Nonlinear Control

Lecture2: Lyapunov Stability

Lyapunov Stability

- Overview
  - Definitions
  - Stability analysis
    - Lyapunov Linearization Method
    - Lyapunov Direct Method
      - ✓ Local Stability
      - ✓ Global Stability
    - Invariant Set
  - Lyapunov Function Generation
    - Krosovskii Methods
    - Variable Gradient Method
  - Lyapunov Based Controller Design

Lyapunov Stability

- Stability Definitions
  - Consider the closed-loop autonomous system:

$$\dot{x} = f(x) \qquad (3.1)$$

  - Where $f : D \to \mathbf{R}^n$ is locally Lipschitz.
  - The equilibrium point is @ origin: $\mathbf{x}^* = \mathbf{0}.$

Definition 3.1 *The equilibrium point $x = 0$ of (3.1) is*

- *stable if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that*

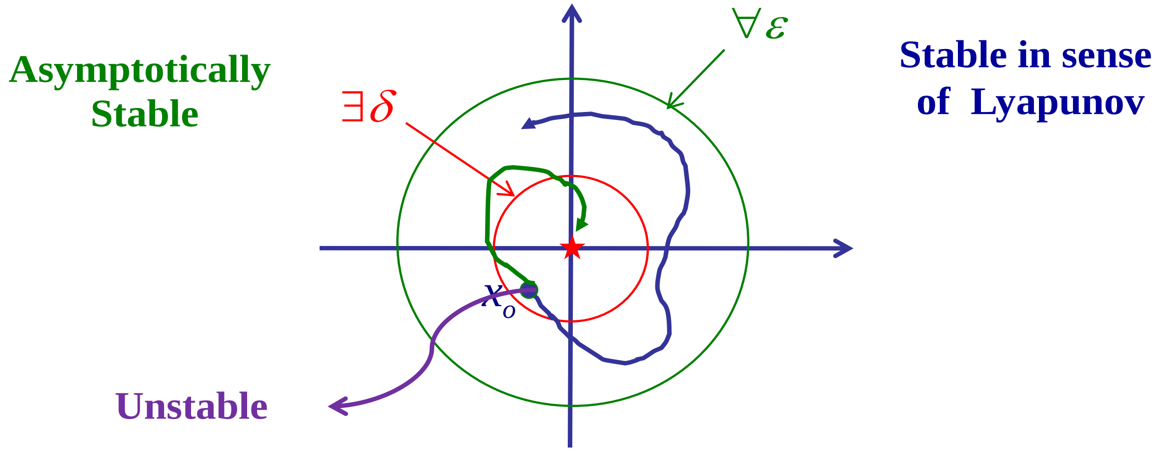
$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall\, t \geq 0$$

- *unstable if not stable.*
- *asymptotically stable if it is stable and δ can be chosen such that*

$$\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0$$

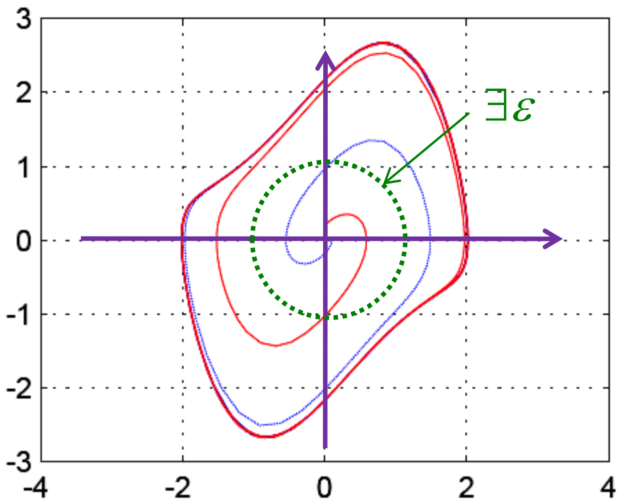
Lyapunov Stability

- Stability Definitions
  - Stability in sense of Lyapunov:
    - The system trajectory can be kept arbitrary close to the equilibrium point.
  - Geometric Representation

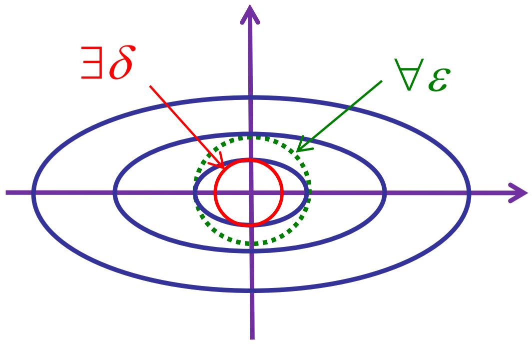


Lyapunov Stability

- Stability Definitions
  - Example: Van der Pol
    - $\exists \varepsilon$ the trajectories diverges
    - Unstable Eq. Point
    - Stable Limit Cycle



  - Example: Pendulum
    - $\forall \varepsilon \rightarrow \exists \delta \ni$ starting from inside δ the trajectory remains in ε
      - Stable (not asymptotically)



Lyapunov Stability

- Stability Definitions
  - Exponential Stability

Definition : *An equilibrium point* $\mathbf{0}$ *is exponentially stable if there exist two strictly positive numbers* α *and* λ *such that*

$$\forall t > 0, \quad \| \mathbf{x}(t) \| \leq \alpha \| \mathbf{x}(0) \| e^{-\lambda t} \tag{3.9}$$

in some ball $\mathbf{B}_r$ *around the origin.*

  - Global Stability

Definition : *If asymptotic (or exponential) stability holds for any initial states, the equilibrium point is said to be asymptotically (or exponentially) stable in the large. It is also called globally asymptotically (or exponentially) stable.*

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Lyapunov Stability

- Stability Analysis
  - Linearization
    - Lyapunov *indirect* or *linearization* method

Theorem : *Let $x = 0$ be an equilibrium point for the nonlinear system*

$$\dot{x} = f(x)$$

where $f : D \rightarrow R^n$ is continuously differentiable and D is a neighborhood of the origin. Let

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=0}$$

Then,

1. *The origin is asymptotically stable if $Re\lambda_i < 0$ for all eigenvalues of A.*
2. *The origin is unstable if $Re\lambda_i > 0$ for one or more of the eigenvalues of A.*

✓ If eq. point is non-hyperbolic → Inconclusive!

Lyapunov Stability

- Stability Analysis
  - Linearization
    - Example 1: Consider the system $\dot{x} = ax^3$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = 3ax^2\big|_{x=0} = 0$$

✓ The eigenvalue is on imaginary axis → Inconclusive.

    - Example 2: Pendulum

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2 \end{aligned}$$

✓ The eq. points are @ (0,0) and (π,0).

✓ Jacobian:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\left(\frac{g}{l}\right)\cos x_1 & -\left(\frac{k}{m}\right) \end{bmatrix}$$

Lyapunov Stability

- Stability Analysis
  - Linearization
    - Example 2: Pendulum (cont.)
      - ✓ For (0,0) Eq. point :

$$A = \frac{\partial f}{\partial x}\bigg|_{x=0} = \begin{bmatrix} 0 & 1 \\ -\left(\frac{g}{l}\right) & -\left(\frac{k}{m}\right) \end{bmatrix} \Longrightarrow \lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{k}{m}\right)^2 - \frac{4g}{l}}$$

        - Eigenvalues Hurwitz → Asymptotically Stable
      - ✓ For (π,0) eq. point.
        - Change variables $z_1 = x_1 - \pi, \; z_2 = x_2$
        - Chack Jacobian @ z=0

$$\tilde{A} = \frac{\partial f}{\partial x}\bigg|_{x_1=\pi, x_2=0} = \begin{bmatrix} 0 & 1 \\ \left(\frac{g}{l}\right) & -\left(\frac{k}{m}\right) \end{bmatrix} \Longrightarrow \lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{k}{m}\right)^2 + \frac{4g}{l}}$$

        - One of the eigenvalues is not Hurwitz
          → Unstable

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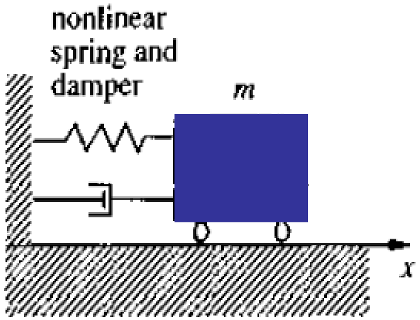
Lyapunov Stability

- Stability Analysis
  - Direct Method, The Philosophy
    - Mathematical extension of a physical observation:
      - ✓ If the total energy is continuously dissipating
      - ✓ Then the system (Linear or Nonlinear) must settle down to an equilibrium point.
    - Example: Mass with nonlinear spring-damper
      - ✓ Consider the system:

$$m\ddot{x} + b\dot{x}|\dot{x}| + k_o x + k_1 x^3 = 0$$



      - ✓ hardening spring + nonlinear damping
      - ✓ Is the resulting motion stable?

Lyapunov Stability

- Direct Method, The Philosophy
  - Examine the total energy

$$V(\mathbf{x}) = \frac{1}{2} m\dot{x}^2 + \int_0^x (k_o x + k_1 x^3)\, dx = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} k_o x^2 + \frac{1}{4} k_1 x^4$$

- Physical observations:
  - zero energy corresponds to the equilibrium point ($\mathbf{x} = \mathbf{0}$, $\dot{\mathbf{x}} = \mathbf{0}$)
  - asymptotic stability implies the convergence of mechanical energy to zero
  - instability is related to the growth of mechanical energy
- Stability is related to the variation of energy

$$\dot{V}(\mathbf{x}) = m\dot{x}\ddot{x} + (k_o x + k_1 x^3)\,\dot{x} = \dot{x}\,(-b\dot{x}|\dot{x}|) = -b|\dot{x}|^3$$

✓ The energy of the system is continuously dissipating toward zero

✓ The motion is converging to eq. point.

Lyapunov Stability

- Direct Method, The Philosophy
  - The energy function has three properties:
    - $V(x)$ is a scalar function
    - $V(x)$ is strictly positive except @ $(\mathbf{x} = \mathbf{0}, \dot{\mathbf{x}} = \mathbf{0})$
    - $\dot{V}(x)$ is monotonically decreasing.
  - Lyapunov Direct Method is
    - A Mathematical generalization of the above observation
      - ✓ Find a scalar energy-type function
      - ✓ which along system trajectory is continuously decreasing
      - ✓ Then the eq. point is stable.

- Direct Method

Definition : *A scalar continuous function* $V(\mathbf{x})$ *is said to be* *locally positive definite* *if* $V(\mathbf{0}) = 0$ *and, in a ball* $\mathbf{B}_{R_o}$

$$\mathbf{x} \neq \mathbf{0} \quad \Rightarrow \quad V(\mathbf{x}) > 0$$

If $V(\mathbf{0}) = 0$ *and the above property holds over the whole state space, then* $V(\mathbf{x})$ *is said to be* *globally positive definite*.

  - Example: Mass with nonlinear spring-damper
    - Kinetic Energy: $V(\mathbf{x}) = \frac{1}{2} m \dot{x}^2$
      - ✓ Is NOT positive definite, since *V(x)* is zero for nonzero states such as $(x_1=c,\ x_2=0)$.
    - Total Energy: $V(\mathbf{x}) = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k_0 x^2 + \frac{1}{4} k_1 x^4$
      - ✓ Is globally positive definite, since it is everywhere positive except at the origin.

Lyapunov Stability

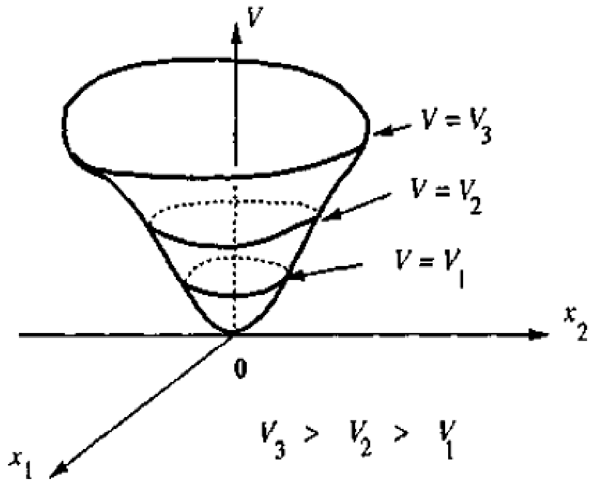
- Direct Method
  - Positive Definiteness
    - Geometrical Representation
    - Negative Definite

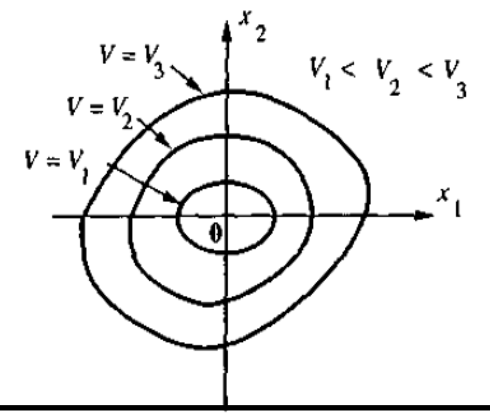
      if $-V(\mathbf{x})$ is positive definite;
    - Positive Semi-Definite

      if $V(0) = 0$ and $V(\mathbf{x}) \geq 0$ for $\mathbf{x} \neq \mathbf{0}$;
    - Time derivative OR

      Derivative along trajectory

$$\dot{V} = \frac{dV(\mathbf{x})}{dt} = \frac{\partial V}{\partial \mathbf{x}} \dot{\mathbf{x}} = \frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x})$$





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    - Lyapunov Direct Method
      - ✓ Local Stability
      - ✓ Global Stability
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    - Krosovskii Methods
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- Stability Analysis
  - Lyapunov Direct Method
    - Local Stability

Theorem : *Let $x = 0$ be an equilibrium point for (3.1) and $D \subset R^n$ be a domain containing $x = 0$. Let $V : D \to R$ be a continuously differentiable function, such that*

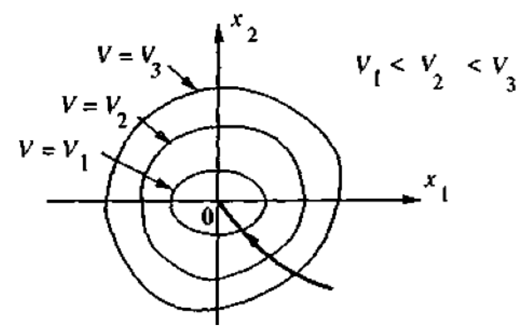
$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } D - \{0\}$$

$$\dot{V}(x) \leq 0 \text{ in } D$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0 \text{ in } D - \{0\}$$

then $x = 0$ is asymptotically stable.



- Proof Idea: (Full proof in Khalil Book page 115)
  - ✓ Lyapunov Surface: $V(x) = c$ for $c>0$.
  - ✓ If $\dot{V}(x) < 0$, then a trajectory crosses a Ly. S. , it moves inside and can never come out again.

- Lyapunov Direct Method
  - Example 1: Pendulum without friction
    - System:
      $$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1 \end{aligned}$$
    - Lyapunov Candidate: $V(x) = \left(\frac{g}{l}\right)(1-\cos x_1) + \frac{1}{2}x_2^2$
      - ✓ How??! (Total Energy)
      - ✓ It is positive definite in the domain $-2\pi < x_1 < 2\pi$
    - Lyapunov Function?
      - ✓ Derivative along trajectory:
        $$\dot{V}(x) = \left(\frac{g}{l}\right)\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = \left(\frac{g}{l}\right) x_2 \sin x_1 - \left(\frac{g}{l}\right) x_2 \sin x_1 = 0$$
      - ✓ Eq. point is stable.
      - ✓ But not asymptotically stable!
      - ✓ Trajectory starting @ Ly. S. $V(x) = c$, remain on it.

- Lyapunov Direct Method
  - Example 2: Pendulum with viscous friction
    - System:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\left(\frac{g}{l}\right)\sin x_1 - \left(\frac{k}{m}\right)x_2 \end{aligned}$$

    - Lyapunov Candidate: $V(x) = \left(\frac{g}{l}\right)(1-\cos x_1) + \frac{1}{2}x_2^2$
      - ✓ The same as Ex1. (Total Energy)
    - Lyapunov Function?
      - ✓ Derivative along trajectory:

$$\dot{V}(x) = \left(\frac{g}{l}\right)\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = -\left(\frac{k}{m}\right)x_2^2$$

      - ✓ Positive Semi-definite: zero irrespective of x_1
      - ✓ Only stable but not asymptotically stable!
      - ✓ Phase portrait and linearization method → Asy. Stable.
  - Lyapunov direct conditions are only sufficient!

- Lyapunov Direct Method
  - Example 2: Pendulum with viscous friction
    - Use another Lyapunov Candidate:

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^T P x + \left(\frac{g}{l}\right)(1-\cos x_1) \\ &= \tfrac{1}{2}[x_1\ x_2]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left(\frac{g}{l}\right)(1-\cos x_1) \end{aligned}$$

    - Lyapunov Function?
      - ✓ *V(x) > 0* if $p_{11} > 0;\ p_{22} > 0;\ p_{11}p_{22} - p_{12}^2 > 0$
    - Derivative along trajectory:

$$\begin{aligned} \dot{V}(x) &= \left(\frac{g}{l}\right)(1-p_{22})x_2 \sin x_1 - \left(\frac{g}{l}\right)p_{12}x_1 \sin x_1 \\ &\quad + \left[p_{11} - p_{12}\left(\frac{k}{m}\right)\right]x_1 x_2 + \left[p_{12} - p_{22}\left(\frac{k}{m}\right)\right]x_2^2 \end{aligned}$$

      - ✓ If $p_{12} = 0.5(k/m)$, then

$$\dot{V}(x) = -\frac{1}{2}\left(\frac{g}{l}\right)\left(\frac{k}{m}\right)x_1 \sin x_1 - \frac{1}{2}\left(\frac{k}{m}\right)x_2^2$$

becomes neg-def. over the domain $D = \{x \in R^2 \mid |x_1| < \pi\}$

- Lyapunov Direct Method
  - Example 3: Consider the first-order differential equation

$$\dot{x} = -g(x)$$

where $g(x)$ is locally Lipschitz on $(-a, a)$ and satisfies

$$g(0) = 0; \quad xg(x) > 0, \quad \forall\, x \neq 0, \ x \in (-a, a)$$

- Lyapunov Candidate:
  - ✓ How??! (Total Energy) $V(x) = \int_0^x g(y)\, dy$
  - ✓ It is positive definite in the domain $D = (-a, a)$
- Lyapunov Function?
  - ✓ Derivative along trajectory:

$$\dot{V}(x) = \frac{\partial V}{\partial x}[-g(x)] = -g^2(x) < 0, \ \forall\, x \in D - \{0\}$$

  - ✓ The eq. point is Asymptotically stable

- Lyapunov Direct Method
  - Example 4: Consider the following system:

$$\dot{x}_1 = x_1 (x_1^2 + x_2^2 - 2) - 4 x_1 x_2^2$$

$$\dot{x}_2 = 4 x_1^2 x_2 + x_2 (x_1^2 + x_2^2 - 2)$$

- The eq. point is @ origin.
- Lyapunov Candidate:

$$V(x_1, x_2) = x_1^2 + x_2^2$$

✓ Derivative along trajectory:

$$\dot{V} = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2)$$

✓ It is negative definite in a ball: $x_1^2 + x_2^2 < 2$

✓ The eq. point is asymptotically stable.

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- Stability Analysis
  - Lyapunov Direct Method
    - Global Stability

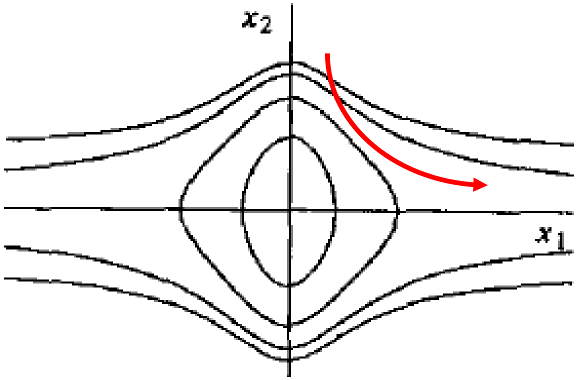
Theorem : *Let $x = 0$ be an equilibrium point for (3.1). Let $V : R^n \to R$ be a continuously differentiable function such that*

$$V(0) = 0 \text{ and } V(x) > 0, \ \forall\, x \neq 0$$

Radial Unboundedness $\|x\| \to \infty \Rightarrow V(x) \to \infty$

$$\dot{V}(x) < 0, \ \forall\, x \neq 0$$

then $x = 0$ is globally asymptotically stable.

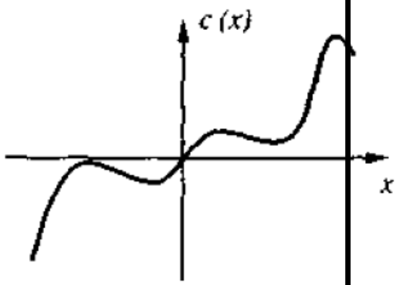


Radial Unboundedness

For small *c* the Ly. Surfaces *V(x)=c* are closed, but for large c the Ly. S. are not closed, then the trajectory may diverge.

- Lyapunov Direct Method
  - Example 5: System as in Ex3: $\dot{x} + c(x) = 0$

    In which, *c(0) =0* and $x\,c(x) > 0 \quad \text{for} \quad x \neq 0$



    - Lyapunov Candidate: $V = x^2$
      - ✓ It is positive definite in the whole space
      - ✓ It is radially unbounded
    - Lyapunov Function?
      - ✓ Derivative along trajectory: $\dot{V} = 2\,x\,\dot{x} = = -2\,x\,c(x)$
      - ✓ Hence: $\dot{V} < 0$ as long as $x \neq 0$
      - ✓ Hence, the origin is globally asymptotically stable.
    - Typical Examples
      - ✓ $\dot{x} = -x^3$ OR $\dot{x} = \sin^2 x - x$ $(\sin^2 x \leq |\sin x| < |x|.)$
      - ✓ $x\,c(x) = x^4 > 0$ and $x\,c(x) = x^2 - x\sin^2 x > x^2 - x|x| > 0$

- Lyapunov Direct Method
  - Example 6: Consider the following system:

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2)$$

$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2)$$

    - The eq. point is @ origin.
    - Lyapunov Candidate:

$$V(\mathbf{x}) = x_1^2 + x_2^2$$

      - ✓ Derivative along trajectory:

$$\dot{V}(\mathbf{x}) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = -2(x_1^2 + x_2^2)^2$$

      - ✓ It is negative definite everywhere,
      - ✓ It is radially unbounded,
      - ✓ The eq. point is globally asymptotically stable .

- Lyapunov Direct Method
  - Remarks:
    1. Use total energy as the first Lyapunov candidate, but don't limit yourself to that.

    2. Many Lyapunov functions exist for a system. If V is a Lyapunov function, so is $V_1 = \rho V^{\alpha}$.

    3. Lyapunov theorems are sufficient theorems, if a Lyapunov candidate doesn't work, search for another one!

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- Invariant Set Theorems:
  - Asymptotic stability needs $\dot{V}(x) < 0$
    - In many systems we reach to $\dot{V}(x) \leq 0$
    - Use invariant set to prove asymptotic stability

Definition : *A set **G** is an invariant set for a dynamic system if every system trajectory which starts from a point in **G** remains in **G** for all future time.*

- Examples of invariant sets
  - ✓ An equilibrium point
  - ✓ A limit cycle
  - ✓ Any trajectory
  - ✓ The domain of attraction of an eq. point or a limit cycle
  - ✓ The whole state space

- Invariant Set Theorem:
  - Krasovskii - Lasalle's Theorem

Theorem : (Local Invariant Set Theorem) *Consider an autonomous system of the form (3.2), with* $\mathbf{f}$ *continuous, and let* $V(\mathbf{x})$ *be a scalar function with continuous first partial derivatives. Assume that*

- *for some* $l > 0$, *the region* Ω_l *defined by* $V(\mathbf{x}) < l$ *is bounded*
- $\dot{V}(\mathbf{x}) \leq 0$ *for all* $\mathbf{x}$ *in* Ω_l

Let $\mathbf{R}$ *be the set of all points within* Ω_l *where* $\dot{V}(\mathbf{x}) = 0$, *and* $\mathbf{M}$ *be the largest invariant set in* $\mathbf{R}$. *Then, every solution* $\mathbf{x}(t)$ *originating in* Ω_l *tends to* $\mathbf{M}$ *as* $t \to \infty$.

- ✓ The function V does not need to be positive definite!
- ✓ The set Ω_l is called a compact set.
- ✓ largest invariant set means the union of all invariant sets.
- ✓ This theorem introduces the notion of Region of Attraction.
- ✓ Can be used for Eq. point, limit cycle, or any invariant set.

- Invariant Set Theorem:
  - Local Asymptotic Stability

Corollary 3.1 *Let $x = 0$ be an equilibrium point for (3.1). Let $V : D \to R$ be a continuously differentiable positive definite function on a domain D containing the origin $x = 0$, such that $\dot{V}(x) \le 0$ in D. Let $S = \{x \in D \mid \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution. Then, the origin is asymptotically stable.* ◇

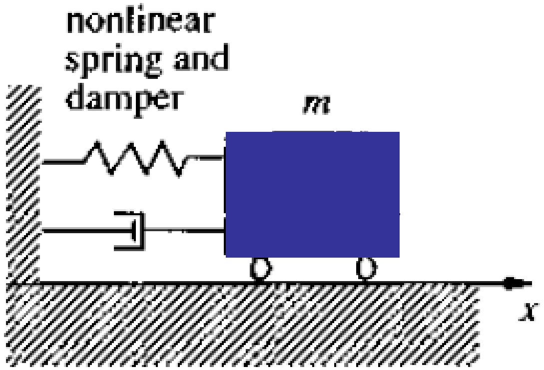
  - Global Asymptotic Stability

Corollary 3.2 *Let $x = 0$ be an equilibrium point for (3.1). Let $V : R^n \to R$ be a continuously differentiable, radially unbounded, positive definite function such that $\dot{V}(x) \le 0$ for all $x \in R^n$. Let $S = \{x \in R^n \mid \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution. Then, the origin is globally asymptotically stable.* ◇

    - No Compact Set
    - No sign of Region of Attraction

- Invariant Set Theorem:
  - Example 1: Mass with nonlinear spring-damper
    - System dynamics

$$m\ddot{x} + b\dot{x}|\dot{x}| + k_o x + k_1 x^3 = 0$$



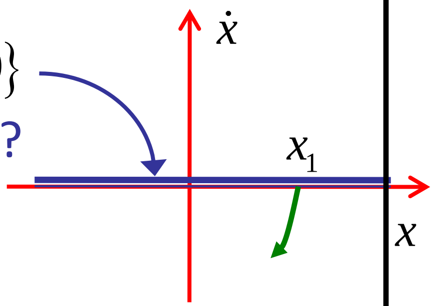
    - Lyapunov Function

$$V(\mathbf{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_0 x^2 + \frac{1}{4}k_1 x^4$$

$$\dot{V}(\mathbf{x}) = -b|\dot{x}|^3 \le 0$$

✓ The set R where $\dot{V}(\mathbf{x}) = 0$ is $R = \{(x, \dot{x}) \mid \dot{x} = 0\}$

✓ Is the largest invariant set in R, $M = \{(0,0)\}$?



Suppose any arbitrary point of R, such as (x_1,0) is also in M.

Any trajectory passing through this point must satisfy:

$\ddot{x} = -(k_o/m)x - (k_1/m)x^3 \ne 0.$ hence the trajectory moves out from R.

✓ The equilibrium point is asymptotically stable.

- Invariant Set Theorem:
  - Example 2:
    - System dynamics:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -g(x_1) - h(x_2)$$

      - ✓ In which,

$$g(0) = 0, \quad yg(y) > 0, \quad \forall\, y \neq 0, \; y \in (-a, a)$$
$$h(0) = 0, \quad yh(y) > 0, \quad \forall\, y \neq 0, \; y \in (-a, a)$$

    - Eq. point @ origin.
    - Lyapunov Candidate: $V(x) = \int_0^{x_1} g(y)\, dy + \frac{1}{2} x_2^2$
      - ✓ In the domain $D = \{x \in R^2 \mid -a < x_i < a\}$ is positive definite.
    - Lyapunov function derivative:

$$\dot{V}(x) = g(x_1)x_2 + x_2[-g(x_1) - h(x_2)] = -x_2 h(x_2) \leq 0$$

      - ✓ Positive semi-definite, needs invariant set Theorem.

- Invariant Set Theorem:
  - Example 2: (cont.)
    - Characterize the Set R where:

$$\dot{V}(x) = 0 \;\Rightarrow\; x_2 h(x_2) = 0 \;\Rightarrow\; x_2 = 0, \;\; \text{since} \; -a < x_2 < a$$

    - Hence, $R = \{(x_1, x_2) \,|\, x_2 = 0\}$
    - Show that M includes only origin:
      - ✓ Suppose x(t) is a trajectory belonging to R, then

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow g(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

      - ✓ Hence, the solution to this trajectory is only the origin.
    - The equilibrium point in asymptotically stable.

- Invariant Set Theorem:
  - Example 3: Region of Attraction
    - System dynamics:

$$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 2) - 4x_1 x_2^2$$

$$\dot{x}_2 = 4x_1^2 x_2 + x_2(x_1^2 + x_2^2 - 2)$$

    - Eq. point is @ origin.
    - Lyapunov candidate: $V(x_1, x_2) = x_1^2 + x_2^2$
      - ✓ for $l=2$, the region Ω defined by $V(x) < 2$ is a compact set.
    - Lyapunov derivative: $\dot{V} = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 2)$
      - ✓ For set Ω the derivative is always negative except @ origin.
    - The set R includes only the origin.
      - ✓ Invariant Set Theorem conditions hold.
      - ✓ The eq. point is locally asymptotically stable.
      - ✓ The Region of Attraction is Ω a circle with radius $r = \sqrt{2}$.

- Invariant Set Theorem:
  - Example 4: Attractive Limit Cycle
    - System dynamics

$$\dot{x}_1 = x_2 - x_1^7 [x_1^4 + 2 x_2^2 - 10]$$

$$\dot{x}_2 = - x_1^3 - 3 x_2^5 [x_1^4 + 2 x_2^2 - 10]$$

    - There exist an invariant set: $x_1^4 + 2 x_2^2 = 10$
      - ✓ Since, its derivative is zero on the set.

$$\frac{d}{dt}(x_1^4 + 2 x_2^2 - 10) = -(4 x_1^{10} + 12 x_2^6)(x_1^4 + 2 x_2^2 - 10) = 0$$

    - On the invariant set:
      - ✓ Simplified system dynamics
      - ✓ Invariant set is a limit cycle

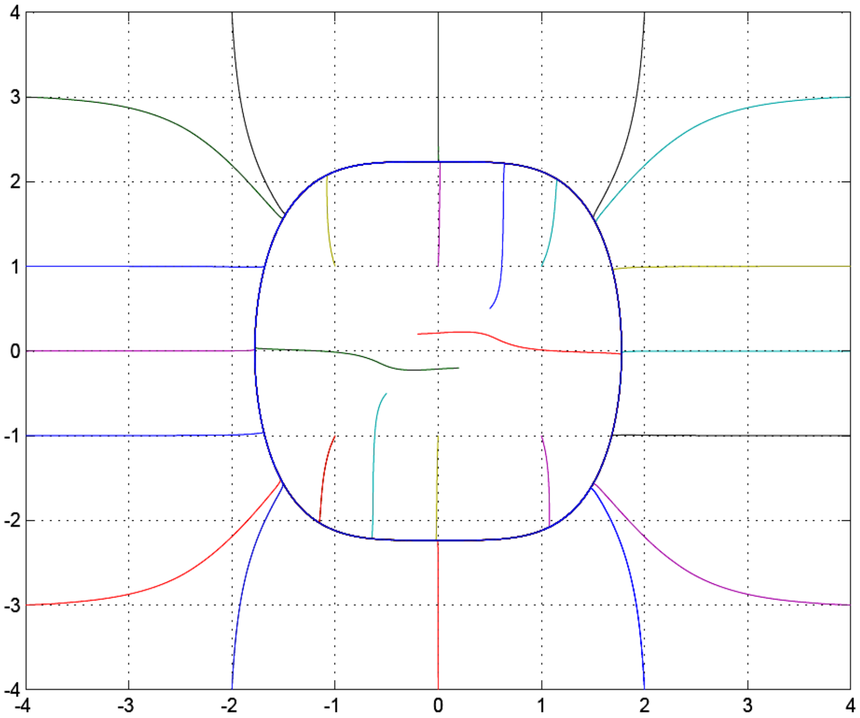
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = - x_1^3$$

    - Is the limit cycle attractive?
      - ✓ Lyapunov candidate: $V = (x_1^4 + 2 x_2^2 - 10)^2$
      - ✓ Physical insight: distance to the limit cycle.

- Invariant Set Theorem:
  - Example 4: Attractive Limit Cycle (cont.)
    - For any $l > 0$,
      - ✓ the region Ω defined by $V(x) < l$ is a compact set.
    - Lyapunov function derivative:
      - ✓ from before, $\dot{V} = -8\,(x_1^{10} + 3\,x_2^{6})\,(x_1^{4} + 2\,x_2^{2} - 10)^2$
      - ✓ $\dot{V}(\mathbf{x}) < 0$ everywhere except at
        - $x_1^4 + 2\,x_2^2 = 10$ or The limit Cycle.
        - $x_1^{10} + 3\,x_2^6 = 0$ The Eq. point @ origin.
      - ✓ The eq. point at origin is unstable.
    - From invariant set theorem, all the trajectories converge to the limit cycle.

- Invariant Set Theorem:
  - Example 4: Attractive Limit Cycle (cont.)
    - Phase portrait:



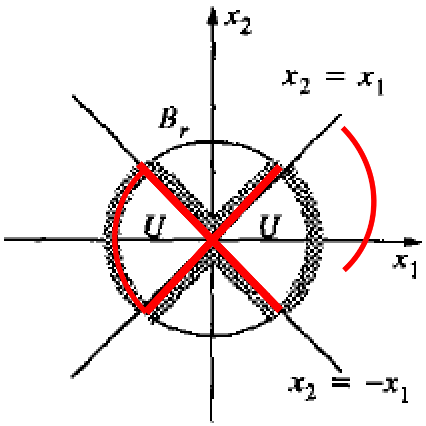
- Linear Time Invariant (LTI) Systems:
  - Consider an LTI system $\dot{x} = Ax$
    - State Transition Matrix: $x(t) = e^{At} x_o$
  - Stability of an LTI system
    - Hurwitz $\forall \lambda_i, \quad \Re(\lambda_i) < 0$
    - Lyapunov Equation
      - ✓ Theorem: A matrix *A* is Hurwitz, *iff* for any given positive definite matrix *Q*, there exists a positive definite matrix *P* that satisfies the Lyapunov Equation: $A^T P + PA = -Q$
        - Sketch of proof: Lyapunov candidate $V(x) = x^T P x$
        - Derivative: $\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x = -x^T Q x$
        - Lyapunov Equation $PA + A^T P = -Q$

Instability Theorem

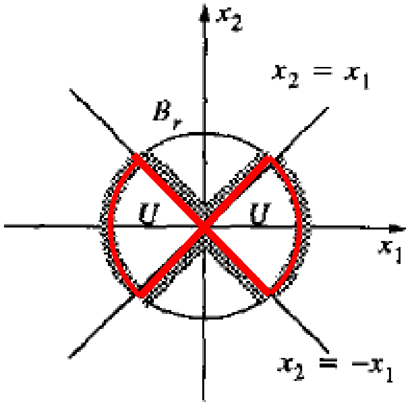
Theorem : *Let $x = 0$ be an equilibrium point for (3.1). Let $V : D \to R$ be a continuously differentiable function such that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrarily small $\|x_0\|$. Define a set U as in (3.8) and suppose that $\dot{V}(x) > 0$ in U. Then, $x = 0$ is unstable.* ◇

$$U = \{x \in B_r \mid V(x) > 0\} \qquad (3.8)$$

- Example 1:
  - For $V(x) = (x_1^2 - x_2^2)$
    - ✓ $V(0) = 0,\ V(x_o) > 0$ in the hatched area:
    - ✓ The Region U is the hatched area
  - For Instability it is sufficient to have
    - ✓ $\exists x_o \ni \quad V(x_o) > 0$, and
    - ✓ $\forall x \in U \quad \dot{V}(x) > 0$



- Instability Theorem
  - Example 2:
    - System dynamics: $$\begin{aligned}\dot{x}_1 &= x_1 + g_1(x)\\ \dot{x}_2 &= -x_2 + g_2(x)\end{aligned}$$
      - ✓ in which, $|g_i(x)| \le k\|x\|_2^2$
    - Eq. point @ origin, since $g_i(0) = 0$
    - Consider: $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$
      - ✓ The set *U* is as shown
      - ✓ For a point inside *U* $V(x_o) > 0$



    - Derivative along trajectory

$$\dot{V}(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x)$$

      - ✓ But, $|x_1 g_1(x) - x_2 g_2(x)| \le \sum_{i=1}^{2} |x_i| \cdot |g_i(x)| \le 2k\|x\|_2^3$
      - ✓ Hence: $\dot{V}(x) \ge \|x\|_2^2 - 2k\|x\|_2^3 = \|x\|_2^2(1 - 2k\|x\|_2)$
        - ✓ For a Ball $B_r \subset D$ and $r < 1/2\dot{k}$, $\dot{V}(x) > 0$ ➡ Unstable

Lyapunov Stability

- Overview
  - Definitions
  - Stability analysis
    - Lyapunov Linearization Method
    - Lyapunov Direct Method
      - ✓ Local Stability
      - ✓ Global Stability
    - Invariant Set
  - Lyapunov Function Generation
    - Krosovskii Methods
    - Variable Gradient Method
  - Lyapunov Based Controller Design

- Lyapunov Function Generation
  - Krasovskii Method

Theorem :(Krasovskii) *Consider the autonomous system defined by* (3.1), *with the equilibrium point of interest being the origin. Let* $\mathbf{A}(\mathbf{x})$ *denote the Jacobian matrix of the system, i.e.,*

$$\mathbf{A}(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$$

If the matrix $\mathbf{F} = \mathbf{A} + \mathbf{A}^T$ *is negative definite in a neighborhood* $\mathbf{\Omega}$, *then the equilibrium point at the origin is asymptotically stable. A Lyapunov function for this system is*

$$V(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})\,\mathbf{f}(\mathbf{x})$$

If $\mathbf{\Omega}$ *is the entire state space and, in addition,* $V(\mathbf{x}) \rightarrow \infty$ *as* $\|\mathbf{x}\| \rightarrow \infty$, *then the equilibrium point is globally asymptotically stable.*

- Lyapunov Function Generation
  - Krasovskii Method
    - Example: Consider the system

$$\dot{x}_1 = -6x_1 + 2x_2$$
$$\dot{x}_2 = 2x_1 - 6x_2 - 2x_2^3$$

✓ The Jacobian:

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} -6 & 2 \\ 2 & -6 - 6x_2^2 \end{bmatrix} \qquad \mathbf{F} = \mathbf{A} + \mathbf{A}^T = \begin{bmatrix} -12 & 4 \\ 4 & -12 - 12x_2^2 \end{bmatrix}$$

✓ F is negative definite for the whole space.

✓ Lyapunov Function

$$V(\mathbf{x}) = \mathbf{f}^T(\mathbf{x})\,\mathbf{f}(\mathbf{x}) = (-6x_1 + 2x_2)^2 + (2x_1 - 6x_2 - 2x_2^3)^2$$

✓ It is Radially unbounded

$$V(\mathbf{x}) \to \infty \quad \text{as} \quad \|\mathbf{x}\| \to \infty$$

✓ The Eq. point is globally asymptotically stable.

- Lyapunov Function Generation
  - Krasovskii Method

Theorem : (Generalized Krasovskii Theorem) *Consider the autonomous system defined by (3.1), with the equilibrium point of interest being the origin, and let* $\mathbf{A}(\mathbf{x})$ *denote the Jacobian matrix of the system. Then, a sufficient condition for the origin to be asymptotically stable is that there exist two symmetric positive definite matrices* $\mathbf{P}$ *and* $\mathbf{Q}$, *such that* $\forall \mathbf{x} \neq \mathbf{0}$, *the matrix*

$$\mathbf{F}(\mathbf{x}) = \mathbf{A}^T \mathbf{P} + \mathbf{P}\,\mathbf{A} + \mathbf{Q}$$

is negative semi-definite in some neighborhood $\mathbf{\Omega}$ *of the origin. The function* $V(\mathbf{x}) = \mathbf{f}^T \mathbf{P} \mathbf{f}$ *is then a Lyapunov function for the system. If the region* $\mathbf{\Omega}$ *is the whole state space, and if in addition,* $V(\mathbf{x}) \to \infty$ *as* $\|\mathbf{x}\| \to \infty$, *then the system is globally asymptotically stable.*

- Proof Idea:

$$\dot{V} = \frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) = \mathbf{f}^T \mathbf{P} \mathbf{A}(\mathbf{x}) \mathbf{f} + \mathbf{f}^T \mathbf{P} \mathbf{A}^T(\mathbf{x}) \mathbf{P} \mathbf{f} = \mathbf{f}^T \mathbf{F} \mathbf{f} - \mathbf{f}^T \mathbf{Q} \mathbf{f}$$

✓ If $F < 0$ and $Q > 0$, then $\dot{V}(x) < 0$

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- Lyapunov Function Generation
  - Variable Gradient Method
    - Search backward, start with $\dot{V}(x) < 0$, then find *V(x)*.
    - Procedure:
      - ✓ Suppose *g(x)* is the gradient of *V(x)*: $g(x) = \nabla V = (\partial V / \partial x)^T$
      - ✓ Derivative of *V(x)* along trajectory:

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) = g^T(x) f(x)$$

      - ✓ Choose *g(x)* such that $\dot{V}(x) < 0$ while, $V(x) > 0$.
      - ✓ For g(x) to be gradient of a scalar function:

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}, \quad \forall\ i, j = 1, \ldots, n$$

      - ✓ Under this constraint choose g(x) such that $g^T(x) f(x) < 0$

- Lyapunov Function Generation
  - Variable Gradient Method
    - Procedure (cont.):
      - ✓ Then generate *V(x)* by integration

$$V(x) = \int_0^x g^T(y)\, dy = \int_0^x \sum_{i=1}^n g_i(y)\, dy_i$$

      - ✓ The integration can be taken along any path, but usually it is taken along the principal axes:

$$\begin{aligned} V(x) &= \int_0^{x_1} g_1(y_1, 0, \ldots, 0)\, dy_1 + \int_0^{x_2} g_2(x_1, y_2, 0, \ldots, 0)\, dy_2 \\ &\quad + \cdots + \int_0^{x_n} g_n(x_1, x_2, \ldots, x_{n-1}, y_n)\, dy_n \end{aligned}$$

      - ✓ Leave some parameters of *g(x)* undetermined, and try to choose them to ensure that *V(x)* positive.

- Variable Gradient Method
  - Example 1:
    - Consider the system: $$\dot{x}_1 = x_2$$ $$\dot{x}_2 = -h(x_1) - ax_2$$
      - ✓ where, *a>0, h(0)=0* and $yh(y)>0, \forall y \in (-b,c)$
    - To ensure $\dot{V}(x) < 0 \rightarrow g^T(x) f(x) < 0$
      - ✓ $\dot{V}(x) = g_1(x)x_2 - g_2(x)[h(x_1) + ax_2] < 0, \quad \text{for } x \neq 0$
    - The Lypunov function is:
      - ✓ $V(x) = \int_0^x g^T(y)\, dy > 0, \quad \text{for } x \neq 0$
    - Let us try
      - ✓ $g(x) = \begin{bmatrix} \alpha(x)x_1 + \beta(x)x_2 \\ \gamma(x)x_1 + \delta(x)x_2 \end{bmatrix}$
    - Gradient condition
      - ✓ $\frac{\partial g_1}{\partial x_2} = \frac{\partial g_2}{\partial x_1} \Rightarrow \beta(x) + \frac{\partial \alpha}{\partial x_2}x_1 + \frac{\partial \beta}{\partial x_2}x_2 = \gamma(x) + \frac{\partial \gamma}{\partial x_1}x_1 + \frac{\partial \delta}{\partial x_1}x_2$

- Variable Gradient Method
  - Example 1: (cont.)
    - Derivative of Ly. f.

$$\dot{V}(x) = \alpha(x)x_1x_2 + \beta(x)x_2^2 - a\gamma(x)x_1x_2 - a\delta(x)x_2^2 - \delta(x)x_2h(x_1) - \gamma(x)x_1h(x_1)$$

✓ To cancel cross terms $\alpha(x)x_1 - a\gamma(x)x_1 - \delta(x)h(x_1) = 0$

✓ Therefore, $\dot{V}(x) = -[a\delta(x) - \beta(x)]x_2^2 - \gamma(x)x_1h(x_1)$

✓ To simplify assign β, γ, and δ to be constant but keep α(x)

- From gradient condition

✓ $$\beta(x) + \frac{\partial\alpha}{\partial x_2}x_1 + \frac{\partial\beta}{\partial x_2}x_2 = \gamma(x) + \frac{\partial\gamma}{\partial x_1}x_1 + \frac{\partial\delta}{\partial x_1}x_2$$

✓ $\alpha(x) = \alpha(x_1)$ and $\beta = \gamma$.

✓ $$g(x) = \begin{bmatrix} \alpha(x)x_1 + \beta(x)x_2 \\ \gamma(x)x_1 + \delta(x)x_2 \end{bmatrix} \Rightarrow g(x) = \begin{bmatrix} a\gamma x_1 + \delta h(x_1) + \gamma x_2 \\ \gamma x_1 + \delta x_2 \end{bmatrix}$$

- Variable Gradient Method
  - Example 1: (cont.)
    - Integrate g(x) to get the Ly. f.

$$V(x) = \int_0^{x_1} [a\gamma y_1 + \delta h(y_1)]\, dy_1 + \int_0^{x_2} (\gamma x_1 + \delta y_2)\, dy_2$$

$$= \tfrac{1}{2} a\gamma x_1^2 + \delta \int_0^{x_1} h(y)\, dy + \gamma x_1 x_2 + \tfrac{1}{2}\delta x_2^2 = \tfrac{1}{2} x^T P x + \delta \int_0^{x_1} h(y)\, dy$$

✓ in which, $P = \begin{bmatrix} a\gamma & \gamma \\ \gamma & \delta \end{bmatrix}$

- Choose $\delta > 0$ and $0 < \gamma < a\delta$ to ensure $V(x) > 0$ and $\dot{V}(x) < 0$

✓ For example $\gamma = ak\delta$ for $0 < k < 1$

✓ Then, $V(x) = \frac{\delta}{2} x^T \begin{bmatrix} ka^2 & ka \\ ka & 1 \end{bmatrix} x + \delta \int_0^{x_1} h(y)\, dy$

✓ $V(x) > 0$ and $\dot{V}(x) < 0$ for $D = \{x \in \mathbf{R}^2 \mid -b < x_1 < c\}$

✓ The eq. point is asymptotically stable.

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- Lyapunov Based Controller Design
  - Example: Robotic Manipulator
    - Physically derived Lyapunov function
    - System dynamics $\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \tau$
    - Controller $\tau = -\mathbf{K}_D\dot{\mathbf{q}} - \mathbf{K}_P\mathbf{q} + \mathbf{g}(\mathbf{q})$
    - Lyapunov Candidate
      - ✓ Total Energy $V = \frac{1}{2}[\dot{\mathbf{q}}^T\mathbf{H}\dot{\mathbf{q}} + \mathbf{q}^T\mathbf{K}_p\mathbf{q}]$
    - Lyapunov Function Derivative
      - ✓ Power of the external forces

$$\dot{V} = \dot{\mathbf{q}}^T(\tau - \mathbf{g}) + \dot{\mathbf{q}}^T\mathbf{K}_P\mathbf{q}$$

      - ✓ Used control law

$$\dot{V} = -\dot{\mathbf{q}}^T\mathbf{K}_D\dot{\mathbf{q}}$$

      - ✓ Lasalle: Global Asymptotically stable

- Lyapunov Based Controller Design
  - Design Idea:
    - Consider a Lyapunov candidate
    - Stability:
      - ✓ Design the control law as a nonlinear function to ensure negative definiteness of the Ly. F. Derivative.
    - Performance:
      - ✓ Rate of decay is related to the time performance.
    - Base of many nonlinear controller designs:
      - ✓ Back-stepping
      - ✓ Sliding mode control
      - ✓ Lyapunov redesign
      - ✓ . . .

- Lyapunov Based Controller Design
  - Example 2: Regulation
    - System dynamics: $\ddot{x} - \dot{x}^3 + x^2 = u$
    - Objective
      - ✓ Push the trajectories toward origin.
    - Consider the controller as: $u = u(x, \dot{x})$
    - Lyapunov Candidate: $V = 1/2(x^2 + \dot{x}^2)$
    - Derivative: $\dot{V} = \dot{x}(x + \dot{x}^3 - x^2 + u)$
    - Design u such that $\dot{V}(x) \le 0$:
      - ✓ For example, $\dot{V} \equiv -K\,\dot{x}^2 \rightarrow u = -x + x^2 - K\,\dot{x} - \dot{x}^3$
      - ✓ Stability: Lasalle → asymptotically stable eq. point.
      - ✓ Performance: increase K to have faster response.
      - ✓ Controller is not unique.

Lyapunov Stability

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