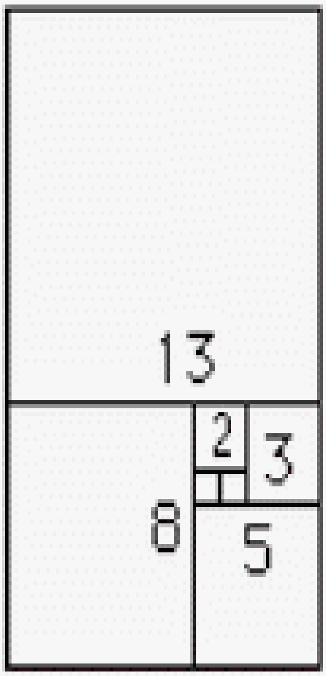
Fibonacci Numbers, $F_n = F_{n-2} + F_{n-}$

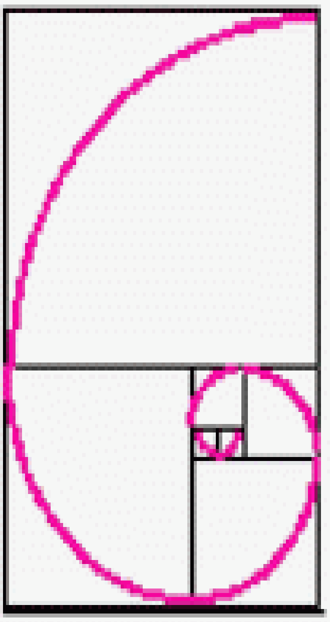
$F_1 = F_2 = 1$

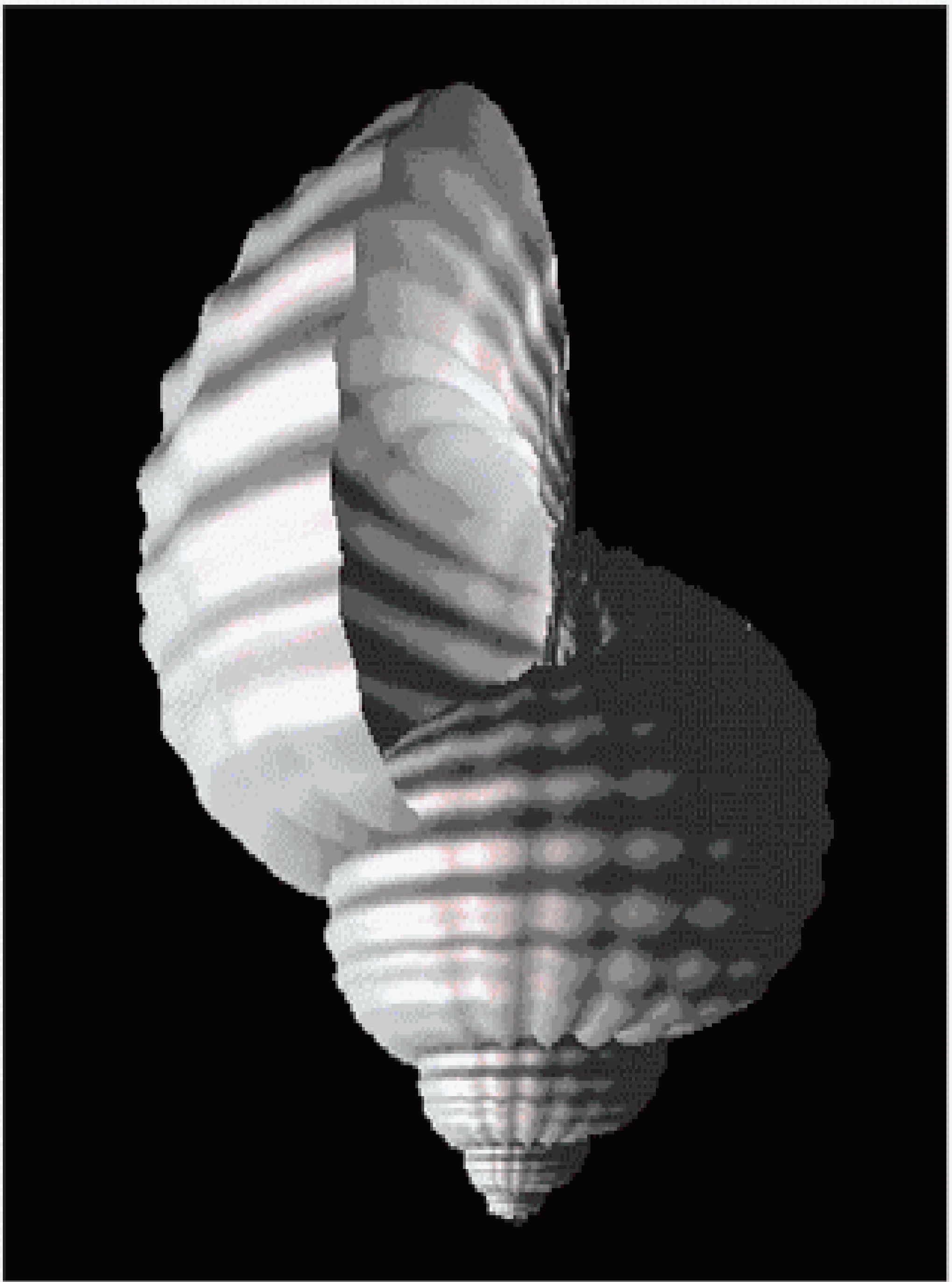


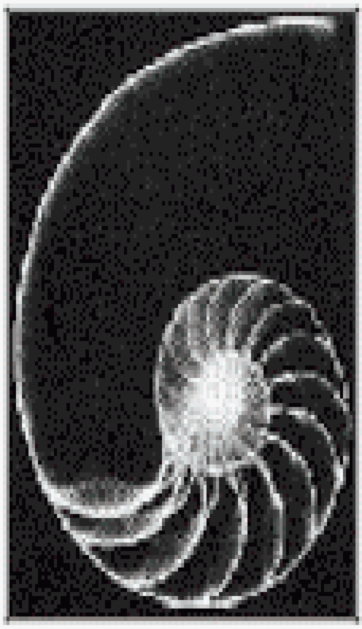
Golden Ratio

$$g = F_{n+1} / F_n = 1.62$$

and the Beauty of Life







Fibonacci Numbers and the Golden Section

This is the **Home page** for the Fibonacci numbers, the Golden section and the Golden string.

The **Fibonacci numbers** are **0, 1, 1, 2, 3, 5, 8, 13, ...** (add the last two to get the next)

The **golden section numbers** are **±0·61803 39887... and ±1·61803 39887...**

The **golden string** is **1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 ...**
a sequence of 0s and 1s which is closely related to the Fibonacci numbers and the golden section.

There is a large amount of information at this site (more than 200 pages if it was printed), so if all you want is a quick introduction then the first link takes you to **an introductory page** on the Fibonacci numbers and where they appear in Nature.

The rest of this page is a brief introduction to all the web pages at this site on **Fibonacci Numbers** the **Golden Section** and the **Golden String** together with ***their many applications***.

What's New? 7 June 2001

A recent back-up error means that I've just lost all emails sent to me during March and April.
Please can you re-send your email if you've had no reply - sorry!

Fibonacci Numbers and Golden sections in Nature

- Fibonacci Numbers and Nature

  Fibonacci and the original problem about rabbits where the series first appears, the family trees of cows and bees, the golden ratio and the Fibonacci series, the Fibonacci Spiral and sea shell shapes, branching plants, flower petal and seeds, leaves and petal arrangements, on pineapples and in apples, pine cones and leaf arrangements. All involve the Fibonacci numbers - and here's how and why.

- The Golden section in Nature

  Continuing the theme of the first page but with specific reference to **why** the golden section appears in nature. NEW Now with a Geometer's Sketchpad dynamic demonstration.

The Puzzling World of Fibonacci Numbers

A pair of pages with plenty of playful problems to perplex the professional and the part-time puzzler!

The Easier Fibonacci Puzzles page

has the Fibonacci numbers in brick wall patterns, Fibonacci bee lines, seating people in a row and the Fibonacci numbers again, giving change and a game with match sticks and even with electrical resistance and lots more puzzles all involve the Fibonacci numbers!

The Harder Fibonacci Puzzles page

still has problems where the Fibonacci numbers are the answers - well, all but ONE, but WHICH one? If you know the Fibonacci Jigsaw puzzle where rearranging the 4 wedge-shaped pieces makes an additional square appear, did you know the *same* puzzle can be rearranged to make a different shape where a square now *disappears*? For these puzzles, I do not know of any simple explanations of ***why*** the Fibonacci numbers occur - and **that's the real puzzle - can you supply a simple reason why??**

The Intriguing Mathematical World of Fibonacci and Phi

The golden section numbers are also written using the greek letters Phi Φ and phi φ.

The Mathematical Magic of the Fibonacci numbers

looks at the patterns in the Fibonacci numbers themselves, the Fibonacci numbers in Pascal's Triangle and using Fibonacci series to generate all right-angled triangles with integers sides based on Pythagoras Theorem.
Impress your friends with a simple Fibonacci numbers trick!
There are many investigations for you to do to find patterns for yourself as well as a complete list of...

The first 500 Fibonacci numbers...

completely factorised up to Fib(300) and all the prime Fibonacci numbers are identified.

A Formula for the Fibonacci numbers

Is there a direct formula to compute Fib(n) just from n? Yes there is!

This page shows several and why they involve Phi and phi - the golden section numbers.

Fibonacci bases and other ways of representing integers

We use base 10 (decimal) for written numbers but computers use base 2 (binary). What happens if we use the Fibonacci numbers as the column headers?

The Golden Section - the Number and Its Geometry

The golden section is also called the golden ratio, the golden mean and the divine proportion. It is closely connected with the Fibonacci series and has a value of (√5 - 1)/2 which is 0·61803... which we call phi on these pages. It has some interesting properties such as 1/phi is the same as 1+phi and we call this value Phi= (√5 + 1)/2.

Two pages are devoted to its applications in Geometry - first in flat (or two dimensional) geometry and then in the solid geometry of three dimensions.

Fantastic Flat Phi Facts

See some of the unexpected places that the golden section (Phi) occurs in Geometry and in Trigonometry: pentagons and decagons, paper folding and Penrose Tilings where we phind phi phrequently!

The Golden Geometry of the Solid Section or Phi in 3 dimensions

The golden section occurs in the most symmetrical of all the three-dimensional solids - the Platonic solids. What are the best shapes for fair dice? Why are there only 5?

The next pages are about the number Phi = 1·61803.. itself and its close cousin phi = 0·61803... .

Phi's Fascinating Figures - the Golden Section number

All the powers of Phi are just whole multiples of itself plus another whole number. Did you guess that these multiples *and* the whole numbers are, of course, the Fibonacci numbers again? Each power of Phi is the sum of the previous two - just like the Fibonacci numbers too.

Introduction to Continued Fractions An optional page that expands on the idea of a continued fraction introduced in the Phi's Fascinating Figures page.

Phigits and Base Phi Representations

We have seen that using a *base* of the Fibonacci Numbers we can represent all integers in a binary-like way. Here we show there is an interesting way of representing *all* integers in a binary-like fashion but using only powers of Phi instead of powers of 2 (binary) or 10 (decimal).

The Golden String

The golden string also referred to as the Infinite Fibonacci Word or the Fibonacci Rabbit sequence.

Fibonacci Rabbit Sequence

There is another way to look at Fibonacci's Rabbits problem that gives an infinitely long sequence of 1s and 0s, which we will call the Fibonacci Rabbit sequence:-

1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 ...

which is a close relative of the golden section and the Fibonacci numbers. You can hear the Golden sequence as a Quicktime movie track too!

The Fibonacci Rabbit sequence is an example of a **fractal**.

Fibonacci - the Man and His Times

Who was Fibonacci?

Here is a brief biography of Fibonacci and his historical achievements in mathematics, and how he helped Europe replace the Roman numeral system with the "algorithms" that we use today.

Also there is a guide to some memorials to Fibonacci to see in Pisa, Italy.

More Applications of Fibonacci Numbers and Phi

The Fibonacci numbers in a formula for Pi (III)

There are several ways to compute pi (3·14159 26535 ..) accurately. One that has been used a lot is based on a nice formula for calculating which angle has a given tangent, discovered by James Gregory. His formula together with the Fibonacci numbers can be used to compute pi. This page introduces you to all these concepts from scratch.

- Fibonacci Forgeries

  Sometimes we find series that for quite a few terms look exactly like the Fibonacci numbers, but, when we look a bit more closely, they aren't - they are Fibonacci Forgeries.
  Since we would not be telling the truth if we said they *were* the Fibonacci numbers, perhaps we should call them **Fibonacci Fibs** !!

- The Lucas Numbers

  Here is a series that is very similar to the Fibonacci series, **the Lucas series**, but it starts with 2 and 1 instead of Fibonacci's 0 and 1. It sometimes pops up in the pages above so here we investigate it some more and discover its properties.
  It ends with a number trick which you can use "to impress your friends with your amazing calculating abilities" as the adverts say. It uses facts about the golden section and its relationship with the Fibonacci and Lucas numbers.

  - The first 100 Lucas numbers and their factors

    together with some suggestions for investigations you can do.

- The Golden Section In Art, Architecture and Music

  The golden section has been used in many designs, from the ancient **Parthenon** in Athens (400BC) to Stradivari's violins. It was known to artists such as **Leonardo da Vinci** and musicians and composers, notably **Bartók** and **Debussy**. This is a different page to those above, being concerned with speculations about where the golden section both does and does not occur in art, architecture and music. All the other pages are factual and verifiable - the material here is a often a matter of opinion - but interesting nevertheless!

Links and References

- Fibonacci, Phi and Lucas numbers Formulae

  A reference page of over 100 formulae and equations showing the properties of these series.
  NEW Now available in PDF format (96K) for which you will need the free Acrobat PDF Reader or plug-in.

- Links and references

  Links to other sites on Fibonacci numbers and the Golden section together with references to books and articles.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Awards for this WWW site

Each icon is a link to lists of other Award winning sites that opens in a new window. Check them out!

The Knot a Braid of Links Project at Camel designated this page a cool math site of the week for 22-28 November 1998 (now available via in the Kabol Database search engine).

This site is listed in the BBC Education Web Guide (January 1999).

LÄNKSKAFFERIET The Link Larder [in Swedish], part of the Swedish Schoolnet.

StudyWeb has given Academic Excellence Awards to four pages at this site: The Fibonacci numbers in a formula for Pi, The Fibonacci numbers and Nature, Introduction to Continued Fractions and Who was Fibonacci?

Links^{2}Go has designated The Fibonacci numbers in a formula for Pi as a **Key Resource** on the topic of **Constants**.
Other citations

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Hosted by the Department of Computing of Surrey University in Guildford in the county of Surrey in the UK where the author was a Lecturer in the Mathematics and Computing departments for many years.

 R.Knott@surrey.ac.uk updated: 21 March 2001

Fibonacci Numbers and Nature

This page has been split into TWO PARTS.

This, the **first**, looks at the Fibonacci numbers and why they appear in various "family trees" and patterns of spirals of leaves and seeds.

The second page then examines why the golden section is used by nature in some detail, including animations of growing plants.

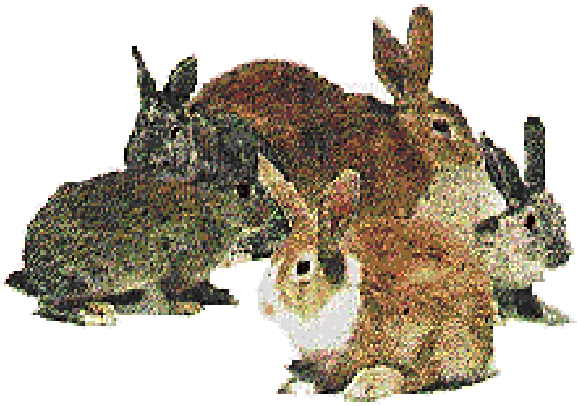
Contents of this Page

The line means there is a Things to do investigation at the end of the section.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Fibonacci's Rabbits

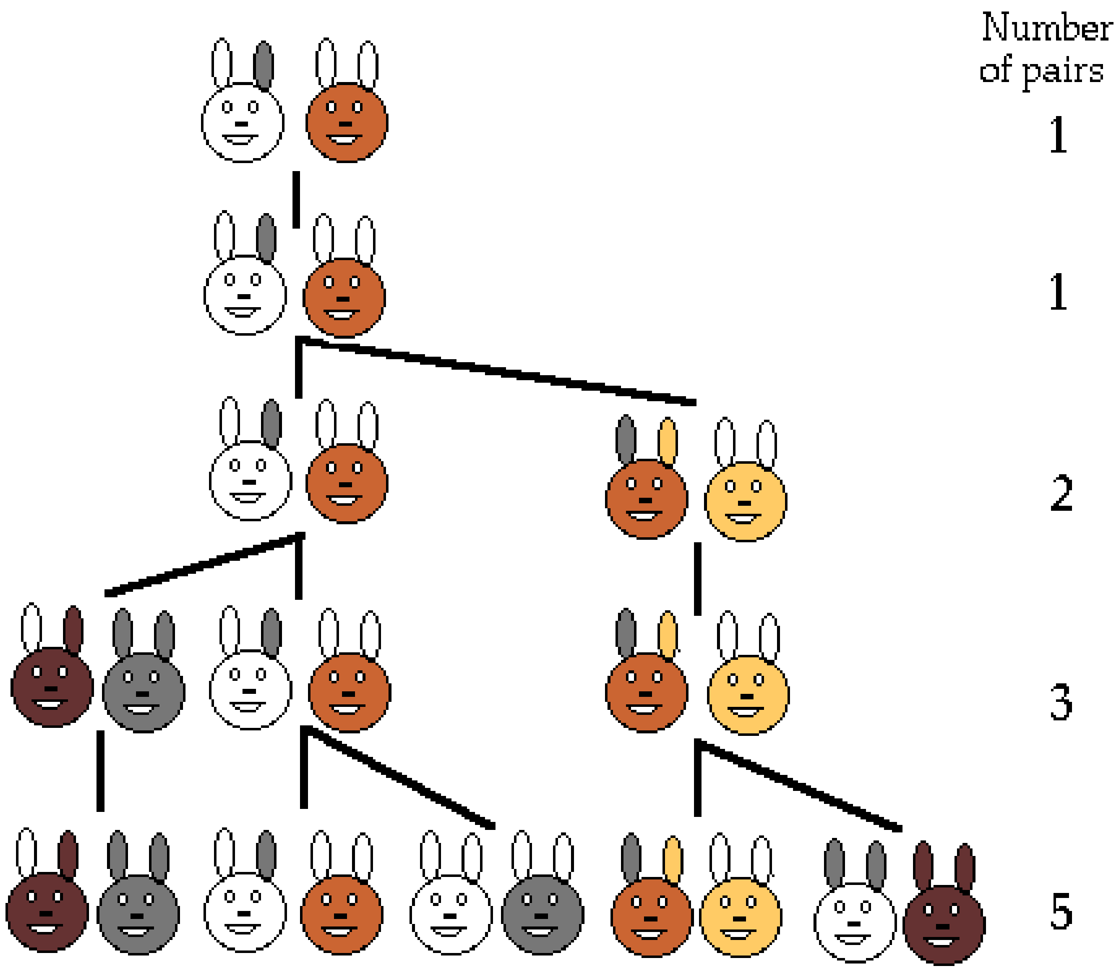
The original problem that Fibonacci investigated (in the year 1202) was about how fast rabbits could breed in ideal circumstances.



Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits. Suppose that our rabbits **never die** and that the female **always** produces one new pair (one male, one female) **every month** from the second month on. The puzzle that Fibonacci posed was...

How many pairs will there be in one year?

1. At the end of the first month, they mate, but there is still one only 1 pair.
2. At the end of the second month the female produces a new pair, so now there are 2 pairs of rabbits in the field.
3. At the end of the third month, the original female produces a second pair, making 3 pairs in all in the field.
4. At the end of the fourth month, the original female has produced yet another new pair, the female born two months ago produces her first pair also, making 5 pairs.

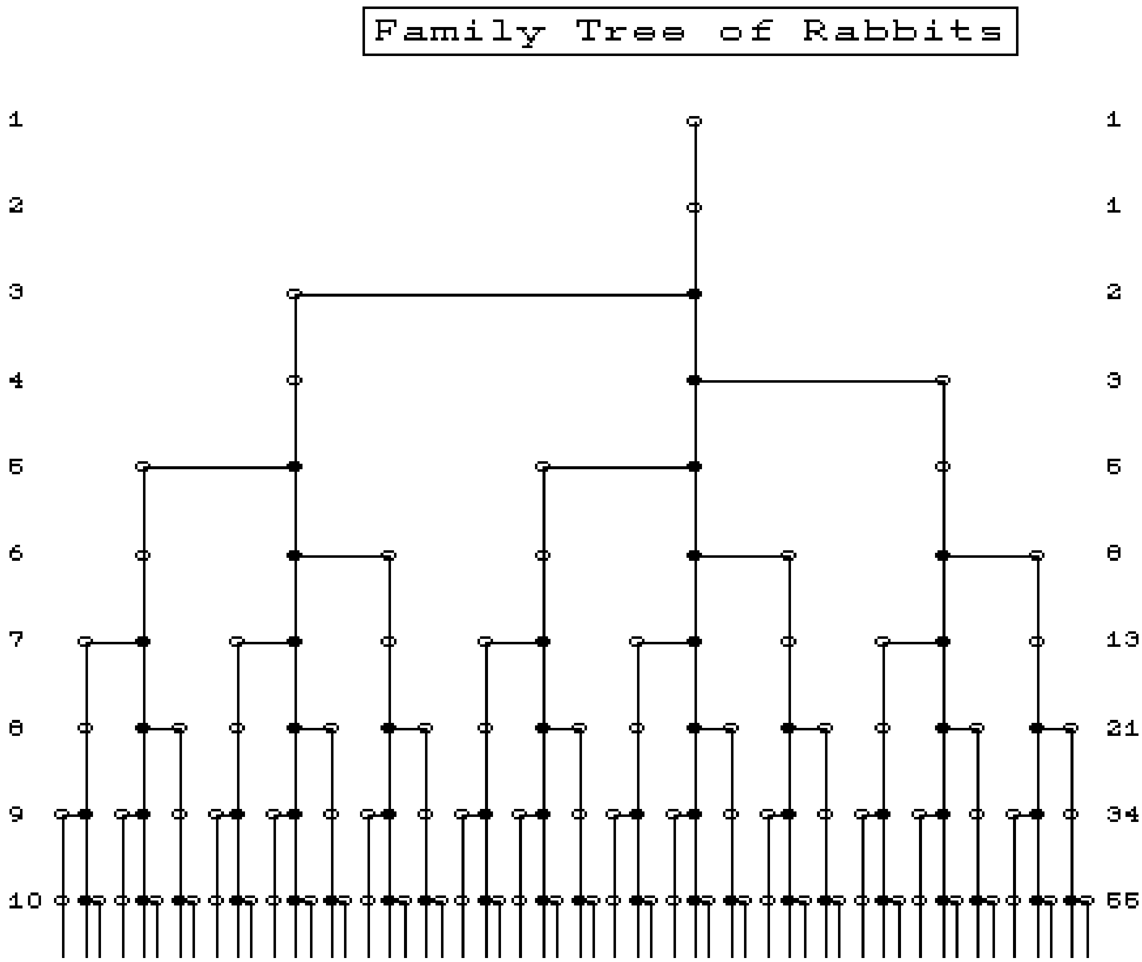


The number of pairs of rabbits in the field at the start of each month is 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

Can you see how the series is formed and how it continues? If not, look at the answer!

The first 100 Fibonacci numbers are here and some questions for you to answer.

Now can you see **why** this is the answer to our Rabbits problem? If not, here's why.
Another view of the Rabbit's Family Tree:



0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

The Rabbits problem is not very realistic, is it?

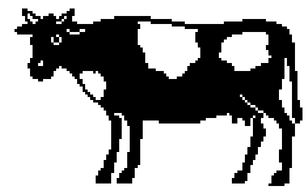
It seems to imply that brother and sisters mate, which, genetically, leads to problems. We can get round this by saying that the female of each pair mates with any male and produces another pair.
Another problem which again is not true to life, is that each birth is of exactly two rabbits, one male and one female.

Dudeney's Cows

The English puzzlist, Henry E Dudeney (1857 - 1930) wrote several excellent books of puzzles. In one of them he adapts Fibonacci's Rabbits to cows, making the problem more realistic in the way we observed above. He gets round the problems by noticing that really, it is only the females that are interesting - er - I mean the *number* of females!

He changes months into years and rabbits into bulls (male) and cows (females) in problem 175 in his book **536 puzzles and Curious Problems** (1967, Souvenir press):

> If a cow produces its first she-calf at age two years and after that produces another single she-calf every year, how many she-calves are there after 12 years, assuming none die?



This is a better simplification of the problem and quite realistic now.

But Fibonacci does what mathematicians often do at first, simplify the problem and see what happens - and the series bearing his name does have *lots* of other interesting and practical applications as we see later. So let's look at another real-life situation that is exactly modelled by Fibonacci's series - honeybees.

Honeybees, Fibonacci numbers and Family trees

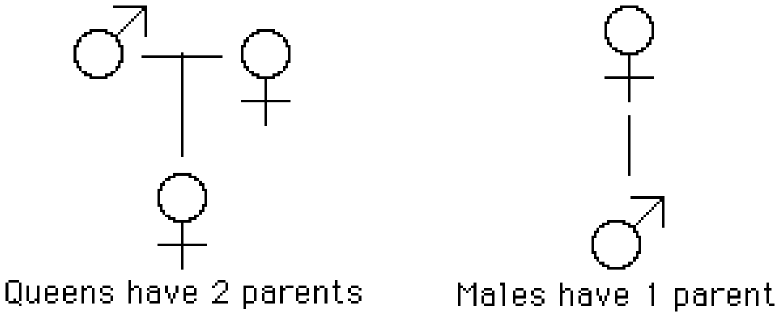
There are over 30,000 species of bees and in most of them the bees live solitary lives. The one most of us know best is the honeybee and it, unusually, lives in a colony called a hive and they have an unusual Family Tree. In fact, there are many unusual features of honeybees and in this section we will show how the Fibonacci numbers count a honeybee's ancestors (in this section a "bee" will mean a "honeybee"). First, some unusual facts about honeybees such as: not all of them have two parents!

- In a colony of honeybees there is one special female called the **queen**.
- There are many **worker** bees who are female too but unlike the queen bee, they produce no eggs.
- There are some **drone** bees who are male and do no work.

Males are produced by the queen's unfertilised eggs, so male bees only have a mother but no father!

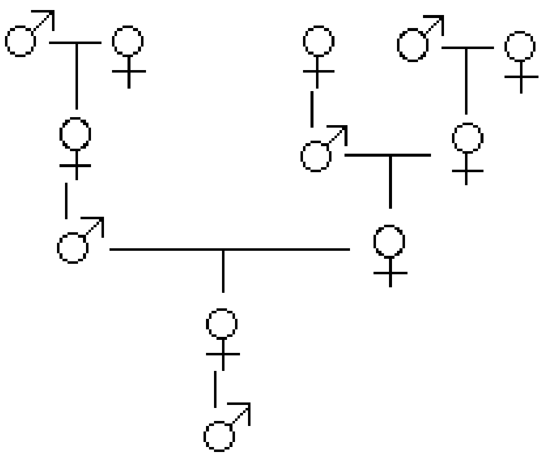
- All the females are produced when the queen has mated with a male and so have two parents.

Females usually end up as worker bees but some are fed with a special substance called **royal jelly** which makes them grow into queens ready to go off to start a new colony when the bees form a **swarm** and leave their home (a **hive**) in search of a place to build a new nest.



So female bees have 2 parents, a male and a female whereas male bees have just one parent, a female.

Here we follow the convention of Family Trees that *parents appear above their children*, so the latest generations are at the bottom and the higher up we go, the older people are. Such trees show all the *ancestors* (predecessors, forebears, antecedents) of the person at the bottom of the diagram. We would get quite a different tree if we listed all the *descendants* (progeny, offspring) of a person as we did in the rabbit problem, where we showed all the descendants of the original pair.



Let's look at the family tree of a male drone bee.

1. He had **1** parent, a female.
2. He has **2** grand-parents, since his mother had two parents, a male and a female.
3. He has **3** great-grand-parents: his grand-mother had two parents but his grand-father had only one.
4. How many great-great-grand parents did he have?

Again we see the Fibonacci numbers :

Number of	parents:	grand-parents:	great-grand-parents:	great,great grand parents:	gt,gt,gt grand parents:
of a MALE bee:	1	2	3	5	8
of a FEMALE bee:	2	3	5	8	13

The Fibonacci Sequence as it appears in Nature by S.L.Basin in *Fibonacci Quarterly*, vol 1 (1963), pages 53 - 57.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Things to do

1. Make a diagram of your own family tree. Ask your parents and grandparents and older relatives as each will be able to tell you about particular parts of your family tree that other's didn't know. It can be quite fun trying to see how far back you can go. If you have them put old photographs of relatives on a big chart of

your Tree (or use photocopies of the photographs if your relatives want to keep the originals). If you like, include the year and place of birth and death and also the dates of any marriages.

2. A brother or sister is the name for someone who has the same two parents as yourself. What is a *half-brother* and *half-sister*? Describe *a cousin* but use simpler words such as *brother, sister, parent, child*?
   Do the same for *nephew* and *niece*. What is *a second cousin*? What do we mean by *a bother-in-law, sister-in-law, mother-in-law, etc*?
   Grand- and *great-* refer to relatives *or your parents*. Thus a *grand-father* is a father of a parent of yours and *great-aunt* or *grand-aunt* is the name given to an aunt of your parent's.

   Make a diagram of Family Tree Names so that "Me" is at the bottom and "Mum" and "Dad" are above you. Mark in "brother", "sister", "uncle", "nephew" and as many other names of (kinds of) relatives that you know. It doesn't matter if you have no brothers or sisters or nephews as the diagram is meant to show the relationships and their names.
   [If you have a friend who speaks a foreign language, ask them what words they use for these relationships.]
3. What is the name for *the wife of a parent's brother*?
   Do you use a different name for the sister of your parent's?
   In law these two are sometimes distinguished because one is a *blood relative* of yours and the other is not, just a relative through marriage.
   Which do you think is the *blood relative* and which the relation because of marriage?
4. How many parents does everyone have?
   So how many grand-parents will you have to make spaces for in your Family tree?
   Each of them also had two parents so how many great-grand-parents of yours will there be in your Tree?
   ..and how many great-great-grandparents?
   What is the pattern in this series of numbers?
   If you go back one generation to your parents, and two to your grand-parents, how many entries will there be 5 generations ago in your Tree? and how many 10 generations ago?

   The Family Tree of humans involves a different sequence to the Fibonacci Numbers. What is this sequence called?
5. Looking at your answers to the previous question, your friend Dee Duckshun says to you:

- **You have 2 parents.**
- **They each have two parents, so that's 4 grand-parents you've got.**
- **They also had two parents each making 8 great-grand-parents in total ...**
- **... and 16 great-great-grand-parents ...**
- **... and so on.**
- **So the farther back you go in your Family Tree the more people there are.**
- **It is the same for the Family Tree of *everyone* alive in the world today.**
- **It shows that the farther back in time we go, the more people there must have been.**
- **So it is a logical deduction that the population of the world *must* be getting smaller and smaller as time goes on!**

Is there an error in Dee's argument? If so, what is it? Ask your maths teacher or a parent if you are not sure of the answer!

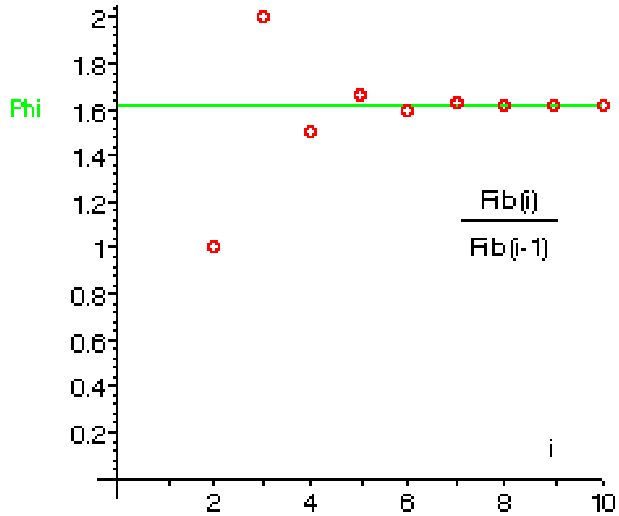
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Fibonacci numbers and the Golden Number

If we take the ratio of two successive numbers in Fibonacci's series, (1, 1, 2, 3, 5, 8, 13, ..) and we divide each by the number before it, we will find the following series of numbers:

$^1/_1 = 1$, $^2/_1 = 2$, $^3/_2 = 1{\cdot}5$, $^5/_3 = 1{\cdot}666...$, $^8/_5 = 1{\cdot}6$, $^{13}/_8 = 1{\cdot}625$, $^{21}/_{13} = 1{\cdot}61538...$

It is easier to see what is happening if we plot the ratios on a graph:



The ratio seems to be settling down to a particular value, which we call **the golden ratio** or **the golden number**. It has a value of approximately **1·61804** , although we shall find an even more accurate value on [a later page [this link opens a new window]](#) .

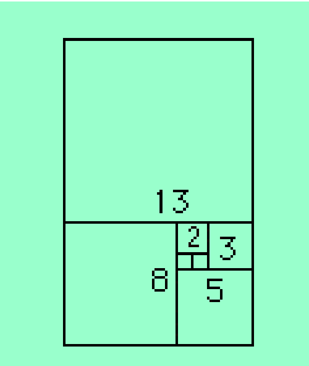
Things to do

- What happens if we take the ratios the other way round i.e. we divide each number by the one *following* it: 1/1, 1/2, 2/3, 3/5, 5/8, 8/13, ..?
  Use your calculator and perhaps plot a graph of these ratios and see if anything similar is happening compared with the graph above.
  You'll have spotted a fundamental property of this ratio when you find the limiting value of the new series!

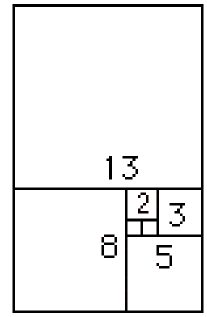
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

The **golden ratio** 1·618034 is also called the **golden section** or the **golden mean** or just the **golden number**. It is often represented by a greek letter **Phi** ϕ. The closely related value which we write as **phi** with a small "p" is just the decimal part of Phi, namely 0·618034.

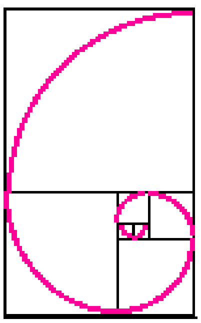
The Fibonacci Rectangles and Shell Spirals



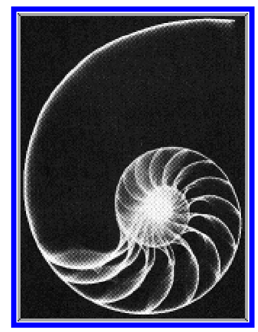
We can make another picture showing the Fibonacci numbers 1,1,2,3,5,8,13,21,.. if we start with two small squares of size 1 next to each other. On top of both of these draw a square of size 2 (=1+1).



We can now draw a new square - touching both a unit square and the latest square of side 2 - so having sides 3 units long; and then another touching both the 2-square and the 3-square (which has sides of 5 units). We can continue adding squares around the picture, **each new square having a side which is as long as the sum of the latest two square's sides**. This set of rectangles whose sides are two successive Fibonacci numbers in length and which are composed of squares with sides which are Fibonacci numbers, we will call the **Fibonacci Rectangles**.



The next diagram shows that we can draw a spiral by putting together quarter circles, one in each new square. This is a spiral (the **Fibonacci Spiral**). A similar curve to this occurs in nature as the shape of a snail shell or some sea shells. Whereas the Fibonacci Rectangles spiral increases in size by a factor of Phi (1.618..) in a *quarter of a turn* (i.e. a point a further quarter of a turn round the curve is 1.618... times as far from the centre, and this applies to *all* points on the curve), the Nautilus spiral curve takes a *whole turn* before points move a factor of 1.618... from the centre.



Click on the shell picture (a slice through a Nautilus shell) to expand it.

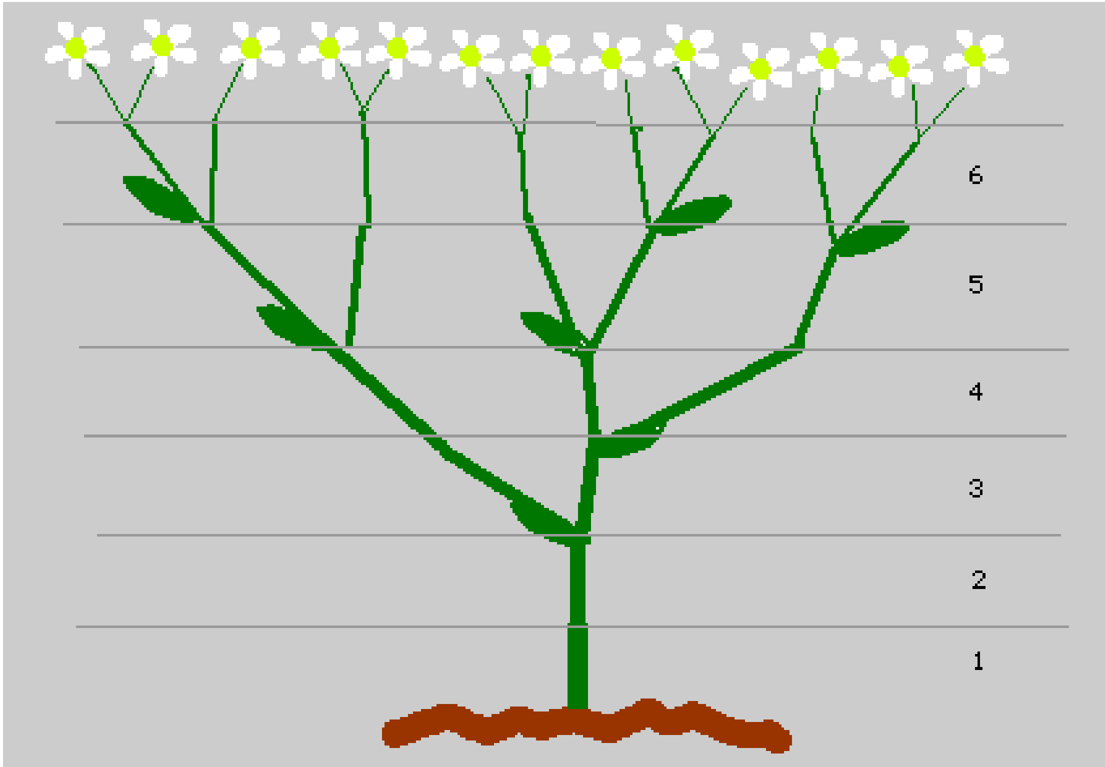
These spiral shapes are called Equiangular or Logarithmic spirals. The links from these terms contain much more information on these curves and pictures of computer-generated shells.

Reference

The Curves of Life Theodore A Cook, Dover books, 1979, ISBN 0 486 23701 X.
A Dover reprint of a classic 1914 book.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Fibonacci Numbers and Branching Plants



One plant in particular shows the Fibonacci numbers in the number of "growing points" that it has. Suppose that when a plant puts out a new shoot, that shoot has to grow two months before it is strong enough to support branching. If it branches every month after that at the growing point, we get the picture shown here.

A plant that grows very much like this is the "sneezewort": Achillea ptarmica.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Petals on flowers

On many plants, the number of petals is a Fibonacci number:
buttercups have 5 petals; lilies and iris have 3 petals; some delphiniums have 8; corn marigolds have 13 petals; some asters have 21 whereas daisies can be found with 34, 55 or even 89 petals.

The links here are to various flower and plant catalogues:

- the Dutch Flowerweb's searchable index called Flowerbase.
- The Helsinki Internet Directory for Botany has a wealth of information of all aspects of Botany and includes a gigantic section on Images with links to sites about plants all over the world.
  Try searching it to see where you can spot the golden section occurring and the Fibonacci numbers.
- The US Department of Agriculture's Plants Database containing over 1000 images, plant information and searchable database.

3 petals: lily, iris

Often lilies have 6 petals formed from two sets of 3.

5 petals: buttercup, wild rose, larkspur, columbine (aquilegia)

The humble buttercup has been bred into a multi-petalled form.

8 petals: delphiniums

13 petals: ragwort, corn marigold, cineraria,

21 petals: aster, black-eyed susan, chicory

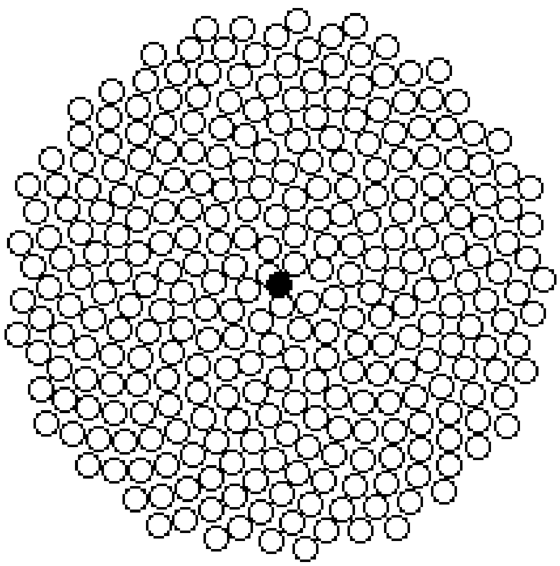
34 petals: plantain, pyrethrum

55, 89 petals: michaelmas daisies, the asteraceae family

Some species are very precise about the number of petals they have - eg buttercups, but others have petals that are very near those above, with the average being a Fibonacci number.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Seed heads



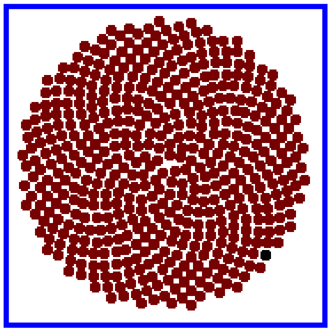
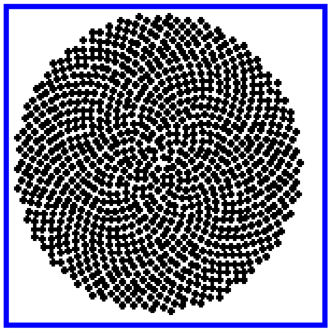
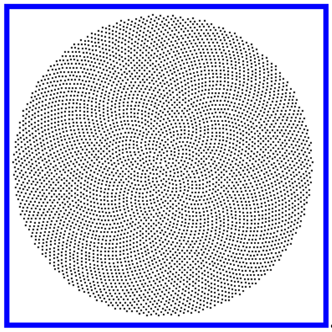
Fibonacci numbers can also be seen in the arrangement of seeds on flowerheads. Here is a diagram of what a large sunflower or daisy might look if magnified. The centre is marked with a black dot.

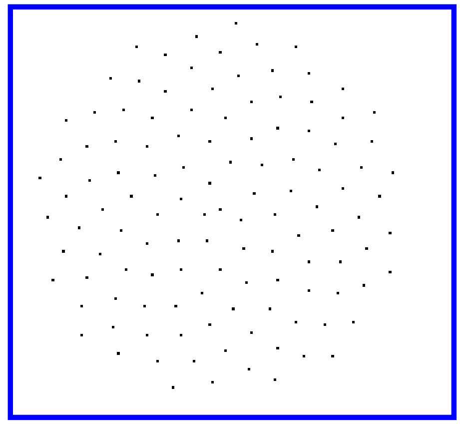
You can see that the seeds seem to form spirals curving both to the left and to the right. If you count those spiralling to the right at the edge of the picture, there are 34. How many are spiralling the other way? You will see that these two numbers are neighbours in the Fibonacci series.

The same happens in real seed heads in nature. The reason seems to be that this forms an **optimal packing** of the seeds so that, no matter how large the seedhead, they are uniformly packed, all the seeds being the same size, no crowding in the centre and not too sparse at the edges.

If you count the spirals near the centre, in both directions, they will both be Fibonacci numbers. The spirals are patterns that the eye sees, "curvier" spirals appearing near the centre, flatter spirals (and more of them) appearing the farther out we go.

Here are some more pictures of 500, 1000 and 5000 seeds - click on them for the full picture:

  .



Click on the image on the right for a Quicktime animation of 120 seeds appearing from a single central growing point. Each new seed is just phi (0·618) of a turn from the last one (or, equivalently, there are Phi (1·618) seeds per turn). The animation shows that, no matter how big the seed head gets, the seeds are always equally spaced. At all stages the Fibonacci Spirals can be seen.

The same pattern shown by these dots (seeds) is followed if the dots then develop into leaves or branches or petals. Each dot only moves out directly from the central stem in a straight line.

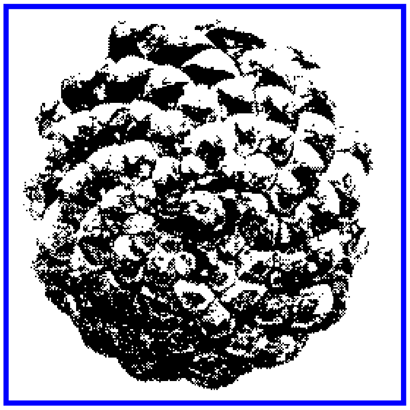
This process models what happens in nature when the "growing tip" produces seeds in a spiral fashion. The only active area is the growing tip - the seeds only get bigger once they have appeared.

[This animation was produced by Maple. If there are N seeds in one frame, then the newest seed appears nearest the central dot, at 0·618 of a turn from the angle at which the last appeared. A seed which is i frames "old" still keeps its original angle from the exact centre but will have moved out to a distance which is the square-root of i.]

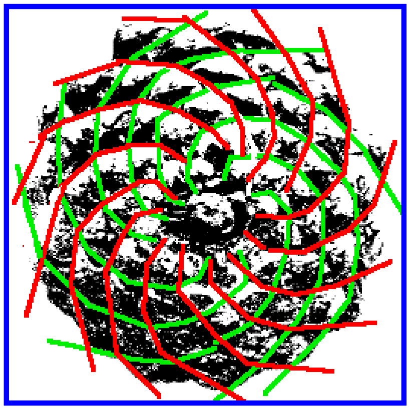
Note that you will not *always* find the Fibonacci numbers in the number of petals or spirals on seed heads etc., although they often come close to the Fibonacci numbers.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Pine cones



Pine cones show the Fibonacci Spirals clearly. Here is a picture of a pinecone seen from its base (sorry the quality is a bit poor) and another with the spirals emphasised: red in one direction and green in the other.[Click on the images to enlarge them.]



Things to do

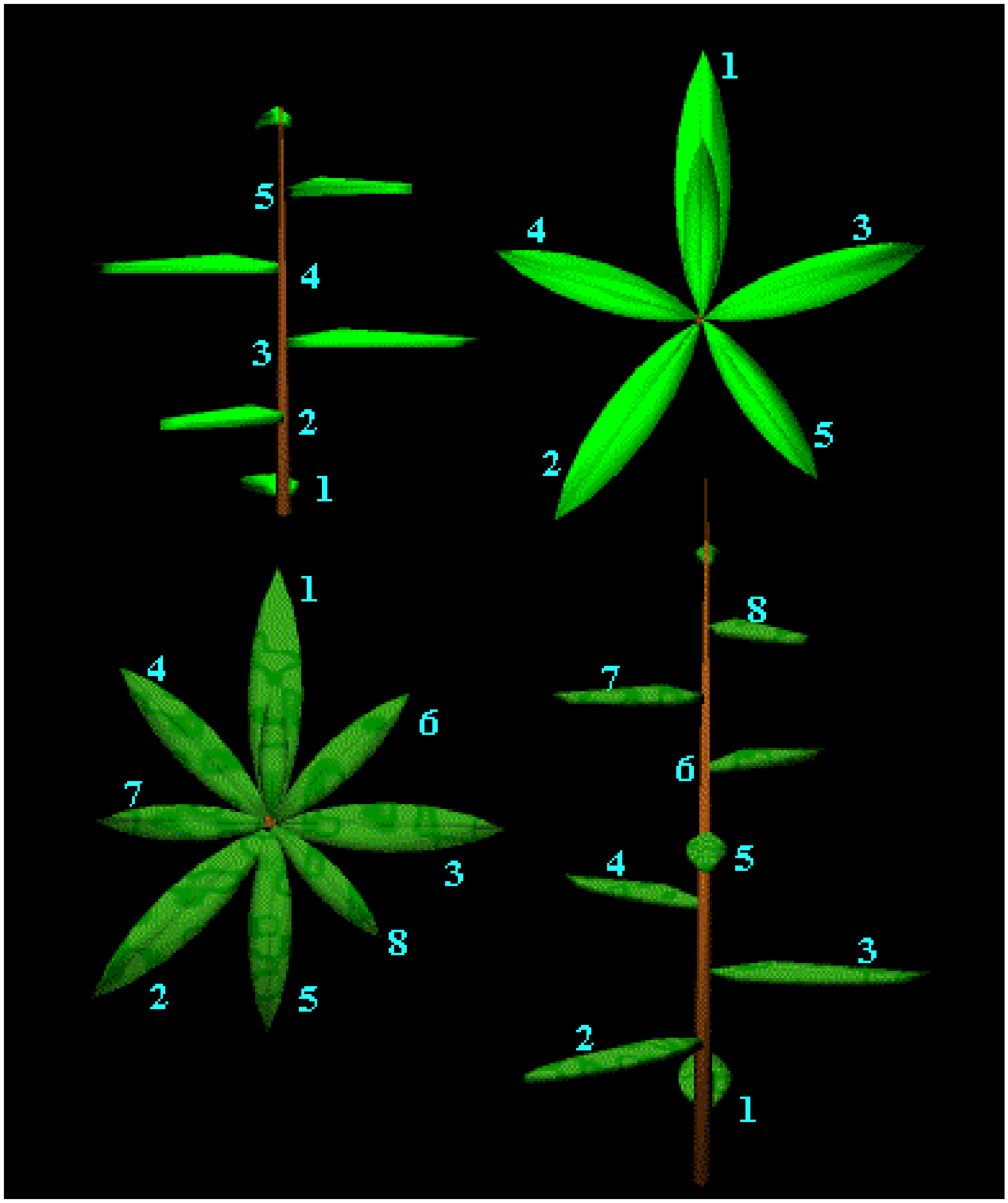
- How many red spirals are there?
- How many green?
- Collect some pine cones for yourself and count the spirals in both directions.
  A tip: Soak the cones in water so that they close up and counting the spirals is easier.
  a. Does the number of spirals differ for each kind of tree/cone or not?
  b. Are all the cones identical in that the steep spiral (the one with most spiral arms) goes in the same direction?
- What about **a pineapple**? Can you spot the same spirala pattern? How many spirals are there in each direction?

You will occasionally find pine cones with do not have a Fibonacci number of spirals in one or both directions. Sometimes this is due to deformities produced by disease or pests. For instance, a large collection of pine cones of different kinds of Californian pine cones was studied by Brother Alfred Brousseau and reported in *The Fibonacci Quarterly* vol 7 (1969) pages 525 - 532 in an article entitled **Fibonacci Statistics in Conifers**. He also found that there were as many with the steep spiral (the one with more arms) going to the left as to the right.

Pineapples and Fibonacci Numbers P B Onderdonk *The Fibonacci Quarterly* vol 8 (1970), pages 507, 508.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Leaf arrangements



Also, many plants show the Fibonacci numbers in the arrangements of the leaves around their stems. If we look down on a plant, the leaves are often arranged so that leaves above do not hide leaves below. This means that each gets a good share of the sunlight and catches the most rain to channel down to the roots as it runs down the leaf to the stem.
The computer generated ray-traced picture here is created by my brother, Brian, and here's another, based on an African violet type of plant, whereas this has lots of leaves.

Leaves per turn

The Fibonacci numbers occur when counting both the number of times we go around the stem, going from leaf to leaf, as well as counting the leaves we meet until we encounter a leaf directly above the starting one.

If we count in the other direction, we get a different number of turns for the same number of leaves.

The number of turns in each direction and the number of leaves met are **three consecutive Fibonacci numbers**!

For example, in the top plant in the picture above, we have **3** clockwise rotations before we meet a leaf directly above the first, passing **5** leaves on the way. If we go anti-clockwise, we need only **2** turns. Notice that 2, 3 and 5 are consecutive Fibonacci numbers.
For the lower plant in the picture, we have **5** clockwise rotations passing **8** leaves, or just **3** rotations in the anti-clockwise direction. This time 3, 5 and 8 are consecutive numbers in the Fibonacci sequence.
We can write this as, for the top plant, **3/5 clockwise rotations per leaf** (or 2/5 for the anticlockwise direction). For the second plant it is **5/8 of a turn per leaf** (or 3/8).

Leaf arrangements of some common plants

The above are computer-generated "plants", but you can see the same thing on real plants. One estimate is that 90 percent of all plants exhibit this pattern of leaves involving the Fibonacci numbers.

Some common trees with their Fibonacci leaf arrangement numbers are:

1/2 elm, linden, lime, grasses
1/3 beech, hazel, grasses, blackberry
2/5 oak, cherry, apple, holly, plum, common groundsel
3/8 poplar, rose, pear, willow
5/13 pussy willow, almond

where n/t means there are n leaves in t turns or n/t leaves per turn.

Cactus's spines often show the same spirals as we have already seen on pine cones, petals and leaf arrangements, but they are much more clearly visible. Charles Dills has noted that the Fibonacci numbers occur in Bromeliads and his Home page has links to lots of pictures.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Things to do WITH VEGETABLES AND FRUIT

- Take a look at a **cauliflower** next time you're preparing one:
  1. First look at it:
     - The florets are arranged in spirals, just like the seed heads and leaves above.
     - Count the number of florets at some fixed distance from the centre. The number in one direction and in the other will be Fibonacci numbers, as we've seen here.
     - Take a closer look at a single floret. It's a mini cauliflower! Each has its own little florets all arranged in spirals. If you can, count the spirals in both directions, and they'll be Fibonacci numbers (but you expected that!).
  2. Then, when cutting off the florets, try this:
     - start at the bottom and take off the largest floret, cutting it off parallel to the main "stem".
     - Find the next on up the stem. It'll be about 0·618 of a turn round (in one direction). Cut it off in the same way.
     - Repeat, as far as you like and..
     - Now look at the stem. Where the florets are rather like a pinecone or pineapple. The florets were arranged in spirals up the stem. Counting them again shows the

Fibonacci numbers.

- Try the same thing for **broccoli**.
- **Chinese leaves** and **lettuce** are similar but there is no proper stem for the leaves. Instead, carefully take off the leaves, from the outermost first, noticing that they overlap and there is usually only one that is the outermost each time. You should be able to find some Fibonacci number connections.
- Look for the Fibonacci numbers in fruit.
  1. What about a **banana**? Count how many "flat" surfaces it is made from - is it 3 or perhaps 5? When you've peeled it, cut it in half (as if breaking it in half, not lengthwise) and look again. Surprise! There's a Fibonacci number.
  2. What about an **apple**? Instead of cutting it from the stalk to the opposite end (where the flower was), ie from "North pole" to "South pole", try cutting it along the "Equator". Surprise! there's your Fibonacci number!
  3. Try a Sharon fruit (which is like an orange-coloured tomato).
  4. Where else can you find the Fibonacci numbers in fruit and vegetables? Why not email me with your results and the best ones will be put on the Web here or links added to your own web pages.

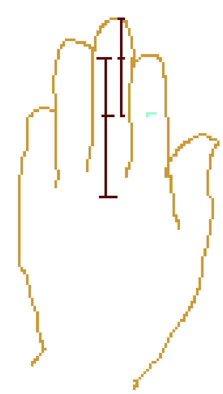
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Fibonacci Fingers?

Look at your own hand:

You have ...

- **2** hands each of which has ...
- **5** fingers, each of which has ...
- **3** parts separated by ...
- **2** knuckles



Is this just a coincidence or not?????

However, if you measure the lengths of the bones in your finger (best seen by slightly bending the finger) does it look as if the ratio of the longest bone in a finger to the middle bone is Phi? What about the ratio of the middle bone to the shortest bone (at the end of the finger) - Phi again? Can you find any ratios in the lengths of the fingers that looks like Phi? ---or does it look as if it could be any other similar ratio also?

Why not measure your friends' hands and gather some statistics? I'd be interested in your results if you want to email them to me.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

A quote from Coxeter on Phyllotaxis

Finally, note that, although the Fibonacci numbers and golden section seem to appear in many situations in nature, they are not the only such numbers. H S M Coxeter, in his **Introduction to Geometry** (1961, Wiley, page 172) - see the references at the foot of this page - has the following important quote:

> *it should be frankly admitted that in some plants the numbers do not belong to the sequence of f's [Fibonacci numbers] but to the sequence of g's [Lucas numbers] or even to the still more anomalous sequences*
>
> *3,1,4,5,9,... or 5,2,7,9,16,...*
>
> *Thus we must face the fact that phyllotaxis is really not a universal* ***law*** *but only a fascinatingly prevalent* ***tendency****.*

He cites A H Church's **The relation of phyllotaxis to mechanical laws**, Williams and Norgate, London, 1904, plates XXV and IX as examples of the Lucas and the latter two sequences and plates V, VII, XIII and VI as examples of the Fibonacci numbers on sunflowers.

The Lucas numbers are formed in the same way as the Fibonacci numbers - by adding the latest two to get the next, but instead of starting at 0 and 1 [Fibonacci numbers] they start with 2 and 1 [the Lucas numbers]. The other two sequences he states above have other pairs of starting values but then proceed with the same rule as the Fibonacci numbers.

An interesting fact is that, for ALL series that are formed from adding the latest two numbers to get the next, and, starting from ANY two values (bigger than zero), the ratio of successive terms will ALWAYS tend to Phi!

So Phi is a more universal constant than the Fibonacci series itself.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

References on Fibonacci and Golden Section

Key

means the reference is to a book (and any link will take you to more information about the book and an on-line site from which you can purchase it);

means the reference is to an article in a magazine or a paper in a scientific periodical.

indicates a link to another web site.

Excellent books which cover similar material to that which you have found on this page are produced by Trudi Garland and Mark Wahl:

Mathematical Mystery Tour by Mark Wahl, 1989, is full of many mathematical investigations, illustrations, diagrams, tricks, facts, notes as well as guides for teachers using the material. It is a great resource for your own investigations.

Books by Trudi Garland:

Fascinating Fibonaccisby Trudi Hammel Garland.
This is a really excellent book - suitable for all, and especially good for teachers seeking more material to use in class.

Trudy is a teacher in California and has some more information on her book. (You can even Buy it online now!)
She also has published several posters, including one on the golden section suitable for a classroom or your study room wall.
You should also look at her other Fibonacci book too:
Fibonacci Fun: Fascinating Activities with Intriguing Numbers Trudi Hammel Garland - a book for teachers. Click on the book image and you can buy it online now.

Sex ratio and sex allocation in sweat bees (Hymenoptera: Halictidae) D Yanega, in *Journal of Kansas Entomology Society*, volume 69 Supplement, 1966, pages 98-115.
Because of the imbalance in the family tree of honeybees, the ratio of male honeybees to females is not 1-to-1. This was noticed by Doug Yanega of the Entomology Research Museum at the University of California. In the article above, he correctly deduced that the number of females to males in the honeybee community will be around the golden-ratio Phi = 1.618033.. .

On the Trail of the California Pine, Brother Alfred Brousseau, *Fibonacci Quarterly*, vol 6, 1968, pages 69 - 76;
on the authors summer expedition to collect examples of all the pines in California and count the number of spirals in both directions, all of which were neighbouring Fibonacci numbers.

Why Fibonacci Sequence for Palm Leaf Spirals? in *The Fibonacci Quarterly* vol 9 (1971), pages 227 - 244.

Fibonacci System in Aroids in *The Fibonacci Quarterly* vol 9 (1971), pages 253 - 263. The Aroids are a family of plants that include the Dieffenbachias, Monsteras and Philodendrons.

Other WWW links on Phyllotaxis, the Fibonacci Numbers and Nature

- Alan Turing

  one of the Fathers of modern computing (who lived here in Guildford during his early school years) was interested in many aspects of computers and Artificial Intelligence (AI) well before the electronic stored-program computer was developed enough to materialise some of his ideas. One of his interests (see his Collected Works) was Morphogenesis, the study of the growing shapes of animals and plants.

  The book Alan Turing: The Enigma by Andrew Hodges is an enjoyable and readable account of his life and work on computing as well as his contributions to solving the German war-time code which used a machine called "Enigma".
  Unfortunately this book is now out of print, but click on the book-title link and Amazon.com will see if they can find a copy for you with no obligation.

- The most irrational number

  One of the American Maths Society (AMS) web site's **What's New in Mathematics** regular monthly columns. This one is on the Golden Section and Fibonacci Spirals in plants.

- Phyllotaxis

  An interactive site for the mathematical study of plant pattern formation for university biology students at Smith College. Has a useful gallery of pictures showing the Fibonacci spirals in various plants.

the Fibonacci Home Page

There are no earlier topics - this is the first.

WHERE TO NOW?

The next page on this topic is ...
The golden section in nature

The next Topic is...
The Puzzling World of Fibonacci Numbers

Fibonacci Numbers and Nature - Part 2
Why is the Golden section the "best" arrangement?

Contents of this Page

The line means there is a Things to do investigation at the end of the section.

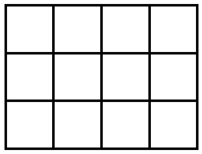
On the first page on the Fibonacci Numbers and Nature we saw that the Fibonacci numbers appeared in (idealised) rabbit, cow and bee populations, and in the arrangements of petals round a flower, leaves round branches and seeds on seed-heads and pinecones and in everyday fruit and vegetables.
We explained why they appear in the rabbit, cow and bee populations but what about the other appearances that we see around us in nature? The answer relates to why Phi appears so often in plants and the Fibonacci numbers appear because the eye "sees" the Fibonaci numbers in the spirals of seedheads, leaf arrangements and so on, and we looked at this on the previous Fibonacci Numbers in Nature page.
So we ask...

Why does nature like using Phi in so many plants?

The answer lies in **packings** - the best arrangement of objects to minimise wasted space.

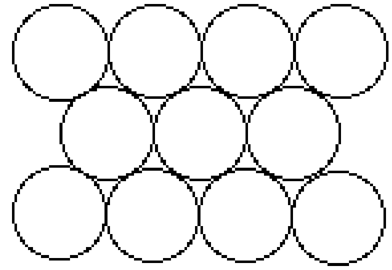
Packings

If you were asked what was the best way to pack objects your answer would depend on the **shape** of the objects since....



...square objects would pack most closely in a square array,

whereas round objects pack better in a hexagonal arrangement....



So why doesn't nature use one of these? Seeds are round (mostly), so why don't we see hexagonal arrangments on seedheads?
Although hexagonal symmetry IS the best packing for circular seeds, it doesn't answer the question of how leaves should be arranged round a stem or how to pack flower-heads (which are circular because that is the shape that encloses maximum area for minimum edge) with seeds *that grow in size*.

What nature seems to use is **the same pattern** to place seeds on a seedhead as it used to arrange petals around the edge of a flower AND to place leaves round a stem. What is more, ALL of these maintain their efficiency **as the plant continues to grow** and that's a lot to ask of a single process!

So just how *do* plants grow to maintain this optimality of design?

The Meristem and Spiral growth patterns

Botanists have shown that plants grow from a single tiny group of cells right at the tip of any growing plant, called the **meristem**. There is a separate meristem at the end of each branch or twig where new cells are formed. Once formed, they grow in size, but new cells are only formed at such growing points. Cells earlier down the stem expand and so the growing point rises.

Also, these cells grow in a spiral fashion, as if the stem turns by an angle and then a new cell appears, turning again and then another new cell is formed and so on.

These cells may then become a new branch, or perhaps on a flower become petals and stamens.

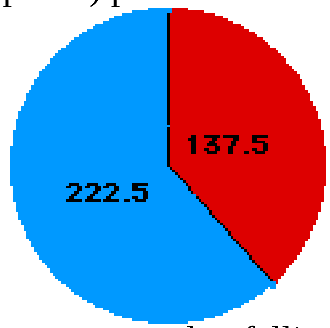
The amazing thing is that **a single fixed angle can produce the optimal design no matter how big the plant grows**. So, once an angle is fixed for a leaf, say, that leaf will least obscure the leaves below and be least obscured by any future leaves above it. Similarly, once a seed is positioned on a seedhead, the seed continues out in a straight line pushed out by other new seeds, but retaining the original angle on the seedhead. No matter how large the seedhead, the seeds will **always** be packed uniformly on the seedhead.

And all this can be done with a single fixed angle of rotation between new cells?
Yes! This was suspected by people as early as the last century. The principle that a single angle produces uniform packings no matter how much growth appears after it was only proved mathematically in 1993 by Douady and Couder, two french mathematicians.

You will have already guessed what the fixed angle of turn is - it is Phi cells per turn or phi turns per new cell.

Why does Phi appear in nature?

The arrangements of leaves is the same as for seeds and petals. All are placed at 0·618034.. leaves, (seeds, petals) per turn. In terms of degrees this is 0·618034 of 360° which is 222·492...°.However, we tend to



"see" the smaller angle which is (1-0·618034)x360 = 0·381966x360 = 137·50776..°. When we look at properties of Phi and phi on a later page, we shall see that

$$1\text{-phi} = \text{phi}^2 = \text{Phi}^{-2}$$

If there are Phi (1·618...) leaves per turn (or, equivalently, phi=0·618... turns per leaf), then we have the best packing so that each leaf gets the maximum exposure to light, casting the least shadow on the others. This also gives the best possible area exposed to falling rain so the rain is directed back along the leaf and down the stem to the roots. For flowers or petals, it gives the best possible exposure to insects to attract them for pollination.
The whole of the plant seems to produce its leaves, flowerhead petals and then seeds based upon the golden number.

And why do the Fibonacci numbers appear as leaf arrangements and as the number of spirals on seedheads?

The Fibonacci numbers form the best *whole number approximations* to the golden number, which we examined in greater detail on the first Fibonacci in Nature page.

Let's now try and show just why phi is the best angle to use in the next few sections of this page.

Why is the Golden section the "best" number?

The links in this section are to **Quicktime animations**. They are worth viewing as they show the dynamics of what might happen if seeds were not placed with a phi-angle between them.

Why not 0·6 of a turn per seed or 0·5 or 0·48 or 1·6 or some other number?

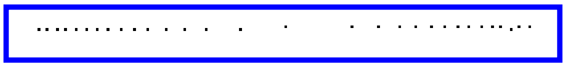
First we can agree that turning 0·6 of a turn is exactly the same as turning 1·6 turns or 2·6 turns or even 12·6 turns because the position of the point looks the same. So we can **ignore the whole number part of a turn and only examine the fractional part**.

Also, since a 0·6 of a turn in one direction is the same as 0·4 of a turn in the other, we could limit our investigation to **turns which are less than 0·5** too. However sometimes it will be easier to talk of fractions of a turn which are bigger than 0·5 or even that are bigger than 1, but **the only *important* part of the number is the fractional part.**

So, in terms of seeds - which develop into fruit - what is a *fruit*-ful numbers? Which has the best properties as a turning angle for our meristem? It *turns* out that numbers which are simple fractions are not good choices, as we see in the next section.

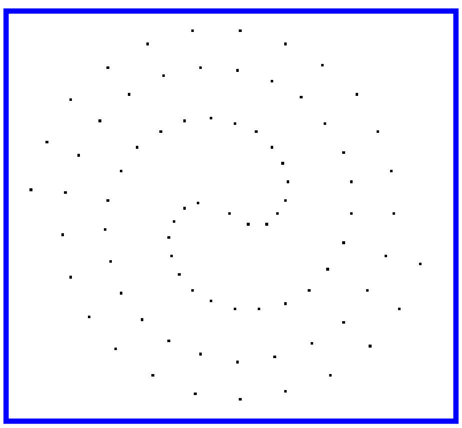
Why exact fractions are fruitless!

Let's first see what happens with a simple number such as 0·5 turns per seed.



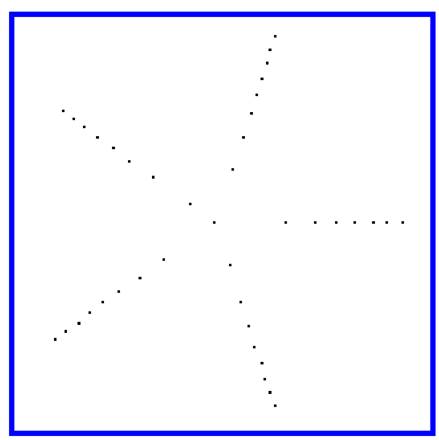
Since 0·5=1/2 we get just 2 "arms" and the seeds use the space on the seedhead very inefficiently: the seedhead is long and floppy. The picture is a link to an animation where you can see the new seeds appearing at the centre as the older ones continue growing outwards in a straight line from the central growing point (where the new seed cells appear).

A circular seedhead is more compact and would have better mechanical strength and so be better able to withstand wind and heavy rain.



Here is 0·48 of a turn between seeds.
[The picture is again a link to an animation.]
The seeds seem to be sprayed from two revolving "arms". This is because 0·48 is very close to 0·5 and a half-turn between seeds would mean that they would just appear on alternate sides, in a straight line. Since 0·48 is a bit *less than* 0·5, the "arms" seem to rotate backwards a bit each time.

So if we has 0·52 seeds per turn, we would be a little in advance of half a turn and the final pattern would be a mirror-image (as if we had used 1-0·52=0·48 seeds per turn but turning in the *opposite* direction).



What do you think will happen with 0·6 of a turn between successive seeds?
Did you expect it to be so different?
Notice how the seeds are not equally spaced, but fairly soon settle down to 5 "arms". Why?

Because 0·6=3/5 so every 3 turns will have produced **exactly 5** seeds and the sixth seed will be at the same angle as the first, the seventh in the same (angular) position as the second and so on. The seeds appearing at every third arm, in turn, round and round the 5 arms. So we count 3-of-the-5 (3/5) to find the next "arm" where a seed will appear.

> **If we try 1·6 or 2·6 or 3·6** can you see that we will get the same animation since the extra whole turns do not affect where the seeds are placed?

So what seems to be important is just the **fractional part of our seeds-per-turn value** and we can ignore the whole number part. There is another value that will give the same animation too. What is it? Well, if we went 0·6 of a turn in the other direction, it is equivalent to going 1-0·6=0·4 of a turn between seeds. So also would be 1·4, 1·4, 3·4 and so on.

Here's what happens if we have a value closer to phi(0·6180339..), namely 0·61. You'll notice that it is better, but that there are still large gaps between the seeds nearest the centre, so the space is not best used. This is also equivalent to using 1·61, 2·61, etc. and also to 1-0·61=0·39 and therefore to 1·39 and 2·39 and so on.

In fact, any number which can be written as an exact ratio (a **rational number**) would *not* be good as a turn-per-seed angle.
If we use p/q as our angle-turn-between-successive-turns, then we will end up with q straight arms, the seeds being placed every p-th arm. [This explains why 0·6=3/5 has 5 arms and the seeds appear at every third arm, going round and round.]

The rational answer is the irrationals!

So what is a "good" value? One that is NOT an exact ratio since very large seed heads will eventually end up with seeds in straight lines.
Numbers which cannot be expressed exactly as a ratio are called **irrational numbers** (ir-ratio-nal) and this description applies to such values as √2, Phi, phi, e, pi and any multiple of them too.

You'll notice that the e(2·71828...) animation has 7 arms since its turns-per-seed is (two whole turns plus) 0·71828... of a turn, which is a bit more than 5/7(=0·71428..).
A similar thing happens with pi(3·14159..) since the fraction of a turn left over after 3 whole turns is 0·14159 and is close to 1/7=0·142857.. . It is a little **less**, so *the "arms" bend in the opposite direction* to that of e's (which were a bit **more** than 5/7).
These rational numbers are called **rational approximations** to the real number value.
If we take more and more seeds, the spirals alter and we get better and better approximations to the irrational value.

What is "the best" irrational number?

One that never settles down to a rational approximation for very long. The mathematical theory is called **CONTINUED FRACTIONS**.
The simplest such number is that which is expressed as P=1+1/(1+1/(1+1/(...) or, its reciprocal p=1/(1+1/(1+1/(...))).
P is just 1+1/P, or P^2=P+1.
p is just 1/(1+p) so p^2+p=1.
We will see later that these are just definitions of Phi (P) and phi (p) (and their negatives)!
The exact value of Phi is **(√5 + 1)/2**
and of phi is **(√5 – 1)/2**.
Both are irrational numbers whose rational approximations are ...

```
phi:   1/1,  1/2,  2/3,  3/5,  5/8,  8/13,  13/21, ...
Phi:   1/1,  2/1,  3/2,  5/3,  8/5,  13/8,  21/13, ...
```

which is why you see the Fibonacci spirals in the seed heads!

Here is another quicktime movie which shows various turns-per-seed values near phi (0·61803) showing that there are always gaps towards the outer edge of the "seedhead" and that phi gives the best value for all sizes of flowerhead.

NEW Try this **Geometer's Sketchpad** active demonstration which lets you alter the inter-seed angle at will (and animate it) to see just why the golden section angle produces the best packing.
Geometer's Sketchpad is available for 30 days free trial, for PC and Apple Mac.

Things to do

- If you have Maple, use this Maple program to try other angles and make some animations for yourself.
- The "rational approximations" to real numbers are better seen if, instead of producing seeds at the centre, we keep adding them round the outside - that is, along the square-root spiral which has equation **R=√A** where R is the (radial) distance of a point from the origin, and A its angle turn (from the 0 angle direction). Use the Maple program to "grow plants" that will find good rational approximations to a decimal fraction of your choice. For example, Pi as the angle of rotation between seeds, shows 7 arms clearly after only 100 seeds, gets confused at about 500 seeds but by 1000 shows a better approximation - there are 113 "arms", seeds being grown every 16 showing that a better approximation for Pi is 3+16/113=335/113.

  - What about approximations to sqrt(3) or sqrt(5)?
  - Take sqrt(3) and plot lots of "seeds".

What sequence of approximations do you get? You should be able to answer this if you plot 500 seeds.

- Now convert each approximation into a continued fraction. What pattern in the numbers in the continued fraction emerges?
- Try to prove that the pattern continues indefinitely, by proving its value is sqrt(3).

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Links and References

Phyllotaxis

The technical term for the study of the arrangements of leaves and of seedheads in plants is **phyllotaxis**.

An important technical paper about phi and its optimal properties for plant growth can be found in **Phyllotaxis as a self-organised growth process** by Stephane Douady and Yves Couder, pages 341 to 352 in *Growth Patterns in Physical Sciences and Biology*, (editor J M Garcia-Ruiz *et al*), Plenum press, 1993.

A history of the study of phyllotaxis by I Adler, D Barabe, R V Jean in *Annals of Botany*, 1997, Vol.80, No.3, pp.231-244.

A better way to construct the Sunflower head in *Mathematical Biosciences* volume 44, (1979) pages 145 - 174.

Fibonacci Numbers in Nature

Here are some not-too technical papers about the maths which justifies the occurrence of the Fibonacci numbers in nature:

A H Church **On the relation of Phyllotaxis to Mechanical Laws**, published by Williams and Norgat, London 1904.

E E Leppik, **Phyllotaxis, anthotaxis and semataxis** *Acta Biotheoretica* Vol 14, 1961, pages 1-28.

F J Richards **Phyllotaxis: Its Quantitative Expression and Relation to growth in the Apex** *Phil. Trans. Series B* Vol 235, 1951, pages 509-564.

D'Arcy W Thompson **On Growth and Form** Dover Press 1992.
This is the complete edition! (Click on the title-link for more information and to order it now.)
There is also an abridged version from Cambridge University press (more information and order it on line

via the title-link.)

T A Davis, **Fibonacci Numbers for Palm Foliar Spirals** *Acta Botanica Neelandica*, Vol 19, 1970, pages 236-243.

T A Davis **Why Fibonacci Sequence for Palm Leaf Spirals?**, *Fibonacci Quarterly*, Vol 9, 1971, pages 237-244.

The Algorithmic Beauty of Plants by P Prusinkiewicz, and A Lindenmayer, published by Springer-Verlag (Second printing 1996) is an astounding book of wonderful images and patterns in plant shapes as well as algorithms for modelling and simulation by computer. (For more information and how to order it online use the title-link).
Related to this book is:

The Algorithmic Beauty of Sea Shells (Virtual Laboratory) in hardback by Hans Meinhardt, Przemyslaw Prusinkiewicz, Deborah R. Fowler (more information and order it online via this title-link).

The Curves of Life: Being an Account of Spiral Formations and Their Application to Growth in Nature, to Science, and to Art Sir Theodore A Cook, Dover books, 1979, ISBN 0 486 23701 X.
A Dover reprint of a classic 1914 book. (More information and you can order it online via the title-link.)

Also see H S M Coxeter's **Introduction to Geometry**, published by Wiley, in its Wiley Classics Library series, 1989, ISBN 0471504580, especially chapter 11 on Phyllotaxis. (More information and order it online via the title-link.)

WWW Links

Eddy Levin has invented a wonderful golden-section measuring tool, like a pair of dividers or callipers and he has a page of examples of it in use showing the golden section on flowers, insects, leaves etc that's well worth looking at. Click on his "Dental" link and you can see that, as a dentist, he sees the golden section every day in the arrangement and width of human teeth too!

Fibonacci Home Page

There are no earlier topics: this is the first.

The Fibonacci Numbers in Nature

WHERE TO NOW??

This is the last page on this topic.

The next Topic is...
The Puzzling World of Fibonacci Numbers

 R.Knott@surrey.ac.uk 31 October 2000

Easier Fibonacci puzzles

All these puzzles except one (which??) have the Fibonacci numbers as their answers.
So now you have the puzzle and the answer - so what's left? Just the *explanation of why* the Fibonacci numbers are the answer - that's the *real* puzzle!!

Puzzles on **this page** have fairly straight-forward and simple explanations as to why their solution invovles the Fibonacci numbers;.
Puzzles on **the next page** are harder to explain but they still have the Fibonacci Numbers as their solutions. So does a *simple explanation* exist for any of them?

Contents of this Page

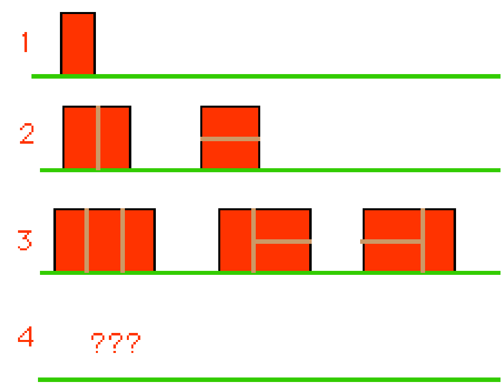
Puzzles that are simply related to the Fibonacci numbers....

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Fibonacci numbers and Brick Wall Patterns

If we want to build a brick wall out of the usual size of brick which has a length twice as long as its height, and if our wall is to be **two units tall**, we can make our wall in a number of patterns, depending on how long we want it:

There's just one wall pattern which is 1 unit wide - made by putting the brick on its end.
There are 2 patterns for a wall of length 2: two side-ways bricks laid on top of each other and two bricks long-ways up put next to each other.
There are three patterns for walls of length 3.
How many patterns can you find for a wall of length 4?
How may different patterns are there for a wall of length 5?



Look at the **number** of patterns you have found for a wall of length 1, 2, 3, 4 and 5.
Does anything seem familiar?
Can you find a reason for this?
Show me an example of **why** the Fibonacci numbers are the answer

Variation - use Dominoes

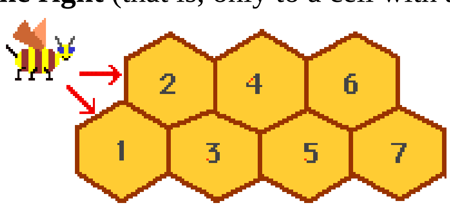
A domino is formed from two squares. In this variation of the Brick Wall puzzle, we are not interested in the spots on the dominoes, just their shape. If you like, turn the dominoes over with the spots underneath so that they all look the same.

Start by placing n dominoes flat on a table, face down, and turn them so that all are in the "tall" or "8" position (as opposed to the "wide" or "oo" orientation). Pack them neatly together to make a rectangle.
Take the same number of dominoes and, using this rectangle as the picture to aim at in a jigsaw puzzle, see how many other flat patterns you can make which have exactly this shape. This time dominoes can be placed in either the tall or wide direction in your design.
Make a table of the patterns you have found and the number of patterns possible using 1 domino (easy!), 2 dominoes, 3 dominoes, and so on, not forgetting to include the original rectangle design too.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Making a bee-line with Fibonacci numbers

Here is a picture of a bee starting at the end of some cells in its hive. It can **start at either cell 1 or cell 2** and **moves only to the right** (that is, only to a cell with a higher number in it).



There is only one path to cell 1, but
two ways to reach cell 2: directly or via cell 1.
For cell 3, it can go 123, 13, or 23, that is, there are three different paths.
How many paths are there from the start to cell number n?

The answer is again the Fibonacci numbers. Can you explain why?

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Chairs in a row: 1

This time we have n chairs in a row and a roomful of people.

If you've ever been to a gathering where there are teachers present, you will know they *always* talk about their school/college (boring!). So we will insist that *no two teachers should sit next to each other* along a row of seats and count *how many ways we can seat n people*, if some are teachers (who cannot be next to each other) and some are not. The number of seating arrangements is always a Fibonacci number:

1 chair or — **2 ways**

2 chairs or or — **3 ways**

since we do not allow

3 chairs , , , or — **5 ways**

this time , and are not allowed.

You can write the sequences using **T** for Teacher and **N** for Normal, oops, I mean Not-teacher !!

There will always be a Fibonacci number of sequences for a given number of chairs, if no two teachers are allowed to sit next to each other!

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Chairs in a Row: 2

This variation is a little friendlier to teachers.

Everyone, teacher or not, must **not sit on their own**, but a teacher must be next to another teacher or the teacher will be blue, and a non-teacher must be next to a non-teacher or she will be red-faced with embarrassment!

So we can have ... ... since the two teachers have the other teacher next to them. The non-teacher on the right of these 3 will now also need another non-teacher on his other side so that he too is not left on his own.

A special *extra*condition in this puzzle is that **any seating arrangement must also start with a teacher**!

1 chair: - — **0 ways**

2 chairs: — **1 way**

3 chairs: — **1 way**

4 chairs: or — **2 ways**

5 chairs: or or — **3 ways**

There will always be a Fibonacci number of arrangements *if* we start with a teacher.

What happens if we start with a non-teacher always?
What happens if we have no restriction on the first seat?
The answers to these two questions also involve the Fibonacci numbers too!!

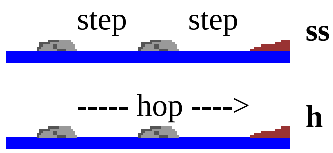
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Stepping Stones

Some stepping stones cross a small river. How many ways back to the bank are there if you are standing on the n-th stone? You can either step on to the next stone or else hop over one stone to land on the next.

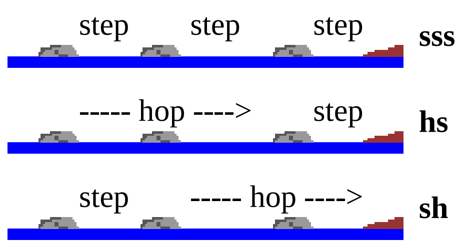
If you are on stone number 1, you can only step (s) on to the bank: 1 route.

If you are on stone 2, you can either step on to stone 1 and then the bank (step, step or ss)
OR you can hop directly onto the bank (h):



2 sequences

From stone 3, you can step, step, step (sss) or else hop over stone 2 and then step (hs) or else step on to stone 2 and then hop over stone 1 to the bank (sh):



3 sequences

Why are the Fibonacci numbers appearing?
[With thanks to Michael West for bringing this puzzle to my attention.]

`0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..`

Fibonacci numbers for a change!

Some countries have coins or notes of value 1 and 2. For instance, in Britain we have coins with values 1 penny (1p) and 2 pence (2p). The USA has 1 cent and 5 cent coins but not a 2 cents coin, but it does have ten dollar and twenty dollar bills ($10, $20). This problem uses coins or notes of values 1 and 2.
If we have just 1p and 2p coins, in how many ways can we make up a given amount of money with just these two coins? For instance:-

```
1p = 1p                        -- only one way but
2p = 1p+1p or 2p               -- two ways, and
3p = 1p+1p+1p or 1p+2p or 2p+1p -- three ways
```

Since we are letting 1p+2p and 2p+1p be different solutions, then we are interested in the order that the coins are given also.
You will have guessed how many ways there are to make up 4p and the general answer by now!
But the challenge is: can you explain **why** the Fibonacci numbers appear yet again?

Follow up: What if we are interested in **collections** of coins rather than **sequences**? Here 1p+2p is the *same* collection as 2p+1p. How many collections are there? If the coins sum to n pence, these are called **partitions of n** and have many applications.

Can you find a simple link between answers to the Change puzzle and your answers to the Stepping Stones puzzle?

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

No one!

Your younger sister is playing with her colouorange rods. They are of various lengths, from single ones (length 1 which are cubes) which are orange, length 2 are magenta, length 3 are blue and so on.

length 1	
length 2	
length 3	
length 4	
length 5	
...	...

However, her brother has just taken all the length 1 rods (the orange cubes) to play with but has left her with all the rest.

So in how many ways can she make a line of length N if there are no rods of length 1?

For a line of length 3, she can use only a rod of length 3.
But for a line of length 4, she can use either a rod of length 4 or else two rods of length 2.
When it comes to making a line of length 5, she has several ways of doing it:

one rod of length 5:	
a rod of length 3 followed by one of length 2:	
OR she could put the rod of length 2 first and the 3-rod after it:	

We can summarise this as follows: 5 = 2 + 3 = 3 + 2 and we can collect the possibilities in a table which just uses numbers:

- **length 3** = 3
- **length 4** = 2 + 2
- **length 5** = 5 = 2 + 3 = 3 + 2

So what we are doing is listing sums where the number ONE must not appear in the sum. The order of the numbers matters so that 2+3 is not the same sum as 3+2 in this problem.

Technically, the collection of sums which total a given value N are called the **partitions of N**.
Here we are finding all the partitions of N that do not use the number 1.
It will always be a Fibonacci number!

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Telephone Trees

This problem is about the best way to pass on news to lots of people using the telephone.

We could just phone everyone ourselves, so 14 people to share the news with would take 14 separate calls. Suppose each call takes just 1 minute, then we will be on the phone at least 14 minutes (if everyone answers their phone immediately).

Can we do better than this? We could use the speakers on the phone - the "hands free" facility which puts the sound out on a speaker rather than through the handset so that others in the room can hear the call too. For the sake of a puzzle, let's suppose that 2 people hear each call. That would halve the number of calls I need to make. My 14 calls now reduces to 7.

Can we do better still?
Well, we could ask each person who receives a call to not only put the call through the loudspeakers but also to do some phoning too. So if two people hear the message, they could each phone two others and pass it on *in the same way* and so on. Here's what it looks like if I have 14 people to phone in this system as the calls "cascade". In the first minute, my first call is heard by A and B. A's call is heard by both C and D; B's call by E and F, and so on as in this diagram:

```
                                         me
                             /------------^-----------\
first minute                A                          B
                       /----^----\               /-----^----\
second minute         C           D             E            F
                   /--^--\     /--^--\      /--^--\      /--^--\
third minute      G       H   I       J    K       L    M       N
```

So all 14 people have heard the news in only 3 minutes! [This is an example of *recursion* - applying the same optimizing principle at *all* levels of a problem.]

Can we do even better than this?
Yes - if all the people got together in one room, it would only take one minute! So let's assume that I cannot get everyone together and I have to use the phone.

Now here is your puzzle. The phones in my company are rather old and do not have an external speaker (and no "conference call" facility) - only one person can hear each call. So I decide that I will phone only **two** people using two separate calls. I shall give them the news and then ask that they *do the same* and phone just **two** more people only. What is the shortest time that the news can pass to 14 people?

1. Draw the *cascade tree* of telephone calls, or the *telephone tree* for this problem. It begins like this:

```
                                         me
                             /------------^----\
first minute                A                   \
                       /----^----\               \
second minute         C           \               B
                   /--^--\         \            /--^--\
third minute      D       \         E          F       \
               /--^-\       \     /--^--\    /--^--\     \
              ...   ...     ... ...    ... ...    ...    ...
```

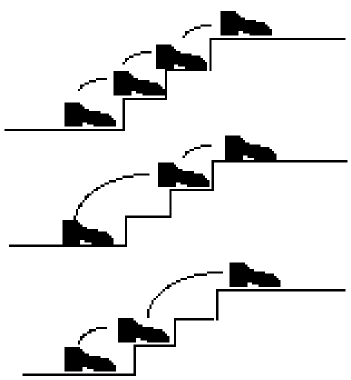
   How does the tree continue?
2. What is the maximum number of people in the office that could hear the news within N minutes using this method? Why is the answer related to the Fibonacci numbers?

Inspired by Joan Reinthaler's **Discrete Mathematics is Already in the Classroom - But It's Hiding** in *Discrete Mathematics in Schools*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Volume 36, 1997, pages 295-299.
This is a great book!

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Leonardo's Leaps

I try and take the stairs rather than the elevator whenever I can so that I get a little more exercise these days. If I'm in a hurry, I can leap two stairs at once otherwise it's the usual one stair at a time. If I mix these two kinds of action - **step** onto the next or else **leap** over the next onto the following one - then in how many different ways can I get up a flight of n steps?



For example, for 3 stairs, I can go

1: **step-step-step**
or else
2: **leap-step**
or finally
3: **step-leap**

...a total of 3 ways to climb 3 steps.

How many ways are there to climb a set of 4 stairs? 5 stairs? n stairs? Why?
Adapted from
Applied Combinatorics (Third Edition) by A Tucker, Wiley, 1995, Example 2, pages 280-281.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Fix or Flip?

Permutations are re-arrangements of a sequence of items into another order. For instance, we can permute D,B,C,A into A,B,C,D.

```
before: DBCA
after : ABCD
```

Here the D has swopped places with the A whilst the B and C have not moved.
In general, since we can place A in any of the 4 places, leaving 3 places for B (4x3=12 ways to place A and B) and so C can go in any of the remaining 2 places (so D has 1 choice left), then there are 4x3x2=24 permutations of 4 objects.
In general, there are nx(n-1)x...x3x2 permutations of n objects.

Suppose we restrict how we may move (permute) an object to
either **fix** it, leaving it in the same position
or **flip** it with a neighbour - two items next to each other swop places (they cannot now be moved again).

However, not all permutations are made of just these two kinds of transformation. Here are 4 examples of permutations on 4 objects: A, B, C and D:

before: ABCD after : DBCA	This is not a **fix-or-flip** permutation since the A and D have moved more than 1 place.
before: ABCD after : ABCD	However, this *is* since nothing has moved - all 4 items were **fix**ed!
before: ABCD after : BACD	B and A have **flip**ped and C and D remain **fix**ed and so this is a **fix-or-flip** permutation.
before: ABCD after : BADC	All objects have been flipped with a neighbour.

For 3 objects, ABC, we have 3x2x1=6 permutations:

```
before: ABC    ABC    ABC    ABC    ABC    ABC
after : ABC    ACB    BAC    BCA    CAB    CBA
```

Only the first three are **fix-or-flip** permutations. In the fourth A has moved more than 1 place and in the last two C has moved 2 places.
How many fix-or-flip permutations are there for 4 objects? for 5? for n objects? Why?

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Two heads are better than one?

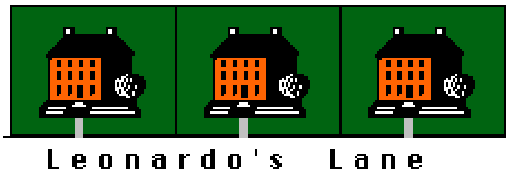
Usually, if using a coin to make a decision, it is something like "Heads I win, Tails you lose" !!
What about tossing a coin until *two* heads appear one after the other?
If we toss a coin twice, then there are four possible outcomes:
TT, TH, HT and HH
In only **1** of these four do we get two heads.
What happens if we have to wait for *exactly three tosses* before we get two heads?
This time the possibilities are
TTT, TTH, THT, HTH, HTT, and THH
Note that we do not have HHT or HHH as we would have got two heads after only 2 tosses which was covered earlier. So there is again just **1** way to get two heads appearing, H on the second and H on the third toss.
How many ways are there if HH appears on the 3rd-and-4th tosses? TTTT, TTTH, TTHT, TTHH, THTT, THTH, HTHH, HTHT, HTTH, HTTT.
This time we find **2** sequences.
Can you find a method of generating *all* the sequences of n coin-tosses that do not have HH until the last two tosses?
Can you find a formula for how many of these will end in HH?

OPTIONAL EXTRA!!! What about the number of sequences of n coin tosses that end with three Heads together? Does this have any relationship to the Fibonacci numbers?

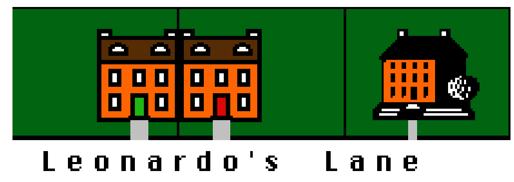
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Leonardo's Lane

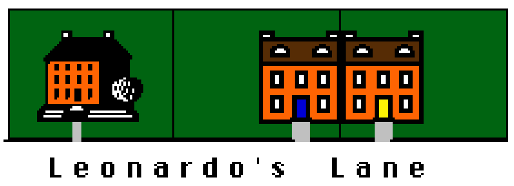
This puzzle was suggested by Paul Dixon, a mathematics teacher at Coulby Newham School, Middlesbrough.
A new estate of houses is to be built on one side of a street - let's call it **Leonardo's Lane**. The houses are to be of two types: a single house (a detached house) or two houses joined by a common wall (called "a pair of semi-detached houses" in the UK) which take up twice the frontage on the lane as a single house.
For instance, if just 3 houses could be fitted on to the plot of land in a row, we could suggest:



DDD: Three detached houses



SD: a pair of semi's first followed by a detached house



DS: a detached house followed by a pair of semi's

If you were the architect and there was space for just n dwellings on the Lane of just the two kinds mentioned above, what combinations could you use along the lane?

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Boat Building NEW

[Suggested by Dmitry Portnoy (7th grade)]

A boat building company makes two kinds of boat:
a **canoe**, which takes a month to make and
a **sailing dinghy** and they two months to build.

The company only has enough space to build one boat at a time but it does have plenty of customers waiting for a boat to be built.

Suppose the area where the boats are built has to be closed for maintenance soon:

- if it is closed after one more months work, the builders can only build one boat - a canoe - before then. Let's write this plan as **C**;
- if it is to be closed after 2 months work, it can EITHER build 2 canoes (**CC**) OR ELSE build one dinghy (**D**), so there are two plans to choose from;
- if it closed in three months time, it could make 3 canoes (CCC) or a dinghy followed by a canoe (**DC**) or a canoe and then a dinghy (**CD**); so there are three choices of plan.
- What choices are there if it closed after 4 months?

- ... or after 5 months?
- ... or after n months?

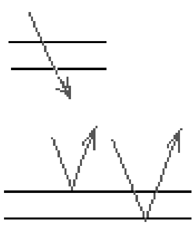
You can adapt this puzzle:

1. .. to larger boats: patrol boats taking a year to build or container ships which take two years to make
2. .. or you can make the problem smaller, and consider *model boats*, a small kit taking one month on your desk or a larger kit taking two months.

How many more ideas can you come up with for a similar puzzle? NEW

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Pause for a little reflection

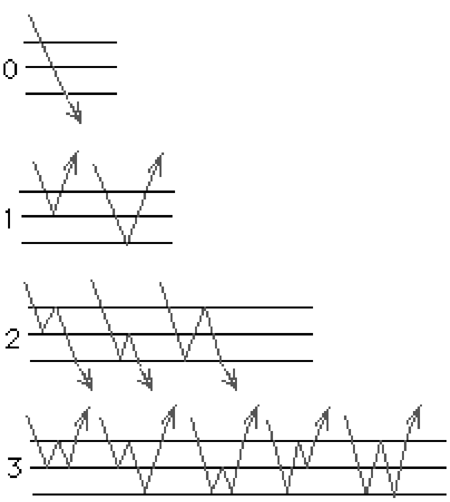


If you look at a window of one sheet of flat, clear glass, what's on the other side is quite clear to see. But if you look through the same piece of glass when it is dark on the other side, for instance into a shop window when the shop is dark, you can see your own reflection. This time the clear glass is behaving like a mirror.
If you look *very* closely, you will see that your reflection is actually doubled - there are two images of your face side by side. This is because your image is not only reflected off the top surface of the glass but also gets reflected from the *other side of the glass* too - which is called **internal reflection**.

So a natural question is *what happens if we have double glazing* which has two sheets of glass separated by an air gap, that is, 4 reflecting surfaces?
Hang on a minute ... what about *three surfaces*?? Let's look at that first!

For three surfaces (for example two sheets of glass resting on each other) what happens depends on whether we are looking *through* both sheets of glass (the rays of light come in on one side of the window but exit from the other) or whether we are looking at our own reflection from the sheets (the rays of light enter and leave from the same side of the window).

We can ignore the reflection off the top surface - the light bounces off and we get one reflection. The other cases are the interesting ones - where all the reflections are *internal reflections*. In other words, the light rays must have actually penetrated the glass and we can get reflections from one or perhaps both or even none of the *two internal surfaces*. We may even get more reflections as the light bounces off the surfaces again and again, some of the light escaping each time.



The diagram here shows the possible reflections ordered by the *number of internal reflections*, starting with none (the light goes straight through) to a single internal reflection (from either of the *internal* surfaces so there are two cases) and then exactly two internal reflections and finally we have shown 3 internal reflections.

If you reflect on this, you'll notice that the Fibonacci numbers seem to be making themselves clearly visible (groan!). ***Why?***

[Advanced puzzle: *What does happen with 4 reflecting surfaces in a double glazed window?*]

Reflections across Two and Three Glass Plates by V E Hoggatt Jr and Marjorie Bicknell-Johnson in *The Fibonacci Quarterly*, volume 17 (1979), pages 118 - 142.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

A Puzzle about Puzzles

This is a puzzle about puzzles - the puzzle is to design your own puzzle!!
You might have noticed that quite a few of the puzzles above are really "the same" but the names and situations are changed a bit. It is fairly easy to see how **Leonardo's Leaps** is the same as the 1p and 2p **coin change puzzle** and also it is just **Leonardo's Lane** but slightly disguised.
So...

can you devise your own puzzle where the answer is the Fibonacci numbers?

The reason the puzzles above are "the same" is that the explanation of the solution of each of them involves the Fibonacci (recurrence) Rule:

$$F(n) = F(n-1) + F(n-2)$$

together with the "initial conditions" that F(0)=0 and F(1)=1

Your puzzle should be based around this relationship.

Do you want to see your name on this page?

Please do email me with any new variations that you find. You can then share your idea with all the other readers of this page. Let's see how big a collection we can build!

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

More Links and References

- The Amazing Mathematical Object Factory has an interesting section on *Fibonacci Numbers* which contains explanations for some of the puzzles on this page and the relationships between them.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Where to Now?

Fibonacci Home Page

The Fibonacci Numbers and Golden Section in Nature

This is the first page on Fibonacci Puzzles.
Harder Fibonacci Puzzles

The next Topic is...
The Mathematical World of Fibonacci and Phi

The Fibonacci Puzzles page has been divided into two. Here is the SECOND part with puzzles a bit harder than those of the FIRST part which you are recommended to browse through first!

Harder Fibonacci Puzzles

All these puzzles except one (which??) have the Fibonacci numbers as their answers.
So now you have the puzzle and the answer - so what's left? Just the *explanation of why* the Fibonacci numbers are the answer - that's the *real* puzzle!!

The Fibonaci puzzles are split into **two sections**: those with fairly straight-forward and simple explanations as to why the answer is the Fibonacci numbers are on the Easier Fibonacci Puzzles page.

CONTENTS of THIS Page

This page contains the second set where it is not so simple to explain why their answers involve the Fibonacci numbers. Does a **simple** explanation exist? If you find a simple explanation please email me and let me know as I'd like to share the simpler solutions on these pages.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Pennies for your thoughts - Part 1

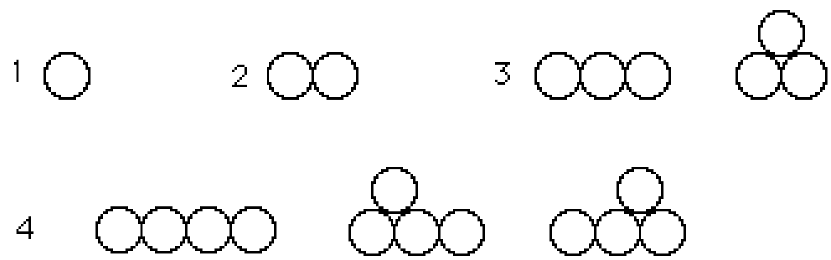
Here are two puzzles which are identical - but we count the solutions in two different ways. Each involves arranging pennies (coins) in rows.

The puzzle here is that **only one of these two puzzles involves the Fibonacci number series!** The other puzzle does not but just *begins* with a few of the Fibonacci numbers and then becomes something different. *One of these puzzles is a fraud, a Fibonacci forgery*. So which is the *real* Fibonacci puzzle?

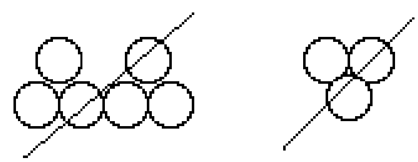
Arrange pennies in rows under these two conditions:

1. each penny must touch the **next** in its row
2. each penny except ones on the bottom row touches *two* pennies on the row **below**.

There is just 1 pattern with one penny,
and 1 with two pennies
but 2 for three pennies
and 3 with four pennies as shown here:-



The first condition means that there are **no gaps** in any row and the second means that **upper rows are smaller than lower ones**.
The following arrangements are not proper combinations for 6 pennies because the first has a gap in one row and the second has a penny which is not on the bottom row and is not touching *two* beneath it.

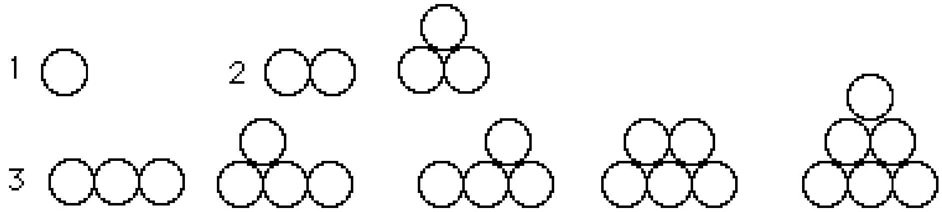


**If there are P(n) such arrangements for n pennies,
are the P(n) numbers always Fibonacci numbers?**

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Pennies for your thoughts - Part 2

This puzzle is the same as the previous one and again seems to involve the Fibonacci numbers - or does it?
The puzzle is exactly the same, but P(n) now counts *the number of arrangements which have n pennies* ***on the bottom row***.



Here there is only 1 arrangement with 1 penny on the bottom row so P(1)=1 and
2 arrangements with two on the bottom row, P(2)=2
and 5 patterns with a bottom row of three coins P(3)=5.
What happened to 3? F(4)=3 is missing! You can check that P(4)=13, so P(n) is clearly *not* the same as the Fibonacci series since F(4)=3 and F(6)=8 are missing. This time the question is

Are the P(n) numbers the *alternate* Fibonacci numbers:

```
     i : 0 1 2 3 4 5  6  7  8
Fib(i): 1 1 2 3 5 8 13 21 34...
  P(n): 1   2   5   13     ?
     n : 1   2   3    4     5
```

Which one of these two Pennies puzzles is the forgery (it does *not* continue with a pattern of Fibonacci numbers after some point) and which one genuinely always has Fibonacci numbers of arrangements?

[With thanks to Wendy Hong for brining these two puzzles to my attention.]

References

Richard K Guy, **The Second Strong Law of Small Numbers** in *The Mathematics Magazine*, **Vol 63** (1990), pages 3-21, Examples 45 and 46.

The first Pennies puzzle above was mentioned by F. C. Auluck in **On some new types of partitions associated with Generalised Ferrers graphs** in *Proceedings of the Cambridge Philosophical Society*, **47** (1951), pages 679-686 (examples 45 and 46).

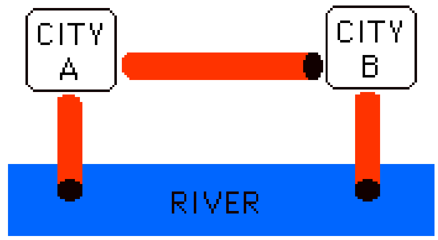
0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Water Treatment Plants puzzle

Cities along a river discharge cleaned-up water from sewage treatment plants. It is more efficient to have treatment plants running at maximum capacity and less-used ones switched off for a period. So each city has its own treatment plant by the river and also a pipe to its neighbouring city upstream and a pipe to the next city downstream along the riverside.
At each city's treatment plant there are three choices:

- **either** process any water it may receive from one neighbouring city, together with its own dirty water, discharging the cleaned-up water into the river;

- **or** send its own dirty water, plus any from its downstream neighbour, along to the upstream neighbouring city's treatment plant (provided that city is not already using the pipe to send it's dirty water downstream);
- **or** send its own dirty water, plus any from the upstream neighbour, to the downstream neighbouring city's plant, if the pipe is not being used.



The choices above ensure that

- every city must have its water treated somewhere and
- at least one city must discharge the cleaned water into the river.

Let's represent a city discharging water into the river as "V" (a downwards flow), passing water onto its neighbours as ">" (to the next city on its right) or else "<" (to the left). When we have several cities along the river bank, we assign a symbol to each (V, < or >) and list the cities symbols in order.
For example, **two** cities, A and B, can

- each treat their own sewage and each discharges clean water into the river. So A's action is denoted V as is B's and we write "VV" ;
- or else city A can send its sewage along the pipe (to the right) to B for treatment and discharge, denoted ">V" ;
- or else city B can send its sewage to (the left to) A, which treats it with its own dirty water and discharges (V) the cleaned water into the river. So A discharges (V) and B passes water to the left (<), and we denote this situation as "V<".

We could not have "><" since this means A sends its water to B and B sends its own to A, so both are using the same pipe and this is not allowed. Similarly "<<" is not possible since A's "<" means it sends its water to a non-existent city on its left.

So we have just **3** possible set-ups that fit the conditions:-

```
        A    B          A>>>B        A<<<B
        V    V              V        V
        V    V              V        V
 RIVER~ ~ ~ ~ ~       ~ ~ ~ ~ ~      ~ ~ ~ ~ ~RIVER
          "VV"          ">V"           "V<"
```

Now suppose that we have more than two cities along the river back:-

```
       _______       _______       ____
  A  (_______()  B  (_______()  C  (____ ...   <- pipes connecting cities
  ||             ||             ||             <- pipes discharging into river
  ()             ()             ()
~ ~ ~ ~ ~ ~ ~R~I~V~E~R~ ~ ~ ~ ~ ~ ~ ~
```

1. What are the **eight** set-ups possible for 3 cities?
2. If S(n) is the number of set-ups for n cities, then S(1)=1 and we have just shown that S(2)=3 and S(3)=8. But this does *not* look like the Fibonacci numbers!
   What is S(4)? What is S(5)?
3. What is the relationship between the S-numbers here and the Fibonacci series!?

See **Fibonacci Numbers and Water Pollution Control** R A Deininger in *Fibonacci Quarterly*, Vol 10, No 3, 1972, pages 299-300 and page 302.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Wythoff's game

The Fibonacci numbers provide a winning strategy for playing a game with two piles of matches (or counters or coins etc), first described by W A Wythoff in 1906.

Players take it in turns to remove some matches (at least one) from EITHER only one pile OR ELSE an equal number from both piles. The players can decide how large each heap will be before the game starts and the winner is the one who takes the LAST match. A complete heap can be removed as your move if you like. This is not to be recommended however, since your opponent can do the same on the next move and so will win by taking the last match! This leads to the idea of "safe configurations", that is, ones from which it is possible to force a win, no matter what your opponent does.

For further details, see
T. H. O'Beirne *Puzzles and Paradoxes*, Dover press, 1965, chapter 8.
Ball, W.W.R. and Coxeter, H.S.M. Mathematical Recreations and Essays, 13th edition, Dover Publications, 1987. A great classic with plenty to keep you amused and enthused on Maths - definitely worth buying! (You can order it online via the title-link.)

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Non-neighbour Groups

How often have the list of names in your class been read out in alphabetical order, or you have been asked to line up in alphabetical order for a fire-practice or when the results of a test are given out? The trouble with this is that you are always next to the same one or two people that are on either side of you in the alphabetical order - your

alphabetical neighbours. You will have got to know them quite well over the course of a year, so this puzzle is about meeting other people who are not your alphabetical neighbours.
Suppose that part of the class is needed for a particular task or game. Let's also say that the group should contain no *alphabetical neighbours* in it, so it gives *everyone* in the group a chance to team up with new people.
In how many ways can you choose such a group from a class of N students?

For instance, if there are 3 people in the class, let's label them according to their position when in the alphabetical order, so they are 1, 2 and 3.

The puzzle is to select a group from the class
with **no pair of successive numbers (positions) in the group**.

So if 1 is in the group, then 2 cannot be and 3 may be or not; so we have the groups:
{1} and {1,3}

If 2 is in the group then, since both 1 and 3 are 2's alphabetical neigbours, then that group will consist of 2 alone!
{2}
If 3 is in the group then 2 cannot be and 1 may be. But remember that the group with 3 and 1 in it has already been included above! So we have the following possible *new* groups with 3 in:
{3}

All the possible groups of non-neighbours are:
{1,3} {1} {2} {3} {}
Did you notice that the group {} with *nobody in it* is a non-neighbour group too? So from a class of 3 people, there are 5 ways to pick a group consisting solely of non-neighbours. How many are there in a class of size 4? or 5? or 6? Why? **On THe Number of Fibonacci Partitions of a Set** Helmut Prodinger *Fibonacci Quarterly*

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

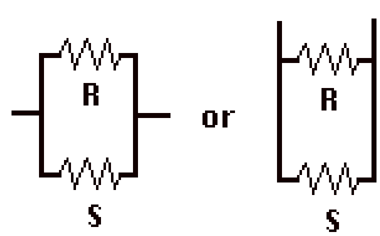
A ladder of resistors

Basic principles

*If we have two electrical resistances of R ohms and S ohms **in series**: then the combined resistance is just R+S ohms.*

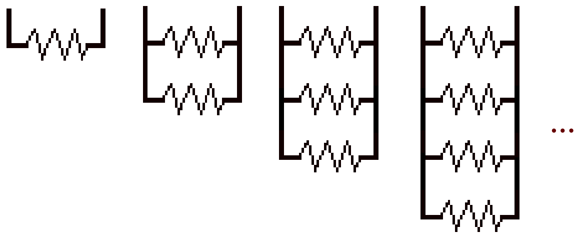
*You'll remember that if we have 2 resistances R and S **in parallel**: then the combined resistance, T, is given by*

$$\frac{1}{T} = \frac{1}{R} + \frac{1}{S}$$



Ladder Problem 1

Suppose we extend the pattern of parallel resistors into longer and longer ladders, by putting a 1 ohm resistor between two wires and then keep adding single ohm resistors in parallel. What is the total resistance?



In the diagram above, the 2 resistor ladder has two 1-ohm resistors in parallel so their combined resistance R is given by the equation:

```
1/R   = 1/1   + 1/1 = 2          so    R=1/2
```

For the 3 resistor ladder, we have combined the 2 resistor ladder with another resistor of 1-ohm, in parallel, so the combined resistance S here is

```
1/S = 1/(1/2) + 1/1 = 2+1 = 3   so   S=1/3
```

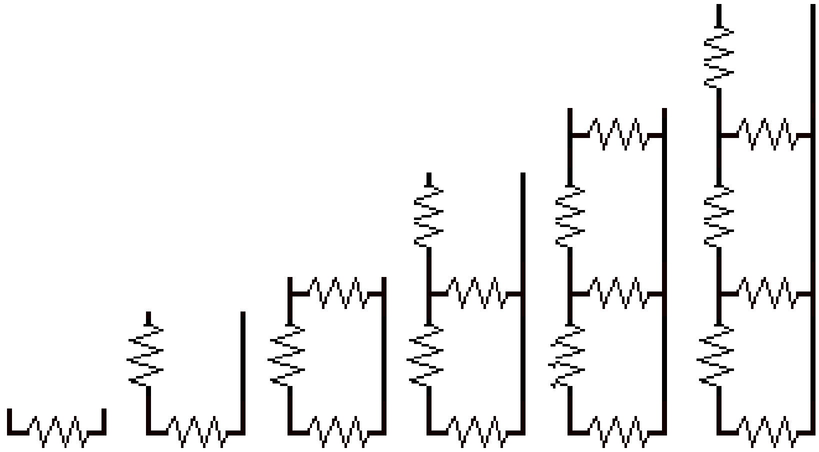
Try computing the overall resistance for yourself for 4 resistors, then with 5 and 6.

What pattern are you getting for the combined resistance?

Can you prove that your pattern always holds?

Ladder Problem 2

Now try it with the following pattern of resistances, where one of the legs of the ladder also has resistance of 1-ohm and we alternately add a resistor on a side leg and then on a rung:



The first ladder has a single resistor so is 1 ohm.
The second ladder has two resistors in series, so the combined resistance is 2.
The third ladder has a 1 ohm resistor in parallel with the second ladder (2 ohms), so the combined resistance S of 1 ohm and 2 ohms in parallel is

$$1/S = 1/1 + 1/2 = 3/2 \quad \text{ie } S=2/3$$

Similarly, the next ladder has a 1 ohm resistor in series with the previous ladder, so its total resistance is 1+2/3=5/3.

What about the next two ladders? What is the general pattern now?

Again, can you prove that your pattern will always hold?

Ladder Problem 3

Try making a ladder where the only resistances are DOWN ONE SIDE and there is no resistance on the "rungs". What pattern do you get now?

Ladder Problem 4

*Replace the **resistors** with **capacitors** in Ladder Problem 2.*
What pattern do you get now?
[Suggested by Bhushit Joshipura.]

References on the Resistance Ladders

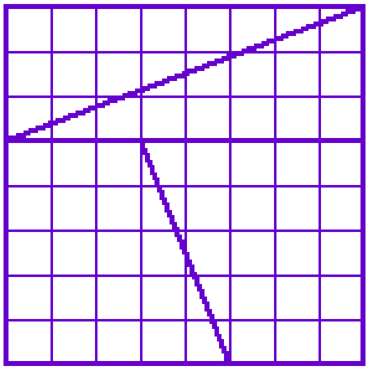
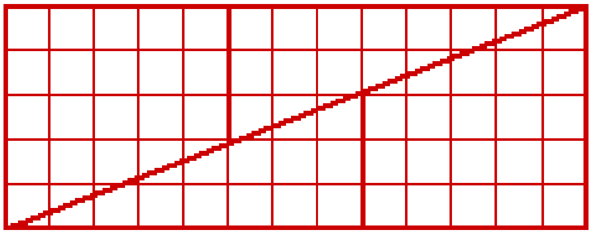
***The Golden Ratio in an Electrical Network**, J Wlodarski in The Fibonacci Quarterly Vol 9 (1971) pages 188 and pg 194.*

***Generalisation of Modified Morgan-Voyce Polynomials**, Fibonacci Quarterly Vol 38 No 1, 2000, pgs 8-16.*

An advanced mathematical article dealing with resistors, capactors and inductors.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

A Fibonacci Jigsaw puzzle
or How to Prove 64=65

*The 8-by-8 blue square in the diagram here can be cut up into 4 pieces that, when rearranged, make the red 5-by-13 rectangle. But the blue square contains 8x8=**64** little squares whereas the red rectangle contains 5x13=**65**. Where has the extra square come from?*

This puzzle can be repeated with other consecutive Fibonacci numbers,

replace 5, 8 and 13 by 8, 13 and 21 or by 3, 5 and 8

If you look at the "8, 13, 21" jigsaw, the square is 13x13=169 but this time the rectangle is 8x21=168 so we have lost a square this time! Sometimes there is a square extra, sometimes a square goes missing.

Not convinced? Try this demonstration

Try cutting out the pieces as shown and rearranging them yourself if you are not sure the puzzle "works".
It works even better as a class demonstration using an overhead projector: photocopy the square with its grid lines onto an overhead projector transparency, cut out the shapes and show them as a square on the screen, then rearrange it into the rectangle, carefully lining up the grid lines to "show I'm not cheating"!

But what is the explanation?

Hints:

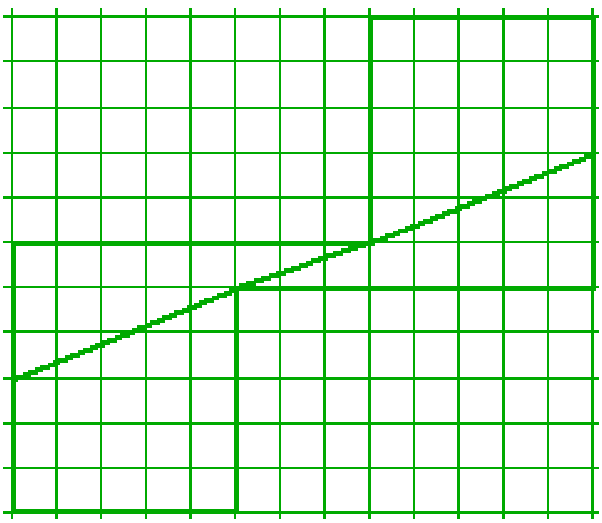
1. *What is the formula behind these puzzles?*
   For any three consecutive Fibonacci numbers: F(n-1), F(n) and F(n+1), it relates $F(n)^2$ to F(n-1)F(n+1); what is it?
   Perhaps you can try to prove it is always true.
2. *Now look carefully at one of the jigsaw puzzles. Is it really what it seems? Try taking **a different angle** on the problem - perhaps **looking at it from a tangent**.*

Edward Wakeling in [Rediscovered Lewis Carroll Puzzles] Dover, 1995, says that this puzzle was found in Lewis Carroll's papers (the original is now kept at Princeton University) and that this puzzle goes back to Schlomilch, 1868.

Martin Gardner's [Mathematics, Magic and Mystery] a 1956 Dover book, is a book with magic tricks and how the mathematics behind them makes them work. It has two chapters on such Geometrical Vanishes and has a full explanation of this and other puzzles. He also traces its origins back to Sam Loyd (senior) who presented it to the American Chess congress (using an 8-by-8 chessboard) in 1858, ten years before Wakeling's reference to Schlomilch in the reference above. However this also appears not to be the earliest refrerence...

David Wells in [The Penguin Book of Curious and Interesting Puzzles] (Penguin, 1997) in Puzzle 143 traces its origin back to William Hooper in Rational Recreations of 1774.

The same puzzle but losing a square or How to Prove 64=63!!



The blue jigsaw of area 64 little squares, when re-arranged into the red positions with 65 little squares, had seemingly gained a square.

Here is another arrangement. This time the blue puzzle's pieces have been re-arranged as shown here in green and now it loses a square -- there are two 5-by-6 rectangles + 3 squares in a row joining them, making a total area of 63!

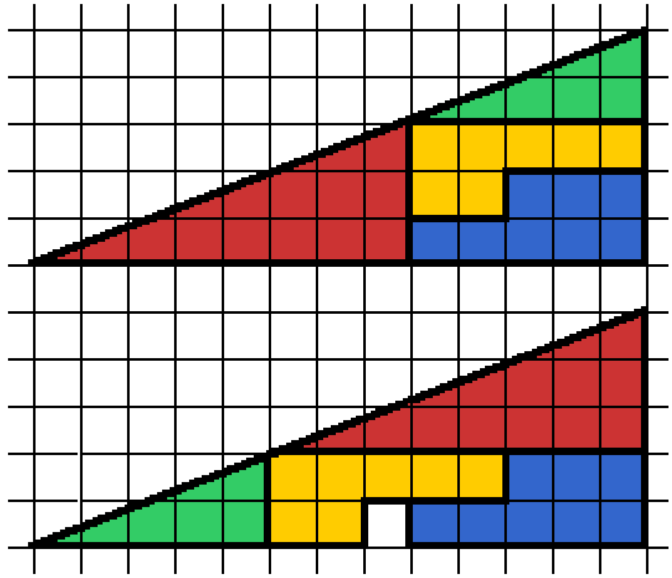
So what's happened this time???

*The second version comes from Henry E Dudeney's **536 Puzzles and Curious Problems** (which has been edited by Martin Gardner) 1967, Souvenir Press; Problems 352 and 353 and their answers*

*Martin Gardner's **Mathematics, Magic and Mystery** a 1956 Dover book (mentioned in the first version of this puzzle) says that Sam Loyd junior (who adopted his father's name and continued his father's puzzle columns) was the first to discover this new reduced-square version. This book has a good explanation of how the two puzzles work and that the Fibonacci numbers produce other sizes of puzzle with identical variations of an additional and missing single square. He shows how other generalised Fibonacci sequences (i.e. starting with two other numbers rather than 0 and 1) can be used to devise variations where any number of squares can be made to appear and disappear, together with many other kinds of geometrical dissection puzzles. If you like the puzzles on these two Web pages, you'll enjoy this book too with number, handkerchief and card puzzles based on mathematics.*

NEW Yet another Fibonacci Jigsaw Puzzle!

Roy Nauw of Kloetinge, the Netherlands found another Fibonacci Puzzle. His lecturer, Floor van Lamoen, mentioned it on the Geometry Puzzles newsgroup (archived at Math Forum) and it is copied here with Roy's permission (and my thanks to them both).



It is made up of 4 pieces,

- *a smaller green triangle with height **2** and base **5** ;*
- *a larger red triangle with height **3** and base **8**;*
- *an orange L-shape of height **2** and width **5**;*
- *a blue L-shape of the same width and height but a different shape.*

The two L-shaped pieces fit together to make a 3-by-5 rectangle. They can all be arranged into a 13-by-5 triangle as shown here. Rearranging the 4 pieces shows the triangle has a square missing!

Where does the hole come from?

What's the answer this time and how is it connected with the Fibonacci Numbers?

The puzzle will work with a green triangle height 1 base 3 and a red triangle height 2 base 5, and two straight pieces (1-by-3) that make up a 2-by-3 rectangle. Rearanging them this time makes the small rectangle 1 square smaller this time so the two straight pieces have to overlap.
Similarly, using triangles of height 3 base 8 and height 5 base 13 the rectangle again loses one square.

	small green triangle		**large red triangle**		**rectangle green width red height**	**rectangle red width green height**	**Rectangle Area Difference**
	height	**base**	**height**	**base**	**height x base = Area**	**height x base = Area**	
smaller puzzle	1	3	2	5	2 x 3 = 6	1 x 5 = 5	-1
puzzle above	2	5	3	8	3 x 5 = 15	2 x 8 = 16	+1
larger puzzle	3	8	5	13	5 x 8 = 40	3 x 13 = 39	-1
larger puzzle	5	13	8	21	8 x 13 = 104	5 x 21 = 105	+1

`0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..`More`..`

More Links and References

The Amazing Mathematical Object Factory has an interesting section on Fibonacci Numbers which contains explanations for some of the puzzles on this page and the relationships between them.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Where to Now?

Fibonacci Home Page

The Fibonacci Numbers and Golden Section in Nature

The Easier Fibonacci Puzzles
This is the last page of Fibonacci Puzzles.

The next Topic is...
The Mathematical World of Fibonacci and Phi

The Mathematics of the Fibonacci series

Take a look at the Fibonacci Numbers List or, better, open another window in your Browser, then you can refer to this page and the list together.

Contents

The line means there is a Things to do investigation at the end of the section.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Patterns in the Fibonacci Numbers

Cycles in the Fibonacci numbers

Here are some patterns people have already noticed:

- There is a cycle in the **units** column - the cycle of units digits (**0,1,1,2,3,5,8,**1**3,**2**1,**3**4,**5**5,...**) repeats from n=60 and again every 60 values.
- There is also a cycle in the **last two** digits, repeating (**00, 01, 01, 02, 03, 05, 08, 13, ...**) from n=300 with a cycle of length 300.
- For the last **three** digits, the cycle length is 1,500
- for the last **four** digits,the cycle length is 15,000 and
- for the last **five** digits the cycle length is 150,000
- and so on...

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Factors of Fibonacci Numbers

There are some fascinating and simple patterns in the Fibonacci numbers when we consider their factors. You might like to click here to open a new browser window which shows the first 100 Fibonacci numbers and their factors. It will be helpful in the following investigations:

Things to do

1. Where are the **even Fibonacci Numbers**?
   Write down the *index numbers* i where Fib(i) is even.
   Do you notice a pattern?
   Write down the pattern you find as clearly as you can *first* in words and *then* in mathematics. Notice that 2=F(3) also.
2. Now find where there are Fibonacci numbers which are **multiples of 3.**
   and again write down the pattern you find in words and then in mathematics.
   Again notice that 3=F(4).
3. What about the **multiples of 5**? These are easy to spot because they end with **0** or **5.**
   Again, write down the pattern you find.
4. You can try and spot the multiples of 8, if you like now.
   Why 8? Because we have found the multiples of 2, then 3, then 5 and now 8 is the next Fibonacci number!
5. Do you think your patterns also have a pattern? That is, for *any Fibonacci Number F* can you tell me where you think all its multiples will appear in the whole list of Fibonacci Numbers?

The above investigations should help you to understand **the general rule**:

Every k-th Fibonacci number is a multiple of F(k)

or, expressed mathematically,

F(nk) is a multiple of F(k) for all values of n and k=1,2,...

This means that if the subscript is composite (not a prime) then so is that Fibonacci number (with one exception - can you find it?) So we now deduce that

Any prime Fibonacci number must have a subscript which is prime

(with one little exception - can you find it? Hint: you won't have to search far **for** it .)

A Primer For the Fibonacci Numbers: Part IX M Bicknell and V E Hoggatt Jr in *The Fibonacci Quarterly* Vol 9 (1971) pages 529 - 536 has several proofs that F(k) divides exactly into F(nk): using the Binet Formula; by mathematical induction and using generating functions.

Fibonacci Primes

Unfortunately, the converse is not always true: that is, it is **not** true that if a subscript is prime then so is that Fibonacci number. The first case to show this is the 19th position (and 19 is prime) but F(19)=4181 and F(19) is *not* prime because 4181=113x37. In fact, a search using Maple finds that the list of index numbers, i, for which Fib(i) is prime begins as follows:

i	3	4	5	7	11	13	17	23	29	43	47	83	131	137	359	431	433	449	...
Fib(i)	2	3	5	13	89	233	1597	28657	514229	433494437	10 digits	17 digits	28 digits	29 digits	75 digits	90 digits	91 digits	94 digits	...

Now you should be able to spot the **odd one out**: that one number, i, which is *not a prime* in the list above, even though Fib(i) is.

Two Prime Curios

G. L. Honaker Jr. pointed me to two curious oddities about the Fibonacci numbers and prime number. a Prime Curio that the number of primes less than 144, which is a Fibonacci number, is 34, another Fibonacci number. He asks:

Can this happen with two larger Fibonacci numbers?

I pass this question on to you - can it? The link to the Prime Curio page uses the notation that π(N) means "the number of primes between 1 and N" and includes N too if N is prime. (See also a graph of this function.) Since the prime numbers begin

2, 3, 5, 7, 11, 13, 17, ...

then π(8)=4 (there are 4 primes between 1 and 8, namely 2, 3, 5 and 7) and π(11)=5.
There are some smaller values, too:

π(2) = 1
π(3) = 2
π(5) = 3
π(21) = 8

More Links and References on Prime Numbers

There is a complete list of all Fibonacci numbers and their factors up to the 1000-th Fibonacci and 1000-th Lucas numbers and partial results beyond that on Blair Kelly's Factorisation pages

Chris Caldwell's Prime Numbers site has a host of information.

There is a nice Primes Calculator at Princeton University's web site.

Factorization of Fibonacci Numbers D E Daykin and L A G Dresel in *The Fibonacci Quarterly*, vol 7 (1969) pages 23 - 30 and 82 gives a method of factorising a Fib(n) for composite n using the "entry point" of a prime, that is, the index of the first Fibonacci number for which prime p is a factor.

`0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..`

Benford's Law and initial digits

[With thanks to Robert Matthews of *The Sunday Telegraph* for suggesting this topic.]

Having looked at the *end digits* of Fibonacci numbers, we might ask

Are there any patterns in the *initial digits* of Fibonacci numbers?

What are the chances of a Fibonacci number beginning with "1", say? or "5"? We might be forgiven for thinking that they probably are all the same - each digit is equally likely to start a randomly chosen Fibonacci number. You only need to look at the Table of the First 100 Fibonacci numbers or use Fibonacci Calculator to see that this is not so. Fibonacci numbers seem far more likely to start with "1" than any other number. The next most popular digit is "2" and "9" is the least probable!

This law is called **Benford's Law** and appears in many tables of statistics. Other examples are a table of populations of countries, or lengths of rivers. About one-third of countries have a population size which begins with the digit "1" and very few have a population size beginning with "9".

Here is a table of the initial digits as produced by the Fibonacci Calculator:

```
Initial digit frequencies of fib(i) for i from 1 to 100:
    Digit:    1   2   3   4   5   6   7   8   9
Frequency:   30  18  13   9   8   6   5   7   4  100 values
  Percent:   30  18  13   9   8   6   5   7   4
```

What are the frequencies for the first 1000 Fibonacci numbers or the first 10,000? Are they settling down to fixed values (percentages)? Use the Fibonacci Calculator to collect the statistics. According to Benford's Law, large numbers of items lead to the following statistics for starting figures for the Fibonacci numbers as well as some natural phenomena

Digit:	1	2	3	4	5	6	7	8	9
Percentage:	30	18	13	10	8	7	6	5	5

Things to do

1. Look at a table of sizes of countries. How many countries areas begin with "1"? "2"? etc.
2. Use a table of population sizes (perhaps of cities in your country or of countries in the world). It doesn't matter if the figures are not the latest ones. Does Benford's Law apply to their initial digits?
3. Look at a table of sizes of lakes and find the frequencies of their initial digits.
4. Using the Fibonacci Calculator make a table of the first digits of powers of 2. Do they follow Benford's Law? What about powers of other numbers?
5. Some newspapers give lists of the prices of various stocks and shares, called "quotations". Select a hundred or so of the quotations (or try the first hundred on the page) and make a table of the distribution of the leading digits of the prices. Does it follow Benford's Law?
6. What other sets of statistics can you find which do show Benford's Law? What about the number of the house where the people in your class live? What about the initial digit of their home telephone number?
7. Generate some random numbers of your own and look at the leading digits.
   You can buy 10-sided dice (bi-pyramids) or else you can cut out a decagon (a 10-sided polygon with all sides the same length) from card and label the sides from 0 to 9. Put a small stick through the centre (a used matchstick or a cocktail stick or a small pencil or a ball-point pen) so that it can spin easily and falls on one of the sides at random. (See the footnote about dice and spinners on the "The Golden Geometry of the Solid Section or Phi in 3 dimensions" page, for picture and more details.)
   Are all digits equally likely or does this device show Benford's Law?
8. Use the random number generator on your calculator and make a table of leading-digit frequencies. Such functions will often generate a "random" number between 0 and 1, although some calculators generate a random value from 0 to the maximum size of number on the calculator. Or you can use the random number generator in the Fibonacci Calculator to both generate the values and count the initial digit frequencies, if you like.
   Do the frequencies of leading digits of random values conform to Benford's Law?
9. Measure the height of everyone in your class to the nearest centimetre. Plot a graph of their heights. Are all heights equally likely? Do their initial digits conform to Benford's Law? Suppose you did this for everyone in your school. Would you expect the same distribution of heights?
10. What about repeatedly tossing five coins all at once and counting the number of heads each time?

```
What if you did this for 10 coins, or 20?
What is the name of this distribution (the shape of the frequency graph)?
```

When does Benford's Law apply?

Random numbers are equally likely to begin with each of the digits 0 to 9. This applies to randomly chosen real numbers or randomly chosen integers.

Randomly chosen real numbers
: If you stick a pin at random on a ruler which is 10cm long and it will fall in each of the 10 sections 0cm-1cm, 1cm-2cm, etc with the same probability. Also, if you look at the initial digits of the points chosen (so that the initial digit of 0.02cm is 2 even though the point is in the 0-1cm section) then each of the 9 values from 1 to 9 is as likely as any other value.

Randomly chosen integers
: This also applies if we choose random integers.
Take a pack of playing cards and remove the jokers, tens, jacks and queens, leaving in all aces up to 9 and the kings. Each card will represent a different digit, with a king representing zero. Shuffle the pack and put the first 4 cards in a row to represent a 4 digit integer. Suppose we have King, Five, King, Nine. This will represent "0509" or the integer 509 whose first digit is 5. The integer is as likely to begin with 0 (a king) as 1 (an ace) or 2 or any other digit up to 9. But if our "integer" began with a king (0), then we look at the next "digit".
These have the same distribution as if we had chosen to put down just 3 cards in a row instead of 4. The first digits all have the same probability again. If our first two cards had been 0, then we look at the third digit, and the same applies again.
So if we ignore the integer 0, any randomly chosen (4 digit) integer begins with 1 to 9 with equal probability. (This is not quite true of a row of 5 or more cards if we use an ordinary pack of cards - why?)

So the question is, why does this *all-digits-equally-likely* property **not** apply to the first digits of each of the following:

- the Fibonacci numbers,
- the Lucas numbers,
- populations of countries or towns
- sizes of lakes
- prices of shares on the Stock Exchange

Whether we measure the size of a country or a lake in square kilometres or square miles (or square anything), does not matter - Benford's Law will still apply.
So when is a number *random*? We often meant that we cannot predict the next value. If we toss a coin, we can never predict if it will be Heads or Tails if we give it a reasonably high flip in the air. Similarly, with throwing a dice - "1" is as likely as "6". Physical methods such as tossing coins or throwing dice or picking numbered balls from a rotating drum as in Lottery games are always unpredictable.

The answer is that the Fibonacci and Lucas Numbers are governed by a **Power Law**.
We have seen that Fib(i) is round($Phi^i/\sqrt{5}$) and Lucas(i) is round(Phi^i). Dividing by sqrt(5) will merely adjust the scale - which does not matter. Similarly, rounding will not affect the overall distribution of the digits in a large sample.

Basically, Fibonacci and Lucas numbers are **powers of Phi**. Many natural statistics are also governed by a power law - the values are related to B^i for some base value B. Such data would seem to include the sizes of lakes and populations of towns as well as non-natural data such as the collection of prices of stocks and shares at any one time. In terms of natural phenomena (like lake sizes or heights of mountains) the larger values are rare and smaller sizes are more common. So there are very few large lakes, quite a few medium sized lakes and very many little lakes. We can see this with the Fibonacci numbers too: there are 11 Fibonacci numbers in the range 1-100, but only one in the next 3 ranges of 100 (101-200, 201-300, 301-400) and they get increasingly rarer for large ranges of size 100. The same is true for any other size of range (1000 or 1000000 or whatever).

Things to do

1. Type a power expression in the *Eval(i)=* box, such as **pow(1.2,i)** and give a range of i values from i=1 to i=100. Clicking the *Initial digits* button will print the leading digit distribution.
   Change 1.2 to any other value. Does Benford's Law apply here?
2. Using *Eval(i)*=**randint(1,100000)** with an i range from 1 to 1000 (so that 1000 separate random integers are generated in the range 1 to 100000) shows that the leading digits are all equally likely.

Benford's Law for Fibonacci and Lucas Numbers, L. C. Washington, *The Fibonacci Quarterly* vol. 19, 1981, pages 175-177.

The original reference: **The Law of Anomalous Numbers** F Benford, (1938) *Proceedings of the American Philosophical Society* vol 78, pages 551-572.

The Math Forum's archives of the History of Mathematics discussion group have an email from Ralph A. Raimi (July 2000) about his research into Benford's Law. It seems that Simon Newcomb had written about it much earlier, in 1881, in **American Journal of Mathematics** volume 4, pages 39-40. The name **Benford** is, however, the one that is commonly used today for this law.

MathTrek by Ivars Peterson (author of **The Mathematical Tourist** and **Islands of Truth**) the editor of Science News Online has produced this very good, short and readable introduction to Benford's Law.

M Schroeder **Fractals, Chaos and Power Laws**, Freeman, 1991, ISBN 0-7167-2357-3. This is an interesting book but some of the mathematics is at first year university level (mathematics or physics degrees), unfortunately, and the rest will need sixth form or college level mathematics beyond age 16. However, it is still good to browse through. It has only a passing reference to Benford's Law: *The Peculiar Distribution of the Leading Digit* on page 116.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

The Fibonacci Numbers in Pascal's Triangle

```
                 col   :  0 1 2 3 4 ...
                 ------+---------------
     1               0 |  1
    1 1          r   1 |  1 1             each number
   1 2 1         o   2 |  1 2 1           is the sum of
  1 3 3 1        w   3 |  1 3 3 1         the one above it and
 1 4 6 4 1           4 |  1 4 6 4 1       the one to the above-left.
    ...             ...   ...             eg 6 is 3+3 from row above.
```

Each entry in the triangle on the left is the sum of the two numbers either side of it but in the row above. A blank space can be taken as "0" so that each row starts and ends with "1".

Pascal's Triangle has lots of uses including

Calculating probabilities.
If you throw n coins randomly onto a table then the chance of getting H heads among them is the entry in row N, col H

divided by 2^n:
for instance, for 3 coins, n=3 so we use row 3:
3 heads: H=3 is found in **1**way (HHH)
2 heads: H=2 can be got in **3**ways (HHT, HTH and THH)
1 head: H=1 is also found in **3**possible ways (HTT, THT, TTH)
0 heads: H=0 (ie all Tails) is also possible in just **1**way: TTT

- **Finding terms in a Binomial expansion: $(a+b)^n$**
EG. $(a+b)^3 = \mathbf{1}a^3 + \mathbf{3}a^2b + \mathbf{3}ab^2 + \mathbf{1}b^3$

Can you find the Fibonacci Numbers in Pascal's Triangle?

Hints:

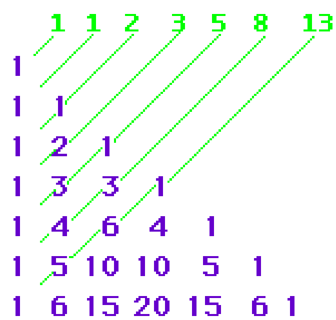
$$Fib(n)=\sum_{k=1}^{n}\binom{n-k}{k-1}$$

- The answer is in the formula on the right:
  where the big brackets with two numbers vertically inside them are a special mathematical notation for the entry in Pascal's triangle on row **`n-k-1`** and column **`k`**

  Or, an equivalent formula is:

$$Fib(n)=\sum_{k=0}^{n-1}\binom{n-k-1}{k}$$

- If that still doesn't help, then this animated diagram might:



Why do the Diagonals sum to Fibonacci numbers?

It is easy to see that the diagonal sums really are the Fibonacci numbers if we remember that each number in Pascal's triangle is the sum of two numbers in the row above it (blank spaces count as zero), so that 6 here is the sum of the two 3's on the row above:

```
col   :  0 1 2 3 4 ...
------+---------------
    0 |  1
r   1 |  1  1           each number
o   2 |  1  2  1         is the sum of
w   3 |  1  3  3  1       the one above it and
    4 |  1  4  6  4  1     the one to the above-left.
    5 |  1  5 10 10  5  1
    6 |  1  6 15 20 15  6  1
```

The numbers in any diagonal row are therefore formed from adding numbers in the previous two diagonal rows as we see here where all the blank spaces are zeroes and where we have introduced an extra column of zeros which we will use later:

```
col   :     0  1  2  3  4  ...
------+----------------
      |     0 1  <-- the first two diagonal sums
           / /
    0 |  0  1
           /
r   1 |  0  1  1     5=sum of green numbers
                       8=sum of blue numbers
o   2 |  0  1  2  1      13=sum of red numbers
                   \ |
w   3 |  0  1  3  3  1
                \ |
    4 |  0  1  4  6  4  1
             \ |
    5 |  0  1  5 10 10  5  1
          \ |
    6 |  0  1  6 15 20 15  6  1

    7 |  0  ...
```

Notice that the **GREEN** numbers are on one diagonal and the **BLUE** ones on the next. The sum of all the green numbers is 5 and all the blue numbers add up to 8.

Because all the numbers in Pascal's Triangle are made the same way - by adding the two numbers *above and to the left* ***on the row above***, then we can see that each red number is just the sum of a green number and a blue number and we use up all the blue and green numbers to make all the red ones.

The sum of all the red numbers is therefore the same as the sum of all the blues and all the greens: 5+8=13!

The general principle that we have just illustrated is:

> The *sum* of the numbers on one diagonal is the sum of the numbers on the previous two diagonals.

If we let D(i) stand for the sum of the numbers on the Diagonal that starts with one of the extra zeros at the beginning of row i, then

$$D(0)=0 \text{ and } D(1)=1$$

are the two initial diagonals shown in the table above. The green diagonal sum is D(5)=5 (since its extra initial zero is in row 5) and the blue diagonal sum is D(6) which is 8. Our red diagonal is D(7) = 13 = D(6)+D(5).

We also have shown that this is always true: one diagonals sum id the sum of the previous two diagonal sums, or, in terms of our D series of numbers:

$$D(i) = D(i-1) + D(i-2)$$

But...

$$D(0) = 1$$
$$D(1) = 1$$
$$D(i) = D(i-1) + D(i-2)$$

is exactly the definition of the Fibonacci numbers! So D(i) is just F(i) and

the sums of the diagonals in Pascal's Triangle are the Fibonacci numbers!

Another arrangement of Pascal's Triangle

By drawing Pascal's Triangle with all the rows moved over by 1 place, we have a clearer arrangement which shows the Fibonacci numbers as sums of columns:

```
    0  1  2  3  4  5  6  7  8  9
 0 1  .  .  .  .  .  .  .  .  .
 1 .  1  1  .  .  .  .  .  .  .
 2 .  .  1  2  1  .  .  .  .  .
 3 .  .  .  1  3  3  1  .  .  .
 4 .  .  .  .  1  4  6  4  1  .
 5 .  .  .  .  .  1  5 10 10  5
 6 .  .  .  .  .  .  1  6 15 20
 7 .  .  .  .  .  .  .  1  7 21
 8 .  .  .  .  .  .  .  .  1  8
 9 .  .  .  .  .  .  .  .  .  1
   1  1  2  3  5  8 13 21 34 55  ... <- sums of columns
```

This table can be explained by referring to one of the (Easier) Fibonacci Puzzles - the one about Fibonacci for a Change. It asks how many ways you can pay n pence (in the UK) using only 1 pence and 2 pence coins. The order of the coins matters, so that 1p+2p will pay for a 3p item and 2p+1p is counted as a different answer. [We now have a new **two pound coin** that is increasing in circulation too!]
Here are the answers for paying up to 5p using only 1p and 2p coins:

1p	**2p**	**3p**	**4p**	**5p**
1p	2p 1p+1p	1p+2p 2p+1p 1p+1p+1p	2p+2p 1p+1p+2p 1p+2p+1p 2p+1p+1p 1p+1p+1p+1p	1p+2p+2p 2p+1p+2p 2p+2p+1p 1p+1p+1p+2p 1p+1p+2p+1p 1p+2p+1p+1p 2p+1p+1p+1p 1p+1p+1p+1p+1p
1 way	2 ways	3 ways	5 ways	8 ways

Let's look at this another way - arranging our answers according to **the number of 1p and 2p coins we use**. Columns will represent all the ways of paying the amount at the head of the column, as before, but now the rows represent **the number of coins in the solutions**:

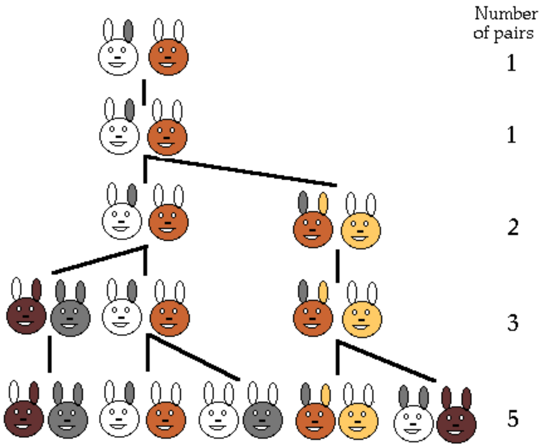
cost:	**1p**	**2p**	**3p**	**4p**	**5p**
1 coin:	1p	2p			
2 coins:		1p+1p	1p+2p 2p+1p	2p+2p	
3 coins:			1p+1p+1p	1p+1p+2p 1p+2p+1p 2p+1p+1p	1p+2p+2p 2p+1p+2p 2p+2p+1p
4 coins:				1p+1p+1p+1p	2p+1p+1p+1p 1p+1p+1p+2p 1p+1p+2p+1p 1p+2p+1p+1p
5p:					1p+1p+1p+1p+1p

If you count the number of solutions in each box, it will be exactly the form of Pascal's triangle that we showed above!

Fibonacci's Rabbit Generations and Pascal's Triangle

Here's another explanation of how the Pascal triangle numbers sum to give the Fibonacci numbers, this time explained in terms of our original rabbit problem.

Let's return to Fibonacci's rabbit problem and look at it another way. We shall be returning to it several more times yet in these pages - and each time we will discover something different!



We shall make a family tree of the rabbits but this time we shall be interested only in the **females and ignore any males** in the population. If you like, in the diagram of the rabbit pairs shown here, assume that the rabbit on the left of each pair is male (say) and so the other is female. Now ignore the rabbit on the left in each pair!
We will assume that **each mating produces exactly one female** and perhaps some males too but we only show the females in the diagram on the left. Also in the diagram on the left we see that each individual rabbit appears several times. For instance, the original brown female was mated with a while male and, since they never die, they both appear once on every line.
Now, in our new family tree diagram, **each female rabbit will appear *only once***. As more rabbits are born, so the Family tree grows adding a new entry for each newly born female.

As in an ordinary human family tree, we shall show parents above a line of all their children.
Here is a fictitious human family tree with the names of the relatives shown for a person marked as **ME**:

```
       Grandpa Grandma Grandma  Grandpa
         Abel===Mabel  Freda=====Fred
             |               |
             |               |     Aunty    Aunt   Uncle
 Uncle Bob---Dad============Mum----Jane-----Hayley=Clement
                 |                                |
   sister-in-law |brother     sister              |
            Joan===John---ME---Jean        Cousin--Cousin
               |                           Sonny     Gale===Gustof
      nephew Dan--niece Pam
```

The diagram shows that:
Grandpa Abel and Grandma Mabel are the parents of my Dad and
Grandma Freda and Grandpa Fred are the parents of my Mum.
Bob is my Dad's brother and
my Mum has two sisters, my aunts Hayley and Jane.
Aunt Hayley became Hayley Weather when she married Clement Weather.
They have two children, my cousins Sonny Weather and Gale Weather.
Gale married Gustof Wind and so is now Gale Wind.
My brother John and his wife Joan have two children,
my nephew Dan and my niece Pam.

In this family tree of human relationships, the === joins people who are parents or signifies a marriage.
In our rabbit's family tree, rabbits don't marry of course, so we just have the vertical and horizontal lines:

The vertical line |
points from a mother (above) to the oldest daughter (below);
the horizontal line -
is drawn between sisters from the oldest on the left down to the youngest on the right;
the small letter r
represents a young female (a little rabbit) and
the large letter R
shows a mature female (a big Rabbit) who can and does mate every month, producing one new daughter each time.

As in Fibonacci's original problem (in its variant form that makes it a bit more realistic) we assume none die and that each month every mature female rabbit always produces a babies of which exactly one is a female. Here is the Rabbit Family tree as if grows month by month for the first 8 months:

```
    M o n t h
1   2   3   4     5       6          7                8
r   R   R   R     R       R          R                R
        |   |     |       |          |                |
        r   R_r   R_R_r   R___R_R_r  R_____R___R_R_r  R_________R_____R___R_R_r
                  |       |   |      |     |   |      |         |     |   |
                  r       R_r r      R_R_r R_r r      R___R_R_r R_R_r R_r r
                                     |                |   |     |
                                     r                R_r r     r
```

So in **month 2**, our young female (r of month 1) becomes mature (R) and mates.
In **month 3**, she becomes a parent for the first time and produces her first daughter, shown on a line below - a new generation.
In **month 4**, the female born in month 3 becomes mature (R) and also her mother produces another daughter (r).
In **month 5**, the original female produces another female child added to the end of the line of the generation of her daughters, while the daughter born the previous month (the second in the line) becomes mature. Also the first daughter produces her own first daughter, so in month 5 the original female becomes a grand-mother and we have started a third line - the third generation.
The Family tree is shown for the first 8 months as more females are added to it. We can see that our original female becomes a great-grandmother in month 7 when a fourth line is added to the Family tree diagram - a fourth generation!

Have you spotted the Pascal's triangle numbers in the Rabbit's Family Tree?

The numbers of rabbits in each generation, that is, along each level (line) of the tree, are the Pascal's triangle numbers that add up to give each Fibonacci number - the total number of (female) rabbits in the Tree. In month n there are a total of F(n) rabbits, a number made up from the entry in row (n-k) and column (k-1) of Pascal's triangle for each of the levels (generations) k from 1 to n. In other words, we are looking at this formula and explaining it in terms of generations, the original rabbit forming generation 1 and her daughters being generation 2 and so on:

$$\mathrm{Fib}(n)=\sum_{k=1}^{n}\binom{n-k}{k-1}$$

Remember that the rows and columns of Pascal's triangle in this formula begin at 0!
For example, in month 8, there are 4 levels and the number on each level is:

```
  M o n t h  8:
Level 1:  1 rabbit  which is Pascal's triangle row 7=8-1 and column 0=1-1
Level 2:  6 rabbits which is Pascal's triangle row 6=8-2 and column 1=2-1
Level 3: 10 rabbits which is Pascal's triangle row 5=8-3 and column 2=3-1
Level 4:  4 rabbit  which is Pascal's triangle row 4=8-4 and column 3=4-1
```

When k is bigger than 4, the column number exceeds the row number in Pascal's Triangle and all those entries are 0.

```
                                         SUM is F(8)=21
col   :  0  1  2  3  4  5  6  7  8 /9 ...
------+-------------------------/-----
    0 |  1  0  0  0  0  0  0  0  0...
r   1 |  1  1  0  0  0  0  0  0  0...
o   2 |  1  2  1  0  0  0  0  0  0...
w   3 |  1  3  3  1  0  0  0  0  0...
    4 |  1  4  6  4  1  0  0  0  0...
    5 |  1  5 10 10  5  1  0  0  0...
    6 |  1  6 15 20 15  6  1  0  0...
    7 |  1  7 21 35 35 21  7  1  0...
    8 |  1  8 28 56 70 56 28  8  1...
   ...   ...
```

The general pattern for month n and level (generation) k is
Level k: is Pascal's triangle row n-k and column k-1 For month n we sum all the generations as k goes from 1 to n (but half of these will be zeros).

Things to do

- Make a diagram of your own family tree. How far back can you go? You will probably have to ask your relatives to fill in the parts of the tree that you don't know, so take your tree with you on family visits and keep extending it as you learn about your ancestors!
- Start again and draw the Female Rabbit Family tree, extending it month by month. Don't distinguish between r and R on the tree, but draw the newly born rabbits *using a new colour for each month* or, instead of using lots of colours, you could just put a number by each rabbit showing in which month it was born.
- If you tossed a coin 10 times, how many possible sequences of Heads and Tails could there be in total (use Pascal's Triangle extending it to the row numbered 10)?
  In how many of these are there 5 heads (and so 5 tails)? What is the probability of tossing 10 coins and getting exactly 5 heads therefore - it is *not* 0·5! Draw up a table for each *even* number of coins from 2 to 10 and show the probability of getting exactly half heads and half tails for each case. What is happening to the probability as the number of coins gets larger?
- Draw a *histogram* of the 10^{th} row of Pascal's triangle, that is, a bar chart, where each column on the row numbered 10 is hown as a bar whose height is the Pascal's triangle number. Try it again for tow 20 if you can (or use a Spreadsheet on your computer). The shape that you get as the row increases is called **a Bell curve** since it looks like a bell cut in half. It has many uses in *Statistics* and is a very important shape.
- Make a **Galton Quincunx**.
  This is a device with lots of nails put in a regular hexagon arrangement. Its name derives from the Latin word *quincunx* for the X-like shape of the spots on the 5-face of a dice:

```
 \  ooo  /  Galton's Quincunx               Quincunx:
```

```
              \ooooo/                                      o   o
               \ooo/  funnel to direct the balls             o
                \o/   directly on to                       o   o
               / . \  the topmost nail
              / . . \
             / . . . \
            / . . . . \       rows
           / . . . . . \       of
          / . . . . . . \     nails
         / . . . . . . . \
        / . . . . . . . . \
        | | | | | | | | | |
        | | | | |o| | | | |  Containers to collect the
        | | | |o|o|o|o| | |  balls as they fall through
        |o| |o|o|o|o|o|o| |
        -------------------
```

The whole board is tilted forward slightly so that the top is raised off the table a little. When small balls are poured onto the network of nails at the top, they fall through, bouncing either to the right or to the left and so hit another nail on the row below. Eventually they fall off the bottom row of nails and are caught in containers.

If you have a lot of nails and a lot of little balls (good sources for these are small steel ball-bearings from a bicycle shop or ping-pong balls for a large version or even dried peas or other cheap round seeds from the supermarket) then they end up forming a shape in the containers that is very much like the Bell curve of the previous exploration.
You will need to space the nails so they are as far apart as about one and a half times the width of the balls you are using.

Programming the Quincunx:

> You could try simulating this experiment on a computer using its random number generator to decide on which side of a nail the ball bounces. If your "random" function generates numbers between 0 and 1 then, if such a value is between 0 and 0.5 the ball goes to the left and if above 0.5 then it bounces to the right. Do this several times for each ball to simulate several bounces.

Thinks.com have a *great* Java version of the Quincunx, called **Ball Drop** which illustrates what your Quincunx will do.

- Let's see how the curve of the last two explorations, **the Bell curve might actually occur in some real data sets.**
  Measure the height of each person in your class and plot a graph similar to the containers above, labelled with heights to the nearest centimetre, each container containing one ball for each person with that height. What shape do you get? Try adding in the results from other classes to get one big graph.
  This makes a good practical demonstration for a Science Fair or Parents' Exhibition or Open Day at your school or college. Measure the height of each

person who passes your display and "add a ball" to the container which represents their height. What shape do you get at the end of the day?

- **What else could you measure?**
  - The weight of each person to the nearest pound or nearest 500 grams;
  - their age last birthday;
    but remember some people do not like disclosing their age or knowing too accurately their own weight!
  - house or apartment number (what range of values should you allow for? In the USA this might be up to several thousands!)
  - the last 3 digits of their telephone number;

  or try these data sets using coins and dice:
  - the total number when you add the spots after throwing 5 dice at once;
  - the number of heads when you toss 20 coins at once.

  Do **all** of these give the Bell curve for large samples?
  If not, why do you think some do and some don't?
  Can you decide beforehand which will give the Bell curve and which won't? If a distribution is not a Bell curve, what shape do you think it will be? How can mathematics help?
- Write out the first few powers of 11. Do they remind you of Pascal's triangle? Why? Why does the Pascal's triangle pattern break down after the first few powers?
  (Hint: consider $(a+b)^m$ where a=10 and b=1).
- To finish, let's return to a **human family tree.** Suppose that the probability of each child being male is exactly 0.5. So half of all new babies will be male and half the time female. If a couple have 2 children, what are the four possible sequences of children they can have? What is it if they have 3 children? In what proportion of the couples that have 3 children will all 3 children be girls? Suppose a couple have 4 children, will is the probability now that all 4 will be girls?

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

The Fibonacci Series as a Decimal Fraction

Have a look at this decimal fraction:

0·0112359550561...

It looks like it begins with the Fibonacci numbers, 0, 1, 1, 2, 3 and 5 and indeed it does if we express it as:

```
0·0             +
   1            +
    1           +
     2
      3
       5
        8
        13
         21
```

```
                 34
                  55
                   89
                   144
                     ...
          --------------------
          0·011235955056179...
```

What is the value of this decimal fraction?

It can be expressed as

```
       0/10 + 1/100 + 1/1000 + 2/10^4 + 3/10^5 + ...
  or, using powers of 10 and replacing the Fibonacci numbers by F(i):
     F(0)/10^1 + F(1)/10^2 + F(2)/10^3 + ... + F(n-1)/10^n + ...
  or, if we use the negative powers of 10 to indicate the decimal fractions:
     F(0)10^-1 + F(1)10^-2 + F(2)10^-3 + ... + F(n-1)10^-n + ...
```

To find the value of the decimal fraction we look at a generalization, replacing 10 by **x**.

Let P(x) be the polynomial in x whose coefficients are the Fibonacci numbers:

```
    P(x)=  0   +   1 x^2 +  1  x^3 + 2x^4 + 3x^5 + 5x^6 + ...
 or P(x)=F(0)x + F(1)x^2 + F(2)x^3 +   ...               ...+F(n-1)x^n + ...
```

To avoid confusion between the variable x and the multiplication sign x, we will represent multiplication by *: The decimal fraction 0.011235955... above is just

```
0*(1/10) + 1*(1/10)^2 + 1*(1/10)^3 + 2*(1/10)^4 + 3*(1/10)^5 + ... + F(n-1)*(1/10)^n + ...
```

which is just P(x) with x taking the value (1/10), which we write as P(1/10).

Now here is the interesting part of the technique!
We now write down xP(x) and x^2P(x) because these will "move the Fibonacci coefficients along":

```
    P(x)=F(0)x + F(1)x^2 + F(2)x^3 + F(3)x^4 + ... +F(n-1)x^n + ...
   xP(x)=F(0)x^2 + F(1)x^3 + F(2)x^4 + ... +F(n-2)x^n + ...
  x^2P(x)=F(0)x^3 + F(1)x^4 + ... +F(n-3)x^n + ...
```

We can align these terms up so that all the same powers of x are in the same column (as we would do when doing ordinary decimal arithmetic on numbers) as follows:

```
   P(x)=F(0)x + F(1)x^2 + F(2)x^3 + F(3)x^4 + ... +F(n-1)x^n + ...
  xP(x)=        F(0)x^2 + F(1)x^3 + F(2)x^4 + ... +F(n-2)x^n + ...
 x^2P(x)=                 F(0)x^3 + F(1)x^4 + ... +F(n-3)x^n + ...
```

We have done this so that each Fibonacci number in P(x) is aligned with the two previous Fibonacci numbers. Since the sum of the two previous numbers always equals the next in the Fibonacci series, then, when we take them away, the result will be zero - the terms will vanish!
So, if we take away the last two expressions (for xP(x) and x^2P(x)) from the first equation for P(x), the right-hand side will simplify since all but the first few terms vanish, as shown here:

```
         P(x)=F(0)x + F(1)x^2 + F(2)x^3 + F(3)x^4 + ... +F(n-1)x^n + ...
        xP(x)=        F(0)x^2 + F(1)x^3 + F(2)x^4 + ... +F(n-2)x^n + ...
      x^2P(x)=                  F(0)x^3 + F(1)x^4 + ... +F(n-3)x^n + ...
 (1-x-x^2)P(x)=F(0)x +(F(1)-F(0))x^2 + (F(2)-F(1)-F(0))x^3+...
```

Apart from the first two terms, the general term, which is just the coefficient of x^n, becomes F(n)-F(n-1)-F(n-2) and, since F(n)=F(n-1)+F(n-2) all but the first two terms become zero which is why we wrote down xP(x) and x^2P(x):
$(1-x-x^2)P(x) = x^2$

$$P(x) = \frac{x^2}{1-x-x^2} = \frac{1}{x^{-2}-x^{-1}-1}$$

So now our fraction is just P(1/10), and the right hand side tells us its exact value:
1 / (100-10-1) = **1/89** = 0·0112358...
From our expression for P(x) we can also deduce the following:

```
            10/89 = 0·112359550561...
```

If x=1/100, we have

```
      P(1/100) = 0·00 01 01 02 03 05 08 13 21 34 55 ... = 1/(10000-100-1) = 1/9899
  and
         100/9899 = 0·010102030508132134 55...
```

and so on.

Things to do

Can you find exact fractions for the following where all continue with the Fibonacci series terms?

- 10102.0305081321...
- 0.0010010020030050080 13...
- 1.001002003005008013...
- 0.001002003005008013...
- 0.00010001000200030005000800 13...
- Expand these fractions and say how they are related to the Fibonacci numbers:
  $\frac{10}{89}$

  $\frac{10}{71}, \frac{90}{71}$

  $\frac{2}{\quad}, \frac{999}{\quad}, \frac{1001}{\quad}$

995999 995999 995999

References

The Decimal Expansion of 1/89 and related Results, *Fibonacci Quarterly*, Vol 19, (1981), pages 53-55.
Calvin Long solves the general problem for *all* Fibonacci-type sequences i.e. G(0)=c, G(1)=d are the two starting terms and G(i) = a G(i-1) + b G(i-1) defines all other values for integers a and b. For our "ordinary" Fibonacci sequence, a=b=1 and c=d=1. He gives the exact fractions for any base B (here B=10 for decimal fractions) and gives conditions when the fraction exists (i.e. when the series does not get too large too quickly so that we do have a genuine fraction).

A Complete Characterization of the Decimal Fractions that can be Represented as SUM($10^{-k(i+1)}$F(ai)) where F(ai) is the ai^{th} Fibonacci number Richard H Hudson and C F Winans, *Fibonacci Quarterly*, 1981, Vol 19, pp 414 - 421.
This article examines all the decimal fractions where the terms are F(a), F(2a), F(3a) taken k digits at a time in the decimal fraction.

A Primer For the Fibonacci Numbers: Part VI, V E Hoggatt Jr, D A Lind in *Fibonacci Quarterly*, vol 5 (1967) pages 445 - 460
is a nice introduction to Generating Functions (a polynomial in x where the coefficients of the powers of x are the members of a particular series). It is readable and not too technical. There is also a list of formulae for all kinds of generating functions, which, if we substitute a power of 10 for x, will give a large collection of fractions whose decimal expansion is , for example:

- the Lucas Numbers (see this page at this site) e.g. 1999/998999
- the squares of the Fibonacci numbers e.g. 999000/997998001
- the product of two neighbouring Fibonacci numbers e.g. 1000/997998001
- the cubes of the Fibonacci numbers e.g. 997999000/996994003001
- the product of three neighbouring Fibonacci numbers e.g. 2000000000/996994003001
- every k^{th} Fibonacci number e.g. 1000/997001 or 999000/997001
- etc

Scott's Fibonacci Scrapbook, Allan Scott in *Fibonacci Quarterly* vol 6 number 2, (April 1968), page 176
is a follow-up article to the one above, extending the generating functions to Lucas cubes and Fibonacci fourth and fifth powers.
Note there are ***several corrections to these equations*** on page 70 of vol 6 number 3 (June 1968).

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

A Fibonacci Number Trick

Here is a little trick you can perform on friends which seems to show that you have amazing mathematical powers. We explain how it works after showing you the trick.
Here is Alice performing the trick on Bill:

Alice: Choose any two numbers you like, Bill, but not too big as you're going to have to do some adding yourself. Write them as if you are going to add them up and I'll, of course, be looking the other way!
Bill: OK, I've done that.

Bill chooses 16 and 21 and writes them one under the other:
16
21

Alice: Now add the first to the second and write the sum underneath to make the third entry in the column.
Bill: I don't think I'll need my calculator just yet.... Ok, I've done that.

Bill writes down 37 (=16+21) under the other two:
16
21

37

Alice: Right, now add up the second and your new number and again write their sum underneath. Keep on doing this, adding the number you have just written to the number before it and putting the new sum underneath. Stop when you have 10 numbers written down and draw a line under the tenth.

There is a sound of lots of buttons being tapped on Bill's calculator!

Bill: OK, the ten numbers are ready.

Bills column now looks like this:

```
  16
  21
  37
  58
  95
 153
 248
 401
 649
1050
----
```

Alice: Now I'll turn round and look at your numbers and write the sum of all ten numbers straight away!

She turns round and almost immediately writes underneath: 2728.

Bill taps away again on his calculator and is amazed that Alice got it right in so short a time [gasp!]

So how did Alice do it?

The sum of all ten numbers is just **eleven times the fourth number from the bottom**. Also, Alice knows the quick method of multiplying a number by eleven. The fourth number from the bottom is 248, and there is the quick and easy method of multiplying numbers by 11 that you can easily do in your head:

Starting at the right, just copy the last digit of the number as the last digit of your product. Here the last digit of 248 is 8 so the product also ends with 8 which Alice writes down:	... **248** 401 649 1050 ---- 8
Now, continuing in 248, keep adding up from the right each number and its neighbour, in pairs, writing down their sum as you go. If ever you get a sum bigger than 10, then write down the units digit of the sum and remember to carry anything over into your next pair to add. Here the pairs of 248 are (from the right) 4+8 and then 2+4. So, next to the 8 Alice thinks "4+8=12" so she writes 2 and *remembers there is an extra one to add on to the next pair*:	... **248** 401 649 1050 ---- 28
Then 2+4 is 6, **adding the one carried** makes 7, so she writes 7 on the left of those digits already written down:	... **248** 401 649 1050 ---- 728

Finally copy down the left hand digit (plus any carry). Alice sees that the left digit is 2 which, because there is nothing being carried from the previous pair, becomes the left-hand digit of the sum. The final sum is therefore 2728 = 11 x 248 .	... **248** 401 649 1050 ---- 2728

Why does it work?

You can see how it works using algebra and by starting with A and B as the two numbers that Bill chooses.
What does he write next? Just A+B in algebraic form.
The next sum is B added to A+B which is A+2B.
The other numbers in the column are 2A+3B, 3A+5B, ... up to 21A+34B.

```
  A
       B
  A +  B
  A + 2B
 2A + 3B
 3A + 5B
 5A + 8B
 8A +13B
13A +21B
21A +34B
--------
55A +88B
```

If you add these up you find the total sum of all ten is 55A+88B.
Now look at the fourth number up from the bottom. What is it?
How is it related to the final sum of 55A+88B?

So the trick works by a special property of adding up exactly ten numbers from a Fibonacci-like sequence and will work for any two starting values A and B!

Perhaps you noticed that the multiples of A and B were the Fibonacci numbers? This is part of a more general pattern which is the first investigation of several to spot new patterns in the Fibonacci sequence in the next section.

Another Number Pattern

Dave Wood has found another number pattern that we can prove using the same method.

He notices that

```
f(10)-f(5)  is   55 -   5 which is  50  or   5 tens and  0;

f(11)-f(6)  is   89 -   8 which is  81  or   8 tens and  1;

f(12)-f(7)  is  144 -  13 which is  131 or  13 tens and  1.
```

It looks like the differences seem to be 'copying' the Fibonacci series in the tens and in the units columns.
If we continue the investigation we have:

```
f(13)-f(8)  is  233 -  21 which is  212 or  21 tens and  2;

f(14)-f(9)  is  377 -  34 which is  343 or  34 tens and  3;

f(15)-f(10) is  610 -  55 which is  555 or  55 tens and  5;

f(16)-f(11) is  987 -  89 which is  898 or  89 tens and  8;

f(17)-f(12) is 1597 - 144 which is 1453 or 144 tens and 13;
```

From this point on, we have to borrow a ten in order to make the 'units' have the 2 digits needed for the next Fibonacci number. Later we shall have to 'borrow' more, but the pattern still seems to hold.

In words we have:

Any Fibonacci number when we take away the Fibonacci number 5 before it is ten times that number we took away PLUS the Fibonacci number ten before it

In mathematical terms, we can write this as:

```
Fib(n) - Fib(n-5) = 10 Fib(n-5) + Fib(n-10)
```

A Proof
That the pattern always holds is found by extending the table we used in the **Why does it work** section of the Number Trick above:

```
  A
       B
  A +  B
  A + 2B
 2A + 3B
 3A + 5B
 5A + 8B
 8A +13B
13A +21B
21A +34B
34A +55B
```

We can always write *any* Fibonacci number Fib(n) as 34A+55B because, since the Fibonacci series extends backwards infinitely far, we just pick A and B as the two numbers that are 10 and 9 places before the one we want.

Now let's look at that last line: 34A +55B.
It is almost 11 times the number 5 rows before it:

```
11 x (3A+5B) = 33A+55B,
```

and it is equal to it if we add on an extra A, i.e. the number ten rows before the last one:

```
34A + 55B = 11 (3A+5B) + A
```

Putting this in terms of the Fibonacci numbers, where the 34A+55B is F(n) and 3A+5B is "the Fibonacci number 5 before it", or Fib(n-5) and A is "the Fibonacci number 10 before it" or Fib(n-10), we have:

```
      34A + 55B = 11 (3A+5B) +  A
or
      Fib(n)    = 11 Fib(n-5) + Fib(n-10)
```

We rearrange this now by taking Fib(n-5) from both sides and we have:

```
      Fib(n) - Fib(n-5) = 10 Fib(n-5) + Fib(n-10)
```

which is just what Dave Wood observed!

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

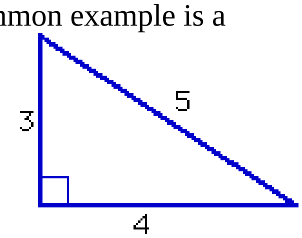
Fibonacci Numbers and Pythagorean Triangles

A **Pythagorean Triangle** is a right-angled triangle with sides which are whole numbers.

In any right-angled triangle with sides s and t and longest side (hypotenuse) h, **the Pythagoras Theorem** applies:

$$s^2 + t^2 = h^2$$

However, for a Pythagorean triangle, we also want the sides to be integers (whole numbers) too. A common example is a triangle with sides s=3, t=4 and h=5:



We can check Pythagoras theorem as follows:

$s^2 + t^2$
$= 3^2 + 4^2$
$= 9 + 16$
$= 25 = 5^2 = h^2$

Here is a list of some of the smaller Pythagorean Triangles:

s	t	h	*=primitive
3	4	5	*
6	8	10	2x(3,4,5)
5	12	13	*
9	12	15	3x(3,4,5)
8	15	17	*
12	16	20	4x(3,4,5)
7	24	25	*
15	20	25	5x(3,4,5)

10	24	26	2x(5,12,13)
20	21	29	*
6	30	34	2x(8,15,17)
18	24	36	6x(3,4,5)

Here is another longer list of Triples generated using Autograph from Oundle School, Peterborough, UK.

You will see that some are just magnifications of smaller ones where all the sides have been doubled, or trebled for example. The others are "new" and are usually called **primitive Pythagorean triangles**.
Any Pythagorean triangle is either primitive or a multiple of a primitive and this is shown in the table above. Primitive Pythagorean triangles are a bit like *prime numbers* in that every integer is either prime or a multiple of a prime.

Using the Fibonacci Numbers to make Pythagorean Triangles

There is an easy way to generate Pythagorean triangles using 4 Fibonacci numbers. Take, for example, the 4 Fibonacci numbers:

1, 2, 3, 5

Let's call the first two a and b. Since they are from the Fibonacci series, the next is the sum of the previous two: a+b and the following one is b+(a+b) or a+2b:-

a	b	a+b	a+2b
1	2	3	5

You can now make a Pythagorean triangle as follows:

1. Multiply the two middle or inner numbers (here 2 and 3 giving 6);
2. Double the result (here twice 6 gives **12**). This is one side, s, of the Pythagorean Triangle.
3. Multiply together the two outer numbers (here 1 and 5 giving **5**). This is the second side, t, of the Pythagorean triangle.
4. The third side, the longest, is found by adding together the *squares* of the inner two numbers (here 2^2=4 and 3^2=9 and their sum is 4+9=**13**). This is the third side, h, of the Pythagorean triangle.

We have generated the 12, 5,13 Pythagorean triangle, or, putting the sides in order, the **5, 12, 13** triangle this time.

> Try it with 2, 3, 5 and 8 and check that you get the Pythagorean triangle: 30, 16, 34.
> Is this one primitive?

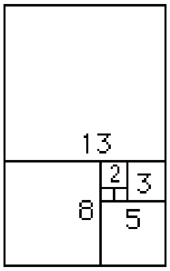
In fact, this process works for **any two numbers a and b**, not just Fibonacci numbers. The third and fourth numbers are found using **the Fibonacci rule**: add the latest two values to get the next.
Four such numbers are part of a *generalised Fibonacci series* which we could continue for as long as we liked, just as we did for the (real) Fibonacci series.

All Pythagorean triangles can be generated in this way by choosing suitable starting numbers a and b!

Connections in Mathematics: An Introduction to Fibonacci via Pythagoras E A Marchisotto in *Fibonacci Quarterly*, vol 31, 1993, pages 21 - 27.
This article explores many ways of introducing the Fibonacci numbers in class starting from the Pythagorean triples, with an extensive Appendix of references useful for the teacher and comparing different approaches. Highly recommended!
Pythagorean Triangles from the Fibonacci Series C W Raine in *Scripta Mathematica* vol 14 (1948) page 164.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Maths from the Fibonacci Spiral diagram



Let's look again at the Fibonacci squares and spiral that we saw in the Fibonacci Spiral section of the Fibonacci in Nature page.

Wherever we stop, we will always get a rectangle, since the next square to add is determined by the longest edge on the current rectangle. Also, those longest edges are just the sum of the latest two sides-of-squares to be added. Since we start with squares of sides 1 and 1, this tells us why the squares sides are the Fibonacci numbers (the next is the sum of the previous 2).

Also, we see that each rectangle is a jigsaw puzzle made up of all the earlier squares to form a rectangle. All the squares and all the rectangles have sides which are Fibonacci numbers in length. What is the mathematical relationship that is shown by this pattern of squares and rectangles? We express each rectangle's area as a sum of its component square areas:

The diagram shows that

$$1^2+1^2+2^2+3^2+5^2+8^2+13^2=13\times21$$

and also, the smaller rectangles show:

$$1^2+1^2=1\times2$$
$$1^2+1^2+2^2=2\times3$$
$$1^2+1^2+2^2+3^2=3\times5$$
$$1^2+1^2+2^2+3^2+5^2=5\times8$$
$$1^2+1^2+2^2+3^2+5^2+8^2=8\times13$$

This picture actually is a convincing proof that the pattern will work for any number of squares of Fibonacci numbers that we wish to sum. They always total to the largest Fibonacci number used in the squares multiplied by the next Fibonacci number. That is a bit of a mouthful to say - and to understand - so it is better to express the relationship in the language of mathematics:

$$1^2 + 1^2 + 2^2 + 3^2 + ... + F(n)^2 = F(n)F(n+1)$$

and it is true for ANY n from 1 upwards.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

..and now it's your turn!

Things to do

Here are some series that use the Fibonacci numbers. Compute a few terms and see if you can spot the pattern, ie guess the formula for the general term and write it down mathematically:

- F(1), F(1)+F(2), F(1)+F(2)+F(3), ... = 1, 2, 4, 7, 12, 20, ...

  Keun Young Lee, a student at the Glenbrook North High School in Chicago, told me of a generalization of this. Can you spot it too?

  What is F(k)+F(k+1)+...+F(n)?

  eg 5+8+13 (k=5 and n=7) is 26

3+5+8+13+21 (k=4 and n=8) is 50.
This problem will be the same as the first problem here if you let k=1 and this is a useful check on your formula.

- F(1), F(1)+F(3), F(1)+F(3)+F(5), ... = 1, 3, 8, 21, ...
- F(2), F(2)+F(4), F(2)+F(4)+F(6), ... = 1, 4, 12, 33, ...
- F(1)+F(4), F(2)+F(5), F(3)+F(6), ... = 4, 6, 10, 16, ...
- F(1)+F(5), F(2)+F(6), F(3)+F(7), ... = 6, 9, 15, 24, ...
- $F(1)^2+F(2)^2$, $F(2)^2+F(3)^2$, $F(3)^2+F(4)^2$, ...= 2, 5, 13, 34, ...
- Can you find a connection between the terms of:
  1x3, 2x5, 3x8, 5x13, ... , F(n-1)xF(n+1), ...
  and the following series?
  2x2, 3x3, 5x5, 8x8, ... , F(n)xF(n), ...
  The connection was first noted by Cassini (1625-1712) in 1680 and is called **Cassini's Relation** (see Knuth, **The Art of Computer Programming**, Volume 1:*Fundamental Algorithms*, section 1.2.8).
- Try choosing different small values for a and b and finding some more Pythagorean triangles.
  Tick those triangles that are primitive and out a cross by those which are multiples (of a primitive triangle).
  Can you find the simple condition on a and b that tells us when the generated Pythagorean triangle is primitive? [Hint: the condition has two parts: i) what happens if both a and b have a common factor? ii) why are no primitive triangles generated if a and b are both odd?].
- Find all 16 primitive Pythagorean triangles with all 3 sides less than 100.
  Use your list to generate *all* Pythagorean triangles with sides smaller than 100.
  How many are there in all?

  [Optional extra part: Can you devise a method to find which a and b generated a given Pythagorean triangle?
  Eg Given Pythagorean triangle 9,40,41 (and we can check that $9^2 + 40^2 = 41^2$), how do we calculate that it was generated from the values a=1, b=4?]

If you don't know how to begin, or get stuck,
look at the Hints and Tips page to get you going!

So try them for yourself. This is where Mathematics becomes more of an Art than a Science, since you are relying on your intuition to "spot" the pattern. No one is quite sure where this ability in humans comes from. It is not easy to get a computer to do this (although Maple is quite good at it) - and it may be something specifically human that a computing machine can never really copy, but no one is sure at present. Here are two references if you want to explore further the arguments and ideas of why an electronic computer may or may not be able to mimic a human brain:

Prof Roger Penrose's book **Shadows of the Mind** published in 1994 by Oxford Press makes interesting reading on this subject.
An on-line Journal, Psyche has many articles and reviews of this book in Volume 2.

Dr. Math has some interesting replies to questions about the Fibonacci series and the Golden section together with a few more formulae for you to check out.

S. Vajda, **Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications**, Halsted Press (1989).

This is a wonderful book - now out of print - which is full of formulae on the Fibonacci numbers and Phi. Do try and find it in your local college or university library. It is well worth dipping in to if you are studying maths at age 16 or beyond!

Mathematical Mystery Tour by Mark Wahl, 1989, is full of many mathematical investigations, illustrations, diagrams, tricks, facts, notes as well as guides for teachers using the material. It is a great resource for your own investigations.

Fibonacci Home Page

The Puzzling World of the Fibonacci Numbers

WHERE TO NOW?

The first 500 Fibonacci Numbers

A Formula for Fibonacci Numbers

The next Topic is...

The Golden Section - the Number and Its Geometry

The Fibonacci numbers

Contents of this Page

.. and, if you want more ...

- Fibonacci numbers 101-300, factorised
- Fibonacci Numbers 301-500, not factorised)
- There is a complete list of all Fibonacci numbers and their factors up to the 1000-th Fibonacci and 1000-th Lucas numbers and partial results beyond that on Blair Kelly's Factorization pages

The Fibonacci series

is formed by adding the latest two numbers to get the next one, starting from 0 and 1:

```
0 1 --the series starts like this.
0+1=1 so the series is now
0 1 1
  1+1=2 so the series continues...
0 1 1 2 and the next term is
    1+2=3 so we now have
0 1 1 2 3  and it continues as follows ...
```

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ...

N E W (May 1999)
Try this Fibonacci Calculator, written in JavaScript,
if you are using Microsoft Interner Explorer 4.0 or later OR Netsacpe Navigator or Communicator version 4.0 or later.

It can find Fib(2000) exactly - all 418 digits - in about 50 seconds on an Apple Macintosh PowerBook G3 series 266MHz computer.

It can find the first few digits of even higher numbers, instantly, such as the twenty-millionth Fibonacci number, F(20,000,000) which begins **285439828...** and has over ***4 million digits***!

The (recurrence) formula for these Fibonacci numbers is:

```
F(0)=0, F(1)=1, F(n)=F(n-1)+F(n-2) for n>1.
```

and an explicit formula for F(n) just in terms of n (not previous terms) is given on a later page.

The first 100 Fibonacci numbers, completely factorised

If a number has no factors except 1 and itself, then it is called **a prime number**.
The factorizations here are produced by Maple with the command

```
with(combinat);
seq(lprint(n,`:`,fibonacci(n),`=`,ifactor(fibonacci(n))),n=1..100);
```

and then reformatted slightly:

The first 100 Fibonacci numbers

n F(n)=factorization

1 : 1 = 1
2 : 1 = 1
3 : 2 = 2 Prime
4 : 3 = 3 Prime
5 : 5 = 5 Prime
6 : 8 = 2^3
7 : 13 = 13 Prime
8 : 21 = 3 x 7
9 : 34 = 2 x 17
10 : 55 = 5 x 11
11 : 89 = 89 Prime
12 : 144 = 2^4 x 3^2
13 : 233 = 233 Prime
14 : 377 = 13 x 29
15 : 610 = 2 x 5 x 61
16 : 987 = 3 x 7 x 47
17 : 1597 = 1597 Prime
18 : 2584 = 2^3 x 17 x 19
19 : 4181 = 37 x 113
20 : 6765 = 3 x 5 x 11 x 41
21 : 10946 = 2 x 13 x 421
22 : 17711 = 89 x 199
23 : 28657 = 28657 Prime
24 : 46368 = 2^5 x 3^2 x 7 x 23
25 : 75025 = 5^2 x 3001

26 : 121393 = 233 x 521
27 : 196418 = 2 x 17 x 53 x 109
28 : 317811 = 3 x 13 x 29 x 281
29 : 514229 = 514229 Prime
30 : 832040 = 2^3 x 5 x 11 x 31 x 61
31 : 1346269 = 557 x 2417
32 : 2178309 = 3 x 7 x 47 x 2207
33 : 3524578 = 2 x 89 x 19801
34 : 5702887 = 1597 x 3571
35 : 9227465 = 5 x 13 x 141961
36 : 14930352 = 2^4 x 3^3 x 17 x 19 x 107
37 : 24157817 = 73 x 149 x 2221
38 : 39088169 = 37 x 113 x 9349
39 : 63245986 = 2 x 233 x 135721
40 : 102334155 = 3 x 5 x 7 x 11 x 41 x 2161
41 : 165580141 = 59369 x 2789
42 : 267914296 = 2^3 x 13 x 29 x 211 x 421
43 : 433494437 = 433494437 Prime
44 : 701408733 = 3 x 43 x 89 x 199 x 307
45 : 1134903170 = 2 x 5 x 17 x 61 x 109441
46 : 1836311903 = 139 x 461 x 28657
47 : 2971215073 = 2971215073 Prime
48 : 4807526976 = 2^6 x 3^2 x 7 x 23 x 47 x 1103
49 : 7778742049 = 13 x 97 x 6168709
50 : 12586269025 = 5^2 x 11 x 101 x 151 x 3001
51 : 20365011074 = 2 x 1597 x 6376021
52 : 32951280099 = 3 x 233 x 521 x 90481
53 : 53316291173 = 953 x 55945741
54 : 86267571272 = 2^3 x 17 x 19 x 53 x 109 x 5779
55 : 139583862445 = 5 x 89 x 661 x 474541
56 : 225851433717 = 3 x 7^2 x 13 x 29 x 281 x 14503
57 : 365435296162 = 2 x 37 x 113 x 797 x 54833
58 : 591286729879 = 59 x 514229 x 19489
59 : 956722026041 = 353 x 2710260697
60 : 1548008755920 = 2^4 x 3^2 x 5 x 11 x 31 x 41 x 61 x 2521
61 : 2504730781961 = 555003497 x 4513
62 : 4052739537881 = 557 x 3010349 x 2417
63 : 6557470319842 = 2 x 13 x 17 x 421 x 35239681
64 : 10610209857723 = 3 x 7 x 47 x 1087 x 2207 x 4481
65 : 17167680177565 = 5 x 233 x 14736206161
66 : 27777890035288 = 2^3 x 89 x 199 x 19801 x 9901
67 : 44945570212853 = 269 x 1429913 x 116849
68 : 72723460248141 = 3 x 67 x 1597 x 63443 x 3571
69 : 117669030460994 = 2 x 137 x 829 x 18077 x 28657
70 : 190392490709135 = 5 x 11 x 13 x 29 x 71 x 911 x 141961
71 : 308061521170129 = 46165371073 x 6673
72 : 498454011879264 = 2^5 x 3^3 x 7 x 17 x 19 x 23 x 107 x 103681

73 : 806515533049393 = 86020717 x 9375829
74 : 1304969544928657 = 73 x 149 x 54018521 x 2221
75 : 2111485077978050 = 2 x 5^2 x 61 x 230686501 x 3001
76 : 3416454622906707 = 3 x 37 x 113 x 29134601 x 9349
77 : 5527939700884757 = 13 x 89 x 4832521 x 988681
78 : 8944394323791464 = 2^3 x 79 x 233 x 521 x 859 x 135721
79 : 14472334024676221 = 157 x 92180471494753
80 : 23416728348467685 = 3 x 5 x 7 x 11 x 41 x 47 x 1601 x 3041 x 2161
81 : 37889062373143906 = 2 x 17 x 53 x 109 x 4373 x 19441 x 2269
82 : 61305790721611591 = 370248451 x 59369 x 2789
83 : 99194853094755497 = 99194853094755497 Prime
84 : 160500643816367088 = 2^4 x 3^2 x 13 x 29 x 83 x 211 x 281 x 421 x 1427
85 : 259695496911122585 = 5 x 1597 x 3415914041 x 9521
86 : 420196140727489673 = 433494437 x 6709 x 144481
87 : 679891637638612258 = 2 x 173 x 3821263937 x 514229
88 : 1100087778366101931 = 3 x 7 x 43 x 89 x 199 x 263 x 307 x 881 x 967
89 : 1779979416004714189 = 1069 x 1665088321800481
90 : 2880067194370816120 = 2^3 x 5 x 11 x 17 x 19 x 31 x 61 x 181 x 541 x 109441
91 : 4660046610375530309 = 13^2 x 233 x 159607993 x 741469
92 : 7540113804746346429 = 3 x 139 x 461 x 275449 x 28657 x 4969
93 : 12200160415121876738 = 2 x 557 x 4531100550901 x 2417
94 : 19740274219868223167 = 6643838879 x 2971215073
95 : 31940434634990099905 = 5 x 37 x 113 x 761 x 67735001 x 29641
96 : 51680708854858323072 = 2^7 x 3^2 x 7 x 23 x 47 x 769 x 1103 x 3167 x 2207
97 : 83621143489848422977 = 193 x 389 x 3084989 x 361040209
98 : 135301852344706746049 = 13 x 29 x 97 x 599786069 x 6168709
99 : 218922995834555169026 = 2 x 17 x 89 x 197 x 18546805133 x 19801
100: 354224848179261915075 = 3 x 5^2 x 11 x 41 x 101 x 151 x 401 x 570601 x 3001

[There is a complete list of all Fibonacci numbers and their factors up to the 1000-th Fibonacci and 1000-th Lucas numbers and partial results beyond that on Blair Kelly's site.]

A Fibonacci Calculator

Here is aFibonacci Calculator which opens in a separate window. It calculates thousands of Fibonacci numbers exactly and millions upon millions to the first few digits!

the Fibonacci Home Page

Mathematical Magic of the Fibonacci Numbers

The Puzzling World of the Fibonacci Numbers

WHERE TO NOW??

The next topic is...
The Golden Section - the Number and Its Geometry

Fibonacci numbers 101-300

Fibonacci Numbers 301-500

A Formula for the Fibonacci numbers

 R.Knott@surrey.ac.uk *updated:* 24 January 2000

A formula for Fib(n)

Contents of this Page

The line means there is a Things to do investigation at the end of the section.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Binet's Formula for the nth Fibonacci number

We have only defined the nth Fibonacci number in terms of the two before it:

the n-th Fibonacci number is the sum of the (n-1)th and the (n-2)th.

So to calculate the 100th Fibonacci number, for instance, we need to compute all the 99 values before it first - quite a task, even with a calculator!

A natural question to ask therefore is:

Can we find a formula for F(n) which involves only n and does not need any other (earlier) Fibonacci values?

Yes! It involves our golden section number Phi and its reciprocal phi:

Here it is:

$$\text{Fib}(n) = \frac{\text{Phi}^n - (-\text{Phi})^{-n}}{\sqrt{5}} = \frac{\text{Phi}^n - (-\text{phi})^n}{\sqrt{5}}$$

where Phi = 1·61803 39887 49894 84820 45868 34365 63811 77203 09179 80576 ... .

The next version uses just one of the golden section values: Phi, and all the powers are positive:

$$\text{Fib}(n) = \frac{\text{Phi}^n - \frac{(-1)^n}{\text{Phi}^n}}{\sqrt{5}}$$

Since phi is the name we use for 1/Phi on these pages, then we can remove the fraction in the numerator here and make it simpler, giving the second form of the formula at the start of this section.

We can also write this in terms of $\sqrt{5}$ since Phi = $\frac{1+\sqrt{5}}{2}$ and –phi = $\frac{1-\sqrt{5}}{2}$:

$$\text{Fib}(n) = \frac{\Phi^n - (-\phi)^n}{\Phi + \phi} = \frac{\Phi^n - (-\phi)^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

If you prefer values in your formulae, then here is another form:-

$$\text{Fib}(n) = \frac{1.6180339..^n - (-0.6180339..)^n}{2.236067977..}$$

This is a surprising formula since it involves square roots and powers of Phi (an irrational number) but it always gives an *integer* for all (integer) values of n!

Here's how it works:

```
Let X=  Phi^n  =(1·618..)^n
and Y=(-Phi)^-n=(-1·618..)^-n=(-0·618..)^n then we have:
```

n:	X=Phi^n :	Y=(-Phi)$^{-n}$:	X-Y:	(X-Y)/sqrt(5):
0	1	1	0	**0**
1	1·618033989	-0·61803399	2·23606798	**1**
2	2·618033989	0·38196601	2·23606798	**1**
3	4·236067977	-0·23606798	4·47213595	**2**
4	6·854101966	0·14589803	6·70820393	**3**
5	11·09016994	-0·09016994	11·18033989	**5**
6	17·94427191	0·05572809	17·88854382	**8**

7	29·03444185	-0·03444185	29·06888371	**13**
8	46·97871376	0·02128624	46·95742753	**21**
9	76·01315562	-0·01315562	76·02631123	**34**
10	122·9918694	0·00813062	122·9837388	**55**
..	....	....	..	

You might want to look at two ways to prove this formula: the first way is very simple and the second is more advanced and is for those who are already familiar with matrices.

Since phi is less than one in size, its powers decrease rapidly. We can use this to derive the following simpler formula for the n-th Fibonacci number F(n):

F(n) = round(Phi^n / √5)

where the round function gives the nearest integer to its argument.

n:	Phi^n/sqrt(5)	..rounded
0	0·447213595	0
1	0·723606798	1
2	1·170820393	1
3	1·894427191	2
4	3·065247584	3
5	4·959674775	5
6	8·024922359	8
7	12·98459713	13
8	21·00951949	21
9	33·99411663	34
10	55·00363612	55
..	...	..

Notice how, as n gets larger, the value of Phi^n/√5 is almost an integer.

Things to do

1. What then is F(100) according to this formula? You may choose to write a computer program for this, or use a package (such as Mathematica or Maple) which lets you work out very long integers exactly, or you can just get an approximate value on your calculator.
2. How many digits does F(100) have? (the approximate value on your calculator should tell you). Check your answer with the list of Fibonacci numbers.
3. Look at the following line from the last Table above:

n:	Phi^n=X:	(-Phi)$^{-n}$=Y:	X-Y:	(X-Y)/√(5):
1	1·618033989	-0·61803399	2·23606798	**1**

You might nave noticed that we didn't ADD the X and Y values to get 1·618..-0·618..=1 directly but instead we subtracted and divided by sqrt(5).
Let's see what happens if we do just ADD the X and Y columns:
(a) Add a new column to the table above which is X+Y. Fill it in and you'll notice something very surprising - *another integer series that is not the Fibonacci numbers!!* These numbers are called the **Lucas Numbers** and they also have some similar properties to the Fibonacci numbers and are covered in another page at this site (see Fibonacci Home page).
(b) Can you spot the rule whereby the latest two Lucas numbers are used to generate the next Lucas number?

Historical Note - Binet's Formula or de Moivre's?

Many authors say that this formula was discovered by J. P. M. Binet (1786-1856) in 1843 and so call it **Binet's Formula**.
Don Knuth in **The Art of Computer Programming**, Volume 1 *Fundamental Algorithms*, section 1·2.8, says that A de Moivre (1667-1754) had written about this formula more than 100 years before Binet, in 1730, and had indeed found a method for finding formula for any general series of numbers formed in a similar way to the Fibonacci series. Like many results in Mathematics, it is often not the original discoverer who gets the glory of having their name attached to the result, but someone later!

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

How many digits does a number have?

Using the display on your calculator

One of the questions above asks you to use your calculator to find out how many digits are in a number. When the number gets too big for the calculator's display, it shows the first few digits and a little "exponent" which says how to move the decimal place from where it is shown to it true place - negative means move it to the left, otherwise move it to the right from where it is shown in the display.
So Phi^{20}/sqrt(5) on my calculator is 6765·000029 and Fib(20)=6765.
But Phi^{60}/sqrt(5) shows as 1·548008755 ^{12} where the little figures at end are the "exponent", that is, the true value is $1{\cdot}548008755 \times 10^{12}$. If we move the decimal point 12 places to the right (putting in 0s for the missing digits), we get:

1548008755000. and the correct value for Fib(60) is
1548008755920

> So the exponent, when positive, has told us how many digits there are in the number calculated, showing just the first few of the digits if not all of them will fit into the display window!

Similarly, phi^{60} is just $1/Phi^{60}$ which we've just calculated. Using the "1/x" button on my calculator when it is showing the value above gives: $6{\cdot}459911784^{-13}$ meaning $6{\cdot}459911784 \times 10^{-13}$. This time we must move the decimal place to the *left* since the exponent is negative and we must move it 13 places. This gives 0·00000000000064511784 as the value for phi^{60} - quite small!

Using the LOG button on your calculator

But how can we ***calculate*** the number of digits in a given whole number?
This section shows how to use the LOG button on your calculator to find out how *long* a number is.

Returning to the investigation above where you calculated F(100), this number is usually too big for most calculators to compute, but we can find how long it is as follows, using the simplified formula:

$$F(n) = round(Phi^n / \sqrt{5})$$

[This very nearly gives the correct value of F(n) since the part of the formula we have omitted is very small indeed for large n.]

The LOG button on your calculator can be used to compute how long a number is, that is, how many decimal digits it has.

- This is the "logarithm to base 10". Another button, usually labelled LN is the "logarithm to base e".
- Take the LOG of any 3-digit number and the answer should be "2 point something".
  Try with any 4-digit number and you get a LOG of "3.something". So,
  the number of digits in any integer is just 1+ the whole-part of its LOG.
- LOGs have useful properties such as:
  if we ADD LOGS we find the length of the PRODUCT of the original numbers;
  if we SUBTRACT LOGS we find the length of the QUOTIENT (DIVISION).
  So the LOG of x^2 is just 2 times LOG x and
  the LOG of x^3 = 3 LOG x and
  the LOG of $\sqrt{x}$ = (LOG x)/2 and so on.

How many digits are there in Fib(n)?

So, now you have enough information to answer the question:

How many digits has F(1000)?

Computing LOG ($Phi^{1000} / \sqrt{5}$) is the same as computing
1000*LOG(Phi) - (LOG $\sqrt{5}$) = 1000*LOG Phi - (LOG 5)/2.
So 1+the whole number part of your answer is the number of digits in F(1000).
In fact, you can find the first few digits by using the rest of the LOG answer, but I'll leave that for you to figure

out, giving you the hint that the "opposite" (the inverse) function to LOG(n) is 10^n.

> There is a PUMAS (Practical Uses of Maths and Science) page by Kim Aaron, of the Jet Propulsion Lab, entitled "Just what is a log anyway?" It shows how Kim has found many practical uses of logarithms as a working engineer.
> This page is designed for middle school students, but teachers will also find it well worth checking out too!

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Calculating the next Fibonacci number directly

Suppose we have evaluated Fib(100) and we want to know the next value: Fib(101). Do we have to use Binet's formula again? Well we could do, of course, but here is a short-cut.

There is also a formula that, given one Fibonacci number, returns the *next Fibonacci number* directly, calculating it in terms only of the previous value (ie not needing the value before as well).

If x is the value of F(n) then

$$F(n+1) = \text{floor}\left(\{x+1+\sqrt{[5 x^2]}\}/2 \right)$$

The "floor" function floor(a) means "the next integer below a or a itself if a is an integer". For positive values, it means "rub out anything after the decimal point". The name comes from the picture of a building with floors at levels 0, 1, 2, etc (say 10 metres tall) and also some below ground labelled -1, -2, -3, etc. If we now hold an object at height "a" and let go, what "floor" will it land on?

```
floor( 2·5)= 2   floor( 2)= 2     floor( 2·99)= 2     floor( 2·00001)= 2
floor(-2·5)=-3   floor(-2)=-2     floor(-2·99)=-3     floor(-2·00001)=-3
```

Here's an example of the "next Fibonacci" formula using a small value of n to check it works:

```
Since F(5)=5 then F(6)=floor(  (5+1+sqrt(5x25))/2  )
                      =floor(  (6  +sqrt( 125))/2  )
                      =floor(  (6  +  11·180  )/2  )
                      =floor(        8·59          )
                      =8
```

which is correct!

Here are two more examples.

Proving that this formula is correct

You can easily evaluate F(0) and F(1) by this formula and see that they give 0 and 1 respectively. Then, if you are familiar with **proof by induction** you can show that, supposing the formula is true for F(n-1) and F(n) then it *must* also be true for F(n+1) by showing that adding the formula's expressions for F(n) and F(n-1) gives the formula's expression for F(n+1).

Other ways of proving it involve results about **recurrence relations** and how to solve them, which are very like solving differential equations, except that they deal with integer values not real number values. This is in University level courses on Pure or Discrete Mathematics.

[For the university level mathematician, there is an interesting HAKMEM note on a fast way of computing Fibonacci numbers and its applications.]

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Binet's Formula for negative n?

Earlier on this page we looked at **Binet's formula** for the Fibonacci numbers:

$$\textbf{Fib(n)} = \{ \text{Phi}^n - (-\text{phi})^n \}/\sqrt{5}$$

Here Phi=1·6180339... and phi = 1/Phi = Phi-1 = ($\sqrt{5}$-1)/2 = 0·6180339... .

We only used this formula for **positive whole values of n** and found - surprisingly - it only gives integer results. Well perhaps it was not so surprising really since the formula is supposed to be define the Fibonacci numbers which *are* integers; but it *is* surprising in that this formula involves the square root of 5, Phi and phi which are all *irrational numbers* i.e. cannot be expressed exactly as the ratio of two whole numbers.

Suppose we try *negative* whole numbers for n in Binet's formula.
The formula extends the definition of the Fibonacci numbers F(n) to negative n.
In fact, if we try to extend the Fibonacci series backwards, still keeping to the rule that a Fibonacci number is the sum of the two numbers on its LEFT, we get the following:

```
     n :   ... -6 -5 -4 -3 -2 -1 0  1  2  3  4  5  6  ...
  Fib(n):   ... -8  5 -3  2 -1  1 0  1  1  2  3  5  8  ...
```

and this is consistent with Binet's formula for negative whole values of n.
So we can think of Fib(n) being defined an *all* integer values of n, both positive and negative and the Fibonacci series extending infinitely far in both the positive and negative directions.

Binet's formula for non-integer values of n?

This section is optional and at an advanced level i.e. post 16 years education.
Take me back to the Fibonacci Home page now
or learn about square roots of negative numbers in what follows!

Well now we've tried negative values for n, why not try fractional or other non-whole values for n?
This doesn't make sense in terms of numbers in a series (there is a 2^{nd} and a 3^{rd} term and even perhaps a -2^{nd} term but what can we possibly mean by a $2{\cdot}5^{th}$ term for instance)??
However, this *could* give us some interesting (mathematical) insights into the whole-number terms which are our familiar Fibonacci series.

Complex Numbers

The trouble is that in Binet's formula:

$$\mathbf{Fib(n) = \{ Phi^{n} - (-phi)^{n} \}/\sqrt{5}}$$

the second term $\mathbf{(-phi)^n}$ means we have to find the n-th power of a *negative* number: -phi and n is not a whole number. If n was 0·5 for instance, meaning sqrt(-phi), then we are taking the *square-root of a negative value* which is "impossible".

Mathematicians have already extended the real-number system to cover such "imaginary" numbers. They are called **complex numbers** and have two parts A and B, both normal real numbers: a **real** part, A, and an **imaginary** part, B. The imaginary part is a multiple of the basic "imaginary" quantity $\sqrt{(-1)}$, denoted **i**. So complex numbers are written as x + **i** y or x + y **i** or sometimes as x + **I** y or even more simply as (x,y).

Applications of Complex numbers

To me it is still surprising that such "imaginary" numbers - or numbers involving the imaginary quantity that is the square root of a negative number - have very practical applications in the real world.
For instance, **electrical engineers** have already found many applications for such "imaginary" or complex numbers. Whereas ***resistance*** can be described by a real number often measured in ohms, complex numbers are used for the ***inductance*** and ***capacitance***, so they have very practical uses!
Electrical engineers tend to use **j** rather than **i** when writing complex numbers.

Mathematicians find uses for complex numbers in solving equations:

- Every equation of the form Ax+B=0 has a solution which is a **fraction**: namely X=-B/A if A and B are integers. These are called *linear equations* where A and B are, in general, any real numbers.
- Every equation of the form $Ax^2 + Bx + C=0$ has either one or two solutions IF we allow complex numbers for x. (Here A is not zero or we just get a linear equation.)
  For instance $x^2=2$ has two solutions:
  +sqrt(2) and -sqrt(2)

but $x^2=0$ has just one solution namely x=0.
Note that $x^2=-2$ has two solutions too:

x=sqrt(-2)=**i**sqrt(2) and x=-sqrt(-2)=-**i**sqrt(2)

- Every equation of the form $Ax^3 + Bx^2 + Cx + D = 0$ has at most 3 solutions again allowing x to be a complex number if necessary.

This leads to a beautiful theorem about solving equations which are sums of (real number multiples of) powers of x, called **polynomials in x**:

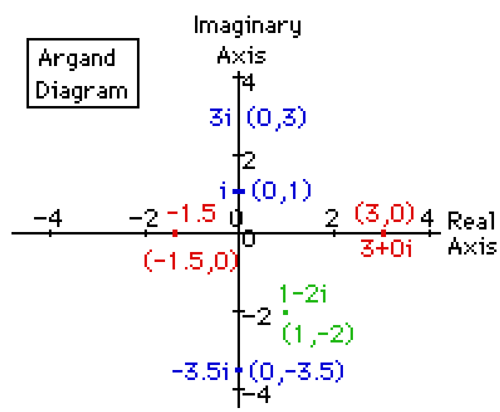
If the highest power of x in a polynomial is n then there are at most n complex number solutions which make the polynomial's value zero

Complex Numbers and Their Applications by F J Budden, Longman's 1968, is now out of print but is an excellent introduction to the fascinating subject of complex numbers and their applications.

Argand Diagrams

Writing (x,y) for a complex numbers suggests we might be able to **plot complex numbers on a graph**, the x distance being the real part of a complex number and the y height being its complex part.

Such plots are called **Argand diagrams** after J. R. Argand (1768-1822).



We can plot an individual point such as 1 - 2**i** as the point (1,-2). Numbers which are real have zero as their complex part so the real number 3 is the same as the complex number 3 + 0 **i** and has "coordinates" (3,0). The real number -1·5 is the same as -1·5 + 0 **i** or (-1·5,0).
In general, the real number *r* is the complex number *r* + 0 **i** and is plotted at (r,0) on the Argand diagram.
In fact, all the real values are already in the graph along the **x axis** also called **the real axis**.
Numbers which are *purely imaginary* have a real-part of zero and so are of the form 0+y**i** always lying exactly on the y axis (**the imaginary axis**).

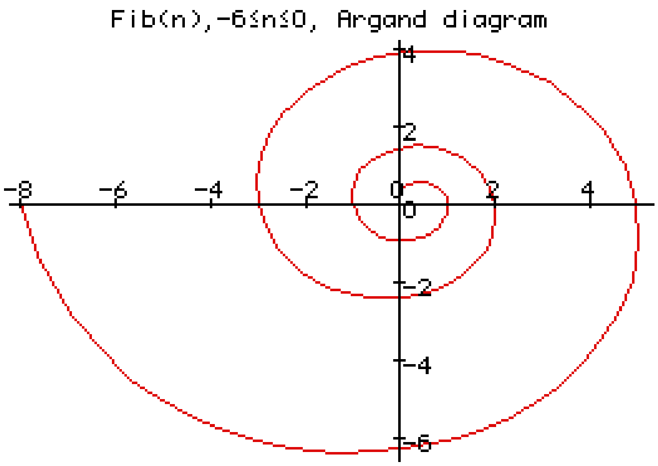
Plotting functions on an Argand Diagram

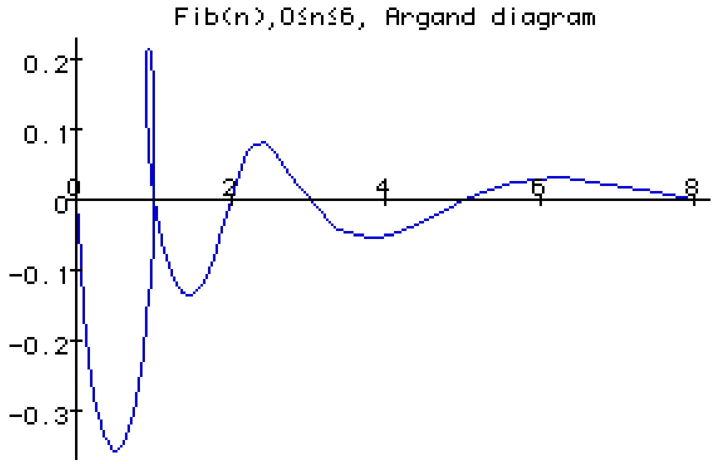
We can plot a *complex function* on an Argand diagram, that is, a function whose values are complex numbers. This is where Binet's formula comes in since it will give us complex numbers as n (now a real number) varies over the real numbers.

So what happens if we plot a graph of F(n) described by Binet's formula, plotting the results on an Argand diagram?

The blue plot is for positive values of n from 0 to 6. Note how this curve crosses the x axis (representing the "real part of the complex number") at the Fibonacci numbers, 0, 1, 2, 3, 5 and 8. But there is a loop so it crosses the axis *twice* at x=1, and we really do get the whole Fibonacci sequence 0,1,1,2,3,5,8.. as the crossing points.
The red plot is of negative values of n from -6 to 0. It also crosses the x axis at the values -8, 5, -3, 2, -1, 1 and 0

corresponding to the Fibonacci numbers F(-6), F(-5), F(-4), F(-3), F(-2), F(-1) and F(0).





Spirals?

- Note that the red spiral for negative values of n is NOT an equiangular or logarithmic spiral that we found in sea-shells on the Fibonacci in Nature page. This is because where the curve crosses the x axis at 1 and next at 2, so the distance from the origin has doubled, but the next crossing is not at 4 (which would mean another doubling as required for a logarithmic spiral) but at 5.
- If you adjust the width of your browser window, you should be able to see both curves side by side. Now it looks as if the two curves are made from the same 3-dimensional spiral spring-shape, a bit like a spiral bed-spring in cartoons, getting narrower towards one end. The red curve seems to be looking down the centre of the three-dimensional spring and the blue one looking at the same spring shape but from the side. Comparing the two diagrams shows even the heights of the loops are the same!
  I haven't yet found an explanation for this - can you find one? [Let me know if you do!]

  The plots were produced using Maple's parametric plotting provided with its built-in "plot" function:

```
Phi:=(sqrt(5)+1)/2;phi:=(sqrt(5)-1)/2;
f:=n->(Phi^n-(-phi)^n)/sqrt(5);
plot([Re(f(n)),Im(f(n)),n=-6..0],color=RED,
     title=`Fib(n),-6²n²0, Argand diagram`);
plot([Re(f(n)),Im(f(n)),n=0..6],color=BLUE,
     title=`Fib(n),0²n²6, Argand diagram`);
```

Kurt Papke has a Web page with a Java applet to show the two Argand diagrams but animating the formula that f(n)=f(n-1)+f(n-2) for *any real value n*. The complex numbers f(n), f(n-1) and f(n-2) can be illustrated as **vectors**, and so the formula f(n)=f(n-1)+f(n-2) becomes a vector equation showing that the vector f(n-1) added to (followed by) the vector f(n-2) gives the same length-and-direction-movement as the vector f(n).
Kurt has an excellent 3D version of the spiral that you can rotate on the screen (using a Java applet) AND one also for the Lucas numbers formula!

For a different complex function based on Binet's formula, see the following two articles where they both introduce the factor $\mathbf{e^{i\,n\,\phi}}$ which is 1 when n is an integer:

Argand Diagrams of Extended Fibonacci and Lucas Numbers, F J Wunderlich, D E Shaw, M J Hones *Fibonacci Quarterly*, vol 12 (1974), pages 233 - 234;

An Extension of Fibonacci's Sequence P J deBruijn, *Fibonacci Quarterly* vol 12 (1974) page 251 - 258;

References on Complex Numbers

Complex Numbers are included in some (UK based) Mathematics syllabuses at Advanced level (school/college examinations taken at about age 17). Here are some books relating to different Advanced level Examination Boards syllabus entries on Complex Numbers:

GCE A level Maths: Complex Numbers A. Nicolaides,ISBN: 1872684270, 1995.

Nuffield Advanced Mathematics: Complex Numbers and Numerical Analysis June 1994, Longman, ISBN: 0582099846.

School Maths Project 16-19: Complex Numbers Cambridge, 1992, ISBN: 0521426529.

Fibonacci Home Page

The Mathematical Magic of the Fibonacci Numbers

The Puzzling World of the Fibonacci Numbers

WHERE TO NOW???

The next Topic is...
The Golden Section - the Number and Its Geometry

Fibonacci bases and other ways of representing integers

Using the Fibonacci numbers to represent integers

Contents of This Page

The symbol means a **Things to do** ends that section.

Our decimal system

The way we write our numbers is based on a system of tens - the **decimal system**. Each column is worth ten times the one on its right so that the columns indicate powers of ten:

```
... 1000 100 10 1
       3   6  0 7 = three thousand, six hundred (no tens) and seven
```

Since each column is TEN times the one on its right, we need ten symbols to represent the ten values in each column: 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, called **digits**.

Each positive number has a unique representation in the decimal system. Why use 10? The reason is almost certainly that early writing systems were based on counting using the fingers. [Our word **digit** comes from the Latin for finger.] Tally systems were ways of putting marks or notches in wooden sticks (tally sticks) and they can be read more easily if grouped in batches of 5 or 10 for convenience.

Other bases

What if we used another power or **base** rather than ten?

Binary

Using powers of 2, we have the **binary** system, used in almost all computers. Here the columns are labelled with the powers of 2, and there are just 2 **b**inary dig**its**, 0 and 1, called **bits**.

```
... 16 8 4 2 1
        1 0 1 1
      = 8 + 2+1 = eleven
```

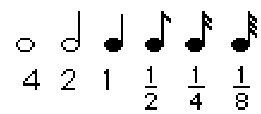
In order to distinguish 11 (eleven) from 11 in another base, we will put the base as a $_{\text{subscript}}$ (or sometimes in brackets) after the representation to avoid confusion. So 1011 in binary is 11 in base 10 is written as:

$1011_2 = 11_{10}$ Note that the base number is always written as a decimal.

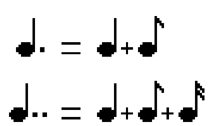
In the next section we will see that the binary system is used in musical notation.

Musical Notation

If a crotchet is taken as unit time (one beat), then the semibreve is 4 beats , thye minim 2, a crotchet is, as we assumed, 1, a quaver $^1/_2$, a semiquaver $^1/_4$ and demisemiquaver is $^1/_8$. They are written in musical notation as shown here:



A **dot** is placed after a note to add on *one half* of its value. So a dotted crotchet is a crotchet plus a quaver and has a duration of 1·5 time units; two dots after a crotchet give a duration of 1 + 1/2 + 1/4 = 1·75 units.



F.J Budden in *An Introduction to Number Scales and Computers*, Longmans, 1965, page 65, says he thinks the record number of dots is 4 in Verdi's *Requiem* in the Rex Tremenda. It is useful when a long note is followed by a quick note and the next note is "on the beat".

Binary fractions are written using column headings as follows:

```
... 8  4  2  1  ·  1/2  1/4  1/8 ...
```

So 1/4 = $0{\cdot}01_2$ and 3/8 = $0{\cdot}011_2$ since it is 1/4+1/8.

In binary, a dot after a crotchet adds a one in a fractional column:

crotchet = 1
crotchet dot = $1{\cdot}1_2$
crotchet dot dot = $1{\cdot}11_2$
crotchet dot dot dot = $1{\cdot}111_2$

and so on.

More bases

Base 8 is called **octal** and is presumably used by intelligent octopuses (or should that be *octopii*)!
It uses "digits" 0, 1, 2, 3, 4, 5, 6 and 7.
Base 3 is **ternary** and uses only 0, 1 and 2.
Here is **one hundred** expressed in all the bases from 2 to 9:

$$1100100_2 = 10201_3 = 1210_4 = 400_5 = 244_6 = 202_7 = 144_8 = 121_9 = 100_{10}$$

Base 2 is called **binary**,
Base 3 is called **ternary**,
Base 4 is called **quaternary**,
Base 5 is called **quinary**,
Base 6 is called **senary**,
Base 7 is called **septenary**,
Base 8 is called **octonary** or **octal**,
Base 9 is called **nonary**,
Base 10 is called **denary** or **decimal**.

What about bases bigger than 10?

There is no logical reason why we cannot use any integer bigger than zero for a base. (Think about base 1: what do the columns represent? What is 2 in base 1? What is 3? What is 7? This corresponds to a very early system of numbering, where notches were put on sticks or knots tied in strings.) The only problem is what to use to represent 10 or more in a **single** column? We need a single symbol for each value from 0 to B-1 in base B.

Usually the capital letters, A, B, C, etc, are used which take us up to base 36 (using the 10 digits and the 26 letters) - after that, it's up to you!

Continuing our list of ways of representing one hundred in different bases, we have:

10_{10} = A 11_{10} = B 12_{10} = C   and so on.

Here is *one hundred* again, this time expressed in some bases bigger than ten:

$100_{10} = 91_{11} = 84_{12} = 79_{13} = 72_{14} = 6A_{15}$

Base 11 is called **undenary**,
Base 12 is called **duodenary or duodecimal**,
Base 16 is called **hexadecimal**.

Chad Lake at the University of Utah has a nice page on what he calls the **Snake Algorithm** for converting from one base to another. It is a web page for a course he gave at Indiana University.

The Fibonacci base system

What if we labelled the columns with the **Fibonacci numbers** instead of powers of 10? We follow the usual conventions of larger column sizes being on the LEFT:

... 13 8 5 3 2 1

We represent number representations in this system by putting $_{\textbf{Fib}}$ after the representation: eg:

```
      8 5 3 2 1
ten = 1 0 0 1 0Fib = 8 + 2
```

Digits in the Fibonacci system

This time it is not clear what digits we should use in the columns. For instance, there are many ways to represent the value ten in this system as well as in the example above:

$$
\begin{aligned}
10(10) &= 2 \times 5 = 2000_{Fib} \\
&= 5 + 3 + 2 = 1110_{Fib} \\
&= 3 \times 3 + 1 = 301_{Fib} \\
&= 10 \times 1 = A_{Fib}
\end{aligned}
$$

Usually a number representation system is most useful if it has a **unique reprentation of every integer**. Here we don't, but we can get a single distinctive way of representing all integers if we use **only the digits 0 and 1** together with the rule that **no two ones can occur next to each other**. This last condition is because the sum of any two consecutive Fibonacci numbers is just the following Fibonacci number, so we can always replace ..011.. by ..100.. .

To convince yourself that every number can be represented in this system, write down the Fibonacci representations of all the numbers from 1 to 40. It starts as follows:

```
Decimal Fibonacci
       ..85321
      0      0
      1      1
      2     10
      3    100
```

```
 4    101
 5   1000
 6   1001
 7   1010
 8  10000
 9  10001
10  10010
11  10100
12  10101
13 100000
14 100001
15 100010
16 100100
17 100101
18 101000
19 101001
20 101010
```

We can also call this the **Fibonaccimal** system (pronounced fib-on-arch-i-mal) as Marijke van Gans does because decimal refers to Base 10.

An Application of the Fibonacci Number Representation

There are approximately 8 kilometres in 5 miles. Since both of these are Fibonacci numbers then there are approximately Phi (1.618..) kilometres in 1 mile and phi (0.618..) miles in 1 kilometres.

The real figure is more like 1.6093.. kilometres in 1 mile. This comes from the *precise* definition of 1 inch equals 2.54 centimetres exactly, and 100,000 centimetres make 1 kilometre. In the imperial system, 36 inches are 1 yard and 1760 yards are 1 mile.

Replacing each Fibonacci number by the *one before it* has the effect of reducing it by approximately 0.618 (phi) times (the ratio of a Fibonacci number to the one before it is nearly phi).

So to **convert 13 kilometres to miles**, replace 13 by the previous Fibonacci number, 8, and 13 kilometres is about 8 miles. Similarly, 5 kilometres is about 3 miles and 2 kilometres is about 1 mile.

Now suppose we want to convert 20 kilometres to miles where 20 is not a Fibonacci number. We can express 20 as a **sum of Fibonacci numbers** and convert each number separately and then add them up.

Thus 20 = 13 + 5 + 2.
Using ≈ to stand for ***approximately equals*** and replacing 13 by 8, 5 by 3 and 2 by 1, we have

$$\begin{aligned} 20 \text{ kms} &= 13 + 5 + 2 \text{ kilometres} \\ &\approx 8 + 3 + 1 \text{ miles} \\ &= 12 \text{ miles.} \end{aligned}$$

To **convert miles to kilometres**, we write the number of miles as a sum of Fibonacci numbers and then replace each by the next *larger* Fibonacci number:

$$\begin{aligned} 20 \text{ miles} &= 13 + 5 + 2 \text{ miles} \\ &\approx 21 + 8 + 3 \text{ kilometres} \\ &= 32 \text{ miles.} \end{aligned}$$

There is no need to use *the* Fibonacci Representation of a number, which uses the fewest Fibonacci numbers, but you can use any combination of numbers that add to the number you are converting. For instance, 40 kilometres is 2×20 and we have just seen that 20 kms is 12 miles. So 40 kms is $2 \times 12 = 24$ miles approximately.

[With thanks to Paul V S Townsend for reminding me of this application.]

Things to do

- A few years ago, the speed limit in USA was 55 mph (miles per hour). What would that be in kph (kilometres per hour)?
- The speed limit on UK motorways is 70 mph. What is this in kph?
  What is the speed limit in built up areas (30 mph) in kph?
- The current train speed record of 552 kph was set on April 14 1999 in Japan.
  What is the equivalent speed in mph using the Fibonacci method?
  What is the equivalent speed in mph using the conversion factor of 1.6093 km per mile?

Reference

Concrete Mathematics (2nd edition) by Graham, Knuth and Patashnik, Addison-Wesley, section 6.6.

An easy way to Multiply

The Egyptian system - using Doubling...

The Egyptians had an easy way to multiply two integers which involved only doubling numbers and adding - no multiplication tables to learn and no need for a calculator (except to do the addition).
For example, 19 x 65. We write the two numbers at the head of two columns, choosing one column to keep doubling and the other to keep halving (ignoring remainders), until the halving column reaches 1:

halve	double	odd?
19	65	+
9	130	+
4	260	
2	520	
1	1040	+

Any row whose **halving** column entry was **odd** is marked (here with +) and we **add the marked values from the doubling column**. In our example 65+130+1040=1235 which is the product of 19 and 65.
The method works because if we represent 19 in the **binary system** we have 16+2+1=10011(2). So we want 19x65=(16+2+1)x65=16*56 + 2*65 + 1*65. ie, the 1st, 2nd and 5th values

Things to do

- Check that if you halve the 65 column and double the 19 column the method still works.
- Try the Egyptian method on 32x65.
- Try it on 31x65.

The Fibonacci system

A similar system uses the Fibonacci representation to replace each doubling of the Egyptian method with an addition.

Let's take the same example: 19x65.
This time we take just one number - say 65 - as the head of the right hand column, the left column starting with 1. The second row has 2 on the left and we double 65 to get 130 on the right. Now each successive row is the sum of the previous TWO entries above it, taking each column separately. So since we started with 1 and 2 on the left we will get 3,5,8,... that is, the Fibonacci numbers on the left hand side. Stop when we can find a Fibonacci number which is bigger than the other number in the product - here 19:

```
  1   65 +
  2  130
  3  195
  5  325 +
  8  520
 13  845 +
 21
```

We mark the rows this time by finding those entries in the left column that add up to 19. There many be several ways to do this selection but any will do. Here we have chosen 13+5+1. If we add up the right hand entries on these rows we have: 65+325+845=1235 which is again 19x65.

Things to do

- Try it the other way round, starting with 19 and stop when the Fibonacci number exceeds 65.
- Try the same multiplications as above: 32x65 and 31x65.
- Look up the article where this idea was first presented:
  Fibonacci, Lucas and the Egyptians by S La Barbera in *The Fibonacci Quarterly*, Vol 9, 1971, pages 177-187.

Patterns in the Fibonacci representations

Patterns in the columns - the Rabbit sequence

In base 10, if we list all the integers from 1, then there are patterns in the columns:

Decimal patterns

Column 1 (units) cycles through all the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 repeatedly;

Column 2 (tens) cycles through all the digits but each digit occurs ten times;
Column 3 (hundreds) is the same but each digits occurs 100 times;
and so on.

Fibonacci Representations patterns

Is there a pattern in the columns of the Fibonacci numbers in the Table above?
Yes there is!
It is based on the Rabbit sequence which now includes the initial 0.
The pattern in column one is derived from the rabbit sequence where
every "1" in the rabbit sequence has been replaced by "10":-
The rabbit sequence:

```
0101101011011010110 10...
```

becomes:

```
  0 1  0 1  1  0 1  0 1  1  0 1  1  0 1  0 1  1  0 1  0 ...
  0 10 0 10 10 0 10 0 10 10 0 10 10 0 10 0 10 10 0 10 0 ...
```

which is **column 1** above, read downwards.

[NB This is exactly the same as if we flipped the bits (1 changes to 0 and 0 to 1) in the Rabbit sequence (without its initial zero)!! However, there is a pattern in the other columns which is better seen with the description above.]

What about **column 2** of the Fibonacci representations?
This is derived similarly:
every "1" in the rabbit sequence is replaced by "100" and
every "0" is replaced by "00".

```
  0  1   0  1   1   0  1   0  1   1   0  1   1   0  ... Rabbit Sequence
  00 100 00 100 100 00 100 00 100 100 00 100 100 00 ... Column 2
```

where column 2 in the Table of Fibonacci representations is read downwards.

For column 3, replace "0" by "000" and "1" by "11000"
For column 4, replace "0" by "00000" and "1" by "11100000"
For column 5, replace "0" by "00000000" and "1" by "1111100000000"

The same pattern follows for all the columns:
Column i the just the rabbit sequence with "0" replaced by F(i) 0s and "1" replaced by F(i-1) 1s followed by F(i) 0s.

The number of 1s in a Fibonacci Representation

What is the least number of Fibonacci numbers that sum to a given n?
This is the number of 1s in the Fibonaci representation, since the description given above guarantees the least number

of Fibonacci's and is also called the **minimal Fibonacci representation**. Here we repeat the Fibonacci Representation table from above but now include the number of 1's in each representation:

```
   Decimal Fibonacci 1s count
          ..85321
         1       1       1
         2      10       1
         3     100       1
         4     101       2
         5    1000       1
         6    1001       2
         7    1010       2
         8   10000       1
         9   10001       2
        10   10010       2
        11   10100       2
        12   10101       3
```

From the table, we can see that the number of numbers with a Fibonacci representation of a given length is a Fibonacci number:
There is 1 of length 1,
there is 1 of length 2,
there are 2 of length 3,
there are 3 of length 4,
there are 5 of length 5,...

Here is a more compact list of the number of 1s in the (minimal) Fibonacci representation of the first few whole numbers :

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	...
1	1	1	2	1	2	2	1	2	2	2	3	1	2	2	2	3	2	3	3	1	2	2	2	3	2	3	3	2	3	3	3	4	...

If we split this list into sublists corresponding to the different lengths of Fibonacci representations we have the following:

$1=1_{Fib}$	1,
$2=10_{Fib}$	1,
$3=100_{Fib}$,$4=101_{Fib}$	1,2
$5=1000_{Fib}$, $6=1001_{Fib}$, $7=1010_{Fib}$	1,2,2
8, 9, 10, 11 and 12	1,2,2,2,3
13 to 20	1,2,2,2,3,2,3,3
21 to 33	1,2,2,2,3,2,3,3,2,3,3,3,4
34 to 54	1,2,2,2,3,2,3,3,2,3,3,3,4,2,3,3,3,4,3,4,4
...	...

It is quite easy to see where this pattern comes from: Each time we put a 1 at the start of our Fibonacci representations and then copy the earlier patterns. For example, 8, 9, 10, 11 and 12 are 8+0, 8+1, 8+2, 8+3 and 8+4.

Can you see any patterns in these sequences?
It seems that each sequence starts off the following sequence.
Can you discover how the remainder of each is formed, that is, the part that follows (the copy of) the previous sequence? It is not quite the sequence before, but, *one added to all the items of the sequence before*:

> Start with 1 and 1.
> The next sequence is the preceding one followed by adding one to the sequence before the preceding one.

Since each sequence in the list above starts off the following one, it defines a unique **infinite sequence**.

Generalised Fibonacci Series in the Fibonacci System

This section was suggested by Marijke van Gans.

If we take any Fibonacci-type series **starting with *any* two numbers of your choice**, -let's call them A and B - and the series continues in the same fashion as the Fibonacci series (by adding the latest two numbers to get the next) then the series is:

A, B, A+B, A+2B, 2A+3B, ...

The interesting part is left for you to discover for yourself in the following questions:

Things to do

- What series of numbers do you get if we start with the following:
  1. 2 and 3 (A=2 and B=3)
  2. 3 and 5
  3. Can you think of other pairs which give the "same" answers as questions 1 and 2 above?
  4. 3 and 4
  5. 1 and 5
  6. Try some others starting pairs of your own.
- Extend the A-B-series above:
  The next term is 3A+5B. What are the next 3 terms? What do you notice about the multiples of A and B?
- Pick one of your Generalised FIbonacci series from above (take at least the first eight numbers).
  Express these 8 or more numbers as Fibonaccimals.
  What do you notice about the pattern in the Fibonaccimals?
  Try it for several more of the series above. Does the same thing happen?

The reason for this behaviour is found in a Theorem that

Any Generalised Fibonacci series has successive terms whose ratio tends to Phi in the long run, no matter what the two starting numbers are.

So the behaviour you spotted is like the rule in Base Ten - to multiply by the Base (10) just shift the numbers one place to the left.

On a later page we will investigate what happens if instead of using the Fibonacci numbers as column headers we use *powers of Phi* (1.61803..), ie base Phi.

A Primer for the Fibonacci Numbers: Part XIIby V E Hoggatt Jr, N Cox, M Bicknell in *Fibonacci Quarterly*, vol 11 (1973), pages 317 -331
is a useful introduction to results in this area, but for post-18 mathematics students.

Fibonacci Home Page

The Fibonacci Numbers in Art, Music and Architecture

A Formula for the Fibonacci numbers

WHERE TO NOW???

This is the last page on this topic.

The next topic is...
The Golden Section - the Number and Its Geometry

The Golden section ratio: Phi

Contents of this Page

The line means there is a Things to do investigation at the end of the section.

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What is the golden section (or Phi)?

We will call the Golden Ratio (or Golden number) after a greek letter,**Phi** (Φ) here, although some writers and mathematicians use another Greek letter, **tau** (τ). Also, we shall use **phi** (note the lower case p) for a closely related value.

A simple definition of Phi

There are *just two numbers that remain the same when they are squared* namely **0** and **1**. Other numbers get bigger and some get smaller when we square them:

Squares that are bigger	Squares that are smaller
2^2 is 4	1/2=0·5 and $0{\cdot}5^2$ is 0·25=1/4
3^2 is 9	1/5=0·2 and $0{\cdot}2^2$ is 0·04=1/25
10^2 is 100	1/10=0·1 and $0{\cdot}1^2$ is 0·01=1/100

One definition of Phi (the golden section number) is that

to square it you just add 1

or, in mathematics:

$$\mathbf{Phi^2 = Phi + 1}$$

In fact, there are *two* numbers with this property, one is Phi and another is closely related to it when we write out some of its decimal places.

Here is a mathematical derivation (or proof) of the two values. You can skip over this to the answers at the foot of this paragraph if you like.

Multiplying both sides by Phi gives a quadratic equation:

$$\mathbf{Phi^2 = Phi + 1} \text{ or}$$
$$\mathbf{Phi^2 - Phi - 1 = 0}$$

We can solve this quadratic equation to find two possible values for Phi as follows:

- First note that $(Phi - {}^1/_2)^2 = Phi^2 - Phi + {}^1/_4$
- Using this we can write $Phi^2 - Phi - 1$ as $(Phi - {}^1/_2)^2 - {}^5/_4$
  and since $Phi^2 - Phi - 1 = 0$ then $(Phi - {}^1/_2)^2$ must equal ${}^5/_4$
- Taking square-roots gives $(Phi - {}^1/_2) = +\sqrt{({}^5/_4)}$ or $-\sqrt{({}^5/_4)}$.
- so $Phi = {}^1/_2 + \sqrt{({}^5/_4)}$ or ${}^1/_2 - \sqrt{({}^5/_4)}$.
- We can simplify this by noting that $\sqrt{({}^5/_4)} = \sqrt{5}/\sqrt{4} = \sqrt{5}/_2$
- The two values of Phi are therefore:
- $\mathbf{{}^1/_2 + \sqrt{5}/_2}$ **and** $\mathbf{{}^1/_2 - \sqrt{5}/_2}$

Use your calculator to see that the values of these two numbers are 1·6180339887... and –0·6180339887...

Did you notice that **their decimal parts are identical**?
We will name the first value **Phi** and the second **– phi** using the first letter to tell us if we want the *bigger* value (Phi) 1·618... or the *smaller* one (phi) 0·618... .

Note that Phi is just 1+phi. As a little practice at algebra, use the expressions above to show that phi times Phi is exactly 1. Here is a summary of what we have found already that we will find very useful in what follows:

$$\text{Phi phi} = 1,\ \text{Phi} - \text{phi} = 1,\ \text{Phi} + \text{phi} = \sqrt{5}$$

Phi = 1.6180339..	phi = 0.6180339..
Phi = 1 + phi	phi = Phi – 1
Phi = 1/phi	phi = 1/Phi
$\text{Phi}^2 = \text{Phi} + 1$	$(-\text{phi})^2 = -\text{phi} + 1$ or $\text{phi}^2 = 1 - \text{phi}$
$\text{Phi} = (\sqrt{5} + 1)/2$	$\text{phi} = (\sqrt{5} - 1)/2$

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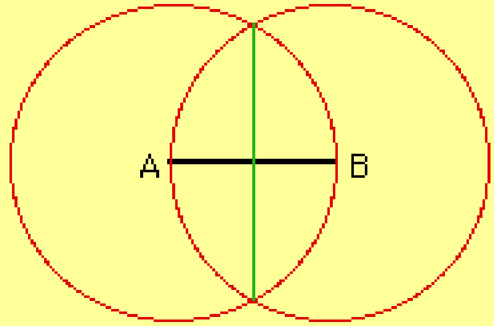
A bit of history...

Euclid, the Greek mathematician who lived from about 365BC to 300BC wrote the *Elements* which is a collection of 13 books on Geometry (written in Greek originally). It was the most important mathematical work until this century, when Geometry began to take a lower place on school syllabuses, but it has had a major influence on mathematics.

It starts from basic definitions called axioms or "postulates" (self-evident starting points). An example is the fifth axiom that

there is only one line parallel to another line through a given point.

From these Euclid develops more results (called *propositions*) about geometry which he proves based purely on the axioms and previously proved propositions using logic alone. The propositions involve constructing geometric figures using a straight edge and compasses only so that we can only draw straight lines and circles.



For instance, **Book 1, Proposition 10** ***to find the exact centre of any line AB***

1. Put your compass point on one end of the line at point A.
2. Open the compasses to the other end of the line, B, and draw the circle.
3. Draw another circle in the same way with centre at the other end of the line.
4. This gives two points where the two red circles cross and, if we join these points, we have a (green) straight line at 90 degrees to the original line which goes through its exact centre.

In **Book 6, Proposition 30**, Euclid shows how to ***divide a line in mean and extreme ratio*** which we would call "finding the golden section G point on the line".

```
<-------- 1 --------->
A           G       B
      g        1-g
```

Euclid used this phrase to mean *the ratio of the smaller part of this line, GB to the larger part AG (ie the ratio GB/AG) is the SAME as the ratio of the larger part, AG, to the whole line AB (ie is the same as the ratio AG/AB)*. If we let the line AB have unit length and AG have length g (so that GB is then just 1–g) then the definition means that

$$\frac{GB}{AG} = \frac{AG}{AB} \quad \text{or} \quad \frac{1-g}{g} = \frac{g}{1} \quad \text{so that } 1-g=g^2$$

Notice that earlier we defined Phi^2 as Phi+1 and here we have $g^2 = 1–g$ or $g^2+g=1$.
We can solve this in the same way as for Phi and we find that

$$g = \frac{-1+\sqrt{5}}{2} \text{ or } g = \frac{-1-\sqrt{5}}{2}$$

So there are two numbers which when added to their squares give 1. For our geometrical problem, g is a positive number so the first value is the one we want. This is our friend **phi** also equal to Phi–1 (and the other value is merely –**Phi**).

It seems that this ratio had been of interest to earlier Greek mathematicians, especially Pythagoras (580BC - 500BC) and his "school".

Things to do

1. Suppose we labelled the parts of our line as follows:

```
A_______________G_________B
       x             1
```

so that AB is now has length 1+x. If Euclid's "division of AB into mean and extreme ratio" still applies to point G, what quadratic equation do you now get for x? What is the value of x?

Links on Euclid and his "Elements"

From Clarke University comes D Joyce's exciting project making Euclid's *Elements* interactive using Java applets.

Phi and the Egyptian Pyramids?

The **Rhind Papyrus** of about 1650 BC is one of the oldest mathematical works in existence, giving methods and problems used by the ancient Babylonians and Egyptians. It includes the solution to some problems about pyramids but it does not mention anything about the golden ratio Phi.

The ratio of the length of a face of the Great Pyramid (from centre of the bottom of a face to the apex of the pyramid) to the distance from the same point to the exact centre of the pyramid's base square is about 1·6. It is a matter of debate whether this was "intended" to be the golden section number or not.
According to Elmer Robinson (see the reference below), using the average of eight sets of data, says that "the theory that the perimeter of the pyramid divided by twice its vertical height is the value of pi" fits the data much better than the theory above about Phi.
The following references will explain circumstantial evidence for and against:

The golden section in The Kings Tomb in Egy pt.

How to Find the "Golden Number" without really trying Roger Fischler, *Fibonacci Quarterly*, 1981, Vol 19, pp 406 - 410

Case studies include the Great Pyramid of Cheops and the various theories propounded to explain its dimensions, the golden section in architecture, its use by Le Corbusier and Seurat and in the visual arts. He concludes that several of the works that purport to show Phi was used are, in fact, fallacious and "without any foundation whatever".

The Fibonacci Drawing Board Design of the Great Pyramid of Gizeh Col. R S Beard in *Fibonacci Quarterly* vol 6, 1968, pages 85 - 87;

has three separate theories (only one of which involves the golden section) which agree quite well with the dimensions as measured in 1880.

A Note on the Geometry of the Great Pyramid Elmer D Robinson in *The Fibonacci Quarterly* vol 20 (1982) page 343

shows that the theory involving pi fits much better than the one regarding Phi.

George Markowsky's **Misconceptions about the Golden ratio** in *The College Mathematics Journal* Vol 23, January 1992, pages 2-19.

This is readable and well presented. You may or may not agree with *all* that Markowsky says, but this is a very good article that tries to debunk a simplistic and unscientific "cult" status being attached to Phi, seeing it where it really is not! He has some convincing arguments that Phi does not occur in the measurements of the Egyptian pyramids.

Other names for Phi

Euclid (365BC - 300BC) in his "Elements" calls dividing a line at the 0.6180399.. point **dividing a line in the extreme and mean ratio.** This later gave rise to the name **golden mean**.

There are no extant records of the Greek architects' plans for their most famous temples and buildings (such as the Parthenon). So we do not know if they deliberately used the golden section in their architectural plans. The American mathematician Mark Barr used the Greek letter **phi** (ϕ) to represent the golden ratio, using the initial letter of the Greek Phidias who used the golden ratio in his sculptures.

Luca Pacioli (also written as Paccioli) wrote a book called *De Divina Proportione* (**The Divine Proportion**) in 1509. It contains drawings made by Leonardo da Vinci of the 5 Platonic solids. It was probably Leonardo (da Vinci) who first called it the **sectio aurea** (Latin for **the golden section**).

Today, mathematicians also use the Greek letter tau (τ), the initial letter of *tome* which is the Greek work for "cut" as well as phi.

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Phi to 2000 decimal places

Phi has the value $\frac{\sqrt{5}+1}{2}$ and phi is $\frac{\sqrt{5}-1}{2}$.

Both have identical fractional parts after the decimal point. Both are also *irrational* which means that

- They cannot be written as M/N for any whole numbers M and N;
- their decimal fraction parts have no pattern in their digits, that is, they never end up repeating a fixed cycle of digits as do all rational values (which are expressed as M/N for some whole numbers M and N).

Later on this page we will show why Phi and phi cannot be written as exact fractions. There is another surprise in store later when we find which fractions are the best approximations to Phi.

Here is the decimal value of Phi to 2000 places grouped in blocks of 5 decimal digits. The value of phi is the same but begins with 0·6.. instead of 1·6.. .
Read this as ordinary text, in lines across, so Phi is 1·61803398874...)

										Dps:
1·61803	39887	49894	84820	45868	34365	63811	77203	09179	80576	**50**
28621	35448	62270	52604	62818	90244	97072	07204	18939	11374	**100**
84754	08807	53868	91752	12663	38622	23536	93179	31800	60766	
72635	44333	89086	59593	95829	05638	32266	13199	28290	26788	**200**
06752	08766	89250	17116	96207	03222	10432	16269	54862	62963	
13614	43814	97587	01220	34080	58879	54454	74924	61856	95364	**300**
86444	92410	44320	77134	49470	49565	84678	85098	74339	44221	
25448	77066	47809	15884	60749	98871	24007	65217	05751	79788	**400**
34166	25624	94075	89069	70400	02812	10427	62177	11177	78053	
15317	14101	17046	66599	14669	79873	17613	56006	70874	80710	**500**
13179	52368	94275	21948	43530	56783	00228	78569	97829	77834	
78458	78228	91109	76250	03026	96156	17002	50464	33824	37764	
86102	83831	26833	03724	29267	52631	16533	92473	16711	12115	
88186	38513	31620	38400	52221	65791	28667	52946	54906	81131	
71599	34323	59734	94985	09040	94762	13222	98101	72610	70596	
11645	62990	98162	90555	20852	47903	52406	02017	27997	47175	
34277	75927	78625	61943	20827	50513	12181	56285	51222	48093	
94712	34145	17022	37358	05772	78616	00868	83829	52304	59264	
78780	17889	92199	02707	76903	89532	19681	98615	14378	03149	
97411	06926	08867	42962	26757	56052	31727	77520	35361	39362	**1000**
10767	38937	64556	06060	59216	58946	67595	51900	40055	59089	
50229	53094	23124	82355	21221	24154	44006	47034	05657	34797	
66397	23949	49946	58457	88730	39623	09037	50339	93856	21024	
23690	25138	68041	45779	95698	12244	57471	78034	17312	64532	
20416	39723	21340	44449	48730	23154	17676	89375	21030	68737	
88034	41700	93954	40962	79558	98678	72320	95124	26893	55730	
97045	09595	68440	17555	19881	92180	20640	52905	51893	49475	
92600	73485	22821	01088	19464	45442	22318	89131	92946	89622	
00230	14437	70269	92300	78030	85261	18075	45192	88770	50210	
96842	49362	71359	25187	60777	88466	58361	50238	91349	33331	
22310	53392	32136	24319	26372	89106	70503	39928	22652	63556	
20902	97986	42472	75977	25655	08615	48754	35748	26471	81414	

51270 00602 38901 62077 73224 49943 53088 99909 50168 03281
12194 32048 19643 87675 86331 47985 71911 39781 53978 07476
15077 22117 50826 94586 39320 45652 09896 98555 67814 10696
83728 84058 74610 33781 05444 39094 36835 83581 38113 11689
93855 57697 54841 49144 53415 09129 54070 05019 47754 86163
07542 26417 29394 68036 73198 05861 83391 83285 99130 39607
20144 55950 44977 92120 76124 78564 59161 60837 05949 87860
06970 18940 98864 00764 43617 09334 17270 91914 33650 13715 **2000**

Phi to 10,000,000 places!

Simon Plouffe of Simon Fraser University notes that Greg J Fee programmed a method of his to compute the golden ratio (Phi) to ten million places in December 1996. He used Maple and it took about 30 minutes on a 194MHz computer. Have a look at the first part with 15,000 decimal places. The rest are organised in several files which you can investigate using this index.

Phi's value **in binary** to 500 places is:

1·10011 11000 11011 10111 10011 01110 01011 11111 01001 01001
11110 00001 01011 11100 11100 11100 11000 00001 10000 00101 **100**
11001 11011 01110 01000 00110 10000 01000 01000 00100 01001
11011 01011 11110 01110 10001 00111 00100 10100 01111 11000 **200**
01101 10001 10101 00001 00011 10100 00110 00001 10001 11010
01010 10010 01110 11001 11111 10000 10110 00101 01001 11101 **300**
00100 11110 11011 11111 00000 01101 00011 10000 01000 10110
11010 11011 11110 00110 00001 00111 11110 00000 01100 01000 **400**
01101 11100 00100 10010 10000 10000 00001 10000 00000 01011
00000 11101 01100 10010 11101 00100 00001 11100 11001 10101 **500**

Neither the decimal form of Phi, nor the binary one nor *any* other base have any ultimate repeating pattern in their digits.

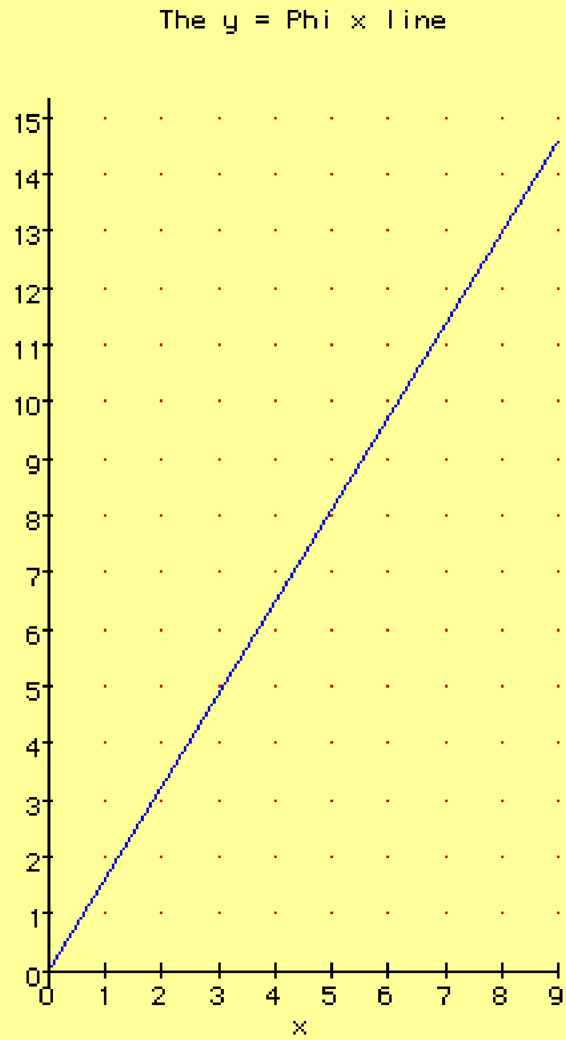
1·61803 39887 49894 84820 45868 34365 63811 77203 09179 80576 ..More..

Phi and the Fibonacci numbers

On the Fibonacci and Nature page we saw a graph which showed that the ratio of successive Fibonacci numbers gets closer and closer to Phi.

Here is the connection the other way round, where we can discover the Fibonacci numbers arising from the number Phi.

The graph on the right shows a line whose gradient is Phi, that is the line



y = Phi x = 1·6180339.. x

Since Phi is not the ratio of any two integers, the graph will *never go through any points of the form (i,j) where i and j are whole numbers* - apart from one trivial exception - can you spot it?
So we can ask
What are the nearest integer-coordinate points to the Phi line?
Let's start at the origin and work up the line.
The first is (0,0) of course, so here ARE two integers i=0 and j=0 making the point (i,j) exactly on the line! In fact ANY line y=kx will go through the origin, so that is why we will ignore this point as a "trivial exception" (as mathematicians like to put it).
The next point close to the line looks like (0,1) although (1,2) is nearer still. The next nearest seems even closer: (2,3) and (3,5) even closer again. So far our sequence of "integer coordinate points close to the Phi line" is as follows: (0,1), (1,2), (2,3), (3,5)
What is the next closest point? and the next? Surprised? The coordinates are successive **Fibonacci numbers**!

Let's call these the **Fibonacci points**. Notice that the ratio y/x for each Fibonacci point (x,y) gets closer and closer to Phi=1·618... but the interesting point that we see on this graph is that

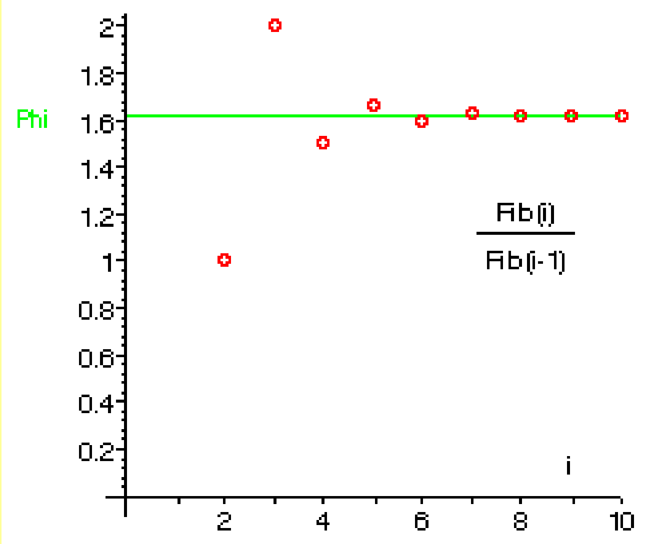
the Fibonacci points are the **closest** points to the Phi line.

1·61803 39887 49894 84820 45868 34365 63811 77203 09179 80576 ..More..

The Ratio of neighbouring Fibonacci Numbers

tends to Phi

On the Fibonacci Numbers and Nature page we saw that the ratio of two neighbouring Fibonacci numbers soon settled down to a particular value near 1·6:



In fact, the exact value is Phi and, the larger the two Fibonacci numbers, the closer their ratio is to Phi. Why? Here we show how this happens.

The basic Fibonacci relationship is

F(i+2) = F(i+1) + F(i)	*The Fibonacci relationship*

The graph shows that the ratio F(i+1)/F(i) seems to get closer and closer to a particular value, which for now we will call X.
If we take three neighbouring Fibonacci numbers, F(i), F(i+1) and F(i+2) then, for very large values of i, the ratio of F(i) and F(i+1) will be almost the same as F(i+1) and F(i+2), so let's see what happens if both of these are the same value: X.

$$\frac{F(i+1)}{F(i)} = \frac{F(i+2)}{F(i+1)} = X$$

But, using the *The Fibonacci relationship* we can replace F(i+2) by F(i+1)+F(i) and then simplify the resulting fraction a bit, as follows:

$$\begin{aligned}\frac{F(i+2)}{F(i+1)} &= \frac{F(i+1) + F(i)}{F(i+1)} \\ &= \frac{F(i+1)}{F(i+1)} + \frac{F(i)}{F(i+1)} \\ &= 1 + \frac{F(i)}{F(i+1)}\end{aligned}$$

So, putting in this new format of F(i+2)/F(i+1) back into the equation for X, we have:

$$X = \frac{F(i+1)}{F(i)} = 1 + \frac{F(i)}{F(i+1)}$$

But the last fraction is just 1 + 1/X, so now we have an equation purely in terms of X:

$$\mathbf{X} = \frac{F(i+1)}{F(i)} = 1 + \frac{F(i)}{F(i+1)} = \mathbf{1} + \frac{\mathbf{1}}{\mathbf{X}}$$

Multiplying both sides by X gives:

$$X = 1 + \frac{1}{X}$$

$$X^2 = X + 1$$

But we have seen this equation before in A simple definition of Phi so know that X is, indeed, exactly Phi!

Remember, this supposed that the ratio of two pairs of neighbours in the Fibonacci series was the same value. This only happens "in the limit" as mathematicians say. So what happens is that, as the series progresses, the ratios get closer and closer to this limiting value, or, in other words, the ratios get closer and closer to Phi the further down the series that we go.

But there are *two* values that satisfy $X^2 = X + 1$ aren't there?

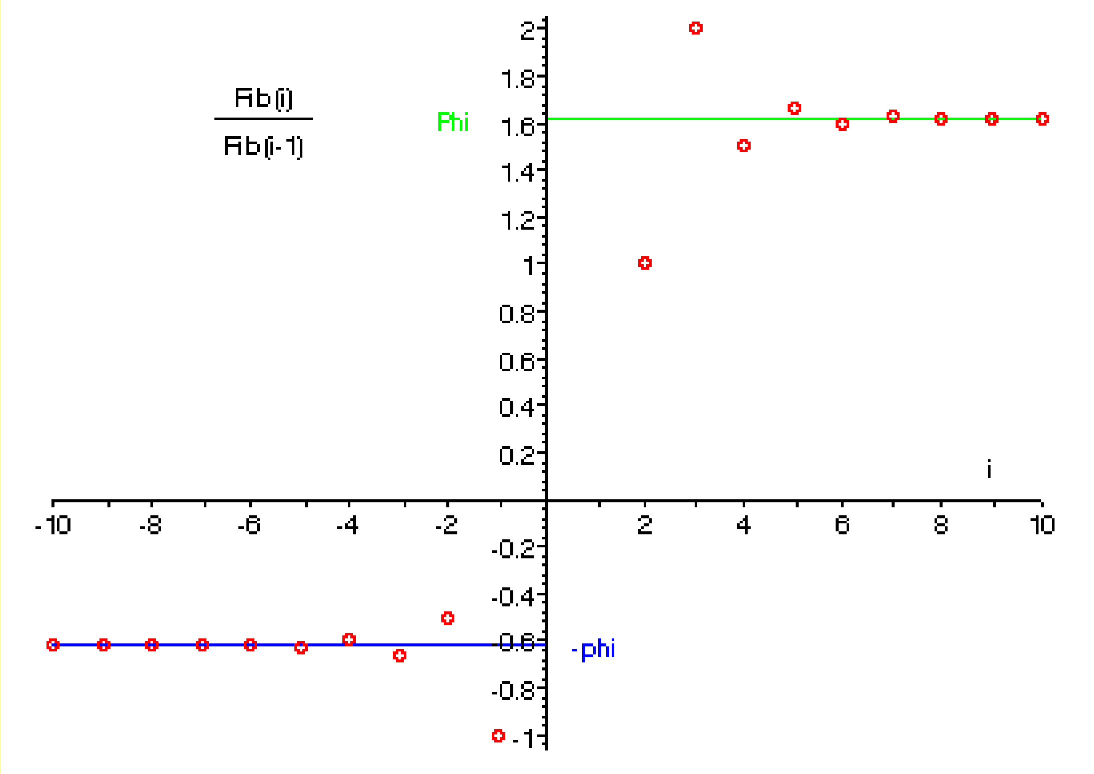
Yes, there are. The other value, –phi which is –0·618... is revealed if we *extend the Fibonacci series backwards*. We still maintain the same *Fibonacci relationship* but we can find numbers *before 0* and still keep this relationship:

i	...	–10	–9	–8	–7	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7	8	9	10	...
Fib(i)	...	–55	34	–21	13	–8	5	–3	2	–1	1	0	1	1	2	3	5	8	13	21	34	55	...

When we use this complete Fibonacci series and plot the ratios F(i)/F(i–1) we see that the ratios on the left-hand side of 0 are

$$\frac{1}{-1} = -1, \frac{-1}{2} = -0.5, \frac{2}{-3} = -0.666.., \frac{-3}{5} = -0.6, \frac{5}{-8} = -0.625, ...$$

Plotting these shows both solutions to $X^2 = X + 1$:-



Another definition of Phi

We defined Phi to be (one of the two values given by)

$$Phi^2 = Phi+1$$

Suppose we divide both sides of this equation by Phi:

$$Phi = 1 + 1/Phi$$

Here is another definition of Phi **- that number which is 1 more than its reciprocal**
(the reciprocal of a number is 1 over it so that, for example, the reciprocal of 2 is 1/2 and the reciprocal of 9 is 1/9).

A formula for Phi using a continued fraction

Look again at the last equation:

$$Phi = 1 + 1/Phi$$

This means that wherever we see "Phi" we can substitute (1 + 1/Phi).
But we see Phi on the right hand side, so lets substitute it in there!

$$Phi = 1 + 1/(1 + 1/Phi)$$

In fact, we can do this again and again and get:

```
Phi = 1 + ______1_______   = 1 + 1/( 1 + 1/( 1 + 1/( 1 +.. )))
           1 + ____1____
```

```
1 +   1
    1 + ..
```

This unusual expression is called a **continued fraction** since we continue to form fractions underneath fractions underneath fractions.

This continued fraction has a big surprise in store for us....

Phi is not a fraction

But Phi *is* a fraction .. it is $(\sqrt{5} + 1)/2$.

Here, by *a fraction* we mean a number fraction such as $^2/_3$ or $-^{17}/_{24}$ or $^{12}/_7$ or $^8/_{12}$. The first two here are *proper fractions* since they are less than 1; the third can also be written as $1^5/_7$, which has a whole part (1) and a fraction part ($^5/_7$) so it is a *mixed number*; the fourth is not *in its lowest terms* since it is the same as $^2/_3$ which *is* in its lowest terms since there is no simpler representation of this quantity. Also 5.61 is a fraction, *a decimal fraction* since it is 561/100, the ratio of a whole number and a power of ten. Strictly, all whole numbers can be written as fractions if we make the denominator (the part below the line) equal to 1! However, we commonly use the word *fraction* when there really *is* a fraction in the value.

Mathematicians call all these fractional (and whole) numbers ***ratio*nal numbers** because they are the ratio of two whole numbers and it is these number fractions that we will mean by *fraction* in this section.

It may seem as if all number can be written as fractions - but this is, in fact, false. There are numbers which are not the ratio of any two whole numbers, eg $\sqrt{2}$=1.41421356... , π=3.14159..., *e*=2.71828... . Such values are called **ir-ratio-nal** since they cannot be represented as a ratio of two whole numbers (ie a fraction). A simple consequence of this is that their decimal fraction expansions go on for ever and never repeat at any stage!

Any and every *fraction*has a decimal fraction expansion that either

- stops as in, for example, 1/8 = 0.125 exactly or else
- eventually gets into a repeating pattern that repeats for ever eg 5/12 = 0.416666666... or 3/7 = 0.428571 428571 428571 ...

Can we write Phi as a fraction?

The answer is "No!" and there is a surprisingly simple proof of this. Here it is. [This proof was given in the *Fibonacci Quarterly*, volume 13, 1975, page 32, in **A simple Proof that Phi is Irrational** by J Shallit and later corrected by D Ross - see below.]

Suppose we could write Phi as A/B where A and B are two integers. If this was possible then we can choose the simplest form for Phi and write Phi=p/q (p and q are integers again) but now p and q will have no factors in common. What we now show is that this leads to a logical contradiction. The only assumption we have made is that Phi can be written as a fraction and, since this will lead to a logical impossibility, then this assumption must be wrong - i.e. Phi *cannot* be written as a fraction.

The definition of Phi (and also of –phi) is that it satisfies the equation

$$\text{Phi}^2 - \text{Phi} = 1 \qquad (*)$$

So, if we are assuming that Phi can be written as p/q, we substitute this in:

$$(p/q)^2 - p/q = 1$$

Since q is not zero, we can multiply through by q^2 to get:

$$p^2 - pq = q^2 \qquad (**)$$

but we can factorise the left hand side, so

$$p(p - q) = q^2$$

Since the left hand side has a factor of p then so must the right hand side. In other words p is a factor of q^2.
Since we said that p and q had no factor in common - except 1 which is a factor common to all numbers - then p must be 1.

Note there is an error in the paper quoted above, which is corrected in the next paragraph and in **A Letter to the Editor** from David Ross in *Fibonacci Quarterly* vol 13 (1975) page 198.

Also, be re-arranging the equation marked (**) above, we have:

$$\begin{aligned} p^2 &= q^2 + pq \\ &= q(q + p) \end{aligned}$$

so q, being a factor of the right-hand side must also be a factor of the left-hand side, which is p^2. But again, since p and q have no common factor except 1 then q also must be 1 too!

Here is the contradiction if both p and 1 are 1, then p/q is 1 and this does *not* satisfy our original equation

for Phi, the one marked (*).
So we have a logical impossibility **if** we assume Phi can be written as a proper fraction.
The only possibility that logical allows therefore is that Phi *cannot* be written as a proper fraction - Phi is irrational.

Rational Approximations to Phi

If no fraction can be the *exact* value of Phi, what fractions are *good approximations* to Phi?

The answer lies in the continued fraction for Phi that we saw earlier on this page.
If we stop the continued fraction for Phi at various points, we get values which approximate to Phi:

$$\text{Phi} = 1 \qquad \text{Phi} = 1 + \frac{1}{1} = 2 \qquad \text{Phi} = 1 + \frac{1}{1 + \frac{1}{1}} = \frac{3}{2} \qquad \text{Phi} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}} = \frac{5}{3}$$

The next approximation is always 1 plus 1-over-the-previous-approximation (shown in green).
Did you notice that this series of fractions is just the **ratios of successive Fibonacci numbers** - surprise!

The proper mathematical term for these fractions which are formed from stopping a continued fraction for Phi at various points is the **convergents** to Phi. The series of convergents is

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \ldots$$

Why do Fibonacci numbers occur in the convergents?

This is an optional section where we show exactly why the Fibonacci numbers appear in the successive approximations (the convergents) above. Skip to the next section if you like!

The convergents start with 1/1 which is F(1)/F(0)
where F(n) represents the n-th Fibonacci number.

To get from one fraction to the next, we saw that we just take the reciprocal of the fraction and add 1:
so the next one after F(1)/F(0) is

$$1 + \frac{1}{F(1)/F(0)} = 1 + \frac{F(0)}{F(1)} = \frac{F(1)+F(0)}{F(1)}$$

But the Fibonacci numbers have the property that two successive numbers add to give the next, so F(1)+F(0)=F(2) and our next fraction can be written as

$$1 + \frac{1}{F(1)/F(0)} = 1 + \frac{F(0)}{F(1)} = \frac{F(1)+F(0)}{F(1)} = \frac{F(2)}{F(1)}$$

So starting with the ratio of the first two Fibonacci numbers the next convergent to Phi is the ratio of the **next** two Fibonacci numbers.

This always happens:
if we have F(n)/F(n–1) as a convergent to Phi, then the next convergent is F(n+1)/F(n).

We will get all the ratios of successive Fibonacci numbers as values which get closer and closer to Phi.

> **You can find out more about *continued fractions* and how they relate to *splitting a rectangle into squares* and also to *Euclid's algorithm* on the Introduction to Continued Fractions page at this site.**

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Other ways to find Phi using your calculator

Here are two more interesting ways to find it.

Calculator Method 1: Invert and Add 1

Enter 1 to start the process.
Take its reciprocal (the **1/x** button). Add 1.
Take its reciprocal. Add 1.
Take its reciprocal. Add 1.
...

Keep repeating these two operations (take the reciprocal, add 1) and you will find that soon the display does not alter and settles down ("converging" as mathematicians call it) to a particular value, namely 1.61803... .

In fact, you can start with many values but not all (for instance 0 or -1 will cause problems) and it will still converge to the same value: Phi.

Why?

The formula Phi=1+1/Phi shows us where the two instructions come from.
To start, we note that the simplest approximation to the continued fraction above is just 1.

Things to do

1. In Calculator method 1, 0 causes a problem because we cannot take its reciprocal.
   So if x is -1, when we take its reciprocal ($^{1}/_{-1}$ = -1) and add 1 we get 0. So 0 and -1 are bad choices since they don't lead to Phi.
   What value of x will give -1? And what value of x would give that value?
   Can you find a whole series of numbers which, in fact, do not lead to Phi with Calculator method 1?
   [Thanks to Warren Criswell for this problem.]

Calculator Method 2: Add 1 and take the square-root

Here is another way to get Phi on your calculator.

> Enter any number (whole or fractional) but it must be bigger than –1.
> Add 1. Take its square root.
> Add 1. Take its square root.
> Add 1. Take its square root.
> ...

Keep repeating these two instructions and you will find it too converges to Phi.

Why?

This time we have used the other definition of Phi, namely

$$Phi^2 = Phi + 1$$

or, taking the square root of both sides:

$$Phi=\sqrt{(Phi+1)}$$

Can you see why we must start with a number which is not smaller (i.e. is not more negative) than –1?

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Similar numbers

Robert Kerr Baxter wrote to me about other numbers that have the Phi property that when you square them their decimal parts remain the same:

Phi = 1.618033.. and Phi^2 is 2.618033..

Phi has the value $\frac{\sqrt{5} + 1}{2}$

Rob had noticed that this happens if we replace the √5 with √13 or √17 or √21 and so on. The series of number here is 5, (9), 13, 17, 21, (25), 29, ... which are the numbers that are 1 more than the multiples of 4. The numbers 9 and 25 are in brackets because they are perfect squares, so taking their square roots gives a whole number - in fact, an *odd* number - so when we add 1 and divide the result by two we just get a whole number with .00000... as the decimal part.

Why does this happen?

Algebra can come to our help here and it is a nice application of "Solving Quadratics" that we have already seen in the first section on this page.

We want to find a formula for the numbers (x, say) "that have the same decimal part as their squares". So, if we *subtract* x from x^2, the result will be a *whole number* because the decimal parts were identical. Let's call this difference N, remembering that it is a whole number.
So

the difference between x^2 and x is N, a whole number

is a description of these numbers in words. We can write this in the language of mathematics as follows:

$x^2 – x = N$ or, adding x to both sides: $x^2 = N + x$

and we can "solve" it in exactly the same way as we did for Phi's quadratic: $x^2 = 1 + x$. The formula for x this time is

$$x = \frac{1 \pm \sqrt{(1 + 4N)}}{2}$$

You can see that, under the square-root sign, we have 1 plus a multiple of 4 which gives the series:

N:	1	2	3	4	5	...
1+4N:	5	9	13	17	21	...

just as Rob had found.

For example: if we choose N=5, then the number x (that increases by exactly 5 when squared) is

$$x = \frac{1 \pm \sqrt{(1 + 4\times5)}}{2} = \frac{1 \pm \sqrt{21}}{2} = \mathbf{2.791287847}.. \text{ and } x^2 = \mathbf{7.791287847}... = 5 + x$$

Checking we see that the square of this x is exactly N (i.e. 5) more than the original number x.

Another example: take Phi, which is $(1 + \sqrt{5})/2$ or $(1 + \sqrt{(1+4\times1)})/2$ so that N=1. Thus we can "predict" that Phi squared will be (N=)1 more than Phi itself and, indeed, Phi=1.618033.. and Phi^2=2.618033.. . We can do the same for other whole number values for N.

More generally: There is nothing in the maths of this section that prevents N from being *any number*, for instance 0·5 or π. Suppose N is pi (π=3.1415926535...). We can find the number x that, when squared, increases by exactly π! It is

$$x = \frac{1 \pm \sqrt{(1 + 4\pi)}}{2} = \frac{1 \pm \sqrt{12.566370614...}}{2} = \mathbf{2\cdot3416277185...}$$

$$\text{and } x^2 = \mathbf{5\cdot483220372}... = 2\cdot3416277185... + 3.1415926535...$$

Things to do

1. Make a table of the first few numbers similar to Phi in this way, starting with Phi and its square.
2. We have only used the + sign in the formula for x above, giving positive values of x.
   What negative values of x are there, that is negative numbers which, when squared (becoming positive) have exactly the same decimal fraction part?
3. What is the number that can be squared by just adding 0·5?
4. Is there *an upper limit* to the size of N?
   Can you use the formula to find *two* numbers that increase by one million (1,000,000) when squared?
5. Can N be *negative*?
   a. For instance, can we use the formula to find a number (as we have seen, there are two of them) that *is 0·5 smaller when it is squared?*
   b. What about a number that decreases by 1 when it is squared?
   c. Is there a *lower limit for the value of N?*

We look at some **other numbers similar to Phi but in a different way** on the (optional) Continued Fractions page. This time we find numbers which are like the Golden Mean, Phi, in that their decimal fraction parts are the same *when we take their reciprocals, ie find 1/x*. They are called the Silver Means.

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Fibonacci Home Page

This is the first page on this topic.

The Mathematical Magic of the Fibonacci Numbers

Where to now???

The next page on this Topic is...

Flat Phi Facts

The Golden String

 R.Knott@surrey.ac.uk 7 June 2001

Fascinating Flat Facts about Phi

On this page we meet some of the marvellous flat (that is, two dimensional) geometry facts related to the golden section number Phi. A following page turns our attention to the solid world of 3 dimensions.

Contents of this Page

The line means there is a **Things to do** investigation at the end of the section.

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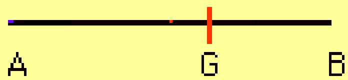
Phi and 2-dimensional geometry

Let's start by showing how to construct the golden section points on any line: first a line phi (0·618..) times as long as the original and then a line Phi (1·618..) times as long.

Constructing the golden section: phi

If we have a line with end-points A and B, how can we find the point which divides it at the golden section point?
We can do this using **compasses** for drawing circles and a **set-square** for drawing lines at right-angles to other lines, and we **don't need a ruler at all** for measuring lengths!

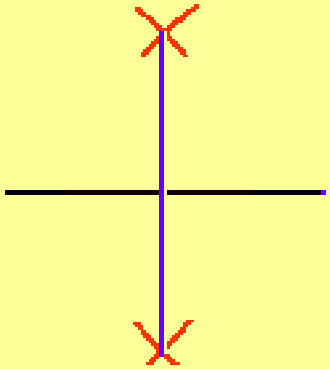
(In fact we can do it with just the compasses, but how to do it without the set-square is left as an exercise for you.)



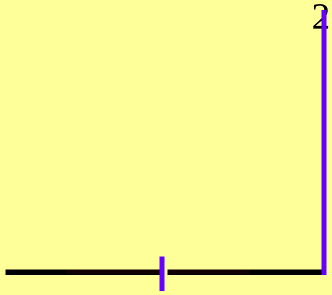
We want to find a point G between A and B so that AG:AB = phi (0·61803...) by which we mean that G is phi of the way along the line. This will also mean that the smaller segment GB is 0·61803.. times the size of the longer segment AG too.

$$\frac{AG}{AB} = \frac{GB}{AG} = \text{phi} = 0{\cdot}618033.. = \frac{\text{sqrt}(5)-1}{2}$$

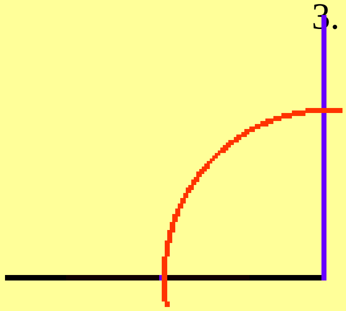
Here's how to construct point G using set-square and compasses only:



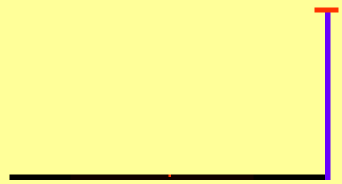
1. First we find the mid point of AB. To do this **without a ruler**, put your compasses on one end, open them out to be somewhere near the other end of the line and draw a semicircle over the line AB. Repeat this at the other end of the line **without altering the compass size**. The two points where the semicircles cross can then be joined and this new line will cross AB at its mid point.

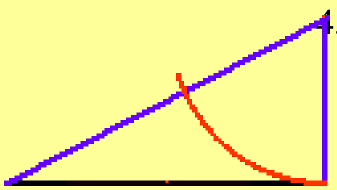


2. Now we are going to draw a line half the length of AB at point B, but at right-angles to the original line. This is where you use the set-square (but you CAN do this just using your compasses too - how?). So first draw a line at right angles to AB at end B.

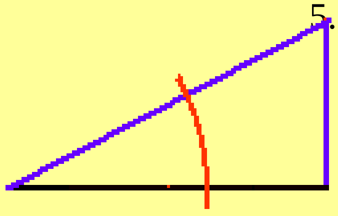


3. Put your compasses on B, open them to the mid-point of AB and draw an arc to find the point on your new line which is half as long as AB. Now you have a new line at right angles to the original and exactly half as long as the original line.

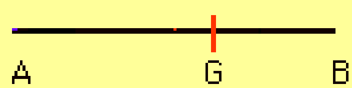




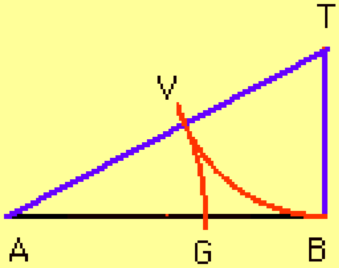
4. Join the point just found to the other end of the original line (A) to make a triangle. Putting the compass point at the top point of the triangle and opening it out to point B (so it has a radius along the right-angle line) mark out a point on the diagonal which will also be half as long as the original line.



5. Finally, put the compass point at point A, open it out to the new point just found on the diagonal and mark a point the same distance along the original line. This point is now divides the original line AB into two parts, where the longer part AG is phi (0·61803..) times as long as the original line AB.



Why does this work?



It works because, if we call the top point of the triangle T, then BT is half as long as AB. So suppose we say AB has length 1. Then BT will have length 1/2. We can find the length of the other side of the triangle, the diagonal AT, by using Pythagoras' Theorem:

$$AT^2 = AB^2 + BT^2$$

i.e.

$$AT^2 = 1^2 + (1/2)^2$$

$$AT^2 = 1 + 1/4 = 5/4$$

Now, taking the square-root of each side gives:

$$AT = (\sqrt{5})/2$$

Point V was drawn so that TV is the same length as TB = AB/2 = 1/2.
So AV is just AT - TV = $(\sqrt{5})/2 - 1/2$ = phi.
The final construction is to mark a point G which is same distance (AV) along the original line (AB) which we do using the compasses.

So AG is phi times as long as AB!

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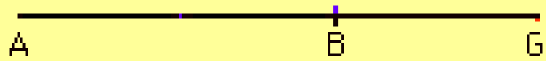
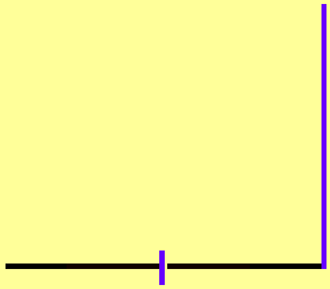
Constructing the golden section: Phi

This time we find a point outside of our line segment AB so that the new point defines a line which is Phi (1·618..) times as long as the original one.

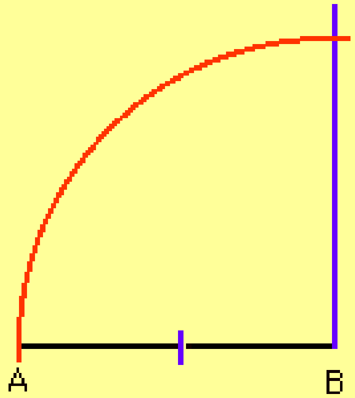


Here's how to find the new line Phi times as long as the original:

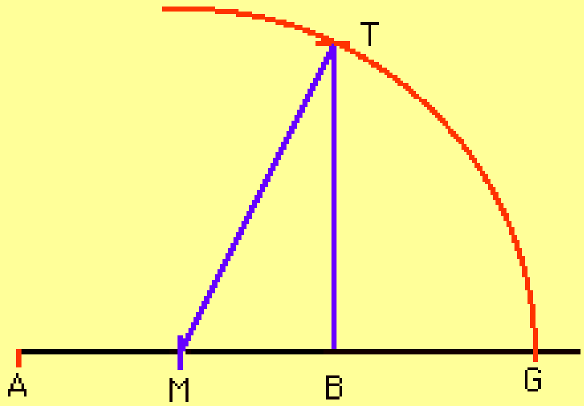
1. First **repeat the steps 1 and 2 above** so that we have found the mid-point of AB and also have a line at right angles at point B.



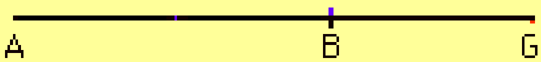
2. Now place the compass point on B and open them out to touch A so that you can mark a point T on the vertical line **which is as long as the original line**.



3. Placing the compass point on the mid-point M of AB, open them out to the new point T on the vertical line and draw an arc on the original line **extended past point B** to a new point G.



4. The line AG is now Phi times as long as the original line AB.



Why does this work?
If you followed the reasoning for why the first construction (for phi) worked, you should find it quite easy to prove that AG is Phi times the length of AB, that is, that AG = (sqrt(5)/2 + 1/2) times AB.

> Hint:
> Let AB have length 1 again and so AM=MB=1/2. Since BT is now also 1, how long is MT? This is the same length as MG so you can now find out how long AG is since AG=AM+MG.

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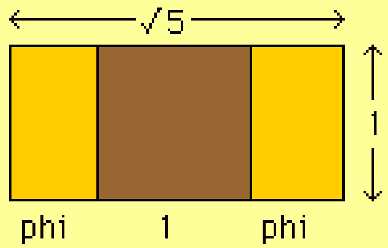
Phi and the Root-5 Rectangle

If we draw a rectangle which is 1 unit high and √5 long, that is, about 2·236 units long, we can draw a square in it, which, if we place it centrally, will leave two rectangles left over. Each of these will be phi=0·618.. units wide and, of course, 1 unit high.

Since we already know that the ratio of 1 to phi(=0·618) is the same as Phi(=1·618..)to 1, then the two rectangles are **Golden** Rectangles.

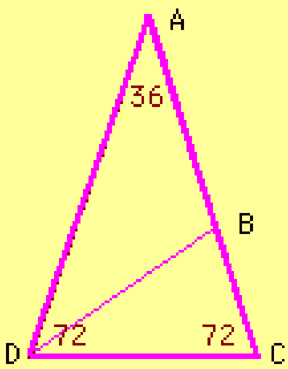


This is nicely illustrated on Ironheart Armoury's Root Rectangles page where he shows how to construct all the rectangles with width any square root, starting from a square.

This rectangle is supposed to have been used by some artists as it is another pleasing rectangular shape, like the golden rectangle itself.

Pentagons and Pentagrams

We can prove that AB:BC is the golden ratio:



In this diagram, the triangle ACD is isosceles, since the two sides, AC and AD, are equal as are the two angles ADC and ACD.
[Also, angles ADC and ACD are twice angle CAD.]

If we bisect the base angle at D by a line from D to point B on AC then we have the angles as shown. BDC is then an isosceles triangle so CD=BD.

Since ABD is also isosceles, then DB=AB also, so CD=BD=AB.

We also note that triangles BCD and CDA are similar since their angles are equal. AB=CD so

$$\frac{AB}{BC} = \frac{CD}{BC}$$

which is the ratio of the lengths of the long side to the base in a 36°-72°-72° triangle.
In the 36°-72°-72° degree triangle ADC, it is the same as the ratio of AC to CD, so:

$$\frac{CD}{BC} = \frac{AC}{CD}$$

We have shown that CD=AB so now

$$\frac{CD}{BC} = \frac{AC}{CD} = \frac{AC}{AB}$$

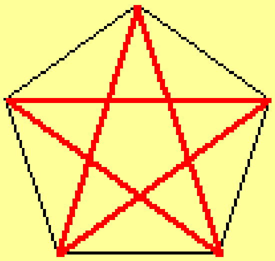
Putting these equalities together we have:

$$\frac{AB}{BC} = \frac{CD}{BC} = \frac{AC}{CD} = \frac{AC}{AB} = r, \text{ say}$$

and we have called this ratio r.

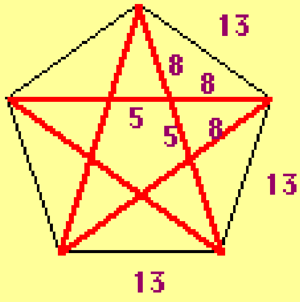
If we let BC be of length 1, then we have AB=r (since AB/BC=r above) and AC=AB+BC=1+r, or: r/1=(1+r)/r, ie $r^2=1+r$, the equation which defined the golden ratio (and a negative quantity, but lengths are positive).

Pentagrams contain this triangle



If we look at the way a pentagram is constructed, we can see there are lots of lines divided in the golden ratio: Since the points can be joined to make a pentagon, the golden ratio appears in the pentagon also and the relationship between its sides and the diagonals (joining two non-adjacent points).

The reason is that Phi has the value 2 cos (pi/5) where the angle is described in radians, or, in degrees, Phi=2 cos (36°).
[See below for more angles whose sines and cosines involve Phi!]



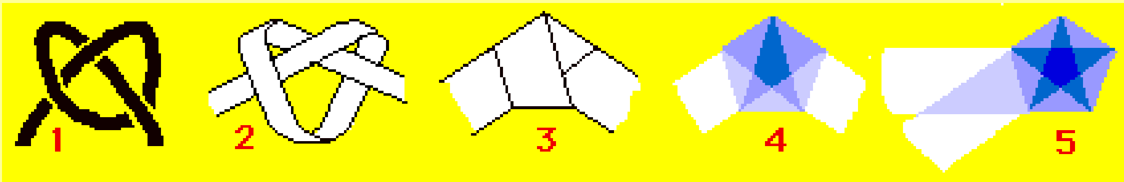
Since the ratio of a pair of consecutive Fibonacci numbers is roughly equal to the golden section, we can get an approximate pentagon and pentagram using the Fibonacci numbers as lengths of lines:

There is another flatter triangle inside the pentagon here. Has this any golden sections in it? Yes! We see where further down this page, but first, a quick and easy way to make a pentagram without measuring angles or using compasses:

Making a Paper Knot to show the Golden Section in a Pentagon

Here's an easy method to show the golden section by making a **Knotty Pentagram**; it doesn't need a ruler and it doesn't involve any maths either!
Take a length of paper from a roll - for instance the type that supermarkets use to print out your bill - or cut off a strip of paper a couple of centimetres wide from the long side of a piece of paper. If you tie a knot in the strip and put a strong light behind it, you will see a pentagram with all lines divided in golden ratios.

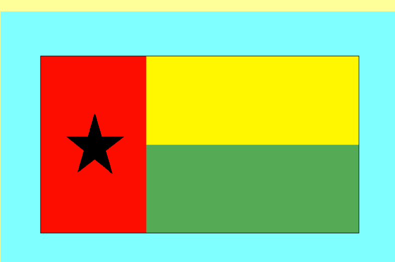
This is my favourite method since it involves a Knot(t)!



Here are 5 pictures to help (well it is a pentagram so I had to make 5 pictures!) - although it really is easy once you practice tying the knot!

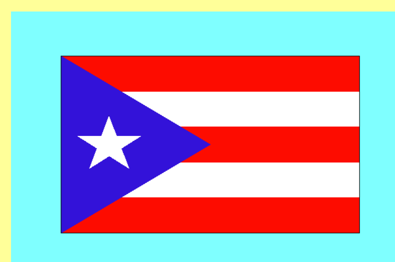
1. As you would tie a knot in a piece of string ...
2. ... gently make an over-and-under knot, rolling the paper round as in the diagram.
3. (This is the slightly tricky bit!)
   Gently pull the paper so that it tightens and you can crease the folds as shown to make it lie perfectly flat.
4. Now if you hold it up to a bright light, you'll notice you *almost* have the pentagram shape - one more fold reveals it ...
5. Fold the end you pushed through the knot back (creasing it along the edge of the pentagon) so that the two ends of the paper almost meet. The knot will then hang like a medal at the end of a ribbon. Looking through the knot held very close of a desk-light or table lamp will show a perfect pentagram, just like the (red) diagram above.

Flags of the World and Pentagram stars



Here are two flags with just one 5-pointed star: Guinea-Bissau (left) and Puerto Rico (right). They are part of a larger (but incomplete) collection in Australia.

How many five-pointed stars are there on the USA flag? Why?



Things to do

1. Many countries have a flag which contains the 5-pointed star above.
   To help, try the Flags Of The World alphabetical list of countries or use this map of the world
2. Some countries have a flag with a star which does not have 5 points: Which country has a six-pointed star in its flag?
3. Find all those countries with a flag which has a star of more than 6 points.

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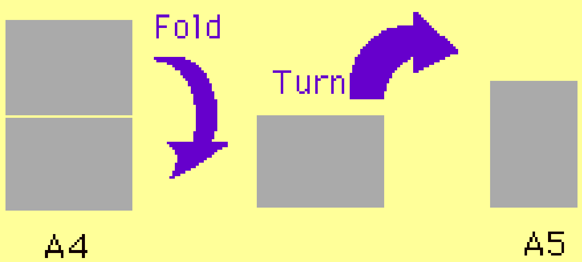
The shape of a piece of paper

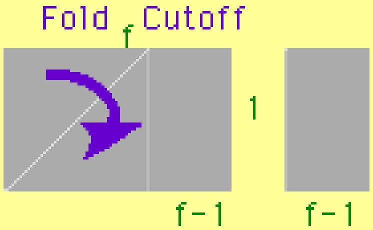
Modern paper sizes have sides that are in the ratio 1:sqrt(2). This means that they can be folded in half and the two halves are still *exactly the same shape*. Here is an explanation of why this is so:

"A" series Paper



Take a sheet of A4 paper.
Fold it in half from top to bottom.
Turn it round and you have a smaller sheet of paper of exactly the same shape as the original, but half the area, called A5.
Since its area is exactly half the original, its sides are sqrt(1/2) of the originals, or, an A4 sheet has sides sqrt(2) times bigger than a sheet of A5.
Do this on a large A3 sheet and you get a sheet of size A4.
The sides must be in the ratio of 1:sqrt(2) since if the original sheet has the shorter side of length 1 and the longer side of length s, then when folded in half the short-to-longer-side ratio is now s/2:1.
By the two sheets being of the same shape, we mean that the ratio of the short-to-long side is the same. So we have:
1/s = s/2 /1 which means that s^2=2 and so s is sqrt(2).

Fibonacci paper



If we take a sheet of paper and fold a corner over to make a square at the top and then cut off that square, then we have a new smaller piece of paper.

If we want the smaller piece to have the same shape as the original one, then, if the longer side is length f and the short side length 1 in the original shape, the smaller one will have shorter side of length f-1 and longer side of length 1.
So the ratio of the sides must be the same in each if they have the same shape: we have 1/f = (f-1)/1 or, f^2-f=1 which is exactly the equation from which we derived Phi.
Thus if the sheets are to have the same shape, their sides must be in the ratio of 1 to Phi, or, the sides are approximately two successive Fibonacci numbers in length!

Here is another site on paper sizes.

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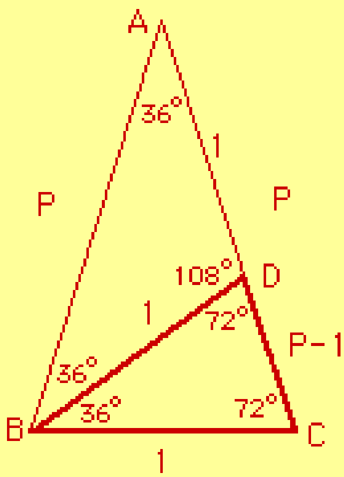
Phi and the Pentagon Triangle



Earlier we saw that the triangle shown here occurs in the pentagon and decagon. If the shorter side is of unit length and we say the longer side has length P, we can calculate P, the ratio of the longer to shorter sides of this "sharply pointed" isosceles triangle (i.e. two sides of the triangle are equal and therefore two of its angles are also). We do this by introducing a point D on side AC.
We choose it so that it makes BD of length 1 also, so BCD is isosceles too.
So we can write in its angles (BDC = 72° also leaving 180°-72°-72°=36° for angle DBC). In other words

Triangle BCD is the same shape as triangle ABC

since their angles are equal. We also see that BD halves the 72 degree angle ABC, so ABD has two angles equal and it too is isosceles. This means that sides AD and BD are equal too, so AD is of length 1 also.
Now we deduce that BD is of length P-1 since AC is of length P and AB is of length 1.
All we have done is justify the numbers and angles on the diagram here.

Now to calculate P!

Since BCD is the same shape as ABC, their sides are in the same ratios.
So the longer-side-to-shorter-side ratio in BCD is also P, i.e.

```
BD/DC=1/(P-1)= P
```

```
or          1      = P(P-1) = P2-P
or          0      = P2-P-1
```

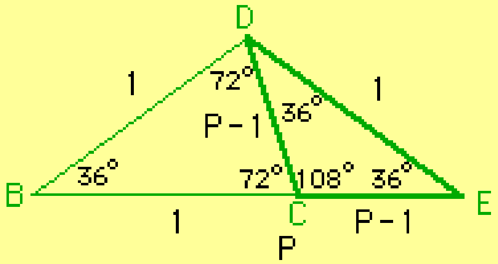
Refer back to the Fibonacci and Geometry section above where we solved this equation to get

$$P=(1+\sqrt{5})/2 = 1\cdot6180339...$$
or
$$P=(1-\sqrt{5})/2 = -0\cdot6180339..$$

Clearly a side of negative length does not apply here, so the first value is the unique value of P, the unique ratio of the sides of the triangle ABC.

So we see **P was just Phi all along!**

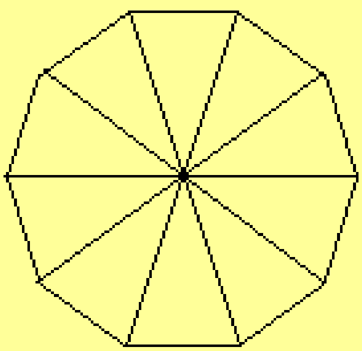
Phi and another Isosceles triangle



If we copy the BCD triangle from the red diagram above (the 36°-72°-72° triangle), and put another triangle on the side as we see in this green diagram, we are again using P=Phi as above and get a similar shape - another isosceles triangle - but a "flat" triangle.
The red triangle of the pentagon has angles 72°, 72° and 36°, this green one has 36°, 36°, and 72°.
Again the ratio of the shorter to longer sides is Phi, but the two equal sides here are the shorter ones (they were the longer ones in the "sharp" triangle).

These two triangles are the basic building shapes of Penrose tilings (see the section mentioned previously for more references). They are a 2-dimensional analogue of the golden section and make a very interesting study in their own right. They have many relationships with both the Fibonacci numbers and Phi.

Decagons



Here is a decagon - a 10-sided regular polygon with all its angles equal and all its sides the same length - which has been divided into 10 triangles.
Because of its symmetry, all the triangles have two sides that are the same length and so the two other angles in each triangle are also equal.
In each triangle, what is the size of the angle at the centre of the decagon?
We now know enough to identify the triangle since we know one angle and that the two sides surrounding it are equal. *Which triangle on this page is it?*

From what we have already found out about this triangle earlier, we can now say that

> The radius of a circle through the points of a decagon is Phi times as long as the side of the decagon.

This follows directly from Euclid's **Elements** Book 13, Proposition 9.

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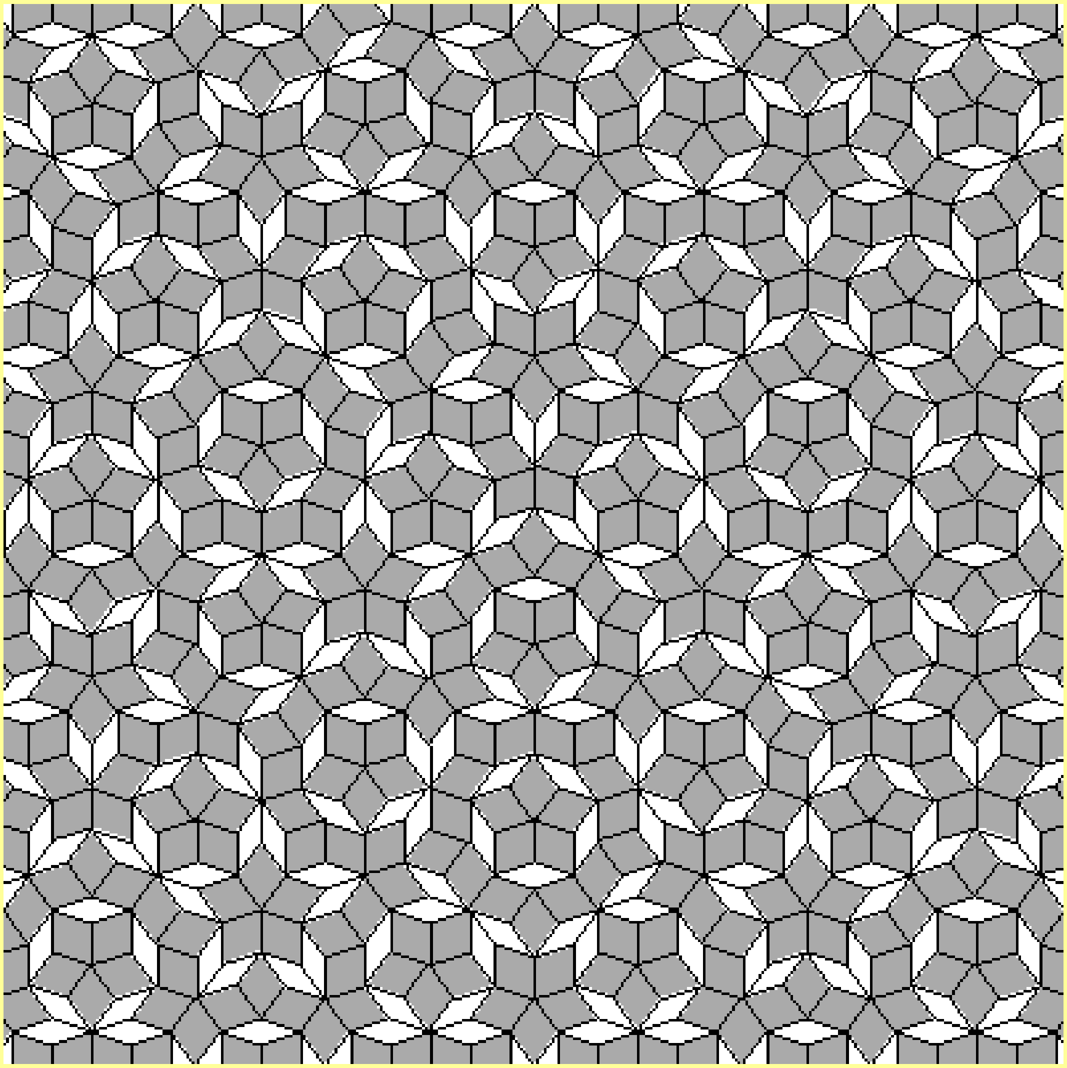
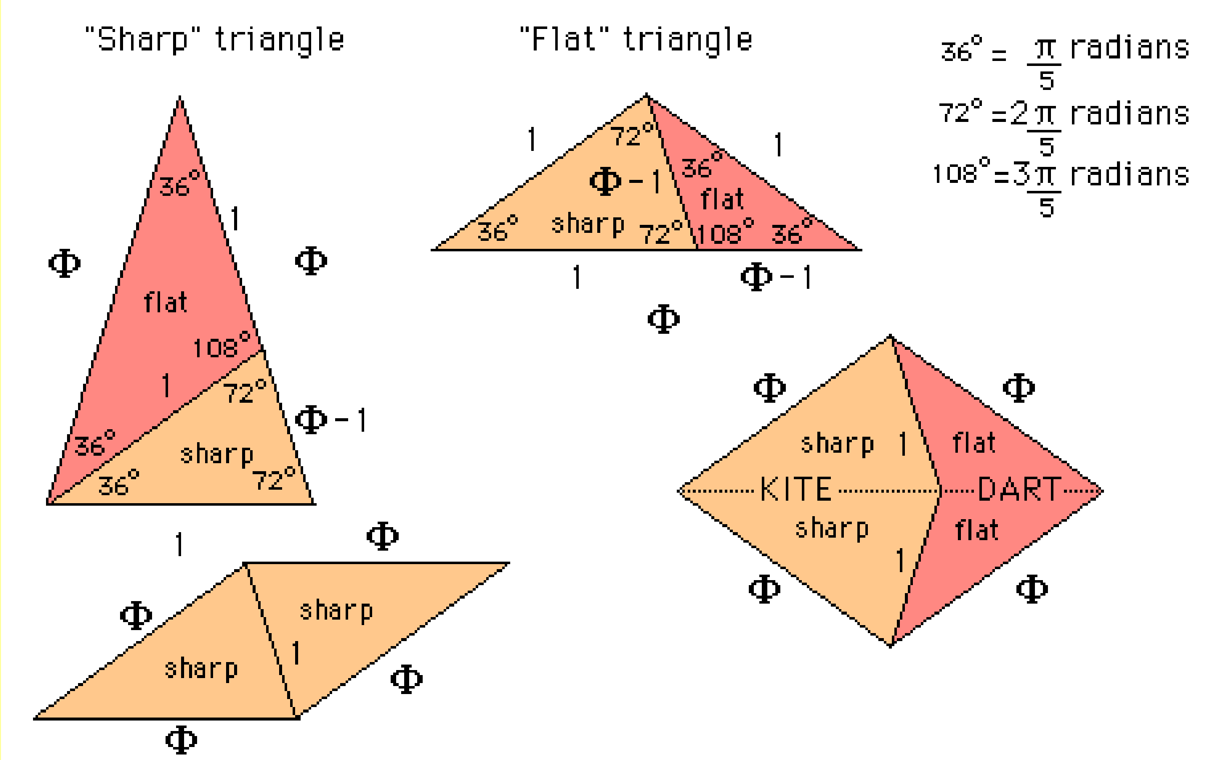
Penrose tilings



Recently, Prof Roger Penrose has come up with some tilings that exhibit five-fold symmetry yet which do not repeat themselves for which the technical term is **aperiodic** or **quasiperiodic**. When they appear in nature in crystals, they are called **quasicrystals**. They were thought to be impossible until fairly recently.
There is a lot in common between Penrose's tilings and the Fibonacci numbers.
The picture above is made up of two shapes of *rhombus* or *rhombs* - that is, "pushed over squares" where each shape has all sides of the same length. The two rhombs are made from glueing two of the flat

pentagon triangles together along their long sides and the other from glueing two of the sharp pentagon triangles together along their short sides.



This picture is part of the Hypercard stack developed by me (Ron Knott) available from this site. [Download 156K binhex file.] The tiling picture was made with Quasitiler 3.0 which is a web-based tool and its link mentions more references to Penrose tilings.
A floor has been tiled with Penrose Rhombs at Wadham College at Oxford University.

I plan more to follow here, but in the meantime, here are some interesting links to the Penrose tilings at other sites.

- The Golden section and Penrose Tilings .
- Here are some ready-to-photocopy Penrose tiles for you to photocopy and cut-out and experiment with making tiling patterns.
- Puzzles to buy from Pentaplex (UK)
- Penrose's rhombs (a fat and a thin diamond) tilings.

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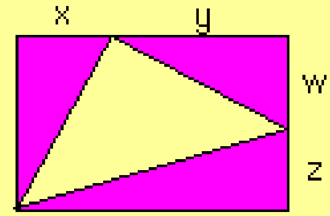
A Rectangle-Triangle dissection Problem



The problem is, given a rectangle, to cut off **three triangles from the corners** of the rectangle so that **all three triangles have the same area**. Or, expressed another way, to find a triangle inside a given rectangle (any rectangle) which when it is removed from the rectangle leaves three triangles *of the same area*.
As shown here, the area of the leftmost triangle is $x(w+z)/2$.
The area of the top-right triangle is $yw/2$.
The area of the bottom triangle is $(x+y)z/2$.
Making these equal means:

$$x(w+z) = yw \text{ and } x(w+z)= z(x+y).$$

The first equation tells us that $x = yw/(w+z)$.
The second equation, when we multiply out the brackets and cancel the zx terms on each side, tells us that $xw=zy$. This means that $y/x=w/z$.

Putting this in other words, we have our first deduction that

Both sides of the rectangle are divided in the same proportion.

Returning to $xw=zy$, we put $x = yw/(w+z)$ into it giving $yw^2/(w+z)=zy$.
We can cancel y from each side and rearrange it to give $w^2 = z^2 + zw$.
If we divide by z^2 we have a quadratic equation in w/z.
Let $X=w/z$ then $X^2 = 1 + X$.
The positive solution of this is $X = Phi$, that is, $w = z\ Phi$. Since we have already seen that $y/x=w/z$ then:

Each side of the rectangle is divided in the same ratio
This ratio is Phi = 1·6180339... ie 1:1·618 or 0·618:1.

The Golden Section strikes again!

This puzzle appeared in J A H Hunter's **Triangle Inscribed In a Rectangle** in *The Fibonacci Quarterly*, Vol 1, 1963, page 66.
A follow-up article by H E Huntley entitled **Fibonacci Geometry** in volume 2 (1964) of the *Fibonacci Quarterly* on page 104 shows that, if the rectangle is itself a golden rectangle (the ratio of the longer side to the shorter one is Phi) then the triangle is both isosceles and right-angled!

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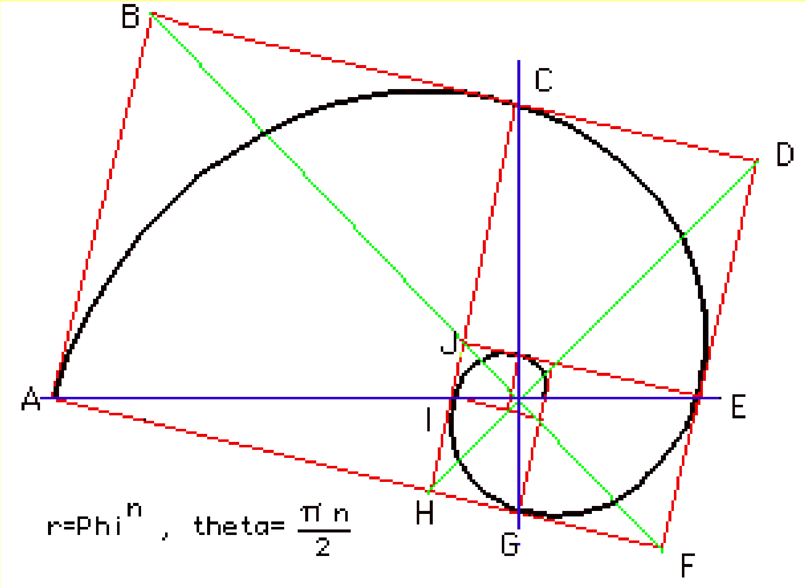
The Golden Spiral

On the Fibonacci Numbers and Golden Section in Nature page, we looked at a spiral formed from squares whose sides had Fibonacci numbers as their lengths.

This section answers the question:

What is the equation of the Golden spiral?



The Golden section squares are shown in red here, the axes in blue and all the points of the squares lie on the green lines, which pass through the origin (0,0).
Also, the blue (axes) lines and the green lines are each separated from the next by 45° exactly.
The large rectangle ABDF is the same shape as CDFH, but is phi times as large. Also it has been rotated by a quarter turn. Similarly with CDFH and HJEF. This applies to all the golden rectangles in the diagram.
So to transform OE (on the x axis) to OC (on the y axis), and indeed any point on the spiral to another point on the spiral, we expand lengths by phi times for every rotation of 90°: that is, we change (r,theta) to (r Phi,theta+Pi/2) (where, as usual, we express angles in radian measure, not degrees).

So if we say E is at (1,0), then C is at (Phi,Pi/2), A is at (Phi^2, Pi), and so on.
Similarly, G is at (phi,-pi/2), and I is at (phi^2, -pi) and so on because phi is 1/Phi.

The points on the spiral are therefore summarised by:

$$r = Phi^n \text{ and } theta = n\ Pi/2$$

If we eliminate the n in the two equations, we get a single equation that all the points on the spiral satisfy:

$$r = Phi^{2\ theta / Pi}$$

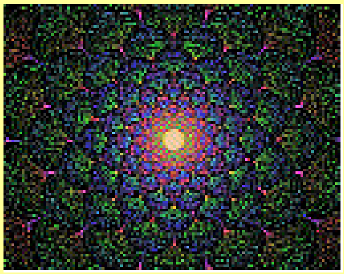
or

$$r = M^{theta} \text{ where } M = Phi^{2/Pi}$$

Such spirals, where the distance from the origin is a constant to the power of the angle, are called **equiangular spirals**, that is, a line from the origin to any point on the curve always finds (the tangent to) the curve meeting it at the *same angle*.
Coxeter states that:

> This true spiral is closely approximated by the artificial spiral formed by circular quadrants inscribed in the successive squares, as in [the figure above]. (But the true spiral cuts the sides of the squares at very small angles, instead of touching them.)

The above is adapted from H S M Coxeter's book **Introduction to Geometry**, 1961, page 165.]

 Ned May has generated some beautiful pictures based on Fibonacci Spirals using Visual Basic.

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Trigonometry and Phi

What is *trigonometry*?
We can answer this by looking at the origin of the word **trigonometry**.
Words ending with **-metry** are to do with *measuring* (from the greek word *metron* meaning "measurement"). (What do you think that *thermometry* measures? What about *geometry*? Can you think of any more words ending with -metry?)
Also, the **-gon** part comes from the greek *gonia*) meaning *angle*. It is derived from the greek word for "knee" which is *gony*.

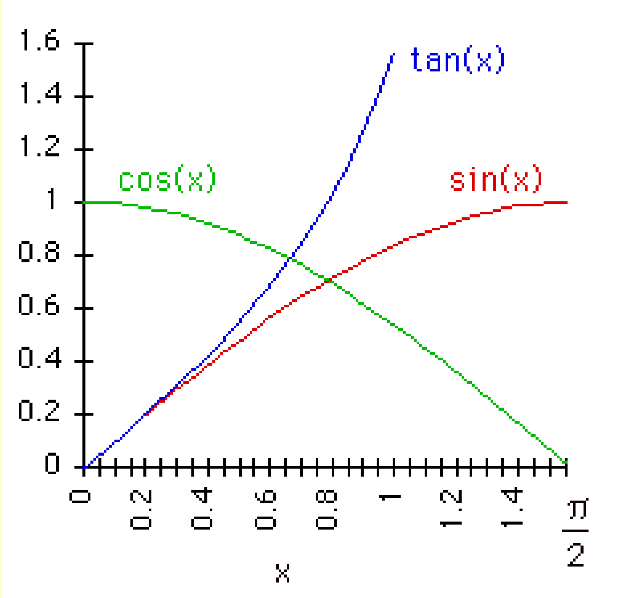
The prefix **tri-** is to do with *three* as in tricycle (a three-wheeled cycle), trio (three people), trident (a three-pronged fork).
Similarly, **quad** means 4, **pent** 5 and **hex** six as in the following:

- a (five-sided and) five-angled shape is a **penta-gon** meaning literally *five-angles* and
- a six angled one is called a **hexa-gon** then we could call
- a four-angled shape a **quadragon**
  (but we don't - using the word **quadrilateral** instead which means "four-**sided**") and
- a three-angled shape would be a **tria-gon**
  (but we call it a **triangle** instead)
  "Trigon" was indeed the old english word for a triangle.

So *trigonon* means "three-angled" or, as we would now say in English, "tri-angular" and hence we have *tri-gonia-metria* meaning "the measurement of triangles".
With thanks to proteus of softnet for this information.

Phi and Trig graphs

Here are the graphs of three familiar trigonometric functions: sin x, cos x and tan x in the region of x from 0 to pi/2 (radians) = 90°:



The graphs meet at

- the origin, where tan x = sin x
- in the middle, where sin x = cos x ie where tan x = 1 or x = 45° = pi/4 radians
- at another angle where tan x = cos x

What angle is at the third meeting point?

tan x = cos x and, since tan x = sin x / cos x, we have:

sin x =(cos x)^2

=1-(sin x)^2 because (sin x)^2+(cos x)^2=1.

or

(sin x)^2 + sin x = 1

and solving this as a quadratic in sin x, we find

sin x = (-1+√5)/2

or sin x = (-1-√5)/2

The second value is negative and our graph picture is for positive x, so we have our answer:

the third point of intersection is the angle whose sine is Phi-1=0·6180339...=phi which is about 0·66623943.. radians or 38·1727076..°

On our graph, we can say that the intersection of the green and blue graphs (cos(x) and tan(x)) is where the red graph (sin(x)) has the value phi [i.e. at the x value of the point where the line y=phi meets the sin(x) curve].

Is there any significance in the value of tan(x) where tan(x)=cos(x)?

Yes. It is √phi = √0·618033988... = 0·786151377757.. .

Things to do

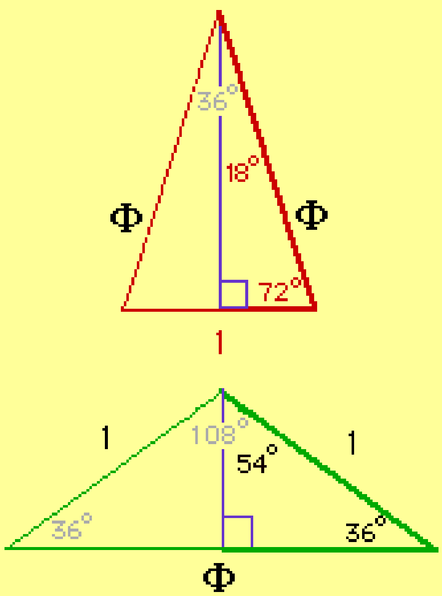
1. Extend the graph above to include
   i. **sec(x)** defined as 1/cos(x)
   ii. **cosec(x)** defined as 1/sin(x)

iii. **cot(x)** defined as 1/tan(x).
   Find the points of intersection of these with themselves and with the other 3 trig functions.
2. In your graph of the question above, can you find these values at points of intersection?
   a. Phi = 1.618033...
   b. √Phi = 1.2720196495...
   c. √2 = 1.414213562...
3. In your answer to the previous question, can you *prove* that the points of intersection really are the exact values given above?

Some Results in Trigonometry, Brother L Raphael, *The Fibonacci Quarterly* vol 8 (1970) pages 371 and 392.

Other angles related to Phi

Look again at the sharp and flat triangles of the pentagon that we saw above. If we divide each in half, we have right angled triangles with sides 1 and Phi/2 around the 36° angle in the flat triangle and sides 1/2 and Phi around the 72° angle in the sharp triangle. So:



$$\cos(72°) = \cos\left(\frac{2\pi}{5}\right) = \sin(18°) = \sin\left(\frac{\pi}{10}\right) = \frac{\phi}{2} = \frac{1}{2\Phi}$$

$$\cos(36°) = \cos\left(\frac{\pi}{5}\right) = \sin(54°) = \sin\left(\frac{3\pi}{10}\right) = \frac{\Phi}{2} = \frac{1}{2\phi}$$

We have sin(18°) but what about cos(18°)? This has a somewhat more awkward expression as:

$$\cos(18°) = \frac{\sqrt{\text{Phi}\,\sqrt{5}}}{2}$$

Now we know the sin and cos of both 30° and 18° we can find the sin and cos of their difference using:

$$\cos(A–B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

and get:

$$\cos(12°) = \frac{-\phi + \sqrt{3\Phi\sqrt{5}}}{4}$$

AAAAgh! as Snoopy might have said.

Is there a neater (that is, a simpler) expression? Perhaps you can find one. Let me know if you do and it will be added here with your name!

> This form of cos(12°) is derived from the expression on page 42 of
> **Roots of (H-L)/15 Recurrence Equations in Generalized Pascal Triangles** by C Smith and V E Hoggatt Jr. in *The Fibonacci Quarterly* vol 18 (1980) pages 36-42.

What about other angles? From an equilateral triangle cut in half we can easily show that:

$$\cos(60°) = \quad \sin(30°) = \frac{1}{2}$$

$$\cos(30°) = \quad \sin(60°) = \frac{\sqrt{3}}{2}$$

and from a 45-45-90 degree triangle we can derive:

$$\cos(45°) = \sin(45°) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

and not forgetting, of course:

$$\sin(0°) = \cos(90°) = 0$$
$$\sin(90°) = \cos(0°) = 1$$

Can you find any more angles that have an *exact* expression (not necessarily involving Phi or phi)? Let me know what you find and let's get a list of them here.

A Purely Trigonometric Formula for Fib(n)

These formulae can lead us to a way of writing Binet's Formula:

$$\text{Fib}(n) = \frac{\text{Phi}^n - (-\text{Phi})^{-n}}{\sqrt{5}} = \frac{\text{Phi}^n - (-\text{phi})^n}{\sqrt{5}}$$

purely in terms of trig. functions. First we have:

$$\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{3\pi}{5}\right) = \frac{\sqrt{5}}{4} \quad \text{and} \quad \sin\left(\frac{3\pi}{5}\right)\sin\left(\frac{9\pi}{5}\right) = -\frac{\sqrt{5}}{4}$$

and so Binet's formula above (in its second form) becomes:

$$\text{Fib}(n) = \frac{2^{n+2}}{5}\left(\cos^n\left(\frac{\pi}{5}\right)\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{3\pi}{5}\right) + \cos^n\left(\frac{3\pi}{5}\right)\sin\left(\frac{3\pi}{5}\right)\sin\left(\frac{9\pi}{5}\right) \right)$$

or, if you prefer degrees rather than radians:

$$\text{Fib}(n) = \frac{2^{n+2}}{5}\left(\cos^n(36°)\sin(36°)\sin(108°) + \cos^n(108°)\sin(108°)\sin(-36°) \right)$$

Can you see how this is just Binet's form re-written?

See **Fibonacci in Trigonometric Form** Problem B-374 proposed and solved by F Stern in *The*

Fibonacci Quarterly vol 17 (1979) page 93 where another form is also given.

Phi and Powers of Pi

There is a simple (infinite) series for calculating the cosine and the sine of an angle where the angle is expressed in radians. See Radian Measure (the link opens in a new window - close it to return here) for a fuller explanation.
Basically, instead of 360 *degrees* in a full turn there are 2π *radians*. The radian measure makes many trigonometric equations simpler and so it is the preferred unit of measuring angles in mathematics.

If *angle x is measured in radians* then

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Here, **n!** means *the factorial of n* which means ***the product of all the whole numbers from 1 to n***.
For example, 4! means 1x2x3x4 which is 24.

So, using the particular angles above in sin(pi/10) and cos(pi/5) we have formulae for phi (ϕ) and Phi (Φ) in terms of powers of pi (π):-

$$\begin{aligned} \phi &= 2\sin\left(\frac{\pi}{10}\right) \\ &= 2\left(\frac{\pi}{10} - \frac{\pi^3}{10^3 3!} + \frac{\pi^5}{10^5 5!} - \frac{\pi^7}{10^7 7!} + \dots\right) \\ &= \frac{\pi}{5} - \frac{\pi^3}{3{,}000} + \frac{\pi^5}{6{,}000{,}000} - \frac{\pi^7}{25{,}200{,}000{,}000} + \dots \end{aligned}$$

$$\begin{aligned} \Phi &= 2\cos\left(\frac{\pi}{5}\right) \\ &= 2\left(1 - \frac{\pi^2}{5^2 2!} + \frac{\pi^4}{5^4 4!} - \frac{\pi^6}{5^6 6!} + \frac{\pi^8}{5^8 8!} - \dots\right) \\ &= 2 - \frac{\pi^2}{25} + \frac{\pi^4}{7{,}500} - \frac{\pi^6}{5{,}625{,}000} + \frac{\pi^8}{7{,}875{,}000{,}000} - \dots \end{aligned}$$

In the upper formula, going to up to the pi^9 term only will give phi to 9 decimal places whereas stopping at the pi^8 term in the lower formula will give Phi to 7 decimal places.

These two formula easily lend themselves as an iterative method for a computer program (i.e. using a loop) to compute Phi and phi. To compute the next term from the previous one, multiply it by $(pi/5)^2$ [or $(pi/10)^2$ for phi] and divide by two integers to update the factorial on the bottom, remembering to add the next term if the previous one was subtracted and vice versa. Finally multiply your number by 2.

You will need and an accurate value of Pi. Here is Pi to 102 decimal places:

```
3. 14159 26535 89793 23846 26433 83279 50288 41971 69399 37510
   58209 74944 59230 78164 06286 20899 86280 34825 34211 70679
   82..
```

With thanks to John R Goering for suggesting this connection between Phi and pi.

1·61803 39887 49894 84820 45868 34365 63811 77203 09179 80576 ..More..

Links to other sites

- The golden section, geometry, Penrose tilings by Rashomon has some more pictures of Penrose tilings.
- Steve Finch's page about the Golden section starts with the material on these pages but he also has some interesting results about the Fibonacci spiral and some truly remarkable formulae of Srinivasan Ramanujan, the famous Indian mathematician who died in 1920. The formulae relate **e**=2·71828.. , **pi**=3·14159.. and **Phi**=1·6180339.. and, like me, you can just admire them if you can't understand them!
- Kyungsoon Jeon at the University of Georgia has an excellent article about Phi and the Fibonacci series and how to investigate it using a Spreadsheet.
- Domingo Gómez Morín's Isosceles-Fibonacci partition page shows how to construct points on any line AB which divide it into AB/2, AB/3, AB/5, AB/8, and so on, where the denominators are the Fibonacci numbers.

1·61803 39887 49894 84820 45868 34365 63811 77203 09179 80576 ..More..

Fibonacci Home Page

The Golden Section - the Number and Its Geometry

The Mathematical Magic of the Fibonacci Numbers

Where to now?

The next topic is...
The Golden String

Phi in 3 dimensions

 R.Knott@surrey.ac.uk last update:7 June 2001

Some Solid (Three-dimensional) Geometrical Facts about the Golden Section

Having looked at the flat geometry (two dimensional) of the number Phi, we now find it in the most symmetrical of the three-dimensional solids - the Platonic Solids.

Contents of this Page

The line means there is a **Things to do** investigation at the end of the section.

1·61803 39887 49894 84820 45868 34365 63811 77203 09179 80576 ..More..

Phi and 3-dimensional geometry

The five regular solids (where "regular" means all sides are equal and all angles are the same and all the faces are identical) are called the five Platonic solids after the Greek philosopher and mathematician, Plato. Euclid also wrote about them. For more information on these two famous Greeks, see the footnote.

Dice shapes

What shapes make the best dice?

We need to make sure all the faces are the same shape and that all the angles and sides are equal, or some faces will be favoured more than others and so our dice will be "unfair".
The dice you usually find today are cube-shaped - 6 square faces, all angles are right-angles and all sides are the same length.

[There *are* other shapes that make fair dice if we relax these conditions a little. Can you guess what they are? See the footnote for the answers.]

There are only FIVE fair-dice-shapes altogether if we strictly insist on the following conditions:

all sides are equal in length and
all angles are equal so that
all the faces are identical in shape and size

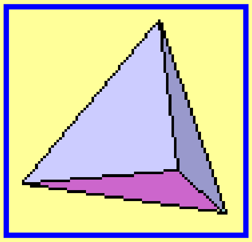
Coordinates and other statistics of the 5 Platonic Solids

They are the tetrahedron, cube (or hexahedron), octahedron, dodecahedron and icosahedron.

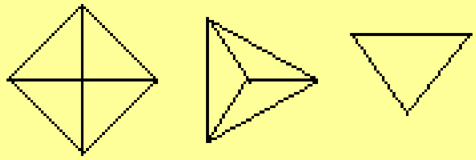
Their names come from the number of faces (hedron=face in Greek and its plural is hedra). tetra=4, hexa=6, octa=8, dodeca=12 and icosa=20.

Clicking on the image gets you an animation of the object

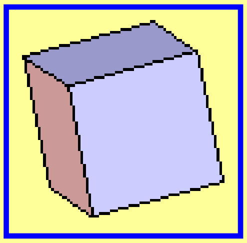
The Tetrahedron



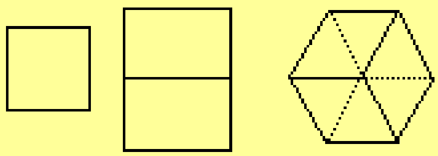
A **tetrahedron** of edge length sqrt(8) has coordinates
`(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1).`
4 points, 6 edges, 4 faces.

Other views: 

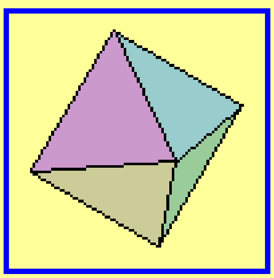
The Cube or Hexahedron



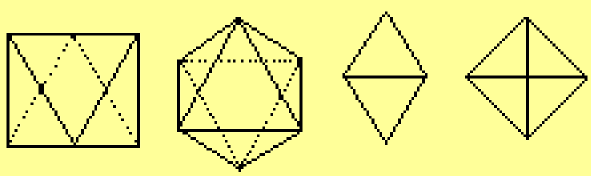
A **cube** (or **hexahedron**) of edge length 2 has coordinates:
`(1, 1, 1), (1, 1, -1), (1, -1, 1), (1, -1, -1),`
`(-1, 1, 1), (-1, 1, -1), (-1, -1, 1), (-1, -1, -1).`
8 points, 12 edges, 6 faces.

Other views: 

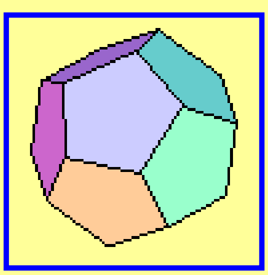
The Octahedron



An **octahedron** of edge length sqrt(2) has coordinates
`(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1),`
`(0, 0, -1).`
6 points, 12 edges, 8 faces.

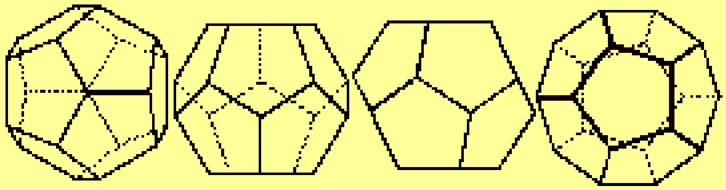
Other views: 

The Dodecahedron

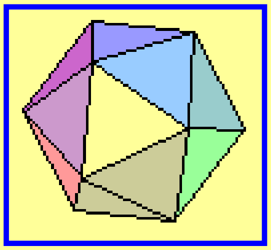


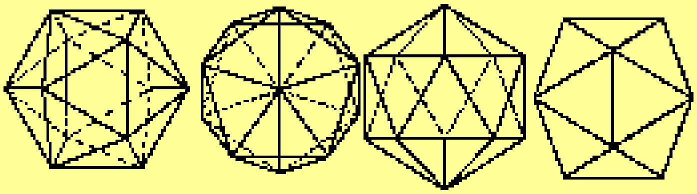
The **dodecahedron** of side 2/Phi has coordinates
`(0, phi, Phi), (0, phi, -Phi), (0, -phi, Phi), (0, -phi, -Phi),`
`(Phi, 0, phi), (Phi, 0, -phi), (-Phi, 0, phi), (-Phi, 0, -phi),`
`(phi, Phi, 0), (phi, -Phi, 0), (-phi, Phi, 0), (-phi, -Phi, 0),`
`(1, 1, 1), (1, 1, -1), (1, -1, 1), (1, -1, -1),`

(-1, 1, 1), (-1, 1, -1), (-1, -1, 1), (-1, -1, -1).
20 points, 30 edges, 12 faces. where Phi=1·61803.. and phi=1/Phi=Phi-1=0·61803....

Other views: 

The Icosahedron

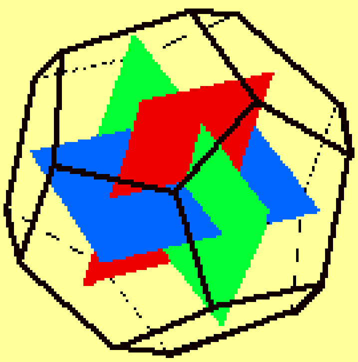

The **icosahedron** of side 2 is defined by coordinates
(0, 1, Phi), (0, -1, Phi), (0, 1, -Phi), (0, -1, -Phi),
(Phi, 0, 1), (Phi, 0, -1), (-Phi, 0, 1), (-Phi, 0, -1),
(1, Phi, 0), (1, -Phi, 0), (-1, Phi, 0), (-1, -Phi, 0).
where Phi is the golden ratio (1·61803..).
12 points, 30 edges, 20 faces.

Other views: 

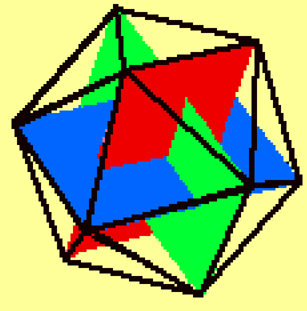
The Dual of a Solid

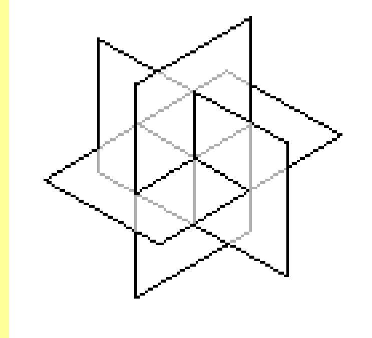
There are two more important relationships between the dodecahedron and the icosahedron. First, the mid-points of the faces of the dodecahedron define the points on an icosahedron and the mid-points of the faces of an icosahedron define a dodecahedron. The same is true of the cube and the octahedron. If we try it with a tetrahedron, we just get another tetrahedron. Each is called the **dual** of the other solid where the number of edges in each pair is the same, but the number of faces of one is the number of points of the other, and vice-versa.

Golden sections in the Dodecahedron, Icosahedron and Octahedron



If we join mid-points of the dodecahedron's faces, we can get three rectangles all at right angles to each other. What's more, they are Golden Rectangles since their edges are in the ratio 1 to Phi.
The same happens if we join the vertices of the icosahedron since it is the dual of the dodecahedron.





Using these *golden rectangles* it is easy to see that the coordinates of the icoshedron are as given above since they are:
(0,± 1, ± phi), (± phi, 0, ± 1), (± 1, ± phi, 0) .

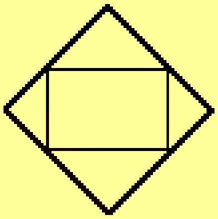
Things to do

1. Here is an interesting way to make a **model of an icosahedron** based on the three golden rectangles intersecting as in the picture above:
   - Cut out three golden rectangles. One way to do this is to take three postcards or other thin card and cut them so they are 10cm by 16.2cm.
   - In the centre of each, make a cut parallel to the longest side which is as long as the shortest side of a card.
     The three cards will be slotted through these slits to make the arrangement in the picture above. To do this, on one of the cards extend the cut to one of the edges.

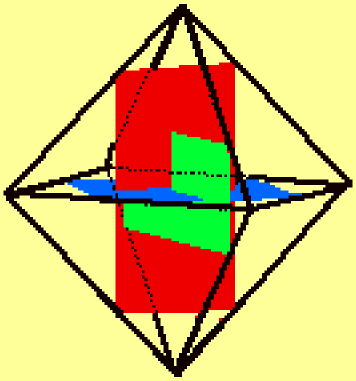
```
+--------------+ Make  and one   +-------------+
!              ! two       of    !             !
!    ======    ! of        these !   ===========
!              ! these           !             !
+--------------+                 +-------------+
```

   - Assemble the cards so that they look like the picture here of the red, green and blue rectangles. [This is a nice little puzzle itself!] You may want to put pices of sticky-tape where two cards meet just to make it a bit more stable.
   - Now you can make an icosahedron by joining the corners of the rectangles by gluing cotton so that it looks like the picture above.
   - It will be quite delicate, so tape another piece of cotton to one of the short edges of one of the cards and hang it up like a mobile!
2. If you are good at coordinate geometry or like a challenge, then show that the 12 points of the icoshedron divide the edges of the octahedron in the ratio Phi:1 (or 1:phi if you like) where the octahedron has vertices at:
   ($\pm Phi^2$, 0 0), (0, $\pm Phi^2$, 0), (0, 0, $\pm Phi^2$)
   [from H S M Coexter's book **Introduction to Geometry**, 1961, page 163.]

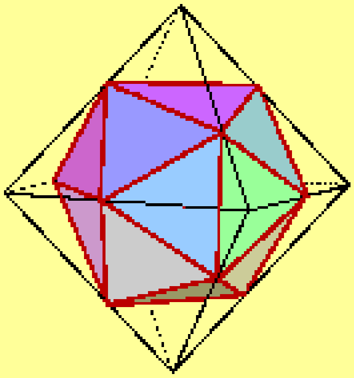
An Icosahedron in an Octahedron



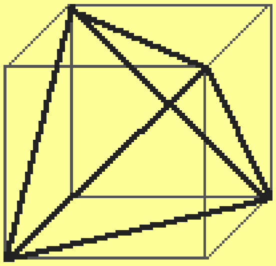
Using the same three golden rectangles at right-angles to each other, we can also make an octahedron.
If we put a square as shown around each rectangle, the squares will also be at right angles to each other and form the edges of an octahedron.



Now if we join the "golden-section points" forming the corners of our three rectangles (and now on both the edges of an octahedron and also forming the vertices of an icoshedron as we saw above), we can see how to fit an icosahedron into an octahedron - and the process involves golden sections!

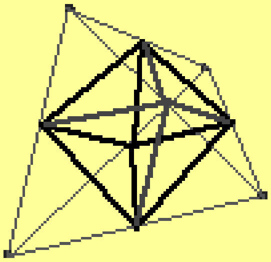


Here are some more Platonic-solids-within-Platonic-solids:



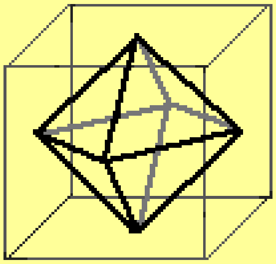
A Tetrahedron in a Cube

Select one corner of a cube and join it to the *opposite* corner on each face.



An Octahedron in a Tetrahedron

Join the mid-point of each edge to any other edge mid-point where the connecting line lies on one face of the tetrahedron.



An Octahedron in a Cube

Join the mid-points of faces: if two faces are next to each other at a corner, then their mid-points can be joined.

The Greeks, Kepler and the Five Elements

The Greeks saw great significance in the existence of just 5 Platonic solids and they related them to the 4 ELEMENTS (fire, earth, air and water) that they thought everything was made from. Together with the UNIVERSE, they associated each with a particular solid.
The astronomer and mathematician, Kepler (1571-1630), shown here as a link to the History of Mathematics web site at St Andrews University, Scotland, justified this as follows:

> Of the 5 solids, the **tetrahedron**has the smallest volume for its surface area and the **icosahedron**the largest; they therefore show the properties of *dryness*and *wetness*respectively and so correspond to FIRE and WATER.
> The **cube**, standing firmly on its base, corresponds to the stable EARTH but the **octahedron**which rotates freely when held by two opposite vertices, corresponds to the mobile AIR.
> The **dodecahedron**corresponds to the UNIVERSE because the zodiac has 12 signs (the constellations of stars that the sun passes through in the course of one year) corresponding to the 12 faces of the dodecahedron.

Kepler called the golden section "the division of a line into extreme and mean ratio", as did the Greeks. He wrote the following about it:

> **"Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel."**
> Johannes Kepler, (1571-1630)

Raoul Martens recommends an article in German on Kepler's interest in the Platonic solds: **Die kosmische Funktion des Goldenen Schnitts** by Theodor Landscheidt in *Sterne, Mond, Kometen, Bremen und die Astronomie* zum 75. Jahrestag der Olbers-Gesell-schaft Bremen e.V. Verlag H. M. Hauschild, Bremen 1995.

1·61803 39887 49894 84820 45868 34365 63811 77203 09179 80576 ..More..

Quasicrystals and Phi

On the Flat Phi page, we saw that the two triangles that appear in the pentagram and pentagon were used by Roger Penrose to design tiling patterns with five-fold symmetry called Penrose tilings. Is there a three-dimensional analogue of those two-dimensional tilings? The answer, thought to be impossible until Penrose's work of the early 1970's showed that there could be structures that filled space (in the same way that tilings fill planes) that have five-fold symmetry.

Are any Platonic solids space-filling?

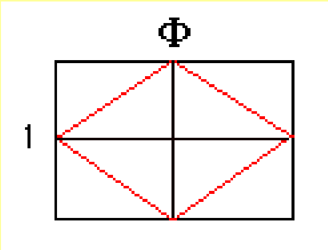
Yes, since identical copies of a cube can be stacked to fill a volume of space as large as we like with no gaps. The same is true of the tetrahedron and the octahedron, but the icosahedron and the dodecahedron cannot be used to fill space. This is analgous to trying to tile a plane with pentagons - they leave odd gaps that are not pentagonal. Both the dodecahedron and the icosahedron exhibit five-fold symmetry too. To see this, look back at the sectios above on the Icosahedron and Dodecaehdron and you wil find that, in the "other views" each has a view with five-fold symmetry. These views correspond to looking along an axis through the centre of the solids which have five-fold symmetry.

Quasicrystals

Penrose found that there are two simple shapes that you can use to fill a space as large as you like and which have five-fold axes of symmetry. The shapes are built from 6 flat faces which are , that is, shapes with all sides of equal length (like a square) and which has oppopsite sides paralled (again like a square), but which does not have all its angles equal - so they are diamond shaped (rhombs, rhombuses). The Penrose tiling shown on the Flat Phi page is also made from two rhombuses and fills theplane with a five-fold symmetric pattern.
For the solid shapes, the faces are all diamonds (rhombs) but not the ones used in the Penrose tiling and pentagons and pentagrams. The surprising relationship that holds for these new rhombuses is that

the ratio of the two diagonals of the diamonds (rhombuses) is Phi!

So this is a different rhomb from the Penrose rhombs and we shall call it the **golden rhomb**.



This makes the semi-angles (half the angles inside the rhombus) have tangents of Phi and phi so the angles of the rhombus are 2x31·717474..° = 2x0·55357435889^r and 2x58·282525588° = 2x1·0172219674^r.
[The angles in the rhombs in the Penrose tiling are 2/5 pi and 3/5 pi (72° and 108°) in one and 1/5 pi and 4/5 pi (36° and 144°) in the other.]

The two solids are similar to a cube but the faces are golden rhombs. The first shape is made by attaching three golden rhombs at their shorter angles in the same way as three squares meet at a corner of a cube. A duplicate is made and the two fit together to make a six-sided shape like a slanted cube. This is called a **prolate rhombohedron**.
The other shape is made by joining three golden rhombs together in the same way but at the larger angles this time. A duplicate of this is again fitted to make a different six-sided cube-like shape. This is called an **oblate rhombohedron**.
The two shapes look like cubes leaning over to one side.
Take a large number of one of these shapes and you can indeed fill as large a space as you like with them. When stacking cubes or octahedra, all the shapes are aligned identically (look identical, with the same

orientation). When we use a rhombohedron, some must be turned round to fit in with others. These also occur in nature, although only discovered since the 1950's and, because they are not quite as symmetrical as crystals, as called **quasi-crystals**.

Do quasicrystals occur in nature too?

Yes they do and a large number of substances have now been identified with such structures.

Crystals, the most symmetrical structures (with identical orientation for all the building blocks) are seen in sugar and salt as well as diamonds and quartz. Quasicrystals are an unsuspected new state of matter, sharing some of the properties of crystals and also on non-crystalline matter (such as glass). In 1984 the "impossible" five-fold symmetry was observed in an aluminiun-manganese alloy (Al_6Mn) and the term quasicrystal was invented for it in:

D Shechtman, I Blech, D Gratias, J W Cahn **Metallic phase with long-range orientational order and no translational symmetry** *Physics Review Letters* 1984, Vol 53, pages 1951-1953.

References and Links

See H S M Coxeter, Regular Polytopes, (Third Ed) 1973, Dover, pages 52-53 is a very popular book at an amazingly low price - well worth getting!

H S M Coxeter, Introduction to Geometry, 1961, John Wiley, (is a classic! See especially section 11.2: *De Divina Proportione*.

The classic and encyclopedic book on tilings is Grunbaum and Shepard's **Tilings and Patterns** Freeman and Co, 1986. It is well worth dipping into just to admire the pictures and patterns as the maths in it can be a bit scary!

Fractals, Chaos and Power Laws, M Schroeder, W H Freeman publishers, 1991. This is another fascinating book with much on self-similar sequences and patterns, Fibonacci and Phi. I have found myself dipping into this book time and time again. There is a chapter on the forbidden five-fold symmetry and its relation to the Fibonacci rabbits. (More information and you can order itonline via the title-link.)

- Robert Conroy has a page with lots of wire-frame pictures of other three-dimensional structures that are related to the Icoshedron and Dodecahendron.
- If your browser has a VRML plug-in, then check out this polyhedron site with over 700 polyhedra to manipulate on-screen!

Footnotes

Footnote 1: On Plato and Euclid

The Greeks from Euclid (365BC-300BC) and before knew that there were only 5 solid shapes with all sides equal, all angles equal, so that the faces are regular polygons.

Plato

They were also mentioned by the Greek philosopher Plato (428BC-348BC). He established an Academy in Greece and the motto over the entrance was

> Let no one ignorant of geometry enter here

As a philosopher, he held the view that mathematical objects "really" existed so that they are **discovered** by mathematicians (in the same way that new continents are discovered by explorers) rather than **invented** in the way that the TV or computer were invented. Plato believed that mathematics provided the best training for thinking about science and philosophy. The five regular solids are named "Platonic Solids" today after Plato.

Euclid

The most famous ancient book on geometry was written by Euclid (pronounced "You - klid") who lived from 365 BC to 300 BC and worked at the Library at Alexandria in Egypt, the foremost centre of learning in the world at that time. Actually, the book was a collection of 13 volumes, called *The Elements* and was the collected knowledge on geometry, superbly arranged and logically presented. It was the standard mathematics text book in Europe for centuries because it trained the reader to think logically, only relying on results that could be proved logically from self-evident starting points (axioms).
Here are some **axioms**:

> Things that are equal to the same thing are equal to each other.
> The whole is greater than the part.
> It is possible to draw a circle with any point as centre and with any radius.
> It is possible to draw a straight line between any two points.

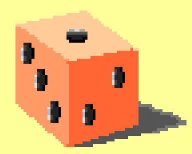
From these, Euclid proved **theorems** such as

> The angles in a triangle add up to two right angles.

One of Euclid's aims in his *Elements* seems to be to prove that there were only 5 solid (i.e. 3-dimensional) objects with all sides equal and all angles equal, and this occupies the final (13th) book of the Elements.

Footnote 2: Shapes for Fair Dice

We saw above that the Greeks knew of the 5 shapes that make fair dice. The Romans used a cubic dice and this is the one we most often use today.



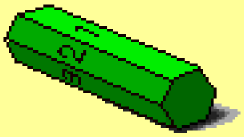
> *Should we say* ***one die*** *or* ***one dice****?*
> *The dictionary says that die is singular and dice is its plural form, so we ought to speak of* ***throwing a die or two dice****.*
> *These days the plural word* ***dice*** *is often used for one die and the dictionary recognises this also.*
> *A popular gambling game from at least Roman times involved throwing dice and is also called* ***casting the dice****. Some of the Roman soldiers "cast lots" for the clothes of Jesus at his crucifiction. Today we still use the phrase* ***the die is cast****. I used to think this phrase meant that a mould (US spelling=mold) had been made since we also read of someone being* ***cast in the heroic mould*** *as if they had been molten metal poured into a mould from which they solidify into a heroic shape. However I was wrong and it is just another use of the word* ***die****.*
> *The real meaning of the phrase* ***the die is cast*** *is that a dice (one!) has been thrown (cast) meaning that, as in a game of chance, "the outcome is now fixed, the decision is made".*
>
> *In these pages, I shall stick to the popular and common use, and make* ***dice*** *refer to the singular as well as the plural.*

From the Platonic solids that we saw above, we have dice of

4 sides : the tetrahedron
6 sides: the cube (or hexahedron)
8 sides: the octahedron
12 sides: the dodecahedron
20 sides: the icosahedron

There are other shapes if we don't insist that all the sides are the same length OR we allow 2-D shapes, but which still are **fair dice** - i.e. each number on a face is as likely as any other number to turn up.

If we let sides be different lengths, we can have a **prism** which is like a new (unsharpened) pencil with flat sides. Often pencils have just 6 flat sides, and we roll the pencil so that any side is likely to be face

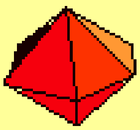


up. We can imagine a pencil with 8 sides, or 7 or even 27. If we have an odd number of sides, no one face is "up" (consider a triangular cross-sectioned pencil for instance, with just 3 choices of side). Here we may agree to use the side that the pencil lands on.

The other range of shapes is the **spinner** that comes with some boxed games. Here we have a flat

polygon with all sides of the same length (to make it fair). This was not in our list of Platonic solids because it is not a *solid* - it is just a flat 2-dimensional shape. However, we *can* have any number of sides and each is equally likely to be the side the spinner lands on, so it is fair.

Bi-pyramids as dice



Putting both of the above shapes together, we get a dice which is two n-gon-al pyramids, joined at their bases (the n-gons) to form a double pyramid or **bi-pyramid**. The picture shows a 12-sided dice formed from two 6-sided pyramids joined at their hexagonal bases. Perhaps we should call it a **bi-hexahedral** dice.

If we used pentagons then the bi-pyramidal dice would be 10-sided. It would be useful for generating random numbers up to 10.
By using two of them, say a red one for **tens digits** and a green one for **units digits**, we can roll random numbers from the hundred values between 00 and 99. If we added a blue one also, then we can get up to 999, and so on.

The advantage of the bi-pyramidal dice is that *there is always a side on top* no matter how the dice lands.

Iso-hedral shapes

Here is Ed Pegg Jr.'s complete list of ALL the 3-D dice shapes which have every face identical.
It includes all our 5 **Platonic solids**, and, since it also includes those where not every edge is the same length, it includes the **bi-pyramids** too. Every face is identical to every other face, so all the faces have exactly the same polygonal shape, but some edges have different lengths to others. There are others apart from the Platonic solids and the bi-pyramids and are some pretty weird too!
The common feature is that all of them would make good dice.
Since every face is the same, they are called **iso-hedral**.

[Back to the main text.]

1·61803 39887 49894 84820 45868 34365 63811 77203 09179 80576 ..More..

Fibonacci Home Page

The Mathematical Magic of the Fibonacci Numbers

Flat Phi Facts

WHERE TO NOW???

Phi's Fascinating Figures

The next topic is...
The Golden String

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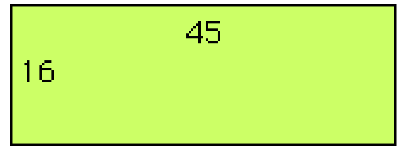
An Introduction to Continued Fractions

Continued fractions are just another way of writing fractions. They have some interesting connections with a jigsaw-puzzle problem of splitting a rectangle up into squares and also with one of the oldest algorithms known to Greek mathematicians of 300 BC - Euclid's Algorithm - for computing the greatest divisor common to two numbers (gcd).

Contents of this Page

The line means there is a Things to do investigation at the end of the section.

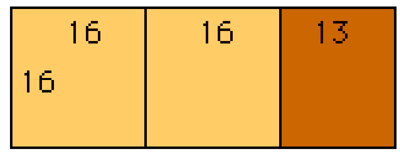
A jigsaw puzzle: splitting rectangles into squares



Suppose we have a rectangle which is 45 units by 16. We shall use this to express 45/16 as a **continued fraction** since at present 45/16 is just a **simple fraction**.

Looking at the rectangle the other way, its sides are in the ratio 16/45. We shall use this change of view when expressing 45/16 as a continued fraction. 45/16 is 2 lots of 16, with 13 left over, or, in terms of ordinary fractions:

$$\frac{45}{16} = \frac{16+16+13}{16} = 2 + \frac{13}{16}$$



In terms of the picture, we have just cut off squares from the rectangle until we have another rectangular bit remaining. There are 2 squares (of side 16) and a 13 by 16 rectangle left over.

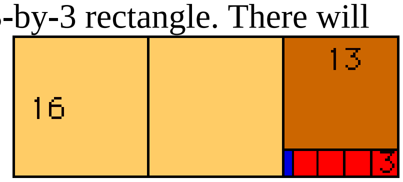
Now, suppose we do the same with the 13-by-16 rectangle, viewing it the other way round, so it is 16 by 13 (so we are expressing 16/13 as a whole number part plus a fraction left over). In terms of the mathematical notation we have:

$$\frac{45}{16} = \frac{16+16+13}{16} = 2 + \frac{13}{16} = 2 + \frac{1}{16/13}$$

Repeating what we did above but on 16/13 now, we see that there is just 1 square to cut off of side 16, with a 3-by-13 rectangle left over, expressing 13/3 as a whole-number-plus-fraction:

$$\frac{45}{16} = 2 + \frac{13}{16} = 2 + \frac{1}{16/13} = 2 + \cfrac{1}{1 + \cfrac{3}{13}}$$

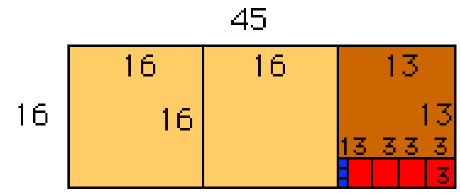
Notice how we have *continued to use fractions* and how the maths ties up with the picture.
Now we do the same thing on the left-over 3-by-13 rectangle, but looking at it as a 13-by-3 rectangle. There will be 4 squares (of side 3) and a rectangle 1-by-3 left over:



$$\frac{45}{16} = 2 + \cfrac{1}{1 + \cfrac{3}{13}} = 2 + \cfrac{1}{1 + \cfrac{1}{13/3}} = 2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{3}}}$$

Now we have ended up with an exact number of squares in a rectangle, with nothing left over so we cannot split it down any more.

$$\frac{45}{16} = 2 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{3}}}$$



In the rectangle of the rectangle, we can relate the geometry to the arithmetic as follows:
we see **2** orange squares (16 by 16), **1** brown square (13 by 13), **4** red squares (3 by 3) leaving a blue rectangle of size 1 by 3 (or you can think of this as **3** blue squares of size 1 by 1): the numbers are 2, 1, 4 and 3, as seen in the continued fraction above.

The General form of a Continued Fraction

We can do the same to any fraction, P/Q (P and Q are whole, positive numbers) expressing it in the form of a **continued fraction** as follows:

$$\frac{P}{Q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}}} = a_0+1/(a_1+1/(a_2+1/(..)))$$

where a_0, a_1, a_2, etc are all whole numbers. If P/Q is less than 1, then a_0 will be 0.

The fractional form that we have derived is called **the continued fraction**.

There is no need to draw the rectangles-as-squares pictures each time, unless you want to, because we can merely look at the numbers. If the fraction is less than 1, we use its reciprocal and then we can split it into a whole-number part plus another fraction which will be less than 1 and repeat. We stop when the fraction has a numerator *or* a denominator of 1.
Take for instance, 7/30. It is already less than 1 so we start off by writing it as

```
7/30    = 0 + 1/(30/7)
```

and then we apply the method of the last paragraph:

```
7/30    = 0 + 1/(4 + 2/7)
           = 0 + 1/(4 + 1/(7/2))
           = 0 + 1/(4 + 1/(3 + 1/2))
           = 0 + 1/(4 + 1/(3 + 1/(1 + 1/1)))
```

Either of the last two lines is a valid continued fraction form for 7/30.

The List Notation for Continued Fractions

We can write down any continued fraction such as

$$P/Q = a_0 + 1/(a_1 + 1/(a_2 + 1/(a_3 + ...)))$$

just as a list of the a's:

$$P/Q = [a_0; a_1, a_2, a_3, ...]$$

The first number is the whole number part of the fraction, so we separate from the other coefficients by using a semi-colon (;) after it.
For the continued fractions used above, we can now write them as:

```
45/16   = [2;1,4,3]
7/30 = [0;4,3,2]
```

If the first number in the list is 0, then the numerical value is less one. For instance, one half is:

```
1/2     = [0;2]
```

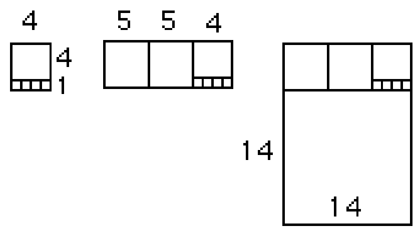
Also, there is a simple way to find the reciprocal of a continued fraction, for instance 16/45, since its list form is 0 + 1/(45/16), so we have:

```
16/45=[0;2,1,4,3]
```

If its list form begins with a zero already, as in 1/2 = [0,2], then its reciprocal is found by removing the 0 from the start of the list:

```
2 = [2]
```

Things to do

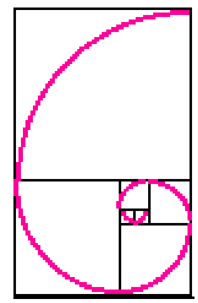
1. Express the following as continued fractions:
   1. 41/13
   2. 125/37
   3. 5/12
2. 

   The three rectangles in the picture are split into squares.
   Assuming that the smallest sized square has sides of length 1, what is the ratio of the two sides of each of the three rectangles?

What is the length of each of the rectangle's sides if the smallest squares have sides of length 2?

3. In the continued fraction for 45/16 = [2; 1, 4, 3], let's shall see what happens when change the final 3 to another number. Can you spot the pattern?
   Convert the following to proper fractions:
   - [2; 1, 4, 4]
   - [2; 1, 4, 5]
   - [2; 1, 4, 6]
   - [2; 1, 4, 7]
   - [2; 1, 4, n]

   How is your pattern related to the proper fraction for [2; 1,4]?
4. Convert these pairs of continued fractions into a single proper fraction:
   - [0; 1,2,3] and [0; 1,2,2,1]
   - [1; 1,2] and [1; 1,1,1]
   - [3; 2] and [3; 1,1]

   What is the general principle here?
5. Here is the Fibonacci Spiral from the Fibonacci Numbers in Nature page:
   
   If the smallest squares have sides of length 1, what continued fraction does it correspond to?
   What proper fraction is this?
6. Convert the successive Fibonacci number ratios into continued fractions. You should notice a striking similarity in your answers.
   1. 1/1
   2. 2/1
   3. 3/2
   4. 5/3
   5. 8/5

   If the ratio of consecutive Fibonacci numbers gets closer and closer to Phi, what do you think the continued fraction might be for Phi=1·618034... which is what the above fractions are tending towards?
7. The last question made fractions from neighbouring Fibonacci numbers. Suppose we take next-but-one pairs for our fractions, eg

   $\frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{5}{13}, \frac{8}{21}, \frac{13}{34},$ etc.

   - Convert each of these to continued fractions, expressing them in the list form. What pattern do you notice?

Continued Fractions and Euclid's GCD Algorithm

Euclid's GCD algorithm

One of the often studied algorithms in computing science is *Euclid's Algorithm for finding the greatest common divisor (gcd) of two numbers*. The *greatest common divisor* (often just abbreviated to **gcd**) is also called *the highest common factor* (or just **hcf**). It is intimately related to continued fractions, but this is hardly ever mentioned in computing science text books. Here we try to show you the link and introduce a visual way of seeing the algorithm at work as well as giving an alternative look into continued fractions.

So let's look again at the calculations we did above for 45/16.

```
45 = 2x16 + 13 :   45 as a multiple of 16 has 13 left over
16 = 1x13 +  3 :   16 as a multiple of 13 has  3 left over
13 = 4x 3 +  1 :   13 as a multiple of  3 has  1 left over
 3 = 3x 1 +  0 :    3 is a multiple of  1 exactly.

 L = Nx S +  R
```

The bold figures (N) are our continued fraction numbers. The L column is the Longest side of each rectangle that we encountered with S the Shortest side and R being the Remainder.
The method shown here is

- precise, and
- works for any two numbers in place of 45 and 13, and
- it always terminates since each time the numbers are reduced until eventually we reach 1.

These are the three conditions necessary for an **algorithm -** a method that a computer can carry out automatically and which eventually stops.

Euclid (a Greek mathematicians and philosopher who lived from about 300 BC to 260 BC) describes this algorithm in Propositions 1 and 2 of Book 7 of **The Elements**, although it was probably known to the Babylonian and Egyptian mathematicians of 3000-4000 BC also.
If we try it with other numbers, the *final non-zero remainder* is the greatest number that is an exact divisor of both our original numbers (the greatest common divisor) **-** here it is 1.

> Given *any* two numbers, they each have 1 as a divisor so there will **always be a greatest common divisor of any two (positive) numbers** and it will be at least 1.

Using Lists of Divisors to find the GCD

Here are the divisors of 45 and of 16:

```
  45 has divisors  1, 3, 5, 9, 15 and 45
  16 has divisors  1, 2, 4, 8 and 16
```

So the largest number in *both* of these lists is just 1.

Let's take a fraction such as 168/720. It is not in its lowest terms because we can find an equivalent fraction which uses simpler numbers. Since both 168 and 720 are even, then 168/720 is the same (size) as 84/360. This fraction too can be reduced, and perhaps the new one will be reducible too. So can we find the **largest number to divide into both** numerator 168 and denominator 720 and get to the simplest form immediately?
However, first, let's try to find the largest number to divide into both 168 and 720 directly:
Find the lists of the divisors of 168 and of 720 and pick the largest number in both lists:

```
168 has divisors 1, 2, 3, 4, 6, 7, 8, 12, 14, 21, 24, 28, 42, 56, 84 and 168
720 has divisors 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 30, 36,
                 40, 45, 48, 60, 72, 80, 90, 120, 144, 180, 240, 360 and 720
```

Phew! - that took some work!
Now we just need to find the largest number in both lists. A bit of careful searching soon reveals that it is 24. So 24 is the greatest common divisor (gcd) of 168 and 720. You will often see statements such as this written as follows:

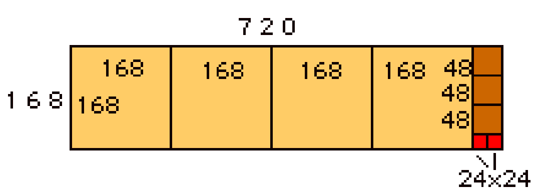
gcd(168, 720) = 24

> **The importance of the gcd of a and b is that it tells us *how to put the fraction a/b into its simplest form* which is to divide the top and the bottom by the gcd. The resulting fraction will be the simplest form possible. So**
>
> $$\frac{168}{720} = \frac{168 \div 24}{720 \div 24} = \frac{7}{30} \quad \text{and} \quad \text{similarly } \frac{720}{168} = \frac{30}{7} = 4 + \frac{2}{7}$$

Euclid's algorithm is here applied to 720 and 168: Just keep dividing and noting remainders so that the larger number 720 is 4 lots of the smaller number 168 with 48 left over. Now repeat on the smaller number (168) and the remainder (48) and so on:

```
720 = 4x168 + 48
168 = 3x 48 + 24
 48 = 2x 24 + 0
```

so the last multiple before we reach the zero is 24, just as we found above but with rather less effort this time!



Here is a rectangle 720 by 168 split up into squares, as above. Note how the quotients 4, 3 and 2 are shown in the picture and also that the gcd is 24 (the side of the smallest squares):

And here is 720/168 expressed as a continued fraction:

$$\frac{720}{168} = 4 + \frac{48}{168} = 4 + \cfrac{1}{\frac{168}{48}} = 4 + \cfrac{1}{3 + \frac{24}{48}} = 4 + \cfrac{1}{3 + \frac{1}{2}}$$

Things to do

1. For each of the fractions in the previous Things To Do section, use Euclid's algorithm to check your answers.
2. There is **another simple way to find gcd's** which takes more work than Euclid's method but is quicker than enumerating all the divisors. It involves expressing the two numbers as **powers of prime factors**, for instance:

   $720 = 2^4 \times 3^2 \times 5^1$
   $168 = 2^3 \times 3^1 \times 7^1$

   - First re-write these so that *the same prime numbers* appear in *both* lists, using a-prime-to-the-power-of-0 if necessary.
     For instance, there are no 7's in the primes product for 720, so, since $7^0=1$, we introduce an extra factor of **$\times 7^0$**. In the same way we can introduce $\times 5^0$ into the product for 168. Now both lists contains exactly the same primes: 2, 3, 5 and 7:

     $720 = 2^4 \times 3^2 \times 5^1 \times 7^0$
     $168 = 2^3 \times 3^1 \times 5^0 \times 7^1$

3. Since there must be 2's in the gcd of 720 and 168, how many twos do we need for the *greatest* factor which divides both?
   What about the number of 3's? and 5's? and 7's?
   So the greatest common divisor has the form:
   $$2^a \times 3^b \times 5^c \times 7^d$$
   What numbers stand in place of the letters?
4. What is the general principle for computing the gcd, given two numbers expressed as powers of the same primes?
5. What is the greatest common divisor of 24 and 18 (call it G)? What is the gcd of 24, 18 and 30? How is it related to the gcd of G and 30? [This is Proposition 3 of Euclid's **The Elements**, Book 7.]

Continued Fractions for decimal fractions?

If we look at irrational numbers (numbers which cannot be written exactly as a fraction) we will find no pattern in their decimal fractions. For instance, here is √2 to 200 decimal places:

```
1·41421 35623 73095 04880 16887 24209 69807 85696 71875 37694
  80731 76679 73799 07324 78462 10703 88503 87534 32764 15727
  35013 84623 09122 97024 92483 60558 50737 21264 41214 97099
  93583 14132 22665 92750 55927 55799 95050 11527 82060 57147
  ...
```

Indeed, it is not too difficult to show that, if any decimal fraction ever repeats, then it must be a proper fraction, that is **a rational number -** see the references section at the foot of this page.
The converse is also true, i.e. that every rational number has a decimal fraction that either stops or eventually repeats the same cycle of digits over and over again for ever.

But what about *continued fractions* for irrational numbers?

There is a pleasant surprise here since square-roots have repeating patterns in their *continued fraction* forms.

Terminating Decimal fractions

If we have a terminating decimal fraction, such as 1·53 then we can always represent it as a proper fraction by using a denominator which is a big enough power of 10.
For instance, 1·53 is just 153/100.
Similarly 3·456 is just 3456/1000
and 0.00075 is 75/100000.
Since they are fractions, we can now use Euclid's algorithm to express them as continued fractions and so their list of integers in the continued fraction will eventually end.

Continued fractions for square-roots

But what about a continued fraction for √2? Since it's decimal fraction never ends, and it is not possible to write it as a fraction, how can we convert it to a continued fraction?
Algebra can come to our assistance here.
To express √2 as a continued fraction, we know its value is bigger than 1 so we will write it as:

```
       sqrt(2) = 1 + 1/x
```

[We use 1/x so that x will be bigger than one.] All we have to do now is find x!
So let's rearrange this equation to find the value of x:

```
 (sqrt(2) - 1) = 1/x  so
       x = 1/(sqrt(2) - 1)
```

There is a useful technique for simplifying fractions with square-roots in the denominator, to get a whole number in the denominator: Here we will multiply the top and bottom of the fraction by (√2 + 1):

```
          1                  sqrt(2)+1    sqrt(2)+1
x = --------- = --------------------- = --------- = sqrt(2)+1
    sqrt(2)-1   (sqrt(2)-1)(sqrt(2)+1)      2-1
```

But we know sqrt(2) = 1 + 1/x, so we have:

```
x=sqrt(2)+1=1 + 1/x + 1 = 2 + 1/x
```

By substituting 2 + 1/x wherever we see x, we now have our continued fraction for x:

$$x = 2 + \frac{1}{x} = 2 + \frac{1}{2 + \frac{1}{x}} = 2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{x}}} = \ldots$$

So now we can express √2 as a continued fraction, *which goes on for ever but which has a simple pattern for its components*:

$$\sqrt{2} = 1 + \frac{1}{x} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \ldots}}}}$$

In terms of our list notation, we would write:

```
√2 = [1; 2, 2, 2, 2, 2, 2, ...]
```

It turns out that ALL square roots have similar infinite repeating patterns in their continued fractions, but for the details, you will need to look at books on Number Theory. Here are some more. What patterns can you spot? To find out more, look at the books in the References section below.

```
√2 = [1; 2, 2, 2, 2, 2, 2, 2, 2, ... ] = [1] then repeat [2]
√3 = [1; 1, 2, 1, 2, 1, 2, 1, 2, ... ] = [1] then repeat [1,2]
√4 = [2]
√5 = [2; 4, 4, 4, 4, 4, 4, 4, 4, ... ] = [2] then repeat [4]
√6 = [2; 2, 4, 2, 4, 2, 4, 2, 4, ... ] = [2] then repeat [2,4]
√7 = [2; 1, 1, 1, 4, 1, 1, 1, 4, ... ] = [2] then repeat [1,1,1,4]
√8 = [2; 1, 4, 1, 4, 1, 4, 1, 4, ... ] = [2] then repeat [1,4]
√9 = [3]
√10= [3; 6, 6, 6, 6, 6, 6, 6, 6, ... ] = [3] then repeat [6]
√11= [3; 3, 6, 3, 6, 3, 6, 3, 6, ... ] = [3] then repeat [3,6]
```

√12= [3; 2, 6, 2, 6, 2, 6, 2, 6, ...] = [3] then repeat [2,6]

You can produce these by a computer program as follows:

- Find the square root as a real number,
- then express it as a fraction over a large power of 10,
- next use Euclid's algorithm to find the entries in the continued fraction list.

Here is a table of the square-roots of all numbers from 2 to 100:

√n	[a; Period]
√2	1; 2
√3	1; 1,2
√4	2;
√5	2; 4
√6	2; 2,4
√7	2; 1,1,1,4
√8	2; 1,4
√9	3;
√10	3; 6
√11	3; 3,6
√12	3; 2,6
√13	3; 1,1,1,1,6
√14	3; 1,2,1,6
√15	3; 1,6
√16	4;
√17	4; 8
√18	4; 4,8
√19	4; 2,1,3,1,2,8
√20	4; 2,8
√21	4; 1,1,2,1,1,8
√22	4; 1,2,4,2,1,8
√23	4; 1,3,1,8
√24	4; 1,8
√25	5;
√26	5; 10
√27	5; 5,10
√28	5; 3,2,3,10
√29	5; 2,1,1,2,10

√n	[a; Period]
√51	7; 7,14
√52	7; 4,1,2,1,4,14
√53	7; 3,1,1,3,14
√54	7; 2,1,6,1,2,14
√55	7; 2,2,2,14
√56	7; 2,14
√57	7; 1,1,4,1,1,14
√58	7; 1,1,1,1,1,1,14
√59	7; 1,2,7,2,1,14
√60	7; 1,2,1,14
√61	7; 1,4,3,1,2,2,1,3,4,1,14
√62	7; 1,6,1,14
√63	7; 1,14
√64	8;
√65	8; 16
√66	8; 8,16
√67	8; 5,2,1,1,7,1,1,2,5,16
√68	8; 4,16
√69	8; 3,3,1,4,1,3,3,16
√70	8; 2,1,2,1,2,16
√71	8; 2,2,1,7,1,2,2,16
√72	8; 2,16
√73	8; 1,1,5,5,1,1,16
√74	8; 1,1,1,1,16
√75	8; 1,1,1,16
√76	8; 1,2,1,1,5,4,5,1,1,2,1,16
√77	8; 1,3,2,3,1,16

√30	5; 2,10
√31	5; 1,1,3,5,3,1,1,10
√32	5; 1,1,1,10
√33	5; 1,2,1,10
√34	5; 1,4,1,10
√35	5; 1,10
√36	6;
√37	6; 12
√38	6; 6,12
√39	6; 4,12
√40	6; 3,12
√41	6; 2,2,12
√42	6; 2,12
√43	6; 1,1,3,1,5,1,3,1,1,12
√44	6; 1,1,1,2,1,1,1,12
√45	6; 1,2,2,2,1,12
√46	6; 1,3,1,1,2,6,2,1,1,3,1,12
√47	6; 1,5,1,12
√48	6; 1,12
√49	7;
√50	7; 14

√78	8; 1,4,1,16
√79	8; 1,7,1,16
√80	8; 1,16
√81	9;
√82	9; 18
√83	9; 9,18
√84	9; 6,18
√85	9; 4,1,1,4,18
√86	9; 3,1,1,1,8,1,1,1,3,18
√87	9; 3,18
√88	9; 2,1,1,1,2,18
√89	9; 2,3,3,2,18
√90	9; 2,18
√91	9; 1,1,5,1,5,1,1,18
√92	9; 1,1,2,4,2,1,1,18
√93	9; 1,1,1,4,6,4,1,1,1,18
√94	9; 1,2,3,1,1,5,1,8,1,5,1,1,3,2,1,18
√95	9; 1,2,1,18
√96	9; 1,3,1,18
√97	9; 1,5,1,1,1,1,1,1,5,1,18
√98	9; 1,8,1,18
√99	9; 1,18

Things to do

What patterns do you notice in the table of square-roots above?

1. Four easy ones first:
   - What is special about the *first* number of the continued fraction?
   - What is special about the *last* number in the periodic part?
   - Can you spot the connection between these two numbers in each row of the table?
   - What about the other numbers in the periodic part? Is there a pattern to them that they ALL have?
2. Now let's look for patterns in the table as a whole.
   How about the continued fractions for the square-roots of 2, 5, 10, 17 and 26.
   - What pattern do they all have?
   - What is the next number in this sequence of square-roots that has the same pattern?
   - Can you **prove** your results?

The proof is quite easy!
Follow the steps above where we showed [1; 2,2,2,2,2...] was sqrt(2), but replace the 2's by **2n**'s say since the general pattern here is [n; 2n,2n,2n,2n,...].

3. How about this pattern:
   look at the square-roots of 3, 6, 11, 18 and 27.
   - What is the pattern this time? Express the general pattern as a mathematical expression.
   - What is the next square-root with this pattern?
   - Again try to verify your results are *always true*.
4. ..or spot the pattern in these sequences of square-roots:
   - 3, 8, 15, 24 and 35
   - 7, 14, 23, 34 and 47
   - 12, 39 and 84
   - We have now covered the patterns of all the square-roots up to 13. There is another pattern that applies to some of these smaller number's too - what pattern connects the cf lists for the square-roots of :
     6, 12, 20 and 30?
   - So what about 13? What pattern starts with the square-roots of 13, 29 and 53?
   - A pattern which includes sqrt(19) is difficult to spot (well I haven't been able to find one yet - can you?) but what other patterns can you find that cover most of the rest of the numbers up to 100?
     What square-roots are left over?

Was the table above produced by a computer program? Yes! The algorithm is explained in R. B. J. T. Allenby and Ed. Redfern's excellent book **Introduction to Number Theory with Computing**, published by E Arnold in 1989 but now out of print. It is well worth browsing through if you can find a copy in your library! Why not produce your own program and then you can extend the table further, using the values above to check your program (and mine!)

Solving Quadratics with Continued Fractions

Many problems, when modelled in mathematics, involve a quadratic equation - i.e. an equation of the form

$$A x^2 + B x + C = 0$$

where the A, B and C are numbers and we want to find values for x to make the equation true.

For instance, take $x^2 - 5x - 1 = 0$.
Can you think of an x value for which this equation holds? We can rewrite the equation in a different way as:

$$x^2 = 5x + 1$$

and now we can divide both sides by x to get:

$$x = 5 + 1/x$$

This means that wherever we have "x", we can replace it by "5 + 1/x". So we can replace the x in "5 + 1/x" for example to get:

$$x = 5 + \frac{1}{x} = 5 + \cfrac{1}{5 + \cfrac{1}{x}}$$

We can clearly replace the x again and get an infinite (periodic) continued fraction:

$$x = 5 + \cfrac{1}{5 + \cfrac{1}{5 + \cfrac{1}{5 + \dots}}} = [\ 5;\ 5,\ 5,\ 5,\ \dots\ ]$$

The Golden section and a quadratic equation

We have seen several times in the other Fibonacci Web pages at this site (see, for example, Formulae for Phi) that Phi is a root of the quadratic equation $x^2 - x - 1 = 0$.
Rearranging this equation gives $x^2=x + 1$ and so dividing both sides by x (since x is not zero) we have $x = 1 + 1/x$ which leads directly a continued fraction for the (positive) root, the value of x which we called **phi**:

$$x = 1 + 1/x = 1 + 1/(\ 1 + 1/x) = \dots = [\ 1;\ 1,\ 1,\ \dots\ ]$$

Of all continued fractions, this is the simplest.
The mathematician Lagrange (1736-1813) proved *the Continued Fraction Theorem* which says that **a quadratic equation with integer coefficients has a periodic continued fraction for all its real roots**.

Things to do

1. Find the 2 roots and a continued fraction for a root of these quadratic equations:
   a. $x^2 + x = 1$
   b. $x^2 - 2x = 1$
2. What happens if we try to find square-roots using this method, for example, the square root of 2 is a solution to $x^2 - 2 = 0$. Why do we not get a continued fraction this time?
   How does the answer to the second part of the previous question give a continued fraction for sqrt(2)?

The Silver Means

Can we find some more numbers with a pattern in their continued fractions which is like that of the golden mean, Phi? Since Phi as a continued fraction is:

```
Phi = [1; 1,1,1,1,1,... ]
```

then we can look at the numbers whose continued fractions are

```
[2; 2,2,2,2,2, ...]
[3; 3,3,3,3,3, ...]
[4; 4,4,4,4,4, ...]
[5; 5,5,5,5,5, ...]
...
```

These also have some interesting properties and are called the **silver means** since the most marvellous properties of all are for that rather special number we call the **golden mean**! Let's use T(n) for the n-th number in the list above, so that T(1) is just Phi and T(n) = [n; n,n,n,n,n, ...]
so T(n) = n+1/(n+1/(n+..)) or T(n) = n+1/T(n) since the value inside the brackets is just T(n)! So we have a definition of the Silver Means:

> A **silver mean** is a number T(n) which has the property that it is n more than its reciprocal, ie T(n) = n+1/T(n).

Numerical values of the Silver Means

Using the last property can we find values for the silver means? For instance,
T(1) = 1·6180339 = 1 + 1/1·6180339 = 1 + 0·6180339
T(2) = 2·4142135 = 2 + 1/2·4142135 = 2 + 0·4142135
and so on.
Here is one simple way to find the values and all you need is your calculator!

Things to do

1. The values of T(n) are easy to find on your calculator using the same method that we used to discover Phi from its property that it is "1 more than its reciprocal".
   The method is, for example, to find T(2) on your calculator:
   1. Enter any positive number you like.
   2. Press the reciprocal button (to find 1 divided by the displayed number)
   3. Add 2 (or, to find T(n), add n) and write down the result.
   4. Repeat from step 2 as often as you like.

   After just a few key presses, the numbers you write down will be identical and this is the value of T(n) as accurately as your calculator will allow.
   For T(2), you will soon reach 2·414213562.
2. For the value of T(2) here, subtract 1 and square the result. What is the answer?

```
What exact value does this suggest for T(2)?
(You will see the answer in next section!)
```

3. `Use the method above to find numerical values for T(3) and T(4).`

Exact values of the Silver Means

The **Things To Do** suggested to us an exact value for T(2). We could guess values for T(3) and T(4), but they are not easy to spot! So it's *mathematics to the rescue*!

By multiplying both side of the equation T(n)=n+1/T(n) by T(n), we get: $T(n)^2 = nT(n)+1$.

For example, the number [5; 5,5,5,5,5, ...] we have already met above and we found that it had the property that $x^2=5x+1$.
We can solve this quadratic equation or you can just check that there are two values of x with this property:

```
x = (5 + √29)/2 and
x = (5 - √29)/2
```

Since √29 is bigger than 5, then the second is a negative value, but since all our continued fractions are positive (they do not contain a negative number!) then the first is the value of our continued fraction:

```
[5; 5,5,5,5,5, ...] = (5 + √29)/2
```

If we review what we did above, then you will notice that we found

```
√2=[1; 2,2,2,2,2, ...]
```

so we can deduce that

```
[2; 2,2,2,2,2, ...] = 1 + √2
```

Following the same reasoning and including the golden mean also, gives the following pattern:

```
[1; 1,1,1,1,1, ...] = (1 + √5  )/2
[2; 2,2,2,2,2, ...] = (2 + √8  )/2 = 1 + √2
[3; 3,3,3,3,3, ...] = (3 + √13 )/2
[4; 4,4,4,4,4, ...] = (4 + √20 )/2 = 2 + √5
[5; 5,5,5,5,5, ...] = (5 + √29 )/2
[6; 6,6,6,6,6, ...] = (6 + √40 )/2 = 3 + √10
...
```

The following **Things To Do** explores this series and produces some more amazing connections between Phi and the Fibonacci numbers!

Things to do

- What is the next line in the table above for T(7)?
- Express the n-th line, that is T(n) as a formula involving square-roots.

2. T(1) is just Phi.

- T(4) also involves √5. Using the [Table of Properties of Phi](#) express this as a power of Phi.
- T(11) also involves √5. What is T(11)?
- Express T(11) as a power of Phi.
- What is the pattern here? Which powers of Phi are also Silver Means and which silver means are they?
  [Hint: the answer involves the [Lucas numbers](#).]

3. What powers of Phi are missing in the answer to the last question? What are their continued fractions?
4. Express all the powers of Phi in the form (X+Y√5)/2. Find a formula for Phi^n in terms of the Lucas and Fibonacci numbers?

Other numbers with patterns in their CFs

All proper fractions can be expressed as continued fractions using the jigsaw-puzzle technique at the top of this page where we split rectangles up into squares. Such continued fractions will eventually end since they are the ratio of two finite whole numbers.

In the section above, we have seen that expressions involving square-roots can be expressed as continued fractions with repeating patterns in them. Such continued fractions never end, but the pattern keeps repeating for ever.

Are there other numbers that have patterns in their continued fractions?

Yes! In particular, **e** does.

E

"E" is the base of **natural logarithms** and a number which occurs in many places in mathematics. **e** is also the number that this series settles down to eventually:

$(1+{}^1/_2)^2=2{\cdot}25$

$(1+{}^1/_3)^3=2{\cdot}37037..$

$(1+{}^1/_4)^4=2{\cdot}4414..$

$(1+{}^1/_5)^5=2{\cdot}48832,$

$(1+{}^1/_6)^6=2{\cdot}5216..,$

...

that is:

$$e = \lim_{n\to\infty} (1+{}^1/_n)^n$$

Its value to 200 dps is

2·71828 18284 59045 23536 02874 71352 66249 77572 47093 69995
95749 66967 62772 40766 30353 54759 45713 82178 52516 64274
27466 39193 20030 59921 81741 35966 29043 57290 03342 95260
59563 07381 32328 62794 34907 63233 82988 07531 95251 01901 ...

As a continued fraction, it can be written as

$$e - 1 = 1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \dots}}}} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \dots}}}}}$$

The above forms were found by the Swiss mathematician Leonhard Euler (1707-1783). [See Cut-the-Knot for more.]

> Note that the above continued fractions does *not* have **1** as the numerator (the top part) of the fractions so we do *not* write it in its abbreviated form as a list inside square brackets since this is *only* used for the *numerator=1* form.

However, another form for **e** is possible which does have our "standard" form:

e = [2; 1,2,1, 1,4,1, 1,6,1, 1,8,1, 1,10,1, ...]

The pattern continues with .. 1, 2n, 1, ... repeated for ever.
Euler also found the following:

$$\sqrt{e} = [1; 1,1,1, 5,1,1, 9,1,1, 13,1,1, 17,1,1, ...]$$

$\sqrt{e}$ to 200 dps is:

1· 64872 12707 00128 14684 86507 87814 16357 16537 76100 71014
80115 75079 31164 06610 21194 21560 86327 76520 05636 66430
02866 63775 63077 97004 67116 69752 19609 15984 09714 52490
05979 69294 22659 09840 39147 19948 46465 94892 44896 86890 ...

Two other expressions with e that have patterns in their continued fractions are

$$\frac{e - 1}{e + 1} = [0; 2, 6, 10, 14, ...]$$

which is a special case (k=2) of the following:

$$\frac{e^{2/k} - 1}{e^{2/k} + 1} = [0; k, 3k, 5k, 7k, 9k, ...]$$

Substituting 2k for k in the general case doubles all the continued fraction entries ...

$$\frac{e^{1/k} - 1}{e^{1/k} + 1} = [0; 2k, 6k, 10k, 14k, 18k, ...]$$

... and we can substitute 4k for k and quadruple the numbers ...

$$\frac{e^{1/(2k)} - 1}{e^{1/(2k)} + 1} = [0; 4k, 12k, 20k, 28k, 36k, ...]$$

By playing with a computer algebra package (because they do computations to large numbers of decimal places accurately), you can discover more continued fraction patterns involving **e**:

$e^{\frac{1}{2}}$ = [1; 1,1,1, 5,1,1, 9,1,1, 13,1,1, ...]= 1.6487212707001281468486 51
$e^{\frac{1}{3}}$ = [1; 2,1,1, 8,1,1, 14,1,1, 20,1,1, ...]= 1.3956124250860895286281 25
$e^{\frac{1}{4}}$ = [1; 3,1,1, 11,1,1, 19,1,1, 27,1,1, ...]= 1.2840254166877414840734 21
$e^{\frac{1}{5}}$ = [1; 4,1,1, 14,1,1, 24,1,1, 34,1,1, ...]= 1.2214027581601698339210 72
$e^{\frac{1}{n}}$ = [1; n-1,1,1, 3n-1,1,1, 5n-1,1,1, 7n-1,1,1, ...]

e^2 also has a pattern in its continued fraction a property not shared with any other natural number power of e:

e^2 = [7; 2, 1,1,3,18,5, 1,1,6,30,8, 1,1,9,42,11, ...]= 7.3890560989306502272304 27

We can take odd-numbered roots (cube-roots, fifth-roots, seventh-roots, etc) of e^2 and discover another simple pattern:

$e^{\frac{2}{3}}$ = [1; 1,18,7, 1,10,54,16, 1,19,90,25, 1,28,126,34, ...]= 1.9477340410546758566390 21
$e^{\frac{2}{5}}$ = [1; 2,30,12, 1,1,17,90,27, 1,1,32,150,42, 1,1,47,20,57, ...]= 1.4918246976412703178248 53
$e^{\frac{2}{7}}$ = [1; 3,42,17, 1,1,24,126,38, 1,1,45,210,59, 1,1,66,294,80, ...]= 1.3307121974473499773031 851
$e^{\frac{2}{2n+1}}$ = [1; n, 12n+6, 5n+2, 1, 1, 7n+3, 36n+18, 11n+5, 1, 1, 13n+6, 60n+30, 17n+8, 1, 1, 19n+9, 84n+42, 23n+1, ...]

Pi

Compare the above continued fractions involving e with the continued fraction for Pi and for √Pi which begin :

Pi =

[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, 6, 6, 99, 1, 2, 2, 6, 3, 5, 1, 1, 6, 8, 1, 7, 1, 2, 3, 7, 1, 2, 1, 1, 12, 1, 1, 1, 3, 1, 1, 8, 1, 1, 2, 1, 6, 1, 1, 5, 2, 2, 3, 1, 2, 4, 4, 16, 1, 161, 45, 1, 22, 1, 2, 2, 1, 4, 1, 2, ...]

$\sqrt{Pi}$ =

[1; 1, 3, 2, 1, 1, 6, 1, 28, 13, 1, 1, 2, 18, 1, 1, 1, 83, 1, 4, 1, 2, 4, 1, 288, 1, 90, 1, 12, 1, 1, 7, 1, 3, 1, 6, 1, 2, 71, 9, 3, 1, 5, 36, 1, 2, 2, 1, 1, 1, 2, 5, 9, 8, 1, 7, 1, 2, 2, 1, 63, 1, 4, 3, 1, 6, 1, 1, 1, 5, 1, 9, 2, 5, 4, 1, 2, 1, 1, 2, 20, 1, 1, 2, 1, 10, 5, 2, 1, 100, 11, 1, 9, 1, 2, 1, 1, 1, 1, 3, ...]

corrected and verified 28 January 2001

These series are not known to have any pattern in them in contrast to those of e and sqrt(e) above. Why? At present no one knows!

There are other more general forms of continued fraction which do not have denominators which are always 1. This one was found sometime around the year 1655 by William Brouncker:

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \dots}}}}$$

For more on the two continued fractions below, see **An Elegant Continued Fraction for Pi** by L J Large in *American Mathematical Monthly* vol 106, May 1999, pages 456-8.

$$\frac{4}{\phi} = 1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \cfrac{4^2}{9 + \dots}}}} \qquad \phi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \cfrac{7^2}{6 + \cfrac{9^2}{6 + \dots}}}}}$$

Squared Fibonacci Number Ratios

What is the period of the continued fractions of the following numbers?

a. 25/9
b. 64/25
c. 169/64

You might have noticed that in all the fractions, both the numerator (top) and denominator (bottom) are square numbers (in the sequence 1, 4, 9, 16, 25, 36 ,49, 64,...). The numbers that are squared are Fibonacci numbers (starting with 0 and 1 we add the latest two numbers to get the next, giving the series 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...).
The fractions above are the squares of the ratio of successive Fibonacci numbers:

a. 25/9 = $(5/3)^2$ = $(\text{Fib}(5)/\text{Fib}(4))^2$
b. 64/25 = $(8/5)^2$ = $(\text{Fib}(6)/\text{Fib}(5))^2$
c. 169/64 = $(13/8)^2$ = $(\text{Fib}(7)/\text{Fib}(6))^2$
d. ...

There is a simple pattern in the continued fractions of all the fractions in this series.

What other continued fraction patterns in fractions formed from Fibonacci numbers (and the Lucas Numbers 2, 1, 3, 4, 7, 11, 18, 29, 47, ...) can you find?

Continued Fractions of Quadratic Fibonacci Ratios Brother Alfred Brousseau in *The Fibonacci Quarterly* vol 9 (1971) pages 427 - 435.
Continued Fractions of Fibonacci and Lucas Ratios Brother Alfred Brousseau in *The Fibonacci Quarterly* vol 2 (1964) pages 269 - 276.

A link between The Golden string, Continued Fractions and The Fibonacci Series

Suppose we make the golden sequence into a binary number (base 2) so that its columns are interpreted not as (fractional) powers of 10, but as powers of 2:

$$0{\cdot}1011010110\ 1101011010\ 1101101011\ \ldots$$
$$= \mathbf{1}\times 2^{-1} + \mathbf{0}\times 2^{-2} + \mathbf{1}\times 2^{-3} + \mathbf{1}\times 2^{-4} + \mathbf{0}\times 2^{-5} + \mathbf{1}\times 2^{-6} + \ldots$$

It is called the **Rabbit Constant**.
Expressed as a normal decimal fraction, it is

0·70980 34428 61291 3... .

Its value has been computed to 330 decimal places where our Phi is referred to as **tau**.

The surprise in store is what happens if we express this number as a continued fraction. It is

[0; 1, 2, 2, 4, 8, 32, 256,...]

These look like powers of 2 and indeed all of the numbers in this continued fraction are powers of two. So which powers are they? Here is the continued fraction with the powers written in:

$[0; 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, 2^8, ..]$

Surprise! The powers of two are **the Fibonacci numbers**!!!

$[0; 2^{F(0)}, 2^{F(1)}, 2^{F(2)}, \ldots , 2^{F(i)}, \ldots]$

A Series and Its Associated Continued Fraction J L Davison, *Fibonacci Quarterly* vol 63, 1977, pages 29-32.

Two Continued Fractions involving The Fibonacci and the Lucas Numbers

The continued fraction for $\sqrt{5}\,\varphi = \frac{5-\sqrt{5}}{2}$ = 1.3819660112501051518... is [1;2,1,1,1,1,1,1,1,...]

and its convergents are: $1, \frac{3}{2}, \frac{4}{3}, \frac{7}{5}, \frac{11}{8}, \frac{18}{13}, \frac{29}{21},$

The pattern continues with the **Lucas Numbers** on the top and the **Fibonacci Numbers** on the bottom of the convergent's fractions.

Taking the reciprocal of this value, i.e. $\frac{\phi}{\sqrt{5}} = \frac{2}{5-\sqrt{5}}$ = 0.72360679774997896964... = [0;1,2,1,1,1,1,1,1,1,1,...]

we get the Fibonacci numbers on the top and the Lucas numbers on the bottom of the convergents.

The Strong Law of Small Numbers Richard K Guy in *The American Mathematical Monthly*, Vol 95, 1988, pages 697-712, Example 14.

Best Rational Approximations to Real Numbers

Continued fractions can be simplified by cutting them off after a given number of terms. The result - a terminating continued fraction - will give a true fraction, but it will only be an *approximation* to the full value.
It turns out - and we shall not prove this here - that these fractions are "the best possible approximations" to (in this case) the square-root of 2. By "best" here, we mean no closer fraction can be made from smaller numbers in the numerator and denominator.

Approximating Root 2 using Fractions

For instance, earlier we saw that the square-root of 2 is [1; 2,2,2,2,2,...]. So the following sequence of values will give rational approximations to root-2:

Shortened CF		Fraction	Value	Error
[1]	= 1	= 1	= 1	-0.4142135..
[1;2]	= 1+1/2	= 3/2	= 1.5	+0.0857864..
[1;2,2]	= 1+1/(2+1/2)	= 7/5	= 1.4	-0.0142135..
[1;2,2,2]	= 1+1/(2+1/(2+1/2))	= 17/12	= 1.416666..	+0.0024531..
[1;2,2,2,2]	= 1+1/(2+1/(2+1/(2+1/2)))	= 41/29	= 1.4137931..	-0.0004204..
[1;2,2,2,2,2]		= 99/70	= 1.4142857..	+0.0000721..

There are some intriguing patterns in the numerators and denominators of the successive fractions in the table above, which I leave you to explore on your own.

Best Fractions for Pi

To find a continued fraction for Pi, take any number of decimal places of Pi and express this as a decimal fraction: eg

Pi = 3.1415926535 = 31415926535 / 10000000000

Then express the fraction as a continued fraction:

31415926535/10000000000 = [3; 7, 15, 1, 292, 1, 1, 278, 1, 1, 1, 9, ...]

So, what are the best rational approximations to Pi?

3
: The nearest whole number.

3+1/7 = **22/7** = 3.142857 = pi+0.00126.
: This is the value everyone knows from school, 22/7. It is a good approximation for Pi, accurate to one-eighth of one percent.

3+1/(7+1/15) = **333/106** = 3.1415094.. = pi-0.00008..,

3+1/(7+1/(15+1/1)) = **355/113** = 3.14159292.. = pi+0.000000266...
: This value is easy to remember - think of the first three odd numbers written down twice: 113355, then split it in the middle to form two three-digit numbers, 113 355, and put the larger number above the smaller!

3+1/(7+1/(15+1/(1+1/292))) = **103993/33102** =3.1415926530.. = pi-0.00000000057..
: This is the next convergent to pi. It corresponds to a term in the CF that is a large number so it gives a particularly good approximation to pi. It is over 400 times more accurate than the previous one (355/113), but this time the numbers involved are not so easy to remember!

So to express a number as a continued fraction means we can determine the best rational approximations to any desired degree. The larger the terms, the better will be the approximation.

An Application to the Solar System

An application of this is if we wish to make two cog wheels where one rotates root-2 times faster than the other. Since cog wheels have a whole number of teeth round their rims, one can only revolve at a fixed fraction of the rate of the other.
We could have 7 cogs on one and 5 on the other, or 17 and 12 cogs would give a closer approximation. From the last line in the table, if we allow ourselves up to 100 teeth on a cog, then the best approximation to root-2 is given by 99 teeth and 70, with an error of only 0.007%.

Such fractions would be useful to know if you were building a clockwork model of the Solar System (called an **orrery**) where you wanted the planets to revolve around a central Sun and accurately represent the period of revolution (a "year") for every planet.

The "most irrational number"

From the examples above, we see that our rational approximations get better if we have large numbers in the continued fraction of the value we are approximating.
So the "hardest" number to make "rational" would be one with the smallest terms, namely, all ones. This is Phi - the golden section number!
The best rational approximations to Phi are just the ratios of successive Fibonacci numbers.
So Ian Stewart, and others, have called Phi "the most irrational number" because of this. But I prefer to call it the "least irrational number" because it is so easy to approximate it with fractions!!

Links and References

WWW Links

More on continued fractions from Calvin College
its history, theory, applications and a bibliography.

References to articles and books

C. Kimberling, **A visual Euclidean algorithm** in *Mathematics Teacher*, vol 76 (1983) pages 108-109.
is the earliest reference I have found to the Rectangle Jigsaw approach to continued fractions.

Introduction to Number Theory with Computing by R B J T Allenby and E Redfern
1989, Edward Arnold publishers, ISBN: 0713136618
is an excellent book on continued fractions and lots of other related and interesting things to do with numbers and suggestions for programming exercises and explorations using your computer.

The Higher Arithmetic by Harold Davenport,
Cambridge University Press, (7th edition) 1999, ISBN: 0521422272
is an enjoyable and readable book about Number Theory which has an excellent chapter on Continued Fractions and proves some of the results we have found above. (More information and you can order it online via the title-link.)
Beware though! We have used [a,b,c,d,...]=X/Y as our concise notation for a continued fraction but Davenport uses [a,b,c,d,..] to mean the **numerator only**, that is, just the X part of the (ordinary) fraction!

Introduction to the Theory of numbers by G H Hardy and E M Wright
Oxford University Press, 1980, ISBN: 0198531710
is a classic but definitely at mathematics undergraduate level. It takes the reader through some of the fundamental results on continued fractions. Surprisingly, it doesn't have an Index, but there is a Web page Index to editions 4 and 5 that you may find useful.

Continued Fractions by A Y Khinchin, ISBN: 0 486 69630 8
This is a Dover book (Sept 1997), well produced, slim and cheap, but it is quite formal and abstract, so probably only of interest to serious mathematicians!

A Limited Arithmetic on Simple Continued Fractions, C T Long and J H Jordan, *Fibonacci Quarterly*, Vol 5, 1967, pp 113-128;

A Limited Arithmetic on Simple Continued Fractions - II, C T Long and J H Jordan, *Fibonacci Quarterly*, Vol 8, 1970, pp 135-157;

A Limited Arithmetic on Simple Continued Fractions - III, C T Long, *Fibonacci Quarterly*, Vol 19, 1981, pp 163-175;

Three articles on continued fractions with a single repeated digit or a pair of repeated digits or with a single different digit followed by these patterns.

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Fibonacci Home Page

Phi's Fascinating Figures

The previous Topic is...
The Golden Section

WHERE TO NOW???

This is the only page on this Topic.

Phigits and Base Phi Representations (Optional)

The next topic is...
The Golden String

Using Powers of Phi to represent Integers (Base Phi)

If you have already looked at the page where we showed how to represent integers using the Fibonacci numbers, and you have also read about the numerical properties of powers of Phi then this page takes you a stage further - writing the integers in base Phi!

Contents of this Page

The line means there is a Things to do investigation at the end of the section.

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Powers of Phi

Here is part of the table of numerical properties of powers Phi:

```
Remember: Phi = 1·6180339...
     and  phi = 0·6180339... = Phi-1 = 1/Phi

Phi                   real
power    A + B phi    value
...
Phi^5  = 8 + 5 phi = 11·090169..
Phi^4  = 5 + 3 phi =  6·8541019..
Phi^3  = 3 + 2 phi =  4·2360679..
Phi^2  = 2 + 1 phi =  2·6180339..
Phi^1  = 1 + 1 phi =  1·6180339..
Phi^0  = 1 + 0 phi =  1·0000000..
Phi^-1 = 0 + 1 phi =  0·6180339..
Phi^-2 = 1 - 1 phi =  0·3819660..
Phi^-3 =-1 + 2 phi =  0·2360679..
Phi^-4 = 2 - 3 phi =  0·1458980..
Phi^-5 =-3 + 5 phi =  0·0901699..
...
```

We can capture these relationships precisely in a formula:

$$\text{Phi}^n = \text{Fib}(n+1) + \text{Fib}(n)\ \text{phi}$$

[It is not difficult to prove (by Induction) that this formula is indeed true.] This formula applies to negative n as well, if we extend the Fibonacci series backwards:

..., -8, 5, -3, 2, -1, 1, 0, 1, 1,2, 3, 5, 8, ...

where we still have the **Fibonacci property**:

$$\text{Fib}(n) = \text{Fib}(n-1) + \text{Fib}(n-2)$$

but it now holds ***for all values of n, positive, zero and negative***!

Another property of this extended Fibonacci series of numbers is that

$$\begin{aligned}\text{Fib}(-n) &= -\ \text{Fib}(n), \text{ for even n and}\\ &= \text{Fib}(n), \text{ for odd n.}\end{aligned}$$

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Integers as sums of powers of Phi

In the table of powers of phi above, you will have noticed that the same multiples of Phi occur, sometimes positive and sometimes negative. For example, **2 phi** occurs in both $\text{Phi}^3 = 3 + 2$ phi and $\text{Phi}^{-3} = -1 + 2$ phi. If we *subtract* these two powers, the *multiples* of phi will disappear and leave us with an integer.
Similarly, **3 phi** occurs in both $\text{Phi}^4 = 5 + 3$ phi and $\text{Phi}^{-4} = 2 - 3$ phi. If we *add* these two powers, again the *multiples* of phi will cancel out and leave an integer.

Here are some more examples:

$$\begin{aligned}
\text{Phi}^1 + \text{Phi}^{-2} &= (1 + 1\ \text{phi}) + (1 - 1\ \text{phi}) = 2\\
\text{Phi}^2 + \text{Phi}^{-2} &= (2 + 1\ \text{phi}) + (1 - 1\ \text{phi}) = 3\\
\text{Phi}^3 - \text{Phi}^{-3} &= (3 + 2\ \text{phi}) - (-1 + 2\ \text{phi}) = 5\\
\text{Phi}^4 + \text{Phi}^{-4} &= (5 + 3\ \text{phi}) + (2 - 3\ \text{phi}) = 7
\end{aligned}$$

So we have expressed the integers 2, 3, 5 and 7 as **a sum of powers of Phi**.

If we also use $Phi^0 = 1$, then we can ***add 1*** ($=Phi^0$) to those numbers above and so represent 3, 4, 6 and 8 as a sum of powers of Phi.

We can also add combinations of these numbers and get other ones too. In all of them, we are writing the integer as a sum of *different* powers of Phi.

$$4 = 3 + 1 = (Phi^2 + Phi^{-2}) + Phi^0$$
$$8 = 7 + 1 = (Phi^4 + Phi^{-4}) + Phi^0$$
$$9 = 2 + 7 = (Phi^1 + Phi^{-2}) + (Phi^4 + Phi^{-4})$$
$$10 = 3 + 7 = (Phi^2 + Phi^{-2}) + (Phi^4 + Phi^{-4})$$

This reminds us of expressing numbers as :

- sums of powers of 2 (binary), or
- sums of powers of 3 (ternary), or
- sums of powers of eight (octal) and, of course, the usual way using
- sums of powers of 10 (decimal)!

All the above are powers of an integer (2, 3, 8 or 10) but the really unusual thing here is that we are taking powers of *Phi, an **irrational** number and adding them to get a purely whole number*!

A natural question now is:

Are all integers representable as sums of powers of phi?

The answer is **Yes!** The number n is just n + 0 Phi !!!
So let's rephrase the question...
What we *really* meant to ask was how to do this **using *only* powers of Phi and *not repeating any power* in the sum** (which is what we did in the examples above).

Things to do

1. $1 = Phi^0$ and
   $1 = Phi^{-1}+ Phi^{-2}$ and
   $1 = Phi^{-1}+ Phi^{-3} + Phi^{-4}$
   How many more ways to represent 1 can you find? Remember that no power of Phi can be used more than *once*!
2. Try to express each of the following numbers as a sum of *different* powers of Phi each power occurring no more than once.
   You could check your answers in two ways:
   - on your calculator to see if you are approximately right but a better way (that is, more precise) is
   - to use the exact values by translating all the powers of Phi into sums of integers and *multiples of Phi* using the formula $Phi^n = Fib(n+1) + Fib(n)\ phi$ so that you can check that all the multiples cancel out:
     * 5 as the sum of 2 and 3
     * 5 as the sum of 4 and 1
       (use your answers to the first question using different representations of 1)
     * 6
     * 6 again, but find a different answer this time
     * 9 Find THREE different answers!

```
    * 10
    * 11
    * 12
    * each of the numbers from 13 to 20
```

3. Of your representations of number 6 in the previous question, which answer has **the fewest powers of Phi**?
4. Find a table of answers for all the values from 1 to 20 but all your answers should have the fewest number of powers in them.

From your answers to the above questions, it may look like many numbers can be expressed in Base Phi. Do you think that ALL whole numbers can be?

> **If you do**, how would you try to *convince* someone of this?
> **If you do not**, which integer do you think does NOT have a Base Phi representation? (Are you sure?)

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Base Phi Representations

Let's use what we learned on the Fibonacci Bases Page to write down our sums-of-distinct-powers-of-Phi representations of a number. As in decimal notation, the columns represent the powers of the Base, but for us the base is Phi, not 10. We have negative powers of Phi as well as positive ones, so, just as in decimal fractions, we need a "point" to separate the positive powers of Phi from the negative ones.

So if 1·25 in decimal means

```
 3 2 1 0 . -1 -2  <-- powers of 10
       1 .  2  5  = 1 + 2x10-1  + 5x10-2
```

$$1.25 = 1 + 2\times10^{-1} + 5\times10^{-2}$$

then

$$2 = Phi^{1} + Phi^{-2}$$

so 2 in Base Phi is

```
  3  2  1  0  .  -1  -2  -3    <-- powers of Phi
        1  0  .   0   1
```

which we write as $2=10{\cdot}01_{Phi}$ to indicate that it is a Base Phi representation.

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Reducing the number of 1's in a Base Phi Representation

We haven't used much of the theory about Fibonacci numbers yet (those formulae further up this page). There are some interesting and relevant facts in the Formula for powers of Phi that we saw on the Phi's Fascinating Figures page. One of these was

$$Phi^n = Phi^{n-1} + Phi^{n-2}$$

This tells us that, **if ever we find two consecutive 1's in a Base Phi representation, we can replace them by an additional one in the column to the left**

For instance,

$$3 = 2 + 1 = 10{\cdot}01_{Phi} + 1{\cdot}0_{Phi} = 11{\cdot}01_{Phi}$$

but we can replace the two consecutive 1's by a 1 in the phi^2 column:

$$3 = 100{\cdot}01_{Phi}$$

Let's call this the **Reducing 1's Process**.

What happens if we have three 1s next to each other?
There will always be two consecutive ones that have a zero on their left, so start with those. This will replace the two ones by zeros. We can always start with the leftmost pair of ones and then repeat the Reducing 1's Process on the new form if necessary.

Repeatedly applying the Reducing 1's process means that we can reduce a Base Phi representation until eventually we have no pairs of consecutive 1's

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Expanding the number of 1's in a Base Phi Representation

What if we get more than one of a certain power of Phi?
The solution here is to use the same formula but *backwards*, that is, **replacing a 1 by 1's in the two columns to the right**. So that, whenever we have

```
...100... we can replace it by ...011...
```

Let's call this **the Expanding 1's Process**.

EG $2 = 1+1 = 1{\cdot}0_{Phi}+1{\cdot}0_{Phi}$ Expanding the second 1·0 into 0·11:
$= 1{\cdot}0_{Phi}+0{\cdot}11_{Phi}$ Now we can add without getting more than 1 in any column:
$= 1{\cdot}11_{Phi}$ and we are ready to apply the Reducing 1's process:
$=10{\cdot}01_{Phi}$

Things to do

1. Write 3 as 2+1 and reduce it to its minimal form (no two consecutive 1's).
2. Try it for 4 = 3+1.

3. Look through your answers to the earlier questions and re-write your Table of Base Phi representations so that all the numbers from 1 to 20 have no two consecutive 1's.

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Minimal Base Phi Representations

You might like to convince yourself that, by successively adding 1's, if necessary applying the Expanding 1's process, then

we can always find a way of representing ANY integer as sum of distinct powers of Phi.

By applying the Reducing 1's process as often as necessary, we can then

always find a base Phi representation that has the minimum number of 1's
and no two of them will be next to each other.

Using the digits 0 and 1 only, we can express every integer as a sum of some powers of Phi

Things to do (Difficult!)

1. How unusual is this property? Could we express every integer as sum of powers of √2? (The answer is easy if you think about even powers of √2)
2. What about powers of e or π or some other *irrational* value which has no integer power giving an integer?

Other names for Base Phi

Let us call our representations of an integer n as a sum of *different* powers of Phi the **Base Phi representation of n**. Other names that have been suggested are

- **Phigital**: compare with digital for Base Ten;
- **Phinary**: compare with Binary since we are also using just the digits 0 and 1 but to base Phi [with thanks to Marijke van Gans for this term];
- expressing a number in **Phigits**[With thanks to Prof Jose Glez-Regueral of Madrid for mentioning this one.]

Links and References

This material originally appeared in an article by George Bergman, in the **Mathematics Magazine** 1957, Vol 31, pages 98-110, where he also gives pencil-and-paper methods of doing arithmetic in Base Phi.

C. Rousseau **The Phi Number System Revisited** in *Mathematics Magazine* 1995, Vol 68, pages 283-284.

Oleksiy Stakhov leads a group of Slavonic mathematicians who investigate the applications of Fibonacci and Phi number systems for instance representing numbers in a computer rather than the familiar binary system. He has published a book on this: Computer Arithmetic based on Fibonacci Numbers and Golden Section: New Information and Arithmetic Computer Foundations and his web site has lots more information on it.

Fibonacci Home Page

The Mathematical Magic of the Fibonacci Numbers

Phi's Fascinating Figures

WHERE TO NOW???

This is the last page on this Topic.

The next topic is...
The Golden String

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The Fibonacci Rabbit sequence

Other names for the Rabbit Sequence are the Golden Sequence because, as we shall see, it is closely related to the golden section numbers Phi (=1·6180339..) and phi=(0·6180339..).

Contents

The line means there is a Things to do investigation at the end of the section.

Fibonacci Numbers and the Rabbit sequence

This page is all about a remarkable sequence of 0s and 1s which is intimately related to the Fibonacci numbers and to Phi:

1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 ...

First we re-examine Fibonacci's original Rabbit problem and see how it can generate an infinite sequence of two symbols and in a later section we see how the same sequence is very simply related to Phi also.

Lining up the Rabbits

If we return to Fibonacci's original problem - about the rabbits (see the Fibonacci home page if you want to remind yourself) then we start with a single **N**ew pair of rabbits in the field. Call this pair N for "new".

```
Month 0:
      N
```

Next month, the pair become **M**ature, denoted by "M".

```
Month 0:  1:
      N   M
```

The following month, the M becomes "MN" since they have produced a new pair (and the original pair also survives).

```
Month 0:  1:  2:
      N   M   M
              N
```

The M of month 2 become MN again and the N of month 2 has become M, so month 3 is: "MNM"

```
Month 0:   1:   2:   3:
      N -  M -  M -  M
              \    \  N
                N -  M
```

The next month it is "MNMMN".

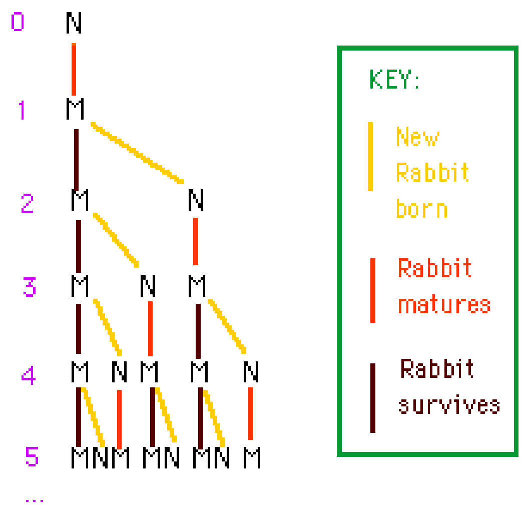
The general rule is

> **replacing every M in one month by MN in the next** and **similarly replace every N by M**.

Hence MNM goes to MN M MN .

We have now got a collection of sequences of M's and N's which begins:

```
0: N              =N

1: M              =M

2: M      N       =MN

3: M   N  M       =MNM

4: M N M  M N     =MNMMN

5: MNM MN MNM     =MNMMNMNM
    ...             ...
```



Compare this with the picture we had of the Rabbit Family Tree where sometimes M is replaced by NM and sometimes by MN.

We often use 1s and 0s for this sequence, so here we have replaced M by **1** and N by **0**:

---

1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 ...

Another way to generate The Rabbit sequence

We can make the rabbit sequence for month x by taking the sequence from month x-1 and writing it out again, following it by a copy of the sequence of month x-2.
So, starting from N and M the next is M (last month) followed by N (the previous month) giving MN.
The next will be MN followed by M = MNM
and the one after that is MNM followed by MN = MNMMN.

From this definition we can see that
each monthly sequence is the start of the following month's sequence.

This means that (after the first sequence which begins with N), there is really just one infinitely long sequence, which we call **the rabbit sequence** or **the golden sequence** or **the golden string**.

---

1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 ...

Computers use The Rabbit sequence!

In this section we show how the definition of the Fibonacci numbers leads us directly to the Fibonacci Rabbit sequence, but this time we use 0s and 1s instead of Ms and Ns.
We see how a computer actually carries out the evaluation of a Fibonacci number using the Rabbit sequence secretly behind the scenes!

We can write a computer program to compute the Fibonacci numbers using the recursive definition:

```
f(0)=0
f(1)=1
f(n)=f(n-1)+f(n-2) for n>1
```

We will be interested in how the computer is evaluating a call of f on a number n - in particular, what are the actual numbers added (and in what order) when computing f(n). The third line of the definition means that to compute f(n) we first need to compute f(n-1) as a separate computation and then remember its result so that, when we have then computed f(n-2) - another separate computation - we can add the two values to find f(n). The first line of the definition means that

to compute f(0)

the program function immediately returns the answer **0**.

The second line of the definition means that

to compute f(1)

the computer again immediately returns the answer **1**.

We will examine the calls to the function **f** and represent them in diagrams of "calling sequences" so that we have the following diagram for f(0):

```
f(0)
  0
```

to show that

a call of f(0) is replaced by (gets expanded to) 0

Similarly,

```
f(1)
  1
```

shows that f(1) gets expanded to 1, shown on the line below it, using the function definition given above.

What happens for larger values of n?

To compute f(2)

since n>1 we will be using the third line of the definition

f(n)=f(n-1)+f(n-2)

For f(2), n is 2 so we need to compute f(1)+f(0).

First f(1) is computed, giving **1** and then we compute *and add on* f(0), which is recomputed as **0**. The pattern of calls of f when computing f(2) is therefore shown in our calling sequence diagram as follows:

```
f(2)
f(1)+f(0)
  1    0
```

To compute f(3)

the function tells us to call f(2) and f(1) to compute f(2)+f(1). f(2) is called first, repeating the above computations and eventually returning 1+0=1 and *after this* f(1) is called, returning 1, so the final result of (1+0)+1=2 is returned.
In this case, the calling sequence in the computer again forms a "tree":

```
f(3)
f(2)......+f(1)
f(1)+f(0)    1
  1    0
```

Note that the actual additions performed are 1+0+1, and that these numbers appear the lower end of the "branches" in the "calling tree".

A note on trees in computing

In computing science such tree diagrams are very useful and they appear in many different situations. The natural way to represent them is as above, where the "root" from which the "tree" grows is at the top (since we read from top down a page of text) and so the ends of the "branches" - often called "leaves" - appear at the lowest level! So our trees are *antipodean* i.e *Australian* since they grow upside-down!

For f(4)
the calling sequence tree is f(3) as in the last calling tree diagram but now inculding the call of f(2) since f(4)=f(3)+f(2):

```
f(4)
f(3)............+f(2)
f(2)......+f(1)  f(1)+f(0)
f(1)+f(0)    1     1    0
  1    0
```

so the actual addition performed is

1+0+1+1+0

If we consider **further calls of f(n) for n=5 and above**
then since f(n)=f(n-1)+f(n-2), each tree begins with the previous tree [used to compute f(n-1)] and is followed by the whole of the tree before that, namely for f(n-2).

For instance, here is the calling tree for f(5) which starts with f(4) and, on the right, we include f(3):

```
f(5)
f(4)......................+f(3)
f(3)............+f(2)       f(2)......+f(1)
f(2)......+f(1)  f(1)+f(0)  f(1)+f(0)    1
f(1)+f(0)    1     1    0     1    0
  1    0
```

The actual additions this time are

1+0+1+1+0+1+0+1=5

You should now be able to see that the sequence of 0's and 1's used in the additions is defined as follows: let's let **s(n)** stand for the sequence of 0's and 1's used in computing **f(n)** so that:

s(0)=0
s(1)=1
s(n)=s(n-1) "followed by" s(n-2)

so we have

```
                                    number of
                                    0s  1s in s(n):
s(0)=0                          =0  1   0
s(1)=1                          =1  0   1
s(2)=1+0                        =1  1   1
s(3)=1+0+1                      =2  1   2
s(4)=1+0+1+1+0                  =3  2   3
s(5)=1+0+1+1+0+1+0+1            =5  3   5
s(6)=1+0+1+1+0+1+0+1+1+0+1+1+0  =8  5   8
...
```

and we see s(n) gives a sequence of additions involving 0s and 1s which defined the Fibonacci numbers.

There is no "last" sequence in the s(n) series but we see that a unique sequence of infintely many 0's and 1's is defined by this process and is the one we call the **the Fibonacci Rabbit sequence** or **the Golden Sequence**.

The number of additions when computing f(n)

When computing f(n) by the recursive formula at the start of this section:

f(0)=0; f(1)=1; f(n)=f(n-1)+f(n-2) for n<0 or n>1

it takes longer to compute the larger values. This is because the computer is doing a lot of recalculation as we have just seen above. So we can ask

*How much work does it take to compute **f(n)**?*

This is measured by the number of additions performed.
We have already written out the actual additions in the table above, up to s(6). Let's look at it again and count the number of addition operations this time:

```
                                    number of +'s
s(0)=0                          0
s(1)=1                          0
s(2)=1+0                        1
s(3)=1+0+1                      2
s(4)=1+0+1+1+0                  4
s(5)=1+0+1+1+0+1+0+1            7
s(6)=1+0+1+1+0+1+0+1+1+0+1+1+0  12
...
```

What is the pattern in the series 0,0,1,2,4,7,12,...?
Let's call this the A series (for Additions):

n:	0	1	2	3	4	5	6	..
A(n):	0	0	1	2	4	7	12	..

We can see some information by just looking at the recursion formula:

f(n) = f(n-1)+ f(n-2)

so

> A(n) is the number of additions in computing F(n-1)
> PLUS the number of additions in computing F(n-1)
> PLUS 1 in order to add f(n-1) to f(n-2)

or, using the A(i) notation for 'the number of additions in computing f(i)':

A(n) = A(n-1) + A(n-2) + 1; A(0)=0; A(1)=0

This is now a complete (recursive) definition of A. We can now use it to find A(7), the number of additions needed to compute f(7) (=13).
It is A(6)+A(5)+1 or 7+12+1 which is 20.
Here are a few more values:

n:	0	1	2	3	4	5	6	7	8	9	10
A(n):	0	0	1	2	4	7	12	20	33	54	89

There is another of the Fibonacci surprises here. Though the numbers are not the Fibonacci numbers, they have a similar method of construction (add the last two and then add 1). Have you noticed how the A series is related to the Fibonacci numbers themselves? The answer....

The A numbers are just 1 less than a Fibonacci number:

n:	0	1	2	3	4	5	6	7	8	9	10
A(n):	0	0	1	2	4	7	12	20	33	54	88
f(n+1):	1	1	2	3	5	8	13	21	34	55	89

So

A(n) = f(n+1) - 1

This means that the work needed to compute f(n) is measured by f(n+1) because we can ignore the 'minus 1' as it is insignificant when f(n) is large.

With thanks to Aaron Goh for suggesting this section.

---

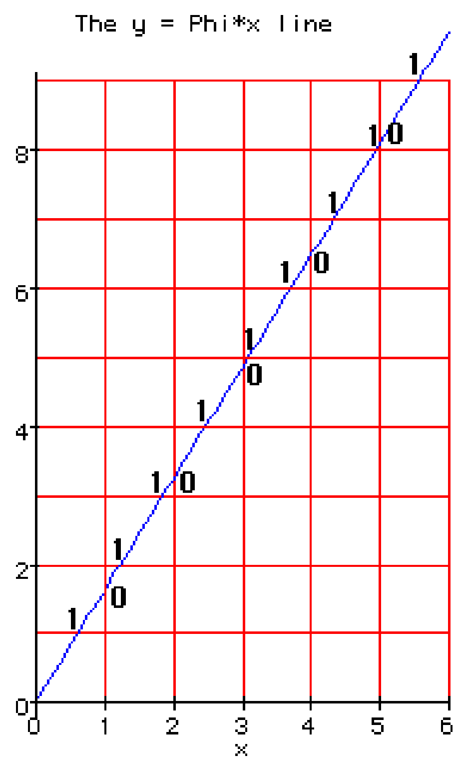
1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 ...

Phi and the Rabbit sequence

Our "golden" sequence has many remarkable properties that involve the golden section.

The Phi line Graph

If we draw the line **y = Phi x** on a graph, (ie a line whose gradient is Phi) then we can see the Rabbit sequence directly.



Where the Phi line crosses a horizontal grid line (y=1, y=2, etc) we write **1** by it on the line and where the Phi line crosses a vertical grid line (x=1, x=2, etc) we record a **0**.

Now as we travel along the Phi line from the origin, we meet a sequence of 1s and 0s - the Rabbit sequence again!

---

1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 ...

The following sections explore this relationship using functions such as "the next integer below" (the **floor** function) and "the next integer above" (the **ceiling** function) which will tell us which grid-line we have just crossed.

The rabbit sequence defined using the whole part of Phi multiples

If we take the number Phi, which we have seen is closely related to the Fibonacci series, then it leads to another simple definition of the rabbit sequence.
With the definitions above, we have to find all the preceding bits (Ms or Ns) to find which letter occurs in place i in the sequence. Using Phi=1·618034... we can compute it directly:
If we let M = 1 and N=0 then thre rabbit sequence is 101101... and:

```
rabbit(i)=trunc((i+1)*Phi)-trunc(i*Phi) -1  OR
rabbit(i)=trunc((i+1)*phi)-trunc(i*phi)
where Phi=(sqrt(5)+1)/2=1·618034... and phi=Phi-1=(sqrt(5)-1)/2=0·618034...
```

"Trunc(x)" is the function which just forgets anything after a decimal point in x.

To see how this works, look at this table:

```
i     i*Phi      trunc(i*Phi) diff diff-1 RabSeq
1     1.618034..    1
                                2     1    M
2     3.223606..    3
                                1     0    N
3     4.854101..    4
                                2     1    M
4     6.472135..    6
                                2     1    M
5     8.090169..    8
                                1     0    N
6     9.708203..    9
                                2     1    M
7    11.326237..   11     ...
```

where diff is the difference between the trunc item of the row above and the row following with 1=M and 0=N.

Things to do

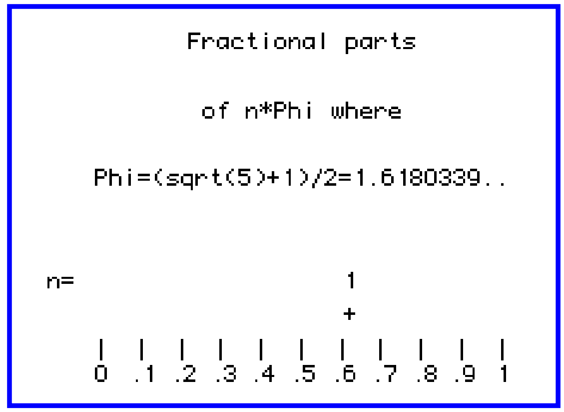
1. Try extending the table for a few more rows.
2. Use phi=Phi-1 instead of Phi in the table but don't subtract 1 from the diffs.

The rabbit sequence defined using the fractional parts of Phi multiples

Here is another method to generate the Rabbit sequence but this time using the bits we threw away above - the fractional parts of the multiples of Phi!

```
i     i*Phi      frac(i*Phi)    R or L?
1     1·618034..    0·618034..
2     3·223606..    0·223606..    L
3     4·854101..    0·854101..    R
4     6·472135..    0·472135..    L
5     8·090169..    0·090169..    L
6     9·708203..    0·708203..    R
7    11·326237..    0·326237..    L
...
"R or L?" means that the fractional part on that line=frac(i*Phi)
is moRe or Less than the fractional value on the line above=frac((i-1)*Phi)
```

An alternative way to generate the sequence of R and L is to look at this Quicktime movie of the fractional parts of the first 56 multiples of Phi (click on the picture).



Does the point move to the **R**ight of the previous one or to the **L**eft.

(Tip: Use the slider on the Quicktime movie frame to advance the picture one frame at a time.)

Things to do

1. Note that sometimes a new point will be plotted further to the right than any previous one (i.e. its fractional part will be larger than any before it). What multiples of Phi result in these "furthest out" points?
2. What multiples correspond to those points plotted furthest to the left?

The rabbit sequence and the "spectrum" of Phi

If we look again at the multiples of Phi, but this time concentrate on the whole number part of the multiples, we find another extraordinary relationship.
The "whole number part" of x is floor(x) so we are looking at floor(i*Phi) for i=1,2,3,.. .
The numbers in the series {trunc(i*Phi)} for i=1,2,... tell us exactly where the 1s (or Ms) appear in the Rabbit sequence!

```
i               : 1   2 3   4   5 6     7  8     ..
trunc(i*Phi)    : 1   3 4   6   8 9    11 12     ..  Position of 1's below:
                    2     5   7     10       13  ..
Rabbit sequence: 1 0 1 1 0 1 0 1 1  0  1  1  0  ..
```

The sequence of truncated multiples of a real number R is called the **spectrum of R**.

---

1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 ...

Things to do

1. Find the first few numbers in the spectrum of phi=Phi-1=0·618034 using your calculator. Some numbers in this spectrum are repeated and others are not. How do the repeated numbers relate to the rabbit sequence and how do the others?
2. What is the significance of the numbers in the spectrum of phi^2=2·618034... when regarded as index numbers of the Rabbit sequence?
3. Look at the differences between the numbers in th3e spectrum of Phi=1·618034. Do you recognize the sequence of differences?

The first 2000 bits of the Rabbit Sequence

```
1011010110 1101011010 1101101011 0110101101 0110110101   50
1010110110 1011011010 1101011011 0101101101 0110101101  100
1010110101 1011010110 1101011010 1101101011 0101101101
0110110101 1010110110 1011011010 1101011011 0101101011  200
0110101101 1010110101 1011010110 1101011010 1101101011
0101101101 0110110101 1010110110 1011010110 1101011011  300
0101101011 0110101101 1010110101 1011010110 1011011010
1101101011 0101101101 0110101101 1010110110 1011010110  400
1101011011 0101101011 0110101101 0110110101 1011010110
1011011010 1101101011 0101101101 0110101101 1010110110  500

1011010110 1101011010 1101101011 0110101101 0110110101
1011010110 1011011010 1101011011 0101101101 0110101101
1010110110 1011010110 1101011010 1101101011 0110101101
0110110101 1010110110 1011011010 1101011011 0101101101
0110101101 1010110101 1011010110 1101011010 1101101011
0101101101 0110110101 1010110110 1011011010 1101011011
0101101011 0110101101 1010110101 1011010110 1101011010
1101101011 0101101101 0110110101 1010110110 1011010110
1101011011 0101101011 0110101101 1010110101 1011010110
1011011010 1101101011 0101101101 0110101101 1010110110

1011010110 1101011011 0101101011 0110101101 0110110101
1011010110 1011011010 1101101011 0101101101 0110101101
1010110110 1011010110 1101011010 1101101011 0110101101
0110110101 1011010110 1011011010 1101011011 0101101101
0110101101 1010110110 1011010110 1101011010 1101101011
0110101101 0110110101 1010110110 1011011010 1101011011
0101101101 0110101101 1010110101 1011010110 1101011010
1101101011 0101101101 0110110101 1010110110 1011011010
1101011011 0101101011 0110101101 1010110101 1011010110
1101011010 1101101011 0101101101 0110110101 1010110110

1011010110 1101011011 0101101011 0110101101 1010110101
1011010110 1011011010 1101101011 0101101101 0110110101
1010110110 1011010110 1101011011 0101101011 0110101101
0110110101 1011010110 1011011010 1101101011 0101101101
0110101101 1010110110 1011010110 1101011010 1101101011
0110101101 0110110101 1011010110 1011011010 1101011011
0101101101 0110101101 1010110110 1011010110 1101011010
1101101011 0110101101 0110110101 1010110110 1011011010
1101011011 0101101101 0110101101 1010110101 1011010110
1101011010 1101101011 0101101101 0110110101 1010110110 2000
```

Hear the golden sequence too

The first 100 notes of the sequence are encoded in the sound track of a Quicktime movie made into notes with every "1" converted to an A note (220Hz) and every "0" into the A an octave higher (440Hz) played at about 5 notes per second (so the track lasts about 20 seconds), in a 467K file.

The rhythm is quite fascinating - hypnotic even - and it seems to have a definite beat that keeps changing and keeping your attention.

---

1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 ...

Does the Golden String ever repeat?

You can use your browser to explore the non-repeating properties of the Fibonacci Rabbit sequence.
The Golden String page contains the digits so that, by re-sizing the Browser page you will get the same number of digits per line and you can see the repetitions in the lines. The best "matches" (when lines look most alike) are when there are a Fibonacci number of digits per line (but by now you probably expected that!). Have a go and experiment for yourself.

---

1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 ...

Fractals

There is a lot of interest currently in **Fractals**. A Fractal is a shape or sequence or system that is infinite and contains a copy of itself within itself. Such pictures or series are called self-replicating or self-generating.

Our golden string contains copies of itself inside it. To see this we first show another way in which we can write down the golden string.

Another way to make the Fibonacci Rabbit sequence

Above, we started with M and then replaced M by MN. From then on, we repeatedly replace M by MN and each N by M which was the process whereby we made the Fibonacci rabbit sequence at the top of this page.

Combining this with the fact that each time we replace all the letters and get a new string, the fact that the old string is the start of the new string, then we have the following simple method of generating the golden sequence (we use 1 for M and 0 for N so that it gives the list of bits above):

1. Start by writing **`10`** (which stands for MN above) and point to the second symbol, the **`0`**, with your left hand.

Keep your right hand ready to add some more symbols at the end of the same sequence.

2. Use the symbol pointed at by your left hand to determine how to extend the sequence at the right hand end:
   - If the symbol you are pointing at with your left hand is a **1**, then, with your right hand, write **10** (at the end of the string);
   - If your left hand is pointing at a **0** then write **1** with your right hand.

   In both cases, then move your left hand to point to the next symbol along.
3. Repeat the step 2 for as long as you like.

Here is how the process starts, where the ^ indicates the symbol pointed at by our left hand:

```
   10
    ^     We are pointing at 0, so write a 1 at the end,
   101
    ^     and move the left hand on one place on (to point to the new symbol in
fact):
   101
     ^    We are pointing at a 1, so write 10 at the end
   10110
     ^    and move the left hand on one place:
   10110
      ^   We are pointing at a 1 so write 10 at the end
   1011010
      ^   and move the left hand on one place:
   1011010  ...
       ^
```

Here is the **algorithm**

```
   Start with sequence 10, pointing at the 0.
   (Step 1) if pointing at 0
                  then write 1 on to the end of the sequence;
         OR if pointing at 1
                  then write 10 at the end;
   (Step 2) Now point at the next symbol along
   (Step 3) Start again at step 1.
```

and below it is shown as an **animated gif image**:

Since we are writing more symbols than we are "reading", the sequence never ends.

The Golden String contains a copy of itself

The sequence contains a copy of itself since we can apply the above process backwards:

- Start by pointing at the left hand end of the (infinite) Fibonacci rabbit sequence with your left hand and with the right get ready to start writing another series.
- If you are pointing at "10", then write down "1".
  Otherwise you will be pointing at a "1", so write down a "0".
- Move your left hand past the symbols you have just "read" and repeat the previous step as often as you like.

You will find that your right hand is copying the original sequence, but at something like 0·6 of the speed (actually, at 0·618034... of the speed!!).

Things to do

1. Looking at the other ways of generating the Rabbit sequence above, can you adapt them to
   - find another way of writing down the golden string by replacing groups of bits pointed at by your left hand by bits written with your right hand?
   - Use your answer "backwards" to find another way in which the golden string conatinas a complete copy of itself
2. Look at the number of bits read and the number of bits written at each stage. Make a table of these two. What is the ratio between them? Do you notice the Fibonacci numbers appearing? This shows that the ratio of the two (the number of bits used to the number of bits written) will approach phi (0·6180339..).
3. Here is another way to show the Golden sequence contains a copy of itself. We "read" digits with our left hand again, one at a time, and the right hand will hop over one or two digits, crossing off the next digit. Both hands start at the leftmost digit of the golden sequence. The crossed off digits are still "readable" by the left hand when we come to them, by the way.
   If we are pointing at a **1** with the left hand, then hop over TWO digits with the right hand and cross off the next.
   If we are pointing at a **0** then hop over ONE digit with the right hand and cross off the next. [In other words, hop over one more digit than you are looking at and cross off the next.]

   Here's how the process starts:

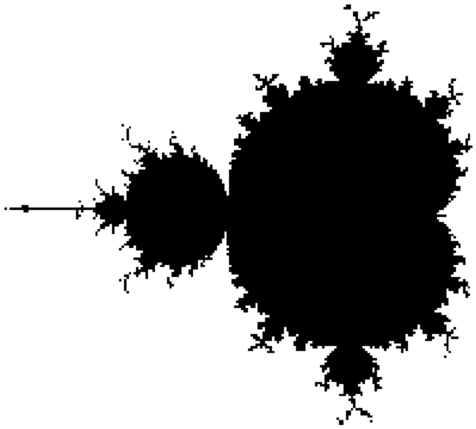
```
^ is left-hand-pointer  and v is the right hand pointer
- indicates a digit hopped over by the right hand
X indicates the digit below is to be crossed off by the right hand
+ is a crossed-out 1 and
8 is a crossed-out 0:
Here is the starting position:
 v
 1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 0 1 1 0 ...
 ^

 - - X
```

```
1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 0 1 1 0 ...
^ Hop over the first two digits and cross off the third

      - X
1 0 + 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 0 1 1 0 ...
  ^   Hop over one and cross off the next

          - - X
1 0 + 1 8 1 0 1 1 0 1 1 0 1 0 1 1 0 1 0 1 1 0 ...
    ^  Hop over two since we are pointing at a (crossed-out) 1

                - - X
1 0 + 1 8 1 0 + 1 0 1 1 0 1 0 1 1 0 1 0 1 1 0 ...
      ^
                      - X
1 0 + 1 8 1 0 + 1 0 + 1 0 1 0 1 1 0 1 0 1 1 0 ...
        ^
                          - - X
1 0 + 1 8 1 0 + 1 0 + 1 8 1 0 1 1 0 1 0 1 1 0 ...
          ^
                                - X
1 0 + 1 8 1 0 + 1 0 + 1 8 1 0 + 1 0 1 0 1 1 0 ...
            ^
                                    - - X
1 0 + 1 8 1 0 + 1 0 + 1 8 1 0 + 1 8 1 0 1 1 0 ...
              ^
We now have:
1 0 + 1 8 1 0 + 1 0 + 1 8 1 0 + 1 8 1 0 + 1 0 ...
and removing the crossed-off digits gives:
1 0   1   1 0   1 0   1   1 0   1   1 0   1 0 ...
```

which is, of course, the original sequence.
We have shown the golden sequence is **self-similar**.

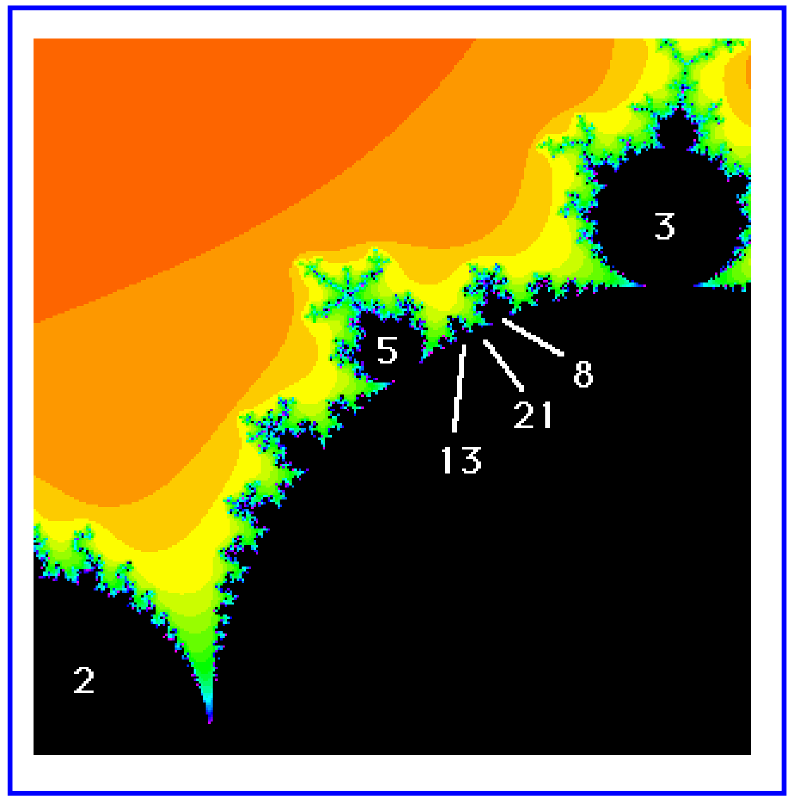
- Continue the process above for some more digits of the golden sequence and check it.
- What do you notice about *the digits we have removed*?

Fibonacci and the Mandelbrot Set



The Mandelbrot set shown here has been written about often in maths books, appears in magazines and posters, greeting cards and wrapping paper and in lots of places on the Net.

A detail from the Mandelbrot set picture is shown here. It is also a link to a page on how the Fibonacci numbers occur in the Mandelbrot Set (at Boston University Mathematics Department).



---

1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 ...

References and Links

M R Schroeder Number Theory in Science and Communication, With Applications in Cryptography, Springer-Verlag, 1990. ISBN 3540158006.
For more on the golden string as well as many reference to the Fibonacci series and the golden section.
Fractals, Chaos and Power Laws by M R Schroeder, 1992, Freeman, ISBN 0 716 72357 3. This is a fascinating book with interesting sections on Phi, the Golden sequence chaos and fractals, and the many places in nature and science where a power law applies (that is, a law of the form $y=a\ x^p$, where y is proportional to a **power** of x) although it is somewhat technical.

Goedel, Escher, Bach, D Hofstadter, Basic Books, (20th Anniversary Edition, 1999), 800 pages, is a fascinating, funny, intriguing book introducing you to the Escher's amazing pictures, Bach's contrapuntal music and the mathematical patterns in his fugues and hwo these illustrate Goedel's foundational theorem proved in the 1930's). See page 137.

The Fibonacci Tree, Hofstadter and the Golden String K P Togneti, G Winley, T van Ravenstein in Applications of Fibonacci Numbers, 3rd International Conference, (editor: A N Phillippou), pages 325-334.

Characterisation of the Set of values f(n)=[n alpha], n=1,2.. by A S Fraenkel, J Levitt, M Shimshoni, in *Discrete Mathematics* Vol 2, 1972, pages 332-345.

Links on Fractals

Here are a few links to help you explore the concept of a Fractal.
[They are not related to the Fibonacci numbers or the golden section or golden string.]

- Xah Lee's Fractal Gallery
  has lots of pictures of fractals
- Fractint
  is *free* and generates fractals on your PC.

---

1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 0 1 1 0 1 1 0 1 0 1 1 0 1 ...

Fibonacci Home Page

The Mathematical World of the Fibonacci and Phi

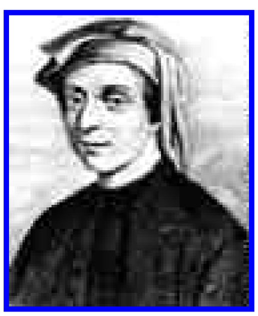
This is the only page on this Topic.

The next Topic is...
Who was Fibonacci?

A brief biographical sketch of Fibonacci, his life, times and mathematical achievements.

Contents of this Page

Who was Fibonacci?



The "greatest European mathematician of the middle ages", his full name was Leonardo of Pisa, or Leonardo Pisano in Italian since he was born in Pisa (Italy), the city with the famous Leaning Tower, about 1175 AD.

Pisa was an important commercial town in its day and had links with many Mediterranean ports. Leonardo's father (Guglielmo Bonaccio) was a kind of customs officer in the North African town of Bugia now called Bougie where wax candles were exported to France. They are still called "bougies" in French, but the town is a ruin today says D E Smith (see below).

So Leonardo grew up with a North African education under the Moors and later travelled extensively around the Mediterranean coast. He would have met with many merchants and learned of their systems of doing arithmetic. He soon realised the many advantages of the "Hindu-Arabic" system over all the others.

D E Smith points out that another famous Italian - St Francis of Assisi (a nearby Italian town) - was also alive at the same time as Fibonacci: St Francis was born about 1182 (after Fibonacci's around 1175) and died in 1226 (before Fibonacci's death commonly assumed to be around 1250).

[The portrait here is a link to the University of St Andrew's site which has more on Fibonacci himself, his life and works.]

His names

He called himself **Fibonacci** [pronounced fib-on-arch-ee or fee-bur-narch-ee] short for **filius Bonacci** which means *son of Bonacci*. Since Fibonacci in Latin is "filius Bonacci" and means "the son of Bonacci", two early writers on Fibonacci (Boncompagni and Milanesi) regard Bonacci as the family name so that Fib-Bonacci is like the English names of Robin-son or John-son. Fibonacci himself wrote both "Bonacci" and "Bonaccii" as well as "Bonacij"! Others think Bonacci may be a kind of nick-name meaning "lucky son" (literally, "son of good fortune").
He is perhaps more correctly called **Leonardo of Pisa** or, using a latinisation of his name, **Leonardo Pisano**. Occasionally he also wrote **Leonardo Bigollo** since, in Tuscany, *bigollo* means *a traveller*.

We shall just call him Fibonacci as do most modern authors, but if you are looking him up in older books, be prepared to see any of the above variations of his name.
D E Smith's History of Mathematics Volume 1, (Dover, 1958 - a reprint of the orignal version from 1923) gives a complete list of other books that he wrote and is a fuller reference on Fibonacci's life and works.

There is another brief biography of Fibonacci which is part of Karen Hunger Pashall's (Virginia University) The art of Algebra from from al-Khwarizmi to Viéte: A Study in the Natural Selection of Ideas if you want to read more about the history of mathematics.

Eight Hundred Years Young by A F Horadam (University of New England) in *The Australian Mathematics Teacher* Vol 31, 1985, pages 123-134, is an interesting and readable article on Fibonacci, his names and origins as well as his mathematical works. He refers to and expands upon the following article...

The Autobiogra[hy of Leonardo Pisano R E Grimm, in *Fibonacci Quarterly* vol 11, 1973, pages 99-104.

Leonard of Pisa and the New Mathematics of the Middle Ages by J and F Gies, Thomas Y Crowell publishers, 1969, 127 pages, is another book with much on the background to Fibonacci's life and work.

Della vita e delle opere di Leonardo Pisano Baldassarre Boncompagni, Rome, 1854 is the only complete printed version of Fibonacci's 1228 edition of *Liber Abbaci*.

The the Math Forum's archives of the History of Mathematics discussion group contain a useful discussion on some of the controversial topics of Fibonacci's names and life (February 1999). Use its **next>>** link to follow the thread of the discussion through its 6 emailed contributions. It talks about the uncertainlty of his birth and death dates and his names. It seems that Fibonacci never referred to himself as "Fibonacci" but this was a nick-name given to him by later writers.

Fibonacci's Mathematical Contributions

Introducing the Decimal Number system into Europe

He was one of the first people to introduce the Hindu-Arabic number system into Europe - the positional system we use today - based on ten digits with its decimal point and a symbol for zero:

```
1  2  3  4  5  6  7  8  9  .    and 0
```

His book on how to do arithmetic in the decimal system, called **Liber abbaci** (meaning *Book of the Abacus* or *Book of Calculating*) completed in 1202 persuaded many European mathematicians of his day to use this "new" system.

The book describes (in Latin) the rules we all now learn at elementary school for adding numbers, subtracting, multiplying and dividing, together with many problems to illustrate the methods:

```
1 7 4 +     1 7 4 -       1 7 4 x     1 7 4 ÷ 28
  2 8         2 8           2 8             is
-----       -----       -------
2 0 2       1 4 6       3 4 8 0 +     6 remainder 6
-----       -----       1 3 9 2
                        -------
                        4 8 7 2
                        -------
```

Let's first of all look at the Roman number system still in use in Europe at that time (1200) and see how awkward it was for arithmetic.

Roman Numerals

The method in use in Europe until then used the Roman numerals:

```
I = 1,
V = 5,
X = 10,
L = 50,
C = 100,
```

```
D = 500 and
M = 1000
```

You can still see them used on foundation stones of old buildings and on some clocks. For instance, 13 would be written as XIII or perhaps IIIX. 2003 would be MMIII or IIIMM. 99 would be LXXXXVIIII and 1998 is MDCCCCLXXXXVIII.

Later, an abbreviation became popular where the *order* of letters *did* matter and, if a single smaller value came *before* the next larger one, it was *subtracted* and if it came after, it was added as usual.
For example, XI means 10+1=1 but IX means 1 less than 10 or 9. 8 is still written as VIII (not IIX).
[Note that in the UK we use a similar system for time when 6:50 is often said as "ten to 7" rather than "6 fifty", similarly for "a quarter to 4" meaning 3:45. In the USA, 6:50 is sometimes referred to as "10 of 7".]
Using this method, 1998 would be written much more compactly as MCMXCVIII but this takes a little more time to interpret: 1000 + (100 less than 1000) + (10 less than 100) + 5 + 1 + 1 + 1.

> *Look out for Roman numerals used as the date a film was made, often recorded on the screen which gives its censor certification or perhaps the very last image of the movie giving credits or copyright information.*

Arithmetic with Roman Numerals

Arithmetic was not easy in the Roman system:

```
CLXXIIII added to XXVIII  is CCII
CLXXIIII less     XXVIII  is CXXXXVI
```

For more on Roman Numerals, see the excellent Frequently Asked Questions on Roman Numerals at Math Forum.

The Decimal Positional System

The system that Fibonacci introduced into Europe came from India and Arabia and used the Arabic symbols 1, 2, 3, 4, 5, 6, 7, 8, 9 with, most importantly, a symbol for zero 0.
With Roman numbers, 2003 could be written as MMIII or, just as clearly, it could be written as IIIMM - the order does not matter since the values of the letters are added to make the number in the original (unabbreviated) system. With the abbreviated system of IX meaning 9, then the order did matter but it seems this sytem was not often used in Roman times.
In the "new system", the order *does* matter *always* since 23 is quite a different number to 32. Also, since the *position* of each digit is important, then we may need a *zero* to get the digits into their correct places

(columns) eg 2003 which has no tens and no hundreds. (The Roman system would have just omitted the values not used so had no need of "zero".)

This *decimal positional system*, as we call it, uses the ten symbols of Arabic origin and the "methods" used by Indian Hindu mathematicians many years before they were imported into Europe. It has been commented that in India, the concept of *nothing* is important in its early religion and philosophy and so it was much more natural to have a symbol for it than for the Latin (Roman) and Greek systems.

"Algorithm"

Earlier the Persian author Abu Ja'far Mohammed ibn Mûsâ al-Khowârizmî had written a book which included the rules of arithmetic for the decimal positional number system, called *Kitab al jabr w'al-muqabala* (Rules of restoration and reduction) dating from about 825 AD. D E Knuth says his name can be translated as *Father of Ja'far, Mohammed, son of Moses, native of the town of Al-Khowârizmî*. He was an astromomer to the caliph at Baghdad (now in Iraq).

- Al-Khowârizmîis the region south and to the east of the Aral Seaaround the town now called **Khiva**(or Urgench) on the Amu Darya river. It was part of the Silk Route, a major trading pathway between the East and Europe. In 1200 it was in Persia but today is in Uzbekistan, part of the former USSR, north of Iran, which gained its independence in 1991.
- Prof Don Knuthhas a picture of a postage stampissued by the USSR in 1983 to commemorate al-Khowârizmî1200 year anniversary of his probable birth date.
- From the title of this book *Kitab* ***al jabr*** *w'al-muqabala*we derive our modern word **algebra**.
- The Persian author's name is commemorated in the word **algorithm**. It has changed over the years from an original European pronunciation and latinisation of *algorism*. Algorithms were known of before Al-Khowârizmî's writings, (for example, Euclid's *Elements*is full of algorithms for geometry, including one to find the greatest common divisor of two numbers called *Euclid's algorithm*today).
- The USA Library of Congress has a list of citationsof Al-Khowârizmîand his works.

Our modern word "algorithm" does not just apply to the rules of arithmetic but means ***any precise set of instructions for performing a computation*** whether this be

- a method followed by humans, for example:
  - a cooking **recipe**;
  - a knitting **pattern**;
  - travel **instructions**;
  - a car **manual page**for example, on how to remove the gear-box;
  - a medical **procedure**such as removing your appendix;
  - a calculation by **human computors**: two examples are:
    - William Shankswho computed the value of pi to 707 decimal places by hand last century

over about 20 years up to 1873 - but he was wrong at the 526-th place when it was checked by desk calculators in 1944!
Earlier Johann Dasehad computed pi correctly to 205 decimal places in 1844 when aged 20 but *this was done completely in his head*just writing the number down after working on it for two months!!

- or **mechanically** by machines (such as placing chips and components at correct places on a circuit board to go inside your TV)
- or **automatically** by electronic computers which store the instructions as well as data to work on.

See D E Knuth, The Art of Computer Programming *Volume 1: Fundamental Algorithms* (now in its Third Edition, 1997)pages 1-2.

There is an English translation of the ".. al jabr .." book: L C Karpinski **Robert of Chester's Latin Translation ... of al-Khowarizmi** published in New York in 1915. [Note the variation in the spelling of "Al-Khowârizmî" here - this is not unusual! Other spellings include al-Khorezmi.]

Ian Stewart's **The Problems of Mathematics** (Oxford) 1992, ISBN: 0-19-286148-4 has a chapter on algorithms and the history of the name: *chapter 21: Dixit Algorizmi*.

The Fibonacci Series

In Fibonacci's book he introduces a problem for his readers to use to practice their arithmetic:-

> *a pair of rabbits are put in a field and, if rabbits take a month to become mature and then produce a new pair every month after that, how many pairs will there be in twelve months time?*

He assumes the rabbits do not escape and none die. The answer involves the series of numbers:

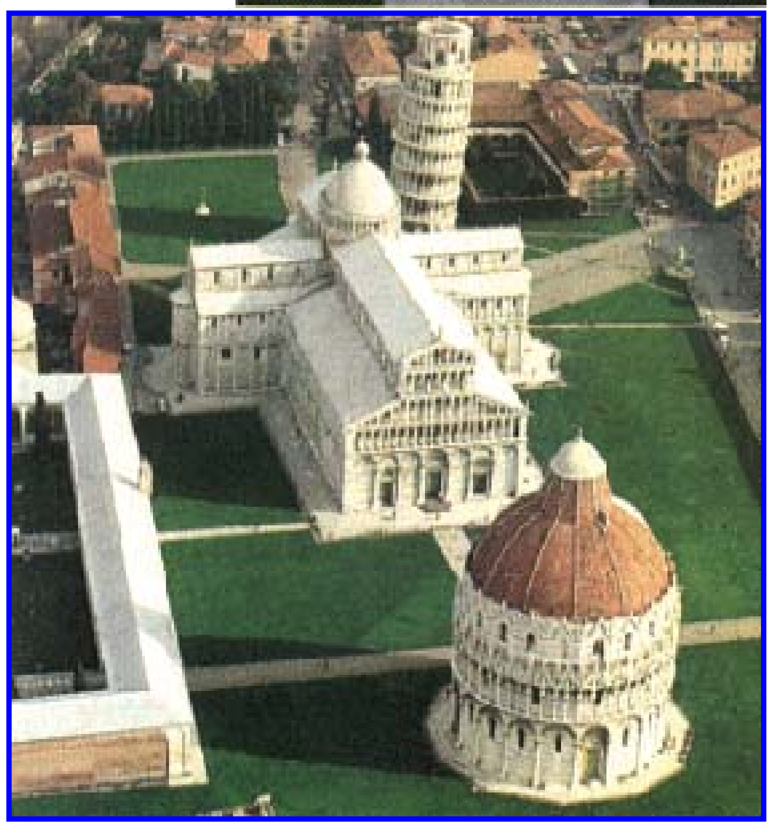
1, 1, 2, 3, 5, 8, 13, 21, ...

but it was the French mathematician Edouard Lucas (1842-1891) who gave the name **Fibonacci numbers** to this series and found many other important applications of them.

Fibonacci memorials to see in Pisa

He died in the 1240's and there is now a statue commemorating him located at the Leaning Tower end of the cemetery next to the Cathedral in Pisa. [With special thanks to Nicholas Farhi, an ex-pupil of Winchester College, for the picture of the statue.]

The picture of Pisa's cathedral and leaning tower is a link to more information on Pisa.



Clark Kimberling, Professor of Mathematics at Evansville University, Indiana, has a Fibonacci biography page. It shows the face of another Fibonacci statute down by the Arno river off the Via Fibonacci.

Fibonacci's Mathematical Books

Leonardo of Pisa wrote 5 mathematical works, 4 as books and one preserved as a letter:

Liber Abbaci, 1202 but revised in 1228.

meaning *The Book of the Abacus* (or *The Book of Calcuating*). One of the problems in this book was the problem about the rabbits in a field which introduced the series 1, 2, 3, 5, 8, ... It was much later (around 1870) that Lucas named this series of numbers after Fibonacci.

Practica geometriae, 1220.

A book on geometry.

Flos, 1225

Liber quadratorum, 1225

The Book of Squares, his largest book.

It was translated into English by L E Sigler and published as **The Book of Squares** in 1987, Academic Press. Another article about this book:

Leonardo of Pisa and his *Liber Quadratorum* by R B McClenon in *American Mathematical Monthly* vol 26, pages 1-8.

A letter to Master Theodorus, around 1225.

Theodorus was a philosopher at the court of the Holy Roman Emporer Frederick II.

There is a very readable outline of the problems in the letter to Master Theodorus in:

Fibonacci's Mathematical Letter to Master Theodorus A F Horodam, *Fibonacci Quarterly* 1991, vol 29, pages 103-107.

The most comprehensive translation of the manuscripts of the 5 works above is:

Scritti di Leonardo Pisano B Boncompagni, 2 volumes, published in Rome in 1857 (vol 1) and 1862 (vol 2).

References to Fibonacci's Life and Times

Leonardo of Pisa and the New Mathematics of the Middle Ages J Gies, F Gies, Crowell press, 1969.

The Autobiography of Leonardo Pisano R E Grimm, in *Fibonacci Quarterly*, vol 11, 1973, pages 99-104 with *corrections* on pages 162 and 168.

800 Years young A F Horodam in *Australian Mathematics Teacher* vol 31, 1975, pages 123-134.

Fibonacci Home Page

The Golden String

WHERE TO NOW???

This is the only page on Who was Fibonacci?

The next topic is...

More Applications of the Fibonacci Numbers and Phi

 R.Knott@surrey.ac.uk Created: 11 March 1998, Latest revision: 14 February 2001

Pi and the Fibonacci Numbers

Surprisingly, there are several formulae that use the Fibonacci numbers to compute Pi (π).
Here's a brief introduction from scratch to all that you need to know to appreciate these formulae.

Contents of this page

The line means there is a Things to doinvestigation at the end of the section.

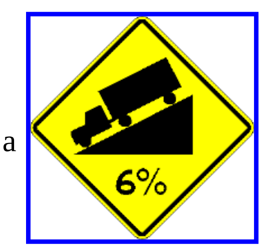
How Pi is calculated

Until very recently there were just two methods used to compute pi, one invented by the Greek mathematician Archimedes, and the other by the Scottish mathematician James Gregory. We'll just look at Gregory's method here.

Measuring the steepness of a hill

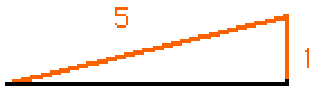
The *steepness* of a hill can be measured in different ways.
It is shown on road signs which indicate a hill and the measure of the steepness is indicated in differing ways from country to country. Some countries measure the steepness by a ratio (eg 1 in 3) and others by a percentage.

The ratio is converted to a decimal to get its percentage, so a slope of "1 in 5" means 1/5 or 20%.
The picture on the road-sign tells us if we are going up a hill or down.
We could say that a 20% rise is a steepness measured as +20% and a 20% fall as a steepness of -20% too.

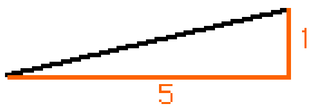


But what does "a slope of 1 in 5" mean?

There are two interpretations.



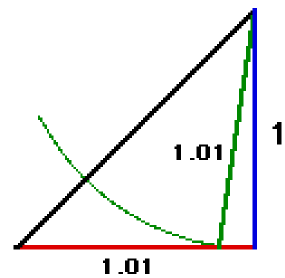
Some people take "1 in 5" to mean **the drop (or rise) of 1 (metres, miles or kilometers) for every 5 (metres, miles, kilometers) travelled *along the road***. In the diagram, the distances are shown in orange.



Others measure it as **the drop or rise per unit distance *travelled horizontally***. A "1 in 5" slope means that I would rise 1 metre for every 5 metres travelled *horizontally*. The same numbers apply if I measure distance in miles or centimeters or any other unit.

In the second interpretation it is easier to calculate the steepness from a map. On the map, take two points where contour lines cross the road. The *contour lines* give the rise or fall in height *vertically* between the two points. Using a ruler and *the scale of the map* you can find the *horizontal* distance between the points but make sure it is in the same units as the horizontal distance! Dividing one by the other gives the ratio measuring the steepness of the road between the two points.

But they look the same slope?
Yes, they do when the slope is "1 in 5" because the difference is very small - about 0·23° in fact.



Here is a slope of 1·01. The green line is 1·01 times as long as the blue height and the red line is too. You can see that they "measure" very different slopes (the green line and the black line are clearly different slopes now).
What do you think a slope of "1 in 1" means in the two interpretations? Only one interpretation will mean a slope of 45° - which one?

So we had better be clear about what *we* mean by slope of a line in mathematics!!

- The first interpretation is called the **sine** of the angle of the slope where we divide the change in height by the **distance along the road (hypotenuse)**.
- The second interpretation is called the **tangent** of the angle of the slope where we divide the change in height by the **horizontal distance**.

The slope of a line in mathematics is ALWAYS taken to mean the *tangent* of the angle of slope.

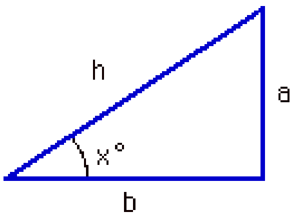
So in mathematics, as on road-signs, we measure the slope by a *a ratio which is just a number*. The higher the number, the steeper the slope. A perfectly "flat" road will have slope 0 in both interpretations. Uphill roads will have a positive steepness and downhill roads will be negative in both interpretations.
In mathematics, a small incline upwards will have slope 0·1 (i.e.10% or 1/10 or *a rise of 1 in 10*)
a road going slightly downhill had slope -0·2 (i.e. 20% or 1/5 or *a fall of 1 in 5*); a fairly steep road uphill will have slope 0·4 (ie 40% or 2/5) and the same road travelled in the other direction (downhill) has the same number, but negative: -0·4
In mathematics, a "1 in 1 " slope will means a metre rise for every metre travelled "along", so the slope is 1:1 = 1/1 = 1 or 45 degrees (upward).

Note that with the other interpretation (using the *sine* of the angle) of **1 in 1** is a rise of 1 metre for every metre *along the road*. This would mean a vertical road (a cliff-face) which is not at all the same thing as a tangent of 1!

Similarly, in mathematics, a slope of -1 would be a hill going downwards at 45 degrees.
In maths, lines can have slopes much steeper than roads designed for vehicles, so our slopes can be anything up to vertical

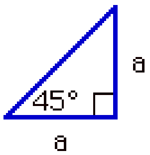
both upwards and downwards. Such a line would have a slope of "infinity".

The *tangent* of an angle



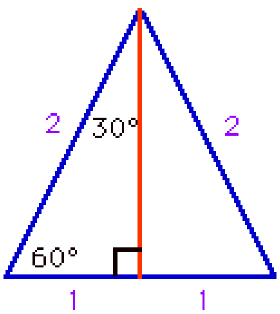
So we can relate the *angle of the slope* to the *ratio of the two sides of the (right-angled) triangle*. This ratio is called the **tangent of the angle**.
In the diagram here, the tangent of angle x is a/b, written:

$$\tan(x) = {}^{a}/_{b}$$



A 45° right-angled triangle has the two sides by the right-angle of equal size, so their ratio is 1, which we write as

$$\tan(45°) = {}^{a}/_{a} = 1$$



If we split an equilateral triangle (ie all sides and all angles are the same) in half, we get a 60°-30°-90° triangle as shown:
We can use **Pythagoras' Theorem** to find the length of the vertical red line. Pythagoras' Theorem says that, in any right-angled triangle with sides a, b and h (h being the hypotenuse which is the longest side - see the first triangle here) then

$$a^2 + b^2 = h^2$$

So, in our split-equilateral triangle with sides of length 2, its height squared must be $2^2-1^2=3$, ie its height is $\sqrt{3}$.
So we have

$$\tan(60°) = \sqrt{3} \text{ and}$$
$$\tan(30°) = {}^{1}/_{\sqrt{3}}$$

The arctan function

If we are given a slope (a tangent of an angle) we may want to find the angle of that slope. This would mean *using the tangent function "backwards"* which in mathematics is called *the inverse* of the tangent function.
It is called the **atan** or **arctan** function so that arctan(t) takes a slope t (a tangent number) and returns the *angle* of a straight line with that slope.

Gregory's Formula for arctan(t)

In 1672, James Gregory (1638-1675) wrote about a formula for calculating the angle given the tangent **t** for angles up to 45° (i.e for tangents or slopes **t** of size up to 1):

$$\arctan(t) = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \ldots$$

Actually, it is not so much a formula as a series, since it goes on for ever.
So we could ask if it will it ever compute an actual value (an angle) if there are always terms to come?
Provided that **t** *is less than 1 in size* then *the terms will get smaller and smaller as the powers of* ***t*** *get higher and higher*. So we can stop after some point confident that the terms missed out contribute an amount too small to alter the amount we have already computed to a certain degree of accuracy. [The question now becomes: "How many terms do I need for a given degree of accuracy?"]

> ***Why must the value of* t *not exceed 1?***

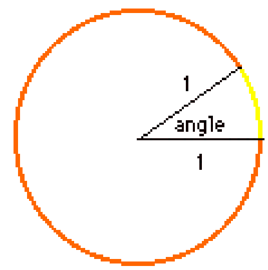
Look at what happens when t is 2, say. t^3 is then 8, the fifth power is 32, the seventh power 128 and so on. Even when we divide by 3,5,7 etc, the values of each term get bigger and bigger (called **divergence**).
The only way that powers can get smaller and smaller (and so the series settles down to a single sum or the series **converges**) is when $t<1$.
For this series, it also gives a sum if $t=1$, but as soon as $t>1$, the series diverges.
Of course t may be negative too. The same applies: the series converges if t is greater than -1 (its size is *less* than 1 if we ignore the sign) and diverges if t is less than -1 (its size is *greater* than 1 if we ignore the sign).
The neatest way to sum this up is to say that

Gregory's series converges if t does not exceed 1 in size (ignoring any minus sign) i.e. $-1 \le t \le 1$.

The error between what we compute for an arctan and what we leave out will be small if we take lots of terms.

The limiting angle that Gregory's Series can be used on has a tangent that is just 1, ie 45 degrees.

Radian measure



First, we note that the angle in Gregory's series is not returned in degrees, but in *radians* which turns out to be the "natural" measure of angles since formulae are much simpler if we use this rather than degrees.
If we draw the angle at the center of a circle of unit radius, then the radian is the **length of the arc** cut off by the angle (hence the "arc" in "arctan": "the arc of an angle whose tangent is...").
So 360 degrees is the whole circumference, that is

360° = 2 Pi radians = 2 Pi^r and halving this gives
180° = Pi radians = Pi^r and
90° = Pi/2 radians = $(\text{Pi}/2)^r$.
Since 60° is a sixth of a full turn (360°) then
60° = 2 Pi/ 6 = Pi/3 radians = $(\text{Pi}/3)^r$ and so
30° = Pi/6 radians = $(\text{Pi}/6)^r$.

Note that, when it does not cause confusion with "raising to the power r" then a^r means "a radians".
A single degree is 1/360 of a full turn of 2 Pi radians so

1° = 2 Pi/360 radians = pi/180 radians

Similarly, 1 radian is 1/(2 Pi) of a full turn of 360 degrees so

1 radian = 360 / (2 Pi) degrees = 180 / Pi degrees.

Using radian measure explains why the inverse-tangent function is also called the **ARCtan** function - it returns the arc angle when given a tangent.

Gregory's series and π

We now have several angles whose tangents we know :-
tan 45° (or π/4 radians) =1, therefore

$$\arctan(1) = \frac{\pi}{4}$$

and if we plug this into Gregory's Series: $\arctan(t) = t - t^3/3 + t^5/5 - t^7/7 + t^9/9 - \dots$ we get the following surprisingly simple and beautiful formula for Pi:

$$\arctan(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

Actually, Gregory never explicitly wrote down this formula but another famous mathematician of the time, Gottfried Leibnitz (1646-1716), mentioned it in print first in 1682, and so this special case of Gregory's series is usually called **Leibnitz Formula for π**.

We can use other angles whose tangent we know too to get some more formulae for Pi. For instance, earlier we saw that tan 60° (orπ/3 radians) = √3 therefore

$$\arctan(\sqrt{3}) = \frac{\pi}{3}$$

So what formula do we get when we use this in Gregory's Series? But wait!!! √3 is bigger than 1, so Gregory's series cannot be used!! The series we would get is not useful since wherever we stop it, the terms left out will always contribute a much larger amount and swamp what we already have. In mathematics we would say that **the sum *diverges***.
Instead let's still use the 30-60-90 triangle, but consider the other angle of 30°. Since tan 30° (or π/6 radians) = 1/√3 which **is** less than 1:

$$\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

The other angle whose tangent we mentioned above gives :

$$\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3\times3\sqrt{3}} + \frac{1}{5\times3^2\sqrt{3}} - \frac{1}{7\times3^3\sqrt{3}} + \dots$$

We can factor out the √3 and get

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\times\left(1 - \frac{1}{3\times3} + \frac{1}{5\times3^2} - \frac{1}{7\times3^3} + \frac{1}{9\times3^4} - \dots\right)$$

or

$$\pi = 2\sqrt{3}\left(1 - \frac{1}{3\times3} + \frac{1}{5\times3^2} - \frac{1}{7\times3^3} + \frac{1}{9\times3^4} - \dots\right) \qquad (**)$$

Using Gregory's Series to calculate π

If you try and work out the value of π/4 from the formula marked as (*) above, you find that the formula, although very pretty (or *elegant* as mathematicians like to say), it is not very useful or practical for calculating pi:

```
1    =   1·0000000000000000000 -
1/3 =   0·3333333333333333333 +
1/5 =   0·2000000000000000000 -
1/7 =   0·1428571428571428571 +
1/9 =   0·1111111111111111111 -
1/11=   0·0909090909090909090 +
1/13=   0·0769230769230769230 -
...
```

In fact, the first **5** terms have to be used before we get to 1/11 which is less than 1/10, that is, before we get a term with a 0 in the first decimal place.
It takes **50** terms before we get to 1/101 which has 0s in the first two decimal places and
500 terms before we get terms with 3 initial zeros.
We would need to compute **five million** terms just to get π/4 to 6 (or 7) decimal places!

This is called a slow "rate of convergence".

The second formula above that is marked (**) that we derived from arctan(1/√3) is *a lot better*

```
1         = 1·000000000000 -
1/9       = 0·111111111111 +
1/45      = 0·022222222222 -
1/189     = 0·005291005291 +
1/729     = 0·001371742112 -
1/2673    = 0·000374111485 +
1/9477    = 0·000105518624 -
1/32805   = 0·000030483158 +
1/111537  = 0·000008965634 -
1/373977  = 0·000002673961 +
1/1240029= 0·000000806432 -
...
```

and after just 10 terms, we are getting zeros in the first 6 places - remember that would have been after at least half a million terms by Leibnitz Formula!
Summing the above and multiplying by 2√3 gives

π = 3·14159 to 5 decimal places

The only problem with the faster formula above is that we need to use √3 and, before calculators were invented, this was tedious to compute.
Can we find some other formulae where there are some nice easy tangent values that we know but which *don't* involve computing square roots? Yes!

Machin's Formula

In 1706, John Machin (1680-1752) found the following formula:

$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

The 239 number is quite large, so we never need very many terms of arctan(1/239) before we've got *lots* of zeros in the initial decimal places. The other term, arctan(1/5) involves easy computations if you are computing terms by hand, since it involves finding reciprocals of powers of 5. In fact, that was just what Machin did, and computed 100 places by hand!

Here are the computations:

```
All computations to 15 decimal places:

arctan(1/5)                              arctan(1/239):
1/5              = 0·200000000000000     1/239          = 0·004184100418410
1/375            =-0·002666666666666     1/40955757     =-0·000000024416591
1/15625          = 0·000064000000000     1/3899056325995= 0·000000000000256
1/546875         =-0·000001828571428
1/17578125       = 0·000000056888889
1/537109375      =-0·000000001861818
1/15869140625    = 0·000000000063015
1/457763671875   =-0·000000000002184
1/12969970703125 = 0·000000000000077
```

```
1/362396240234375=-0·000000000000002

SUMMING:
arctan(1/5)      = 0·1973955598498807  and arctan(1/239) = 0·004184076002074

Putting these in the Machin's formula gives:
        Pi/4=  4xarctan(1/5 )         -   arctan( 1/239 )
 or     Pi  = 16xarctan(1/5 )         - 4xarctan( 1/239 )
            = 16x0·1973955598498807   - 4x0·004184076002074
            = 3·1415926535897922
```

Another two-angle arctan formula for π

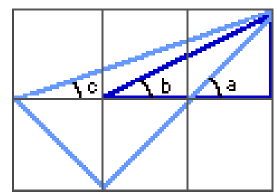
Here's another beautifully simple formula which Euler (1707-1783) wrote about in 1738:

$$\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)$$

It's even more *elegant* when we write pi/4 as **arctan(1)**:

$$\arctan(1) = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)$$

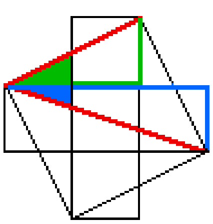
With just a little geometry and the diagram here, you might be able to verify that this formula is indeed correct.



HINTS:

1. What are tan(a), tan(b) and tan(c) from the diagram?
2. The dark blue and light blue triangles are the same shape (why? consider tangents)
3. so which angle in the light-blue triangle is the same as b in the dark blue one?
4. Angles in a triangle add to 180 degrees so what can you say about angle c and a as shown and the new angle equal to b? (ie prove that angle a = angle b + angle c)
5. Express this angle relationship using arctans, since you know their tangents from Hint 1 above.
6. *Eh Voila!*

Here is another diagram which illustrates the relationship even more simply:



The green angle has a tangent of 1/2;
the blue angle has a tangent of 1/3;
together they make the corner angle in red whose tangent is 1.

NOW we are ready for the formula using the Fibonacci Numbers to compute π!

Pi and the Fibonacci Numbers

Now we return to using the Fibonacci numbers to compute π. Euler's formula that we have just proved:

$$\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)$$

is good for computing π since 1/2 and 1/3 are smaller than 1. (The smaller the value of the tangent in Gregory's formula, the quicker the sum converges and the less work we have to do to find pi!)

Things to do

- Use this formula to compute π to a few decimal places *by hand*

Are there any more formulae like it, that is, using two angles whose tangents we know and which add up to 45 degrees (ie π/4 radians whose tangent is 1)?

Yes, here are some (not proved here). Can you spot the pattern?

```
Pi/4 = arctan(1)  and ...
arctan(1)    = arctan(1/2)  + arctan(1/3)

arctan(1/3)  = arctan(1/5)  + arctan(1/8)

arctan(1/8)  = arctan(1/13) + arctan(1/21)

arctan(1/21) = arctan(1/34) + arctan(1/55)
```

We can combine them by putting the second equation for arctan (1/3) into the first to get:

```
Pi/4 = arctan(1)
     = arctan(1/2) + arctan(1/3)
     = arctan(1/2) + arctan(1/5) + arctan(1/8)
```

and then combine this with the third equation for arctan(1/8) to get:

```
Pi/4 = arctan(1/2) + arctan(1/5) + arctan(1/13) + arctan(1/21)
```

You'll have already noticed the **Fibonacci numbers** here. However, not all the Fibonacci numbers appear on the left hand sides. For instance, we have no expansion for arctan(1/5) nor for arctan(1/13).
Only the *even numbered Fibonacci terms* seem to be expanded (F(2)=1, F(4)=3, F(6)=8, F(8)=21, ...):

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

The General Formulae

We have just seen that there are *infinitely many* **formulae for Pi using the Fibonacci numbers**! They are:

```
  Pi/4 = arctan(1)
       = arctan(1/2) + arctan(1/3)
       = arctan(1/2) + arctan(1/5) + arctan(1/8)
       = arctan(1/2) + arctan(1/5) + arctan(1/13) + arctan(1/21)
       = arctan(1/2) + arctan(1/5) + arctan(1/13) + arctan(1/34) + arctan(55)
       =...
```

or, putting these in terms of the Fibonacci numbers:

```
  Pi/4 = arctan(1/Fib(1) )
       = arctan(1/Fib(3)) + arctan(1/(Fib(4))
       = arctan(1/Fib(3)) + arctan(1/Fib(5)) + arctan(1/Fib(6))
       = arctan(1/Fib(3)) + arctan(1/Fib(5)) + arctan(1/Fib(7)) + arctan(1/Fib(8))
       = arctan(1/Fib(3)) + arctan(1/Fib(5)) + arctan(1/Fib(7)) + arctan(1/Fib(9)) +
arctan(1/Fib(10))
       = ...
```

What is the general formula?
It is

$$\arctan\left(\frac{1}{\text{Fib}(2n)}\right) = \arctan\left(\frac{1}{\text{Fib}(2n+1)}\right) + \arctan\left(\frac{1}{\text{Fib}(2n+2)}\right)$$

What happens if we keep on expanding the last term as we have done above?
We get the infinite sum

$$\arctan(1) = \sum_{n=1}^{\infty} \arctan\left(\frac{1}{F(2n+1)}\right)$$

or

```
  arctan(1) = arctan(1/Fib(3)) + arctan(1/Fib(5)) + arctan(1/Fib(7)) + ...
            = arctan(1/2)    + arctan(1/5)    + arctan(1/13)+...
```

which is a special case of the following when k is 1:

$$\arctan\left(\frac{1}{F(2k)}\right) = \sum_{n=k}^{\infty} \arctan\left(\frac{1}{F(2n+1)}\right)$$

Some more formulae for two angles

There are many more angles which have tangents of the form 1/X which are the sum of two other angles with tangents of the same kind. Above we looked at such formulae which only involved the Fibonacci numbers. Here are some more examples:

```
  arctan(1/2) = arctan(1/ 3) + arctan(1/ 7)
  arctan(1/3) = arctan(1/ 4) + arctan(1/13)
  arctan(1/4) = arctan(1/ 5) + arctan(1/21)
```

```
arctan(1/5) = arctan(1/ 6) + arctan(1/31)
arctan(1/5) = arctan(1/ 7) + arctan(1/18)
arctan(1/6) = arctan(1/ 7) + arctan(1/43)
arctan(1/7) = arctan(1/ 8) + arctan(1/57)
arctan(1/7) = arctan(1/ 9) + arctan(1/32)
arctan(1/7) = arctan(1/12) + arctan(1/17)
```

Some Experimental Maths for you to try

Here are some suggestions to see if we can find some *reasons* for the above results, and some *order* in the numbers.

You can use a computer to do the hard work, then you have the fun job of looking for patterns in its results! This is called Experimental Mathematics since we are using the computer as a microscope is used in biology or like a telescope for astronomy. We can find some results that we then have to find a theory or explanation for, except that what we look at is the World of Numbers, not plants or stars.

Things to do

1. Is there a formula of the kind

   arctan(1/X) = arctan(1/Y) + arctan(1/Z)

   for **all positive integers X** (Y and Z also positive integers)? that is, if I give you an X can you *always* find a Y and a Z?

   How would you go about doing a **computer search for numerical values** that look as if they might be true (ie searching through some small values of X, Y and Z and seeing where the value of the left hand side is **almost** equal to the value of the right hand side? [Remember, it could just be that the numbers are really **almost** equal but not exactly equal. However, you have to allow for small errors in your computer's tan and arctan functions, so you almost certainly will **not** get zero exactly even for results which we can prove are true mathematically. This is the central problem of Experimental Maths and show that it never avoids the need for **proving your results**.]

2. Can you spot any **patterns in the numerical results** of your computer search?
3. Can you **prove** that your patterns are always true?
   Try a different approach to the proofs. Since we *have a proof* for the first result (we used the dark blue and light blue triangles in the diagram earlier in this page), can we **extend or generalize the proof method**?
4. Once you have a list of pairs of angles which sum to another, you can use it to generate three angles that sum to another (as we did for 3 then 4 and an infinite number for the arctan(1) series for π above). Eg:

   arctan(1/4) = **arctan(1/5)** + arctan(1/21)
   and **arctan(1/5)** = arctan(1/6) + arctan(1/31)
   and substituting gives
   arctan(1/4) = arctan(1/6) + arctan(1/21) + arctan(1/31)

   Perhaps there are sums of three angles that are NOT generated in this way (ie where any two of the angles do *not* sum to one with a tangent of the form 1/X)? It looks like:

arctan(1/2) = arctan(1/4) + arctan(1/5) + arctan(1/47)

might be one (if, indeed, it is exactly true). If so, how would you go about searching for them numerically?

5. We've only looked at angles whose tangents are of the form 1/N. Perhaps there are some nice formula for expressing angles of the form arctan(M/N) as the sum of angles of the form arctan(1/X)? or even as a sum of other such "rational" tangents, not just reciprocals. What patterns are there here?

   To start you off:
   One such pattern looks like having Y=X+1, that is,

   arctan(1/X) = arctan(1/(X+1)) + arctan(1/Z)

   Here are some results from a computer search (- or are they?!? - see below):
   NB To save space here and also in other mathematical texts, **arctan** is abbreviated further to **atan**.

   atan(1/**2**)=atan(1/**3**)+atan(1/7)
   atan(1/**3**)=atan(1/**4**)+atan(1/13)
   atan(1/**4**)=atan(1/**5**)+atan(1/21)
   atan(1/**5**)=atan(1/**6**)+atan(1/31)
   atan(1/**6**)=atan(1/**7**)+atan(1/43)
   atan(1/**7**)=atan(1/**8**)+atan(1/57)
   atan(1/**8**)=atan(1/**9**)+atan(1/72)

   In fact, **there IS a mistake in one of these 7 lines** because a genuine mathematical pattern is spoilt by one of the results - but which one? Can you find a formula for Z and can you **prove** that it is *always* true?

6. **Tadaaki Ohno**, a mathematics student at the University of Tokyo , Japan, (July 1999) has found a nice method of looking for arctangent relations which depends on factoring numbers. Using the following formula for the tangent of the sum of two angles, a and b:

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

   He transforms it into the problem of finding *integers*x, y and z which satisfy:

$$(x - z)(y - z) = z^2 + 1$$

   (You can derive this expression from the tan(a+b) formula as follows:
   Let tan a = 1/x i.e arctan(1/x) is angle a and let tan b = 1/y so arctan(1/y) is angle b.
   Then a+b = arctan(1/x) + arctan(1/y) = arctan(1/z) so that tan(a+b) = 1/z.
   Put these values in the tan(a+b) formula above and then simplify the right hand side by multiplying top and bottom by xy.
   After rearranging you will then need to add z^2 to both sides and then Tadaaki Ohno's formula appears.)

   So, for instance, if arctan(1/z)= pi/4 and therefore z is 1 then we can find values x and y by solving

$$(x - 1)(y - 1) = 1^2 + 1 = 2$$

   The important things is that **x and y are integers**so we only need to look for integer fractors of 2 and there are only two factors of 2, namely 1 and 2:

   x – 1 = 1 and y – 1 = 2 which gives x = 2 and y = 3

   This is the first two-angle formula that we mentioned earlier that Euler found in 1738:

$$\mathbf{Pi/_4 = arctan(\ ^1/_2\ ) + arctan(\ ^1/_3\ )}$$

The important other part of Tad's proof is that

all two-angle values satisfy this formula.

So we now know that there is *only one way to write arctan(1) as the sum of two angles of the form arctan(1/x) + arctan(1/y)*.

- How does this formula help in answering the first question in this Things To Do section?
- Find *all* the two-angle sums (x and y) for z from 1 to 12.
- **Research Problem** Can you find a similar formula for x, y and z when

  arctan(1/z) = 2 arctan(1/x) + arctan(1/y)

  What about

  arctan(1/z) = 3 arctan(1/x) + arctan(1/y)

  and

  arctan(1/z) = 4 arctan(1/x) + arctan(1/y)

  and, in general,

  arctan(1/z) = k arctan(1/x) + arctan(1/y)

  Tad says he has proved that Machin's formula (which has z=1, x=239 and y=5) is the only solution for k=4.

7. **Research Problems**
   Hwang Chien-lih of Taiwan told me that Stormer proved that there are only four 2-term formulae for arctan(1), including Euler's and Machin's that we have already met:

   arctan(1) = 4 arctan(1/5) – arctan(1/239) discovered Machin in 1706.
   arctan(1) = arctan(1/2) + arctan(1/3), discovered by Euler in 1738
   arctan(1) = 2 arctan(1/2) – arctan(1/7) (discovered by Hermann in 1706?)
   arctan(1) = 2 arctan(1/3) + arctan(1/7) (discovered by Hutton in 1776?)

   He also says the same Stormer found 103 three-term formulae, J W Wrench had found 2 more and Hang Chien-lih has found another. How many are there in total?

If you get some results from these problems, please send them to me - I'd be interested to see what you come up with so I can put your name and your results on this page too. Perhaps you can find some results in the Journals in your University library (not so easy!)? Even if the results you discover for yourself are already known (in books and papers), you'll have done some *real* maths in the meantime. Anyway, perhaps your results really *are* new and your proofs are *much simpler* than those known and we need to let the world know so have a go!

Leroy Quet of Denver, Colorado, has found a proof (here it is) of the real pattern in a simple proof.

More links and References

Links

A brief history of computing pi
at the St Andrews site and well worth looking at.

Jeremy Gilbert's Pi to 10 Million places!
You can search the first 10,000,000 places of Pi for any particular string of numbers eg if your birthday is 4th May,

1982, we can write it as a number such as 04051982 (or 040582 or, if you are American, 050482 or perhaps 820504) or for some other sequences. "999999" occurs no less than 17 times in the first ten million places, the first time being at decimal places 763-768!
Jeremy's page also points to an actual list of all 10 million digits of Pi which you can download. Before you do, however, beware that since each digit is stored as one byte, the file is 10 Megabytes in size! So how about...

University of Exeter has a page of the first 10,000 digits of Pi!

References

- **The Enhancement of Machin's Formula by Todd's Process**
  by Michael Wetherfield is in *Mathematical Gazette*, Vol 80, No 488, July 1996, pages 333-344. It has lots of interesting formula like those above. Since arctan(1/x) appears very often, he uses an alternative notation, arccotan(x) or arcot(x) for short.
  If the tan of angle A is p/q then the cotangent of A is defined to be q/p.
  He further abbreviates arcot(A) to just {A} - note the curly brackets - so that our formula arctan(1)=arctan(1/2)+arctan(1/3) becomes arccot(1) = arccot(2) + arccot(3), or, in his abbreviated notation: {1}={2}+{3}. Nice!
- **More Machin-type identities** *Mathematical Gazette* March 1997, pages 120-121.
  Just after this article in the same issue is ...
- **Machin revisited** *Mathematical Gazette* March 1997, pages 121-123.
- **Some new inverse cotangent identities for pi** *Mathematical Gazette* (1997? or 1998?) pages 459-460.
- **Problem B-218** in the *Fib. Q.*, **10**, 1972, pp 335-336
  gives the sum of the arctans of the reciprocals of the alternate (odd-indexed) Fibonacci numbers from F(2k+1) onwards as the arctan of 1/F(2k). The formula for pi/4 then follows when k=1.
- C W Trigg **Geometric Proof of a Result of Lehmer's**, *Fib. Q.*, **11**, 1973, pp 539-540
  again proves the main formula of this page but using geometric arguments.
- D H Lehmer, **Problem 3801**, *Am Math Month* 1936, pp 580
  here the problem is posed to prove the main formula on this page that the arctans of reciprocals of alternate Fibonacci numbers sum to pi/4. It's proof was given in...
- M A Heaslet **Solution 3801**, *Am Math Month*, 1938, pg 636-7
- D H Lehmer **On arcotangent relations for Pi** *Am Math Month* 1938, pp 657-664
  Here are many formulae involving arctans that sum to pi/4.
  He gives the originators of two of the Fibonacci formula that we derived earlier on this page as
  pi/4 = arctan(1/2)+arctan(1/3) as Euler and
  pi/4 = arctan(1/2)+arctan(1/5)+arctan(1/8) as Daze
- The Joy of Pi D Blatner, 1997,
  is a fun book which will appeal to school students and upward.
- Petr Beckmann's A History of Pi, 1976, St Martins Press
  is a classic, quirky, fun book on Pi and its calculation, with odd and interesting snippets from its history. However, there are errors in one or two of the formulae.

Robert Erra of E.S.I-E-A (Ecole Supérieure d'Informatique- Electronique- Automatique), Paris, has contributed the following references:

- D.H. Lehmer, **On arcotangent relations for Pi** *Amer. Math. Month.* Vol 45, 1938, pp 657-664.
- J. Todd, **A problem on arc tangent relations**, *Amer. Math Month.* Vol 56, 1940, pp 517-528.
- S. Stormer, **Sur l'application de la théorie des nombres entiers complexes** *Archiv for Math. og Naturv.* Vol 19, 1897, pp 1-96,
  The rest of the title is **à la solution en nombres rationnels x1,x2...c1c2... de l'équation: c1 arctan x1+...+ cn arctan xn = k Pi/4**.

This is a long and very interesting article in French which uses what are now called Gaussian integers.

R H Birch, **An algorithm for the construction of arctangent relations**, 1946,
is reprinted in the following book ...

Pi: A Source Book , L Berggren, ISBN: 0 387 94924 0, Springer-Verlag, 1997.

This page is a ***Links*[2]*Go* Key Resource** on the topic of **Constants**.

The icon is a link to a large resource of other excellent pages on Pi.

Fibonacci Home Page

The is the first page on More Applications of the Fibonacci Numbers and Phi.

Fibonacci - the man and His Times

The next topics...
Fibonacci, Phi and Lucas numbers Formulae
Links and References

WHERE TO NOW???

Fibonacci Forgeries

This page is about series that masquerade as the Fibonacci numbers, but, when we examine them carefully, they are forgeries.

Contents of this Page

What is a "Fibonacci Forgery"

Sometimes we find a series of numbers which *looks* as if it is the Fibonacci series, but, when we look at bit further, we discover it isn't! These are the **Fibonacci Forgeries**!

Another formula for the Fibonacci numbers?

Someone suggests to you that the following is another formula for the Fibonacci numbers - is it?

$$G(n) = \text{ceiling}(e^{(n-2)/2}) = \text{ceiling}((\sqrt{e})^{n-2})$$

where the "ceiling" function means "the next integer above" (eg: ceiling(2·1)=3 and ceiling(2·9)=3 also). This is a remarkable formula since we get:

n:	1	2	3	4	5	6	7	8
G(n):	1	1	2	3	5	8	13	21

but it is, in fact, a forgery!

References

R K Guy in **The Second Strong Law of Small Numbers** in *The Mathematics Magazine* (1990), Vol 63, pages 3-20, example 41 adapts an inequality of Larry Hoehn's to get this surprising coincidence.

Things to do

- How far does G(n) go before we no longer get the successive numbers of the Fibonacci series appearing?

A Polynomial formula for Fib(n)?

A **polynomial in x**,P(x), is a sum of various powers of x and their (postive or negative) multiples.
The highest power of x which occurs is called the **degree** of the polynomial. A polynomial of degree 1 is called *linear*;
a polyniomial of degree 2 is called *quadratic*;
a polynomial of degree 3 is called *cubic*; etc.
If the polynomial has an infinite number of powers of x, it is called a *power series*.

A simple polynomial

Here is a simple example of a linear polynomial P(x) which gives the first 3 values of the Fibonacci series, that is, P(1)=1, P(2)=2 and P(3)=3 :

```
P(x) = x
```

but that doesn't give P(4)=5, which is what we want for the *real* Fibonacci series, so this P(x) is a *Fibonacci forgery*.

Another polynomial

Can you find a polynomial Q(x) which gives Q(1)=1, Q(2)=2, Q(3)=3 and Q(4)=5? Here is a cubic polynomial which does that:

$$Q(x)= (x^3 - 6 x^2 + 17 x - 6)/6$$

but Q(5) is... 9 whereas Fib(5)=8, so this is another, but better, forgery! How about...

And another!

Here's an example of a "Fibonacci Forgery" polynomial p(x) for which p(1)=1, p(2)=2, p(3)=3, p(4)=5 and p(6)=8 so that its first **6** values look like the Fibonacci series. However, here, p(7)=a and "a" can be any value you choose!

$$p(x) = [(a-11) x^5 + (160-15a) x^4 + (85 a-865) x^3 + (2180-225 a) x^2 + (274 a - 2424) x - 120 a + 1080] / 120;$$

Or, if you want both 1's at the beginning of your series, then the following version has 1, 1, 2, 3, 5, 8 and then "a" as its first **7** values:

$$p(x) = [(a-8) x^6 + (150-21 a) x^5 + (175 a-1070) x^4 + (3630-735 a) x^3 + (1624 a - 5762) x^2 + (3780 - 1764 a) x + 720 a] / 720;$$

Things to do

- It looks like, given the 4 values P(1),P(2),P(3) and P(4) we can find a degree 3 polynomial for P (ie one whose highest power of x is 3); and given 6 values a degree 5 polynomial and given 7 values a degree 6 polynomial. However, the first 3 values were fitted to a linear (degree 1) polynomial. Is it always true that given N values for P(1) to P(N) then we can find a polynomial P which has degree at most N-1? [Consult your teacher or maths library at college.] If so, how do we calcuate the polynomial?

Right-angled links: a new Forgery?

In the December issue of *The Mathematics Teacher* (ISSN 0025-5769) a letter from Deborah Freedman a student at Framingham High School, MA, USA, conjectured that the following series was the Fibonacci numbers.

> *In how many ways can n segments of equal length be connected in a plane if the beginning of one segment is to be connected to the end of the previous segment at a right-angle? Congruent configurations are to be counted as one.*

From the examples given, we can clear up a few questions left by this definition.
By congruent, she means that a shape can be rotated or reflected and it still counts as the same shape. So for 3 links, there are just two "shapes":

```
   oooo   oooo      o         o          o  o   oooo  oooo   oooo
   o         o      o         o          o  o = o   = o  o =    o
oooo   =     oooo = oooo = oooo    and   oooo   oooo  o  o   oooo
                       o   o
                       o   o
```

The sequences are not to overlap, that is, a later segment cannot lie on top of an earlier one, so that each diagram of n links has exactly n straight line segments in it. Links can cross over (at the ends where they join others) and we can have "squares" in our link chains for example:

```
ooooooo      oooo            ooooooo           ooooooo
   o  o  or o  o      or     o  o  o  but not  o  o  o
   oooo      ooooooo         ooooooo           ooooooo
                o  o            o  o           o  o  o
                oooo            oooo           ooooooo
```

since the last shape cannot be made from 12 links in a single chain (in other words, it cannot be drawn in one go, without taking your pen off the page and without going over any line twice).
Deborah's table of small solutions is:

```
number of                                                    number of
segments                  configurations                       ways
-----------------------------------------------------------------------
  1                             ooo                             1
-----------------------------------------------------------------------
  2                             ooo
                                o                               1
                                o
-----------------------------------------------------------------------
  3                          ooo         o o
                             o           o o                    2
                           ooo           ooo
-----------------------------------------------------------------------
  4                      ooo        ooo o        ooo
                         o            o o        o o            3
                       ooo            ooo        ooo
                       o
                       o
-----------------------------------------------------------------------
  5                   ooo   o      ooo o    ooo ooo      o
                      o     o      o o o      o o        o
                    ooo     ooo o  o ooo      ooo      ooo          5
                    o         o o                      o o
                  ooo         ooo                      ooo
                  o
                ooo
-----------------------------------------------------------------------
  6     ooo  ooo     ooo ooo  o        ooo    ooo o  ooo   ooo
        o      o       o o o  o        o      o o o  o o   o o
      ooo      ooo o   ooo o  ooo ooo  ooo o  ooooo  ooo   ooooo
      o          o o            o o      o o           o     o
    ooo          ooo            ooo      ooo           ooo   o   8
    o
    o
-----------------------------------------------------------------------
 7      ooo  o                 ooo           ooo      o  o     o
        o    o                 o o           o        o  o     o
```

```
      ooo    ooo      o ooo o  o ooo o   ooooo  ooo ooo  ooo ooo
      o        o      o o o o      o o   o o    o o o      o o
    ooo        ooo o  ooo ooo      ooo   ooo    o ooo      ooo
    o            o o
  ooo            ooo

                ooo  ooo       ooo o      o o
                  o  o         o o o      o o                     13
  ooo ooo     ooooo  ooo ooo   ooooo    ooooo   ooooo
  o o o       o o      o o       o      o o     o o o
  ooooo       ooo      ooo       o      ooo     ooooo
-----------------------------------------------------------------------
```

[With thanks to Jeff Myers, Granville High School, OH, USA for part of this table and for pointing out this problem in The Mathematics Teacher.]

How many shapes are there with 7 links? Try it and you'll find **the number of 7-link shapes is 15**. In her listing in the *The Mathematics Teacher* she only gave 13, and missed the following two shapes with 7 links:

```
       ooo
       o o
     ooo ooo        ooo ooo
     o              o o o
   ooo            ooo ooo
```

These were generated by a computer program (in Prolog), so, if my programming is correct, there aren't any more shapes missing. The program also showed that The number of **8-link shapes is 23** and this should be 21 if the Fibonacci numbers were the correct series. **There are 43 9-link shapes** and again this should be 34 if the Fibonacci numbers were involved.

So - Deborah's conjecture looks interesting - that there are Fib(n) shapes that can be made from n links at right-angles with no overlapping and allowing for rotations and reflections, but it is another **forgery!**

[There is an online WWW page for The Mathematics Teacher.]

Links and References

Mark Lewis and John Moore have a page on Fibonacci Forgeries which is a summary of a Scientific American article of May 1995 by Ian Stewart on series that look as if they are the Fibonacci numbers to start with, but which turn out not to be.

References

Richard K Guy in **The Second Strong Law of Small Numbers** in *The Mathematics Magazine* (1990), Vol 63, pages 3-20
mentions the Pennies Puzzle 1 and Pennies Puzzle 2 on the Fibonacci Puzzle page and that only one of them is truly Fibonacci.

This fun paper also has several other Fibonacci Forgeries including ones on partitions of n, rooted trees with one label, the number of disconnected graphs on n+1 vertices and the number of connected graphs on n+2 vertices which have just one cycle.

There are many other forgeries in the paper to do with primes, Catalan numbers, binomial and trinomial numbers, mixing some genuine examples with the forgeries. His whole point is that **There are not enough small numbers to meet the many demands made of them** and so we are bound to be fooled with small examples of a problem if we are not careful!

Fibonacci Home Page

Fibonacci - the man and His Times

The Fibonacci Numbers in formulae for Pi

WHERE TO NOW???

The Lucas Numbers

The next topics...
Fibonacci, Phi and Lucas numbers Formulae
Links and References

The Lucas Numbers

We have seen in earlier pages that there is another series quite similar to the Fibonacci series that often occurs when working with the Fibonacci series. Edouard Lucas (1842-1891) (who gave the name "Fibonacci Numbers" to the series written about by Leonardo of Pisa) studied this second series of numbers: 2, 1, 3, 4, 7, 11, 18, .. called **the Lucas numbers** in his honour. On this page we examine some of the interesting properties of the Lucas numbers themselves as well as looking at its close relationship with the Fibonacci numbers.

Contents

The line means there is a **Things to do** investigation at the end of the section.

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843 ..More..

Other starting values for a "Fibonacci" series

The definition of the Fibonacci series is:

$$F_{n+1} = F_{n-1} + F_n \text{ , if } n>1$$
$$F_0 = 0$$
$$F_1 = 1$$

What if we have the same general rule: *add the latest two values to get the next* but we started with different values instead of 0 and 1?

Things to do

1. The Fibonacci series starts with 0 and 1. What if we started a "Fibonacci" series with 1 and 2, using the same general rule is for the

Fibonacci series proper, so that $F_0 = 1$ and $F_1 = 2$? What numbers follow?

2. What if we started with 2 and 3 so that $F_0 = 2$ and $F_1 = 3$?
3. What other starting values give the same series as the previous two questions?
4. The simplest values to start with are
   0 and 1, or
   1 and 1, or
   1 and 2 or even
   1 and 0 (in this order)
   all of which we recognise as (part of) the Fibonacci series after a few terms.
   The next two simplest numbers are 2 and 1.
   What if we started with 2 and 1 so that $F_0 = 2$ and $F_1 = 1$? Does this become part of the FIbonacci series too?
5. Try some other starting values of your own.
6. Investigate what happens to the *ratio* of successive terms in the series of the earlier questions. We know that for the Fibonacci series, the ratio gets closer and closer to Phi = $(\sqrt{5}+1)/2$. Does it look as (oh dear, I feel a pun coming on: *Lucas*) if *all* the series, no matter what starting values we choose, eventually have successive terms whose ratio is Phi?

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843 ..More..

The Lucas series

The French mathematician, Edouard Lucas (1842-1891), who gave the series of numbers 0, 1, 1, 2, 3, 5, 8, 13, .. the name *the Fibonacci Numbers*. found another similar series: 2, 1, 3, 4, 7, 11, 18, ... The Fibonacci rule of adding the latest two to get the next is kept, but here we begin with 2 and 1 (in this order).
The series, called the **Lucas Numbers** after him, is defined as follows: where we write its members as L_n, for **L**ucas:

$$L_n = L_{n-1} + L_{n-2} \text{ for } n>1$$
$$L_0 = 2$$
$$L_1 = 1$$

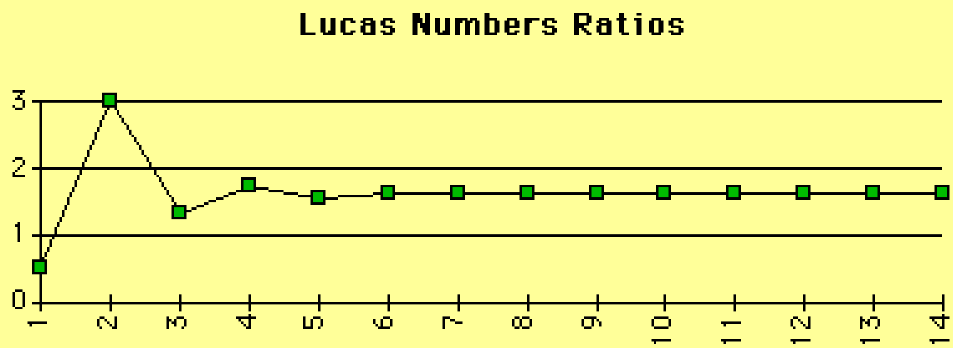
and here are some more values of L_n together with the Fibonacci numbers for comparison:

n:	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**...**
F_n:	0	1	1	2	3	5	8	13	21	34	55	...
L_n:	2	1	3	4	7	11	18	29	47	76	123	...

The Lucas numbers have lots of properties similar to those of Fibonacci numbers and, uniquely among the series you examined in the **Things To Do** section above, the Lucas numbers often occur in various **formulae for the Fibonacci Numbers**. Also, if you look at many formulae for the Lucas numbers, you will find the Fibonacci series is there too. The next section introduces you to some of these equations. So of all the 'general Fibonacci' series, these two seem to be the most important.

For instance, here is the graph of the ratios of successive Lucas numbers:

$\frac{1}{2} = 0{\cdot}5 \quad \frac{3}{1} = 3 \quad \frac{4}{3} = 1{\cdot}333.. \quad \frac{7}{4} = 1{\cdot}75 \quad \frac{11}{7} = 1{\cdot}5714.. \quad \frac{18}{11} = 1{\cdot}6363.. \quad \frac{29}{18} = 1{\cdot}6111.. \quad \frac{47}{29} = 1{\cdot}6206..$



In fact, for *every* series formed by adding the latest two values to get the next, and *no matter what two values we start with* we will **always** end up having terms whose ratio is Phi=1·6180339.. eventually!

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843 ..More..

Two formulae relating the Lucas and Fibonacci numbers

Suppose we add up alternate Fibonacci numbers, $F_{n-1} + F_{n+1}$; that is, what do you notice about the two *Fibonacci* numbers either side of a *Lucas* number in the table below: eg

n:	0	1	2	3	4	5	6	7	8	9	10	...
F_n:	0	1	1	2	3	5	8	13	21	34	55	...
L_n:	2	1	3	4	7	11	18	29	47	76	123	...

Now try your guess on some other Lucas numbers.
This gives our first equation connecting the Fibonacci numbers F(n) to the Lucas numbers L(n):

$$L(n) = F(n-1) + F(n+1) \text{ for all integers } n$$

What about adding alternate Lucas numbers?

n:	0	1	2	3	4	5	6	7	8	9	10	...

F_n:	0	1	1	2	3	5	8	13	21	34	55	...
L_n:	2	1	3	4	7	11	18	29	47	76	123	...

The sum of L(2)=3 and L(4)=7 is *not* F(3)=2 However, try a few more additions in this pattern:

```
L(1)=1 and L(3)= 4 so their sum is  5 whereas F(2)=1;
L(2)=3 and L(4)= 7 so their sum is 10 whereas F(3)=2;
L(3)=4 and L(5)=11 so their sum is 15 whereas F(4)=3;
L(4)=7 and L(6)=18 so their sum is 25 whereas F(5)=5;
```

Have you spotted the pattern?

5 F(n) = L(n-1) + L(n+1) for all integers n

Things to do

a. What about the Fibonacci numbers that are TWO places away from Lucas(n)?

n:	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**...**
F_n:	0	1	1	2	3	5	8	13	21	34	55	...
L_n:	2	1	3	4	7	11	18	29	47	76	123	...

What is the relationship between F(n-2), and F(n+2) that will give L(n)?

b. There is also a relationship between F(n-3) and F(n+3) that gives L(n).

n:	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**...**
F_n:	0	1	1	2	3	5	8	13	21	34	55	...
L_n:	2	1	3	4	7	11	18	29	47	76	123	...

What is it? Write it down as a mathematical formula.

c. .. and between F(n-4) and F(n+4) to give L(n)?

d. Look back at the formula you have just found. Do they work if n is negative (n<0)?

e. Can you write down a *general expression* that relates F(n-k) and F(n+k) to give L(n)?

2. How about the other way round now!

   a. We have already found the relationship between L(n-1) and L(n+1) that gives F(n) - in fact 5F(n) - above.
   What about L(n-2) and L(n+2) to give F(n)?

   b. And now try using L(n-3) and L(n+3) to get F(n).

   c. .. and how can you use L(n-4) and L(n+4) to derive F(n)?

   d. Look back at the formula you have just found. Do they work if n is negative (n<0)?

   e. Can you write down a *general expression* that relates L(n-k) and

L(n+k) to give F(n)?

3. Now - the *really* interesting part!
   Have you spotted a pattern in these patterns?
   If you have, can you write down a mathematical expression which covers ALL the formula found in this Things To Do section?

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843 ..More..

Here is Fibonacci and Lucas Numbers Calculator to help with the investigations on this page. It opens the calculator in a separate window.

A formula for the Lucas Numbers involving Phi and phi

Binet's formula for the Fibonacci numbers in terms of Phi and phi is:

$$\text{Fib}(n) = \frac{\text{Phi}^n - (-\text{phi})^n}{\sqrt{5}}$$

Some alternative forms for this equation are:

$$\text{Fib}(n) = \frac{\Phi^n - (-\phi)^n}{\Phi + \phi} = \frac{\Phi^n - (-\phi)^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

On the Phi's Fascinating Figures page the **Things To Do** in the Numerical Relationships between Phi and its Powers section asked you to investigate what happens when, instead of subtracting the powers of Phi and (-phi) as in the formula for Fib(n) above, we added them:

n:	Phi^n	$(-\text{phi})^n$	$\text{Phi}^n+(-\text{phi})^n$
0	1·000000000	1·000000000	2·000000000
1	1·618033989	-0·618033989	1·000000000
2	2·618033989	0·318196601	3·000000000
3	4·236067978	-0·236067978	4·000000000

Extend this table by a few more rows. Do the values look like they are integers always? What integers do they **Luc-as** if they are (hint!)? Yes! They are the Lucas numbers again:

$$\text{Lucas}(n) = \text{Phi}^n + (-\text{phi})^n$$

Things to do

1. Make a table of the first few powers of Phi=(√5+1)/2=1·618033.. starting at the second power (Phi squared).
   Round each value to the nearest whole number.
   What do you notice? This is an easier method than the formula given above.
2. Take a Fibonacci number, double it and add it to its neighbour on its

right. What do you notice?
Can you *prove* that your observation is *always* true?
[Hint: Use the formula for the Lucas numbers given in terms of the Fibonacci numbers.]

3. Take F_2 and *multiply* it by the Fibonacci number after it:
   $F_2=1$ and $F_3=2$ and 1x2=2.
   Do this with F_4,
   with F_6,
   with F_8 and so on.
   There is a relationship between the new numbers you have found and the Lucas series. What is it?
   [Hint: multiply your number by 5 and see if it is near a number in the Lucas series.]
   Now write the relationship as a mathematical formula.
   [You should be able to prove this one if you keep applying the basic definition of that a Fibonacci number is the sum of the two previous ones and do this several times!]

   > *Optional extra!*
   > Can you *prove* that your formula is always true?
   > This result may help: $F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$

4. If we sum the first k Fibonacci numbers, the answer is *almost* another Fibonacci number. First find the *exact* formula by continuing the pattern below for a few more rows, filling in the gaps marked ? and ! so that the ! values are as small as possible *and ! is the same value on each line*:

   $F_1 = F_? - !$
   $F_1 + F_2 = F_? - !$
   $F_1 + F_2 + F_3 = F_? - !$
   ...

   Now fill in this sentence replacing ? and ! symbols with something more precise:

   > The sum of the first K Fibonacci numbers is !
   > less than the ?-th Fibonacci number.

5. Now try the same pattern as in the previous question, but using L instead of F: *and again @ is to be the same value on each line:*

   $L_1 = L_? - @$

$L_1 + L_2 = L_? - @$
$L_1 + L_2 + L_3 = L_? - @$
...

and so fill in this sentence:

> The sum of the first K Lucas numbers is @ less than the ?-th Lucas number.

6. Compare F_2 with F_1.
   Compare F_4 with F_2.
   Compare F_6 with F_3.
   Compare F_8 with F_4.
   What pattern is emerging?
   [Hint: does one divide exactly into the other?]
   How is this pattern related to the Lucas numbers?
   Now express the pattern as a mathematical equation.
7. We have seen that Lucas Number L(n) is also just **F(n-1)+F(n+1)**.
   So we can ask:
   Is there anything special about **F(n-2)+F(n+2)?**
   Yes! They are all multiples of 3 but can you spot *which* multiples they are, that is, can you fill in this equation:
   F(n-2)+F(n+2) = 3 × ?
   Try the same thing for
   - F(n-3)+F(n+3) = 2 × ?
   - F(n-4)+F(n+4) = 7 × ?
   - F(n-5)+F(n+5)
   - ...

   Can you put all these results into *one* formula:
   F(n-k) + F(n+k) = ?? x ??
   Hint: consider even values of k then look at the odd values of k.
8. Surprisingly, there is a similar formula for the Lucas numbers **L(n-k)+L(n+k)**.
   Repeat the above investigation for this new expression, spotting the patterns for k=1, then k=2, k=3, k=4, and so on, until you can spot the general pattern.

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843 ..More..

A number trick based on Phi, Lucas and Fibonacci numbers!

Here is a trick that you can use to amaze your friends with your (supposed) stupendous calculating powers. All you need to remember is a few Lucas and Fibonacci numbers and you can write down a complicated expression like this:

$$\sqrt[4]{\frac{7+3\sqrt{5}}{2}} - \sqrt[4]{\frac{7-3\sqrt{5}}{2}} = 1$$

You can ask them to verify these formulas on their calculators and they will always work out! The 4 by the √ sign means **the fourth-root**. So if

$2^4 = 16$ "**2 to the fourth is 16**" then

$2 = \sqrt[4]{16}$ "**2 is the fourth-root of 16**"

You will often find a button on your calculator which extracts roots (perhaps marked $\sqrt[y]{x}$) near the button which computes the power of a number (marked x^y). If there is no $\sqrt[y]{x}$ button on your calculator, you can compute $\sqrt[4]{16}$ for instance by computing 1/4 first and using this as the y power with x as 16. This is because

$$\sqrt[y]{x} = x^{1/y}$$

What's the secret?

You will need to learn a few of the early Lucas and Fibonacci numbers and their positions in the sequences:

n:	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**...**
F_n:	0	1	1	2	3	5	8	13	21	34	55	...
L_n:	2	1	3	4	7	11	18	29	47	76	123	...

For the example at the head of this section, I randomly picked the index (column) **4** numbers, F(4)=**3** and L(4)=**7**. We will use these three numbers, **4, 3 and 7** in both expressions. Notice that the first expression has a **plus** inside its 4-th-root-sign whereas the second has a **minus**.
Since the position number, 4, is EVEN, I will use a MINUS sign BETWEEN the two expressions.

Now just substitute your values into this formula:

$$\sqrt[n]{\frac{L(n)+F(n)\sqrt{5}}{2}} \pm \sqrt[n]{\frac{L(n)-F(n)\sqrt{5}}{2}} = 1$$

The SIGN in the middle is + if n is ODD and – if n is EVEN

Here is a Fibonacci and Lucas Numbers Calculator which also generates these expressions for you. Click on the "Amaze me!" button and see a new example every time.

An even more complicated-looking variation!

If you want to make it look even more complicated, choose TWO columns in the table, one for the first expression and one for the second. Here's an example where I use the fifth and ninth columns:

$$\sqrt[5]{\frac{11 + 5\sqrt{5}}{2}} + \sqrt[9]{\frac{76 - 34\sqrt{5}}{2}} = 1$$

The sign in the middle (between the two root-expressions) will depend on the SECOND POSITION (in the example it was 9): if it is ODD (and 9 here is odd), then use PLUS and if it is EVEN put a MINUS sign.

In the new example above, I chose two different positions: 5 for the first expression and 9 for the second.
For the first expression with position=5, I will then use Fib(5)=5 and Lucas(5)=11.
For the second, with position 9, I will use Fib(9)=34 and Lucas(9)=76.
Since 9, the second choice, is ODD, I will put a PLUS sign between the two expressions.

Just substitute your two sets of values: N, Lucas(N) and Fib(N); K, Lucas(K) and Fib(K) in each expression like this, taking care not to mix up your two sets of numbers:

$$\sqrt[n]{\frac{L(n) + F(n)\sqrt{5}}{2}} \pm \sqrt[k]{\frac{L(k) - F(k)\sqrt{5}}{2}} = 1$$

> REMEMBER that the **first** expression always has a **plus(+) inside** the root sign and the **second** always has a **minus (-) inside** its root-sign but the **sign in-between** depends on the **second** (K) value.

Why does it work?

Follow through the suggestions in the following Investigation section and the secret will be revealed!

Things to do

1. (a) See what happens in the first n-th-root expression if we let n=2. The first expression is just:
$$\sqrt{\frac{3 + 1\sqrt{5}}{2}}$$
Use your calculator and find its value.
(b) Now try the second expression with n (or k) =2:
$$\sqrt{\frac{3 - 1\sqrt{5}}{2}}$$
Use your calculator and find this value.
(c) Adding the numbers in (a) and (b) should give 1. Does it?
2. Repeat the above for n=3 finding the two values:
$$\sqrt[3]{\frac{4 + 2\sqrt{5}}{2}} \quad \text{and} \quad \sqrt[3]{\frac{4 - 2\sqrt{5}}{2}}$$
Check that combining them really does give 1, remembering that since n is ODD, you must **subtract**the second from the first, not add it.
3. You can try n=4, if you like:

$$\sqrt[4]{\frac{7 + 3\sqrt{5}}{2}} \quad \text{and} \quad \sqrt[4]{\frac{7 - 3\sqrt{5}}{2}}$$

4. What do you notice about the values of the separate square-, cube- and fourth-roots in all the questions above?
5. Look at the Table of relationships between Phi, phi and √5 and see if you can spot the two expressions in each of questions. So when we take the square-roots in question (1) and the cube-roots in question (2), and the fourth-roots in question (3), what are the results for each expression?
6. Finally, does it matter if we use *different columns of figures* for the two expressions in the trick?

Now you know the secret behind this trick!

With thanks to R. S. (Chuck) Tiberio of Wellesley, MA, USA for pointing out to me the basic relationships that this trick depends upon. He was one of the solvers of the original problem which you can find in:

Problem 402 in **The College Mathematics Journal**, vol. 21, No. 4, September 1990, page 339.

For a similar unlikely-looking collection of identities see:

Incredible Identities by D Shanks in *Fibonacci Quarterly* vol 12 (1974) pages 271 amd 280.

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843 ..More..

The Lucas Numbers in Pascal's Triangle

We found the Fibonacci numbers appearing as sums of "diagonals" in Pascal's Triangle on the Mathematical Patterns in the Fibonacci Numbers page. We can also find the Lucas numbers there too.

Here is the alternative form of Pascal's triangle that we referred to above, with the diagonals re-aligned as columns and the sums of the new columns are the Fibonacci numbers:

	0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
0	1	.	.	.	.	.	.	.	.	.
1	.	1	1	.	.	.	.	.	.	.
2	.	.	1	2	1	.	.	.	.	.
3	.	.	.	1	3	3	1	.	.	.
4	.	.	.	.	1	4	6	4	1	.
5	.	.	.	.	.	1	5	10	10	5
6	.	.	.	.	.	.	1	6	15	20
7	.	.	.	.	.	.	.	1	7	21
8	.	.	.	.	.	.	.	.	1	8

9	.	.	.	.	.	.	.	.	1
	1	**1**	**2**	**3**	**5**	**8**	**13**	**21**	**34**	**55**

To derive the Lucas numbers we still add the columns, but to each number in the column we first **multiply by its column number and divide by its row number**! Here's an example:-
Let's take the third column which, when after the appropriate multiplications and divisions should sum to L(3) which is 4. The lowest number in column 3 is **1** and it is on row 3, so we need:

```
1 x column / row = 1 x 3 / 3 = 1
```

which, in this case, doesn't alter the number by much!
The other number in column 3 is **2** on row 2, so this time we have:

```
2 x column / row = 2 x 3 / 2 = 3
```

Note that for all the numbers in the same column, we will always multiply by the same number - the column number is the same for all of them - but the divisors will alter each time.
Adding the numbers we have derived for this column we have 1+3=4 which is the third Lucas number L(3).

Here is what happens in column 4, starting from the bottom again:-

```
1 x 4 / 4 = 1
3 x 4 / 3 = 4
1 x 4 / 2 = 2
      SUM = 7
```

Here's our revised Pascal's triangle from above showing some of the fractions that we use to derive the Lucas numbers - it shows the pattern in the multipliers and divisors more easily:

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	1									
1		1x1/1=1	1x2/1=2							
2			1x2/2=1	2x3/2=3	1x4/2=2					
3				1x3/3=1	3x4/3=4	3x5/3=5	1x6/3=2			
4					1x4/4=1	4x5/4=5	6x6/4=9	4x7/4= 7	1x8/4= 2	
5						1x5/5=1	5x6/5=6	10x7/5=14	10x8/5=16	...
6							1x6/6=1	6x7/6= 7	15x8/6=20	...
7								1x7/7= 1	7x8/7= 8	...
8									1x8/8= 1	...
		1	**3**	**4**	**7**	**11**	**18**	**29**	**47**	**...**

2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843 ..More..

References

Lucas and Primality Testing Hugh C Williams, Wiley, 1998, ISBN: 0471 14852 0
is a new book (due April 1998) on how to test if a number is prime without factoring it using a technique developed by Edouard Lucas, with modern extensions to his work.
Primality testing has become a focus of modern number-theory and algorithmics research. Our present inability to find prime factors of a number in a fast and efficient way is relied upon in encryption systems - systems which encode information to send over phone lines. Such encryption systems are now built into computer chips in

- cash-card machines which communicate with your bank's central computing service to check your PIN and to verify the transaction;
- electronic cash transfer over the WWW where your browser encodes the message
- credit card transactions when your card is swiped through a machine at the till

Each of these systems must send the information in a secure way, free from tampering by fraudsters.

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The first 100 Lucas Numbers
The Golden Section In Art, Architecture and Music

Dr Ron Knott R.Knott@surrey.ac.uk Created: 18 October 1997 Updated: 7 May 2001

The First 100 Lucas numbers and their factors

The Lucas numbers are defined very similarly to the Fibonacci numbers, but start with 2 and 1 (in this order) rather than the Fibonacci's 0 and 1:

$$L_0 = 2$$
$$L_1 = 1$$
$$L_n = L_{n-1} + L_{n-2} \text{ for } n>1$$

This Maple program was used to produce the table below:

```
lucas:=proc(n) option remember;
# this OPTION makes it very fast even though defined
# by using an inefficient form of recursion
    if n=0 then  2
    elif n=1 then  1
    else lucas(n-1)+lucas(n-2)
    fi
end;

seq(lprint(i,`:`,lucas(i),`=`,ifactor(lucas(i))),i=1..100);
```

and here is the output - a table of the first 100 Lucas numbers and their factors, where the prime numbers are indicated:

n L_n Factors of L_n

```
1 : 1 = 1
2 : 3 = 3    Prime
3 : 4 = 2^2
4 : 7 = 7    Prime
5 : 11 = 11    Prime
6 : 18 = 2*3^2
7 : 29 = 29    Prime
8 : 47 = 47    Prime
9 : 76 = 2^2*19
10 : 123 = 3*41
```

```
11 : 199 = 199    Prime
12 : 322 = 2*7*23
13 : 521 = 521    Prime
14 : 843 = 3*281
15 : 1364 = 2^2*11*31
16 : 2207 = 2207    Prime
17 : 3571 = 3571    Prime
18 : 5778 = 2*3^3*107
19 : 9349 = 9349    Prime
20 : 15127 = 7*2161
21 : 24476 = 2^2*29*211
22 : 39603 = 3*43*307
23 : 64079 = 139*461
24 : 103682 = 2*47*1103
25 : 167761 = 11*101*151
26 : 271443 = 3*90481
27 : 439204 = 2^2*19*5779
28 : 710647 = 7^2*14503
29 : 1149851 = 59*19489
30 : 1860498 = 2*3^2*41*2521
31 : 3010349 = 3010349    Prime
32 : 4870847 = 1087*4481
33 : 7881196 = 2^2*199*9901
34 : 12752043 = 3*67*63443
35 : 20633239 = 11*29*71*911
36 : 33385282 = 2*7*23*103681
37 : 54018521 = 54018521    Prime
38 : 87403803 = 3*29134601
39 : 141422324 = 2^2*79*521*859
40 : 228826127 = 47*1601*3041
41 : 370248451 = 370248451    Prime
42 : 599074578 = 2*3^2*83*281*1427
43 : 969323029 = 6709*144481
44 : 1568397607 = 7*263*881*967
45 : 2537720636 = 2^2*11*19*31*181*541
46 : 4106118243 = 3*275449*4969
47 : 6643838879 = 6643838879    Prime
48 : 10749957122 = 2*769*3167*2207
49 : 17393796001 = 29*599786069
50 : 28143753123 = 3*41*401*570601
51 : 45537549124 = 2^2*919*3469*3571
52 : 73681302247 = 7*103*102193207
53 : 119218851371 = 119218851371    Prime
```

54 : 192900153618 = 2*3^4*107*11128427
55 : 312119004989 = 11^2*199*331*39161
56 : 505019158607 = 47*10745088481
57 : 817138163596 = 2^2*229*9349*95419
58 : 1322157322203 = 3*347*1270083883
59 : 2139295485799 = 709*336419*8969
60 : 3461452808002 = 2*7*23*241*20641*2161
61 : 5600748293801 = 5600748293801   Prime
62 : 9062201101803 = 3*3020733700601
63 : 14662949395604 = 2^2*19*29*211*1009*31249
64 : 23725150497407 = 127*186812208641
65 : 38388099893011 = 11*131*521*24571*2081
66 : 62113250390418 = 2*3^2*43*307*261399601
67 : 100501350283429 = 24994118449*4021
68 : 162614600673847 = 7*23230657239121
69 : 263115950957276 = 2^2*139*461*691*1485571
70 : 425730551631123 = 3*41*281*12317523121
71 : 688846502588399 = 688846502588399   Prime
72 : 1114577054219522 = 2*47*1103*10749957121
73 : 1803423556807921 = 11899937029*151549
74 : 2918000611027443 = 3*81143477963*11987
75 : 4721424167835364 = 2^2*11*31*101*151*18451*12301
76 : 7639424778862807 = 7*1091346396980401
77 : 12360848946698171 = 29*199*9321929*229769
78 : 20000273725560978 = 2*3^2*12280217041*90481
79 : 32361122672259149 = 32361122672259149   Prime
80 : 52361396397820127 = 23725145626561*2207
81 : 84722519070079276 = 2^2*19*62650261*5779*3079
82 : 137083915467899403 = 3*163*800483*350207569
83 : 221806434537978679 = 6202401259*35761381
84 : 358890350005878082 = 2*7^2*23*167*65740583*14503
85 : 580696784543856761 = 11*12760031*1158551*3571
86 : 939587134549734843 = 3*313195711516578281
87 : 1520283919093591604 = 2^2*59*349*947104099*19489
88 : 2459871053643326447 = 47*562418561*93058241
89 : 3980154972736918051 = 179*22235502640988369
90 : 6440026026380244498 = 2*3^3*41*107*10783342081*2521
91 : 10420180999117162549 = 29*521*689667151970161
92 : 16860207025497407047 = 7*9506372193863*253367
93 : 27280388024614569596 = 2^2*3010349*35510749*63799
94 : 44140595050111976643 = 3*563*4632894751907*5641
95 : 71420983074726546239 = 11*191*87382901*41611*9349
96 : 115561578124838522882 = 2*1087*11862575248703*4481

```
97 : 18698256119565069121 = 56678557502141579*3299
98 : 302544139324403592003 = 3*281*61025309469041*5881
99 : 489526700523968661124 = 2^2*19*199*991*1513909*9901*2179
100 : 792070839848372253127 = 7*5738108801*9125201*2161
```

Rules for Primes and Factors of the Fibonacci Numbers

The table of the first 100 Fibonacci numbers had some very interesting properties such as:

F_{nk} is a multiple of F_k

For example:

2 and 4 are both factors of 8:
so F_2=**1** and F_4=**3** should be factors of F_8=**21**

We also saw that, for the Fibonacci numbers,

the Fibonacci number F_n is prime only if n is prime.

apart from F_4 which *is* prime!
[But remember the converse is not always true - just because n is prime does *not* mean that F_n must be prime!]

Do the Fibonacci rules apply to the Lucas numbers?

The same rules do **not** seem to apply to the Lucas numbers above!
For example:

2 and 4 are factors of 8:
but L_2=3 and L_4=7 but L_8=47 *is prime*
so cannot have factors 3 and 7!

So the big question is:

Can you find some other rules that apply to Lucas numbers and their factors?

To help with your investigations, here are the results of a search for prime number among the first 1000 Lucas numbers:

> The only Lucas number which are prime up to L(1000) are L(i) where i=
> 2, 4, 5, 7, 8, 11, 13, 16, 17, 19, 31, 37, 41, 47, 53, 61, 71, 79, 113, 313, 353, 503, 613, 617, 863.
> (Lucas(1000) has 209 digits!)

Cycles in the Lucas numbers?

On the The Mathematics of the Fibonacci Series we saw that the units digits of the Fibonacci numbers repeat in a cycle of length 60 (so that the units digits of F_{60} = the units digits of F_0, and so on for following digits).

- For the Lucas numbers, there is also a cycle of 60 - which is when the **last two digits** repeat in a cycle.
  There is a cycle of units digits in the Lucas numbers, which is much shorter. What is it? How long is it?

Fibonacci - the man and His Times

Fibonacci Home Page
Fibonacci Forgeries!
The Lucas Numbers

WHERE TO NOW???

The Golden Section In Art, Architecture and Music

Fibonacci, Phi and Lucas numbers Formulae
Links and References

Dr Ron Knott R.Knott@surrey.ac.uk Created: 18 October 1997 Updated: 27 November 1998

The Golden Section in Art, Architecture and Music

This section introduces you to some of the occurrences of the Fibonacci series and the Golden Ratio in architecture, art and music.

Contents of this page

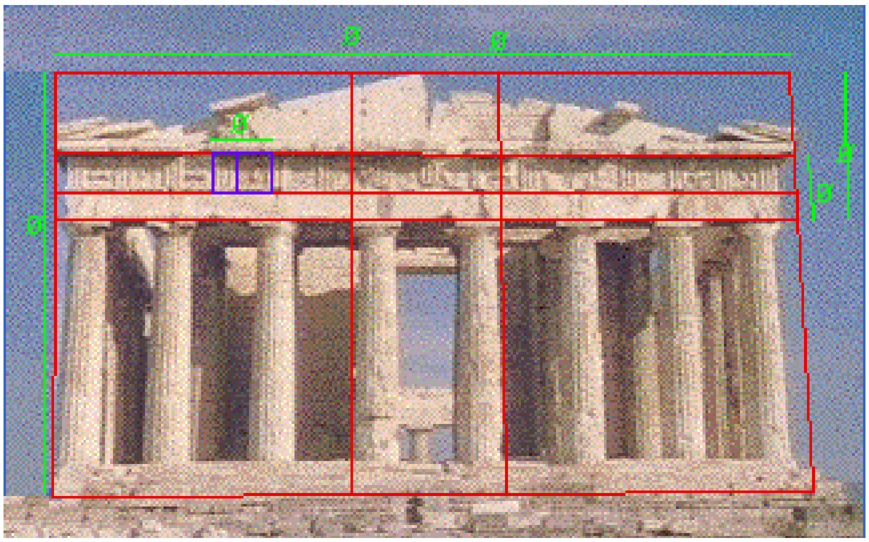
1·61803 39887 49894 84820 45868 34365 63811 77203 09179 80576 ..More..

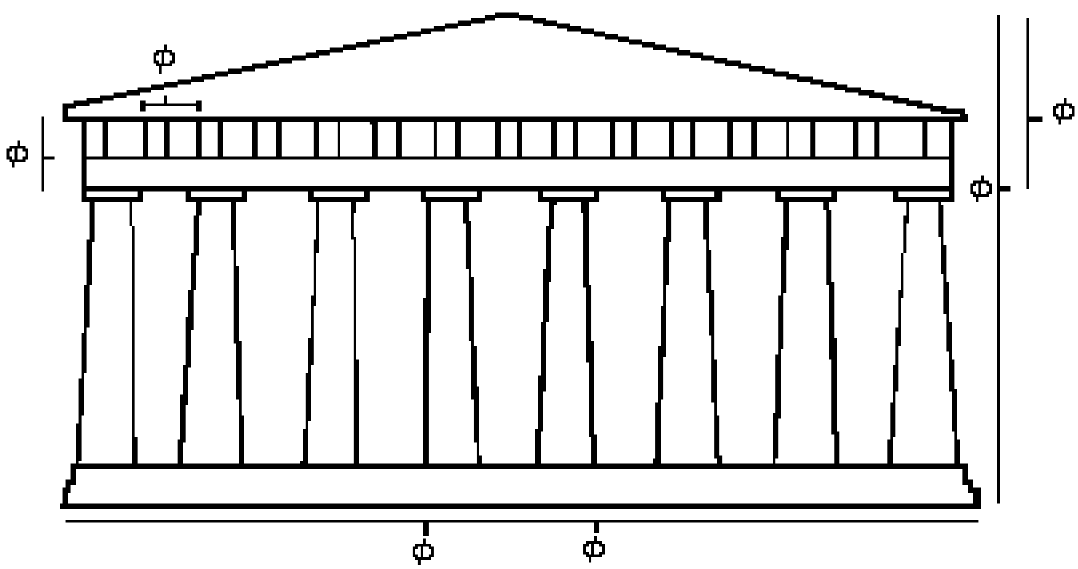
The Golden section in architecture

The Parthenon and Greek Architecture

Even from the time of the Greeks, a rectangle whose sides are in the "golden proportion" (1 : 1.618 which is the same as 0.618 : 1) has been known since it occurs naturally in some of the proportions of the Five Platonic Solids (as we have already seen). This rectangle is supposed to appear in many of the proportions of that famous ancient Greek temple, the Parthenon, in the Acropolis in **Athens**, Greece. (There is a replica of the original building (accurate to one-eighth of an inch!) at Nashville which calls

itself "The Athens of South USA".)
The Acropolis, in the centre of Athens, is an outcrop of rock that dominates this ancient city. Its most famous monument, now largely ruined, is the Parthenon, a temple to the goddess "Athena" built around 430 or 440 BC.
Though no original plans of the temple exist, it appears that the temple was built on a square-root-of-5 rectangle, that is, it is √5 times as long as it is wide. These are also the dimensions of the longest side view of the temple. Also, the front elevation is built on a Golden Rectangle, that is, it is Phi times as wide as it is tall.





Links

There is a wonderful collection of pictures of the Parthenon and the Acropolis at Indiana University's web site.

Modern Architecture

The architect LeCorbusier deliberately incorporated some golden rectangles as the shapes of windows or other aspects of buildings he designed. One of these (not designed by LeCorbusier) is the United Nations building in New York which is L-shaped. Although you will read in some books that "the upright part of the L has sides in the golden ratio, and there are distinctive marks on this taller part which divide the height by the golden ratio", when I looked at photos of the building, I could not find these measurements. The United Nations Headquarters On-line Tour has an aerial view of the building (with thanks to Ralph Bechtolt for alerting me to this link).

Here are three more impressive photographs that you can use for your own investigation (part of the New York SkyscrapersWWW pages).

- The Secretariat building from the visitors entrance (photo by Lawrence A Martin)

[With thanks to Bjorn Smestad of Finnmark College, Norway for mentioning these links.]

Joerg Wiegels of Duesseldorf told me that he was astonished to see the Fibonacci numbers glowing brightly in the night sky on a visit to Turku in Finland. The chimney of the Turku power station has the Fibonacci numbers on it in 2 metre high neon lights! The artist says "it is a metaphor of the human quest for order and harmony among chaos."

Incidentally, in Halifax, Nova Scotia, there are 4 non-cable TV channels and they are numbered 3, 5, 8 and 13! Karl Dilcher reported this coincidence at the Eighth International Conference on Fibonacci Numbers and their Applications in summer 1998.

Architecture links

- An excellent source of **architecture images**is the University of Wisconsin's Library of Art History images- well worth checking out! It has many images of the Parthenon, pictures of its friezes and other details. Use their searcherselecting the Period **Ancient Greece: Classical**and the Site **Athens**. Note: the images cannot be copied or even made into links, only viewed on their page!
- Also see University of Michigan, June Komisar's pageof architectural links. She points to the Great Building Collectionwhich has some excellent photo images on their Parthenon page. Do check this out as they have a FREE 3D viewer to download and lots of buildings in 3D to view. You can take your own virtual walk through the Parthenon!
- There is a link to some nice pictures of Greek temples etc at http://tony.ai/KW/golden.html.
- The golden section in The Kings Tombin Egypt.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

The Golden Section and Art

Luca Pacioli (1445-1517) in his *Divina proportione (On Divine Proportion)* wrote about the **golden section** also called the **golden mean** or **the divine proportion**:

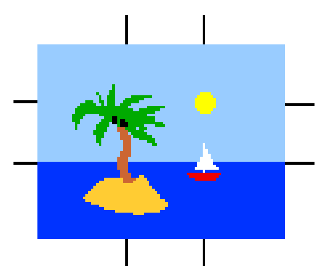
```
A     M        B
| 1-x |   x    |
```

The line AB is divided at point M so that the ratio of the two parts, the smaller to the larger (AM and MB), is the same as the ratio of the larger part (MB) to the whole AB.

> If AB is of length 1 unit, and we let MB have length x, then the definition (in bold) above becomes
> **the ratio of 1-x to x is the same as the ratio of x to 1** or, in symbols:
>
> $$\frac{1-x}{x} = \frac{x}{1} \quad \text{which simplifies to } 1-x = x^2$$
>
> This gives two values for x, $(-1-\sqrt{5})/2$ and $(\sqrt{5}-1)/2$.
> The first is negative, so does not apply here. The second is just phi (which has the same value as 1/Phi and as Phi-1).

Pacioli's work influenced Leonardo da Vinci (1452-1519) and Albrecht Durer (1471-1528) and is seen in some of the work of Georges Seurat, Paul Signac and Mondrian, for instance.

Many books on oil painting and water colour in your local library will point out that it is better to position objects not in the centre of the picture but to one side or "about one-third" of the way across, and to use lines which divide the picture into thirds. This seems to make the picture design more pleasing to the eye and relies again on the idea of the golden section being "ideal".



Leonardo's Art

The Uffizi Gallery's Web site in Florence, Italy, has a virtual room of some of Leonardo da Vinci's paintings. Here are two for you to analyse for yourself. [The pictures are links to the Uffizi Gallery site and the pictures are copyrighted by the Gallery.]

(image: The Annunciation)

is a picture that looks like it is in a frame of 1:sqrt(5) shape (a root-5 rectangle). Print it and measure it - is it a root-5 rectangle? Divide it into a square on the left and another on the right. (If it is a root-5 rectangle, these lines mark out two golden-section rectangles as the parts remaining after a square has been removed). Also mark in the lines across the picture which are 0·618 of the way up and 0·618 of the way down it. Also mark in the vertical lines which are 0·618 of the way along from both ends. You will see that these lines mark out significant parts of the picture or go through important objects. You can then try marking lines that divide these parts into their golden sections too.

This image: Madonna with Child and Saints

is in a square frame. Print it out and mark on it the golden section lines (0·618 of the way down and up the frame and 0·618 of the way across from the left and from the right) and see if these lines mark out significant parts of the picture. Do other sub-divisions look like further golden sections?

Links to Art sources

Links specifically related to the Fibonacci numbers or the golden section (Phi):

- A ray traced imagebased on Fibonacci spirals and rectangles
- the Web Museumpages on Durer, Famous Painting Virtual Exhibition. their long list of famous artists and their works.
- There is a very useful set of mathematical links to Art and Musicweb resources from Mathematics Archivesthat is worth looking at.

Links to **major sources of Art on the Web**:

- Top9.com's List of the top art sources on the webis an excellent place for links to good art sources on the web. Highly recommended!
- The Metropolitan Museum of Artin New York houses more than 2 million works of art.
- The Fine Arts Museums of San Francisco sitehas an Image base of 65,000 works of art. It includes art from Ancient to Modern, from paintings to ceramics and textiles, from all over the world as well as America.
- A Guide to Art Collections in the UK
- Michelangelois famous for his paintings (such as the **ceiling in the Sistine Chapel**) and his sculptures (for instance **David**). This site has links to several sources and images of his works and some links to sites on the golden section.

Using the picture of his **David**sculpture, measure it and see if he has used Phi - eg is the navel ("belly button") 0·618 of the David's height?

Why not visit the Leonardo Museum in the town of Vinci (Italy) itselffrom which town Leonardo is named, of course.
There are many sketches and paintings of Leonardo's at The WebMuseum, Paristoo.

The work of modern artists using the Golden Section

Billie Ruth Sudduthis a North American artist specialising in basket work that is now internationally known. Her designs are based on the Fibonacci Numbers and the golden section - see her web page JABOBs(Just A Bunch Of Baskets). Mathematics Teaching in the Middle Schoolhas a good online introductionto her work (January 1999).

Kees van Prooijenof California has used a similar series to the Fibonacci series - one made from adding the previous three terms, as a basis for his art.

Ned May has generated some beautiful pictures based on Fibonacci Spiralsusing Visual Basic (an example is shown here on the right).

1·61803 39887 49894 84820 45868 34365 63811 77203 09179 80576 ..More..

Fibonacci and Poetry

Martin Gardner, in the chapter "Fibonacci and Lucas Numbers" in "Mathematical Circus" (Penguin books, 1979) mentions Prof George Eckel Duckworth's book **Structural patterns and proportions in Virgil's Aeneid : a study in mathematical composition** (University of Michigan Press, 1962). Duckworth argues that Virgil consciously used Fibonacci numbers to structure his poetry and so did other Roman poets of the time.

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Fibonacci and Music

Trudi H Garland's [see below] points out that on the 5-tone scale (the black notes on the piano), the 8-tone scale (the white notes on the piano) and the 13-notes scale (a complete octave in semitones, with the two notes an octave apart included). However, this is bending the truth a little, since to get both 8 and 13, we have to count the same note twice (C...C in both cases). Yes, it is called an **oct**ave, because we usually sing or play the 8th note which completes the cycle by repeating the starting note "an octave higher" and perhaps sounds more pleasing to the ear. But there are really only *12* different notes in our octave, not 13!

Various composers have used the Fibonacci numbers when composing music - more details in Garland's book.

Golden sections in Violin construction

The section on "The Violin" in The New Oxford Companion to Music, Volume 2, shows how Stradivari was aware of the golden section and used it to place the f-holes in his famous violins.

Baginsky's method of constructing violins is also based on golden sections.

Did Mozart use the Golden mean?

This is the title of an article in the American Scientist of March/April 1996 by Mike Kay. He reports on the analysis of many of Mozart's sonatas and finds they divide into two parts exactly at the golden section point in almost all cases. Was this a conscious choice (his sister said he was always playing with numbers and was fascinated by mathematics) or did he do this intuitively?

The Mathematics Magazine Vol 68 No. 4, pages 275-282, October 1995 has an article by Putz on Mozart and the Golden section in his music.

Beethoven's Fifth

In an interesting little article in **Mathematics Teaching** volume 84 in 1978, Derek Haylock writes about *The Golden Section in Beethoven's Fifth* on pages 56-57.
He finds that the famous opening "motto" appears not only in the first and last bars (bar 601 before the Coda) but also exactly at the golden mean point 0·618 of the way through the symphony (bar 372) and also at the start of the recapitulation which is phi or 0·382 of the way through the piece! He poses the question:

Was this by design or accident?

Bartók, Debussy, Schubert, Bach and Satie

There are some fascinating articles and books which explain how these composers may have deliberately used the golden section in their music:

Duality and Synthesis in the Music of Bela Bartók E Lendvai pages 174-193 of *Module, Proportion, Symmetry, Rhythm* G Kepes (editor), George Brazille, 1966;

Some striking Proportions in the Music of Bela Bartók
in *Fibonacci Quarterly* Vol 9, part 5, 1971, pages 527-528 and 536-537.

Bela Bartók: an analysis of his music
by Erno Lendvai, published by Kahn & Averill, 1971; has a more detailed look at Bartók's use of the golden mean.

Debussy in Proportion - a musical analysis by Roy Howat,
Cambridge Univ. Press,1983, ISBN = 0 521 23282 1. After its first publication in 1986, this book is now (February 2000) back in print.

See also Roy Howat's Web site for more information.

Adams, Coutney S. **Erik Satie and Golden Section Analysis.**
in *Music and Letters*, Oxford University Press,ISSN 0227-4224, Volume 77, Number 2 (May 1996), pages 242-252

Schubert Studies, (editor Brian Newbould) London: Ashgate Press, 1998
has a chapter by Roy Howat *Architecture as drama in late Schubert*, pages 168 - 192, about Schubert's golden sections in his late A major sonata (D.959).

The Proportional Design of J.S. Bach's Two Italian Cantatas, Tushaar Power, *Musical Praxis*, Vol.1, No.2. Autumn 1994, pp.35-46.
This is part of the author's Ph D Thesis *J.S. Bach and the Divine Proportion* presented at Duke University's Music Department in March 2000.

Proportions in Music by Hugo Norden in *Fibonacci Quarterly* vol 2 (1964) pages 219-222
talks about the first fugue in J S Bach's *The Art of Fugue* and shows how both the Fibonacci and Lucas numbers appear in its organisation.

There is a very useful set of mathematical links to Art and Music web resources from Mathematics Archives that is worth looking at.

1·61803 39887 49894 84820 45868 34365 63811 77203 09179 80576 ..More..

A Controversial Issue

There are many books and articles that say that the golden rectangle is **the most pleasing shape** and point out how it was used in the shapes of famous buildings, in the structure of some music and in the design of some famous works of art. Indeed, people such as Corbusier and Bartók have deliberately and consciously used the golden section in their designs.
However, the "most pleasing shape" idea is open to criticism. The golden section as a concept was studied by the Greek geometers several hundred years before Christ, as mentioned on earlier pages at this

site, But the concept of it as a *pleasing* or beautiful shape only originated in the late 1800's and does not seem to have any written texts (ancient Greek, Egyptian or Babylonian) as supporting hard evidence.
At best, the golden section used in design is just **one of several possible "theory of design" methods** which help people structure what they are creating. At worst, some people have tried to elevate the golden section beyond what we can verify scientifically. Did the ancient Egyptians really use it as the main "number" for the shapes of the Pyramids? We do not know. Usually the shapes of such buildings are not truly square and perhaps, as with the pyramids and the Parthenon, parts of the buildings have been eroded or fallen into ruin and so we do not know what the original lengths were. Indeed, if you look at where I have drawn the lines on the Parthenon picture above, you can see that they can hardly be called *precise* so any measurements quoted by authors are fairly rough!

So this page has lots of speculative material on it and would make a good Project for a Science Fair perhaps, investigating if the golden section does account for some major design features in important works of art, whether architecture, paintings, sculpture, music or poetry. It's over to you on this one!

Important article that point out the weaknesses in parts of "the golden-section is the most pleasing shape" theory:

George Markowsky's **Misconceptions about the Golden ratio** in *The College Mathematics Journal* Vol 23, January 1992, pages 2-19.

This is readable and well presented. Perhaps too many people just take the (unsupportable?) remarks of others and incorporate them in their works? You may or may not agree with all that Markowsky says, but this is a good article which tries to debunk a simplistic and unscientific "cult" status being attached to Phi, seeing it where it really is not! This is not to deny that Phi certainly *is* genuinely present in much of botany and the mathematical reasons for this are explained on earlier pages at this site.

How to Find the "Golden Number" without really trying Roger Fischler, *Fibonacci Quarterly*, 1981, Vol 19, pp 406 - 410

Another important paper that points out how taking measurements and averaging them will almost always produce an average near Phi. Case studies are data about the Great Pyramid of Cheops and the various theories propounded to explain its dimensions, the golden section in architecture, its use by Le Corbusier and Seurat and in the visual arts. He concludes that several of the works that purport to show Phi was used are, in fact, fallacious and "without any foundation whatever".

The Fibonacci Drawing Board Design of the Great Pyramid of Gizeh Col. R S Beard in *Fibonacci Quarterly* vol 6, 1968, pages 85 - 87;

has three separate theories (only one of which involves the golden section) which agree quite well with the dimensions as measured in 1880.

Since almost all of the material at this site is about Mathematics, then this page is definitely the odd one out! All the other material is scientifically (mathematically) verifiable and this page (and the final part of the Links page) is the only speculative material on these Fibonacci and Phi pages.

References and Links on the golden section in Music and Art

Key:	
	a book
	an article in a magazine or a paper in an academic journal
	a website

Music

Fascinating Fibonaccis by Trudi Hammel Garland,
Dale Seymours publications, 1987 is an excellent introduction to the Fibonacci series with lots of useful ideas for the classroom. Includes a section on Music.

An example of Fibonacci Numbers used to Generate Rhythmic Values in Modern Music
in *Fibonacci Quarterly* Vol 9, part 4, 1971, pages 423-426;

Links to other Music Web sites

Gamelan music

Gamelan
is the percussion oriented music of Indonesia.

New music
from David Canright of the Maths Dept at the Naval Postgraduate School in Monterey, USA; combining the Fibonacci series with Indonesian Gamelan musical forms.

Some CDs
on Gamelan music of Central Java (the country not the software!).

Other music

The **Fibonacci Sequence**
is the name of a classical music ensemble of internationally famous soloists, who are the musicians in residence at Kingston University (Kingston-upon-Thames,

Surrey, UK). Based in the London (UK) area, their current programme of events is on the Web site link above.

Art

- **A Mathematical History of the Golden Section** ISBN 0486400077.
- **Education through Art** (3rd edition) H Read,
  Pantheon books,1956, pages 14-22;
- **The New Landscape in Art and Science** G Kepes
  P Theobald and Co, 1956, pages 329 and 294;
- H E Huntley's, The Divine Proportion: A study in mathematical beauty,
  ISBN 0-486-22254-3 is a 1970 Dover reprint of an old classic.
- C. F. Linn, **The Golden Mean: Mathematics and the Fine Arts**,
  Doubleday 1974.
- Gyorgy Doczi, The Power of Limits: Proportional Harmonies in Nature, Art, and Architecture
  Shambala Press, (new edition 1994).
- M. Boles, **The Golden Relationship: Art, Math, Nature**, 2nd ed.,
  Pythagorean Press 1987.
  The "Golden Cut" or beauty and design using the golden section, through the eyes of a florist.

Fibonacci Home Page

The Lucas Numbers

← Who was Fibonacci?

WHERE TO NOW???

This is the last page on
More Applications of the
Fibonacci Numbers and Phi.

The next topics...
→ Fibonacci, Phi and Lucas numbers Formulae
→ Links and References

Fibonacci, Lucas, Generalised Fibonacci and Golden section Formulae

Here are about 100 formula involving the Fibonacci numbers, the golden ratio and the Lucas numbers. This forms a major reference page for my Fibonacci Web site where there are many more details, explanations and applications, with puzzles and tricks aimed at secondary school students and teachers as well as interested mathematical enthusiasts.

Contents of This Page

Definitions and Notation

Beware of different golden ratio symbols used by different authors!
At this web site Phi is 1.618033... and phi is 0.618033.. but Vajda(see below) and Dunlap(see below) use a symbol for -0.618033.. .
Where a formula below (or a simple re-arrangement of it) occurs in either Vajda or Dunlap's book, the reference number they use is given. Dunlap's formulae are listed in his Appendix A3. Hoggatt's formula are from his "Fibonacci and Lucas Numbers" booklet. Full bibliographic details are at the end of this page.

As used here	Vajda	Dunlap	Description

floor(x)	[x]	trunc(x), not used for x<0	the nearest integer $\le$ x. **When x>0**, this is "the integer part of x" or "truncate x" i.e. delete any fractional part after the decimal point. 3=floor(3)=floor(3.1)=floor(3.9), -4=floor(-4)=floor(-3.1)=floor(-3.9)
round(x)	$[\, x + \frac{1}{2} \,]$	$\text{trunc}\left(x + \frac{1}{2}\right)$	the nearest integer to x, equivalent to trunc(x+0.5) 3=round(3)=round(3.1), 4=round(3.9), -4=round(-4)=round(-3.9), -3=round(-3.1) 4=round(3.5), -3=round(-3.5)
ceil(x)	-	-	the nearest integer $\ge$ x. 3=ceil(3), 4=ceil(3.1)=ceil(3.9), -3=ceil(-3)=ceil(-3.1)=ceil(-3.9)
$\binom{n}{r}$	$\binom{n}{r}$	$\binom{n}{r}$	$= \frac{n!}{r!\,(n-r)!}$ $_nC_r$; n choose r, the element in row n column r of Pascal's Triangle, the coefficient of x^r in $(1+x)^n$, the number of ways of choosing r objects from a set of n different objects. $n \ge 0$ and $r \ge 0$.

F(i) is the Fibonacci series and L(i) is the Lucas series.

Formula	**Vajda**	**Dunlap**	Comments
F(0) = 0, F(1) = 1, F(n+2) = F(n + 1) + F(n)	-	-	Fibonacci series
L(0) = 2, L(1) = 1, L(n + 2) = L(n + 1) + L(n)	-	-	Lucas series
G(n + 2) = G(n + 1) + G(n)	3	4	Generalised Fibonacci series, G(0) and G(1) needed
Phi $= \frac{\sqrt{5} + 1}{2}$	$=\tau$	$=\tau$,63	Phi and –phi are the roots of $x^2 = x + 1$
phi $= \frac{\sqrt{5} - 1}{2}$	$= -\sigma$	$= -\phi$,65	Dunlap occasionally uses ϕ to represent our phi = 0.61803.., but more frequently he uses ϕ to represent -0.618033..!

Linear Relationships

Linear relationships involve only sums or differences of Fibonacci numbers or Lucas numbers or their multiples.

Formula	Vajda	Dunlap
$F(-n) = (-1)^{n+1} F(n)$	2	5
$L(-n) = (-1)^n L(n)$	4	6
$F(n) + F(n + 3) = 2 F(n + 2)$	-	-
$L(n) + L(n + 3) = 2 L(n + 2)$	-	-
$F(n) + F(n + 4) = 3 F(n + 2)$	-	-
$L(n) + L(n + 4) = 3 L(n + 2)$	-	-
$5 F(n) = L(n - 1) + L(n + 1)$	5	13
$L(n) = F(n + 1) + F(n - 1)$	6, Hoggatt-I8	14
$L(n) = F(n) + 2 F(n - 1)$	-	(32)
$5 F(n) = L(n) + 2 L(n - 1)$	-	-
$L(n) = F(n + 2) - F(n - 2)$	7a	15
$5 F(n) = L(n + 2) - L(n - 2)$	-	-
$2 F(n + 1) = F(n) + L(n)$	7b	16
$2 L(n + 1) = L(n) + 5 F(n)$	-	-
$2 F(n + 2) = 3 F(n) + L(n)$	26	28
$2 L(n + 2) = 3 L(n) + 5 F(n)$	27	29
$L(n) = F(n + 3) - 2 F(n)$	-	31-possible1
$5 F(n) = L(n + 3) - 2 L(n)$	-	-
$L(n) = F(n + 2) - F(n) + F(n - 1)$	-	31-possible2

Basic Golden Ratio Identities

Here Phi is Vajda's and Dunlap's tau (τ). phi here is Vajda's sigma (σ) and Dunlap's ϕ.

Formula	Vajda	Dunlap
Phi phi = 1	page 51(3)	65
Phi / phi = Phi + 1	-	-
Phi + phi = $\sqrt{5}$	-	-

phi / Phi = 1 – phi	-	-
Phi – phi = 1	-	-
Phi = phi + 1 = $\sqrt{5}$ - phi	-	-
phi = Phi – 1 = $\sqrt{5}$ - Phi	-	-
Phi^2 = Phi + 1	page 51(4)	64
phi^2 + phi = 1	page 51(4)	64
$Phi^{n+2} = Phi^{n+1} + Phi^n$	-	-
$phi^n = phi^{n+1} + phi^{n+2}$	-	-

Golden Ratio with Fibonacci and Lucas

Formula	**Vajda**	**Dunlap**
Binet's Formula: ($\sqrt{5}$=Phi–phi) $F(n) = \frac{Phi^n - (-phi)^n}{\sqrt{5}}$	58	69,Hoggatt-page 11
$L(n) = Phi^n + (-phi)^n$	59	70
$F(n) = round\left(\frac{Phi^n}{\sqrt{5}}\right)$,if n≥0	62	71,corrected
L(n) = round(Phi^n),if n≥2	63	72
$F(-n) = round\left(\frac{-(-phi)^{-n}}{\sqrt{5}}\right)$,if n≥0	-	-
L(–n) = round($(-phi)^{-n}$), n≥3	-	-
$F(-n) = (-1)^{n+1}\ round\left(\frac{Phi^n}{\sqrt{5}}\right)$,if n≥0	-	-
L(–n) = round($(-Phi)^n$), n≥3	-	-
F(n + 1) = round(Phi F(n)),if n≥2	64	73
L(n + 1) = round(Phi L(n)),if n≥4	65	74
$F(n+1) - Phi\ F(n) = (-phi)^n$	103b	75

Order 2 Fibonacci and Lucas Relationships

Formula involving squares of Fibonacci or Lucas numbers or a product of a Fibonacci number and Lucas number.

Formula	Vajda	Dunlap
F(2 n) = F(n) L(n)	Vajda-13, Dunlap-17, Hoggatt-17	
$F(2n) = F(n)^2 + 2F(n-1)F(n)$	-	-
$L(2n) = L(n)^2 - 2(-1)^n$	-	-
$F(2n+1) = F(n+1)^2 + F(n)^2$	11	7
$L(n+1)^2 + L(n)^2 = 5F(2n+1)$	-	-
$L(2n+1) = L(n+1)^2 - 5F(n)^2$	-	-
$F(n+2)F(n-1) = F(n+1)^2 - F(n)^2$	12	8
$L(n+2)L(n-1) = L(n+1)^2 - L(n)^2$	-	-
$F(n+1)F(n-1) - F(n)^2 = (-1)^n$	29	9
$L(n+1)L(n-1) - L(n)^2 = -5(-1)^n$	-	-
$L(2n) + 2(-1)^n = L(n)^2$	17c	12
$L(2n) - 2(-1)^n = 5F(n)^2$	23	25
$F(n+1)L(n) = F(2n+1) + (-1)^n$	30,31	27,30
$L(n+1)F(n) = F(2n+1) - (-1)^n$	-	-
$F(2n+1) = F(n+1)L(n+1) - F(n)L(n)$	14	18
$L(2n+1) = F(n+1)L(n+1) + F(n)L(n)$	-	-
$L(n)^2 - 2L(2n) = -5F(n)^2$	22	24
$5F(n)^2 - L(n)^2 = 4(-1)^{n+1}$	24	26
$5(F(n)^2 + F(n+1)^2) = L(n)^2 + L(n+1) = 5F(2n+1)^2$	25a	-
$F(n) = F(m)F(n+1-m) + F(m-1)F(n-m)$	-	10
$F(n)L(m) = F(n+m) + (-1)^m F(n-m)$	15a	19
$L(n)F(m) = F(n+m) - (-1)^m F(n-m)$	15b	20
$5F(m)F(n) = L(n+m) - (-1)^m L(n-m)$	17b	23

$L(n + m) + (-1)^m L(n - m) = L(m) L(n)$	17a	11
$2 F(n + m) = L(m) F(n) + L(n) F(m)$	16a	21
$2 L(n + m) = L(m) L(n) + 5 F(n) F(m)$	-	-
$(-1)^m 2 F(n - m) = L(m) F(n) - L(n) F(m)$	16b	22
$L(n + i) F(n + k) - L(n) F(n + i + k) = (-1)^{n+1} F(i) L(k)$	19a	-
$F(n + i) L(n + k) - F(n) L(n + i + k) = (-1)^n F(i) L(k)$	19b	-
$F(n + i) F(n + k) - F(n) F(n + i + k) = (-1)^n F(i) F(k)$	20a	-
$L(n + i) L(n + k) - L(n) L(n + i + k) = (-1)^{n+1} 5 F(i) F(k)$	20b	-
$F(n)^2 F(m + 1) F(m - 1) - F(m)^2 F(n + 1) F(n - 1) = (-1)^{n-1} F(m + n) F(m - n)$	32	-

Basic G Identities

G(i) is the General Fibonacci series. It has the same recurrence relation as Fibonacci and Lucas, namely **G(n+2) = G(n+1) + G(n) for all integers n (i.e. n can be negative)**, but the "starting values" of G(0) and G(1) can be specified. It therefore is a generalisation of both series and includes them both as special cases. Hoggatt and others use the letter H for series G.

e.g.

- If G(0)=0 and G(1)=1 we have 0,1,1,2,3,5,8,13,.. the Fibonacci series, i.e. G(0,1,i) = F(i);
- G(0)=2 and G(1)=1 gives 2,1,3,4,7,11,18,.. the Lucas series, i.e. G(2,1,i) = L(i);
- G(0)=1 and G(1)=1 gives 1,1,2,3,5,8,13,.. the Fibonacci series again but "moved left one place" i.e. G(1,1,i) = F(i+1).
- G(0,2,i) is 0,2,2,4,6,10,16,26,.. which is the Fibonacci series with all terms doubled, i.e. G(0,2,i) = 2 Fib(i).
- G(3,0,i) is 3,0,3,3,6,9,15,.. which is Fibonacci tripled and shifted right one place: G(3,0,i) = 3 F(i-1).
- G(3,2,i) is 3,2,5,7,12,19,31,.. is new - it is not a multiple of either the Fibonacci or Lucas series values.

Formula	**Refs**
$G(n + 2) = G(n + 1) + G(n)$	Vajda-3, Dunlap-4
$G(n) = G(0) F(n - 1) + G(1) F(n)$	-
$G(-n) = (-1)^n (G(0) F(n + 1) - G(1) F(n))$	

$G(n + m) = F(m - 1) G(n) + F(m) G(n + 1)$	Vajda-8, Dunlap-33
$G(n - m) = (-1)^m (F(m + 1) G(n) - F(m) G(n + 1))$	Vajda-9, Dunlap-34
$L(m) G(n) = G(n + m) + (-1)^m G(n - m)$	Vajda-10a, Dunlap-35
$F(m) (G(n - 1) + G(n + 1)) = G(n + m) - (-1)^m G(n - m)$	Vajda-10b,Dunlap-36
$G(m) F(n) - G(n) F(m) = (-1)^{n + 1} G(0) F(m - n)$	Vajda-21a
$G(m) F(n) - G(n) F(m) = (-1)^m G(0) F(n - m)$	Vajda-21b

Order 2 G Formulae

These formulae include terms which are a product of two G numbers either from the same G series of from two different G series i.e. with different index 0 and 1 values. Where the series may be different they are denoted G and H eg special cases include G = Fibonacci(F) and H = Lucas(L), or they could also be the same series, G=H=F.

Formula	**Vajda**	**Dunlap**
$G(n + i) H(n + k) - G(n) H(n + i + k) = (-1)^n (G(i) H(k) - G(0) H(i + k))$	18	-
$G(n + 1) G(n - 1) - G(n)^2 = (-1)^n (G(1)^2 - G(0) G(2))$	28	-
$\sqrt{5} G(n) = (G(1) + G(0) \text{phi}) \text{Phi}^n + (G(0) \text{Phi} - G(1)) (-\text{phi})^n$	55,56	77

Fibonacci and Lucas Summations

These formulae involve a sum of Fibonacci or Lucas numbers.

Formula	**Vajda**	**Dunlap**
$\sum_{i=0}^{n} F(i) = F(n + 2) - 1$	Hoggatt-11	
$\sum_{i=0}^{n} L(i) = L(n + 2) - 1$	Hoggatt-12	
$\sum_{i=a}^{n} F(i) = F(n + 2) - F(a + 1)$	-	-

$\sum_{i=a}^{n} L(i) = L(n+2) - L(a+1)$	-	-
$\sum_{i=1}^{n} F(2i) = F(2n+1) - 1, n>=1$	Hoggatt-16	
$\sum_{i=1}^{n} L(2i) = L(2n+1) - 1$	-	-
$\sum_{i=1}^{n} F(2i-1) = F(2n) - 1, n>=1$	Hoggatt-15	
$\sum_{i=1}^{n} L(2i-1) = L(2n) - 2$	-	-
$\sum_{i=1}^{n} 2^{n-i} F(i-1) = 2^n - F(n+2)$	37a-variant	42-variant
$\sum_{i=0}^{n} (-1)^i L(n-2i) = 2F(n+1)$	97	54

Formula	**Vajda**	**Dunlap**
$\sum_{i=0}^{\infty} \frac{F(i)}{2^i} = 2$	60	51
$\sum_{i=0}^{\infty} \frac{L(i)}{2^i} = 6$	-	-
$\sum_{i=0}^{\infty} \frac{F(i)}{r^i} = \frac{r}{r^2 - r - 1}$	-	-
$\sum_{i=0}^{\infty} \frac{L(i)}{r^i} = 2 + \frac{r+2}{r^2 - r - 1}$	-	-

$\sum_{i=1}^{\infty} \frac{i\,F(i)}{2^i} = 10$	61	52
$\sum_{i=1}^{\infty} \frac{i\,L(i)}{2^i} = 22$	-	-
$\sum_{i=1}^{\infty} \frac{1}{F(2^i)} = 4 - Phi = 3 - phi$	77-corrected	53-corrected

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Formula	Vajda	Dunlap
$\sum_{i=1}^{2n} F(i)\,F(i-1) = F(2n)^2$	40	45
$\sum_{i=1}^{2n} L(i)\,L(i-1) = L(2n)^2 - 4$	-	-
$\sum_{i=1}^{2n+1} F(i)\,F(i-1) = F(2n+1)^2 - 1$	42	47
$\sum_{i=1}^{2n+1} L(i)\,L(i-1) = L(2n+1)^2 - 5$	-	-
$\sum_{i=0}^{n-1} F(2i+1)^2 = \frac{F(4n) + 2n}{5}$	95	-
$\sum_{i=0}^{n-1} L(2i+1)^2 = F(4n) - 2n$	96	-
$\sum_{i=1}^{n} F(i)^2 = F(n)\,F(n+1)$	45, Hoggatt-13	50

$\sum_{i=1}^{n} L(i)^2 = L(n)\,L(n+1) - 2$	Hoggatt-14	
$\sum_{i=1}^{2n-1} L(i)^2 = 5\,F(2n)\,F(2n-1)$	-	-
$5\sum_{i=0}^{n} F(i)\,F(n-i)$ $\begin{cases} = (n+1)\,L(n) - 2\,F(n+1) \\ = n\,L(n) - F(n) \end{cases}$	98	55
$\sum_{i=0}^{n} L(i)\,L(n-i)$ $\begin{cases} = (n+1)\,L(n) + 2\,F(n+1) \\ = (n+2)\,L(n) + F(n) \end{cases}$	99	56
$\sum_{i=0}^{n} F(i)\,L(n-i) = (n+1)\,F(n)$	100	57
$\sum_{i=1}^{n} L(2i)^2 = F(4n+2) + 2n - 1$	page 70	-

General Summations

Formula	Vajda	Dunlap
$\sum_{i=1}^{n} G(i) = G(n+2) - G(2)$	33	38
$\sum_{i=a}^{n} G(i) = G(n+2) - G(a+1)$	-	-

$\sum_{i=1}^{n} G(2i-1) = G(2n) - G(0)$	34	37
$\sum_{i=1}^{n} G(2i) = G(2n+1) - G(1)$	35	39
$\sum_{i=1}^{n} G(2i) - \sum_{i=1}^{n} G(2i-1) = G(2n-1) + G(0) - G(1)$	36	40
$\sum_{i=1}^{n} 2^{n-i} G(i-1) = 2^{n-1}(G(0) + G(3)) - G(n+2)$	37-variant	41-variant
$\sum_{i=1}^{4n+2} G(i) = L(2n+1) G(2n+3)$	38	43
$\sum_{i=1}^{2n} G(i) G(i-1) = G(2n)^2 - G(0)^2$	39	44
$\sum_{i=1}^{2n+1} G(i) G(i-1) = G(2n+1)^2 - G(0)^2 - G(1)^2 + G(0) G(2)$	41	46
$\sum_{i=1}^{n} G(i+2) G(i-1) = G(n+1)^2 - G(1)^2$	43	48
$\sum_{i=1}^{n} G(i)^2 = G(n) G(n+1) - G(0) G(1)$	44	49
$\sum_{i=0}^{\infty} \frac{G(a, b, i)}{r^i} = a + \frac{a + b r}{r^2 - r - 1}$	Stan Rabinowitz, "Second-Order Linear Recurrences" card, *Generating Function* special case (x=1/r, P=1, Q=-1)	
$\sum_{i=0}^{\infty} \frac{i\, G(a, b, i)}{r^i} = \frac{r (b r^2 - 2 a r + b - a)}{(r^2 - r - 1)^2}$		

Summations with Binomial Coefficients

Formula	Vajda	Dunlap
$\sum_{i=1}^{n} \binom{n-i}{i-1} = F(n)$	-	-
$\sum_{i=0}^{\infty} \binom{n-i-1}{i} = F(n)$	54 corrected	84 corrected
$\sum_{i=0}^{n} \binom{n+1}{i+1} F(i) = F(2n+1) - 1$	50	82
$\sum_{i=0}^{2n} \binom{2n}{i} F(2i) = 5^n F(2n)$	69	85
$\sum_{i=0}^{2n} \binom{2n}{i} L(2i) = 5^n L(2n)$	71	87
$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F(2i) = 5^n L(2n+1)$	70	86
$\sum_{i=0}^{2n+1} \binom{2n+1}{i} L(2i) = 5^{n+1} F(2n+1)$	72	88
$\sum_{i=0}^{2n} \binom{2n}{i} F(i)^2 = 5^{n-1} L(2n)$	73	89
$\sum_{i=0}^{2n} \binom{2n}{i} L(i)^2 = 5^n L(2n)$	75	91

$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F(i)^2 = 5^n F(2n+1)$	74	90
$\sum_{i=0}^{2n+1} \binom{2n+1}{i} L(i)^2 = 5^{n+1} F(2n+1)$	76	92
$\sum_{i=0}^{\infty} 5^i \binom{n}{2i+1} = 2^{n-1} F(n)$	91	-
$\sum_{i=0}^{\infty} 5^i \binom{n}{2i} = 2^{n-1} L(n)$	92	-
With Generalised Fibonacci:		
$\sum_{i=0}^{n} \binom{n}{i} G(i) = G(2n)$	47	80
$\sum_{i=0}^{n} \binom{n}{i} G(p-i) = G(p+n)$	46	79
$\sum_{i=0}^{n} \binom{n}{i} G(p+i) = G(p+2n)$	49	81
$\sum_{i=0}^{n} (-1)^i \binom{n}{i} G(n+p-i) = G(p-n)$	51	83

References

S Vajda, **Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications**, Halsted Press (1989).

This is a wonderful book! Unfortunately, it is now out of print. Vajda packs the book full of formulae on

the Fibonacci numbers and Phi and the Lucas numbers. The whole book develops these formulae step by step, proving each from earlier ones or occasionally from scratch.

R A Dunlap, **The Golden Ratio and Fibonacci Numbers** World Scientific Press, 1997.
An introductory book strong on the geometry and natural aspects of the golden section and which does not dwell overmuch on the mathematical details. Beware - some of the formula in the Appendix are wrong! The formulae on this Web page are corrected versions and have been verified .

V E Hoggatt Jr **Fibonacci and Lucas Numbers** published by and available from The Fibonacci Association, 1969 (Houghton Mifflin). A very good introduction to the Fibonacci and Lucas Numbers written by a founder of the Fibonacci Quarterly.

Back to the Fibonacci Home Page

Further sources of Information on Fibonacci Numbers and the Golden Section

This is a page of WWW links to other sites on Fibonacci numbers and the Golden section in general, together with a list of useful books and articles that are recommended for further reading.

Contents

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Other WWW pages on Fibonacci and his series

About Fibonacci himself (St Andrews University)

Dawson Merrill's Fibonacci and Phi site is excellent and full of useful material and links. I highly recommend it!

ACCESS Indiana's K-12 Teaching and Learning Center has an excellent page Fibonacci, Golden section, Art and Music links that is worth checking out.

Prof. Robert Devaney of Boston University has found the Fibonacci numbers in the Mandelbrot set and it's all to do with those buds on the outside of the set!

The Fibonacci Quarterly is devoted solely to the Fibonacci numbers and their uses. See also the current volume and other books by the Fibonacci Association too.
The early issues of the Fibonacci Quarterly have some useful introductions to the Fibonacci numbers suitable to pre-university (and undergraduate) students and I highly recommend them. The Quarterly

started in 1963 but you may need to hunt through some University and College on-line periodicals catalogues to see who holds current and back copies.
The contents of some recent back copies give you an idea of the kind of papers published which are increasingly now only accessible to professional mathematicians. The earlier volumes (1960s and 1970s) are very readable by anyone who has enjoyed the pages at this site.

The Eighth International Conference on Fibonacci Numbers and their Applications was held June 21 - June 26 1998 in Rochester, New York State, USA. Published as Applications of the Fibonacci Numbers, Volume 8 edited by F T Howard, Kluwer Press, 1999. The Proceedings of previous conferences in this series are available as books:
Applications of Fibonacci Numbers, Volume 7 edited by Gerald E. Bergum, Andreas N. Philippou and Alwyn F. Horadam, Kluwer Press, 1998.
Applications of Fibonacci Numbers, Proceedings of the Sixth International Conference edited by G E Bergum and A N Phillipou, Kluwer Press, 1996.

Dr Math is for secondary schools (US: elementary school and high schools) where you can ask "Dr Math" questions. Search Dr Math's archives to find some answers to previously asked questions about the Fibonacci numbers or the Golden section.

Don Cohen, alias the Mathman has some interesting samples of his workbooks on the Web. His approach to maths I heartily agree with and recommend to you - letting people discover the beauty and fascination of maths for themselves. Do have a look at this site if you're an educator, student or just interested in more maths! [Thanks to Bud Weiss of New York City for this.]

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Useful book references

More fascinating facts on Fibonacci numbers are available in your local library:

> means the whole book is useful and
> indicates an article in a magazine or else a paper in a professional journal where mathematicians and scientists report their latest findings (which may only be available in a college or university library).

Ian Stewart's **Mathematical Recreations** column on page 96 of the January 1995 (vol.272 no.1) issue of *Scientific American*.

The Penguin Dictionary of Curious and Interesting Numbers, by David Wells, Penguin press, (new edition 1998) is full of interesting facts about all sorts of individual numbers. See the entry under 1·6180339887... for more information about Phi and the FIbonacci numbers. This is an excellent book! (More information and you can order it online via the title-link.)

Garth Runion's The Golden Section Dale Seymours publications, 1990, is also an excellent introduction to applications and maths on the Golden section and is very popular especially as a source for classroom work. (More information and you can order it online via the title-link.)

Theoni Pappas, The Joy of Mathematics: Discovering Mathematics All Around You, World Wide Publishers, 1989, ISBN: 0 93317465 9.

J & F Gies, **Leonard of Pisa & the New Mathematics of the Middle Ages**, Thos Cromwell, New York, 1969. F Gies is the author of the entry on Fibonacci numbers in the Encyclopaedia Brittanica.

Martin Gardner's books are always worth looking at. He has covered different aspects of the Fibonacci numbers in several of his books in his own enthusiastic and fascinating style:

- Mathematical Circus, Mathematical Association of America, 1992 , chapter 13. *Fibonacci and Lucas Numbers*
- More mathematical puzzles and diversions, Mathematical Association of America press, ISBN: 0 14013823 4, (revised edition 1997), chapter 8 *Phi: the Golden Ratio*
- **Penrose Tiles to Trapdoor Ciphers**, W H Freeman and Co press, 1988, chapters 1 and 2 on *Penrose Tilings* and also chapter 8 *Wythoff's Nim*

A complete list of his books is available at this Think.com site , with separate links to each book at Amazon.com's on-line bookstore. All of is books are a treasure trove of fun and he writes with a clarity and I guarantee you will be dipping into them again and again.

Books by Trudi Garland:

- Fascinating Fibonaccisby Trudi Hammel Garland.
  Trudy is a teacher in California and has some more information on her book. She also has published several posters, including one on the golden sectionsuitable for a classroom or your study room wall.
  You should also look at her other Fibonacci books too:
- Fibonacci Fun: Fascinating Activities with Intriguing NumbersTrudi Hammel Garland - a book for teachers;
- **Math and Music: Harmonious Connections**by Trudi Hammel Garland, Charity Vaughan Kahn and Katarina Stenstedt .

On the theme of good books for teachers, Math Curse by Jon Scieszka and Lane Smith, published by Viking in 1995, is the story of Mrs Fibonacci and, of course, mentions the Fibonacci series. It is getting good reviews as a book for (US) grades 4 to 8.

Schroeder, Manfred R. Number Theory in Science and Communication, With Applications in Cryptography, Springer-Verlag, 1990. ISBN 3540158006.

This is a fascinating collection of all sorts of applications of Number Theory to many areas of science and technology. It has sections on the Fibonacci Numbers, the Golden section and the Rabbit sequence (also called the Golden String).

S Hildebrandt and A Tromba's The Parsimonious Universe - Shape and Form in the Natural World How scientists and mathematicians have sought the laws of shape of natural forms.

Books available through the Fibonacci Association:
The current volume and previous volumes' indexes (or should it be indices?) of the Fibonacci Quarterly are useful to see the kind of papers that they deal with.

Eric W. Weisstein's **World of Mathematics** list of books on Fibonacci numbers .

Some earlier Proceedings of the Third, Fourth, Fifth and Sixth International Conference on *Fibonacci Numbers and Their Applications* are available as books. The editor of each is A N Philippou. The latest is the Seventh edited by Gerald E. Bergum, Andreas N. Philippou and Alwyn F. Horadam .

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987 ..More..

Current research and speculations

Some speculations about the Fibonacci numbers and some propositions about Phi - not proved, just conjectures, but for your interest!

John Harrisof Canada has been working for over 30 years on some aspects of astronomy - in particular, a rejection of Bode's Law (an ad hoc scheme to explain the mean distances of the planets from the sun). His own research involves Phi to make sense of the statistics of orbits, and it involves Phi! Phi in fact turned out to be the solution to a quadratic equation (Section 3) necessary to determine a log-linear function for the planetary periods. He speculates about the history of this subject - what do you think? [John's pages need some familiarity with logarithms and log graphs as well as astronomical terms such as synodic period.]

Fibonacci Home Page

More Applications of Fibonacci Numbers and Phi

Fibonacci, Phi and Lucas numbers Formulae

This is the last topic.

WHERE TO NOW???

The is the last page of Links and References

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