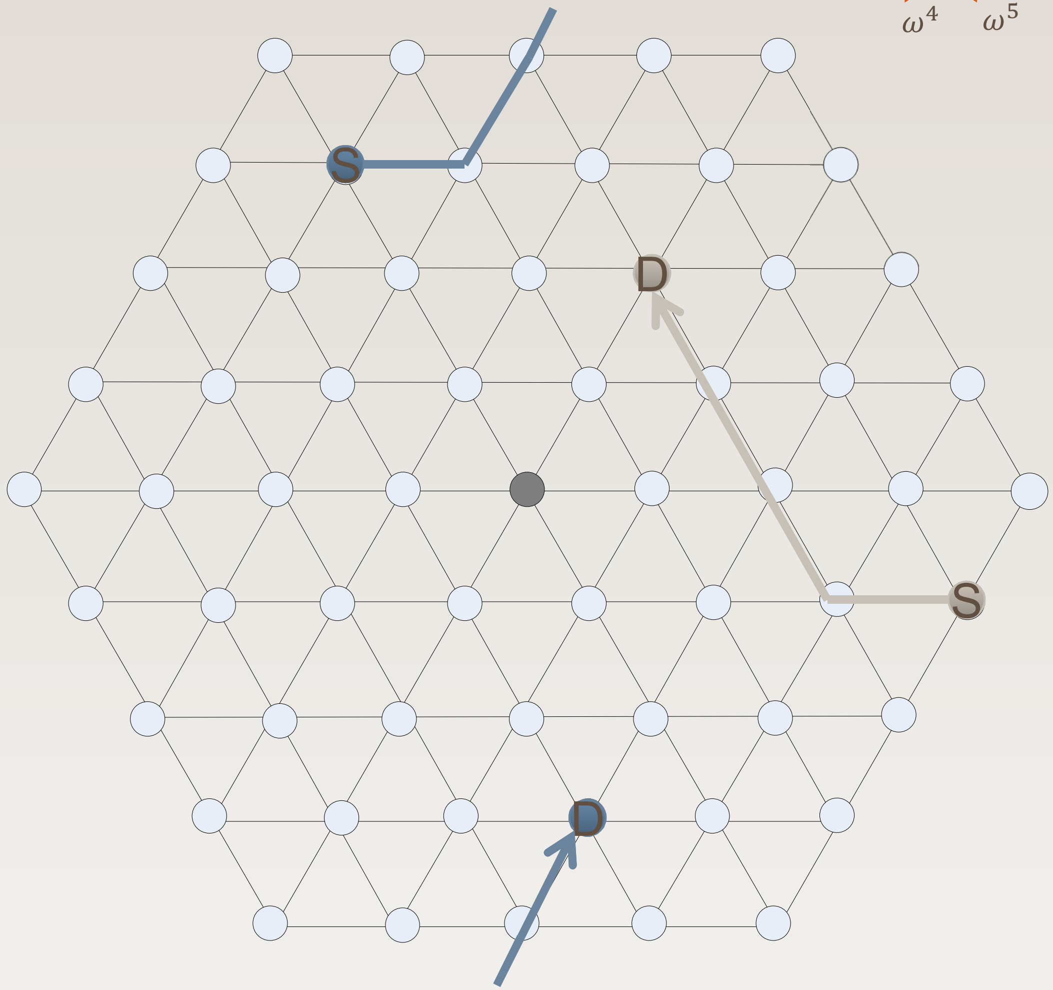
$S = -3 + 3\omega; D = 2 - 3\omega$

$D - S = 5 - 6\omega = \omega^4 + 5\omega^5$

- Modulo α
  - $\omega^4\alpha = 4 - 9\omega$
  - $5 - 6\omega$-$(4 - 9\omega)$=$1 + 3\omega$

Sector 1

- 1 nodes in ω^0, 3 nodes in ω^1



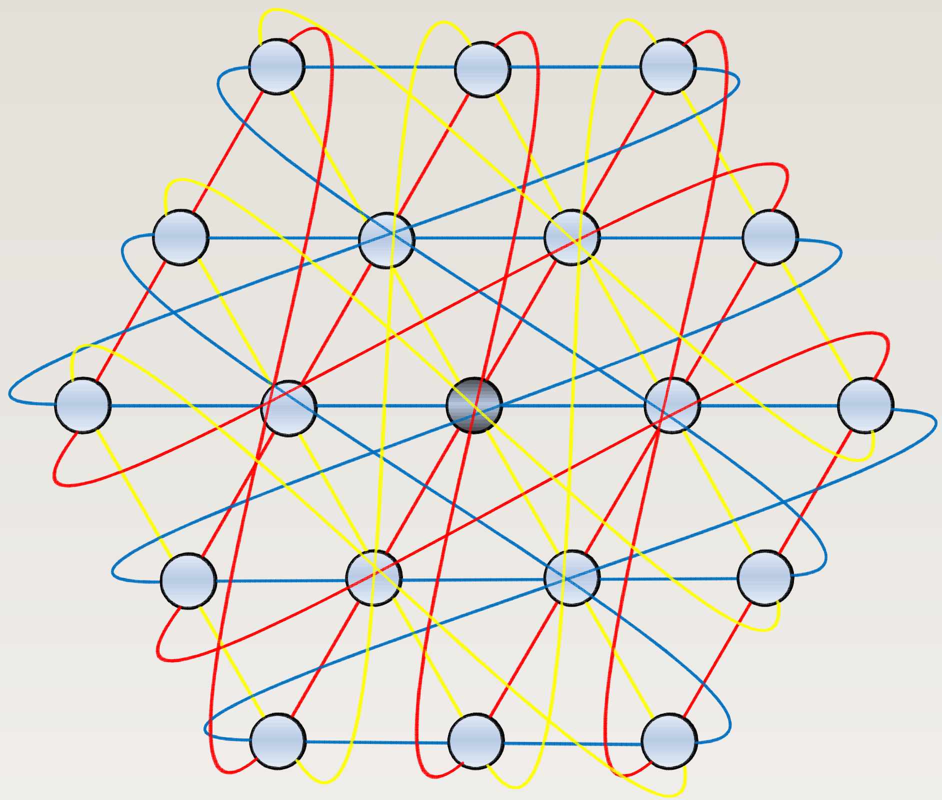
Hamiltonian Decomposition

- $\alpha = a + b\omega$
- $\mathrm{N} = a^2 + ab + b^2$
- $GCD(a, b) = 1$

- Example $\alpha = 2 + 3\omega$

  Start at 0, Go along

  1. $1, 2, 3, \ldots, N - 1$
  2. $\omega, 2\omega, 3\omega, \ldots, (N-1)\omega$
  3. $\omega^2, 2\omega^2, 3\omega^2, \ldots, (N-1)\omega^2$



Multidimensional EJ

$$H_\alpha^{(2)} = H_\alpha \otimes H_\alpha$$

$$= (H_1^N \oplus H_2^N \oplus H_3^N) \otimes (H_4^N \oplus H_5^N \oplus H_6^N)$$

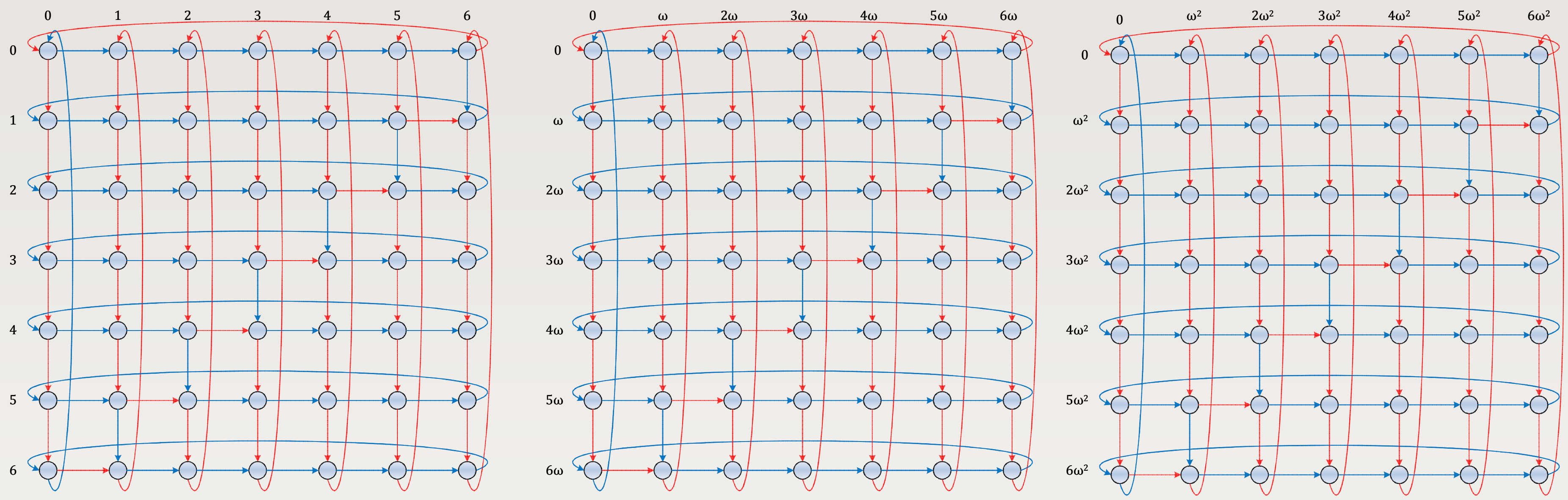
$$= (H_1^N \otimes H_4^N) \oplus (H_2^N \otimes H_5^N) \oplus (H_3^N \otimes H_6^N)$$

$$= T_{N\times N}^1 \oplus T_{N\times N}^2 \oplus T_{N\times N}^3$$

$$= (H_1^{N\times N} \oplus H_2^{N\times N}) \oplus (H_3^{N\times N} \oplus H_4^{N\times N}) \oplus (H_5^{N\times N} \oplus H_6^{N\times N})$$

Hamiltonian Decomposition in 2D EJ

- H_α is Hexagonal network generated by $\alpha = 1 + 2\omega$
- ${H_\alpha}^2 = H_\alpha \times H_\alpha$
  - # of nodes: 49
  - Diameter: 2
  - Degree: 12



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Summary

- Gaussian networks is a generalization of torus networks
- Gaussian networks has much better performance compared to equivalent torus networks
- Hexagonal networks is a special case of EJ networks
- Described efficient communication algorithms

THANK YOU

QUESTIONS?

Shortest Path Routing (2)

Sector	Signs	Shortest Path Routing
1	$x, y \geq 0$	x nodes in ω^0, y nodes in ω^1
2	$x \leq 0, y \geq 0, \|x\| \leq \|y\|$	$\|y\| - \|x\|$ nodes in ω^1, $\|x\|$ nodes in ω^2
3	$x \leq 0, y \geq 0, \|x\| \geq \|y\|$	$\|x\| - \|y\|$ nodes in ω^3, y nodes in ω^2
4	$x, y \leq 0$	$\|x\|$ nodes in ω^3, $\|y\|$ nodes in ω^4
5	$x \geq 0, y \leq 0, \|x\| \leq \|y\|$	$\|y\| - \|x\|$ nodes in ω^4, x nodes in ω^5
6	$x \geq 0, y \leq 0, \|x\| \geq \|y\|$	$\|x\| - \|y\|$ nodes in ω^6, $\|y\|$ nodes in ω^5

How to do mod α operation

- $\alpha = a + bi;\ \beta\ mod\ \alpha = ?$
- $(a + bi)(x + yi) = c + di$
- $(ax - by) + (ay + bx)i = c + di$
  - $(ax - by) = c$
  - $(ay + bx) = d$
- $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$
- Perform
  1. $\beta - (\lfloor x \rfloor + \lfloor y \rfloor i)\alpha$
  2. $\beta - ((\lfloor x \rfloor + 1) + \lfloor y \rfloor i)\alpha$
  3. $\beta - (\lfloor x \rfloor + (\lfloor y \rfloor + 1)i)\alpha$
  4. $\beta - \big((\lfloor x \rfloor + 1) + (\lfloor y \rfloor + 1)i\big)\alpha$
  - Take the min weight vector