CHAPTER 1

Limits and Continuity

> Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

Isaac Newton 1642–1727
from *Principia Mathematica, 1687*

> It was not until Leibniz and Newton, by the discovery of the differential calculus, had dispelled the ancient darkness which enveloped the conception of the infinite, and had clearly established the conception of the continuous and continuous change, that a full productive application of the newly found mechanical conceptions made any progress.

Hermann von Helmholtz 1821–1894

Introduction

Calculus was created to describe how quantities change. It has two basic procedures that are opposites of one another, namely:

- *differentiation,* for finding the rate of change of a given function, and
- *integration,* for finding a function having a given rate of change.

Both of these procedures are based on the fundamental concept of the *limit* of a function. It is this idea of limit that distinguishes calculus from algebra, geometry, and trigonometry, which are useful for describing static situations.

In this chapter we will introduce the limit concept and develop some of its properties. We begin by considering how limits arise in some basic problems.

1.1 Examples of Velocity, Growth Rate, and Area

In this section we consider some examples of phenomena where limits arise in a natural way.

Average Velocity and Instantaneous Velocity

The position of a moving object is a function of time. The average velocity of the object over a time interval is found by dividing the change in the object's position by the length of the time interval.

EXAMPLE 1 **(The average velocity of a falling rock)** Physical experiments show that if a rock is dropped from rest near the surface of the earth, in the first t s it will fall a distance

$$y = 4.9t^2 \text{ m}.$$

(a) What is the average velocity of the falling rock during the first 2 s?

(b) What is its average velocity from $t = 1$ to $t = 2$?

Solution The *average velocity* of the falling rock over any time interval $[t_1, t_2]$ is the change Δy in the distance fallen divided by the length Δt of the time interval:

$$\text{average velocity over } [t_1, t_2] = \frac{\Delta y}{\Delta t} = \frac{4.9t_2^2 - 4.9t_1^2}{t_2 - t_1}.$$

(a) In the first 2 s (time interval $[0, 2]$), the average velocity is

$$\frac{\Delta y}{\Delta t} = \frac{4.9(2^2) - 4.9(0^2)}{2 - 0} = 9.8 \text{ m/s}.$$

(b) In the time interval $[1, 2]$, the average velocity is

$$\frac{\Delta y}{\Delta t} = \frac{4.9(2^2) - 4.9(1^2)}{2 - 1} = 14.7 \text{ m/s}.$$

EXAMPLE 2 How fast is the rock in Example 1 falling (a) at time $t = 1$? (b) at time $t = 2$?

Solution We can calculate the average velocity over any time interval, but this question asks for the *instantaneous velocity* at a given time. If the falling rock had a speedometer, what would it show at time $t = 1$? To answer this, we first write the average velocity over the time interval $[1, 1 + h]$ starting at $t = 1$ and having length h:

$$\text{Average velocity over } [1, 1 + h] = \frac{\Delta y}{\Delta t} = \frac{4.9(1 + h)^2 - 4.9(1^2)}{h}.$$

We can't calculate the instantaneous velocity at $t = 1$ by substituting $h = 0$ in this expression, because we can't divide by zero. But we can calculate the average velocities over shorter and shorter time intervals and see whether they seem to get close to a particular number. Table 1 shows the values of $\Delta y/\Delta t$ for some values of h approaching zero. Indeed, it appears that these average velocities get closer and closer to 9.8 m/s as the length of the time interval gets closer and closer to zero. This suggests that the rock is falling at a rate of 9.8 m/s one second after it is dropped.

Similarly, Table 2 shows values of the average velocities over shorter and shorter time intervals $[2, 2 + h]$ starting at $t = 2$. The values suggest that the rock is falling at 19.6 m/s two seconds after it is dropped.

Table 1. Average velocity over $[1, 1 + h]$

h	$\Delta y/\Delta t$
1	14.7000
0.1	10.2900
0.01	9.8490
0.001	9.8049
0.0001	9.8005

Table 2. Average velocity over $[2, 2 + h]$

h	$\Delta y/\Delta t$
1	24.5000
0.1	20.0900
0.01	19.6490
0.001	19.6049
0.0001	19.6005

In Example 2 the average velocity of the falling rock over the time interval $[t, t + h]$ is

$$\frac{\Delta y}{\Delta t} = \frac{4.9(t + h)^2 - 4.9t^2}{h}.$$

To find the instantaneous velocity (usually just called *the velocity*) at the instants $t = 1$ and $t = 2$, we examined the values of this average velocity for time intervals whose lengths h became smaller and smaller. We were, in fact, finding the *limit of the average velocity as h approaches zero.* This is expressed symbolically in the form

$$\text{velocity at time } t = \lim_{h \to 0} \frac{\Delta y}{\Delta t} = \lim_{h \to 0} \frac{4.9(t + h)^2 - 4.9t^2}{h}.$$

Read "$\lim_{h \to 0} \ldots$" as "the limit as h approaches zero of ..." We can't find the limit of the fraction by just substituting $h = 0$ because that would involve dividing by zero. However, we can calculate the limit by first performing some algebraic simplifications on the expression for the average velocity.

EXAMPLE 3 Simplify the expression for the average velocity of the rock over $[t, t+h]$ by first expanding $(t+h)^2$. Hence, find the velocity $v(t)$ of the falling rock at time t directly, without making a table of values.

Solution The average velocity of the rock over time interval $[t, t+h]$ is

$$\begin{aligned}\frac{4.9(t+h)^2 - 4.9t^2}{h} &= \frac{4.9(t^2 + 2th + h^2 - t^2)}{h} \\ &= \frac{4.9(2th + h^2)}{h} \\ &= 9.8t + 4.9h.\end{aligned}$$

The final form of the expression no longer involves division by h. It approaches $9.8t + 4.9(0) = 9.8t$ as h approaches 0. Thus, t s after the rock is dropped, its velocity is $v(t) = 9.8t$ m/s. In particular, at $t = 1$ and $t = 2$ the velocities are $v(1) = 9.8$ m/s and $v(2) = 19.6$ m/s, respectively.

The Growth of an Algal Culture

In a laboratory experiment, the biomass of an algal culture was measured over a 74-day period by measuring the area in square millimetres occupied by the culture on a microscope slide. These measurements m were plotted against the time t in days and the points joined by a smooth curve $m = f(t)$, as shown in red in Figure 1.1.

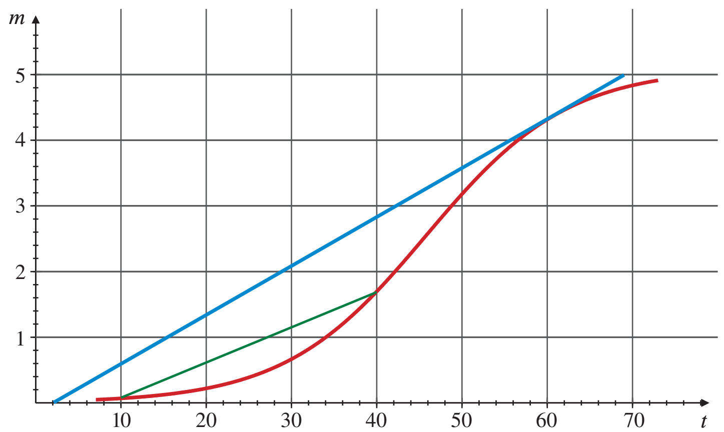


Figure 1.1 The biomass m of an algal culture after t days

Observe that the biomass was about 0.1 mm^2 on day 10 and had grown to about 1.7 mm^2 on day 40, an increase of $1.7 - 0.1 = 1.6$ mm^2 in a time interval of $40 - 10 = 30$ days. The average rate of growth over the time interval from day 10 to day 40 was therefore

$$\frac{1.7 - 0.1}{40 - 10} = \frac{1.6}{30} \approx 0.053 \text{ mm}^2/\text{d}.$$

This average rate is just the slope of the green line joining the points on the graph of $m = f(t)$ corresponding to $t = 10$ and $t = 40$. Similarly, the average rate of growth of the algal biomass over any time interval can be determined by measuring the slope of the line joining the points on the curve corresponding to that time interval. Such lines are called **secant lines** to the curve.

EXAMPLE 4 How fast is the biomass growing on day 60?

Solution To answer this question, we could measure the average rates of change over shorter and shorter times around day 60. The corresponding secant lines become shorter and shorter, but their slopes approach a *limit*, namely, the slope of the **tangent line** to the graph of $m = f(t)$ at the point where $t = 60$. This tangent line is sketched in blue in Figure 1.1; it seems to go through the points $(2, 0)$ and $(69, 5)$, so that its slope is

$$\frac{5-0}{69-2} \approx 0.0746 \text{ mm}^2\text{/d}.$$

This is the rate at which the biomass was growing on day 60.

The Area of a Circle

All circles are similar geometric figures; they all have the same shape and differ only in size. The ratio of the circumference C to the diameter $2r$ (twice the radius) has the same value for all circles. The number π is defined to be this common ratio:

$$\frac{C}{2r} = \pi \qquad \text{or} \qquad C = 2\pi r.$$

In school we are taught that the area A of a circle is this same number π times the square of the radius:

$$A = \pi r^2.$$

How can we deduce this area formula from the formula for the circumference that is the definition of π?

The answer to this question lies in regarding the circle as a "limit" of regular polygons, which are in turn made up of triangles, figures about whose geometry we know a great deal.

Suppose a regular polygon having n sides is inscribed in a circle of radius r. (See Figure 1.2.) The perimeter P_n and the area A_n of the polygon are, respectively, less than the circumference C and the area A of the circle, but if n is large, P_n is *close to* C and A_n is *close to* A. (In fact, the "circle" in Figure 1.2 was drawn by a computer as a regular polygon having 180 sides, each subtending a 2° angle at the centre of the circle. It is very difficult to distinguish this 180-sided polygon from a real circle.) We would expect P_n to approach the limit C and A_n to approach the limit A as n grows larger and larger and approaches infinity.

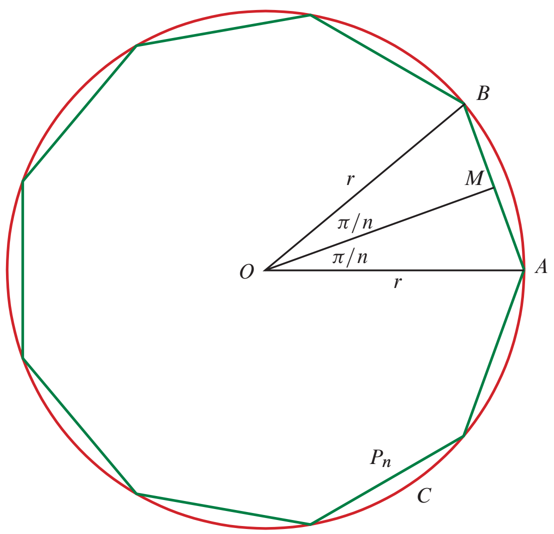


Figure 1.2 A regular polygon (green) of n sides inscribed in a red circle. Here $n = 9$

A regular polygon of n sides is the union of n nonoverlapping, congruent, isosceles triangles having a common vertex at O, the centre of the polygon. One of these triangles, $\triangle OAB$, is shown in Figure 1.2. Since the total angle around the point O is 2π radians (we are assuming that a circle of radius 1 has circumference 2π), the angle AOB is $2\pi/n$ radians. If M is the midpoint of AB, then OM bisects angle AOB. Using elementary trigonometry, we can write the length of AB and the area of triangle OAB in terms of the radius r of the circle:

$$\begin{aligned} |AB| &= 2|AM| = 2r\sin\frac{\pi}{n} \\ \text{area } OAB &= \frac{1}{2}|AB||OM| = \frac{1}{2}\left(2r\sin\frac{\pi}{n}\right)\left(r\cos\frac{\pi}{n}\right) \\ &= r^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}. \end{aligned}$$

The perimeter P_n and area A_n of the polygon are n times these expressions:

$$\begin{aligned} P_n &= 2rn\sin\frac{\pi}{n} \\ A_n &= r^2 n\sin\frac{\pi}{n}\cos\frac{\pi}{n}. \end{aligned}$$

Solving the first equation for $rn\sin(\pi/n) = P_n/2$ and substituting into the second equation, we get

$$A_n = \left(\frac{P_n}{2}\right) r\cos\frac{\pi}{n}.$$

Now the angle $AOM = \pi/n$ approaches 0 as n grows large, so its cosine, $\cos(\pi/n) = |OM|/|OA|$, approaches 1. Since P_n approaches $C = 2\pi r$ as n grows large, the expression for A_n approaches $(2\pi r/2)r(1) = \pi r^2$, which must therefore be the area of the circle.

Remark There is a fundamental relationship between the problem of finding the area under the graph of a function f and the problem of finding another function g whose rate of change is f. It will be explored fully beginning in Chapter 5. As an example, for the falling rock of Example 1–Example 3, the green area A under the graph of the velocity function $v = 9.8t$ m/s and above the interval $[0, t]$ on the t-axis is the area of a triangle of base length t s and height $9.8t$ m/s, and so (see Figure 1.3) is

$$A = \frac{1}{2}(t)(9.8t) = 4.9t^2 \text{ m},$$

which is exactly the distance y that the rock falls during the first t seconds. The rate of change of the area function $A(t)$ (that is, of the distance function y) is the velocity function $v(t)$.

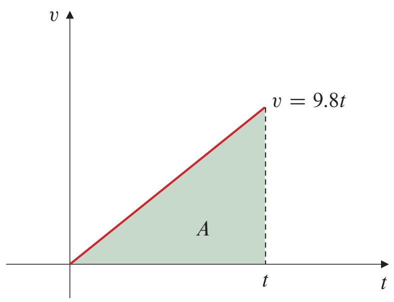


Figure 1.3 $A = \dfrac{1}{2}t\,(9.8t) = 4.9t^2$

EXERCISES 1.1

Exercises 1–4 refer to an object moving along the x-axis in such a way that at time t s its position is $x = t^2$ m to the right of the origin.

1. Find the average velocity of the object over the time interval $[t, t+h]$.
2. Make a table giving the average velocities of the object over time intervals $[2, 2+h]$, for $h = 1, 0.1, 0.01, 0.001$, and 0.0001 s.
3. Use the results from Exercise 2 to guess the instantaneous velocity of the object at $t = 2$ s.
4. Confirm your guess in Exercise 3 by calculating the limit of the average velocity over $[2, 2+h]$ as h approaches zero,

using the method of Example 3.

Exercises 5–8 refer to the motion of a particle moving along the x-axis so that at time t s it is at position $x = 3t^2 - 12t + 1$ m.

5. Find the average velocity of the particle over the time intervals $[1, 2]$, $[2, 3]$, and $[1, 3]$.

6. Use the method of Example 3 to find the velocity of the particle at $t = 1$, $t = 2$, and $t = 3$.

7. In what direction is the particle moving at $t = 1$? $t = 2$? $t = 3$?

8. Show that for any positive number k, the average velocity of the particle over the time interval $[t - k, t + k]$ is equal to its velocity at time t.

In Exercises 9–11, a weight that is suspended by a spring bobs up and down so that its height above the floor at time t s is y ft, where

$$y = 2 + \frac{1}{\pi}\sin(\pi t).$$

9. Sketch the graph of y as a function of t. How high is the weight at $t = 1$ s? In what direction is it moving at that time?

10. What is the average velocity of the weight over the time intervals $[1, 2]$, $[1, 1.1]$, $[1, 1.01]$, and $[1, 1.001]$?

11. Using the results of Exercise 10, estimate the velocity of the weight at time $t = 1$. What is the significance of the sign of your answer?

Exercises 12–13 refer to the algal biomass graphed in Figure 1.1.

12. Approximately how fast is the biomass growing on day 20?

13. On about what day is the biomass growing fastest?

14. The annual profits of a small company for each of the first five years of its operation are given in Table 3.

Table 3.

Year	Profit ($1,000s)
2011	6
2012	27
2013	62
2014	111
2015	174

(a) Plot points representing the profits as a function of year on graph paper, and join them by a smooth curve.

(b) What is the average rate of increase of the annual profits between 2013 and 2015?

(c) Use your graph to estimate the rate of increase of the profits in 2013.

1.2 Limits of Functions

In order to speak meaningfully about rates of change, tangent lines, and areas bounded by curves, we have to investigate the process of finding limits. Indeed, the concept of *limit* is the cornerstone on which the development of calculus rests. Before we try to give a definition of a limit, let us look at more examples.

EXAMPLE 1 Describe the behaviour of the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ near $x = 1$.

Solution Note that $f(x)$ is defined for all real numbers x except $x = 1$. (We can't divide by zero.) For any $x \neq 1$ we can simplify the expression for $f(x)$ by factoring the numerator and cancelling common factors:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \qquad \text{for} \quad x \neq 1.$$

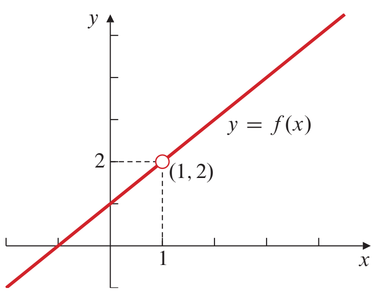


Figure 1.4 The graph of $f(x) = \dfrac{x^2 - 1}{x - 1}$

The graph of f is the line $y = x + 1$ with one point removed, namely, the point $(1, 2)$. This removed point is shown as a "hole" in the graph in Figure 1.4. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ *as close as we want* to 2 by choosing x *close enough* to 1. Therefore, we say that $f(x)$ approaches arbitrarily close to 2 as x approaches 1, or, more simply, $f(x)$ approaches *the limit* 2 as x approaches 1. We write this as

$$\lim_{x \to 1} f(x) = 2 \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

EXAMPLE 2 What happens to the function $g(x) = (1+x^2)^{1/x^2}$ as x approaches zero?

Table 4.

x	$g(x)$
±1.0	2.0000 00000
±0.1	2.7048 13829
±0.01	2.7181 45927
±0.001	2.7182 80469
±0.0001	2.7182 81815
±0.00001	1.0000 00000

Solution Note that $g(x)$ is not defined at $x = 0$. In fact, for the moment it does not appear to be defined for any x whose square x^2 is not a rational number. (Recall that if $r = m/n$, where m and n are integers and $n > 0$, then x^r means the nth root of x^m.) Let us ignore for now the problem of deciding what $g(x)$ means if x^2 is irrational and consider only rational values of x. There is no obvious way to simplify the expression for $g(x)$ as we did in Example 1. However, we can use a scientific calculator to obtain approximate values of $g(x)$ for some rational values of x approaching 0. (The values in Table 4 were obtained with such a calculator.)

Except for the last value in the table, the values of $g(x)$ seem to be approaching a certain number, 2.71828 . . . , as x gets closer and closer to 0. We will show in Section 3.4 that

$$\lim_{x\to0} g(x) = \lim_{x\to0}(1+x^2)^{1/x^2} = e = 2.7\ 1828\ 1828\ 45\ 90\ 45\ldots.$$

The number e turns out to be very important in mathematics.

Observe that the last entry in the table appears to be wrong. This is important. It is because the calculator can only represent a finite number of numbers. The calculator was unable to distinguish $1 + (0.00001)^2 = 1.0000000001$ from 1, and it therefore calculated $1^{10,000,000,000} = 1$. While for many calculations on computers this reality can be minimized, it cannot be eliminated. The wrong value warns us of something called round-off error. We can explore with computer graphics what this means for g near 0. As was the case for the *numerical monster* encountered in Section P.4, the computer can produce rich and beautiful behaviour in its failed attempt to represent g, which is very different from what g actually does. While it is possible to get computer algebra software like Maple to evaluate limits correctly (as we will see in the next section), we cannot use computer graphics or floating-point arithmetic to study many mathematical notions such as limits. In fact, we will need mathematics to understand what the computer actually does so that we can be the master of our tools.

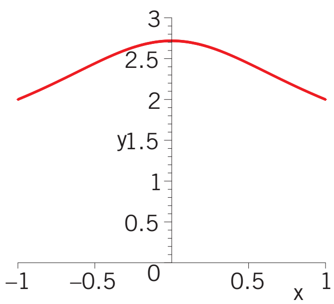


Figure 1.5 The graph of $y = g(x)$ on the interval $[-1, 1]$

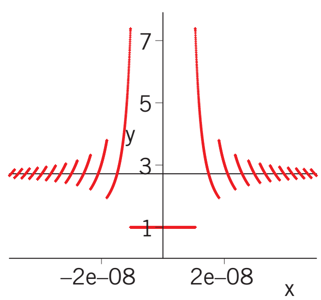


Figure 1.6 The graphs of $y = g(x)$ (colour) and $y = e \approx 2.718$ (black) on the interval $[-5\times10^{-8}, 5\times10^{-8}]$

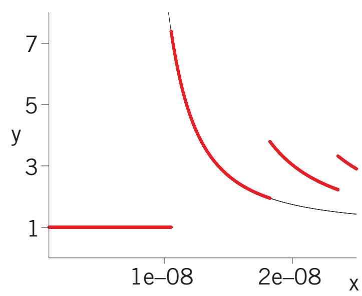


Figure 1.7 The graphs of $y = g(x)$ (colour) and $y = (1+2\times10^{-16})^{1/x^2}$ (black) on the interval $[10^{-9}, 2.5\times10^{-8}]$

Figures 1.5–1.7 illustrate this fascinating behaviour of g with three plots made with Maple using its default 10-significant-figure precision in representing floating-point (i.e., real) numbers. Figure 1.5 is a plot of the graph of g on the interval $[-1, 1]$. The graph starts out at height 2 at either endpoint $x = \pm1$ and rises to height approximately $2.718\cdots$ as x decreases in absolute value, as we would expect from Table 4. Figure 1.6 shows the graph of g restricted to the tiny interval $[-5\times10^{-8}, 5\times10^{-8}]$. It consists of many short arcs decreasing in height as $|x|$ increases, and clustering around the line $y = 2.718\cdots$, and a horizontal part at height 1 between approximately -10^{-8} and 10^{-8}. Figure 1.7 zooms in on the part of the graph to the right of the origin up to $x = 2.5\times10^{-8}$. Note how the arc closest to 0 coincides with the graph of

$y = \left(1 + 2 \times 10^{-16}\right)^{1/x^2}$ (shown in black), indicating that $1 + 2 \times 10^{-16}$ may be the smallest number greater than 1 that Maple can distinguish from 1. Both figures show that the breakdown in the graph of g is not sudden, but becomes more and more pronounced as $|x|$ decreases until the breakdown is complete near $\pm 10^{-8}$.

The examples above and those in Section 1.1 suggest the following *informal* definition of limit.

DEFINITION 1

An informal definition of limit

If $f(x)$ is defined for all x near a, except possibly at a itself, and if we can ensure that $f(x)$ is as close as we want to L by taking x close enough to a, but not equal to a, we say that the function f approaches the **limit** L as x approaches a, and we write

$$\lim_{x \to a} f(x) = L \qquad \text{or} \qquad \lim\nolimits_{x \to a} f(x) = L.$$

This definition is *informal* because phrases such as *close as we want* and *close enough* are imprecise; their meaning depends on the context. To a machinist manufacturing a piston, *close enough* may mean *within a few thousandths of an inch*. To an astronomer studying distant galaxies, *close enough* may mean *within a few thousand light-years*. The definition should be clear enough, however, to enable us to recognize and evaluate limits of specific functions. A more precise "formal" definition, given in Section 1.5, is needed if we want to *prove* theorems about limits like Theorems 2–4, stated later in this section.

EXAMPLE 3 Find (a) $\lim_{x \to a} x$ and (b) $\lim_{x \to a} c$ (where c is a constant).

Solution In words, part (a) asks: "What does x approach as x approaches a?" The answer is surely a.

$$\lim_{x \to a} x = a.$$

Similarly, part (b) asks: "What does c approach as x approaches a?" The answer here is that c approaches c; you can't get any closer to c than by *being* c.

$$\lim_{x \to a} c = c.$$

Example 3 shows that $\lim_{x \to a} f(x)$ can *sometimes* be evaluated by just calculating $f(a)$. This will be the case if $f(x)$ is defined in an open interval containing $x = a$ and the graph of f passes unbroken through the point $(a, f(a))$. The next example shows various ways algebraic manipulations can be used to evaluate $\lim_{x \to a} f(x)$ in situations where $f(a)$ is undefined. This usually happens when $f(x)$ is a fraction with denominator equal to 0 at $x = a$.

EXAMPLE 4 Evaluate:

$$\text{(a)} \lim_{x \to -2} \frac{x^2 + x - 2}{x^2 + 5x + 6}, \quad \text{(b)} \lim_{x \to a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a}, \quad \text{and} \quad \text{(c)} \lim_{x \to 4} \frac{\sqrt{x} - 2}{x^2 - 16}.$$

Solution Each of these limits involves a fraction whose numerator and denominator are both 0 at the point where the limit is taken.

(a) $\displaystyle \lim_{x\to -2} \frac{x^2+x-2}{x^2+5x+6}$ — fraction undefined at $x=-2$. Factor numerator and denominator. (See Section P.6.)

$$= \lim_{x\to -2} \frac{(x+2)(x-1)}{(x+2)(x+3)}$$ Cancel common factors.

$$= \lim_{x\to -2} \frac{x-1}{x+3}$$ Evaluate this limit by substituting $x=-2$.

$$= \frac{-2-1}{-2+3} = -3.$$

(b) $\displaystyle \lim_{x\to a} \frac{\frac{1}{x}-\frac{1}{a}}{x-a}$ — fraction undefined at $x=a$. Simplify the numerator.

$$= \lim_{x\to a} \frac{\frac{a-x}{ax}}{x-a}$$

$$= \lim_{x\to a} \frac{-(x-a)}{ax(x-a)}$$ Cancel the common factor.

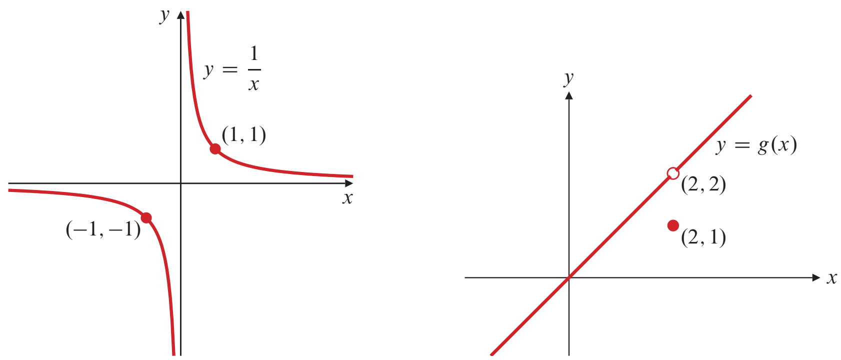
$$= \lim_{x\to a} \frac{-1}{ax} = -\frac{1}{a^2}.$$

(c) $\displaystyle \lim_{x\to 4} \frac{\sqrt{x}-2}{x^2-16}$ — fraction undefined at $x=4$. Multiply numerator and denominator by the conjugate of the expression in the numerator.

$$= \lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x^2-16)(\sqrt{x}+2)}$$

$$= \lim_{x\to 4} \frac{x-4}{(x-4)(x+4)(\sqrt{x}+2)}$$

$$= \lim_{x\to 4} \frac{1}{(x+4)(\sqrt{x}+2)} = \frac{1}{(4+4)(2+2)} = \frac{1}{32}$$



(a) (b)

Figure 1.8

(a) $\displaystyle \lim_{x\to 0} \frac{1}{x}$ does not exist

(b) $\displaystyle \lim_{x\to 2} g(x) = 2$, but $g(2)=1$

> **BEWARE!** Always be aware that the existence of $\lim_{x\to a} f(x)$ does not require that $f(a)$ exist and does not depend on $f(a)$ even if $f(a)$ does exist. It depends only on the values of $f(x)$ for x *near but not equal to* a.

A function f may be defined on both sides of $x=a$ but still not have a limit at $x=a$. For example, the function $f(x)=1/x$ has no limit as x approaches 0. As can be seen in Figure 1.8(a), the values $1/x$ grow ever larger in absolute value as x approaches 0; there is no single number L that they approach.

The following example shows that even if $f(x)$ is defined at $x=a$, the limit of $f(x)$ as x approaches a may not be equal to $f(a)$.

EXAMPLE 5 Let $g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$ (See Figure 1.8(b).) Then

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} x = 2, \qquad \text{although} \quad g(2) = 1.$$

One-Sided Limits

Limits are *unique*; if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$, then $L = M$. (See Exercise 31 in Section 1.5.) Although a function f can only have one limit at any particular point, it is, nevertheless, useful to be able to describe the behaviour of functions that approach different numbers as x approaches a from one side or the other. (See Figure 1.9.)

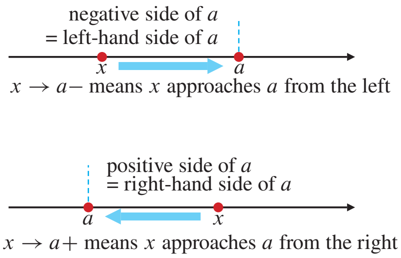


Figure 1.9 One-sided approach

DEFINITION 2

Informal definition of left and right limits

If $f(x)$ is defined on some interval (b, a) extending to the left of $x = a$, and if we can ensure that $f(x)$ is as close as we want to L by taking x to the left of a and close enough to a, then we say $f(x)$ has **left limit** L at $x = a$, and we write

$$\lim_{x \to a-} f(x) = L.$$

If $f(x)$ is defined on some interval (a, b) extending to the right of $x = a$, and if we can ensure that $f(x)$ is as close as we want to L by taking x to the right of a and close enough to a, then we say $f(x)$ has **right limit** L at $x = a$, and we write

$$\lim_{x \to a+} f(x) = L.$$

Note the use of the suffix $+$ to denote approach from the right (the *positive* side) and the suffix $-$ to denote approach from the left (the *negative* side).

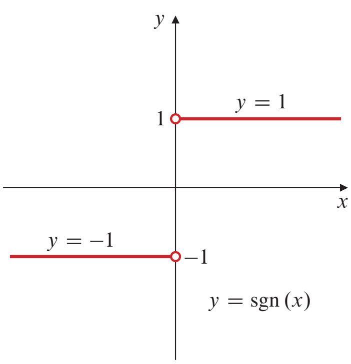


Figure 1.10
$\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist, because $\lim_{x \to 0-} \operatorname{sgn}(x) = -1$, $\lim_{x \to 0+} \operatorname{sgn}(x) = 1$

EXAMPLE 6 The signum function $\operatorname{sgn}(x) = x/|x|$ (see Figure 1.10) has left limit -1 and right limit 1 at $x = 0$:

$$\lim_{x \to 0-} \operatorname{sgn}(x) = -1 \qquad \text{and} \qquad \lim_{x \to 0+} \operatorname{sgn}(x) = 1$$

because the values of $\operatorname{sgn}(x)$ approach -1 (they *are* -1) if x is negative and approaches 0, and they approach 1 if x is positive and approaches 0. Since these left and right limits are not equal, $\lim_{x \to 0} \operatorname{sgn}(x)$ *does not exist.*

As suggested in Example 6, the relationship between ordinary (two-sided) limits and one-sided limits can be stated as follows:

THEOREM 1

Relationship between one-sided and two-sided limits

A function $f(x)$ has limit L at $x = a$ if and only if it has both left and right limits there and these one-sided limits are both equal to L:

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = L.$$

EXAMPLE 7 If $f(x) = \dfrac{|x-2|}{x^2 + x - 6}$, find: $\lim_{x \to 2+} f(x)$, $\lim_{x \to 2-} f(x)$, and $\lim_{x \to 2} f(x)$.

Solution Observe that $|x-2| = x-2$ if $x > 2$, and $|x-2| = -(x-2)$ if $x < 2$. Therefore,

$$\begin{aligned}\lim_{x\to 2+} f(x) &= \lim_{x\to 2+}\frac{x-2}{x^2+x-6} \\ &= \lim_{x\to 2+}\frac{x-2}{(x-2)(x+3)} \\ &= \lim_{x\to 2+}\frac{1}{x+3} = \frac{1}{5},\end{aligned} \qquad \begin{aligned}\lim_{x\to 2-} f(x) &= \lim_{x\to 2-}\frac{-(x-2)}{x^2+x-6} \\ &= \lim_{x\to 2-}\frac{-(x-2)}{(x-2)(x+3)} \\ &= \lim_{x\to 2-}\frac{-1}{x+3} = -\frac{1}{5}.\end{aligned}$$

Since $\lim_{x\to 2-} f(x) \ne \lim_{x\to 2+} f(x)$, the limit $\lim_{x\to 2} f(x)$ does not exist.

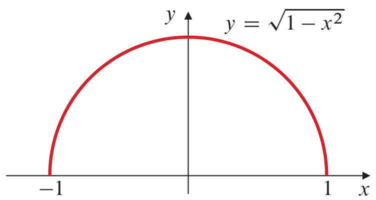


Figure 1.11 $\sqrt{1-x^2}$ has right limit 0 at -1 and left limit 0 at 1

EXAMPLE 8 What one-sided limits does $g(x) = \sqrt{1-x^2}$ have at $x = -1$ and $x = 1$?

Solution The domain of g is $[-1, 1]$, so $g(x)$ is defined only to the right of $x = -1$ and only to the left of $x = 1$. As can be seen in Figure 1.11,

$$\lim_{x\to -1+} g(x) = 0 \qquad \text{and} \qquad \lim_{x\to 1-} g(x) = 0.$$

$g(x)$ has no left limit or limit at $x = -1$ and no right limit or limit at $x = 1$.

Rules for Calculating Limits

The following theorems make it easy to calculate limits and one-sided limits of many kinds of functions when we know some elementary limits. We will not prove the theorems here. (See Section 1.5.)

THEOREM 2

Limit Rules

If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, and k is a constant, then

1. **Limit of a sum:** $\lim_{x\to a}[f(x) + g(x)] = L + M$
2. **Limit of a difference:** $\lim_{x\to a}[f(x) - g(x)] = L - M$
3. **Limit of a product:** $\lim_{x\to a} f(x)g(x) = LM$
4. **Limit of a multiple:** $\lim_{x\to a} kf(x) = kL$
5. **Limit of a quotient:** $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, if $M \ne 0$.

If m is an integer and n is a positive integer, then

6. **Limit of a power:** $\lim_{x\to a}\big[f(x)\big]^{m/n} = L^{m/n}$, provided $L > 0$ if n is even, and $L \ne 0$ if $m < 0$.

If $f(x) \le g(x)$ on an interval containing a in its interior, then

7. **Order is preserved:** $L \le M$

Rules 1–6 are also valid for right limits and left limits. So is Rule 7, under the assumption that $f(x) \le g(x)$ on an open interval extending from a in the appropriate direction.

In words, rule 1 of Theorem 2 says that the limit of a sum of functions is the sum of their limits. Similarly, rule 5 says that the limit of a quotient of two functions is the quotient of their limits, provided that the limit of the denominator is not zero. Try to state the other rules in words.

We can make use of the limits (a) $\lim_{x\to a} c = c$ (where c is a constant) and (b) $\lim_{x\to a} x = a$, from Example 3, together with parts of Theorem 2 to calculate limits of many combinations of functions.

EXAMPLE 9 Find: (a) $\lim_{x \to a} \dfrac{x^2+x+4}{x^3-2x^2+7}$ and (b) $\lim_{x \to 2} \sqrt{2x+1}$.

Solution

(a) The expression $\dfrac{x^2+x+4}{x^3-2x^2+7}$ is formed by combining the basic functions x and c (constant) using addition, subtraction, multiplication, and division. Theorem 2 assures us that the limit of such a combination is the same combination of the limits a and c of the basic functions, provided the denominator does not have limit zero. Thus,

$$\lim_{x \to a} \frac{x^2+x+4}{x^3-2x^2+7} = \frac{a^2+a+4}{a^3-2a^2+7} \qquad \text{provided } a^3-2a^2+7 \neq 0.$$

(b) The same argument as in (a) shows that $\lim_{x \to 2}(2x+1) = 2(2)+1 = 5$. Then the Power Rule (rule 6 of Theorem 2) assures us that

$$\lim_{x \to 2} \sqrt{2x+1} = \sqrt{5}.$$

The following result is an immediate corollary of Theorem 2. (See Section P.6 for a discussion of polynomials and rational functions.)

THEOREM 3

Limits of Polynomials and Rational Functions

1. If $P(x)$ is a polynomial and a is any real number, then

$$\lim_{x \to a} P(x) = P(a).$$

2. If $P(x)$ and $Q(x)$ are polynomials and $Q(a) \neq 0$, then

$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}.$$

The Squeeze Theorem

The following theorem will enable us to calculate some very important limits in subsequent chapters. It is called the *Squeeze Theorem* because it refers to a function g whose values are squeezed between the values of two other functions f and h that have the same limit L at a point a. Being trapped between the values of two functions that approach L, the values of g must also approach L. (See Figure 1.12.)

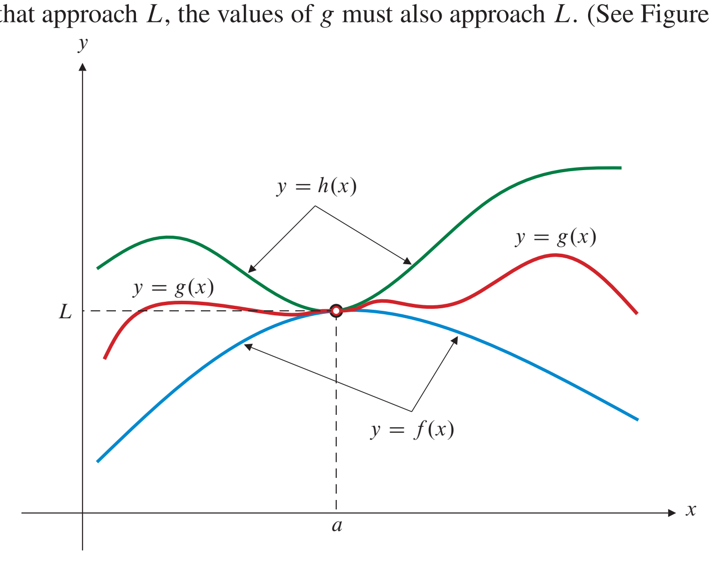


Figure 1.12 The graph of g is squeezed between those of f (blue) and h (green)

THEOREM 4

The Squeeze Theorem

Suppose that $f(x) \le g(x) \le h(x)$ holds for all x in some open interval containing a, except possibly at $x = a$ itself. Suppose also that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L.$$

Then $\lim_{x\to a} g(x) = L$ also. Similar statements hold for left and right limits.

EXAMPLE 10 Given that $3 - x^2 \le u(x) \le 3 + x^2$ for all $x \ne 0$, find $\lim_{x\to 0} u(x)$.

Solution Since $\lim_{x\to 0}(3 - x^2) = 3$ and $\lim_{x\to 0}(3 + x^2) = 3$, the Squeeze Theorem implies that $\lim_{x\to 0} u(x) = 3$.

EXAMPLE 11 Show that if $\lim_{x\to a} |f(x)| = 0$, then $\lim_{x\to a} f(x) = 0$.

Solution Since $-|f(x)| \le f(x) \le |f(x)|$, and $-|f(x)|$ and $|f(x)|$ both have limit 0 as x approaches a, so does $f(x)$ by the Squeeze Theorem.

EXERCISES 1.2

1. Find: (a) $\lim_{x\to -1} f(x)$, (b) $\lim_{x\to 0} f(x)$, and (c) $\lim_{x\to 1} f(x)$, for the function f whose graph is shown in Figure 1.13.

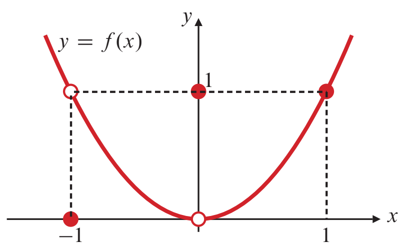


Figure 1.13

2. For the function $y = g(x)$ graphed in Figure 1.14, find each of the following limits or explain why it does not exist.
(a) $\lim_{x\to 1} g(x)$, (b) $\lim_{x\to 2} g(x)$, (c) $\lim_{x\to 3} g(x)$

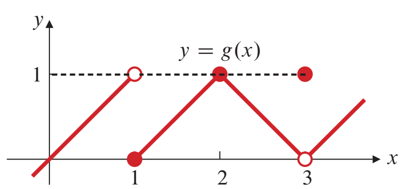


Figure 1.14

In Exercises 3–6, find the indicated one-sided limit of the function g whose graph is given in Figure 1.14.

3. $\lim_{x\to 1-} g(x)$

4. $\lim_{x\to 1+} g(x)$

5. $\lim_{x\to 3+} g(x)$

6. $\lim_{x\to 3-} g(x)$

In Exercises 7–36, evaluate the limit or explain why it does not exist.

7. $\lim_{x\to 4}(x^2 - 4x + 1)$

8. $\lim_{x\to 2} 3(1 - x)(2 - x)$

9. $\lim_{x\to 3} \dfrac{x + 3}{x + 6}$

10. $\lim_{t\to -4} \dfrac{t^2}{4 - t}$

11. $\lim_{x\to 1} \dfrac{x^2 - 1}{x + 1}$

12. $\lim_{x\to -1} \dfrac{x^2 - 1}{x + 1}$

13. $\lim_{x\to 3} \dfrac{x^2 - 6x + 9}{x^2 - 9}$

14. $\lim_{x\to -2} \dfrac{x^2 + 2x}{x^2 - 4}$

15. $\lim_{h\to 2} \dfrac{1}{4 - h^2}$

16. $\lim_{h\to 0} \dfrac{3h + 4h^2}{h^2 - h^3}$

17. $\lim_{x\to 9} \dfrac{\sqrt{x} - 3}{x - 9}$

18. $\lim_{h\to 0} \dfrac{\sqrt{4 + h} - 2}{h}$

19. $\lim_{x\to \pi} \dfrac{(x - \pi)^2}{\pi x}$

20. $\lim_{x\to -2} |x - 2|$

21. $\lim_{x\to 0} \dfrac{|x - 2|}{x - 2}$

22. $\lim_{x\to 2} \dfrac{|x - 2|}{x - 2}$

23. $\lim_{t\to 1} \dfrac{t^2 - 1}{t^2 - 2t + 1}$

24. $\lim_{x\to 2} \dfrac{\sqrt{4 - 4x + x^2}}{x - 2}$

25. $\lim_{t\to 0} \dfrac{t}{\sqrt{4 + t} - \sqrt{4 - t}}$

26. $\lim_{x\to 1} \dfrac{x^2 - 1}{\sqrt{x + 3} - 2}$

27. $\lim_{t\to 0} \dfrac{t^2 + 3t}{(t + 2)^2 - (t - 2)^2}$

28. $\lim_{s\to 0} \dfrac{(s + 1)^2 - (s - 1)^2}{s}$

29. $\lim_{y\to 1} \dfrac{y - 4\sqrt{y} + 3}{y^2 - 1}$

30. $\lim_{x\to -1} \dfrac{x^3 + 1}{x + 1}$

31. $\lim_{x\to 2} \dfrac{x^4-16}{x^3-8}$

32. $\lim_{x\to 8} \dfrac{x^{2/3}-4}{x^{1/3}-2}$

33. $\lim_{x\to 2} \left(\dfrac{1}{x-2} - \dfrac{4}{x^2-4}\right)$

34. $\lim_{x\to 2} \left(\dfrac{1}{x-2} - \dfrac{1}{x^2-4}\right)$

35. $\lim_{x\to 0} \dfrac{\sqrt{2+x^2}-\sqrt{2-x^2}}{x^2}$

36. $\lim_{x\to 0} \dfrac{|3x-1|-|3x+1|}{x}$

The limit $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ occurs frequently in the study of calculus. (Can you guess why?) Evaluate this limit for the functions f in Exercises 37–42.

37. $f(x) = x^2$

38. $f(x) = x^3$

39. $f(x) = \dfrac{1}{x}$

40. $f(x) = \dfrac{1}{x^2}$

41. $f(x) = \sqrt{x}$

42. $f(x) = 1/\sqrt{x}$

Examine the graphs of $\sin x$ and $\cos x$ in Section P.7 to determine the limits in Exercises 43–46.

43. $\lim_{x\to\pi/2} \sin x$

44. $\lim_{x\to\pi/4} \cos x$

45. $\lim_{x\to\pi/3} \cos x$

46. $\lim_{x\to 2\pi/3} \sin x$

47. Make a table of values of $f(x) = (\sin x)/x$ for a sequence of values of x approaching 0, say ± 1.0, ± 0.1, ± 0.01, ± 0.001, ± 0.0001, and ± 0.00001. Make sure your calculator is set in *radian mode* rather than degree mode. Guess the value of $\lim_{x\to 0} f(x)$.

48. Repeat Exercise 47 for $f(x) = \dfrac{1-\cos x}{x^2}$.

In Exercises 49–60, find the indicated one-sided limit or explain why it does not exist.

49. $\lim_{x\to 2-} \sqrt{2-x}$

50. $\lim_{x\to 2+} \sqrt{2-x}$

51. $\lim_{x\to -2-} \sqrt{2-x}$

52. $\lim_{x\to -2+} \sqrt{2-x}$

53. $\lim_{x\to 0} \sqrt{x^3-x}$

54. $\lim_{x\to 0-} \sqrt{x^3-x}$

55. $\lim_{x\to 0+} \sqrt{x^3-x}$

56. $\lim_{x\to 0+} \sqrt{x^2-x^4}$

57. $\lim_{x\to a-} \dfrac{|x-a|}{x^2-a^2}$

58. $\lim_{x\to a+} \dfrac{|x-a|}{x^2-a^2}$

59. $\lim_{x\to 2-} \dfrac{x^2-4}{|x+2|}$

60. $\lim_{x\to 2+} \dfrac{x^2-4}{|x+2|}$

Exercises 61–64 refer to the function

$$f(x) = \begin{cases} x-1 & \text{if } x \le -1 \\ x^2+1 & \text{if } -1 < x \le 0 \\ (x+\pi)^2 & \text{if } x > 0. \end{cases}$$

Find the indicated limits.

61. $\lim_{x\to -1-} f(x)$

62. $\lim_{x\to -1+} f(x)$

63. $\lim_{x\to 0+} f(x)$

64. $\lim_{x\to 0-} f(x)$

65. Suppose $\lim_{x\to 4} f(x) = 2$ and $\lim_{x\to 4} g(x) = -3$. Find:

(a) $\lim_{x\to 4} \big(g(x)+3\big)$ (b) $\lim_{x\to 4} x f(x)$

(c) $\lim_{x\to 4} \big(g(x)\big)^2$ (d) $\lim_{x\to 4} \dfrac{g(x)}{f(x)-1}$.

66. Suppose $\lim_{x\to a} f(x) = 4$ and $\lim_{x\to a} g(x) = -2$. Find:

(a) $\lim_{x\to a} \big(f(x)+g(x)\big)$ (b) $\lim_{x\to a} f(x)\cdot g(x)$

(c) $\lim_{x\to a} 4g(x)$ (d) $\lim_{x\to a} f(x)/g(x)$.

67. If $\lim_{x\to 2} \dfrac{f(x)-5}{x-2} = 3$, find $\lim_{x\to 2} f(x)$.

68. If $\lim_{x\to 0} \dfrac{f(x)}{x^2} = -2$, find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} \dfrac{f(x)}{x}$.

Using Graphing Utilities to Find Limits

Graphing calculators or computer software can be used to evaluate limits at least approximately. Simply "zoom" the plot window to show smaller and smaller parts of the graph near the point where the limit is to be found. Find the following limits by graphical techniques. Where you think it justified, give an exact answer. Otherwise, give the answer correct to 4 decimal places. Remember to ensure that your calculator or software is set for radian mode when using trigonometric functions.

69. $\lim_{x\to 0} \dfrac{\sin x}{x}$

70. $\lim_{x\to 0} \dfrac{\sin(2\pi x)}{\sin(3\pi x)}$

71. $\lim_{x\to 1-} \dfrac{\sin\sqrt{1-x}}{\sqrt{1-x^2}}$

72. $\lim_{x\to 0+} \dfrac{x-\sqrt{x}}{\sqrt{\sin x}}$

73. On the same graph, plot the three functions $y = x\sin(1/x)$, $y = x$, and $y = -x$ for $-0.2 \le x \le 0.2$, $-0.2 \le y \le 0.2$. Describe the behaviour of $f(x) = x\sin(1/x)$ near $x = 0$. Does $\lim_{x\to 0} f(x)$ exist, and if so, what is its value? Could you have predicted this before drawing the graph? Why?

Using the Squeeze Theorem

74. If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$.

75. If $2-x^2 \le g(x) \le 2\cos x$ for all x, find $\lim_{x\to 0} g(x)$.

76. (a) Sketch the curves $y = x^2$ and $y = x^4$ on the same graph. Where do they intersect?

(b) The function $f(x)$ satisfies:

$$\begin{cases} x^2 \le f(x) \le x^4 & \text{if } x < -1 \text{ or } x > 1 \\ x^4 \le f(x) \le x^2 & \text{if } -1 \le x \le 1 \end{cases}$$

Find (i) $\lim_{x\to -1} f(x)$, (ii) $\lim_{x\to 0} f(x)$, (iii) $\lim_{x\to 1} f(x)$.

77. On what intervals is $x^{1/3} < x^3$? On what intervals is $x^{1/3} > x^3$? If the graph of $y = h(x)$ always lies between the graphs of $y = x^{1/3}$ and $y = x^3$, for what real numbers a can you determine the value of $\lim_{x\to a} h(x)$? Find the limit for each of these values of a.

78. What is the domain of $x \sin\dfrac{1}{x}$? Evaluate $\lim_{x\to 0} x \sin\dfrac{1}{x}$.

79. Suppose $|f(x)| \le g(x)$ for all x. What can you conclude about $\lim_{x\to a} f(x)$ if $\lim_{x\to a} g(x) = 0$? What if $\lim_{x\to a} g(x) = 3$?

1.3 Limits at Infinity and Infinite Limits

In this section we will extend the concept of limit to allow for two situations not covered by the definitions of limit and one-sided limit in the previous section:

(i) limits at infinity, where x becomes arbitrarily large, positive or negative;

(ii) infinite limits, which are not really limits at all but provide useful symbolism for describing the behaviour of functions whose values become arbitrarily large, positive or negative.

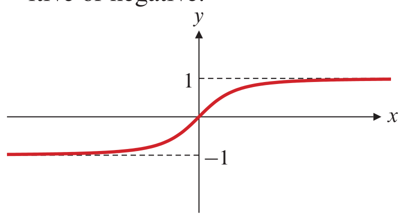


Figure 1.15 The graph of $x/\sqrt{x^2+1}$

Limits at Infinity

Consider the function

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$

whose graph is shown in Figure 1.15 and for which some values (rounded to 7 decimal places) are given in Table 5. The values of $f(x)$ seem to approach 1 as x takes on larger and larger positive values, and -1 as x takes on negative values that get larger and larger in absolute value. (See Example 2 below for confirmation.) We express this behaviour by writing

$$\lim_{x\to\infty} f(x) = 1 \qquad \text{“}f(x) \text{ approaches 1 as } x \text{ approaches infinity.”}$$

$$\lim_{x\to-\infty} f(x) = -1 \qquad \text{“}f(x) \text{ approaches } -1 \text{ as } x \text{ approaches negative infinity.”}$$

Table 5.

x	$f(x) = x/\sqrt{x^2+1}$
$-1{,}000$	-0.9999995
-100	-0.9999500
-10	-0.9950372
-1	-0.7071068
0	0.0000000
1	0.7071068
10	0.9950372
100	0.9999500
$1{,}000$	0.9999995

The graph of f conveys this limiting behaviour by approaching the horizontal lines $y = 1$ as x moves far to the right and $y = -1$ as x moves far to the left. These lines are called **horizontal asymptotes** of the graph. In general, if a curve approaches a straight line as it recedes very far away from the origin, that line is called an **asymptote** of the curve.

DEFINITION 3

Limits at infinity and negative infinity (informal definition)

If the function f is defined on an interval (a, ∞) and if we can ensure that $f(x)$ is as close as we want to the number L by taking x large enough, then we say that $f(x)$ **approaches the limit L as x approaches infinity**, and we write

$$\lim_{x\to\infty} f(x) = L.$$

If f is defined on an interval $(-\infty, b)$ and if we can ensure that $f(x)$ is as close as we want to the number M by taking x negative and large enough in absolute value, then we say that $f(x)$ **approaches the limit M as x approaches negative infinity**, and we write

$$\lim_{x\to-\infty} f(x) = M.$$

Recall that the symbol ∞, called **infinity**, does *not* represent a real number. We cannot use ∞ in arithmetic in the usual way, but we can use the phrase “approaches ∞” to mean “becomes arbitrarily large positive” and the phrase “approaches $-\infty$” to mean “becomes arbitrarily large negative.”

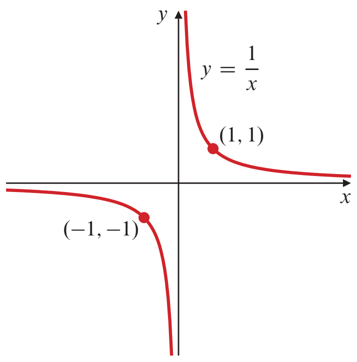


Figure 1.16 $\lim_{x\to\pm\infty} \frac{1}{x} = 0$

EXAMPLE 1 In Figure 1.16, we can see that $\lim_{x\to\infty} 1/x = \lim_{x\to-\infty} 1/x = 0$. The x-axis is a horizontal asymptote of the graph $y = 1/x$.

The theorems of Section 1.2 have suitable counterparts for limits at infinity or negative infinity. In particular, it follows from the example above and from the Product Rule for limits that $\lim_{x\to\pm\infty} 1/x^n = 0$ for any positive integer n. We will use this fact in the following examples. Example 2 shows how to obtain the limits at $\pm\infty$ for the function $x/\sqrt{x^2+1}$ by algebraic means, without resorting to making a table of values or drawing a graph, as we did above.

EXAMPLE 2 Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

Solution Rewrite the expression for $f(x)$ as follows:

$$f(x) = \frac{x}{\sqrt{x^2\left(1+\dfrac{1}{x^2}\right)}} = \frac{x}{\sqrt{x^2}\sqrt{1+\dfrac{1}{x^2}}} \qquad \text{Remember } \sqrt{x^2} = |x|.$$

$$= \frac{x}{|x|\sqrt{1+\dfrac{1}{x^2}}}$$

$$= \frac{\operatorname{sgn} x}{\sqrt{1+\dfrac{1}{x^2}}}, \qquad \text{where } \operatorname{sgn} x = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

The factor $\sqrt{1+(1/x^2)}$ approaches 1 as x approaches ∞ or $-\infty$, so $f(x)$ must have the same limits as $x \to \pm\infty$ as does $\operatorname{sgn}(x)$. Therefore (see Figure 1.15),

$$\lim_{x\to\infty} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to-\infty} f(x) = -1.$$

Limits at Infinity for Rational Functions

The only polynomials that have limits at $\pm\infty$ are constant ones, $P(x) = c$. The situation is more interesting for rational functions. Recall that a rational function is a quotient of two polynomials. The following examples show how to render such a function in a form where its limits at infinity and negative infinity (if they exist) are apparent. The way to do this is to *divide the numerator and denominator by the highest power of x appearing in the denominator.* The limits of a rational function at infinity and negative infinity either both fail to exist or both exist and are equal.

EXAMPLE 3 **(Numerator and denominator of the same degree)** Evaluate $\lim_{x\to\pm\infty} \dfrac{2x^2-x+3}{3x^2+5}$.

Solution Divide the numerator and the denominator by x^2, the highest power of x appearing in the denominator:

$$\lim_{x\to\pm\infty} \frac{2x^2-x+3}{3x^2+5} = \lim_{x\to\pm\infty} \frac{2-(1/x)+(3/x^2)}{3+(5/x^2)} = \frac{2-0+0}{3+0} = \frac{2}{3}.$$

EXAMPLE 4 **(Degree of numerator less than degree of denominator)** Evaluate $\lim_{x\to\pm\infty} \dfrac{5x+2}{2x^3-1}$.

Solution Divide the numerator and the denominator by the largest power of x in the denominator, namely, x^3:

$$\lim_{x\to\pm\infty} \frac{5x+2}{2x^3-1} = \lim_{x\to\pm\infty} \frac{(5/x^2)+(2/x^3)}{2-(1/x^3)} = \frac{0+0}{2-0} = 0.$$

Summary of limits at $\pm\infty$ for rational functions

Let $P_m(x) = a_m x^m + \cdots + a_0$ and $Q_n(x) = b_n x^n + \cdots + b_0$ be polynomials of degree m and n, respectively, so that $a_m \neq 0$ and $b_n \neq 0$. Then

$$\lim_{x\to\pm\infty} \frac{P_m(x)}{Q_n(x)}$$

(a) equals zero if $m < n$,

(b) equals $\dfrac{a_m}{b_n}$ if $m = n$,

(c) does not exist if $m > n$.

The limiting behaviour of rational functions at infinity and negative infinity is summarized at the left.

The technique used in the previous examples can also be applied to more general kinds of functions. The function in the following example is not rational, and the limit seems to produce a meaningless $\infty - \infty$ until we resolve matters by rationalizing the numerator.

EXAMPLE 5 Find $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$.

Solution We are trying to find the limit of the difference of two functions, each of which becomes arbitrarily large as x increases to infinity. We rationalize the expression by multiplying the numerator and the denominator (which is 1) by the conjugate expression $\sqrt{x^2+x}+x$:

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right) &= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+x}-x\right)\left(\sqrt{x^2+x}+x\right)}{\sqrt{x^2+x}+x} \\ &= \lim_{x\to\infty} \frac{x^2+x-x^2}{\sqrt{x^2\left(1+\dfrac{1}{x}\right)}+x} \\ &= \lim_{x\to\infty} \frac{x}{x\sqrt{1+\dfrac{1}{x}}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\dfrac{1}{x}}+1} = \frac{1}{2}.\end{aligned}$$

(Here, $\sqrt{x^2} = x$ because $x > 0$ as $x \to \infty$.)

Remark The limit $\lim_{x\to-\infty}(\sqrt{x^2+x}-x)$ is not nearly so subtle. Since $-x > 0$ as $x \to -\infty$, we have $\sqrt{x^2+x}-x > \sqrt{x^2+x}$, which grows arbitrarily large as $x \to -\infty$. The limit does not exist.

Infinite Limits

A function whose values grow arbitrarily large can sometimes be said to have an infinite limit. Since infinity is not a number, infinite limits are not really limits at all, but they provide a way of describing the behaviour of functions that grow arbitrarily large positive or negative. A few examples will make the terminology clear.

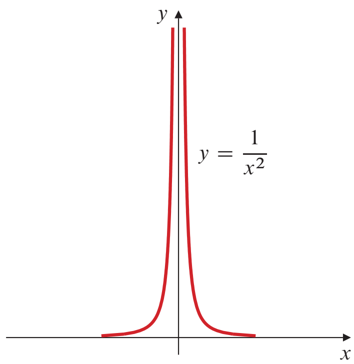


Figure 1.17 The graph of $y = 1/x^2$ (not to scale)

EXAMPLE 6 **(A two-sided infinite limit)** Describe the behaviour of the function $f(x) = 1/x^2$ near $x = 0$.

Solution As x approaches 0 from either side, the values of $f(x)$ are positive and grow larger and larger (see Figure 1.17), so the limit of $f(x)$ as x approaches 0 *does not exist.* It is nevertheless convenient to describe the behaviour of f near 0 by saying that $f(x)$ *approaches* ∞ as x approaches zero. We write

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2} = \infty.$$

Note that in writing this we are *not* saying that $\lim_{x\to 0} 1/x^2$ *exists*. Rather, we are saying that that limit *does not exist because* $1/x^2$ *becomes arbitrarily large near* $x = 0$. Observe how the graph of f approaches the y-axis as x approaches 0. The y-axis is a **vertical asymptote** of the graph.

EXAMPLE 7 **(One-sided infinite limits)** Describe the behaviour of the function $f(x) = 1/x$ near $x = 0$. (See Figure 1.18.)

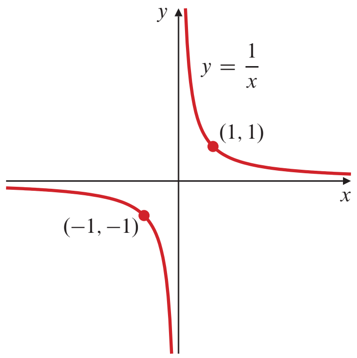


Figure 1.18 $\lim_{x\to 0-} 1/x = -\infty$, $\lim_{x\to 0+} 1/x = \infty$

Solution As x approaches 0 from the right, the values of $f(x)$ become larger and larger positive numbers, and we say that f has right-hand limit infinity at $x = 0$:

$$\lim_{x\to 0+} f(x) = \infty.$$

Similarly, the values of $f(x)$ become larger and larger negative numbers as x approaches 0 from the left, so f has left-hand limit $-\infty$ at $x = 0$:

$$\lim_{x\to 0-} f(x) = -\infty.$$

These statements do not say that the one-sided limits *exist*; they do not exist because ∞ and $-\infty$ are not numbers. Since the one-sided limits are not equal even as infinite symbols, all we can say about the two-sided $\lim_{x\to 0} f(x)$ is that it does not exist.

EXAMPLE 8 **(Polynomial behaviour at infinity)**

(a) $\lim_{x\to\infty} (3x^3 - x^2 + 2) = \infty$ (b) $\lim_{x\to-\infty} (3x^3 - x^2 + 2) = -\infty$

(c) $\lim_{x\to\infty} (x^4 - 5x^3 - x) = \infty$ (d) $\lim_{x\to-\infty} (x^4 - 5x^3 - x) = \infty$

The highest-degree term of a polynomial dominates the other terms as $|x|$ grows large, so the limits of this term at ∞ and $-\infty$ determine the limits of the whole polynomial. For the polynomial in parts (a) and (b) we have

$$3x^3 - x^2 + 2 = 3x^3\left(1 - \frac{1}{3x} + \frac{2}{3x^3}\right).$$

The factor in the large parentheses approaches 1 as x approaches $\pm\infty$, so the behaviour of the polynomial is just that of its highest-degree term $3x^3$.

We can now say a bit more about the limits at infinity and negative infinity of a rational function whose numerator has higher degree than the denominator. Earlier in this section we said that such a limit *does not exist*. This is true, but we can assign ∞ or $-\infty$ to such limits, as the following example shows.

EXAMPLE 9 **(Rational functions with numerator of higher degree)** Evaluate

$$\lim_{x\to\infty} \frac{x^3 + 1}{x^2 + 1}.$$

Solution Divide the numerator and the denominator by x^2, the largest power of x in the denominator:

$$\lim_{x\to\infty} \frac{x^3 + 1}{x^2 + 1} = \lim_{x\to\infty} \frac{x + \frac{1}{x^2}}{1 + \frac{1}{x^2}} = \frac{\lim_{x\to\infty}\left(x + \frac{1}{x^2}\right)}{1} = \infty.$$

A polynomial $Q(x)$ of degree $n > 0$ can have at most n *zeros*; that is, there are at most n different real numbers r for which $Q(r) = 0$. If $Q(x)$ is the denominator of a rational function $R(x) = P(x)/Q(x)$, that function will be defined for all x except those finitely many zeros of Q. At each of those zeros, $R(x)$ may have limits, infinite limits, or one-sided infinite limits. Here are some examples.

EXAMPLE 10

(a) $\lim_{x\to 2}\dfrac{(x-2)^2}{x^2-4} = \lim_{x\to 2}\dfrac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2}\dfrac{x-2}{x+2} = 0.$

(b) $\lim_{x\to 2}\dfrac{x-2}{x^2-4} = \lim_{x\to 2}\dfrac{x-2}{(x-2)(x+2)} = \lim_{x\to 2}\dfrac{1}{x+2} = \dfrac{1}{4}.$

(c) $\lim_{x\to 2+}\dfrac{x-3}{x^2-4} = \lim_{x\to 2+}\dfrac{x-3}{(x-2)(x+2)} = -\infty.$ (The values are negative for $x > 2$, x near 2.)

(d) $\lim_{x\to 2-}\dfrac{x-3}{x^2-4} = \lim_{x\to 2-}\dfrac{x-3}{(x-2)(x+2)} = \infty.$ (The values are positive for $x < 2$, x near 2.)

(e) $\lim_{x\to 2}\dfrac{x-3}{x^2-4} = \lim_{x\to 2}\dfrac{x-3}{(x-2)(x+2)}$ does not exist.

(f) $\lim_{x\to 2}\dfrac{2-x}{(x-2)^3} = \lim_{x\to 2}\dfrac{-(x-2)}{(x-2)^3} = \lim_{x\to 2}\dfrac{-1}{(x-2)^2} = -\infty.$

In parts (a) and (b) the effect of the zero in the denominator at $x = 2$ is cancelled because the numerator is zero there also. Thus a finite limit exists. This is not true in part (f) because the numerator only vanishes once at $x = 2$, while the denominator vanishes three times there.

Using Maple to Calculate Limits

Maple's `limit` procedure can be easily used to calculate limits, one-sided limits, limits at infinity, and infinite limits. Here is the syntax for calculating

$$\lim_{x\to 2}\frac{x^2-4}{x^2-5x+6},\quad \lim_{x\to 0}\frac{x\sin x}{1-\cos x},\quad \lim_{x\to -\infty}\frac{x}{\sqrt{x^2+1}},\quad \lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}},$$

$$\lim_{x\to 0}\frac{1}{x},\quad \lim_{x\to 0-}\frac{1}{x},\quad \lim_{x\to a-}\frac{x^2-a^2}{|x-a|},\quad \text{and}\quad \lim_{x\to a+}\frac{x^2-a^2}{|x-a|}.$$

```
> limit((x^2-4)/(x^2-5*x+6),x=2);
```

$$-4$$

```
> limit(x*sin(x)/(1-cos(x)),x=0);
```

$$2$$

```
> limit(x/sqrt(x^2+1),x=-infinity);
```

$$-1$$

```
> limit(x/sqrt(x^2+1),x=infinity);
```

$$1$$

```
> limit(1/x,x=0); limit(1/x,x=0,left);
```

undefined

$$-\infty$$

```
> limit((x^2-a^2)/(abs(x-a)),x=a,left);
```

$$-2a$$

```
> limit((x^2-a^2)/(abs(x-a)),x=a,right);
```

$$2a$$

Finally, we use Maple to confirm the limit discussed in Example 2 in Section 1.2.

```
> limit((1+x^2)^(1/x^2), x=0); evalf(%);
```

$$e$$

$$2.718281828$$

We will learn a great deal about this very important number in Chapter 3.

EXERCISES 1.3

Find the limits in Exercises 1–10.

1. $\lim_{x\to\infty} \frac{x}{2x-3}$

2. $\lim_{x\to\infty} \frac{x}{x^2-4}$

3. $\lim_{x\to\infty} \frac{3x^3-5x^2+7}{8+2x-5x^3}$

4. $\lim_{x\to-\infty} \frac{x^2-2}{x-x^2}$

5. $\lim_{x\to-\infty} \frac{x^2+3}{x^3+2}$

6. $\lim_{x\to\infty} \frac{x^2+\sin x}{x^2+\cos x}$

7. $\lim_{x\to\infty} \frac{3x+2\sqrt{x}}{1-x}$

8. $\lim_{x\to\infty} \frac{2x-1}{\sqrt{3x^2+x+1}}$

9. $\lim_{x\to-\infty} \frac{2x-1}{\sqrt{3x^2+x+1}}$

10. $\lim_{x\to-\infty} \frac{2x-5}{|3x+2|}$

In Exercises 11–32 evaluate the indicated limit. If it does not exist, is the limit ∞, $-\infty$, or neither?

11. $\lim_{x\to 3} \frac{1}{3-x}$

12. $\lim_{x\to 3} \frac{1}{(3-x)^2}$

13. $\lim_{x\to 3-} \frac{1}{3-x}$

14. $\lim_{x\to 3+} \frac{1}{3-x}$

15. $\lim_{x\to-5/2} \frac{2x+5}{5x+2}$

16. $\lim_{x\to-2/5} \frac{2x+5}{5x+2}$

17. $\lim_{x\to-(2/5)-} \frac{2x+5}{5x+2}$

18. $\lim_{x\to-(2/5)+} \frac{2x+5}{5x+2}$

19. $\lim_{x\to 2+} \frac{x}{(2-x)^3}$

20. $\lim_{x\to 1-} \frac{x}{\sqrt{1-x^2}}$

21. $\lim_{x\to 1+} \frac{1}{|x-1|}$

22. $\lim_{x\to 1-} \frac{1}{|x-1|}$

23. $\lim_{x\to 2} \frac{x-3}{x^2-4x+4}$

24. $\lim_{x\to 1+} \frac{\sqrt{x^2-x}}{x-x^2}$

25. $\lim_{x\to\infty} \frac{x+x^3+x^5}{1+x^2+x^3}$

26. $\lim_{x\to\infty} \frac{x^3+3}{x^2+2}$

! **27.** $\lim_{x\to\infty} \frac{x\sqrt{x+1}\left(1-\sqrt{2x+3}\right)}{7-6x+4x^2}$

28. $\lim_{x\to\infty} \left(\frac{x^2}{x+1}-\frac{x^2}{x-1}\right)$

! **29.** $\lim_{x\to-\infty} \left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)$

! **30.** $\lim_{x\to\infty} \left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)$

31. $\lim_{x\to\infty} \frac{1}{\sqrt{x^2-2x}-x}$

32. $\lim_{x\to-\infty} \frac{1}{\sqrt{x^2+2x}-x}$

33. What are the horizontal asymptotes of $y = \frac{1}{\sqrt{x^2-2x}-x}$? What are its vertical asymptotes?

34. What are the horizontal and vertical asymptotes of $y = \frac{2x-5}{|3x+2|}$?

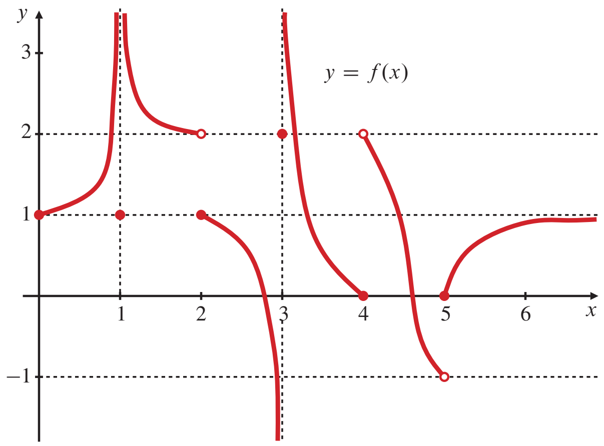


Figure 1.19

The function f whose graph is shown in Figure 1.19 has domain $[0, \infty)$. Find the limits of f indicated in Exercises 35–45.

35. $\lim_{x\to 0+} f(x)$

36. $\lim_{x\to 1} f(x)$

37. $\lim_{x\to 2+} f(x)$

38. $\lim_{x\to 2-} f(x)$

39. $\lim_{x\to 3-} f(x)$

40. $\lim_{x\to 3+} f(x)$

41. $\lim_{x\to 4+} f(x)$

42. $\lim_{x\to 4-} f(x)$

43. $\lim_{x\to 5-} f(x)$

44. $\lim_{x\to 5+} f(x)$

45. $\lim_{x\to\infty} f(x)$

46. What asymptotes does the graph in Figure 1.19 have?

Exercises 47–52 refer to the **greatest integer function** $\lfloor x \rfloor$ graphed in Figure 1.20. Find the indicated limit or explain why it does not exist.

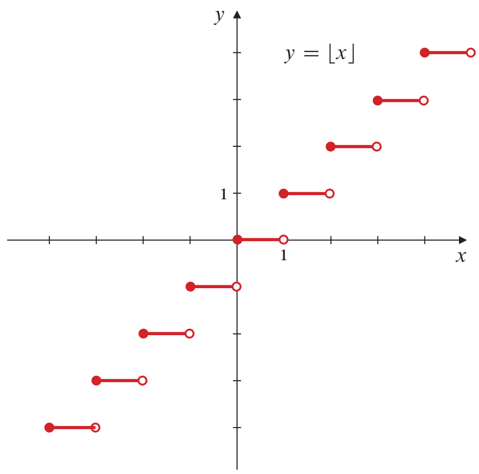


Figure 1.20

47. $\lim_{x\to 3+} \lfloor x \rfloor$

48. $\lim_{x\to 3-} \lfloor x \rfloor$

49. $\lim_{x\to 3} \lfloor x \rfloor$

50. $\lim_{x\to 2.5} \lfloor x \rfloor$

51. $\lim_{x\to 0+} \lfloor 2 - x \rfloor$

52. $\lim_{x\to -3-} \lfloor x \rfloor$

53. Parking in a certain parking lot costs \$1.50 for each hour or part of an hour. Sketch the graph of the function $C(t)$ representing the cost of parking for t hours. At what values of t does $C(t)$ have a limit? Evaluate $\lim_{t\to t_0-} C(t)$ and $\lim_{t\to t_0+} C(t)$ for an arbitrary number $t_0 > 0$.

54. If $\lim_{x\to 0+} f(x) = L$, find $\lim_{x\to 0-} f(x)$ if (a) f is even, (b) f is odd.

55. If $\lim_{x\to 0+} f(x) = A$ and $\lim_{x\to 0-} f(x) = B$, find

(a) $\displaystyle\lim_{x\to 0+} f(x^3 - x)$ (b) $\displaystyle\lim_{x\to 0-} f(x^3 - x)$

(c) $\displaystyle\lim_{x\to 0-} f(x^2 - x^4)$ (d) $\displaystyle\lim_{x\to 0+} f(x^2 - x^4)$.

1.4 Continuity

When a car is driven along a highway, its distance from its starting point depends on time in a *continuous* way, changing by small amounts over short intervals of time. But not all quantities change in this way. When the car is parked in a parking lot where the rate is quoted as "\$2.00 per hour or portion," the parking charges remain at \$2.00 for the first hour and then suddenly jump to \$4.00 as soon as the first hour has passed. The function relating parking charges to parking time will be called *discontinuous* at each hour. In this section we will define continuity and show how to tell whether a function is continuous. We will also examine some important properties possessed by continuous functions.

Continuity at a Point

Most functions that we encounter have domains that are intervals, or unions of separate intervals. A point P in the domain of such a function is called an **interior point** of the domain if it belongs to some *open* interval contained in the domain. If it is not an interior point, then P is called an **endpoint** of the domain. For example, the domain of the function $f(x) = \sqrt{4 - x^2}$ is the closed interval $[-2, 2]$, which consists of interior points in the interval $(-2, 2)$, a left endpoint -2, and a right endpoint 2. The domain of the function $g(x) = 1/x$ is the union of open intervals $(-\infty, 0) \cup (0, \infty)$ and consists entirely of interior points. Note that although 0 is an endpoint of each of those intervals, it does not belong to the domain of g and so is not an endpoint of that domain.

DEFINITION 4

Continuity at an interior point

We say that a function f is **continuous** at an interior point c of its domain if

$$\lim_{x\to c} f(x) = f(c).$$

If either $\lim_{x\to c} f(x)$ fails to exist or it exists but is not equal to $f(c)$, then we will say that f is **discontinuous** at c.

In graphical terms, f is continuous at an interior point c of its domain if its graph has no break in it at the point $(c, f(c))$; in other words, if you can draw the graph through that point without lifting your pen from the paper. Consider Figure 1.21. In (a), f is continuous at c. In (b), f is discontinuous at c because $\lim_{x\to c} f(x) \ne f(c)$. In (c),

f is discontinuous at c because $\lim_{x\to c} f(x)$ does not exist. In both (b) and (c) the graph of f has a break at $x = c$.

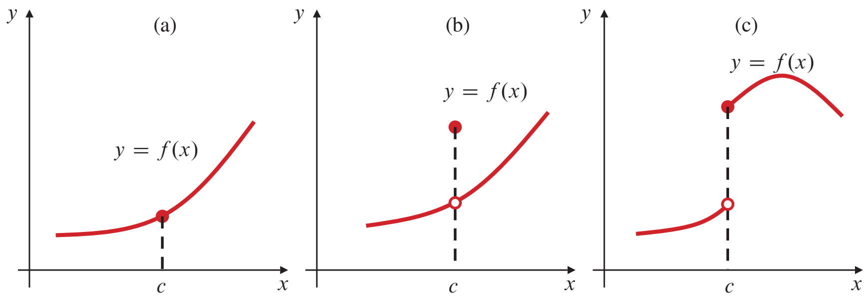


Figure 1.21
(a) f is continuous at c
(b) $\lim_{x\to c} f(x) \neq f(c)$
(c) $\lim_{x\to c} f(x)$ does not exist

Although a function cannot have a limit at an endpoint of its domain, it can still have a one-sided limit there. We extend the definition of continuity to provide for such situations.

DEFINITION 5

Right and left continuity

We say that f is **right continuous** at c if $\lim\limits_{x\to c+} f(x) = f(c)$.

We say that f is **left continuous** at c if $\lim\limits_{x\to c-} f(x) = f(c)$.

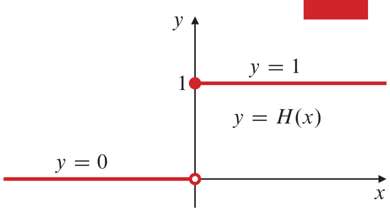


Figure 1.22 The Heaviside function

EXAMPLE 1 The Heaviside function $H(x)$, whose graph is shown in Figure 1.22, is continuous at every number x except 0. It is right continuous at 0 but is not left continuous or continuous there.

The relationship between continuity and one-sided continuity is summarized in the following theorem.

THEOREM 5

Function f is continuous at c if and only if it is both right continuous and left continuous at c.

DEFINITION 6

Continuity at an endpoint

We say that f is continuous at a left endpoint c of its domain if it is right continuous there.
We say that f is continuous at a right endpoint c of its domain if it is left continuous there.

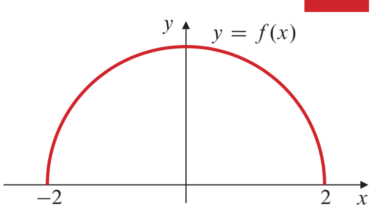


Figure 1.23 $f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain

EXAMPLE 2 The function $f(x) = \sqrt{4 - x^2}$ has domain $[-2, 2]$. It is continuous at the right endpoint 2 because it is left continuous there, that is, because $\lim_{x\to 2-} f(x) = 0 = f(2)$. It is continuous at the left endpoint -2 because it is right continuous there: $\lim_{x\to -2+} f(x) = 0 = f(-2)$. Of course, f is also continuous at every interior point of its domain. If $-2 < c < 2$, then $\lim_{x\to c} f(x) = \sqrt{4 - c^2} = f(c)$. (See Figure 1.23.)

Continuity on an Interval

We have defined the concept of continuity at a point. Of greater importance is the concept of continuity on an interval.

DEFINITION 7

Continuity on an interval

We say that function f is **continuous on the interval** I if it is continuous at each point of I. In particular, we will say that f is a **continuous function** if f is continuous at every point of its domain.

EXAMPLE 3 The function $f(x) = \sqrt{x}$ is a continuous function. Its domain is $[0, \infty)$. It is continuous at the left endpoint 0 because it is right continuous there. Also, f is continuous at every number $c > 0$ since $\lim_{x \to c} \sqrt{x} = \sqrt{c}$.

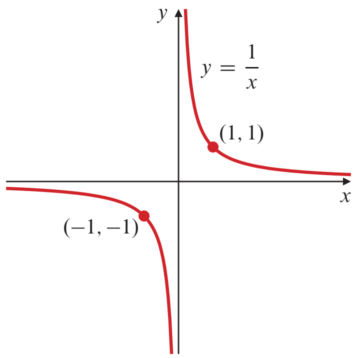


Figure 1.24 $1/x$ is continuous on its domain

EXAMPLE 4 The function $g(x) = 1/x$ is also a continuous function. This may seem wrong to you at first glance because its graph is broken at $x = 0$. (See Figure 1.24.) However, the number 0 is not in the domain of g, so we will prefer to say that g is undefined rather than discontinuous there. (Some authors would say that g is discontinuous at $x = 0$.) If we were to define $g(0)$ to be some number, say 0, then we would say that $g(x)$ is discontinuous at 0. There is no way of defining $g(0)$ so that g becomes continuous at 0.

EXAMPLE 5 The greatest integer function $\lfloor x \rfloor$ (see Figure 1.20) is continuous on every interval $[n, n+1)$, where n is an integer. It is right continuous at each integer n but is not left continuous there, so it is discontinuous at the integers.

$$\lim_{x \to n+} \lfloor x \rfloor = n = \lfloor n \rfloor, \qquad \lim_{x \to n-} \lfloor x \rfloor = n - 1 \neq n = \lfloor n \rfloor.$$

There Are Lots of Continuous Functions

The following functions are continuous wherever they are defined:

(a) all polynomials;

(b) all rational functions;

(c) all rational powers $x^{m/n} = \sqrt[n]{x^m}$;

(d) the sine, cosine, tangent, secant, cosecant, and cotangent functions defined in Section P.7; and

(e) the absolute value function $|x|$.

Theorem 3 of Section 1.2 assures us that every polynomial is continuous everywhere on the real line, and every rational function is continuous everywhere on its domain (which consists of all real numbers except the finitely many where its denominator is zero). If m and n are integers and $n \neq 0$, the rational power function $x^{m/n}$ is defined for all positive numbers x, and also for all negative numbers x if n is odd. The domain includes 0 if and only if $m/n \geq 0$.

The following theorems show that if we combine continuous functions in various ways, the results will be continuous.

THEOREM 6

Combining continuous functions

If the functions f and g are both defined on an interval containing c and both are continuous at c, then the following functions are also continuous at c:

1. the sum $f + g$ and the difference $f - g$;
2. the product fg;
3. the constant multiple kf, where k is any number;
4. the quotient f/g (provided $g(c) \neq 0$); and
5. the nth root $(f(x))^{1/n}$, provided $f(c) > 0$ if n is even.

The proof involves using the various limit rules in Theorem 2 of Section 1.2. For example,

$$\lim_{x\to c}\big(f(x) + g(x)\big) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = f(c) + g(c),$$

so $f + g$ is continuous.

THEOREM 7

Composites of continuous functions are continuous

If $f(g(x))$ is defined on an interval containing c, and if f is continuous at L and $\lim_{x\to c} g(x) = L$, then

$$\lim_{x\to c} f(g(x)) = f(L) = f\left(\lim_{x\to c} g(x)\right).$$

In particular, if g is continuous at c (so $L = g(c)$), then the composition $f \circ g$ is continuous at c:

$$\lim_{x\to c} f(g(x)) = f(g(c)).$$

(See Exercise 37 in Section 1.5.)

EXAMPLE 6 The following functions are continuous everywhere on their respective domains:

(a) $3x^2 - 2x$ (b) $\dfrac{x-2}{x^2-4}$ (c) $|x^2 - 1|$

(d) $\sqrt{x}$ (e) $\sqrt{x^2 - 2x - 5}$ (f) $\dfrac{|x|}{\sqrt{|x+2|}}$.

Continuous Extensions and Removable Discontinuities

As we have seen in Section 1.2, a rational function may have a limit even at a point where its denominator is zero. If $f(c)$ is not defined, but $\lim_{x\to c} f(x) = L$ exists, we can define a new function $F(x)$ by

$$F(x) = \begin{cases} f(x) & \text{if } x \text{ is in the domain of } f \\ L & \text{if } x = c. \end{cases}$$

$F(x)$ is continuous at $x = c$. It is called the **continuous extension** of $f(x)$ to $x = c$. For rational functions f, continuous extensions are usually found by cancelling common factors.

EXAMPLE 7 Show that $f(x) = \dfrac{x^2 - x}{x^2 - 1}$ has a continuous extension to $x = 1$, and find that extension.

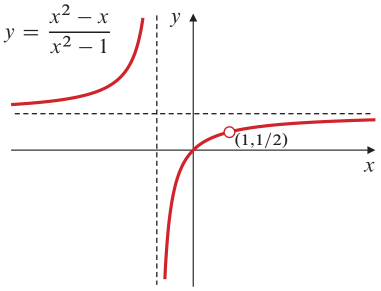


Figure 1.25 This function has a continuous extension to $x = 1$

Solution Although $f(1)$ is not defined, if $x \neq 1$ we have

$$f(x) = \frac{x^2 - x}{x^2 - 1} = \frac{x(x-1)}{(x+1)(x-1)} = \frac{x}{x+1}.$$

The function

$$F(x) = \frac{x}{x+1}$$

is equal to $f(x)$ for $x \neq 1$ but is also continuous at $x = 1$, having there the value $1/2$. The graph of f is shown in Figure 1.25. The continuous extension of $f(x)$ to $x = 1$ is $F(x)$. It has the same graph as $f(x)$ except with no hole at $(1, 1/2)$.

If a function f is undefined or discontinuous at a point a but can be (re)defined at that *single point* so that it becomes continuous there, then we say that f has a **removable discontinuity** at a. The function f in the above example has a removable discontinuity at $x = 1$. To remove it, define $f(1) = 1/2$.

EXAMPLE 8 The function $g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$ has a removable discontinuity at $x = 2$. To remove it, redefine $g(2) = 2$. (See Figure 1.26.)

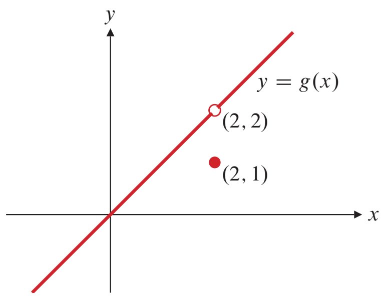


Figure 1.26 g has a removable discontinuity at 2

Continuous Functions on Closed, Finite Intervals

Continuous functions that are defined on *closed, finite intervals* have special properties that make them particularly useful in mathematics and its applications. We will discuss two of these properties here. Although they may appear obvious, these properties are much more subtle than the results about limits stated earlier in this chapter; their proofs (see Appendix III) require a careful study of the implications of the completeness property of the real numbers.

The first of the properties states that a function $f(x)$ that is continuous on a closed, finite interval $[a, b]$ must have an **absolute maximum value** and an **absolute minimum value**. This means that the values of $f(x)$ at all points of the interval lie between the values of $f(x)$ at two particular points in the interval; the graph of f has a highest point and a lowest point.

THEOREM 8

The Max-Min Theorem

If $f(x)$ is continuous on the closed, finite interval $[a, b]$, then there exist numbers p and q in $[a, b]$ such that for all x in $[a, b]$,

$$f(p) \leq f(x) \leq f(q).$$

Thus, f has the absolute minimum value $m = f(p)$, taken on at the point p, and the absolute maximum value $M = f(q)$, taken on at the point q.

Many important problems in mathematics and its applications come down to having to find maximum and minimum values of functions. Calculus provides some very useful tools for solving such problems. Observe, however, that the theorem above merely asserts that minimum and maximum values *exist;* it doesn't tell us how to find them. In Chapter 4 we will develop techniques for calculating maximum and minimum values of functions. For now, we can solve some simple maximum and minimum value problems involving quadratic functions by completing the square without using any calculus.

EXAMPLE 9 What is the largest possible area of a rectangular field that can be enclosed by 200 m of fencing?

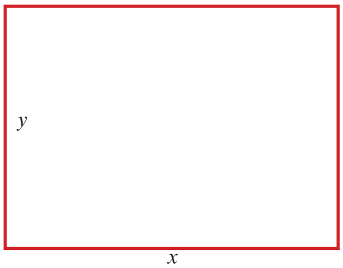


Figure 1.27 Rectangular field: perimeter $= 2x + 2y$, area $= xy$

Solution If the sides of the field are x m and y m (Figure 1.27), then its perimeter is $P = 2x + 2y$ m, and its area is $A = xy$ m^2. We are given that $P = 200$, so $x + y = 100$, and $y = 100 - x$. Neither side can be negative, so x must belong to the closed interval $[0, 100]$. The area of the field can be expressed as a function of x by substituting $100 - x$ for y:

$$A = x(100 - x) = 100x - x^2.$$

We want to find the maximum value of the quadratic function $A(x) = 100x - x^2$ on the interval $[0, 100]$. Theorem 8 assures us that such a maximum exists.

To find the maximum, we complete the square of the function $A(x)$. Note that $x^2 - 100x$ are the first two terms of the square $(x - 50)^2 = x^2 - 100x + 2{,}500$. Thus,

$$A(x) = 2{,}500 - (x - 50)^2.$$

Observe that $A(50) = 2{,}500$ and $A(x) < 2{,}500$ if $x \neq 50$, because we are subtracting a positive number $(x - 50)^2$ from 2,500 in this case. Therefore, the maximum value of $A(x)$ is 2,500. The largest field has area 2,500 m^2 and is actually a square with dimensions $x = y = 50$ m.

Theorem 8 implies that a function that is continuous on a closed, finite interval is **bounded**. This means that it cannot take on arbitrarily large positive or negative values; there must exist a number K such that

$$|f(x)| \le K; \qquad \text{that is,} \qquad -K \le f(x) \le K.$$

In fact, for K we can use the larger of the numbers $|f(p)|$ and $|f(q)|$ in the theorem.

The conclusions of Theorem 8 may fail if the function f is not continuous or if the interval is not closed. See Figures 1.28–1.31 for examples of how such failure can occur.

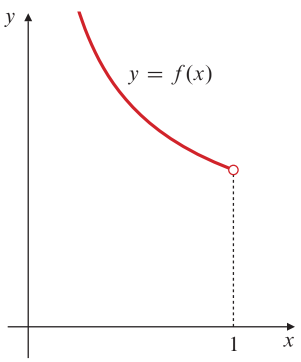


Figure 1.28 $f(x) = 1/x$ is continuous on the open interval $(0, 1)$. It is not bounded and has neither a maximum nor a minimum value

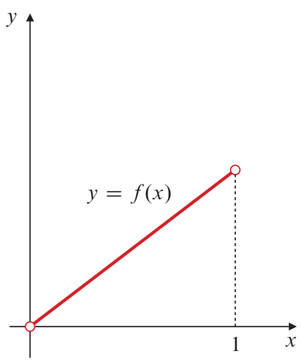


Figure 1.29 $f(x) = x$ is continuous on the open interval $(0, 1)$. It is bounded but has neither a maximum nor a minimum value

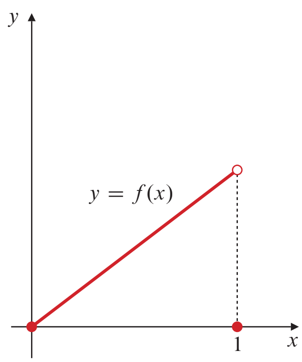


Figure 1.30 This function is defined on the closed interval $[0, 1]$ but is discontinuous at the endpoint $x = 1$. It has a minimum value but no maximum value

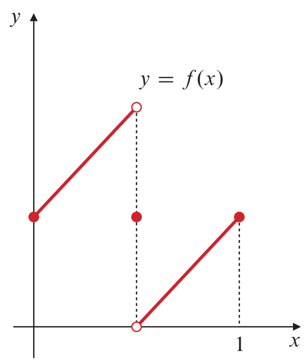


Figure 1.31 This function is discontinuous at an interior point of its domain, the closed interval $[0, 1]$. It is bounded but has neither maximum nor minimum values

The second property of a continuous function defined on a closed, finite interval is that the function takes on all real values between any two of its values. This property is called the **intermediate-value property**.

THEOREM 9

The Intermediate-Value Theorem

If $f(x)$ is continuous on the interval $[a, b]$ and if s is a number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ such that $f(c) = s$.

In particular, a continuous function defined on a closed interval takes on all values between its minimum value m and its maximum value M, so its range is also a closed interval, $[m, M]$.

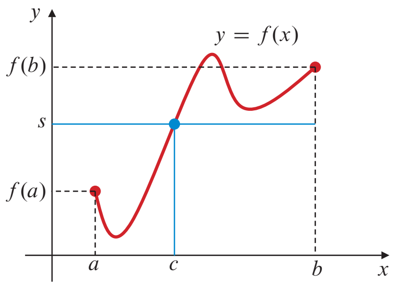


Figure 1.32 The continuous function f takes on the value s at some point c between a and b

Figure 1.32 shows a typical situation. The points $(a, f(a))$ and $(b, f(b))$ are on opposite sides of the horizontal line $y = s$. Being unbroken, the graph $y = f(x)$ must cross this line in order to go from one point to the other. In the figure, it crosses the line only once, at $x = c$. If the line $y = s$ were somewhat higher, there might have been three crossings and three possible values for c.

Theorem 9 is the reason why the graph of a function that is continuous on an interval I cannot have any breaks. It must be **connected**, a single, unbroken curve with no jumps.

EXAMPLE 10 Determine the intervals on which $f(x) = x^3 - 4x$ is positive and negative.

Solution Since $f(x) = x(x^2 - 4) = x(x - 2)(x + 2)$, $f(x) = 0$ only at $x = 0$, 2, and -2. Because f is continuous on the whole real line, it must have constant sign on each of the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$. (If there were points a and b in one of those intervals, say in $(0, 2)$, such that $f(a) < 0$ and $f(b) > 0$, then by the Intermediate-Value Theorem there would exist c between a and b, and therefore between 0 and 2, such that $f(c) = 0$. But we know f has no such zero in $(0, 2)$.)

To find whether $f(x)$ is positive or negative throughout each interval, pick a point in the interval and evaluate f at that point:

Since $f(-3) = -15 < 0$, $f(x)$ is negative on $(-\infty, -2)$.
Since $f(-1) = 3 > 0$, $f(x)$ is positive on $(-2, 0)$.
Since $f(1) = -3 < 0$, $f(x)$ is negative on $(0, 2)$.
Since $f(3) = 15 > 0$, $f(x)$ is positive on $(2, \infty)$.

Finding Roots of Equations

Among the many useful tools that calculus will provide are ones that enable us to calculate solutions to equations of the form $f(x) = 0$ to any desired degree of accuracy. Such a solution is called a **root** of the equation, or a **zero** of the function f. Using these tools usually requires previous knowledge that the equation has a solution in some interval. The Intermediate-Value Theorem can provide this information.

EXAMPLE 11 Show that the equation $x^3 - x - 1 = 0$ has a solution in the interval $[1, 2]$.

Solution The function $f(x) = x^3 - x - 1$ is a polynomial and is therefore continuous everywhere. Now $f(1) = -1$ and $f(2) = 5$. Since 0 lies between -1 and 5, the Intermediate-Value Theorem assures us that there must be a number c in $[1, 2]$ such that $f(c) = 0$.

One method for finding a zero of a function that is continuous and changes sign on an interval involves bisecting the interval many times, each time determining which half of the previous interval must contain the root, because the function has opposite signs at the two ends of that half. This method is slow. For example, if the original interval has length 1, it will take 11 bisections to cut down to an interval of length less than 0.0005 (because $2^{11} > 2{,}000 = 1/(0.0005)$), and thus to ensure that we have found the root correct to 3 decimal places.

EXAMPLE 12 **(The Bisection Method)** Solve the equation $x^3 - x - 1 = 0$ of Example 11 correct to 3 decimal places by successive bisections.

Solution We start out knowing that there is a root in $[1, 2]$. Table 6 shows the results of the bisections.

Table 6. The Bisection Method for $f(x) = x^3 - x - 1 = 0$

Bisection Number	x	$f(x)$	Root in Interval	Midpoint
	1	-1		
	2	5	$[1, 2]$	1.5
1	1.5	0.8750	$[1, 1.5]$	1.25
2	1.25	-0.2969	$[1.25, 1.5]$	1.375
3	1.375	0.2246	$[1.25, 1.375]$	1.3125
4	1.3125	-0.0515	$[1.3125, 1.375]$	1.3438
5	1.3438	0.0826	$[1.3125, 1.3438]$	1.3282
6	1.3282	0.0147	$[1.3125, 1.3282]$	1.3204
7	1.3204	-0.0186	$[1.3204, 1.3282]$	1.3243
8	1.3243	-0.0018	$[1.3243, 1.3282]$	1.3263
9	1.3263	0.0065	$[1.3243, 1.3263]$	1.3253
10	1.3253	0.0025	$[1.3243, 1.3253]$	1.3248
11	1.3248	0.0003	$[1.3243, 1.3248]$	1.3246
12	1.3246	-0.0007	$[1.3246, 1.3248]$	

The root is 1.325, rounded to 3 decimal places.

In Section 4.2, calculus will provide us with much faster methods of solving equations such as the one in the example above. Many programmable calculators and computer algebra software packages have built-in routines for solving equations. For example, Maple's `fsolve` routine can be used to find the real solution of $x^3 - x - 1 = 0$ in $[1, 2]$ in Example 11:

```
> fsolve(x^3-x-1=0,x=1..2);
```

$$1.324717957$$

Remark The Max-Min Theorem and the Intermediate-Value Theorem are examples of what mathematicians call **existence theorems**. Such theorems assert that something exists without telling you how to find it. Students sometimes complain that mathematicians worry too much about proving that a problem has a solution and not enough about how to find that solution. They argue: "If I can calculate a solution to a problem, then surely I do not need to worry about whether a solution exists." This is, however, false logic. Suppose we pose the problem: "Find the largest positive integer." Of course, this problem has no solution; there is no largest positive integer because we can add 1 to any integer and get a larger integer. Suppose, however, that we forget this and try to calculate a solution. We could proceed as follows:

Let N be the largest positive integer.
Since 1 is a positive integer, we must have $N \ge 1$.
Since N^2 is a positive integer, it cannot exceed the largest positive integer.
Therefore, $N^2 \le N$ and so $N^2 - N \le 0$.
Thus, $N(N-1) \le 0$ and we must have $N - 1 \le 0$.
Therefore, $N \le 1$. Since also $N \ge 1$, we have $N = 1$.
Therefore, 1 is the largest positive integer.

The only error we have made here is in the assumption (in the first line) that the problem has a solution. It is partly to avoid logical pitfalls like this that mathematicians prove existence theorems.

EXERCISES 1.4

Exercises 1–3 refer to the function g defined on $[-2, 2]$, whose graph is shown in Figure 1.33.

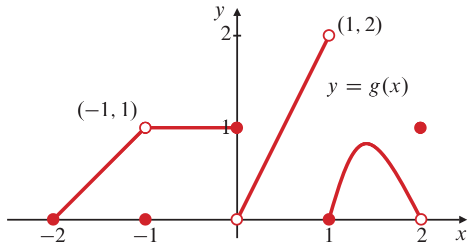


Figure 1.33

1. State whether g is (a) continuous, (b) left continuous, (c) right continuous, and (d) discontinuous at each of the points -2, -1, 0, 1, and 2.
2. At what points in its domain does g have a removable discontinuity, and how should g be redefined at each of those points so as to be continuous there?
3. Does g have an absolute maximum value on $[-2, 2]$? an absolute minimum value?

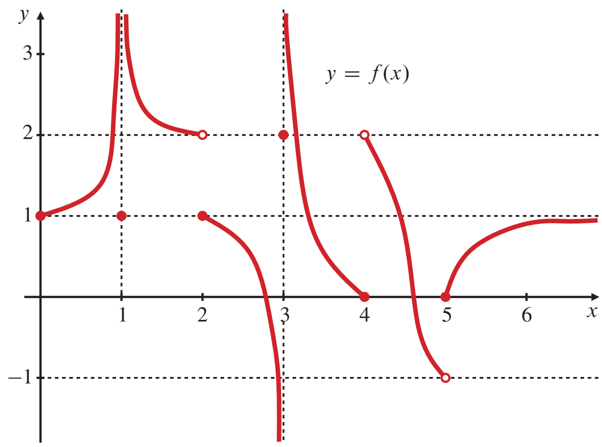


Figure 1.34

4. At what points is the function f, whose graph is shown in Figure 1.34, discontinuous? At which of those points is it left continuous? right continuous?
5. Can the function f graphed in Figure 1.34 be redefined at the single point $x = 1$ so that it becomes continuous there?
6. The function $\operatorname{sgn}(x) = x/|x|$ is neither continuous nor discontinuous at $x = 0$. How is this possible?

In Exercises 7–12, state where in its domain the given function is continuous, where it is left or right continuous, and where it is just discontinuous.

7. $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } x \ge 0 \end{cases}$
8. $f(x) = \begin{cases} x & \text{if } x < -1 \\ x^2 & \text{if } x \ge -1 \end{cases}$
9. $f(x) = \begin{cases} 1/x^2 & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$
10. $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 0.987 & \text{if } x > 1 \end{cases}$
11. The least integer function $\lceil x \rceil$ of Example 11 in Section P.5.
12. The cost function $C(t)$ of Exercise 53 in Section 1.3.

In Exercises 13–16, how should the given function be defined at the given point to be continuous there? Give a formula for the continuous extension to that point.

13. $\dfrac{x^2 - 4}{x - 2}$ at $x = 2$
14. $\dfrac{1 + t^3}{1 - t^2}$ at $t = -1$
15. $\dfrac{t^2 - 5t + 6}{t^2 - t - 6}$ at 3
16. $\dfrac{x^2 - 2}{x^4 - 4}$ at $\sqrt{2}$
17. Find k so that $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ k - x^2 & \text{if } x > 2 \end{cases}$ is a continuous function.
18. Find m so that $g(x) = \begin{cases} x - m & \text{if } x < 3 \\ 1 - mx & \text{if } x \ge 3 \end{cases}$ is continuous for all x.
19. Does the function x^2 have a maximum value on the open interval $-1 < x < 1$? a minimum value? Explain.
20. The Heaviside function of Example 1 has both absolute maximum and minimum values on the interval $[-1, 1]$, but it is not continuous on that interval. Does this violate the Max-Min Theorem? Why?

Exercises 21–24 ask for maximum and minimum values of functions. They can all be done by the method of Example 9.

21. The sum of two nonnegative numbers is 8. What is the largest possible value of their product?
22. The sum of two nonnegative numbers is 8. What is (a) the smallest and (b) the largest possible value for the sum of their squares?
23. A software company estimates that if it assigns x programmers to work on the project, it can develop a new product in T days, where
$$T = 100 - 30x + 3x^2.$$
How many programmers should the company assign in order to complete the development as quickly as possible?
24. It costs a desk manufacturer $\$(245x - 30x^2 + x^3)$ to send a shipment of x desks to its warehouse. How many desks should it include in each shipment to minimize the average shipping cost per desk?

Find the intervals on which the functions $f(x)$ in Exercises 25–28 are positive and negative.

25. $f(x) = \dfrac{x^2 - 1}{x}$
26. $f(x) = x^2 + 4x + 3$
27. $f(x) = \dfrac{x^2 - 1}{x^2 - 4}$
28. $f(x) = \dfrac{x^2 + x - 2}{x^3}$
29. Show that $f(x) = x^3 + x - 1$ has a zero between $x = 0$ and $x = 1$.
30. Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

31. Show that the function $F(x) = (x-a)^2(x-b)^2 + x$ has the value $(a+b)/2$ at some point x.

32. **(A fixed-point theorem)** Suppose that f is continuous on the closed interval $[0, 1]$ and that $0 \le f(x) \le 1$ for every x in $[0, 1]$. Show that there must exist a number c in $[0, 1]$ such that $f(c) = c$. (c is called a fixed point of the function f.) *Hint:* If $f(0) = 0$ or $f(1) = 1$, you are done. If not, apply the Intermediate-Value Theorem to $g(x) = f(x) - x$.

33. If an even function f is right continuous at $x = 0$, show that it is continuous at $x = 0$.

34. If an odd function f is right continuous at $x = 0$, show that it is continuous at $x = 0$ and that it satisfies $f(0) = 0$.

Use a graphing utility to find maximum and minimum values of the functions in Exercises 35–38 and the points x where they occur. Obtain 3-decimal-place accuracy for all answers.

35. $f(x) = \dfrac{x^2 - 2x}{x^4 + 1}$ on $[-5, 5]$

36. $f(x) = \dfrac{\sin x}{6 + x}$ on $[-\pi, \pi]$

37. $f(x) = x^2 + \dfrac{4}{x}$ on $[1, 3]$

38. $f(x) = \sin(\pi x) + x(\cos(\pi x) + 1)$ on $[0, 1]$

Use a graphing utility or a programmable calculator and the Bisection Method to solve the equations in Exercises 39–40 to 3 decimal places. As a first step, try to guess a small interval that you can be sure contains a root.

39. $x^3 + x - 1 = 0$

40. $\cos x - x = 0$

Use Maple's `fsolve` routine to solve the equations in Exercises 41–42.

41. $\sin x + 1 - x^2 = 0$ (two roots)

42. $x^4 - x - 1 = 0$ (two roots)

43. Investigate the difference between the Maple routines `fsolve(f,x)`, `solve(f,x)`, and `evalf(solve(f,x))`, where `f := x^3-x-1=0`. Note that no interval is specified for x here.

1.5 The Formal Definition of Limit

The material in this section is optional.

The *informal* definition of limit given in Section 1.2 is not precise enough to enable us to prove results about limits such as those given in Theorems 2–4 of Section 1.2. A more precise *formal* definition is based on the idea of controlling the input x of a function f so that the output $f(x)$ will lie in a specific interval.

EXAMPLE 1 The area of a circular disk of radius r cm is $A = \pi r^2$ cm^2. A machinist is required to manufacture a circular metal disk having area 400π cm^2 within an error tolerance of ± 5 cm^2. How close to 20 cm must the machinist control the radius of the disk to achieve this?

Solution The machinist wants $|\pi r^2 - 400\pi| < 5$, that is,

$$400\pi - 5 < \pi r^2 < 400\pi + 5,$$

or, equivalently,

$$\sqrt{400 - (5/\pi)} < r < \sqrt{400 + (5/\pi)}$$
$$19.96017 < r < 20.03975.$$

Thus, the machinist needs $|r - 20| < 0.03975$; she must ensure that the radius of the disk differs from 20 cm by less than 0.4 mm so that the area of the disk will lie within the required error tolerance.

When we say that $f(x)$ has limit L as x approaches a, we are really saying that we can ensure that the *error* $|f(x) - L|$ will be less than *any* allowed tolerance, no matter how small, by taking x *close enough* to a (but not equal to a). It is traditional to use ϵ, the Greek letter "epsilon," for the size of the allowable *error* and δ, the Greek letter "delta," for the *difference* $x - a$ that measures how close x must be to a to ensure that the error is within that tolerance. These are the letters that Cauchy and Weierstrass used in their pioneering work on limits and continuity in the nineteenth century.

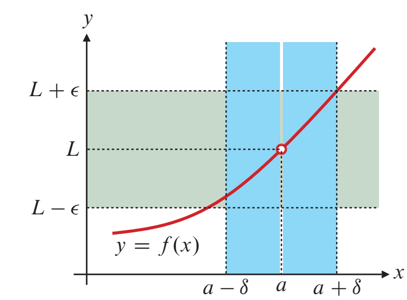


Figure 1.35 If $x \ne a$ and $|x - a| < \delta$, then $|f(x) - L| < \epsilon$

If ϵ is any positive number, *no matter how small*, we must be able to ensure that $|f(x) - L| < \epsilon$ by restricting x to be *close enough to* (but not equal to) a. How close is close enough? It is sufficient that the distance $|x - a|$ from x to a be less than a positive number δ that depends on ϵ. (See Figure 1.35.) If we can find such a δ for any positive ϵ, we are entitled to conclude that $\lim_{x\to a} f(x) = L$.

DEFINITION 8

A formal definition of limit

We say that $f(x)$ **approaches the limit** L as x **approaches** a, and we write

$$\lim_{x\to a} f(x) = L \qquad \text{or} \qquad \lim\nolimits_{x\to a} f(x) = L,$$

if the following condition is satisfied:
for every number $\epsilon > 0$ there exists a number $\delta > 0$, possibly depending on ϵ, such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and

$$|f(x) - L| < \epsilon.$$

Though precise, the above definition is more restrictive than it needs to be. It requires that the domain of f must contain open intervals with right and left endpoints at a. In Section 12.2 of Chapter 12 we will give a new, more general definition of limit for functions of any number of variables. For functions of one variable, it replaces the requirement that f be defined on open intervals with right and left endpoints at a with the weaker requirement that every open interval containing a must contain a point of the domain of f different from a. For now, we prefer the simpler but more restrictive definition given above.

The formal definition of limit does not tell you how to find the limit of a function, but it does enable you to verify that a suspected limit is correct. The following examples show how it can be used to verify limit statements for specific functions. The first of these gives a formal verification of the two limits found in Example 3 of Section 1.2.

EXAMPLE 2 **(Two important limits)** Verify that:
(a) $\lim_{x\to a} x = a$ and (b) $\lim_{x\to a} k = k$ (k = constant).

Solution

(a) Let $\epsilon > 0$ be given. We must find $\delta > 0$ so that

$$0 < |x - a| < \delta \qquad \text{implies} \qquad |x - a| < \epsilon.$$

Clearly, we can take $\delta = \epsilon$ and the implication above will be true. This proves that $\lim_{x\to a} x = a$.

(b) Let $\epsilon > 0$ be given. We must find $\delta > 0$ so that

$$0 < |x - a| < \delta \qquad \text{implies} \qquad |k - k| < \epsilon.$$

Since $k - k = 0$, we can use any positive number for δ and the implication above will be true. This proves that $\lim_{x\to a} k = k$.

EXAMPLE 3 Verify that $\lim_{x\to 2} x^2 = 4$.

Solution Here $a = 2$ and $L = 4$. Let ϵ be a given positive number. We want to find $\delta > 0$ so that if $0 < |x - 2| < \delta$, then $|f(x) - 4| < \epsilon$. Now

$$|f(x) - 4| = |x^2 - 4| = |(x + 2)(x - 2)| = |x + 2||x - 2|.$$

We want the expression above to be less than ϵ. We can make the factor $|x - 2|$ as small as we wish by choosing δ properly, but we need to control the factor $|x + 2|$ so that it does not become too large. If we first assume $\delta \le 1$ and require that $|x - 2| < \delta$, then we have

$$\begin{aligned} |x - 2| < 1 \quad &\Rightarrow \quad 1 < x < 3 \quad \Rightarrow \quad 3 < x + 2 < 5 \\ &\qquad\qquad\qquad\qquad\;\; \Rightarrow \quad |x + 2| < 5. \end{aligned}$$

Hence,

$$|f(x)-4| < 5|x-2| \quad \text{if} \quad |x-2| < \delta \le 1.$$

But $5|x-2| < \epsilon$ if $|x-2| < \epsilon/5$. Therefore, if we take $\delta = \min\{1, \epsilon/5\}$, the *minimum* (the smaller) of the two numbers 1 and $\epsilon/5$, then

$$|f(x)-4| < 5|x-2| < 5 \times \frac{\epsilon}{5} = \epsilon \quad \text{if} \quad |x-2| < \delta.$$

This proves that $\lim_{x\to 2} f(x) = 4$.

Using the Definition of Limit to Prove Theorems

We do not usually rely on the formal definition of limit to verify specific limits such as those in the two examples above. Rather, we appeal to general theorems about limits, in particular Theorems 2–4 of Section 1.2. The definition is used to prove these theorems. As an example, we prove part 1 of Theorem 2, the *Sum Rule*.

EXAMPLE 4 **(Proving the rule for the limit of a sum)** If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, prove that $\lim_{x\to a}\big(f(x)+g(x)\big) = L + M$.

Solution Let $\epsilon > 0$ be given. We want to find a positive number δ such that

$$0 < |x-a| < \delta \quad \Rightarrow \quad \big|\big(f(x)+g(x)\big)-(L+M)\big| < \epsilon.$$

Observe that

$$\begin{aligned} &\big|\big(f(x)+g(x)\big)-(L+M)\big| && \text{Regroup terms.} \\ &\qquad = \big|\big(f(x)-L\big)+\big(g(x)-M\big)\big| && \text{(Use the triangle inequality:} \\ & && |a+b| \le |a|+|b|\text{).} \\ &\qquad \le |f(x)-L| + |g(x)-M|. \end{aligned}$$

Since $\lim_{x\to a} f(x) = L$ and $\epsilon/2$ is a positive number, there exists a number $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \quad \Rightarrow \quad |f(x)-L| < \epsilon/2.$$

Similarly, since $\lim_{x\to a} g(x) = M$, there exists a number $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \quad \Rightarrow \quad |g(x)-M| < \epsilon/2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$, the smaller of δ_1 and δ_2. If $0 < |x-a| < \delta$, then $|x-a| < \delta_1$, so $|f(x)-L| < \epsilon/2$, and $|x-a| < \delta_2$, so $|g(x)-M| < \epsilon/2$. Therefore,

$$\big|\big(f(x)+g(x)\big)-(L+M)\big| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $\lim_{x\to a}\big(f(x)+g(x)\big) = L + M$.

Other Kinds of Limits

The formal definition of limit can be modified to give precise definitions of one-sided limits, limits at infinity, and infinite limits. We give some of the definitions here and leave you to supply the others.

DEFINITION 9

Right limits

We say that $f(x)$ has **right limit** L at a, and we write

$$\lim_{x\to a+} f(x) = L,$$

if the following condition is satisfied:
for every number $\epsilon > 0$ there exists a number $\delta > 0$, possibly depending on ϵ, such that if $a < x < a + \delta$, then x belongs to the domain of f and

$$|f(x) - L| < \epsilon.$$

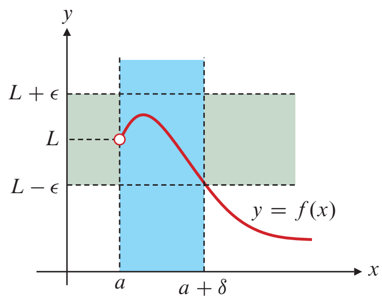


Figure 1.36 If $a < x < a + \delta$, then $|f(x) - L| < \epsilon$

Notice how the condition $0 < |x - a| < \delta$ in the definition of limit becomes $a < x < a + \delta$ in the right limit case (Figure 1.36). The definition for a left limit is formulated in a similar way.

EXAMPLE 5 Show that $\lim_{x\to 0+} \sqrt{x} = 0.$

Solution Let $\epsilon > 0$ be given. If $x > 0$, then $|\sqrt{x} - 0| = \sqrt{x}$. We can ensure that $\sqrt{x} < \epsilon$ by requiring $x < \epsilon^2$. Thus, we can take $\delta = \epsilon^2$ and the condition of the definition will be satisfied:

$$0 < x < \delta = \epsilon^2 \qquad \text{implies} \qquad |\sqrt{x} - 0| < \epsilon.$$

Therefore, $\lim_{x\to 0+} \sqrt{x} = 0.$

To claim that a function f has a limit L at infinity, we must be able to ensure that the error $|f(x) - L|$ is less than any given positive number ϵ by restricting x to be *sufficiently large*, that is, by requiring $x > R$ for some positive number R depending on ϵ.

DEFINITION 10

Limit at infinity

We say that $f(x)$ **approaches the limit** L **as** x **approaches infinity**, and we write

$$\lim_{x\to\infty} f(x) = L,$$

if the following condition is satisfied:
for every number $\epsilon > 0$ there exists a number R, possibly depending on ϵ, such that if $x > R$, then x belongs to the domain of f and

$$|f(x) - L| < \epsilon.$$

You are invited to formulate a version of the definition of a limit at negative infinity.

EXAMPLE 6 Show that $\lim_{x\to\infty} \frac{1}{x} = 0.$

Solution Let ϵ be a given positive number. For $x > 0$ we have

$$\left|\frac{1}{x} - 0\right| = \frac{1}{|x|} = \frac{1}{x} < \epsilon \qquad \text{provided} \qquad x > \frac{1}{\epsilon}.$$

Therefore, the condition of the definition is satisfied with $R = 1/\epsilon$. We have shown that $\lim_{x\to\infty} 1/x = 0.$

To show that $f(x)$ has an infinite limit at a, we must ensure that $f(x)$ is larger than any given positive number (say B) by restricting x to a sufficiently small interval centred at a, and requiring that $x \neq a$.

DEFINITION 11

Infinite limits

We say that $f(x)$ approaches infinity as x approaches a and write

$$\lim_{x \to a} f(x) = \infty,$$

if for every positive number B we can find a positive number δ, possibly depending on B, such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and $f(x) > B$.

Try to formulate the corresponding definition for the concept $\lim_{x \to a} f(x) = -\infty$. Then try to modify both definitions to cover the case of infinite one-sided limits and infinite limits at infinity.

EXAMPLE 7 Verify that $\lim_{x \to 0} \dfrac{1}{x^2} = \infty$.

Solution Let B be any positive number. We have

$$\frac{1}{x^2} > B \qquad \text{provided that} \qquad x^2 < \frac{1}{B}.$$

If $\delta = 1/\sqrt{B}$, then

$$0 < |x| < \delta \quad \Rightarrow \quad x^2 < \delta^2 = \frac{1}{B} \quad \Rightarrow \quad \frac{1}{x^2} > B.$$

Therefore, $\lim_{x \to 0} 1/x^2 = \infty$.

EXERCISES 1.5

1. The length L of a metal rod is given in terms of the temperature T (°C) by $L = 39.6 + 0.025T$ cm. Within what range of temperature must the rod be kept if its length must be maintained within ± 1 mm of 40 cm?
2. What is the largest tolerable error in the 20 cm edge length of a cubical cardboard box if the volume of the box must be within $\pm 1.2\%$ of 8,000 cm^3?

In Exercises 3–6, in what interval must x be confined if $f(x)$ must be within the given distance ϵ of the number L?

3. $f(x) = 2x - 1,\quad L = 3,\quad \epsilon = 0.02$
4. $f(x) = x^2,\quad L = 4,\quad \epsilon = 0.1$
5. $f(x) = \sqrt{x},\quad L = 1,\quad \epsilon = 0.1$
6. $f(x) = 1/x,\quad L = -2,\quad \epsilon = 0.01$

In Exercises 7–10, find a number $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - L|$ will be less than the given number ϵ.

7. $f(x) = 3x + 1,\quad a = 2,\quad L = 7,\quad \epsilon = 0.03$
8. $f(x) = \sqrt{2x + 3},\quad a = 3,\quad L = 3,\quad \epsilon = 0.01$
9. $f(x) = x^3,\quad a = 2,\quad L = 8,\quad \epsilon = 0.2$
10. $f(x) = 1/(x + 1),\quad a = 0,\quad L = 1,\quad \epsilon = 0.05$

In Exercises 11–20, use the formal definition of limit to verify the indicated limit.

11. $\lim_{x \to 1} (3x + 1) = 4$
12. $\lim_{x \to 2} (5 - 2x) = 1$
13. $\lim_{x \to 0} x^2 = 0$
14. $\lim_{x \to 2} \dfrac{x - 2}{1 + x^2} = 0$
15. $\lim_{x \to 1/2} \dfrac{1 - 4x^2}{1 - 2x} = 2$
16. $\lim_{x \to -2} \dfrac{x^2 + 2x}{x + 2} = -2$
17. $\lim_{x \to 1} \dfrac{1}{x + 1} = \dfrac{1}{2}$
18. $\lim_{x \to -1} \dfrac{x + 1}{x^2 - 1} = -\dfrac{1}{2}$
19. $\lim_{x \to 1} \sqrt{x} = 1$
20. $\lim_{x \to 2} x^3 = 8$

Give formal definitions of the limit statements in Exercises 21–26.

21. $\lim_{x \to a-} f(x) = L$
22. $\lim_{x \to -\infty} f(x) = L$
23. $\lim_{x \to a} f(x) = -\infty$
24. $\lim_{x \to \infty} f(x) = \infty$
25. $\lim_{x \to a+} f(x) = -\infty$
26. $\lim_{x \to a-} f(x) = \infty$

Use formal definitions of the various kinds of limits to prove the statements in Exercises 27–30.

27. $\lim_{x\to 1+} \frac{1}{x-1} = \infty$

28. $\lim_{x\to 1-} \frac{1}{x-1} = -\infty$

29. $\lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}} = 0$

30. $\lim_{x\to\infty} \sqrt{x} = \infty$

Proving Theorems with the Definition of Limit

31. Prove that limits are unique; that is, if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, prove that $L = M$. *Hint:* Suppose $L \neq M$ and let $\epsilon = |L - M|/3$.

32. If $\lim_{x\to a} g(x) = M$, show that there exists a number $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |g(x)| < 1 + |M|.$$

(*Hint:* Take $\epsilon = 1$ in the definition of limit.) This says that the values of $g(x)$ are **bounded** near a point where g has a limit.

33. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, prove that $\lim_{x\to a} f(x)g(x) = LM$ (the Product Rule part of Theorem 2). *Hint:* Reread Example 4. Let $\epsilon > 0$ and write

$$\begin{aligned}|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM|\\ &= |(f(x) - L)g(x) + L(g(x) - M)|\\ &\le |(f(x) - L)g(x)| + |L(g(x) - M)|\\ &= |g(x)||f(x) - L| + |L||g(x) - M|\end{aligned}$$

Now try to make each term in the last line less than $\epsilon/2$ by taking x close enough to a. You will need the result of Exercise 32.

34. If $\lim_{x\to a} g(x) = M$, where $M \neq 0$, show that there exists a number $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |g(x)| > |M|/2.$$

35. If $\lim_{x\to a} g(x) = M$, where $M \neq 0$, show that

$$\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{M}.$$

Hint: You will need the result of Exercise 34.

36. Use the facts proved in Exercises 33 and 35 to prove the Quotient Rule (part 5 of Theorem 2): if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, where $M \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

37. Use the definition of limit twice to prove Theorem 7 of Section 1.4; that is, if f is continuous at L and if $\lim_{x\to c} g(x) = L$, then

$$\lim_{x\to c} f(g(x)) = f(L) = f\left(\lim_{x\to c} g(x)\right).$$

38. Prove the Squeeze Theorem (Theorem 4 in Section 1.2). *Hint:* If $f(x) \le g(x) \le h(x)$, then

$$\begin{aligned}|g(x) - L| &= |g(x) - f(x) + f(x) - L|\\ &\le |g(x) - f(x)| + |f(x) - L|\\ &\le |h(x) - f(x)| + |f(x) - L|\\ &= |h(x) - L - (f(x) - L)| + |f(x) - L|\\ &\le |h(x) - L| + |f(x) - L| + |f(x) - L|\end{aligned}$$

Now you can make each term in the last expression less than $\epsilon/3$ and so complete the proof.

CHAPTER REVIEW

Key Ideas

- **What do the following statements and phrases mean?**
  - the average rate of change of $f(x)$ on $[a, b]$
  - the instantaneous rate of change of $f(x)$ at $x = a$
  - $\lim_{x\to a} f(x) = L$
  - $\lim_{x\to a+} f(x) = L, \quad \lim_{x\to a-} f(x) = L$
  - $\lim_{x\to\infty} f(x) = L, \quad \lim_{x\to-\infty} f(x) = L$
  - $\lim_{x\to a} f(x) = \infty, \quad \lim_{x\to a+} f(x) = -\infty$
  - f is continuous at c.
  - f is left (or right) continuous at c.
  - f has a continuous extension to c.
  - f is a continuous function.
  - f takes on maximum and minimum values on interval I.
  - f is bounded on interval I.
  - f has the intermediate-value property on interval I.
- **State as many "laws of limits" as you can.**
- **What properties must a function have if it is continuous and its domain is a closed, finite interval?**
- **How can you find zeros (roots) of a continuous function?**

Review Exercises

1. Find the average rate of change of x^3 over $[1, 3]$.

2. Find the average rate of change of $1/x$ over $[-2, -1]$.

3. Find the rate of change of x^3 at $x = 2$.

4. Find the rate of change of $1/x$ at $x = -3/2$.

Evaluate the limits in Exercises 5–30 or explain why they do not exist.

5. $\lim_{x\to 1} (x^2 - 4x + 7)$

6. $\lim_{x\to 2} \frac{x^2}{1 - x^2}$

7. $\lim_{x\to 1} \frac{x^2}{1 - x^2}$

8. $\lim_{x\to 2} \frac{x^2 - 4}{x^2 - 5x + 6}$

9. $\lim_{x\to 2} \frac{x^2 - 4}{x^2 - 4x + 4}$

10. $\lim_{x\to 2-} \frac{x^2 - 4}{x^2 - 4x + 4}$

11. $\lim_{x\to -2+} \dfrac{x^2-4}{x^2+4x+4}$

12. $\lim_{x\to 4} \dfrac{2-\sqrt{x}}{x-4}$

13. $\lim_{x\to 3} \dfrac{x^2-9}{\sqrt{x}-\sqrt{3}}$

14. $\lim_{h\to 0} \dfrac{h}{\sqrt{x+3h}-\sqrt{x}}$

15. $\lim_{x\to 0+} \sqrt{x-x^2}$

16. $\lim_{x\to 0} \sqrt{x-x^2}$

17. $\lim_{x\to 1} \sqrt{x-x^2}$

18. $\lim_{x\to 1-} \sqrt{x-x^2}$

19. $\lim_{x\to\infty} \dfrac{1-x^2}{3x^2-x-1}$

20. $\lim_{x\to -\infty} \dfrac{2x+100}{x^2+3}$

21. $\lim_{x\to -\infty} \dfrac{x^3-1}{x^2+4}$

22. $\lim_{x\to\infty} \dfrac{x^4}{x^2-4}$

23. $\lim_{x\to 0+} \dfrac{1}{\sqrt{x-x^2}}$

24. $\lim_{x\to 1/2} \dfrac{1}{\sqrt{x-x^2}}$

25. $\lim_{x\to\infty} \sin x$

26. $\lim_{x\to\infty} \dfrac{\cos x}{x}$

27. $\lim_{x\to 0} x \sin\dfrac{1}{x}$

28. $\lim_{x\to 0} \sin\dfrac{1}{x^2}$

29. $\lim_{x\to -\infty} [x+\sqrt{x^2-4x+1}]$

30. $\lim_{x\to\infty} [x+\sqrt{x^2-4x+1}]$

At what, if any, points in its domain is the function f in Exercises 31–38 discontinuous? Is f left or right continuous at these points? In Exercises 35 and 36, H refers to the Heaviside function: $H(x) = 1$ if $x \ge 0$ and $H(x) = 0$ if $x < 0$.

31. $f(x) = x^3 - 4x^2 + 1$

32. $f(x) = \dfrac{x}{x+1}$

33. $f(x) = \begin{cases} x^2 & \text{if } x > 2 \\ x & \text{if } x \le 2 \end{cases}$

34. $f(x) = \begin{cases} x^2 & \text{if } x > 1 \\ x & \text{if } x \le 1 \end{cases}$

35. $f(x) = H(x-1)$

36. $f(x) = H(9-x^2)$

37. $f(x) = |x| + |x+1|$

38. $f(x) = \begin{cases} |x|/|x+1| & \text{if } x \ne -1 \\ 1 & \text{if } x = -1 \end{cases}$

Challenging Problems

1. Show that the average rate of change of the function x^3 over the interval $[a, b]$, where $0 < a < b$, is equal to the instantaneous rate of change of x^3 at $x = \sqrt{(a^2+ab+b^2)/3}$. Is this point to the left or to the right of the midpoint $(a+b)/2$ of the interval $[a, b]$?

2. Evaluate $\lim_{x\to 0} \dfrac{x}{|x-1|-|x+1|}$.

3. Evaluate $\lim_{x\to 3} \dfrac{|5-2x|-|x-2|}{|x-5|-|3x-7|}$.

4. Evaluate $\lim_{x\to 64} \dfrac{x^{1/3}-4}{x^{1/2}-8}$.

5. Evaluate $\lim_{x\to 1} \dfrac{\sqrt{3+x}-2}{\sqrt[3]{7+x}-2}$.

6. The equation $ax^2 + 2x - 1 = 0$, where a is a constant, has two roots if $a > -1$ and $a \ne 0$:

$$r_+(a) = \frac{-1+\sqrt{1+a}}{a} \quad\text{and}\quad r_-(a) = \frac{-1-\sqrt{1+a}}{a}.$$

(a) What happens to the root $r_-(a)$ when $a \to 0$?

(b) Investigate numerically what happens to the root $r_+(a)$ when $a \to 0$ by trying the values $a = 1, \pm 0.1, \pm 0.01, \ldots$. For values such as $a = 10^{-8}$, the limited precision of your calculator may produce some interesting results. What happens, and why?

(c) Evaluate $\lim_{a\to 0} r_+(a)$ mathematically by using the identity

$$\sqrt{A} - \sqrt{B} = \frac{A-B}{\sqrt{A}+\sqrt{B}}.$$

7. TRUE or FALSE? If TRUE, give reasons; if FALSE, give a counterexample.

(a) If $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} (f(x)+g(x))$ does not exist.

(b) If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a} (f(x)+g(x))$ does not exist.

(c) If f is continuous at a, then so is $|f|$.

(d) If $|f|$ is continuous at a, then so is f.

(e) If $f(x) < g(x)$ for all x in an interval around a, and if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then $\lim_{x\to a} f(x) < \lim_{x\to a} g(x)$.

8. (a) If f is a continuous function defined on a closed interval $[a, b]$, show that $R(f)$ is a closed interval.

(b) What are the possibilities for $R(f)$ if $D(f)$ is an open interval (a, b)?

9. Consider the function $f(x) = \dfrac{x^2-1}{|x^2-1|}$. Find all points where f is not continuous. Does f have one-sided limits at those points, and if so, what are they?

10. Find the minimum value of $f(x) = 1/(x-x^2)$ on the interval $(0, 1)$. Explain how you know such a minimum value must exist.

11. (a) Suppose f is a continuous function on the interval $[0, 1]$, and $f(0) = f(1)$. Show that $f(a) = f\left(a+\dfrac{1}{2}\right)$ for some $a \in \left[0, \dfrac{1}{2}\right]$.

Hint: Let $g(x) = f\left(x+\dfrac{1}{2}\right) - f(x)$, and use the Intermediate-Value Theorem.

(b) If n is an integer larger than 2, show that $f(a) = f\left(a+\dfrac{1}{n}\right)$ for some $a \in \left[0, 1-\dfrac{1}{n}\right]$.