Solution Manual

Fundamentals of Communication Systems

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Second Edition

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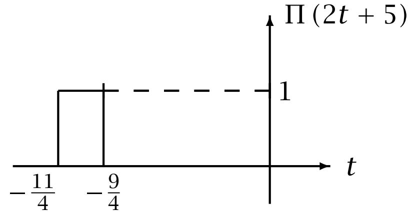
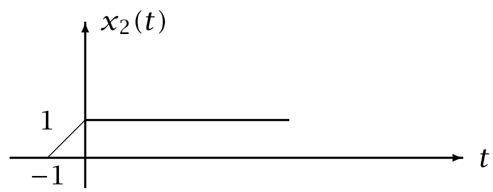
Chapter 2

Problem 2.1

1. $\Pi(2t+5) = \Pi\left(2\left(t+\frac{5}{2}\right)\right)$. This indicates first we have to plot $\Pi(2t)$ and then shift it to left by $\frac{5}{2}$. A plot is shown below:

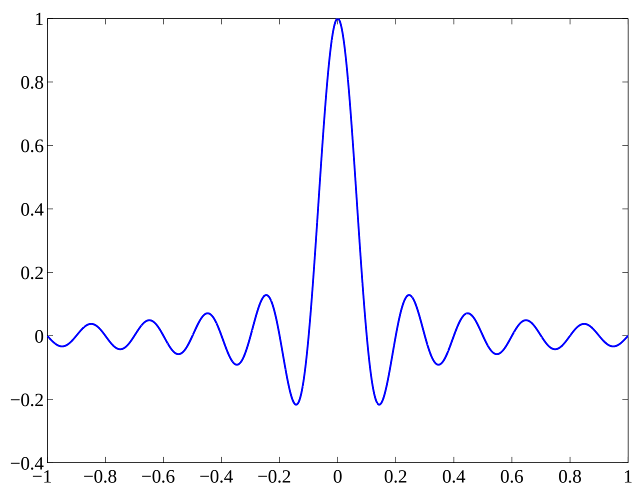


2. $\sum_{n=0}^{\infty} \Lambda(t-n)$ is a sum of shifted triangular pulses. Note that the sum of the left and right side of triangular pulses that are displaced by one unit of time is equal to 1, The plot is given below



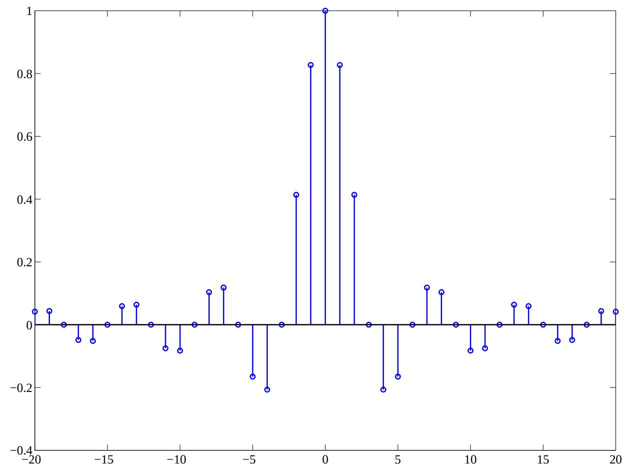
3. It is obvious from the definition of $\operatorname{sgn}(t)$ that $\operatorname{sgn}(2t) = \operatorname{sgn}(t)$. Therefore $x_3(t) = 0$.

4. $x_4(t)$ is $\operatorname{sinc}(t)$ contracted by a factor of 10.

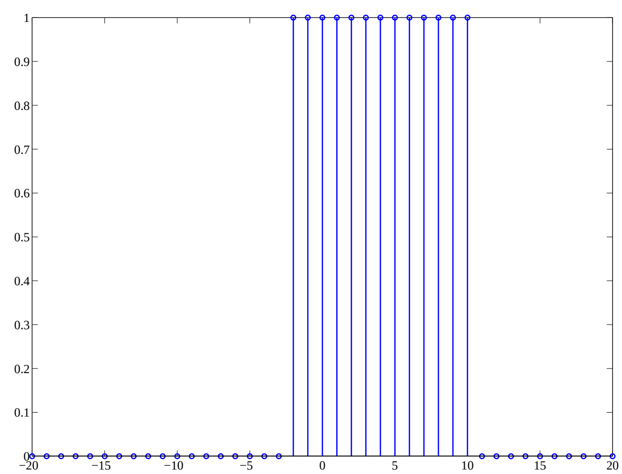


Problem 2.2

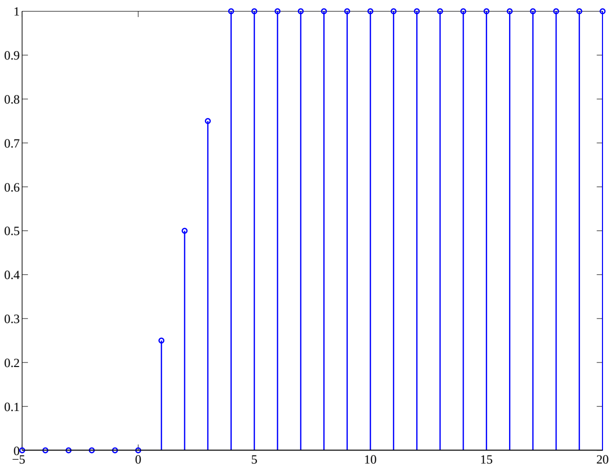
1. $x[n] = \operatorname{sinc}(3n/9) = \operatorname{sinc}(n/3)$.



2. $x[n] = \Pi\left(\frac{\frac{n}{4}-1}{3}\right)$. If $-\frac{1}{2} \le \frac{\frac{n}{4}-1}{3} \le \frac{1}{2}$, i.e., $-2 \le n \le 10$, we have $x[n] = 1$.



3. $x[n] = \frac{n}{4}u_{-1}(n/4) - (\frac{n}{4} - 1)u_{-1}(n/4 - 1)$. For $n < 0$, $x[n] = 0$, for $0 \le n \le 3$, $x[n] = \frac{n}{4}$ and for $n \ge 4$, $x[n] = \frac{n}{4} - \frac{n}{4} + 1 = 1$.



Problem 2.3

$x_1[n] = 1$ and $x_2[n] = \cos(2\pi n) = 1$, for all n. This shows that two signals can be different but their sampled versions be the same.

Problem 2.4

Let $x_1[n]$ and $x_2[n]$ be two periodic signals with periods N_1 and N_2, respectively, and let $N = \text{LCM}(N_1, N_2)$, and define $x[n] = x_1[n] + x_2[n]$. Then obviously $x_1[n+N] = x_1[n]$ and $x_2[n+N] = x_2[n]$, and hence $x[n] = x[n+N]$, i.e., $x[n]$ is periodic with period N.

For continuous-time signals $x_1(t)$ and $x_2(t)$ with periods T_1 and T_2 respectively, in general we cannot find a T such that $T = k_1T_1 = k_2T_2$ for integers k_1 and k_2. This is obvious for instance if $T_1 = 1$ and $T_2 = \pi$. The necessary and sufficient condition for the sum to be periodic is that $\frac{T_1}{T_2}$ be a rational number.

Problem 2.5

Using the result of problem 2.4 we have:

1. The frequencies are 2000 and 5500, their ratio (and therefore the ratio of the periods) is rational, hence the sum is periodic.

2. The frequencies are 2000 and $\frac{5500}{\pi}$. Their ratio is not rational, hence the sum is not periodic.

3. The sum of two periodic discrete-time signal is periodic.

4. The fist signal is periodic but $\cos[11000n]$ is *not* periodic, since there is no N such that $\cos[11000(n+N)] = \cos(11000n)$ for all n. Therefore the sum cannot be periodic.

Problem 2.6

1)

$$x_1(t) = \begin{cases} e^{-t} & t > 0 \\ -e^{t} & t < 0 \\ 0 & t = 0 \end{cases} \Longrightarrow x_1(-t) = \begin{cases} -e^{-t} & t > 0 \\ e^{t} & t < 0 \\ 0 & t = 0 \end{cases} = -x_1(t)$$

Thus, $x_1(t)$ is an odd signal

2) $x_2(t) = \cos\left(120\pi t + \frac{\pi}{3}\right)$ is neither even nor odd. We have $\cos\left(120\pi t + \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)\cos(120\pi t) - \sin\left(\frac{\pi}{3}\right)\sin(120\pi t)$. Therefore $x_{2e}(t) = \cos\left(\frac{\pi}{3}\right)\cos(120\pi t)$ and $x_{2o}(t) = -\sin\left(\frac{\pi}{3}\right)\sin(120\pi t)$. (Note: This part can also be considered as a special case of part 7 of this problem)

3)

$$x_3(t) = e^{-|t|} \Longrightarrow x_3(-t) = e^{-|(-t)|} = e^{-|t|} = x_3(t)$$

Hence, the signal $x_3(t)$ is even.

4)

$$x_4(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases} \Longrightarrow x_4(-t) = \begin{cases} 0 & t \geq 0 \\ -t & t < 0 \end{cases}$$

The signal $x_4(t)$ is neither even nor odd. The even part of the signal is

$$x_{4,e}(t) = \frac{x_4(t) + x_4(-t)}{2} = \begin{cases} \frac{t}{2} & t \geq 0 \\ \frac{-t}{2} & t < 0 \end{cases} = \frac{|t|}{2}$$

The odd part is

$$x_{4,o}(t) = \frac{x_4(t) - x_4(-t)}{2} = \begin{cases} \frac{t}{2} & t \geq 0 \\ \frac{t}{2} & t < 0 \end{cases} = \frac{t}{2}$$

5)

$$x_5(t) = x_1(t) - x_2(t) \Longrightarrow x_5(-t) = x_1(-t) - x_2(-t) = x_1(t) + x_2(t)$$

Clearly $x_5(-t) \neq x_5(t)$ since otherwise $x_2(t) = 0\ \forall t$. Similarly $x_5(-t) \neq -x_5(t)$ since otherwise $x_1(t) = 0\ \forall t$. The even and the odd parts of $x_5(t)$ are given by

$$\begin{aligned} x_{5,e}(t) &= \frac{x_5(t) + x_5(-t)}{2} = x_1(t) \\ x_{5,o}(t) &= \frac{x_5(t) - x_5(-t)}{2} = -x_2(t) \end{aligned}$$

Problem 2.7

For the first two questions we will need the integral $I = \int e^{ax} \cos^2 x dx$.

$$\begin{aligned}
I &= \frac{1}{a}\int \cos^2 x \, de^{ax} = \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a}\int e^{ax}\sin 2x \, dx \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}\int \sin 2x \, de^{ax} \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}e^{ax}\sin 2x - \frac{2}{a^2}\int e^{ax}\cos 2x \, dx \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}e^{ax}\sin 2x - \frac{2}{a^2}\int e^{ax}(2\cos^2 x - 1) \, dx \\
&= \frac{1}{a}e^{ax}\cos^2 x + \frac{1}{a^2}e^{ax}\sin 2x - \frac{2}{a^2}\int e^{ax} \, dx - \frac{4}{a^2}I
\end{aligned}$$

Thus,

$$I = \frac{1}{4+a^2}\left[(a\cos^2 x + \sin 2x) + \frac{2}{a}\right]e^{ax}$$

1)

$$\begin{aligned}
E_x &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_1^2(t)dx = \lim_{T\to\infty}\int_0^{\frac{T}{2}} e^{-2t}\cos^2 t dt \\
&= \lim_{T\to\infty}\frac{1}{8}\left[(-2\cos^2 t + \sin 2t) - 1\right]e^{-2t}\Big|_0^{\frac{T}{2}} \\
&= \lim_{T\to\infty}\frac{1}{8}\left[(-2\cos^2\frac{T}{2} + \sin T - 1)e^{-T} + 3\right] = \frac{3}{8}
\end{aligned}$$

Thus $x_1(t)$ is an energy-type signal and the energy content is $3/8$

2)

$$\begin{aligned}
E_x &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_2^2(t)dx = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-2t}\cos^2 t dt \\
&= \lim_{T\to\infty}\left[\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 t dt + \int_0^{\frac{T}{2}} e^{-2t}\cos^2 t dt\right]
\end{aligned}$$

But,

$$\begin{aligned}
\lim_{T\to\infty}\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 t dt &= \lim_{T\to\infty}\frac{1}{8}\left[(-2\cos^2 t + \sin 2t) - 1\right]e^{-2t}\Big|_{-\frac{T}{2}}^{0} \\
&= \lim_{T\to\infty}\frac{1}{8}\left[-3 + (2\cos^2\frac{T}{2} + 1 + \sin T)e^{T}\right] = \infty
\end{aligned}$$

since $2 + \cos\theta + \sin\theta > 0$. Thus, $E_x = \infty$ since as we have seen from the first question the second integral is bounded. Hence, the signal $x_2(t)$ is not an energy-type signal. To test if $x_2(t)$ is a power-type signal we find P_x.

$$P_x = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 dt + \lim_{T\to\infty}\frac{1}{T}\int_0^{\frac{T}{2}} e^{-2t}\cos^2 dt$$

But $\lim_{T\to\infty} \frac{1}{T}\int_0^{\frac{T}{2}} e^{-2t}\cos^2 dt$ is zero and

$$\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{0} e^{-2t}\cos^2 dt &= \lim_{T\to\infty}\frac{1}{8T}\left[2\cos^2\frac{T}{2}+1+\sin T\right]e^T \\
&> \lim_{T\to\infty}\frac{1}{T}e^T > \lim_{T\to\infty}\frac{1}{T}(1+T+T^2) > \lim_{T\to\infty} T = \infty
\end{aligned}$$

Thus the signal $x_2(t)$ is not a power-type signal.

3)

$$\begin{aligned}
E_x &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x_3^2(t)dx = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}\operatorname{sgn}^2(t)dt = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt = \lim_{T\to\infty} T = \infty \\
P_x &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\operatorname{sgn}^2(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt = \lim_{T\to\infty}\frac{1}{T}T = 1
\end{aligned}$$

The signal $x_3(t)$ is of the power-type and the power content is 1.

4)

First note that

$$\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} A\cos(2\pi ft)dt = \sum_{k=-\infty}^{\infty} A\int_{k-\frac{1}{2f}}^{k+\frac{1}{2f}}\cos(2\pi ft)dt = 0$$

so that

$$\begin{aligned}
\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2\cos^2(2\pi ft)dt &= \lim_{T\to\infty}\frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}}(A^2+A^2\cos(2\pi 2ft))dt \\
&= \lim_{T\to\infty}\frac{1}{2}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 dt = \lim_{T\to\infty}\frac{1}{2}A^2T = \infty
\end{aligned}$$

$$\begin{aligned}
E_x &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}(A^2\cos^2(2\pi f_1t)+B^2\cos^2(2\pi f_2t)+2AB\cos(2\pi f_1t)\cos(2\pi f_2t))dt \\
&= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} A^2\cos^2(2\pi f_1t)dt + \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} B^2\cos^2(2\pi f_2t)dt + \\
&\quad AB\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}[\cos^2(2\pi(f_1+f_2)+\cos^2(2\pi(f_1-f_2)]dt \\
&= \infty+\infty+0 = \infty
\end{aligned}$$

Thus the signal is not of the energy-type. To test if the signal is of the power-type we consider two cases $f_1 = f_2$ and $f_1 \neq f_2$. In the first case

$$\begin{aligned}
P_x &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}(A+B)^2\cos^2(2\pi f_1)dt \\
&= \lim_{T\to\infty}\frac{1}{2T}(A+B)^2\int_{-\frac{T}{2}}^{\frac{T}{2}} dt = \frac{1}{2}(A+B)^2
\end{aligned}$$

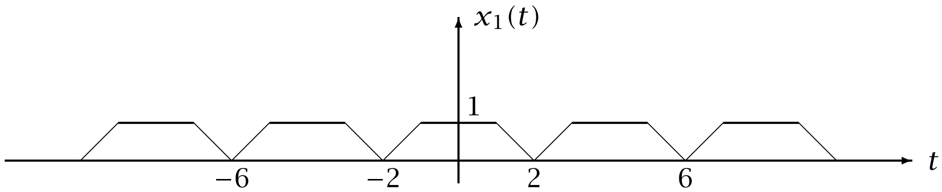
If $f_1 \neq f_2$ then

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} (A^2\cos^2(2\pi f_1 t) + B^2\cos^2(2\pi f_2 t) + 2AB\cos(2\pi f_1 t)\cos(2\pi f_2 t))dt \\ &= \lim_{T\to\infty} \frac{1}{T}\left[\frac{A^2 T}{2} + \frac{B^2 T}{2}\right] = \frac{A^2}{2} + \frac{B^2}{2} \end{aligned}$$

Thus the signal is of the power-type and if $f_1 = f_2$ the power content is $(A+B)^2/2$ whereas if $f_1 \neq f_2$ the power content is $\frac{1}{2}(A^2 + B^2)$

Problem 2.8

1. Let $x(t) = 2\Lambda\left(\frac{t}{2}\right) - \Lambda(t)$, then $x_1(t) = \sum_{n=-\infty}^{\infty} x(t-4n)$. First we plot $x(t)$ then by shifting it by multiples of 4 we can plot $x_1(t)$. $x(t)$ is a triangular pulse of width 4 and height 2 from which a standard triangular pulse of width 1 and height 1 is subtracted. The result is a trapezoidal pulse, which when replicated at intervals of 4 gives the plot of $x_1(t)$.



2. This is the sum of two periodic signals with periods 2π and 1. Since the ratio of the two periods is not rational the sum is not periodic (by the result of problem 2.4)

3. $\sin[n]$ is not periodic. There is no integer N such that $\sin[n+N] = \sin[n]$ for all n.

Problem 2.9

1)

$$P_x = \lim_{T\to\infty}\frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} A^2\left|e^{j(2\pi f_0 t+\theta)}\right|^2 dt = \lim_{T\to\infty}\frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} A^2 dt = \lim_{T\to\infty}\frac{1}{T}A^2 T = A^2$$

Thus $x(t) = Ae^{j(2\pi f_0 t+\theta)}$ is a power-type signal and its power content is A^2.

2)

$$P_x = \lim_{T\to\infty}\frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} A^2\cos^2(2\pi f_0 t+\theta)\,dt = \lim_{T\to\infty}\frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}}\frac{A^2}{2}dt + \lim_{T\to\infty}\frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}}\frac{A^2}{2}\cos(4\pi f_0 t + 2\theta)\,dt$$

As $T\to\infty$, the there will be no contribution by the second integral. Thus the signal is a power-type signal and its power content is $\frac{A^2}{2}$.

3)

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u_{-1}^2(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_0^{\frac{T}{2}} dt = \lim_{T\to\infty} \frac{1}{T}\frac{T}{2} = \frac{1}{2}$$

Thus the unit step signal is a power-type signal and its power content is $1/2$

4)

$$\begin{aligned} E_x &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt = \lim_{T\to\infty}\int_0^{\frac{T}{2}} K^2 t^{-\frac{1}{2}}dt = \lim_{T\to\infty} 2K^2 t^{\frac{1}{2}}\Big|_0^{T/2} \\ &= \lim_{T\to\infty} \sqrt{2}K^2 T^{\frac{1}{2}} = \infty \end{aligned}$$

Thus the signal is not an energy-type signal.

$$\begin{aligned} P_x &= \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt = \lim_{T\to\infty}\frac{1}{T}\int_0^{\frac{T}{2}} K^2 t^{-\frac{1}{2}}dt \\ &= \lim_{T\to\infty}\frac{1}{T}2K^2 t^{\frac{1}{2}}\Big|_0^{T/2} = \lim_{T\to\infty}\frac{1}{T}2K^2(T/2)^{\frac{1}{2}} = \lim_{T\to\infty}\sqrt{2}K^2 T^{-\frac{1}{2}} = 0 \end{aligned}$$

Since P_x is not bounded away from zero it follows by definition that the signal is not of the power-type (recall that power-type signals should satisfy $0 < P_x < \infty$).

Problem 2.10

$$\Lambda(t) = \begin{cases} t+1, & -1 \le t \le 0 \\ -t+1, & 0 \le t \le 1 \\ 0, & \text{o.w.} \end{cases} \qquad u_{-1}(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}$$

Thus, the signal $x(t) = \Lambda(t)u_{-1}(t)$ is given by

$$x(t) = \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ -t+1 & 0 \le t \le 1 \\ 0 & t \ge 1 \end{cases} \Rightarrow x(-t) = \begin{cases} 0 & t \le -1 \\ t+1 & -1 \le t < 0 \\ 1/2 & t = 0 \\ 0 & t > 0 \end{cases}$$

The even and the odd part of $x(t)$ are given by

$$\begin{aligned} x_e(t) &= \frac{x(t)+x(-t)}{2} = \frac{1}{2}\Lambda(t) \\ x_o(t) &= \frac{x(t)-x(-t)}{2} = \begin{cases} 0 & t \le -1 \\ \frac{-t-1}{2} & -1 \le t < 0 \\ 0 & t = 0 \\ \frac{-t+1}{2} & 0 < t \le 1 \\ 0 & 1 \le t \end{cases} \end{aligned}$$

Problem 2.11

1) Suppose that

$$x(t) = x_e^1(t) + x_o^1(t) = x_e^2(t) + x_o^2(t)$$

with $x_e^1(t)$, $x_e^2(t)$ even signals and $x_o^1(t)$, $x_o^1(t)$ odd signals. Then, $x(-t) = x_e^1(t) - x_o^1(t)$ so that

$$\begin{aligned} x_e^1(t) &= \frac{x(t) + x(-t)}{2} \\ &= \frac{x_e^2(t) + x_o^2(t) + x_e^2(-t) + x_o^2(-t)}{2} \\ &= \frac{2x_e^2(t) + x_o^2(t) - x_o^2(t)}{2} = x_e^2(t) \end{aligned}$$

Thus $x_e^1(t) = x_e^2(t)$ and $x_o^1(t) = x(t) - x_e^1(t) = x(t) - x_e^2(t) = x_o^2(t)$

2) Let $x_e^1(t)$, $x_e^2(t)$ be two even signals and $x_o^1(t)$, $x_o^2(t)$ be two odd signals. Then,

$$\begin{aligned} y(t) = x_e^1(t)x_e^2(t) &\implies y(-t) = x_e^1(-t)x_e^2(-t) = x_e^1(t)x_e^2(t) = y(t) \\ z(t) = x_o^1(t)x_o^2(t) &\implies z(-t) = x_o^1(-t)x_o^2(-t) = (-x_o^1(t))(-x_o^2(t)) = z(t) \end{aligned}$$

Thus the product of two even or odd signals is an even signal. For $v(t) = x_e^1(t)x_o^1(t)$ we have

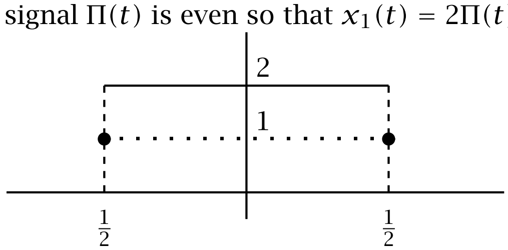
$$v(-t) = x_e^1(-t)x_o^1(-t) = x_e^1(t)(-x_o^1(t)) = -x_e^1(t)x_o^1(t) = -v(t)$$

Thus the product of an even and an odd signal is an odd signal.

3) One trivial example is $t + 1$ and $\frac{t^2}{t+1}$.

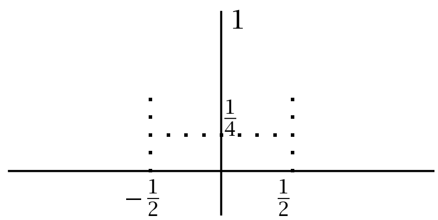
Problem 2.12

1) $x_1(t) = \Pi(t) + \Pi(-t)$. The signal $\Pi(t)$ is even so that $x_1(t) = 2\Pi(t)$

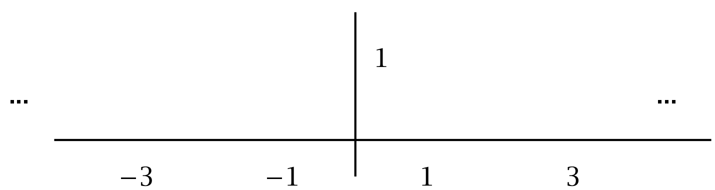


2)

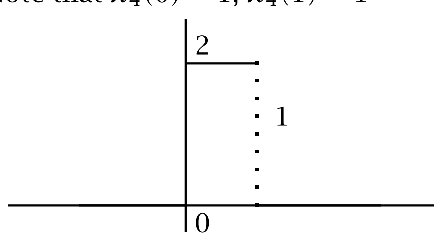
$$x_2(t) = \Lambda(t) \cdot \Pi(t) = \begin{cases} 0, & t < -1/2 \\ 1/4, & t = -1/2 \\ t+1, & -1/2 < t \leq 0 \\ -t+1, & 0 \leq t < 1/2 \\ 1/4, & t = 1/2 \\ 0, & 1/2 < t \end{cases}$$



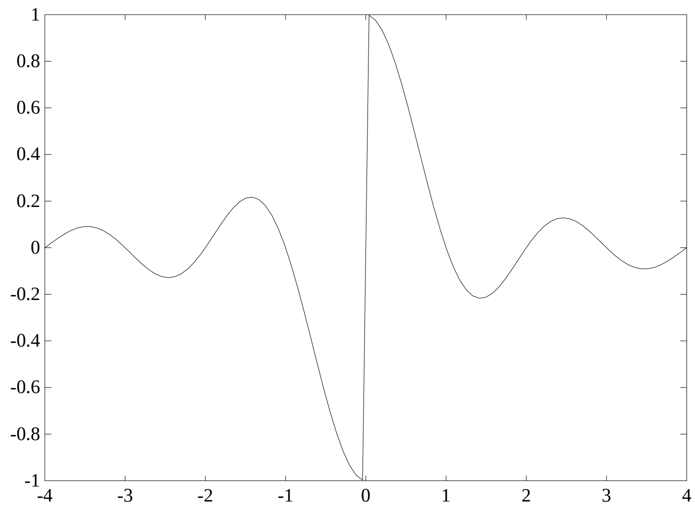
3) $x_3(t) = \sum_{n=-\infty}^{\infty} \Lambda(t-2n)$



4) $x_4(t) = \text{sgn}(t) + \text{sgn}(1-t)$. Note that $x_4(0) = 1$, $x_4(1) = 1$



5) $x_5(t) = \text{sinc}(t)\text{sgn}(t)$. Note that $x_5(0) = 0$.



Problem 2.13

1) The value of the expression $\text{sinc}(t)\delta(t)$ can be found by examining its effect on a function $\phi(t)$ through the integral

$$\int_{-\infty}^{\infty} \phi(t)\text{sinc}(t)\delta(t) = \phi(0)\text{sinc}(0) = \text{sinc}(0)\int_{-\infty}^{\infty} \phi(t)\delta(t)$$

Thus $\text{sinc}(t)\delta(t)$ has the same effect as the function $\text{sinc}(0)\delta(t)$ and we conclude that

$$x_1(t) = \text{sinc}(t)\delta(t) = \text{sinc}(0)\delta(t) = \delta(t)$$

2) $\text{sinc}(t)\delta(t-3) = \text{sinc}(3)\delta(t-3) = 0$.

3)

$$\begin{aligned} x_3(t) &= \Lambda(t) \star \sum_{n=-\infty}^{\infty} \delta(t-2n) \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \Lambda(t-\tau)\delta(\tau-2n)d\tau \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \Lambda(\tau-t)\delta(\tau-2n)d\tau \\ &= \sum_{n=-\infty}^{\infty} \Lambda(t-2n) \end{aligned}$$

4)

$$\begin{aligned} x_4(t) &= \Lambda(t) \star \delta'(t) = \int_{-\infty}^{\infty} \Lambda(t-\tau)\delta'(\tau)d\tau \\ &= (-1)\frac{d}{d\tau}\Lambda(t-\tau)\Big|_{\tau=0} = \Lambda'(t) = \begin{cases} 0 & t < -1 \\ \frac{1}{2} & t = -1 \\ 1 & -1 < t < 0 \\ 0 & t = 0 \\ -1 & 0 < t < 1 \\ -\frac{1}{2} & t = 1 \\ 0 & 1 < t \end{cases} \end{aligned}$$

5) $x_5(t) = \cos\left(2t + \frac{\pi}{3}\right)\delta(3t) = \frac{1}{3}\cos\left(2t + \frac{\pi}{3}\right)\delta(t) = \frac{1}{3}\cos\left(\frac{\pi}{3}\right)\delta(t)$. Hence $x_5(t) = \frac{1}{6}\delta(t)$.

6)

$$x_6(t) = \delta(5t) \star \delta(4t) = \frac{1}{5}\delta(t) \star \frac{1}{4}\delta(t) = \frac{1}{20}\delta(t)$$

7)

$$\int_{-\infty}^{\infty} \text{sinc}(t)\delta(t)dt = \text{sinc}(0) = 1$$

8)

$$\int_{-\infty}^{\infty} \text{sinc}(t+1)\delta(t)dt = \text{sinc}(1) = 0$$

Problem 2.14

The impulse signal can be defined in terms of the limit

$$\delta(t) = \lim_{\tau \to 0} \frac{1}{2\tau}\left(e^{-\frac{|t|}{\tau}}\right)$$

But $e^{-\frac{|t|}{\tau}}$ is an even function for every τ so that $\delta(t)$ is even. Since $\delta(t)$ is even, we obtain

$$\delta(t) = \delta(-t) \Longrightarrow \delta'(t) = -\delta'(-t)$$

Thus, the function $\delta'(t)$ is odd. For the function $\delta^{(n)}(t)$ we have

$$\int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(-t)dt = (-1)^n \int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(t)dt$$

where we have used the differentiation chain rule

$$\frac{d}{dt}\delta^{(k-1)}(-t) = \frac{d}{d(-t)}\delta^{(k-1)}(-t)\frac{d}{dt}(-t) = (-1)\delta^{(k)}(-t)$$

Thus, if $n = 2l$ (even)

$$\int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(-t)dt = \int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(t)dt$$

and the function $\delta^{(n)}(t)$ is even. If $n = 2l + 1$ (odd), then $(-1)^n = -1$ and

$$\int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(-t)dt = -\int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(t)dt$$

from which we conclude that $\delta^{(n)}(t)$ is odd.

Problem 2.15

$$x(t) \star \delta^{(n)}(t) = \int_{-\infty}^{\infty} x(\tau)\delta^{(n)}(t-\tau)\, d\tau$$

The signal $\delta^{(n)}(t)$ is even if n is even and odd if n is odd. Consider first the case that $n = 2l$. Then,

$$x(t) \star \delta^{(2l)}(t) = \int_{-\infty}^{\infty} x(\tau)\delta^{(2l)}(\tau - t)\, d\tau = (-1)^{2l} \frac{d^{2l}}{d\tau^{2l}} x(\tau)\bigg|_{\tau=t} = \frac{d^n}{dt^n} x(t)$$

If n is odd then,

$$\begin{aligned} x(t) \star \delta^{(2l+1)}(t) &= \int_{-\infty}^{\infty} x(\tau)(-1)\delta^{(2l+1)}(\tau - t)\, d\tau = (-1)(-1)^{2l+1} \frac{d^{2l+1}}{d\tau^{2l+1}} x(\tau)\bigg|_{\tau=t} \\ &= \frac{d^n}{dt^n} x(t) \end{aligned}$$

In both cases

$$x(t) \star \delta^{(n)}(t) = \frac{d^n}{dt^n} x(t)$$

The convolution of $x(t)$ with $u_{-1}(t)$ is

$$x(t) \star u_{-1}(t) = \int_{-\infty}^{\infty} x(\tau)u_{-1}(t-\tau)d\tau$$

But $u_{-1}(t-\tau) = 0$ for $\tau > t$ so that

$$x(t) \star u_{-1}(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Problem 2.16

1) Nonlinear, since the response to $x(t) = 0$ is not $y(t) = 0$ (this is a necessary condition for linearity of a system, see also problem 2.21).

2) Nonlinear, if we multiply the input by constant -1, the output does not change. In a linear system the output should be scaled by -1.

3) Linear, the output to any input zero, therefore for the input $\alpha x_1(t) + \beta x_2(t)$ the output is zero which can be considered as $\alpha y_1(t) + \beta y_2(t) = \alpha \times 0 + \beta \times 0 = 0$. This is a linear combination of the corresponding outputs to $x_1(t)$ and $x_2(t)$.
4) Nonlinear, the output to $x(t) = 0$ is not zero.
5) Nonlinear. The system is not homogeneous for if $\alpha < 0$ and $x(t) > 0$ then $y(t) = T[\alpha x(t)] = 0$ whereas $z(t) = \alpha T[x(t)] = \alpha$.

6) Linear. For if $x(t) = \alpha x_1(t) + \beta x_2(t)$ then

$$\begin{aligned} T[\alpha x_1(t) + \beta x_2(t)] &= (\alpha x_1(t) + \beta x_2(t))e^{-t} \\ &= \alpha x_1(t)e^{-t} + \beta x_2(t)e^{-t} = \alpha T[x_1(t)] + \beta T[x_2(t)] \end{aligned}$$

7) Linear. For if $x(t) = \alpha x_1(t) + \beta x_2(t)$ then

$$\begin{aligned} T[\alpha x_1(t) + \beta x_2(t)] &= (\alpha x_1(t) + \beta x_2(t))u(t) \\ &= \alpha x_1(t)u(t) + \beta x_2(t)u(t) = \alpha T[x_1(t)] + \beta T[x_2(t)] \end{aligned}$$

8) Linear. We can write the output of this feedback system as

$$y(t) = x(t) + y(t-1) = \sum_{n=0}^{\infty} x(t-n)$$

Then for $x(t) = \alpha x_1(t) + \beta x_2(t)$

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} (\alpha x_1(t-n) + \beta x_2(t-n)) \\ &= \alpha \sum_{n=0}^{\infty} x_1(t-n) + \beta \sum_{n=0}^{\infty} x_2(t-n)) \\ &= \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

9) Linear. Assuming that only a finite number of jumps occur in the interval $(-\infty, t]$ and that the magnitude of these jumps is finite so that the algebraic sum is well defined, we obtain

$$y(t) = T[\alpha x(t)] = \sum_{n=1}^{N} \alpha J_x(t_n) = \alpha \sum_{n=1}^{N} J_x(t_n) = \alpha T[x(t)]$$

where N is the number of jumps in $(-\infty, t]$ and $J_x(t_n)$ is the value of the jump at time instant t_n, that is

$$J_x(t_n) = \lim_{\epsilon \to 0}(x(t_n + \epsilon) - x(t_n - \epsilon))$$

For $x(t) = x_1(t) + x_2(t)$ we can assume that $x_1(t)$, $x_2(t)$ and $x(t)$ have the same number of jumps and at the same positions. This is true since we can always add new jumps of magnitude zero to the already existing ones. Then for each t_n, $J_x(t_n) = J_{x_1}(t_n) + J_{x_2}(t_n)$ and

$$y(t) = \sum_{n=1}^{N} J_x(t_n) = \sum_{n=1}^{N} J_{x_1}(t_n) + \sum_{n=1}^{N} J_{x_2}(t_n)$$

so that the system is additive.

Problem 2.17

Only if ($\Longrightarrow$)

If the system $\mathcal{T}$ is linear then

$$\mathcal{T}[\alpha x_1(t) + \beta x_2(t)] = \alpha\mathcal{T}[x_1(t)] + \beta\mathcal{T}[x_2(t)]$$

for all α, β and $x(t)$'s. If we set $\beta = 0$, then

$$\mathcal{T}[\alpha x_1(t)] = \alpha\mathcal{T}[x_1(t)]$$

so that the system is homogeneous. If we let $\alpha = \beta = 1$, we obtain

$$\mathcal{T}[x_1(t) + x_2(t)] = \mathcal{T}[x_1(t)] + \mathcal{T}[x_2(t)]$$

and thus the system is additive.

If ($\Longleftarrow$)

Suppose that both conditions 1) and 2) hold. Thus the system is homogeneous and additive. Then

$$\begin{aligned} &\mathcal{T}[\alpha x_1(t) + \beta x_2(t)] \\ &= \mathcal{T}[\alpha x_1(t)] + \mathcal{T}[\beta x_2(t)] \text{ (additive system)} \\ &= \alpha\mathcal{T}[x_1(t)] + \beta\mathcal{T}[x_2(t)] \text{ (homogeneous system)} \end{aligned}$$

Thus the system is linear.

Problem 2.18

1. Neither homogeneous nor additive.
2. Neither homogeneous nor additive.
3. Homogeneous and additive.
4. Neither homogeneous nor additive.
5. Neither homogeneous nor additive.
6. Homogeneous but not additive.
7. Neither homogeneous nor additive.
8. Homogeneous and additive.
9. Homogeneous and additive.

10. Homogeneous and additive.
11. Homogeneous and additive.
12. Homogeneous and additive.
13. Homogeneous and additive.
14. Homogeneous and additive.

Problem 2.19

We first prove that

$$\mathcal{T}[nx(t)] = n\mathcal{T}[x(t)]$$

for $n \in \mathcal{N}$. The proof is by induction on n. For $n = 2$ the previous equation holds since the system is additive. Let us assume that it is true for n and prove that it holds for $n+1$.

$$\begin{aligned}
&\mathcal{T}[(n+1)x(t)] \\
&= \mathcal{T}[nx(t) + x(t)] \\
&= \mathcal{T}[nx(t)] + \mathcal{T}[x(t)] \text{ (additive property of the system)} \\
&= n\mathcal{T}[x(t)] + \mathcal{T}[x(t)] \text{ (hypothesis, equation holds for } n) \\
&= (n+1)\mathcal{T}[x(t)]
\end{aligned}$$

Thus $\mathcal{T}[nx(t)] = n\mathcal{T}[x(t)]$ for every n. Now, let

$$x(t) = my(t)$$

This implies that

$$\mathcal{T}\left[\frac{x(t)}{m}\right] = \mathcal{T}[y(t)]$$

and since $\mathcal{T}[x(t)] = \mathcal{T}[my(t)] = m\mathcal{T}[y(t)]$ we obtain

$$\mathcal{T}\left[\frac{x(t)}{m}\right] = \frac{1}{m}\mathcal{T}[x(t)]$$

Thus, for an arbitrary rational $\alpha = \frac{k}{\lambda}$ we have

$$\mathcal{T}\left[\frac{k}{\lambda}x(t)\right] = \mathcal{T}\left[k\left(\frac{x(t)}{\lambda}\right)\right] = k\mathcal{T}\left[\frac{x(t)}{\lambda}\right] = \frac{k}{\lambda}\mathcal{T}[x(t)]$$

Problem 2.20

Clearly, for any α

$$y(t) = T[\alpha x(t)] = \begin{cases} \frac{\alpha^2 x^2(t)}{\alpha x'(t)} & x'(t) \neq 0 \\ 0 & x'(t) = 0 \end{cases} = \begin{cases} \frac{\alpha x^2(t)}{x'(t)} & x'(t) \neq 0 \\ 0 & x'(t) = 0 \end{cases} = \alpha T[x(t)]$$

Thus the system is homogeneous and if it is additive then it is linear. However, if $x(t) = x_1(t)+x_2(t)$ then $x'(t) = x_1'(t) + x_2'(t)$ and

$$\frac{(x_1(t)+x_2(t))^2}{x_1'(t)+x_2'(t)} \neq \frac{x_1^2(t)}{x_1'(t)} + \frac{x_2^2(t)}{x_2'(t)}$$

for some $x_1(t)$, $x_2(t)$. To see this let $x_2(t) = c$ (a constant signal). Then

$$T[x_1(t)+x_2(t)] = \frac{(x_1(t)+c)^2}{x_1'(t)} = \frac{x_1^2(t)+2cx_1(t)+c^2}{x_1'(t)}$$

and

$$T[x_1(t)] + T[x_2(t)] = \frac{x_1^2(t)}{x_1'(t)}$$

Thus $T[x_1(t)+x_2(t)] \neq T[x_1(t)] + T[x_2(t)]$ unless $c = 0$. Hence the system is nonlinear since the additive property has to hold for every $x_1(t)$ and $x_2(t)$.

As another example of a system that is homogeneous but non linear is the system described by

$$T[x(t)] = \begin{cases} x(t)+x(t-1) & x(t)x(t-1) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Clearly $T[\alpha x(t)] = \alpha T[x(t)]$ but $T[x_1(t)+x_2(t)] \neq T[x_1(t)] + T[x_2(t)]$

Problem 2.21

Any zero input signal can be written as $0 \cdot x(t)$ with $x(t)$ an arbitrary signal. Then, the response of the linear system is $y(t) = \mathcal{L}[0 \cdot x(t)]$ and since the system is homogeneous (linear system) we obtain

$$y(t) = \mathcal{L}[0 \cdot x(t)] = 0 \cdot \mathcal{L}[x(t)] = 0$$

Thus the response of the linear system is identically zero.

Problem 2.22

For the system to be linear we must have

$$\mathcal{T}[\alpha x_1(t) + \beta x_2(t)] = \alpha\mathcal{T}[x_1(t)] + \beta\mathcal{T}[x_2(t)]$$

for every α, β and $x(t)$'s.

$$\begin{aligned} \mathcal{T}[\alpha x_1(t) + \beta x_2(t)] &= (\alpha x_1(t) + \beta x_2(t))\cos(2\pi f_0 t) \\ &= \alpha x_1(t)\cos(2\pi f_0 t) + \beta x_2(t)\cos(2\pi f_0 t) \\ &= \alpha\mathcal{T}[x_1(t)] + \beta\mathcal{T}[x_2(t)] \end{aligned}$$

Thus the system is linear. In order for the system to be time-invariant the response to $x(t-t_0)$ should be $y(t-t_0)$ where $y(t)$ is the response of the system to $x(t)$. Clearly $y(t-t_0) = x(t-t_0)\cos(2\pi f_0(t-t_0))$ and the response of the system to $x(t-t_0)$ is $y'(t) = x(t-t_0)\cos(2\pi f_0 t)$. Since $\cos(2\pi f_0(t-t_0))$ is not equal to $\cos(2\pi f_0 t)$ for all t, t_0 we conclude that $y'(t) \neq y(t-t_0)$ and thus the system is time-variant.

Problem 2.23
1) False. For if $T_1[x(t)] = x^3(t)$ and $T_2[x(t)] = x^{1/3}(t)$ then the cascade of the two systems is the identity system $T[x(t)] = x(t)$ which is known to be linear. However, both $T_1[\cdot]$ and $T_2[\cdot]$ are nonlinear.

2) False. For if

$$T_1[x(t)] = \begin{cases} tx(t) & t \neq 0 \\ 0 & t = 0 \end{cases} \qquad T_2[x(t)] = \begin{cases} \frac{1}{t}x(t) & t \neq 0 \\ 0 & t = 0 \end{cases}$$

Then $T_2[T_1[x(t)]] = x(t)$ and the system which is the cascade of $T_1[\cdot]$ followed by $T_2[\cdot]$ is time-invariant , whereas both $T_1[\cdot]$ and $T_2[\cdot]$ are time variant.

3) False. Consider the system

$$y(t) = T[x(t)] = \begin{cases} x(t) & t \geq 0 \\ 1 & t < 0 \end{cases}$$

Then the output of the system $y(t)$ depends only on the input $x(\tau)$ for $\tau \leq t$ This means that the system is causal. However the response to a causal signal, $x(t) = 0$ for $t \leq 0$, is nonzero for negative values of t and thus it is not causal.

Problem 2.24
1) Time invariant: The response to $x(t - t_0)$ is $2x(t - t_0) + 3$ which is $y(t - t_0)$.
2) Time varying the response to $x(t - t_0)$ is $(t + 2)x(t - t_0)$ but $y(t - t_0) = (t - t_0 + 2)x(t - t_0)$, obviously the two are not equal.
3) Time-varying system. The response $y(t - t_0)$ is equal to $x(-(t - t_0)) = x(-t + t_0)$. However the response of the system to $x(t - t_0)$ is $z(t) = x(-t - t_0)$ which is not equal to $y(t - t_0)$

4) Time-varying system. Clearly

$$y(t) = x(t)u_{-1}(t) \Longrightarrow y(t - t_0) = x(t - t_0)u_{-1}(t - t_0)$$

However, the response of the system to $x(t - t_0)$ is $z(t) = x(t - t_0)u_{-1}(t)$ which is not equal to $y(t - t_0)$

5) Time-invariant system. Clearly

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau \Longrightarrow y(t - t_0) = \int_{-\infty}^{t-t_0} x(\tau)d\tau$$

The response of the system to $x(t - t_0)$ is

$$z(t) = \int_{-\infty}^{t} x(\tau - t_0)d\tau = \int_{-\infty}^{t-t_0} x(v)dv = y(t - t_0)$$

where we have used the change of variable $v = \tau - t_0$.

6) Time-invariant system. Writing $y(t)$ as $\sum_{n=-\infty}^{\infty} x(t-n)$ we get

$$y(t-t_0) = \sum_{n=-\infty}^{\infty} x(t-t_0-n) = T[x(t-t_0)]$$

Problem 2.25

The differentiator is a LTI system (see examples 2.19 and 2.1.21 in book). It is true that the output of a system which is the cascade of two LTI systems does not depend on the order of the systems. This can be easily seen by the commutative property of the convolution

$$h_1(t) \star h_2(t) = h_2(t) \star h_1(t)$$

Let $h_1(t)$ be the impulse response of a differentiator, and let $y(t)$ be the output of the system $h_2(t)$ with input $x(t)$. Then,

$$\begin{aligned} z(t) &= h_2(t) \star x'(t) = h_2(t) \star (h_1(t) \star x(t)) \\ &= h_2(t) \star h_1(t) \star x(t) = h_1(t) \star h_2(t) \star x(t) \\ &= h_1(t) \star y(t) = y'(t) \end{aligned}$$

Problem 2.26

The integrator is is a LTI system (why?). It is true that the output of a system which is the cascade of two LTI systems does not depend on the order of the systems. This can be easily seen by the commutative property of the convolution

$$h_1(t) \star h_2(t) = h_2(t) \star h_1(t)$$

Let $h_1(t)$ be the impulse response of an integrator, and let $y(t)$ be the output of the system $h_2(t)$ with input $x(t)$. Then,

$$\begin{aligned} z(t) &= h_2(t) \star \int_{-\infty}^{t} x(\tau)\,d\tau = h_2(t) \star (h_1(t) \star x(t)) \\ &= h_2(t) \star h_1(t) \star x(t) = h_1(t) \star h_2(t) \star x(t) \\ &= h_1(t) \star y(t) = \int_{-\infty}^{t} y(\tau)\,d\tau \end{aligned}$$

Problem 2.27

The output of a LTI system is the convolution of the input with the impulse response of the system. Thus,

$$\delta(t) = \int_{-\infty}^{\infty} h(\tau)e^{-\alpha(t-\tau)}u_{-1}(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)e^{-\alpha(t-\tau)}d\tau$$

Differentiating both sides with respect to t we obtain

$$\begin{aligned}\delta'(t) &= (-\alpha)e^{-\alpha t}\int_{-\infty}^{t} h(\tau)e^{\alpha\tau}d\tau + e^{-\alpha t}\frac{d}{dt}\left[\int_{-\infty}^{t} h(\tau)e^{\alpha\tau}d\tau\right] \\ &= (-\alpha)\delta(t) + e^{-\alpha t}h(t)e^{\alpha t} = (-\alpha)\delta(t) + h(t)\end{aligned}$$

Thus

$$h(t) = \alpha\delta(t) + \delta'(t)$$

The response of the system to the input $x(t)$ is

$$\begin{aligned}y(t) &= \int_{-\infty}^{\infty} x(\tau)\left[\alpha\delta(t-\tau) + \delta'(t-\tau)\right]d\tau \\ &= \alpha\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)\delta'(t-\tau)d\tau \\ &= \alpha x(t) + \frac{d}{dt}x(t)\end{aligned}$$

Problem 2.28
For the system to be causal the output at the time instant t_0 should depend only on $x(t)$ for $t \le t_0$.

$$y(t_0) = \frac{1}{2T}\int_{t_0-T}^{t_0+T} x(\tau)d\tau = \frac{1}{2T}\int_{t_0-T}^{t_0} x(\tau)d\tau + \frac{1}{2T}\int_{t_0}^{t_0+T} x(\tau)d\tau$$

We observe that the second integral on the right side of the equation depends on values of $x(\tau)$ for τ greater than t_0. Thus the system is non causal.

Problem 2.29
Consider the system

$$y(t) = T[x(t)] = \begin{cases} x(t) & x(t) \neq 0 \\ 1 & x(t) = 0 \end{cases}$$

This system is causal since the output at the time instant t depends only on values of $x(\tau)$ for $\tau \le t$ (actually it depends only on the value of $x(\tau)$ for $\tau = t$, a stronger condition.) However, the response of the system to the impulse signal $\delta(t)$ is one for $t < 0$ so that the impulse response of the system is nonzero for $t < 0$.

Problem 2.30

1. Noncausal: Since for $t < 0$ we do not have $sinc(t) = 0$.

2. This is a rectangular signal of width 6 centered at $t_0 = 3$, for negative t's it is zero, therefore the system is causal.

3. The system is causal since for negative t's $h(t) = 0$.

Problem 2.31

The output $y(t)$ of a LTI system with impulse response $h(t)$ and input signal $u_{-1}(t)$ is

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u_{-1}(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)u_{-1}(t-\tau)d\tau + \int_{t}^{\infty} h(\tau)u_{-1}(t-\tau)d\tau$$

But $u_{-1}(t-\tau) = 1$ for $\tau < t$ so that

$$\int_{-\infty}^{t} h(\tau)u_{-1}(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$$

Similarly, since $u_{-1}(t-\tau) = 0$ for $\tau < t$ we obtain

$$\int_{t}^{\infty} h(\tau)u_{-1}(t-\tau)d\tau = 0$$

Combining the previous integrals we have

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u_{-1}(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$$

Problem 2.32

Let $h(t)$ denote the the impulse response of a differentiator. Then for every input signal

$$x(t) \star h(t) = \frac{d}{dt}x(t)$$

If $x(t) = \delta(t)$ then the output of the differentiator is its impulse response. Thus,

$$\delta(t) \star h(t) = h(t) = \delta'(t)$$

The output of the system to an arbitrary input $x(t)$ can be found by convolving $x(t)$ with $\delta'(t)$. In this case

$$y(t) = x(t) \star \delta'(t) = \int_{-\infty}^{\infty} x(\tau)\delta'(t-\tau)d\tau = \frac{d}{dt}x(t)$$

Assume that the impulse response of a system which delays its input by t_0 is $h(t)$. Then the response to the input $\delta(t)$ is

$$\delta(t) \star h(t) = \delta(t-t_0)$$

However, for every $x(t)$

$$\delta(t) \star x(t) = x(t)$$

so that $h(t) = \delta(t-t_0)$. The output of the system to an arbitrary input $x(t)$ is

$$y(t) = x(t) \star \delta(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\delta(t-t_0-\tau)d\tau = x(t-t_0)$$

Problem 2.33

The response of the system to the signal $\alpha x_1(t) + \beta x_2(t)$ is

$$y_1(t) = \int_{t-T}^{t} (\alpha x_1(\tau) + \beta x_2(\tau))d\tau = \alpha \int_{t-T}^{t} x_1(\tau)d\tau + \beta \int_{t-T}^{t} x_2(\tau)d\tau$$

Thus the system is linear. The response to $x(t - t_0)$ is

$$y_1(t) = \int_{t-T}^{t} x(\tau - t_0)d\tau = \int_{t-t_0-T}^{t-t_0} x(v)dv = y(t - t_0)$$

where we have used the change of variables $v = \tau - t_0$. Thus the system is time invariant. The impulse response is obtained by applying an impulse at the input.

$$h(t) = \int_{t-T}^{t} \delta(\tau)d\tau = \int_{-\infty}^{t} \delta(\tau)d\tau - \int_{-\infty}^{t-T} \delta(\tau)d\tau = u_{-1}(t) - u_{-1}(t - T)$$

Problem 2.34

1)

$$\begin{aligned} e^{-t}u_{-1}(t) \star e^{-t}u_{-1}(t) &= \int_{-\infty}^{\infty} e^{-\tau}u_{-1}(\tau)e^{-(t-\tau)}u_{-1}(t-\tau)d\tau = \int_{0}^{t} e^{-t}d\tau \\ &= \begin{cases} te^{-t} & t > 0 \\ 0 & t < 0 \end{cases} \end{aligned}$$

2)

$$x(t) = \Pi(t) \star \Lambda(t) = \int_{-\infty}^{\infty} \Pi(\theta)\Lambda(t-\theta)d\theta = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Lambda(t-\theta)d\theta = \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \Lambda(v)dv$$

$$\begin{aligned} t \le -\frac{3}{2} \quad &\Longrightarrow \quad x(t) = 0 \\ -\frac{3}{2} < t \le -\frac{1}{2} \quad &\Longrightarrow \quad x(t) = \int_{-1}^{t+\frac{1}{2}} (v+1)dv = (\frac{1}{2}v^2 + v)\Big|_{-1}^{t+\frac{1}{2}} = \frac{1}{2}t^2 + \frac{3}{2}t + \frac{9}{8} \\ -\frac{1}{2} < t \le \frac{1}{2} \quad &\Longrightarrow \quad x(t) = \int_{t-\frac{1}{2}}^{0} (v+1)dv + \int_{0}^{t+\frac{1}{2}} (-v+1)dv \\ &\qquad\qquad = (\frac{1}{2}v^2 + v)\Big|_{t-\frac{1}{2}}^{0} + (-\frac{1}{2}v^2 + v)\Big|_{0}^{t+\frac{1}{2}} = -t^2 + \frac{3}{4} \\ \frac{1}{2} < t \le \frac{3}{2} \quad &\Longrightarrow \quad x(t) = \int_{t-\frac{1}{2}}^{1} (-v+1)dv = (-\frac{1}{2}v^2 + v)\Big|_{t-\frac{1}{2}}^{1} = \frac{1}{2}t^2 - \frac{3}{2}t + \frac{9}{8} \\ \frac{3}{2} < t \quad &\Longrightarrow \quad x(t) = 0 \end{aligned}$$

Thus,

$$x(t) = \begin{cases} 0 & t \leq -\frac{3}{2} \\ \frac{1}{2}t^2 + \frac{3}{2}t + \frac{9}{8} & -\frac{3}{2} < t \leq -\frac{1}{2} \\ -t^2 + \frac{3}{4} & -\frac{1}{2} < t \leq \frac{1}{2} \\ \frac{1}{2}t^2 - \frac{3}{2}t + \frac{9}{8} & \frac{1}{2} < t \leq \frac{3}{2} \\ 0 & \frac{3}{2} < t \end{cases}$$

Problem 2.35

The output of a LTI system with impulse response $h(t)$ is

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Using the first formula for the convolution and observing that $h(\tau) = 0,\ \tau < 0$ we obtain

$$y(t) = \int_{-\infty}^{0} x(t-\tau)h(\tau)d\tau + \int_{0}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{0}^{\infty} x(t-\tau)h(\tau)d\tau$$

Using the second formula for the convolution and writing

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau + \int_{t}^{\infty} x(\tau)h(t-\tau)d\tau$$

we obtain

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

The last is true since $h(t-\tau) = 0$ for $t < \tau$ so that $\int_t^\infty x(\tau)h(t-\tau)d\tau = 0$

Problem 2.36

In order for the signals $\psi_n(t)$ to constitute an orthonormal set of signals in $[\alpha, \alpha + T_0]$ the following condition should be satisfied

$$\langle \psi_n(t), \psi_m(t) \rangle = \int_{\alpha}^{\alpha+T_0} \psi_n(t)\psi_m^*(t)dt = \delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

But

$$\begin{aligned} \langle \psi_n(t), \psi_m(t) \rangle &= \int_{\alpha}^{\alpha+T_0} \frac{1}{\sqrt{T_0}} e^{j2\pi \frac{n}{T_0} t} \frac{1}{\sqrt{T_0}} e^{-j2\pi \frac{m}{T_0} t} dt \\ &= \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} e^{j2\pi \frac{(n-m)}{T_0} t} dt \end{aligned}$$

If $n = m$ then $e^{j2\pi \frac{(n-m)}{T_0} t} = 1$ so that

$$\langle \psi_n(t), \psi_n(t) \rangle = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} dt = \frac{1}{T_0} t \Big|_{\alpha}^{\alpha+T_0} = 1$$

When $n \neq m$ then,

$$\langle \psi_n(t), \psi_m(t) \rangle = \frac{1}{j2\pi(n-m)} e^x \Big|_{j2\pi(n-m)\alpha/T_0}^{j2\pi(n-m)(\alpha+T_0)/T_0} = 0$$

Thus, $\langle \psi_n(t), \psi_n(t) \rangle = \delta_{mn}$ which proves that $\psi_n(t)$ constitute an orthonormal set of signals.

Problem 2.37

1) Since $(a-b)^2 \geq 0$ we have that

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

with equality if $a = b$. Let

$$A = \left[\sum_{i=1}^{n} \alpha_i^2 \right]^{\frac{1}{2}}, \qquad B = \left[\sum_{i=1}^{n} \beta_i^2 \right]^{\frac{1}{2}}$$

Then substituting α_i/A for a and β_i/B for b in the previous inequality we obtain

$$\frac{\alpha_i}{A} \frac{\beta_i}{B} \leq \frac{1}{2} \frac{\alpha_i^2}{A^2} + \frac{1}{2} \frac{\beta_i^2}{B^2}$$

with equality if $\frac{\alpha_i}{\beta_i} = \frac{A}{B} = k$ or $\alpha_i = k\beta_i$ for all i. Summing both sides from $i = 1$ to n we obtain

$$\begin{aligned}
\sum_{i=1}^{n} \frac{\alpha_i \beta_i}{AB} &\leq \frac{1}{2} \sum_{i=1}^{n} \frac{\alpha_i^2}{A^2} + \frac{1}{2} \sum_{i=1}^{n} \frac{\beta_i^2}{B^2} \\
&= \frac{1}{2A^2} \sum_{i=1}^{n} \alpha_i^2 + \frac{1}{2B^2} \sum_{i=1}^{n} \beta_i^2 = \frac{1}{2A^2} A^2 + \frac{1}{2B^2} B^2 = 1
\end{aligned}$$

Thus,

$$\frac{1}{AB} \sum_{i=1}^{n} \alpha_i \beta_i \leq 1 \Rightarrow \sum_{i=1}^{n} \alpha_i \beta_i \leq \left[\sum_{i=1}^{n} \alpha_i^2 \right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} \beta_i^2 \right]^{\frac{1}{2}}$$

Equality holds if $\alpha_i = k\beta_i$, for $i = 1, \ldots, n$.

2) The second equation is trivial since $|x_i y_i^*| = |x_i||y_i^*|$. To see this write x_i and y_i in polar coordinates as $x_i = \rho_{x_i} e^{j\theta_{x_i}}$ and $y_i = \rho_{y_i} e^{j\theta_{y_i}}$. Then, $|x_i y_i^*| = |\rho_{x_i} \rho_{y_i} e^{j(\theta_{x_i} - \theta_{y_i})}| = \rho_{x_i} \rho_{y_i} = |x_i||y_i| = |x_i||y_i^*|$. We turn now to prove the first inequality. Let z_i be any complex with real and imaginary components $z_{i,R}$ and $z_{i,I}$ respectively. Then,

$$\begin{aligned}
\left| \sum_{i=1}^{n} z_i \right|^2 &= \left| \sum_{i=1}^{n} z_{i,R} + j \sum_{i=1}^{n} z_{i,I} \right|^2 = \left(\sum_{i=1}^{n} z_{i,R} \right)^2 + \left(\sum_{i=1}^{n} z_{i,I} \right)^2 \\
&= \sum_{i=1}^{n} \sum_{m=1}^{n} (z_{i,R} z_{m,R} + z_{i,I} z_{m,I})
\end{aligned}$$

Since $(z_{i,R}z_{m,I} - z_{m,R}z_{i,I})^2 \geq 0$ we obtain

$$(z_{i,R}z_{m,R} + z_{i,I}z_{m,I})^2 \leq (z_{i,R}^2 + z_{i,I}^2)(z_{m,R}^2 + z_{m,I}^2)$$

Using this inequality in the previous equation we get

$$\begin{aligned}\left|\sum_{i=1}^n z_i\right|^2 &= \sum_{i=1}^n\sum_{m=1}^n (z_{i,R}z_{m,R} + z_{i,I}z_{m,I})\\ &\leq \sum_{i=1}^n\sum_{m=1}^n (z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}(z_{m,R}^2 + z_{m,I}^2)^{\frac{1}{2}}\\ &= \left(\sum_{i=1}^n (z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)\left(\sum_{m=1}^n (z_{m,R}^2 + z_{m,I}^2)^{\frac{1}{2}}\right) = \left(\sum_{i=1}^n (z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)^2\end{aligned}$$

Thus

$$\left|\sum_{i=1}^n z_i\right|^2 \leq \left(\sum_{i=1}^n (z_{i,R}^2 + z_{i,I}^2)^{\frac{1}{2}}\right)^2 \quad \text{or} \quad \left|\sum_{i=1}^n z_i\right| \leq \sum_{i=1}^n |z_i|$$

The inequality now follows if we substitute $z_i = x_i y_i^*$. Equality is obtained if $\frac{z_{i,R}}{z_{i,I}} = \frac{z_{m,R}}{z_{m,I}} = k_1$ or $\angle z_i = \angle z_m = \theta$.

3) From 2) we obtain

$$\left|\sum_{i=1}^n x_i y_i^*\right|^2 \leq \sum_{i=1}^n |x_i||y_i|$$

But $|x_i|$, $|y_i|$ are real positive numbers so from 1)

$$\sum_{i=1}^n |x_i||y_i| \leq \left[\sum_{i=1}^n |x_i|^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^n |y_i|^2\right]^{\frac{1}{2}}$$

Combining the two inequalities we get

$$\left|\sum_{i=1}^n x_i y_i^*\right|^2 \leq \left[\sum_{i=1}^n |x_i|^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^n |y_i|^2\right]^{\frac{1}{2}}$$

From part 1) equality holds if $\alpha_i = k\beta_i$ or $|x_i| = k|y_i|$ and from part 2) $x_i y_i^* = |x_i y_i^*|e^{j\theta}$. Therefore, the two conditions are

$$\begin{cases} |x_i| = k|y_i| \\ \angle x_i - \angle y_i = \theta \end{cases}$$

which imply that for all i, $x_i = K y_i$ for some complex constant K.

4) The same procedure can be used to prove the Cauchy-Schwartz inequality for integrals. An easier approach is obtained if one considers the inequality

$$|x(t) + \alpha y(t)| \geq 0, \qquad \text{for all } \alpha$$

Then

$$\begin{aligned} 0 &\leq \int_{-\infty}^{\infty} |x(t) + \alpha y(t)|^2 dt = \int_{-\infty}^{\infty} (x(t) + \alpha y(t))(x^*(t) + \alpha^* y^*(t))dt \\ &= \int_{-\infty}^{\infty} |x(t)|^2 dt + \alpha \int_{-\infty}^{\infty} x^*(t) y(t) dt + \alpha^* \int_{-\infty}^{\infty} x(t) y^*(t) dt + |a|^2 \int_{-\infty}^{\infty} |y(t)|^2 dt \end{aligned}$$

The inequality is true for $\int_{-\infty}^{\infty} x^*(t)y(t)dt = 0$. Suppose that $\int_{-\infty}^{\infty} x^*(t)y(t)dt \neq 0$ and set

$$\alpha = -\frac{\int_{-\infty}^{\infty} |x(t)|^2 dt}{\int_{-\infty}^{\infty} x^*(t) y(t) dt}$$

Then,

$$0 \leq -\int_{-\infty}^{\infty} |x(t)|^2 dt + \frac{[\int_{-\infty}^{\infty} |x(t)|^2 dt]^2 \int_{-\infty}^{\infty} |y(t)|^2 dt}{|\int_{-\infty}^{\infty} x(t) y^*(t) dt|^2}$$

and

$$\left| \int_{-\infty}^{\infty} x(t) y^*(t) dt \right| \leq \left[\int_{-\infty}^{\infty} |x(t)|^2 dt \right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} |y(t)|^2 dt \right]^{\frac{1}{2}}$$

Equality holds if $x(t) = -\alpha y(t)$ a.e. for some complex α.

Problem 2.38

1)

$$\begin{aligned} \epsilon^2 &= \int_{-\infty}^{\infty} \left| x(t) - \sum_{i=1}^{N} \alpha_i \phi_i(t) \right|^2 dt \\ &= \int_{-\infty}^{\infty} \left(x(t) - \sum_{i=1}^{N} \alpha_i \phi_i(t) \right) \left(x^*(t) - \sum_{j=1}^{N} \alpha_j^* \phi_j^*(t) \right) dt \\ &= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{\infty} \phi_i(t) x^*(t) dt - \sum_{j=1}^{N} \alpha_j^* \int_{-\infty}^{\infty} \phi_j^*(t) x(t) dt \\ &\quad + \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j^* \int_{-\infty}^{\infty} \phi_i(t) \phi_j^* dt \\ &= \int_{-\infty}^{\infty} |x(t)|^2 dt + \sum_{i=1}^{N} |\alpha_i|^2 - \sum_{i=1}^{N} \alpha_i \int_{-\infty}^{\infty} \phi_i(t) x^*(t) dt - \sum_{j=1}^{N} \alpha_j^* \int_{-\infty}^{\infty} \phi_j^*(t) x(t) dt \end{aligned}$$

Completing the square in terms of α_i we obtain

$$\epsilon^2 = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \left| \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \right|^2 + \sum_{i=1}^{N} \left| \alpha_i - \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt \right|^2$$

The first two terms are independent of α's and the last term is always positive. Therefore the minimum is achieved for

$$\alpha_i = \int_{-\infty}^{\infty} \phi_i^*(t) x(t) dt$$

which causes the last term to vanish.

2) With this choice of α_i's

$$\begin{aligned}\epsilon^2 &= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} \left| \int_{-\infty}^{\infty} \phi_i^*(t)x(t)dt \right|^2 \\ &= \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{N} |\alpha_i|^2\end{aligned}$$

Problem 2.39

1) Using Euler's relation we have

$$\begin{aligned}x_1(t) &= \cos(2\pi t) + \cos(4\pi t) \\ &= \frac{1}{2}\left(e^{i2\pi t} + e^{-j2\pi t} + e^{j4\pi t} + e^{-j4\pi t}\right)\end{aligned}$$

Therefore for $n = \pm 1, \pm 2$, $x_{1,n} = \frac{1}{2}$ and for all other values of n, $x_{1,n} = 0$.

2) Using Euler's relation we have

$$\begin{aligned}x_2(t) &= \cos(2\pi t) - \cos(4\pi t + \pi/3) \\ &= \frac{1}{2}\left(e^{i2\pi t} + e^{-j2\pi t} - e^{j(4\pi t+\pi/3)} - e^{-j(4\pi t+\pi/3)}\right) \\ &= \frac{1}{2}e^{i2\pi t} + \frac{1}{2}e^{-j2\pi t} + \frac{1}{2}e^{-j2\pi/3}e^{j4\pi t} + \frac{1}{2}e^{j2\pi/3}e^{-j4\pi t}\end{aligned}$$

from this we conclude that $x_{2,\pm 1} = \frac{1}{2}$ and $x_{2,2} = x_{2,-2}^* = \frac{1}{2}e^{-j2\pi/3}$, and for all other values of n, $x_{2,n} = 0$.

3) We have $x_3(t) = 2\cos(2\pi t) - \sin(4\pi t) = 2\cos(2\pi t) + \cos(4\pi t + \pi/2)$. Using Euler's relation as in parts 1 and 2 we see that $x_{3,\pm 1} = 1$ and $x_{3,2} = x_{3,-2}^* = j$, and for all other values of n, $x_{3,n} = 0$.

4) The signal $x_4(t)$ is periodic with period $T_0 = 2$. Thus

$$\begin{aligned}x_{4,n} &= \frac{1}{2}\int_{-1}^{1} \Lambda(t)e^{-j2\pi\frac{n}{2}t}dt = \frac{1}{2}\int_{-1}^{1} \Lambda(t)e^{-j\pi nt}dt \\ &= \frac{1}{2}\int_{-1}^{0} (t+1)e^{-j\pi nt}dt + \frac{1}{2}\int_{0}^{1} (-t+1)e^{-j\pi nt}dt \\ &= \frac{1}{2}\left(\frac{j}{\pi n}te^{-j\pi nt} + \frac{1}{\pi^2 n^2}e^{-j\pi nt}\right)\Bigg|_{-1}^{0} + \frac{j}{2\pi n}e^{-j\pi nt}\Bigg|_{-1}^{0} \\ &\quad -\frac{1}{2}\left(\frac{j}{\pi n}te^{-j\pi nt} + \frac{1}{\pi^2 n^2}e^{-j\pi nt}\right)\Bigg|_{0}^{1} + \frac{j}{2\pi n}e^{-j\pi nt}\Bigg|_{0}^{1} \\ &\quad \frac{1}{\pi^2 n^2} - \frac{1}{2\pi^2 n^2}(e^{j\pi n} + e^{-j\pi n}) = \frac{1}{\pi^2 n^2}(1 - \cos(\pi n))\end{aligned}$$

When $n = 0$ then

$$x_{4,0} = \frac{1}{2}\int_{-1}^{1} \Lambda(t)dt = \frac{1}{2}$$

Thus

$$x_4(t) = \frac{1}{2} + 2\sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2}(1 - \cos(\pi n))\cos(\pi n t)$$

5) The signal $x_5(t)$ is periodic with period $T_0 = 1$. For $n = 0$

$$x_{5,0} = \int_0^1 (-t+1)dt = (-\frac{1}{2}t^2 + t)\Big|_0^1 = \frac{1}{2}$$

For $n \neq 0$

$$\begin{aligned}
x_{5,n} &= \int_0^1 (-t+1)e^{-j2\pi n t}dt \\
&= -\left(\frac{j}{2\pi n}te^{-j2\pi n t} + \frac{1}{4\pi^2 n^2}e^{-j2\pi n t}\right)\Big|_0^1 + \frac{j}{2\pi n}e^{-j2\pi n t}\Big|_0^1 \\
&= -\frac{j}{2\pi n}
\end{aligned}$$

Thus,

$$x_5(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n}\sin 2\pi n t$$

6) The signal $x_6(t)$ is real even and periodic with period $T_0 = \frac{1}{2f_0}$. Hence, $x_{6,n} = a_{8,n}/2$ or

$$\begin{aligned}
x_{6,n} &= 2f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0 t)\cos(2\pi n 2 f_0 t)dt \\
&= f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1+2n)t)dt + f_0\int_{-\frac{1}{4f_0}}^{\frac{1}{4f_0}} \cos(2\pi f_0(1-2n)t)dt \\
&= \frac{1}{2\pi(1+2n)}\sin(2\pi f_0(1+2n)t)\Big|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} + \frac{1}{2\pi(1-2n)}\sin(2\pi f_0(1-2n)t)\Big|_{\frac{1}{4f_0}}^{\frac{1}{4f_0}} \\
&= \frac{(-1)^n}{\pi}\left[\frac{1}{(1+2n)} + \frac{1}{(1-2n)}\right]
\end{aligned}$$

Problem 2.40

It follows directly from the uniqueness of the decomposition of a real signal in an even and odd part. Nevertheless for a real periodic signal

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(2\pi\frac{n}{T_0}t) + b_n\sin(2\pi\frac{n}{T_0}t)\right]$$

The even part of $x(t)$ is

$$\begin{aligned} x_e(t) &= \frac{x(t)+x(-t)}{2} \\ &= \frac{1}{2}\left(a_0 + \sum_{n=1}^{\infty} a_n(\cos(2\pi\frac{n}{T_0}t) + \cos(-2\pi\frac{n}{T_0}t)) \right. \\ &\qquad \left. + b_n(\sin(2\pi\frac{n}{T_0}t) + \sin(-2\pi\frac{n}{T_0}t))\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(2\pi\frac{n}{T_0}t) \end{aligned}$$

The last is true since $\cos(\theta)$ is even so that $\cos(\theta)+\cos(-\theta) = 2\cos\theta$ whereas the oddness of $\sin(\theta)$ provides $\sin(\theta)+\sin(-\theta) = \sin(\theta)-\sin(\theta) = 0$.

The odd part of $x(t)$ is

$$\begin{aligned} x_o(t) &= \frac{x(t)-x(-t)}{2} \\ &- \sum_{n=1}^{\infty} b_n\sin(2\pi\frac{n}{T_0}t) \end{aligned}$$

Problem 2.41

1) The signal $y(t) = x(t-t_0)$ is periodic with period $T = T_0$.

$$\begin{aligned} y_n &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t-t_0)e^{-j2\pi\frac{n}{T_0}t}dt \\ &= \frac{1}{T_0}\int_{\alpha-t_0}^{\alpha-t_0+T_0} x(v)e^{-j2\pi\frac{n}{T_0}}(v+t_0)dv \\ &= e^{-j2\pi\frac{n}{T_0}t_0}\frac{1}{T_0}\int_{\alpha-t_0}^{\alpha-t_0+T_0} x(v)e^{-j2\pi\frac{n}{T_0}v}dv \\ &= x_n e^{-j2\pi\frac{n}{T_0}t_0} \end{aligned}$$

where we used the change of variables $v = t - t_0$

2) For $y(t)$ to be periodic there must exist T such that $y(t+mT) = y(t)$. But $y(t+T) = x(t+T)e^{j2\pi f_0 t}e^{j2\pi f_0 T}$ so that $y(t)$ is periodic if $T = T_0$ (the period of $x(t)$) and $f_0 T = k$ for some k in $\mathcal{Z}$. In this case

$$\begin{aligned} y_n &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}e^{j2\pi f_0 t}dt \\ &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)e^{-j2\pi\frac{(n-k)}{T_0}t}dt = x_{n-k} \end{aligned}$$

3) The signal $y(t)$ is periodic with period $T = T_0/\alpha$.

$$\begin{aligned} y_n &= \frac{1}{T}\int_{\beta}^{\beta+T} y(t)e^{-j2\pi\frac{n}{T}t}dt = \frac{\alpha}{T_0}\int_{\beta}^{\beta+\frac{T_0}{\alpha}} x(\alpha t)e^{-j2\pi\frac{n\alpha}{T_0}t}dt \\ &= \frac{1}{T_0}\int_{\beta\alpha}^{\beta\alpha+T_0} x(v)e^{-j2\pi\frac{n}{T_0}v}dv = x_n \end{aligned}$$

where we used the change of variables $v = \alpha t$.

Problem 2.42

$$\begin{aligned} \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)y^*(t)dt &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} \sum_{n=-\infty}^{\infty} x_n e^{\frac{j2\pi n}{T_0}t} \sum_{m=-\infty}^{\infty} y_m^* e^{-\frac{j2\pi m}{T_0}t}dt \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n y_m^* \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} e^{\frac{j2\pi(n-m)}{T_0}t}dt \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n y_m^* \delta_{mn} = \sum_{n=-\infty}^{\infty} x_n y_n^* \end{aligned}$$

Problem 2.43

a) The signal is periodic with period T. Thus

$$\begin{aligned} x_n &= \frac{1}{T}\int_0^T e^{-t}e^{-j2\pi\frac{n}{T}t}dt = \frac{1}{T}\int_0^T e^{-(j2\pi\frac{n}{T}+1)t}dt \\ &= -\frac{1}{T\left(j2\pi\frac{n}{T}+1\right)}e^{-(j2\pi\frac{n}{T}+1)t}\Big|_0^T = -\frac{1}{j2\pi n+T}\left[e^{-(j2\pi n+T)}-1\right] \\ &= \frac{1}{j2\pi n+T}[1-e^{-T}] = \frac{T-j2\pi n}{T^2+4\pi^2n^2}[1-e^{-T}] \end{aligned}$$

If we write $x_n = \frac{a_n - jb_n}{2}$ we obtain the trigonometric Fourier series expansion coefficients as

$$a_n = \frac{2T}{T^2+4\pi^2n^2}[1-e^{-T}], \qquad b_n = \frac{4\pi n}{T^2+4\pi^2n^2}[1-e^{-T}]$$

b) The signal is periodic with period $2T$. Since the signal is odd we obtain $x_0 = 0$. For $n \neq 0$

$$\begin{aligned} x_n &= \frac{1}{2T}\int_{-T}^{T} x(t)e^{-j2\pi\frac{n}{2T}t}dt = \frac{1}{2T}\int_{-T}^{T}\frac{t}{T}e^{-j2\pi\frac{n}{2T}t}dt \\ &= \frac{1}{2T^2}\int_{-T}^{T} te^{-j\pi\frac{n}{T}t}dt \\ &= \frac{1}{2T^2}\left(\frac{jT}{\pi n}te^{-j\pi\frac{n}{T}t} + \frac{T^2}{\pi^2n^2}e^{-j\pi\frac{n}{T}t}\right)\Big|_{-T}^{T} \\ &= \frac{1}{2T^2}\left[\frac{jT^2}{\pi n}e^{-j\pi n} + \frac{T^2}{\pi^2n^2}e^{-j\pi n} + \frac{jT^2}{\pi n}e^{j\pi n} - \frac{T^2}{\pi^2n^2}e^{j\pi n}\right] \\ &= \frac{j}{\pi n}(-1)^n \end{aligned}$$

The trigonometric Fourier series expansion coefficients are:

$$a_n = 0, \qquad b_n = (-1)^{n+1}\frac{2}{\pi n}$$

c) The signal is periodic with period T. For $n = 0$

$$x_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{3}{2}$$

If $n \neq 0$ then

$$\begin{aligned}
x_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{j}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Big|_{-\frac{T}{2}}^{\frac{T}{2}} + \frac{j}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Big|_{-\frac{T}{4}}^{\frac{T}{4}} \\
&= \frac{j}{2\pi n}\left[e^{-j\pi n} - e^{j\pi n} + e^{-j\pi\frac{n}{2}} - e^{-j\pi\frac{n}{2}}\right] \\
&= \frac{1}{\pi n}\sin(\pi\frac{n}{2}) = \frac{1}{2}\text{sinc}(\frac{n}{2})
\end{aligned}$$

Note that $x_n = 0$ for n even and $x_{2l+1} = \frac{1}{\pi(2l+1)}(-1)^l$. The trigonometric Fourier series expansion coefficients are:

$$a_0 = 3, \quad ,a_{2l} = 0, \quad ,a_{2l+1} = \frac{2}{\pi(2l+1)}(-1)^l, \quad ,b_n = 0, \ \forall n$$

d) The signal is periodic with period T. For $n = 0$

$$x_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{2}{3}$$

If $n \neq 0$ then

$$\begin{aligned}
x_n &= \frac{1}{T}\int_0^T x(t)e^{-j2\pi\frac{n}{T}t}dt = \frac{1}{T}\int_0^{\frac{T}{3}} \frac{3}{T}te^{-j2\pi\frac{n}{T}t}dt \\
&\quad + \frac{1}{T}\int_{\frac{T}{3}}^{\frac{2T}{3}} e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{\frac{2T}{3}}^{T}(-\frac{3}{T}t + 3)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{3}{T^2}\left(\frac{jT}{2\pi n}te^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2 n^2}e^{-j2\pi\frac{n}{T}t}\right)\Big|_0^{\frac{T}{3}} \\
&\quad - \frac{3}{T^2}\left(\frac{jT}{2\pi n}te^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2 n^2}e^{-j2\pi\frac{n}{T}t}\right)\Big|_{\frac{2T}{3}}^{T} \\
&\quad + \frac{j}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Big|_{\frac{T}{3}}^{\frac{2T}{3}} + \frac{3}{T}\frac{jT}{2\pi n}e^{-j2\pi\frac{n}{T}t}\Big|_{\frac{2T}{3}}^{T} \\
&= \frac{3}{2\pi^2 n^2}[\cos(\frac{2\pi n}{3}) - 1]
\end{aligned}$$

The trigonometric Fourier series expansion coefficients are:

$$a_0 = \frac{4}{3}, \quad a_n = \frac{3}{\pi^2 n^2}[\cos(\frac{2\pi n}{3}) - 1], \quad b_n = 0, \ \forall n$$

e) The signal is periodic with period T. Since the signal is odd $x_0 = a_0 = 0$. For $n \neq 0$

$$\begin{aligned}
x_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{4}} -e^{-j2\pi\frac{n}{T}t}dt \\
&\quad + \frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} \frac{4}{T} t e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{\frac{T}{4}}^{\frac{T}{2}} e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{4}{T^2}\left(\frac{jT}{2\pi n} t e^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2 n^2} e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{-\frac{T}{4}}^{\frac{T}{4}} \\
&\quad - \frac{1}{T}\left(\frac{jT}{2\pi n} e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{-\frac{T}{2}}^{-\frac{T}{4}} + \frac{1}{T}\left(\frac{jT}{2\pi n} e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{\frac{T}{4}}^{\frac{T}{2}} \\
&= \frac{j}{\pi n}\left[(-1)^n - \frac{2\sin(\frac{\pi n}{2})}{\pi n}\right] = \frac{j}{\pi n}\left[(-1)^n - \operatorname{sinc}(\frac{n}{2})\right]
\end{aligned}$$

For n even, $\operatorname{sinc}(\frac{n}{2}) = 0$ and $x_n = \frac{j}{\pi n}$. The trigonometric Fourier series expansion coefficients are:

$$a_n = 0, \ \forall n, \quad b_n = \begin{cases} -\frac{1}{\pi l} & n = 2l \\ \frac{2}{\pi(2l+1)}[1 + \frac{2(-1)^l}{\pi(2l+1)}] & n = 2l+1 \end{cases}$$

f) The signal is periodic with period T. For $n = 0$

$$x_0 = \frac{1}{T}\int_{-\frac{T}{3}}^{\frac{T}{3}} x(t)dt = 1$$

For $n \neq 0$

$$\begin{aligned}
x_n &= \frac{1}{T}\int_{-\frac{T}{3}}^{0} (\frac{3}{T}t + 2)e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{0}^{\frac{T}{3}} (-\frac{3}{T}t + 2)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{3}{T^2}\left(\frac{jT}{2\pi n} t e^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2 n^2} e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{-\frac{T}{3}}^{0} \\
&\quad - \frac{3}{T^2}\left(\frac{jT}{2\pi n} t e^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2 n^2} e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{0}^{\frac{T}{3}} \\
&\quad + \frac{2}{T}\frac{jT}{2\pi n} e^{-j2\pi\frac{n}{T}t}\Bigg|_{-\frac{T}{3}}^{0} + \frac{2}{T}\frac{jT}{2\pi n} e^{-j2\pi\frac{n}{T}t}\Bigg|_{0}^{\frac{T}{3}} \\
&= \frac{3}{\pi^2 n^2}\left[\frac{1}{2} - \cos(\frac{2\pi n}{3})\right] + \frac{1}{\pi n}\sin(\frac{2\pi n}{3})
\end{aligned}$$

The trigonometric Fourier series expansion coefficients are:

$$a_0 = 2, \quad a_n = 2\left[\frac{3}{\pi^2 n^2}\left(\frac{1}{2} - \cos(\frac{2\pi n}{3})\right) + \frac{1}{\pi n}\sin(\frac{2\pi n}{3})\right], \quad b_n = 0, \ \forall n$$

Problem 2.44

1) $H(f) = 10\Pi(\frac{f}{4})$. The system is bandlimited with bandwidth $W = 2$. Thus at the output of the system only the frequencies in the band $[-2,\ 2]$ will be present. The gain of the filter is 10 for all f in $(-2,\ 2)$ and 5 at the edges $f = \pm 2$.

a) Since the period of the signal is $T = 1$ we obtain

$$\begin{aligned} y(t) &= 10[\frac{a_0}{2} + a_1\cos(2\pi t) + b_1\sin(2\pi t)] \\ &\quad +5[a_2\cos(2\pi 2t) + b_2\sin(2\pi 2t)] \end{aligned}$$

With

$$a_n = \frac{2}{1+4\pi^2n^2}[1-e^{-1}], \qquad b_n = \frac{4\pi n}{1+4\pi^2n^2}[1-e^{-1}]$$

we obtain

$$\begin{aligned} y(t) &= (1-e^{-1})\left[20 + \frac{20}{1+4\pi^2}\cos(2\pi t) + \frac{40\pi}{1+4\pi^2}\sin(2\pi t)\right. \\ &\quad \left. + \frac{10}{1+16\pi^2}\cos(2\pi 2t) + \frac{40\pi}{1+16\pi^2}\sin(2\pi 2t)\right] \end{aligned}$$

b) Since the period of the signal is $2T = 2$ and $a_n = 0$, for all n, we have

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(2\pi\frac{n}{2}t)$$

The frequencies $\frac{n}{2}$ should satisfy $|\frac{n}{2}| \leq 2$ or $n \leq 4$. With $b_n = (-1)^{n+1}\frac{2}{\pi n}$ we obtain

$$\begin{aligned} y(t) &= \frac{20}{\pi}\sin(\frac{2\pi t}{2}) - \frac{20}{2\pi}\sin(2\pi t) \\ &\quad + \frac{20}{3\pi}\sin(\frac{2\pi 3t}{2}) - \frac{10}{4\pi}\sin(2\pi 2t) \end{aligned}$$

c) The period of the signal is $T = 1$ and

$$a_0 = 3, \quad , a_{2l} = 0, \quad , a_{2l+1} = \frac{2}{\pi(2l+1)}(-1)^l, \quad , b_n = 0,\ \forall n$$

Hence,

$$x(t) = \frac{3}{2} + \sum_{l=0}^{\infty} a_{2l+1}\cos(2\pi(2l+1)t)$$

At the output of the channel only the frequencies for which $2l + 1 \leq 2$ will be present so that

$$y(t) = 10\frac{3}{2} + 10\frac{2}{\pi}\cos(2\pi t)$$

d) Since $b_n = 0$ for all n, and the period of the signal is $T = 1$, we have

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi n t)$$

With $a_0 = \frac{4}{3}$ and $a_n = \frac{3}{\pi^2 n^2}[\cos(\frac{2\pi n}{3}) - 1]$ we obtain

$$\begin{aligned} y(t) &= \frac{20}{3} + \frac{30}{\pi^2}(\cos(\frac{2\pi}{3}) - 1)\cos(2\pi t) \\ &\quad + \frac{15}{4\pi^2}(\cos(\frac{4\pi}{3}) - 1)\cos(2\pi 2t) \\ &= \frac{20}{3} - \frac{45}{\pi^2}\cos(2\pi t) - \frac{45}{8\pi^2}\cos(2\pi 2t) \end{aligned}$$

e) With $a_n = 0$ for all n, $T = 1$ and

$$b_n = \begin{cases} -\frac{1}{\pi l} & n = 2l \\ \frac{2}{\pi(2l+1)}[1 + \frac{2(-1)^l}{\pi(2l+1)}] & n = 2l+1 \end{cases}$$

we obtain

$$\begin{aligned} y(t) &= 10 b_1 \sin(2\pi t) + 5 b_2 \sin(2\pi t 2t) \\ &= 10\frac{2}{\pi}(1 + \frac{2}{\pi})\sin(2\pi t) - 5\frac{1}{\pi}\sin(2\pi t 2t) \end{aligned}$$

f) Similarly with the other cases we obtain

$$\begin{aligned} y(t) &= 10 + 10 \cdot 2\left[\frac{3}{\pi^2}(\frac{1}{2} - \cos(\frac{2\pi}{3}) + \frac{1}{\pi}\sin(\frac{2\pi}{3})\right]\cos(2\pi t) \\ &\quad + 5 \cdot 2\left[\frac{3}{4\pi^2}(\frac{1}{2} - \cos(\frac{4\pi}{3}) + \frac{1}{2\pi}\sin(\frac{4\pi}{3})\right]\cos(2\pi 2t) \\ &= 10 + 20\left[\frac{3}{\pi^2} + \frac{\sqrt{3}}{2\pi}\right]\cos(2\pi t) + 10\left[\frac{3}{4\pi^2} - \frac{\sqrt{3}}{4\pi}\right]\cos(2\pi 2t) \end{aligned}$$

2) In general

$$y(t) = \sum_{n=-\infty}^{\infty} x_n H(\frac{n}{T}) e^{j2\pi \frac{n}{T} t}$$

The DC component of the input signal and all frequencies higher than 4 will be cut off.

a) For this signal $T = 1$ and $x_n = \frac{1-j2\pi n}{1+4\pi^2 n^2}(1-e^{-1})$. Thus,

$$\begin{aligned} y(t) &= \frac{1-j2\pi}{1+4\pi^2}(1-e^{-1})(-j)e^{j2\pi t} + \frac{1-j2\pi 2}{1+4\pi^2 4}(1-e^{-1})(-j)e^{j2\pi 2t} \\ &\quad + \frac{1-j2\pi 3}{1+4\pi^2 9}(1-e^{-1})(-j)e^{j2\pi 3t} + \frac{1-j2\pi 4}{1+4\pi^2 16}(1-e^{-1})(-j)e^{j2\pi 4t} \\ &\quad + \frac{1+j2\pi}{1+4\pi^2}(1-e^{-1})je^{-j2\pi t} + \frac{1+j2\pi 2}{1+4\pi^2 4}(1-e^{-1})je^{-j2\pi 2t} \\ &\quad + \frac{1+j2\pi 3}{1+4\pi^2 9}(1-e^{-1})je^{-j2\pi 3t} + \frac{1+j2\pi 4}{1+4\pi^2 16}(1-e^{-1})je^{-j2\pi 4t} \\ &= (1-e^{-1})\sum_{n=1}^{4}\frac{2}{1+4\pi^2 n^2}(\sin(2\pi n t) - 2\pi n\cos(2\pi n t)) \end{aligned}$$

b) With $T = 2$ and $x_n = \frac{j}{\pi n}(-1)^n$ we obtain

$$\begin{aligned} y(t) &= \sum_{n=1}^{8}\frac{j}{\pi n}(-1)^n(-j)e^{j\pi n t} + \sum_{n=-8}^{-1}\frac{j}{\pi n}(-1)^n je^{j\pi n t} \\ &= \sum_{n=1}^{8}\frac{(-1)^n}{\pi n}e^{j\pi n t} + \sum_{n=-8}^{-1} -\frac{1}{\pi n}(-1)^n je^{j\pi n t} \end{aligned}$$

c) In this case

$$x_{2l} = 0, \qquad x_{2l+1} = \frac{1}{\pi(2l+1)}(-1)^l$$

Hence

$$\begin{aligned} y(t) &= \frac{1}{\pi}(-j)e^{j2\pi t} + \frac{1}{3\pi}(-1)(-j)e^{j2\pi 3t} \\ &\quad + \frac{1}{-\pi}(-1)je^{-j2\pi t} + \frac{1}{-3\pi}je^{-j2\pi 3t} \\ &= \frac{1}{2\pi}\sin(2\pi t) - \frac{1}{6\pi}\sin(2\pi 3t) \end{aligned}$$

d) $x_0 = \frac{2}{3}$ and $x_n = \frac{3}{2\pi n^2}(\cos(\frac{2\pi n}{3}) - 1)$. Thus

$$\begin{aligned} y(t) &= \sum_{n=1}^{4}\frac{3}{2\pi n^2}(\cos(\frac{2\pi n}{3}) - 1)(-j)e^{j2\pi n t} \\ &\quad + \sum_{n=-4}^{-1}\frac{3}{2\pi n^2}(\cos(\frac{2\pi n}{3}) - 1)je^{j2\pi n t} \end{aligned}$$

e) With $x_n = \frac{j}{\pi n}((-1)^n - \text{sinc}(\frac{n}{2}))$ we obtain

$$y(t) = \sum_{n=1}^{4}\frac{1}{\pi n}((-1)^n - \text{sinc}(\frac{n}{2})) + \sum_{n=-4}^{-1}\frac{-1}{\pi n}((-1)^n - \text{sinc}(\frac{n}{2}))$$

f) Working similarly with the other cases we obtain

$$\begin{aligned} y(t) &= \sum_{n=1}^{4}\left[\frac{3}{\pi^2 n^2}\left(\frac{1}{2}-\cos(\frac{2\pi n}{3})\right)+\frac{1}{\pi n}\sin(\frac{2\pi n}{3})\right](-j)e^{j2\pi nt} \\ &+ \sum_{n=-4}^{-1}\left[\frac{3}{\pi^2 n^2}\left(\frac{1}{2}-\cos(\frac{2\pi n}{3})\right)+\frac{1}{\pi n}\sin(\frac{2\pi n}{3})\right]je^{j2\pi nt} \end{aligned}$$

Problem 2.45

Using Parseval's relation (Equation 2.2.38), we see that the power in the periodic signal is given by $\sum_{n=-\infty}^{\infty}|x_n|^2$. Since the signal has finite power

$$\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0}|x(t)|^2 dt = K < \infty$$

Thus, $\sum_{n=-\infty}^{\infty}|x_n|^2 = K < \infty$. The last implies that $|x_n| \to 0$ as $n \to \infty$. To see this write

$$\sum_{n=-\infty}^{\infty}|x_n|^2 = \sum_{n=-\infty}^{-M}|x_n|^2 + \sum_{n=-M}^{M}|x_n|^2 + \sum_{n=M}^{\infty}|x_n|^2$$

Each of the previous terms is positive and bounded by K. Assume that $|x_n|^2$ does not converge to zero as n goes to infinity and choose $\epsilon = 1$. Then there exists a subsequence of x_n, x_{n_k}, such that

$$|x_{n_k}| > \epsilon = 1, \qquad \text{for } n_k > N \geq M$$

Then

$$\sum_{n=M}^{\infty}|x_n|^2 \geq \sum_{n=N}^{\infty}|x_n|^2 \geq \sum_{n_k}|x_{n_k}|^2 = \infty$$

This contradicts our assumption that $\sum_{n=M}^{\infty}|x_n|^2$ is finite. Thus $|x_n|$, and consequently x_n, should converge to zero as $n \to \infty$.

Problem 2.46

1) Using the Fourier transform pair

$$e^{-\alpha|t|} \xrightarrow{\mathcal{F}} \frac{2\alpha}{\alpha^2+(2\pi f)^2} = \frac{2\alpha}{4\pi^2}\frac{1}{\frac{\alpha^2}{4\pi^2}+f^2}$$

and the duality property of the Fourier transform: $X(f) = \mathcal{F}[x(t)] \Rightarrow x(-f) = \mathcal{F}[X(t)]$ we obtain

$$\left(\frac{2\alpha}{4\pi^2}\right)\mathcal{F}\left[\frac{1}{\frac{\alpha^2}{4\pi^2}+t^2}\right] = e^{-\alpha|f|}$$

With $\alpha = 2\pi$ we get the desired result

$$\mathcal{F}\left[\frac{1}{1+t^2}\right] = \pi e^{-2\pi|f|}$$

2)

$$\begin{aligned}\mathcal{F}[x(t)] &= \mathcal{F}[\Pi(t-3)+\Pi(t+3)]\\ &= \text{sinc}(f)e^{-j2\pi f3}+\text{sinc}(f)e^{j2\pi f3}\\ &= 2\text{sinc}(f)\cos(2\pi 3f)\end{aligned}$$

3) $\mathcal{F}[\Pi(t/4)] = 4\,\text{sinc}(4f)$, hence $\mathcal{F}[4\Pi(t/4)] = 16\,\text{sinc}(4f)$. Using modulation property of FT we have $\mathcal{F}[4\Pi(t/4)\cos(2\pi f_0 t)] = 8\,\text{sinc}(4(f-f_0)) + 8\,\text{sinc}(4(f+f_0))$.

4)

$$\mathcal{F}[t\text{sinc}(t)] = \frac{1}{\pi}\mathcal{F}[\sin(\pi t)] = \frac{j}{2\pi}\left[\delta(f+\frac{1}{2})-\delta(f-\frac{1}{2})\right]$$

The same result is obtain if we recognize that multiplication by t results in differentiation in the frequency domain. Thus

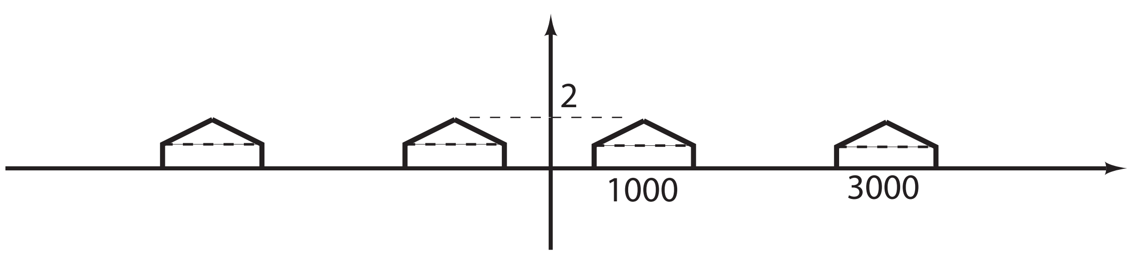
$$\mathcal{F}[t\text{sinc}] = \frac{j}{2\pi}\frac{d}{df}\Pi(f) = \frac{j}{2\pi}\left[\delta(f+\frac{1}{2})-\delta(f-\frac{1}{2})\right]$$

5)

$$\begin{aligned}\mathcal{F}[t\cos(2\pi f_0 t)] &= \frac{j}{2\pi}\frac{d}{df}\left(\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)\right)\\ &= \frac{j}{4\pi}\left(\delta'(f-f_0)+\delta'(f+f_0)\right)\end{aligned}$$

Problem 2.47

$x_1(t) = -x(t)+x(t)\cos(2000\pi t)+x(t)\,(1+\cos(6000\pi t))$ or $x_1(t) = x(t)\cos(2000\pi t)+x(t)\cos(6000\pi t)$. Using modulation property, we have $X_1(f) = \frac{1}{2}X(f-1000)+\frac{1}{2}X(f+1000)+\frac{1}{2}X(f-3000)+\frac{1}{2}X(f+3000)$. The plot is given below:



Problem 2.48

$$\begin{aligned}\mathcal{F}[\frac{1}{2}(\delta(t+\frac{1}{2})+\delta(t-\frac{1}{2}))] &= \int_{-\infty}^{\infty}\frac{1}{2}(\delta(t+\frac{1}{2})+\delta(t-\frac{1}{2}))e^{-j2\pi ft}dt \\ &= \frac{1}{2}(e^{-j\pi f}+e^{-j\pi f}) = \cos(\pi f)\end{aligned}$$

Using the duality property of the Fourier transform:

$$X(f) = \mathcal{F}[x(t)] \Longrightarrow x(f) = \mathcal{F}[X(-t)]$$

we obtain

$$\mathcal{F}[\cos(-\pi t)] = \mathcal{F}[\cos(\pi t)] = \frac{1}{2}(\delta(f+\frac{1}{2})+\delta(f-\frac{1}{2}))$$

Note that $\sin(\pi t) = \cos(\pi t + \frac{\pi}{2})$. Thus

$$\begin{aligned}\mathcal{F}[\sin(\pi t)] &= \mathcal{F}[\cos(\pi(t+\frac{1}{2}))] = \frac{1}{2}(\delta(f+\frac{1}{2})+\delta(f-\frac{1}{2}))e^{j\pi f} \\ &= \frac{1}{2}e^{j\pi\frac{1}{2}}\delta(f+\frac{1}{2})+\frac{1}{2}e^{-j\pi\frac{1}{2}}\delta(f-\frac{1}{2}) \\ &= \frac{j}{2}\delta(f+\frac{1}{2})-\frac{j}{2}\delta(f-\frac{1}{2})\end{aligned}$$

Problem 2.49

a) We can write $x(t)$ as $x(t) = 2\Pi(\frac{t}{4}) - 2\Lambda(\frac{t}{2})$. Then

$$\mathcal{F}[x(t)] = \mathcal{F}[2\Pi(\frac{t}{4})] - \mathcal{F}[2\Lambda(\frac{t}{2})] = 8\text{sinc}(4f) - 4\text{sinc}^2(2f)$$

b)

$$x(t) = 2\Pi(\frac{t}{4}) - \Lambda(t) \Longrightarrow \mathcal{F}[x(t)] = 8\text{sinc}(4f) - \text{sinc}^2(f)$$

c)

$$\begin{aligned}X(f) &= \int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}dt = \int_{-1}^{0}(t+1)e^{-j2\pi ft}dt + \int_{0}^{1}(t-1)e^{-j2\pi ft}dt \\ &= \left(\frac{j}{2\pi f}t+\frac{1}{4\pi^2f^2}\right)e^{-j2\pi ft}\Big|_{-1}^{0} + \frac{j}{2\pi f}e^{-j2\pi ft}\Big|_{-1}^{0} \\ &\quad + \left(\frac{j}{2\pi f}t+\frac{1}{4\pi^2f^2}\right)e^{-j2\pi ft}\Big|_{0}^{1} - \frac{j}{2\pi f}e^{-j2\pi ft}\Big|_{0}^{1} \\ &= \frac{j}{\pi f}(1-\sin(\pi f))\end{aligned}$$

d) We can write $x(t)$ as $x(t) = \Lambda(t+1) - \Lambda(t-1)$. Thus

$$X(f) = \text{sinc}^2(f)e^{j2\pi f} - \text{sinc}^2(f)e^{-j2\pi f} = 2j\text{sinc}^2(f)\sin(2\pi f)$$

e) We can write $x(t)$ as $x(t) = \Lambda(t+1) + \Lambda(t) + \Lambda(t-1)$. Hence,

$$X(f) = \text{sinc}^2(f)(1 + e^{j2\pi f} + e^{-j2\pi f}) = \text{sinc}^2(f)(1 + 2\cos(2\pi f)$$

f) We can write $x(t)$ as

$$x(t) = \left[\Pi\left(2f_0(t - \frac{1}{4f_0})\right) - \Pi\left(2f_0(t - \frac{1}{4f_0})\right)\right]\sin(2\pi f_0 t)$$

Then

$$\begin{aligned} X(f) &= \left[\frac{1}{2f_0}\text{sinc}\left(\frac{f}{2f_0}\right)e^{-j2\pi\frac{1}{4f_0}f} - \frac{1}{2f_0}\text{sinc}\left(\frac{f}{2f_0})\right)e^{j2\pi\frac{1}{4f_0}f}\right] \\ &\quad \star \frac{j}{2}(\delta(f+f_0) - \delta(f+f_0)) \\ &= \frac{1}{2f_0}\text{sinc}\left(\frac{f+f_0}{2f_0}\right)\sin\left(\pi\frac{f+f_0}{2f_0}\right) - \frac{1}{2f_0}\text{sinc}\left(\frac{f-f_0}{2f_0}\right)\sin\left(\pi\frac{f-f_0}{2f_0}\right) \end{aligned}$$

Problem 2.50
(Convolution theorem:)

$$\mathcal{F}[x(t) \star y(t)] = \mathcal{F}[x(t)]\mathcal{F}[y(t)] = X(f)Y(f)$$

Thus

$$\begin{aligned} \text{sinc}(t) \star \text{sinc}(t) &= \mathcal{F}^{-1}[\mathcal{F}[\text{sinc}(t) \star \text{sinc}(t)]] \\ &= \mathcal{F}^{-1}[\mathcal{F}[\text{sinc}(t)] \cdot \mathcal{F}[\text{sinc}(t)]] \\ &= \mathcal{F}^{-1}[\Pi(f)\Pi(f)] = \mathcal{F}^{-1}[\Pi(f)] \\ &= \text{sinc}(t) \end{aligned}$$

Problem 2.51

$$\begin{aligned} \mathcal{F}[x(t)y(t)] &= \int_{-\infty}^{\infty} x(t)y(t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} X(\theta)e^{j2\pi\theta t}d\theta\right)y(t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} X(\theta)\left(\int_{-\infty}^{\infty} y(t)e^{-j2\pi(f-\theta)t}dt\right)d\theta \\ &= \int_{-\infty}^{\infty} X(\theta)Y(f-\theta)d\theta = X(f) \star Y(f) \end{aligned}$$

Problem 2.52

1) Clearly

$$\begin{aligned} x_1(t+kT_0) &= \sum_{n=-\infty}^{\infty} x(t+kT_0-nT_0) = \sum_{n=-\infty}^{\infty} x(t-(n-k)T_0) \\ &= \sum_{m=-\infty}^{\infty} x(t-mT_0) = x_1(t) \end{aligned}$$

where we used the change of variable $m = n - k$.

2)

$$x_1(t) = x(t) \star \sum_{n=-\infty}^{\infty} \delta(t-nT_0)$$

This is because

$$\int_{-\infty}^{\infty} x(\tau) \sum_{n=-\infty}^{\infty} \delta(t-\tau-nT_0)d\tau = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-nT_0)d\tau = \sum_{n=-\infty}^{\infty} x(t-nT_0)$$

3)

$$\begin{aligned} \mathcal{F}[x_1(t)] &= \mathcal{F}[x(t) \star \sum_{n=-\infty}^{\infty} \delta(t-nT_0)] = \mathcal{F}[x(t)]\mathcal{F}[\sum_{n=-\infty}^{\infty} \delta(t-nT_0)] \\ &= X(f)\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f-\frac{n}{T_0}) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_0})\delta(f-\frac{n}{T_0}) \end{aligned}$$

Problem 2.53

1) By Parseval's theorem

$$\int_{-\infty}^{\infty} \text{sinc}^5(t)dt = \int_{-\infty}^{\infty} \text{sinc}^3(t)\text{sinc}^2(t)dt = \int_{-\infty}^{\infty} \Lambda(f)T(f)df$$

where

$$T(f) = \mathcal{F}[\text{sinc}^3(t)] = \mathcal{F}[\text{sinc}^2(t)\text{sinc}(t)] = \Pi(f) \star \Lambda(f)$$

But

$$\Pi(f) \star \Lambda(f) = \int_{-\infty}^{\infty} \Pi(\theta)\Lambda(f-\theta)d\theta = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Lambda(f-\theta)d\theta = \int_{f-\frac{1}{2}}^{f+\frac{1}{2}} \Lambda(v)dv$$

For $f \le -\frac{3}{2} \Longrightarrow T(f) = 0$

For $-\frac{3}{2} < f \le -\frac{1}{2} \Longrightarrow T(f) = \int_{-1}^{f+\frac{1}{2}} (v+1)dv = (\frac{1}{2}v^2 + v)\Big|_{-1}^{f+\frac{1}{2}} = \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8}$

For $-\frac{1}{2} < f \le \frac{1}{2} \Longrightarrow T(f) = \int_{f-\frac{1}{2}}^{0} (v+1)dv + \int_{0}^{f+\frac{1}{2}} (-v+1)dv$

$$= (\frac{1}{2}v^2 + v)\Big|_{f-\frac{1}{2}}^{0} + (-\frac{1}{2}v^2 + v)\Big|_{0}^{f+\frac{1}{2}} = -f^2 + \frac{3}{4}$$

For $\frac{1}{2} < f \le \frac{3}{2} \Longrightarrow T(f) = \int_{f-\frac{1}{2}}^{1} (-v+1)dv = (-\frac{1}{2}v^2 + v)\Big|_{f-\frac{1}{2}}^{1} = \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8}$

For $\frac{3}{2} < f \Longrightarrow T(f) = 0$

Thus,

$$T(f) = \begin{cases} 0 & f \le -\frac{3}{2} \\ \frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8} & -\frac{3}{2} < f \le -\frac{1}{2} \\ -f^2 + \frac{3}{4} & -\frac{1}{2} < f \le \frac{1}{2} \\ \frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8} & \frac{1}{2} < f \le \frac{3}{2} \\ 0 & \frac{3}{2} < f \end{cases}$$

Hence,

$$\begin{aligned} \int_{-\infty}^{\infty} \Lambda(f)T(f)df &= \int_{-1}^{-\frac{1}{2}} (\frac{1}{2}f^2 + \frac{3}{2}f + \frac{9}{8})(f+1)df + \int_{-\frac{1}{2}}^{0} (-f^2 + \frac{3}{4})(f+1)df \\ &\quad + \int_{0}^{\frac{1}{2}} (-f^2 + \frac{3}{4})(-f+1)df + \int_{\frac{1}{2}}^{1} (\frac{1}{2}f^2 - \frac{3}{2}f + \frac{9}{8})(-f+1)df \\ &= \frac{41}{64} \end{aligned}$$

2)

$$\begin{aligned} \int_{0}^{\infty} e^{-\alpha t}\text{sinc}(t)dt &= \int_{-\infty}^{\infty} e^{-\alpha t}u_{-1}(t)\text{sinc}(t)dt \\ &= \int_{-\infty}^{\infty} \frac{1}{\alpha + j2\pi f}\Pi(f)df = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\alpha + j2\pi f}df \\ &= \frac{1}{j2\pi}\ln(\alpha + j2\pi f)\Big|_{-1/2}^{1/2} = \frac{1}{j2\pi}\ln(\frac{\alpha + j\pi}{\alpha - j\pi}) = \frac{1}{\pi}\tan^{-1}\frac{\pi}{\alpha} \end{aligned}$$

3)

$$\begin{aligned} \int_{0}^{\infty} e^{-\alpha t}\cos(\beta t)dt &= \int_{-\infty}^{\infty} e^{-\alpha t}u_{-1}(t)\cos(\beta t)dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{\alpha + j2\pi f}(\delta(f - \frac{\beta}{2\pi}) + \delta(f + \frac{\beta}{2\pi}))dt \\ &= \frac{1}{2}[\frac{1}{\alpha + j\beta} + \frac{1}{\alpha - j\beta}] = \frac{\alpha}{\alpha^2 + \beta^2} \end{aligned}$$

Problem 2.54
Using the convolution theorem we obtain

$$\begin{aligned} Y(f) &= X(f)H(f) = (\frac{1}{\alpha + j2\pi f})(\frac{1}{\beta + j2\pi f}) \\ &= \frac{1}{(\beta-\alpha)}\frac{1}{\alpha + j2\pi f} - \frac{1}{(\beta-\alpha)}\frac{1}{\beta + j2\pi f} \end{aligned}$$

Thus

$$y(t) = \mathcal{F}^{-1}[Y(f)] = \frac{1}{(\beta-\alpha)}[e^{-\alpha t} - e^{-\beta t}]u_{-1}(t)$$

If $\alpha = \beta$ then $X(f) = H(f) = \frac{1}{\alpha + j2\pi f}$. In this case

$$y(t) = \mathcal{F}^{-1}[Y(f)] = \mathcal{F}^{-1}[(\frac{1}{\alpha + j2\pi f})^2] = te^{-\alpha t}u_{-1}(t)$$

The signal is of the energy-type with energy content

$$\begin{aligned} E_y &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} |y(t)|^2 dt = \lim_{T\to\infty}\int_0^{\frac{T}{2}} \frac{1}{(\beta-\alpha)^2}(e^{-\alpha t} - e^{-\beta t})^2 dt \\ &= \lim_{T\to\infty}\frac{1}{(\beta-\alpha)^2}\left[-\frac{1}{2\alpha}e^{-2\alpha t}\Big|_0^{T/2} - \frac{1}{2\beta}e^{-2\beta t}\Big|_0^{T/2} + \frac{2}{(\alpha+\beta)}e^{-(\alpha+\beta)t}\Big|_0^{T/2}\right] \\ &= \frac{1}{(\beta-\alpha)^2}[\frac{1}{2\alpha} + \frac{1}{2\beta} - \frac{2}{\alpha+\beta}] = \frac{1}{2\alpha\beta(\alpha+\beta)} \end{aligned}$$

Problem 2.55

$$x_\alpha(t) = \begin{cases} x(t) & \alpha \le t < \alpha + T_0 \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$X_\alpha(f) = \int_{-\infty}^{\infty} x_\alpha(t)e^{-j2\pi f t}dt = \int_\alpha^{\alpha+T_0} x(t)e^{-j2\pi f t}dt$$

Evaluating $X_\alpha(f)$ for $f = \frac{n}{T_0}$ we obtain

$$X_\alpha(\frac{n}{T_0}) = \int_\alpha^{\alpha+T_0} x(t)e^{-j2\pi\frac{n}{T_0}t}dt = T_0 x_n$$

where x_n are the coefficients in the Fourier series expansion of $x(t)$. Thus $X_\alpha(\frac{n}{T_0})$ is independent of the choice of α.

Problem 2.56

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x(t-nT_s) &= x(t) \star \sum_{n=-\infty}^{\infty} \delta(t-nT_s) = \frac{1}{T_s} x(t) \star \sum_{n=-\infty}^{\infty} e^{j2\pi \frac{n}{T_s} t} \\
&= \frac{1}{T_s} \mathcal{F}^{-1}\left[X(f) \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T_s})\right] \\
&= \frac{1}{T_s} \mathcal{F}^{-1}\left[\sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right) \delta(f - \frac{n}{T_s})\right] \\
&= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right) e^{j2\pi \frac{n}{T_s} t}
\end{aligned}$$

If we set $t = 0$ in the previous relation we obtain Poisson's sum formula

$$\sum_{n=-\infty}^{\infty} x(-nT_s) = \sum_{m=-\infty}^{\infty} x(mT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_s}\right)$$

Problem 2.57

1) We know that

$$e^{-\alpha|t|} \xrightarrow{\mathcal{F}} \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$$

Applying Poisson's sum formula with $T_s = 1$ we obtain

$$\sum_{n=-\infty}^{\infty} e^{-\alpha|n|} = \sum_{n=-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + 4\pi^2 n^2}$$

2) Use the Fourier transform pair $\Pi(t) \to \text{sinc}(f)$ in the Poisson's sum formula with $T_s = K$. Then

$$\sum_{n=-\infty}^{\infty} \Pi(nK) = \frac{1}{K} \sum_{n=-\infty}^{\infty} \text{sinc}(\frac{n}{K})$$

But $\Pi(nK) = 1$ for $n = 0$ and $\Pi(nK) = 0$ for $|n| \geq 1$ and $K \in \{1, 2, \ldots\}$. Thus the left side of the previous relation reduces to 1 and

$$K = \sum_{n=-\infty}^{\infty} \text{sinc}(\frac{n}{K})$$

3) Use the Fourier transform pair $\Lambda(t) \to \text{sinc}^2(f)$ in the Poisson's sum formula with $T_s = K$. Then

$$\sum_{n=-\infty}^{\infty} \Lambda(nK) = \frac{1}{K} \sum_{n=-\infty}^{\infty} \text{sinc}^2(\frac{n}{K})$$

Reasoning as before we see that $\sum_{n=-\infty}^{\infty} \Lambda(nK) = 1$ since for $K \in \{1, 2, \ldots\}$

$$\Lambda(nK) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, $K = \sum_{n=-\infty}^{\infty} \text{sinc}^2(\frac{n}{K})$

Problem 2.58

Let $H(f)$ be the Fourier transform of $h(t)$. Then

$$H(f)\mathcal{F}[e^{-\alpha t}u_{-1}(t)] = \mathcal{F}[\delta(t)] \Longrightarrow H(f)\frac{1}{\alpha + j2\pi f} = 1 \Longrightarrow H(f) = \alpha + j2\pi f$$

The response of the system to $e^{-\alpha t}\cos(\beta t)u_{-1}(t)$ is

$$y(t) = \mathcal{F}^{-1}\left[H(f)\mathcal{F}[e^{-\alpha t}\cos(\beta t)u_{-1}(t)]\right]$$

But

$$\begin{aligned}\mathcal{F}[e^{-\alpha t}\cos(\beta t)u_{-1}(t)] &= \mathcal{F}[\frac{1}{2}e^{-\alpha t}u_{-1}(t)e^{j\beta t} + \frac{1}{2}e^{-\alpha t}u_{-1}(t)e^{-j\beta t}] \\ &= \frac{1}{2}\left[\frac{1}{\alpha + j2\pi(f - \frac{\beta}{2\pi})} + \frac{1}{\alpha + j2\pi(f + \frac{\beta}{2\pi})}\right]\end{aligned}$$

so that

$$Y(f) = \mathcal{F}[y(t)] = \frac{\alpha + j2\pi f}{2}\left[\frac{1}{\alpha + j2\pi(f - \frac{\beta}{2\pi})} + \frac{1}{\alpha + j2\pi(f + \frac{\beta}{2\pi})}\right]$$

Using the linearity property of the Fourier transform, the Convolution theorem and the fact that $\delta'(t) \xrightarrow{\mathcal{F}} j2\pi f$ we obtain

$$\begin{aligned}y(t) &= \alpha e^{-\alpha t}\cos(\beta t)u_{-1}(t) + (e^{-\alpha t}\cos(\beta t)u_{-1}(t)) \star \delta'(t) \\ &= e^{-\alpha t}\cos(\beta t)\delta(t) - \beta e^{-\alpha t}\sin(\beta t)u_{-1}(t) \\ &= \delta(t) - \beta e^{-\alpha t}\sin(\beta t)u_{-1}(t)\end{aligned}$$

Problem 2.59

1) Using the result of Problem 2.50 we have $\text{sinc}(t) \star \text{sinc}(t) = \text{sinc}(t)$.

2)

$$\begin{aligned}y(t) &= x(t) \star h(t) = x(t) \star (\delta(t) + \delta'(t)) \\ &= x(t) + \frac{d}{dt}x(t)\end{aligned}$$

With $x(t) = e^{-\alpha|t|}$ we obtain $y(t) = e^{-\alpha|t|} - \alpha e^{-\alpha|t|}\text{sgn}(t)$.

3)

$$\begin{aligned}y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_0^t e^{-\alpha\tau}e^{-\beta(t-\tau)}d\tau = e^{-\beta t}\int_0^t e^{-(\alpha-\beta)\tau}d\tau\end{aligned}$$

If $\quad \alpha = \beta \Rightarrow y(t) = te^{-\alpha t}u_{-1}(t)$

$$\alpha \neq \beta \Rightarrow y(t) = e^{-\beta t}\frac{1}{\beta-\alpha}e^{-(\alpha-\beta)t}\Big|_0^t u_{-1}(t) = \frac{1}{\beta-\alpha}\left[e^{-\alpha t} - e^{-\beta t}\right]u_{-1}(t)$$

Problem 2.60

Let the response of the LTI system be $h(t)$ with Fourier transform $H(f)$. Then, from the convolution theorem we obtain

$$Y(f) = H(f)X(f) \Longrightarrow \Lambda(f) = \Pi(f)H(f)$$

However, this relation cannot hold since $\Pi(f) = 0$ for $\frac{1}{2} < |f|$ whereas $\Lambda(f) \neq 0$ for $1 < |f| \leq 1/2$.

Problem 2.61

1) No. The input $\Pi(t)$ has a spectrum with zeros at frequencies $f = k$, ($k \neq 0$, $k \in \mathcal{Z}$) and the information about the spectrum of the system at those frequencies will not be present at the output. The spectrum of the signal $\cos(2\pi t)$ consists of two impulses at $f = \pm 1$ but we do not know the response of the system at these frequencies.

2)

$$\begin{aligned}
h_1(t) \star \Pi(t) &= \Pi(t) \star \Pi(t) = \Lambda(t) \\
h_2(t) \star \Pi(t) &= (\Pi(t) + \cos(2\pi t)) \star \Pi(t) \\
&= \Lambda(t) + \frac{1}{2}\mathcal{F}^{-1}\left[\delta(f-1)\text{sinc}^2(f) + \delta(f+1)\text{sinc}^2(f)\right] \\
&= \Lambda(t) + \frac{1}{2}\mathcal{F}^{-1}\left[\delta(f-1)\text{sinc}^2(1) + \delta(f+1)\text{sinc}^2(-1)\right] \\
&= \Lambda(t)
\end{aligned}$$

Thus both signals are candidates for the impulse response of the system.

3) $\mathcal{F}[u_{-1}(t)] = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$. Thus the system has a nonzero spectrum for every f and all the frequencies of the system will be excited by this input. $\mathcal{F}[e^{-at}u_{-1}(t)] = \frac{1}{a+j2\pi f}$. Again the spectrum is nonzero for all f and the response to this signal uniquely determines the system. In general the spectrum of the input must not vanish at any frequency. In this case the influence of the system will be present at the output for every frequency.

Problem 2.62

$$\begin{aligned}
\mathcal{F}[A\sin(\widehat{2\pi f_0 t}+\theta)] &= -j\text{sgn}(f)A\left[-\frac{1}{2j}\delta(f+f_0)e^{j2\pi f\frac{\theta}{2f_0}}+\frac{1}{2j}\delta(f-f_0)e^{-j2\pi f\frac{\theta}{2f_0}}\right] \\
&= \frac{A}{2}\left[\text{sgn}(-f_0)\delta(f+f_0)e^{j2\pi f\frac{\theta}{2f_0}}-\text{sgn}(-f_0)\delta(f-f_0)e^{-j2\pi f\frac{\theta}{2f_0}}\right] \\
&= -\frac{A}{2}\left[\delta(f+f_0)e^{j2\pi f\frac{\theta}{2f_0}}+\delta(f-f_0)e^{-j2\pi f\frac{\theta}{2f_0}}\right] \\
&= -A\mathcal{F}[\cos(2\pi f_0 t+\theta)]
\end{aligned}$$

Thus, $A\sin(\widehat{2\pi f_0 t}+\theta) = -A\cos(2\pi f_0 t+\theta)$

Problem 2.63

Taking the Fourier transform of $\widehat{e^{j2\pi f_0 t}}$ we obtain

$$\mathcal{F}[\widehat{e^{j2\pi f_0 t}}] = -j\text{sgn}(f)\delta(f-f_0) = -j\text{sgn}(f_0)\delta(f-f_0)$$

Thus,

$$\widehat{e^{j2\pi f_0 t}} = \mathcal{F}^{-1}[-j\text{sgn}(f_0)\delta(f-f_0)] = -j\text{sgn}(f_0)e^{j2\pi f_0 t}$$

Problem 2.64

$$\begin{aligned}
\mathcal{F}\left[\widehat{\frac{d}{dt}x(t)}\right] &= \mathcal{F}[x(t)\widehat{\star\,\delta'}(t)] = -j\text{sgn}(f)\mathcal{F}[x(t)\star\delta'(t)] \\
&= -j\text{sgn}(f)j2\pi f X(f) = 2\pi f\text{sgn}(f)X(f) \\
&= 2\pi|f|X(f)
\end{aligned}$$

Problem 2.65

We need to prove that $\widehat{x'(t)} = (\hat{x}(t))'$.

$$\begin{aligned}
\mathcal{F}[\widehat{x'(t)}] &= \mathcal{F}[x(t)\widehat{\star\,\delta'}(t)] = -j\text{sgn}(f)\mathcal{F}[x(t)\star\delta'(t)] = -j\text{sgn}(f)X(f)j2\pi f \\
&= \mathcal{F}[\hat{x}(t)]j2\pi f = \mathcal{F}[(\hat{x}(t))']
\end{aligned}$$

Taking the inverse Fourier transform of both sides of the previous relation we obtain, $\widehat{x'(t)} = (\hat{x}(t))'$

Problem 2.66

1) The spectrum of the output signal $y(t)$ is the product of $X(f)$ and $H(f)$. Thus,

$$Y(f) = H(f)X(f) = X(f)A(f_0)e^{j(\theta(f_0)+(f-f_0)\theta'(f)|_{f=f_0})}$$

$y(t)$ is a narrowband signal centered at frequencies $f = \pm f_0$. To obtain the lowpass equivalent signal we have to shift the spectrum (positive band) of $y(t)$ to the right by f_0. Hence,

$$Y_l(f) = u(f+f_0)X(f+f_0)A(f_0)e^{j(\theta(f_0)+f\theta'(f)|_{f=f_0})} = X_l(f)A(f_0)e^{j(\theta(f_0)+f\theta'(f)|_{f=f_0})}$$

2) Taking the inverse Fourier transform of the previous relation, we obtain

$$\begin{aligned} y_l(t) &= \mathcal{F}^{-1}\left[X_l(f)A(f_0)e^{j\theta(f_0)}e^{jf\theta'(f)|_{f=f_0}}\right] \\ &= A(f_0)x_l(t+\frac{1}{2\pi}\theta'(f)|_{f=f_0}) \end{aligned}$$

With $y(t) = \text{Re}[y_l(t)e^{j2\pi f_0 t}]$ and $x_l(t) = V_x(t)e^{j\Theta_x(t)}$ we get

$$\begin{aligned} y(t) &= \text{Re}[y_l(t)e^{j2\pi f_0 t}] \\ &= \text{Re}\left[A(f_0)x_l(t+\frac{1}{2\pi}\theta'(f)|_{f=f_0})e^{j\theta(f_0)}e^{j2\pi f_0 t}\right] \\ &= \text{Re}\left[A(f_0)V_x(t+\frac{1}{2\pi}\theta'(f)|_{f=f_0})e^{j2\pi f_0 t}e^{j\Theta_x(t+\frac{1}{2\pi}\theta'(f)|_{f=f_0})}\right] \\ &= A(f_0)V_x(t-t_g)\cos(2\pi f_0 t+\theta(f_0)+\Theta_x(t+\frac{1}{2\pi}\theta'(f)|_{f=f_0})) \\ &= A(f_0)V_x(t-t_g)\cos(2\pi f_0(t+\frac{\theta(f_0)}{2\pi f_0})+\Theta_x(t+\frac{1}{2\pi}\theta'(f)|_{f=f_0})) \\ &= A(f_0)V_x(t-t_g)\cos(2\pi f_0(t-t_p)+\Theta_x(t+\frac{1}{2\pi}\theta'(f)|_{f=f_0})) \end{aligned}$$

where

$$t_g = -\frac{1}{2\pi}\theta'(f)|_{f=f_0}, \qquad t_p = -\frac{1}{2\pi}\frac{\theta(f_0)}{f_0} = -\frac{1}{2\pi}\frac{\theta(f)}{f}\bigg|_{f=f_0}$$

3) t_g can be considered as a time lag of the envelope of the signal, whereas t_p is the time corresponding to a phase delay of $\frac{1}{2\pi}\frac{\theta(f_0)}{f_0}$.

Problem 2.67

1) We can write $H_\theta(f)$ as follows

$$H_\theta(f) = \begin{cases} \cos\theta - j\sin\theta & f > 0 \\ 0 & f = 0 \\ \cos\theta + j\sin\theta & f < 0 \end{cases} = \cos\theta - j\text{sgn}(f)\sin\theta$$

Thus,

$$h_\theta(t) = \mathcal{F}^{-1}[H_\theta(f)] = \cos\theta\delta(t) + \frac{1}{\pi t}\sin\theta$$

2)

$$\begin{aligned} x_\theta(t) &= x(t) \star h_\theta(t) = x(t) \star (\cos\theta\delta(t) + \frac{1}{\pi t}\sin\theta) \\ &= \cos\theta x(t) \star \delta(t) + \sin\theta\frac{1}{\pi t} \star x(t) \\ &= \cos\theta x(t) + \sin\theta\hat{x}(t) \end{aligned}$$

3)

$$\begin{aligned} \int_{-\infty}^{\infty} |x_\theta(t)|^2 dt &= \int_{-\infty}^{\infty} |\cos\theta x(t) + \sin\theta\hat{x}(t)|^2 dt \\ &= \cos^2\theta\int_{-\infty}^{\infty}|x(t)|^2 dt + \sin^2\theta\int_{-\infty}^{\infty}|\hat{x}(t)|^2 dt \\ &\quad + \cos\theta\sin\theta\int_{-\infty}^{\infty} x(t)\hat{x}^*(t)dt + \cos\theta\sin\theta\int_{-\infty}^{\infty} x^*(t)\hat{x}(t)dt \end{aligned}$$

But $\int_{-\infty}^{\infty}|x(t)|^2 dt = \int_{-\infty}^{\infty}|\hat{x}(t)|^2 dt = E_x$ and $\int_{-\infty}^{\infty} x(t)\hat{x}^*(t)dt = 0$ since $x(t)$ and $\hat{x}(t)$ are orthogonal. Thus,

$$E_{x_\theta} = E_x(\cos^2\theta + \sin^2\theta) = E_x$$

Computer Problems

Computer Problem 2.1

1) To derive the Fourier series coefficients in the expansion of $x(t)$, we have

$$\begin{aligned} x_n &= \frac{1}{4}\int_{-1}^{1} e^{-j2\pi nt/4}\, dt \\ &= \frac{1}{-2j\pi n}\left[e^{-j2\pi n/4} - e^{j2\pi n/4}\right] \quad (2.1) \\ &= \frac{1}{2}\,\mathrm{sinc}\left(\frac{n}{2}\right) \quad (2.2) \end{aligned}$$

where sinc(x) is defined as

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \quad (2.3)$$

2) Obviously, all the x_n's are real (since $x(t)$ is real and even), so

$$\begin{cases} a_n = \mathrm{sinc}\left(\frac{n}{2}\right) \\ b_n = 0 \\ c_n = \left|\mathrm{sinc}\left(\frac{n}{2}\right)\right| \\ \theta_n = 0, \pi \end{cases} \quad (2.4)$$

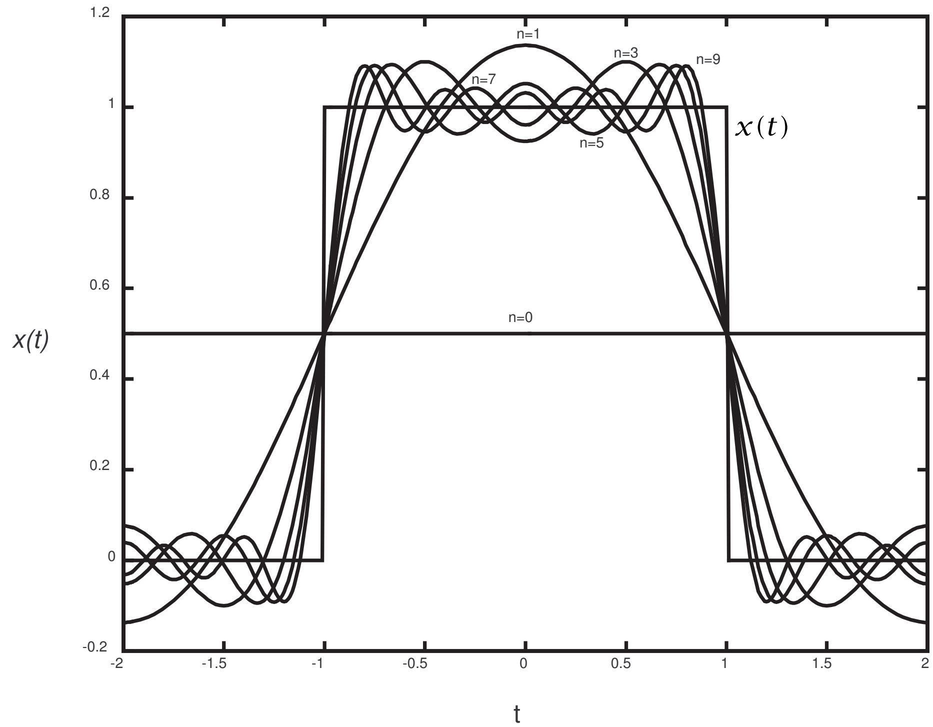


Figure 2.1: Various Fourier series approximations for the rectangular pulse

Note that for even n's, $x_n = 0$ (with the exception of $n = 0$, where $a_0 = c_0 = 1$ and $x_0 = \frac{1}{2}$). Using these coefficients, we have

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} \frac{1}{2} \operatorname{sinc}\left(\frac{n}{2}\right) e^{j2\pi nt/4} \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \operatorname{sinc}\left(\frac{n}{2}\right) \cos\left(2\pi t \frac{n}{4}\right) \end{aligned} \tag{2.5}$$

A plot of the Fourier series approximations to this signal over one period for $n = 0, 1, 3, 5, 7, 9$ is shown in Figure 2.1.

3) Note that x_n is always real. Therefore, depending on its sign, the phase is either zero or π. The magnitude of the x_n's is $\frac{1}{2}\left|\operatorname{sinc}\left(\frac{n}{2}\right)\right|$. The discrete and phase spectrum are shown in Figure 2.2.

Computer Problem 2.2

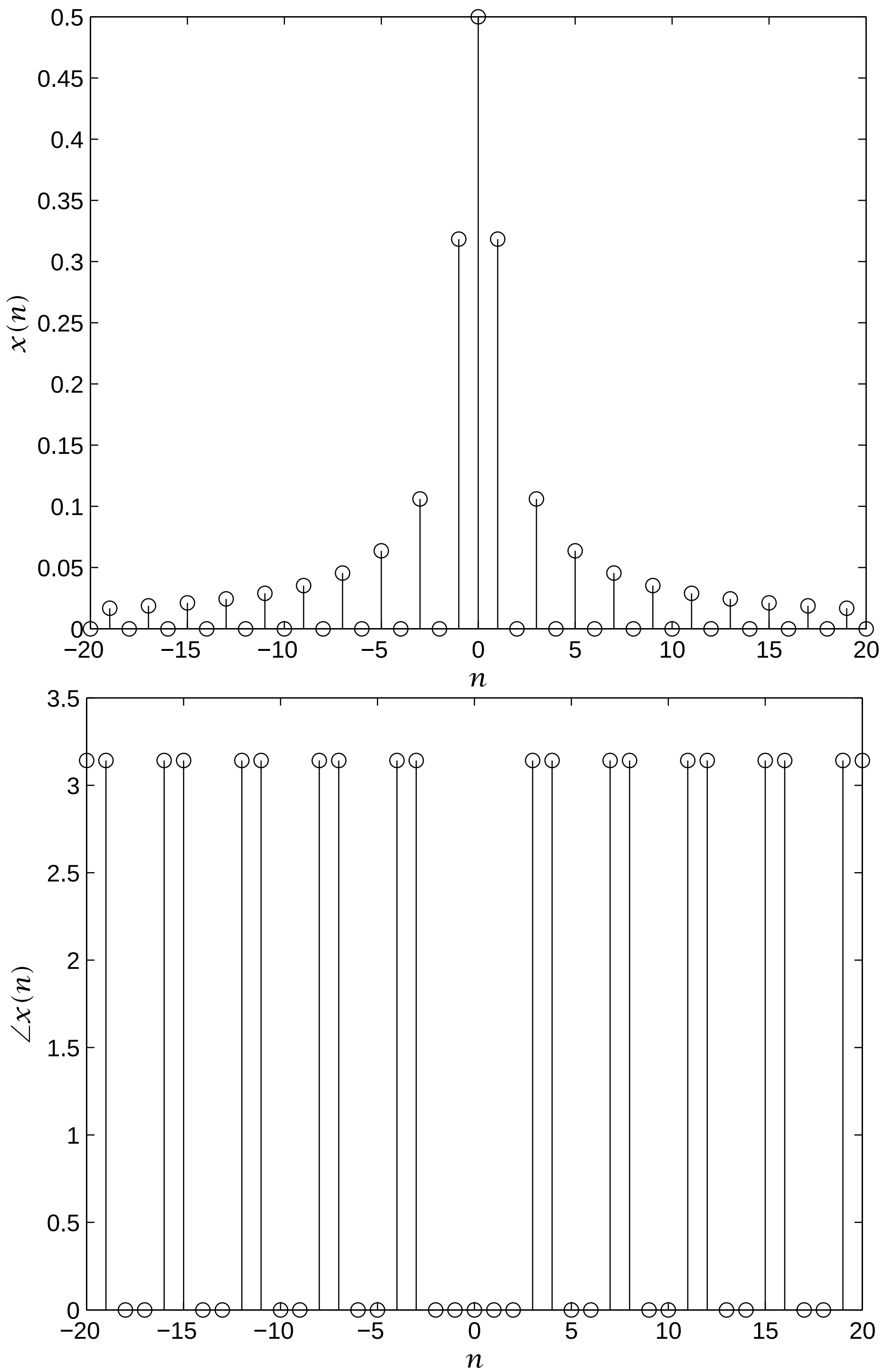


Figure 2.2: The discrete and phase spectrum of the signal

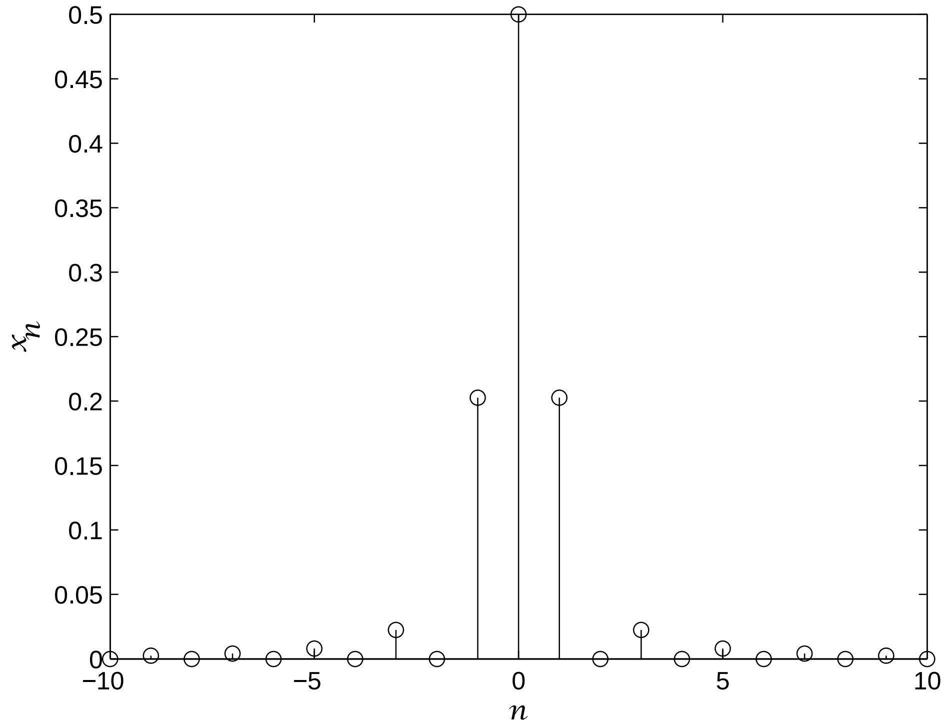


Figure 2.3: The discrete spectrum of the signal

1) We have

$$\begin{aligned} x_n &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j2\pi nt/T_0}dt && (2.6)\\ &= \frac{1}{2}\int_{-1}^{1}\Lambda(t)e^{-j\pi nt}dt && (2.7)\\ &= \frac{1}{2}\int_{-\infty}^{+\infty}\Lambda(t)e^{-j\pi nt}dt && (2.8)\\ &= \frac{1}{2}\mathcal{F}[\Lambda(t)]_{f=n/2} && (2.9)\\ &= \frac{1}{2}\,\text{sinc}^2(\frac{n}{2}) && (2.10)\\ & && (2.11) \end{aligned}$$

where we have used the facts that $\Lambda(t)$ vanishes outside the $[-1,1]$ interval and that the Fourier transform of $\Lambda(t)$ is $\text{sinc}^2(f)$. This result can also be obtained by using the expression for $\Lambda(t)$ and integrating by parts. Obviously, we have $x_n = 0$ for all even values of n except for $n = 0$.
2) A plot of the discrete spectrum of $x(t)$ is presented in Figure 2.3
3) A plot of the discrete spectrum $\{y_n\}$ is presented in Figure 2.4
The MATLAB script for this problem is given next.

```
% MATLAB script for Computer Problem 2.2.
echo on
n=[−20:1:20];
```

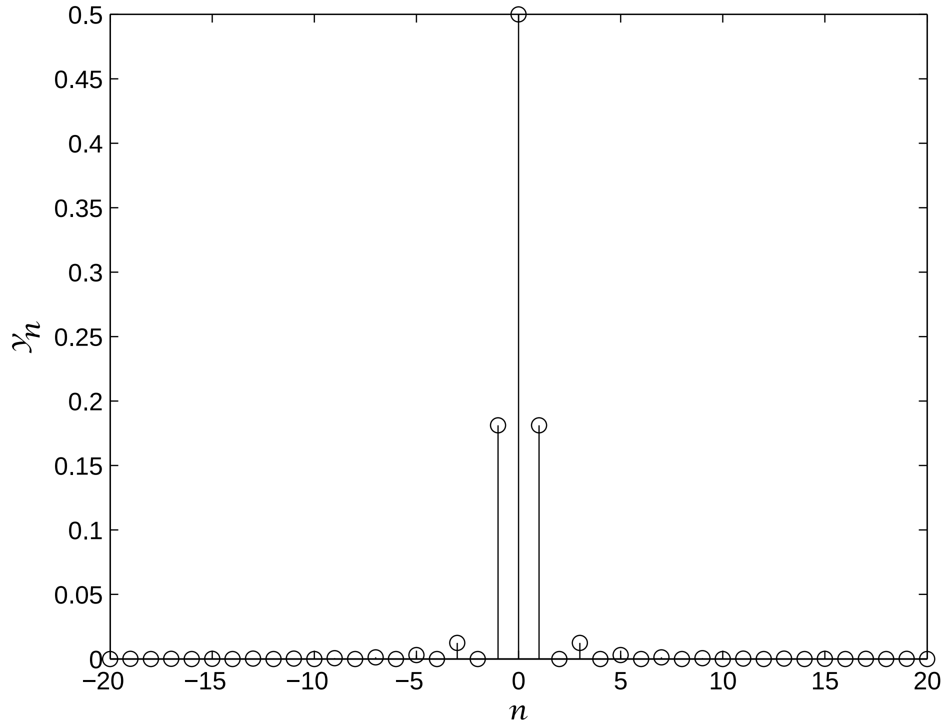


Figure 2.4: The discrete spectrum of the signal

```
% Fourier series coefficients of x(t) vector
x=.5*(sinc(n/2)).^2;
% sampling interval
ts=1/40;
% time vector
t=[−.5:ts:1.5];
% impulse response
fs=1/ts;
h=[zeros(1,20),t(21:61),zeros(1,20)];
% transfer function
H=fft(h)/fs;
% frequency resolution
df=fs/80;
f=[0:df:fs]−fs/2;
% rearrange H
H1=fftshift(H);
y=x.*H1(21:61);
% Plotting commands follow.
```

Computer Problem 2.3

The common magnitude spectrum is presented in Figure 2.5. The two phase spectrum of the two signals plotted on the same axes are given in Figure 2.6.

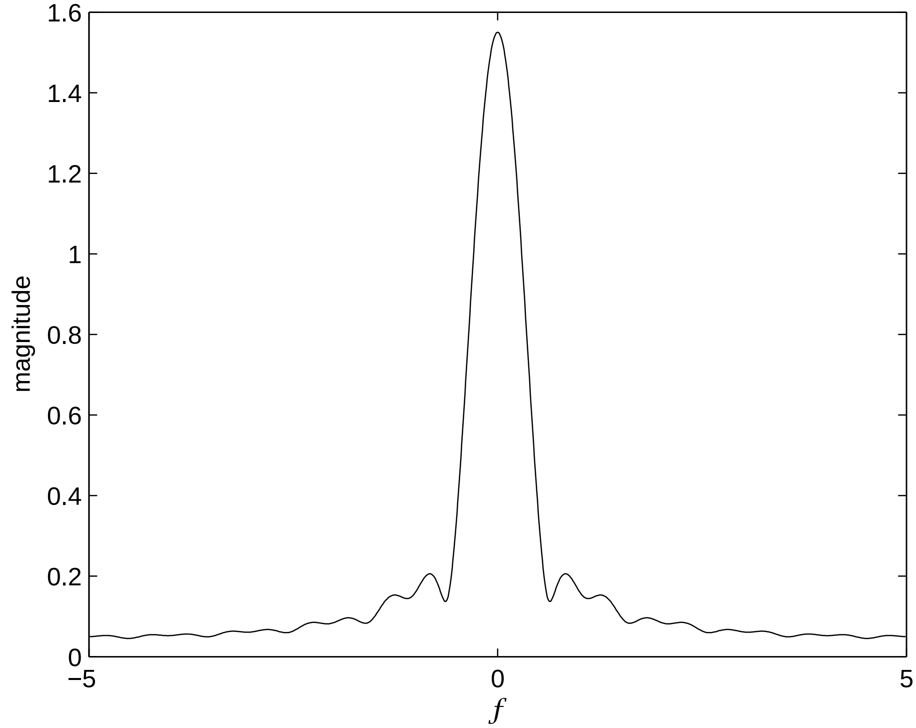


Figure 2.5: The common magnitude spectrum of the signals $x_1(t)$ and $x_2(t)$

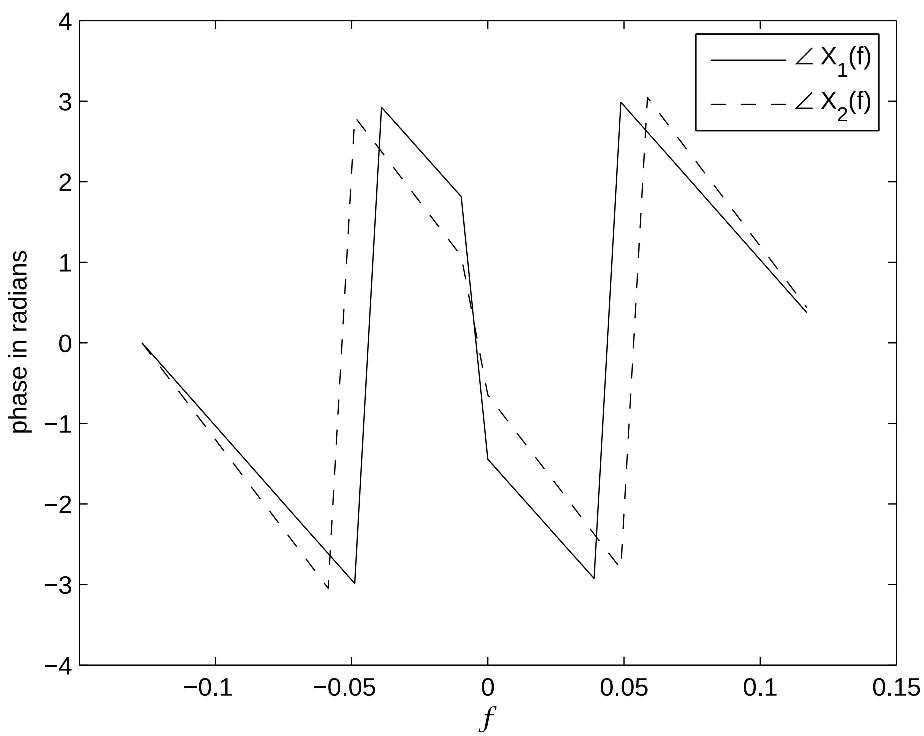


Figure 2.6: The phase spectrum of the signals $\Delta x_1(t)$ and $\Delta x_2(t)$

The MATLAB script for this problem follows.

```
% MATLAB script for Computer Problem 2.3.
df=0.01;
fs=10;
ts=1/fs;
t=[-5:ts:5];
x1=zeros(size(t));
x1(41:51)=t(41:51)+1;
x1(52:61)=ones(size(x1(52:61)));
x2=zeros(size(t));
x2(51:71)=x1(41:61);
[X1,x11,df1]=fftseq(x1,ts,df);
[X2,x21,df2]=fftseq(x2,ts,df);
X11=X1/fs;
X21=X2/fs;
f=[0:df1:df1*(length(x11)-1)]-fs/2;
plot(f,fftshift(abs(X11)))
figure
plot(f(500:525),fftshift(angle(X11(500:525))),f(500:525),fftshift(angle(X21(500:525))),'--')
```

Computer Problem 2.4

The Fourier transform of the signal $x(t)$ is

$$\frac{1}{1+j2\pi f}$$

Figures 2.7 and 2.8 present the magnitude and phase spectrum of the input signal $x(t)$.

2) The fourier transform of the output signal $y(t)$ is

$$y(f) = \begin{cases} \frac{1}{1+j2\pi f} & |f| \le 1.5 \\ 0 & \text{otherwise} \end{cases}$$

The magnitude and phase spectrum of $y(t)$ is given in Figures 2.9 and 2.10, respectively.

3) The inverse Fourier transform of the output signal is parented in Figure 2.11

The MATLAB script for this problem is given next

```
% MATLAB script for Computer Problem 2.4.
df= 0.01;
f = -4:df:4;
x_f = 1./(1+2*pi*i*f);
plot(f, abs(x_f));
figure;
plot(f, angle(x_f));
indH = find(abs(f) <= 1.5);
H_f = zeros(1, length(x_f));
H_f(indH) = cos(pi*f(indH)./3);
y_f = x_f.*H_f;
figure;
```

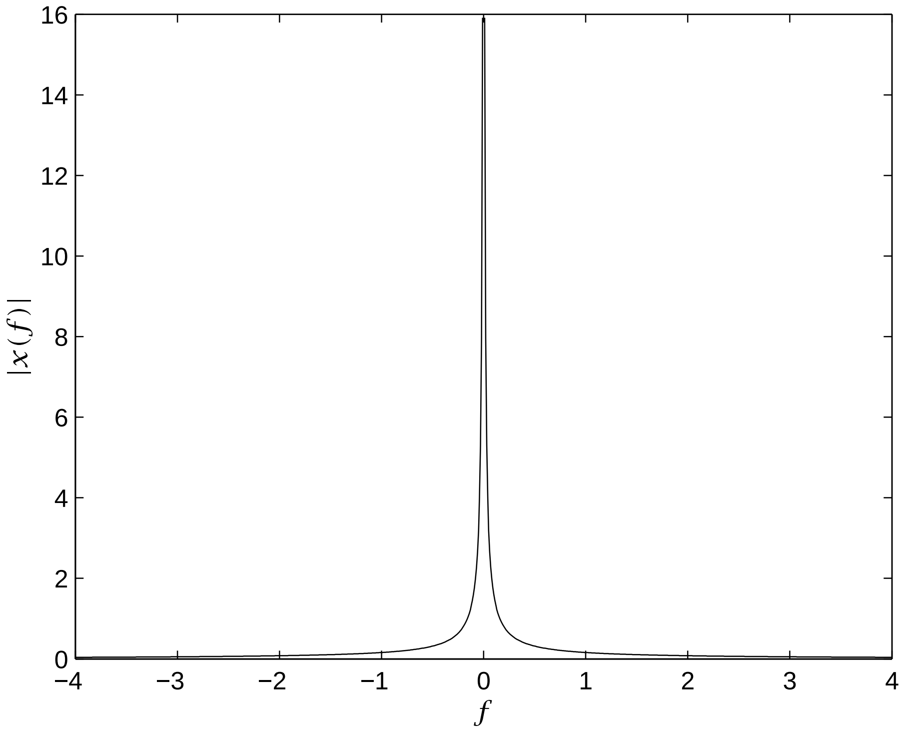


Figure 2.7: Magnitude spectrum of $x(t)$

```
plot(f,abs(y_f));
axis([-1.5 1.5 0 16]);
figure;
plot(f, angle(y_f));

y_f(401) = 10^30;
y_t = ifft(y_f, 'symmetric');
figure;
plot(y_t)
```
20

Computer Problem 2.5

Choosing the sampling interval to be $t_s = 0.001$ s, we have a sampling frequency of $f_s = 1/t_s = 1000$ Hz. Choosing a desired frequency resolution of $df = 0.5$ Hz, we have the following.

1) Plots of the signal and its magnitude spectrum are given in Figures 2.12 and 2.13, respectively. Plots are generated by Matlab.

2) Choosing $f_0 = 200$ Hz, we find the lowpass equivalent to $x(t)$ by using the loweq.m function. Then using fftseq.m, we obtain its spectrum; we plot its magnitude spectrum in Figure 2.14. The MATLAb functions loweq.m and fftseq.m are given next.

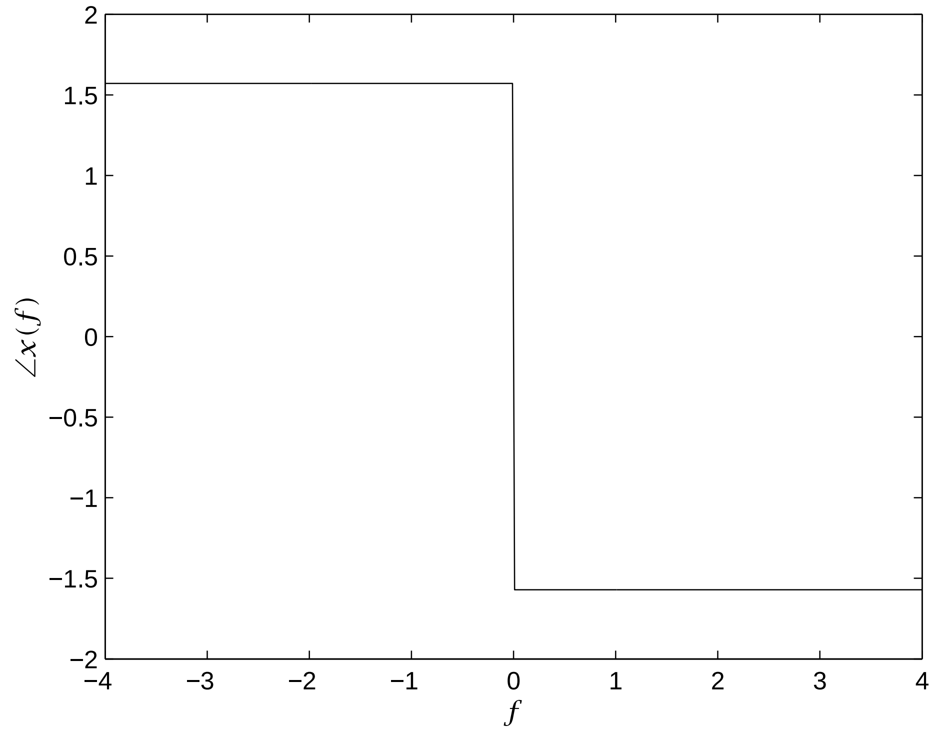


Figure 2.8: Phase spectrum of $x(t)$

```
function [M,m,df]=fftseq(m,ts,df)
%           [M,m,df]=fftseq(m,ts,df)
%           [M,m,df]=fftseq(m,ts)
%FFTSEQ     generates M, the FFT of the sequence m.
%           The sequence is zero-padded to meet the required frequency resolution df.
%           ts is the sampling interval. The output df is the final frequency resolution.
%           Output m is the zero-padded version of input m. M is the FFT.
fs=1/ts;
if nargin == 2
  n1=0;
else
  n1=fs/df;
end
n2=length(m);
n=2^(max(nextpow2(n1),nextpow2(n2)));
M=fft(m,n);
m=[m,zeros(1,n-n2)];
df=fs/n;
```

```
function xl=loweq(x,ts,f0)
%           xl=loweq(x,ts,f0)
%LOWEQ      returns the lowpass equivalent of the signal x
%           f0 is the center frequency.
%           ts is the sampling interval.
```

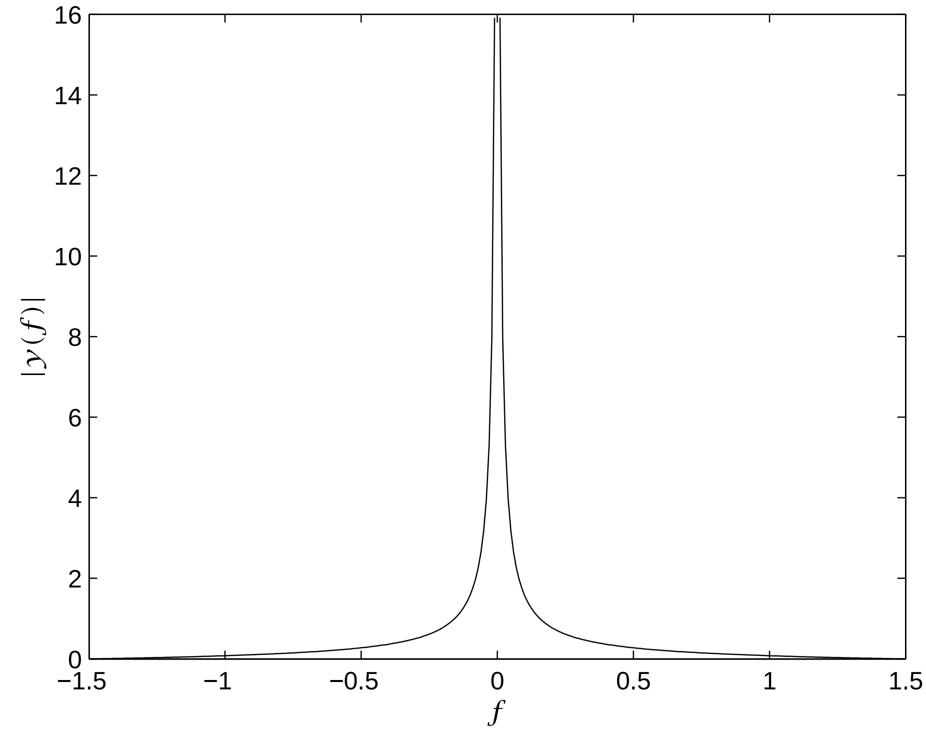


Figure 2.9: Magnitude spectrum of $y(t)$

```
%
t=[0:ts:ts*(length(x)−1)];
z=hilbert(x);
xl=z.*exp(−j*2*pi*f0*t);
```

It is seen that the magnitude spectrum is an even function in this case because we can write

$$x(t) = Re[\text{sinc}(100t)e^{j\times 400\pi t}] \tag{2.12}$$

Comparing this to

$$x(t) = Re[x_l(t)e^{j2\pi\times f_0 t}] \tag{2.13}$$

we conclude that

$$x_l(t) = sinc(100t) \tag{2.14}$$

which means that the lowpass equivalent signal is a real signal in this case. This, in turn, means that $x_c(t) = x_l(t)$ and $x_s(t) = 0$. Also, we conclude that

$$\begin{cases} V(t) = |x_c(t)| \\ \Theta = \begin{cases} 0, & x_c(t) \geq 0 \\ \pi, & x_c(t) < 0 \end{cases} \end{cases} \tag{2.15}$$

Plots of $x_c(t)$ and $V(t)$ are given in Figures 2.15 and 2.16, respectively. Note that choosing f_0 to be the frequency with respect to which $X(f)$ is symmetric result in these figures.

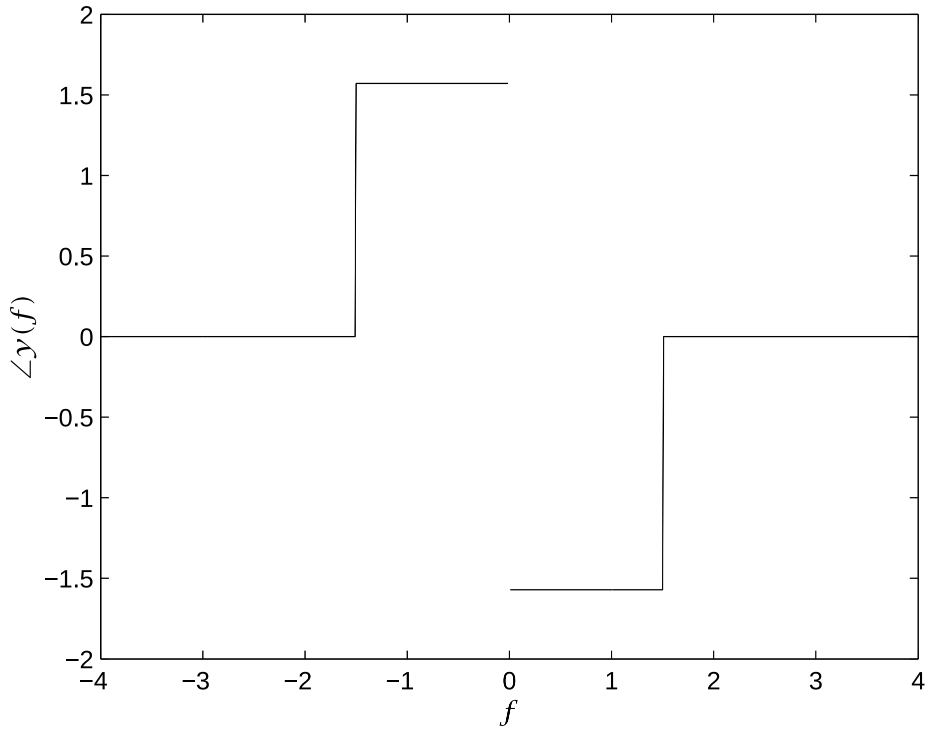


Figure 2.10: Phase spectrum of $y(t)$

Computer Problem 2.6

The Remez algorithm requires that we specify the length of the FIR filter M, the passband edge frequency f_p, the stopband edge frequency f_s, and the ratio δ_2/δ_1. Here, δ_1 and δ_2 denote passband and stopband ripples, respectively. The filter length M can be approximated using

$$\hat{M} = \frac{-20\log_{10}\sqrt{\delta_1\delta_2} - 13}{14.6\Delta f} + 1$$

where Δf is the transition bandwidth $\Delta f = f_s - f_p$

1) Figure 2.17 shows the impulse response coefficients of the FIR filter.

2) Figures 2.18 and 2.19 show the magnitude and phase of the frequency response of the filter, respectively.

The MATLAB script for this problem is given next

```
% MATLAB script for Computer Problem 2.6.
fp = 0.4;
fs = 0.5;
df = fs-fp;
Rp = 0.5;
As = 40;
```

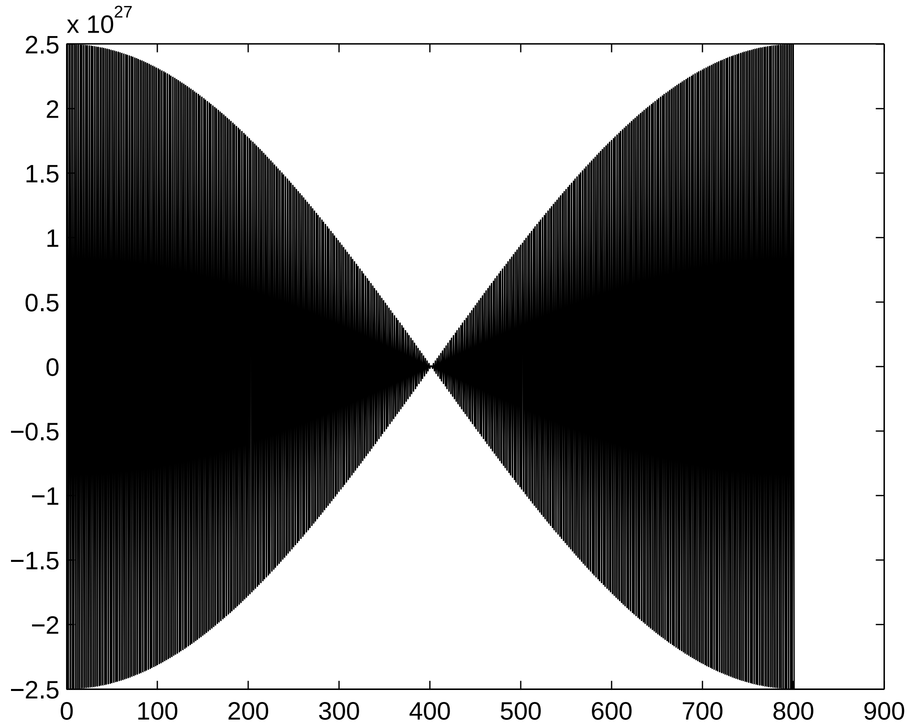


Figure 2.11: Inverse Fourier transform

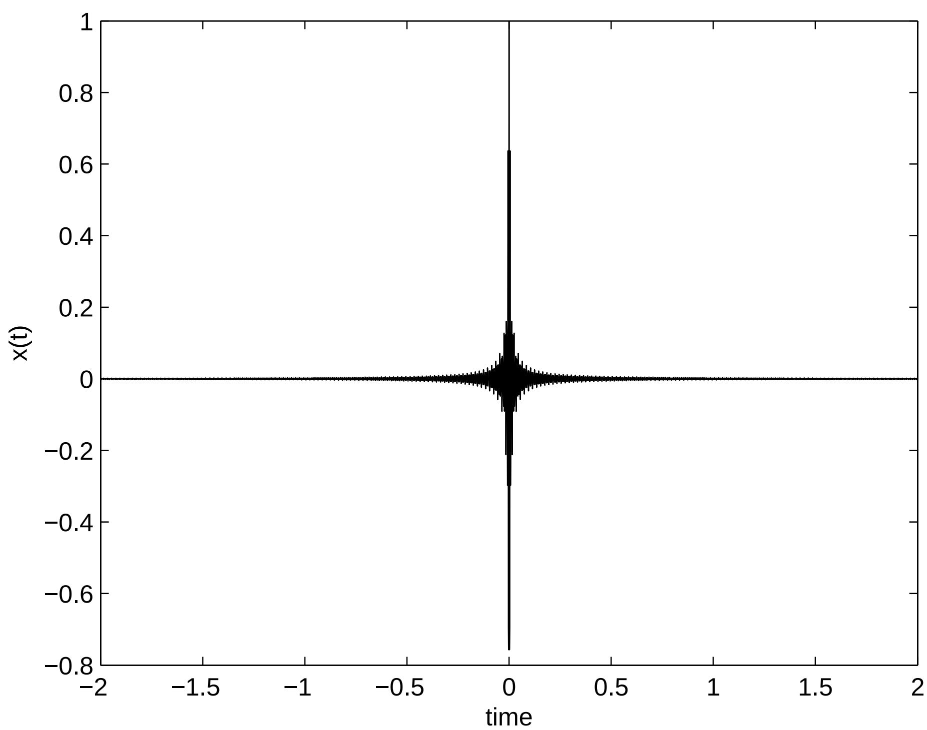


Figure 2.12: The signal $x(t)$

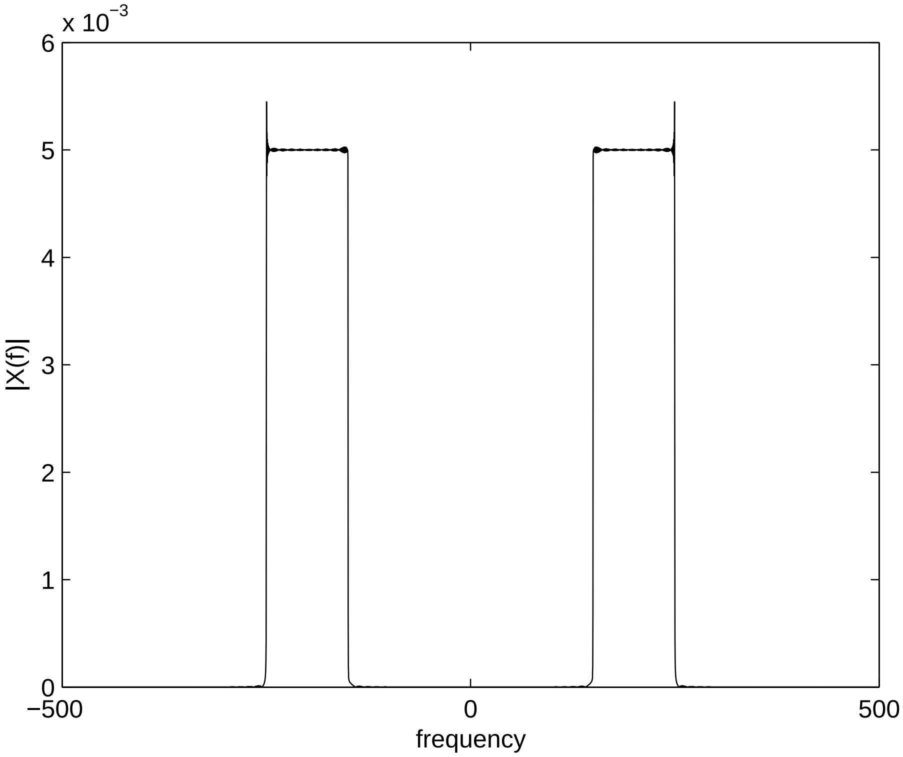


Figure 2.13: The magnitude spectrum of $x(t)$

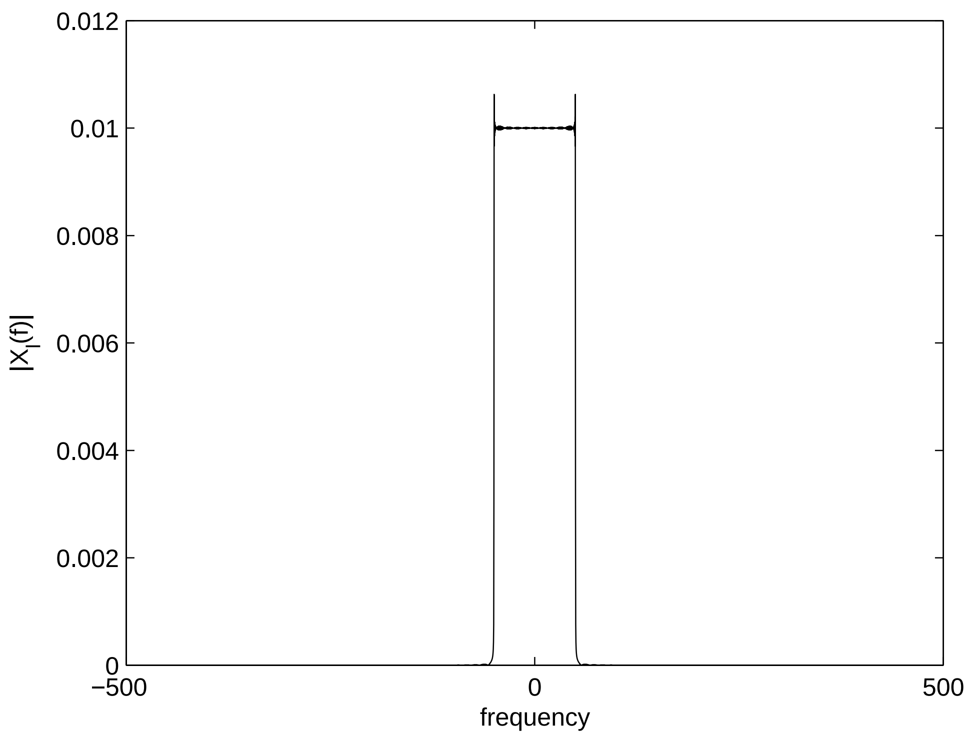


Figure 2.14: The magnitude spectrum of $x_l(t)$

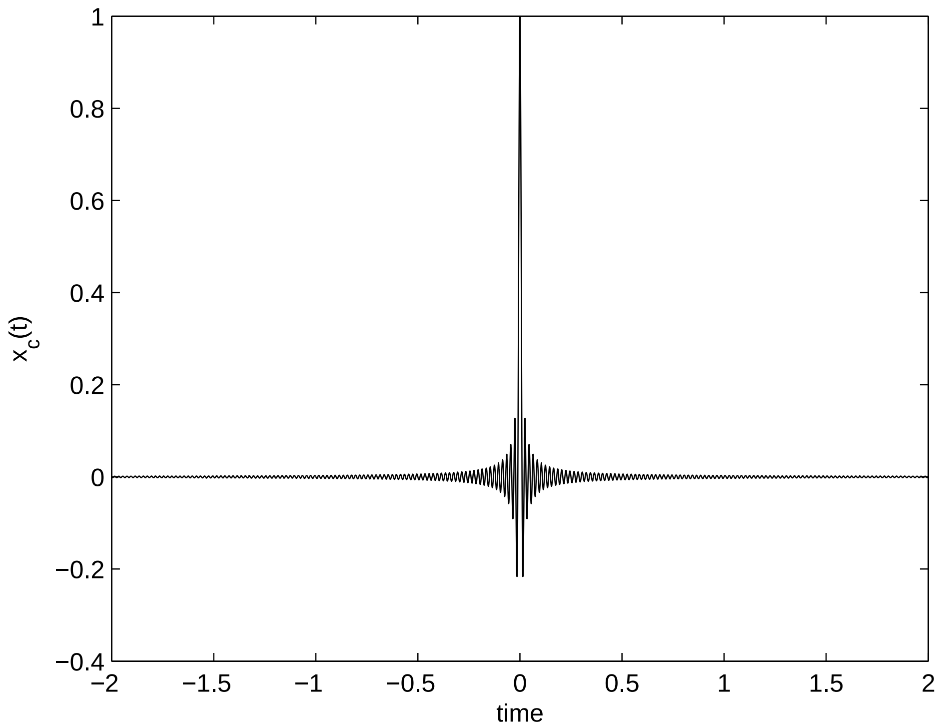


Figure 2.15: The signal $x_C(t)$

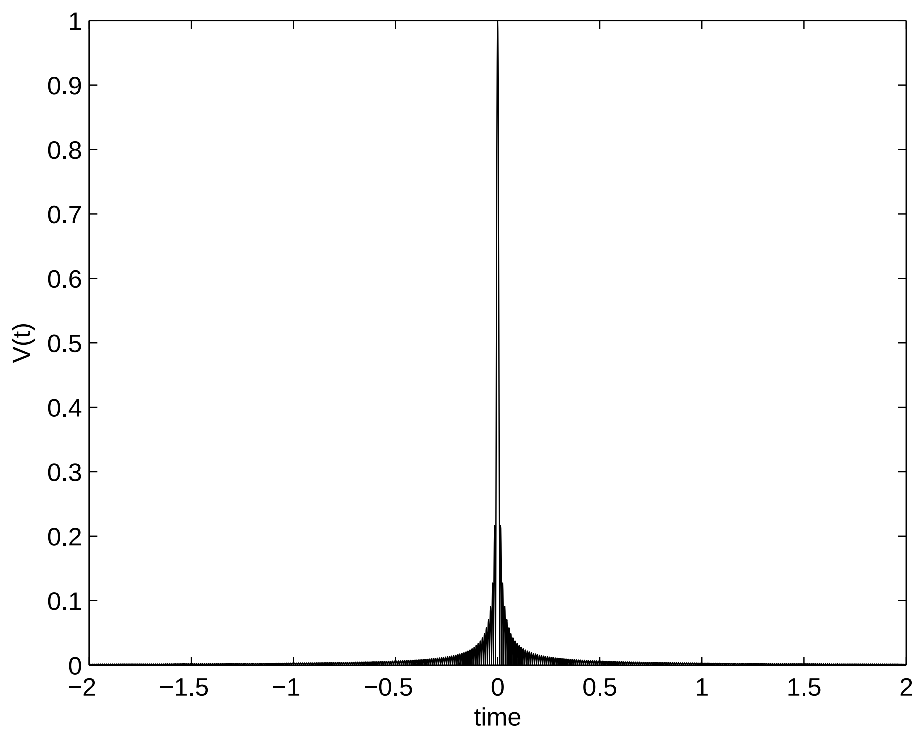


Figure 2.16: The signal $V(t)$

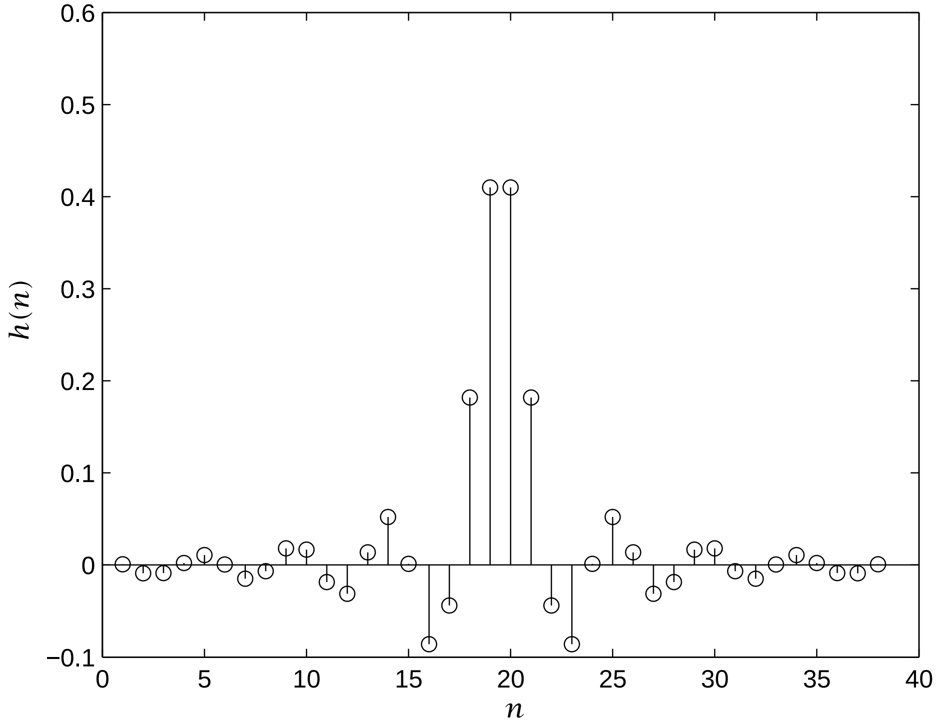


Figure 2.17: Impulse response coefficients of the FIR filter

```
delta1=(10^(Rp/20)−1)/(10^(Rp/20)+1);
delta2=(1+delta1)*(10^(−As/20));
%Calculate approximate filter length
Mhat=ceil((−20*log10(sqrt(delta1*delta2))−13)/(14.6*df)+1);
f=[0 fp fs 1];
m=[1 1 0 0];
w=[delta2/delta1 1];
h=remez(Mhat+20,f,m,w);
[H,W]=freqz(h,[1],3000);
db = 20*log10(abs(H));
% plot results
stem(h);
figure;
plot(W/pi, db)
figure;
plot(W/pi, angle(H));
```

Computer Problem 2.7

1) The impulse response coefficients of the filter is presented in Figure 2.20.

2) The magnitude of the frequency response of the filter is given in Figure 2.21.

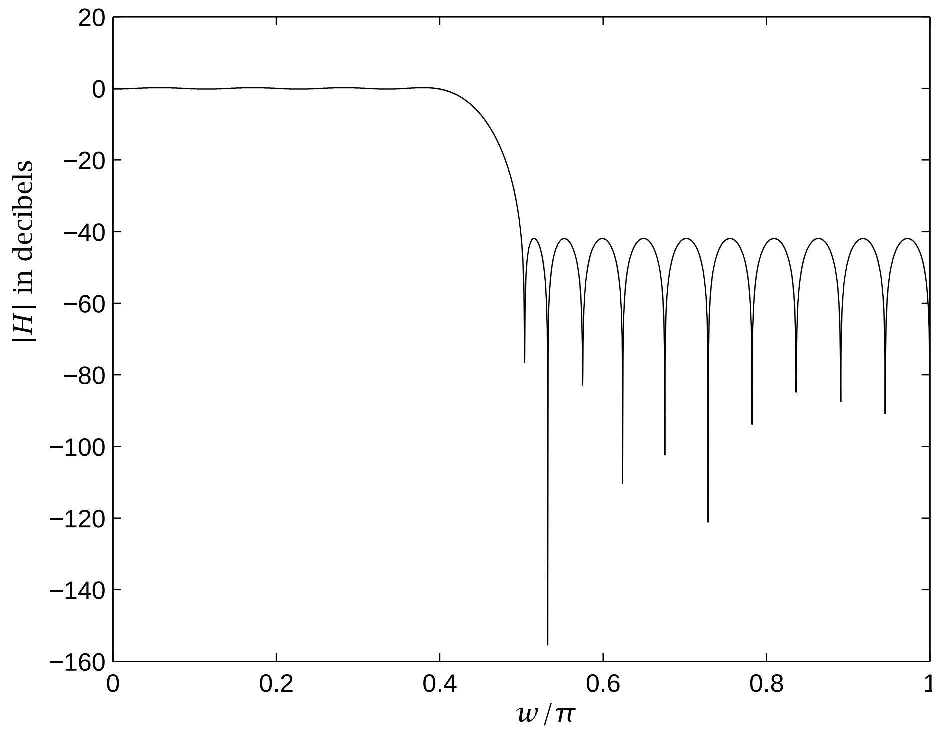


Figure 2.18: Magnitude of the frequency response of the FIR filter

The MATLAB script for this problem is given next

```
% MATLAB script for Computer Problem 2.7.
f=[0 0.01 0.1 0.5 0.6 1];
m=[0 0 1 1  0 0];
delta1 = 0.01;
delta2 = 0.01;
df = 0.1 − 0.01;
Mhat=ceil((−20*log10(sqrt(delta1*delta2))−13)/(14.6*df)+1);
w=[1 delta2/delta1 1];
h=remez(Mhat+20,f,m,w,'hilbert');

[H,W]=freqz(h,[1],3000);
db = 20*log10(abs(H));
% plot results
stem(h);
figure;
plot(W/pi, db)
figure;
plot(W/pi, angle(H));
```

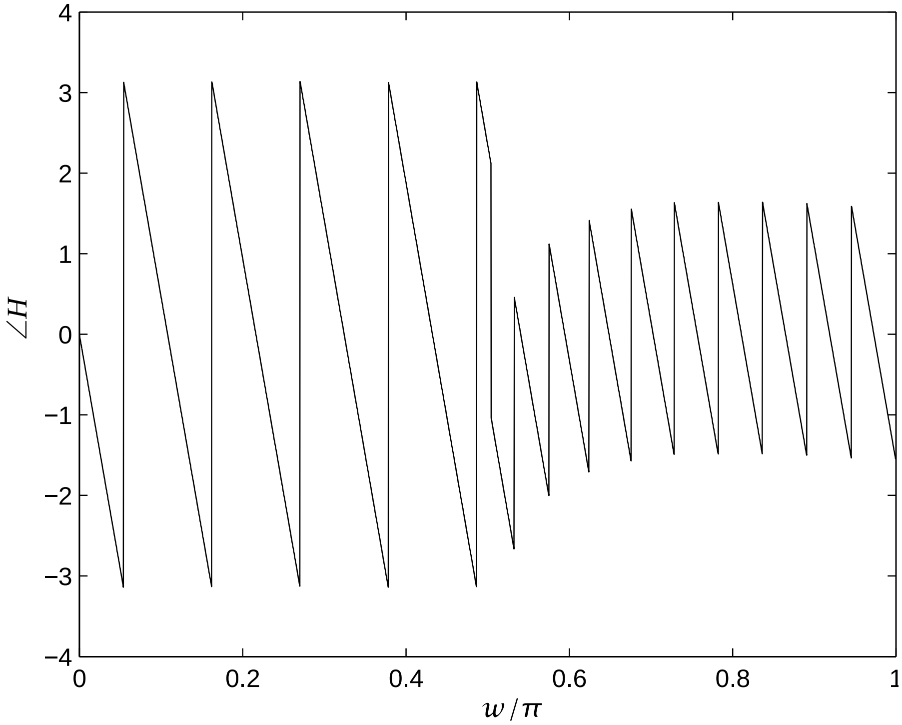


Figure 2.19: Phase of the frequency response of the FIR filter

Computer Problem 2.8

1) The impulse response of the filter is given in Figure 2.22.

2) The magnitude of the frequency response of the filter is presented in Figure 2.23.

3) The filter output $y(n)$ and $x(n)$ are presented in Figure 2.24. It should be noted that $y(n)$ is the derivative of $x(n)$.

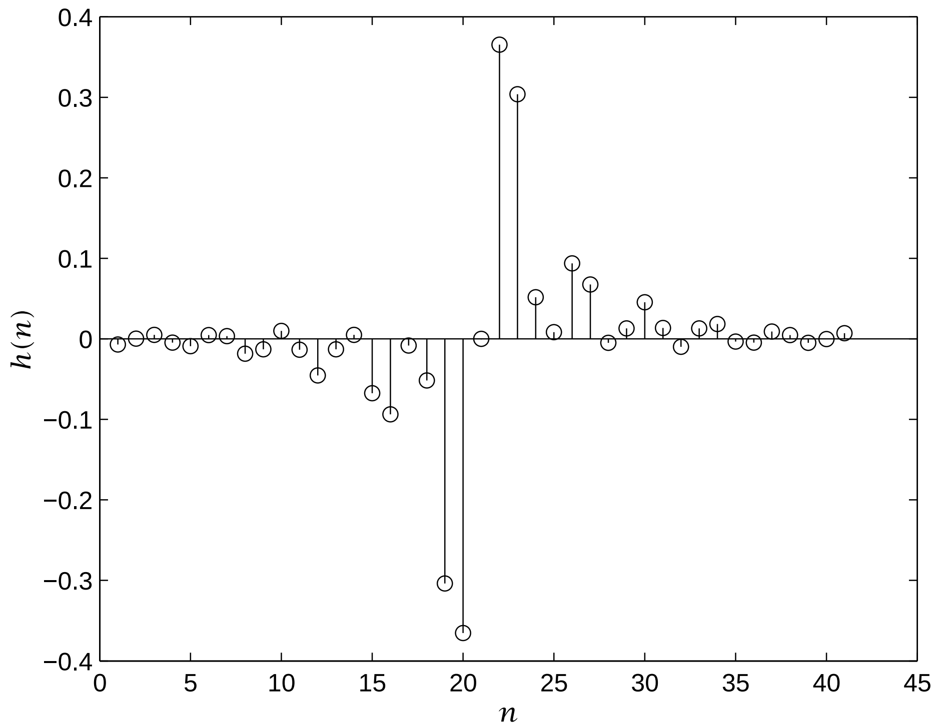


Figure 2.20: The impulse response coefficients of the filter

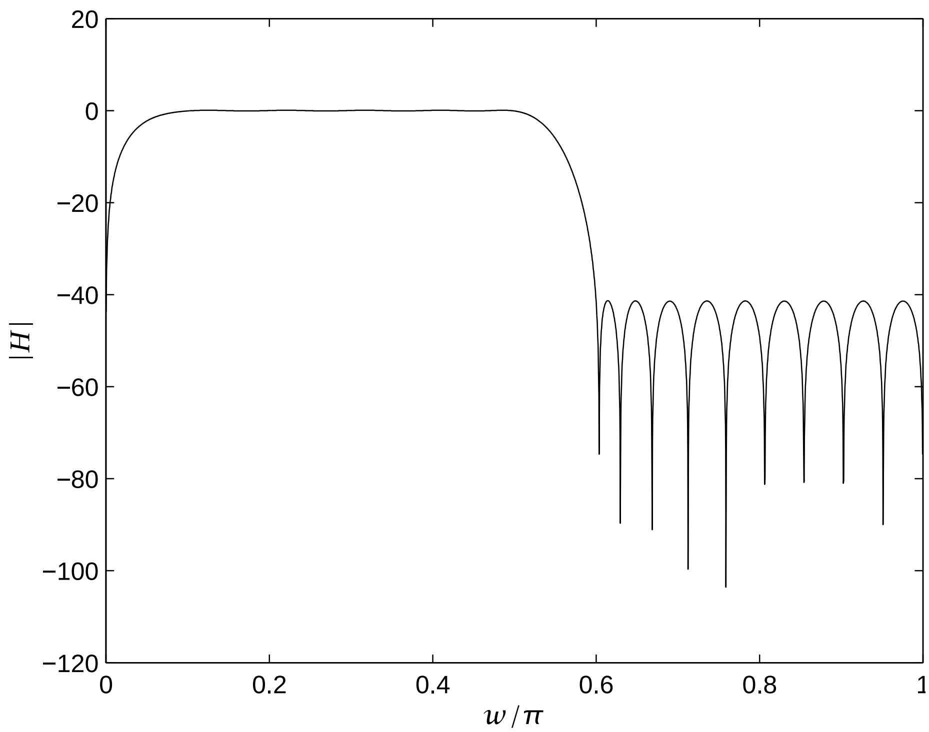


Figure 2.21: The magnitude of the frequency response of the filter

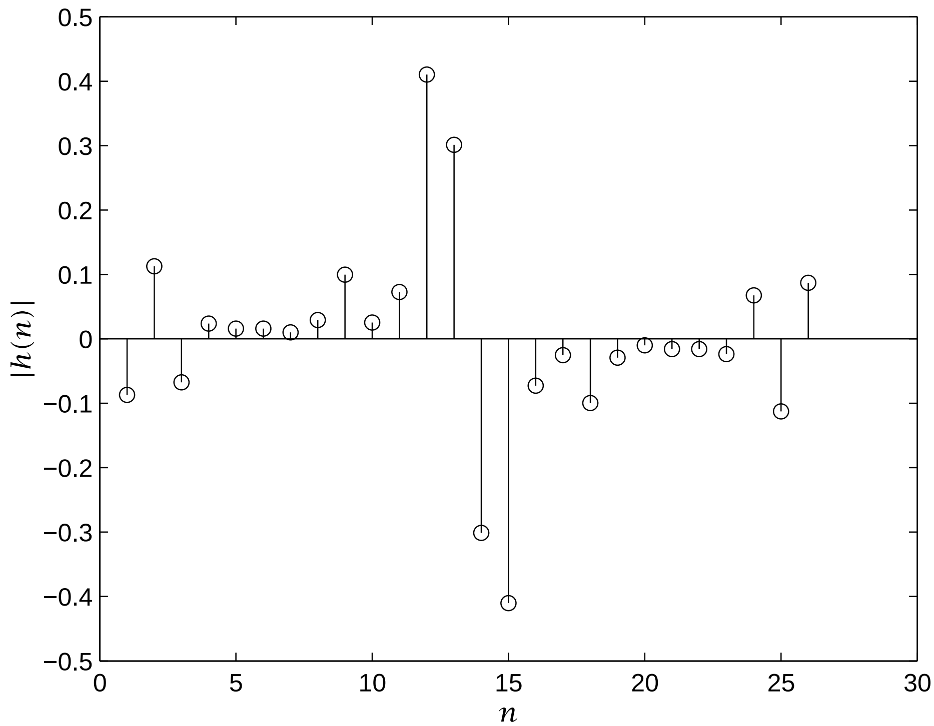


Figure 2.22: Impulse response of the filter

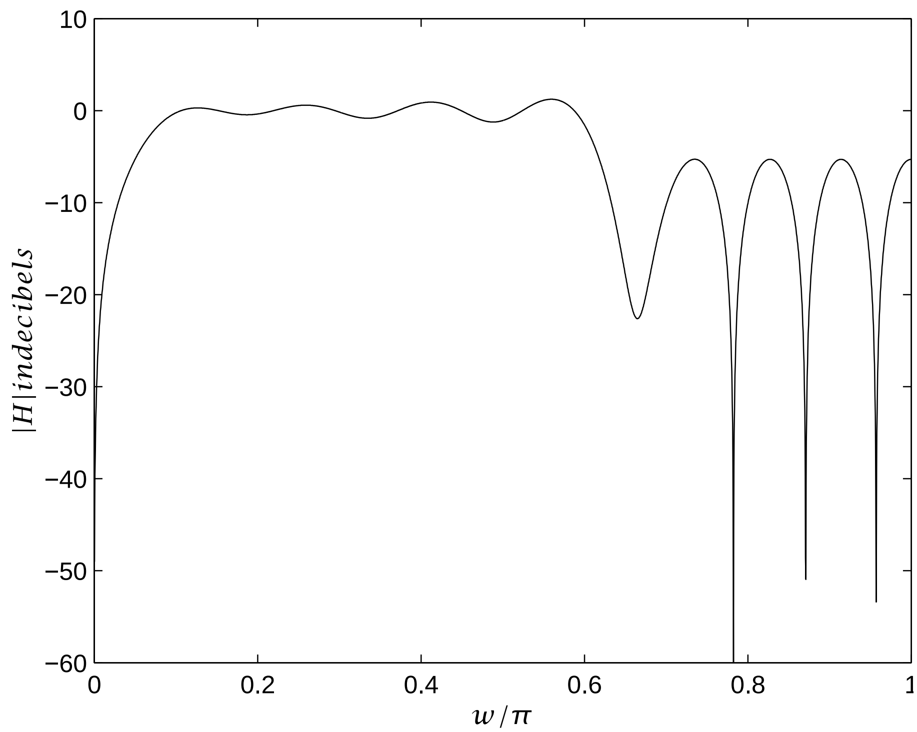


Figure 2.23: Magnitude of the frequency response of the filter

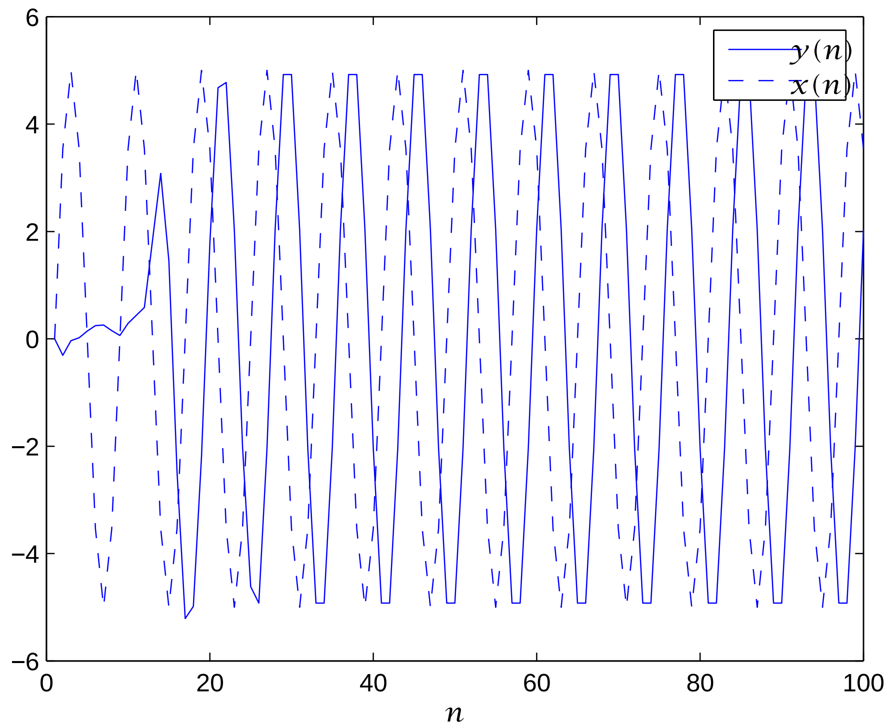


Figure 2.24: Signals $x(n)$ and $y(n)$

Chapter 3

Problem 3.1

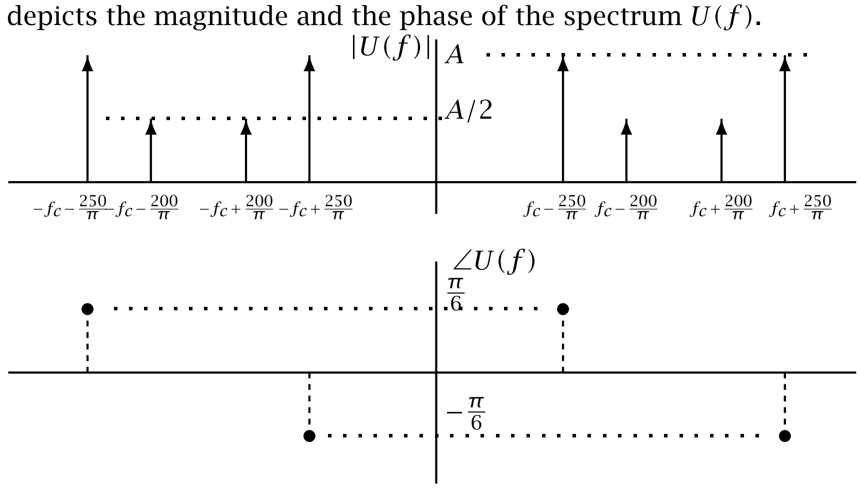
The modulated signal is

$$\begin{aligned}
u(t) &= m(t)c(t) = Am(t)\cos(2\pi 4\times 10^3 t)\\
&= A\left[2\cos(2\pi\frac{200}{\pi}t) + 4\sin(2\pi\frac{250}{\pi}t+\frac{\pi}{3})\right]\cos(2\pi 4\times 10^3 t)\\
&= A\cos(2\pi(4\times 10^3+\frac{200}{\pi})t) + A\cos(2\pi(4\times 10^3-\frac{200}{\pi})t)\\
&\quad +2A\sin(2\pi(4\times 10^3+\frac{250}{\pi})t+\frac{\pi}{3}) - 2A\sin(2\pi(4\times 10^3-\frac{250}{\pi})t-\frac{\pi}{3})
\end{aligned}$$

Taking the Fourier transform of the previous relation, we obtain

$$\begin{aligned}
U(f) &= A\left[\delta(f-\frac{200}{\pi}) + \delta(f+\frac{200}{\pi}) + \frac{2}{j}e^{j\frac{\pi}{3}}\delta(f-\frac{250}{\pi}) - \frac{2}{j}e^{-j\frac{\pi}{3}}\delta(f+\frac{250}{\pi})\right]\\
&\quad \star\frac{1}{2}[\delta(f-4\times 10^3)+\delta(f+4\times 10^3)]\\
&= \frac{A}{2}\left[\delta(f-4\times 10^3-\frac{200}{\pi}) + \delta(f-4\times 10^3+\frac{200}{\pi})\right.\\
&\quad +2e^{-j\frac{\pi}{6}}\delta(f-4\times 10^3-\frac{250}{\pi}) + 2e^{j\frac{\pi}{6}}\delta(f-4\times 10^3+\frac{250}{\pi})\\
&\quad +\delta(f+4\times 10^3-\frac{200}{\pi}) + \delta(f+4\times 10^3+\frac{200}{\pi})\\
&\quad \left.+2e^{-j\frac{\pi}{6}}\delta(f+4\times 10^3-\frac{250}{\pi}) + 2e^{j\frac{\pi}{6}}\delta(f+4\times 10^3+\frac{250}{\pi})\right]
\end{aligned}$$

The next figure depicts the magnitude and the phase of the spectrum $U(f)$.



To find the power content of the modulated signal we write $u^2(t)$ as

$$\begin{aligned}
u^2(t) &= A^2\cos^2(2\pi(4\times 10^3+\frac{200}{\pi})t) + A^2\cos^2(2\pi(4\times 10^3-\frac{200}{\pi})t)\\
&\quad +4A^2\sin^2(2\pi(4\times 10^3+\frac{250}{\pi})t+\frac{\pi}{3}) + 4A^2\sin^2(2\pi(4\times 10^3-\frac{250}{\pi})t-\frac{\pi}{3})\\
&\quad +\text{terms of cosine and sine functions in the first power}
\end{aligned}$$

Hence,

$$P = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} u^2(t)dt = \frac{A^2}{2} + \frac{A^2}{2} + \frac{4A^2}{2} + \frac{4A^2}{2} = 5A^2$$

Problem 3.2

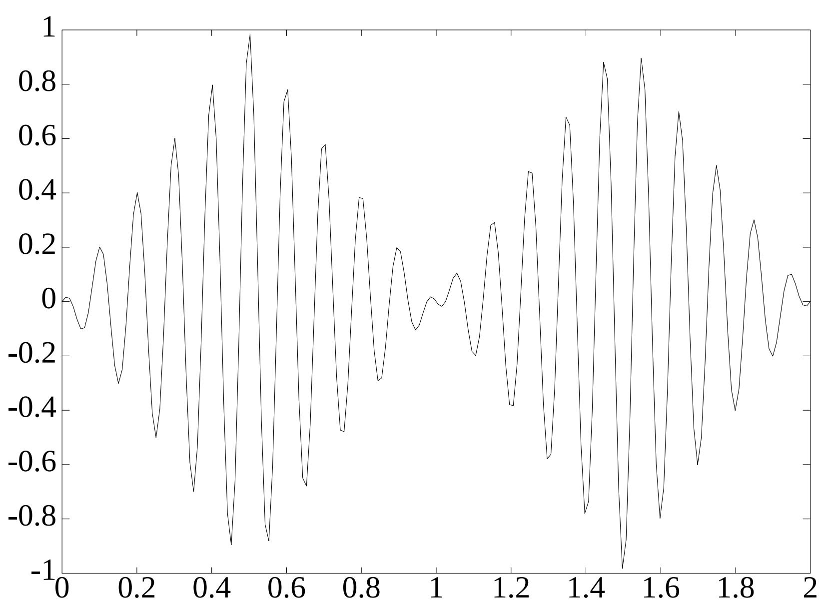
$$u(t) = m(t)c(t) = A(\text{sinc}(t) + \text{sinc}^2(t))\cos(2\pi f_c t)$$

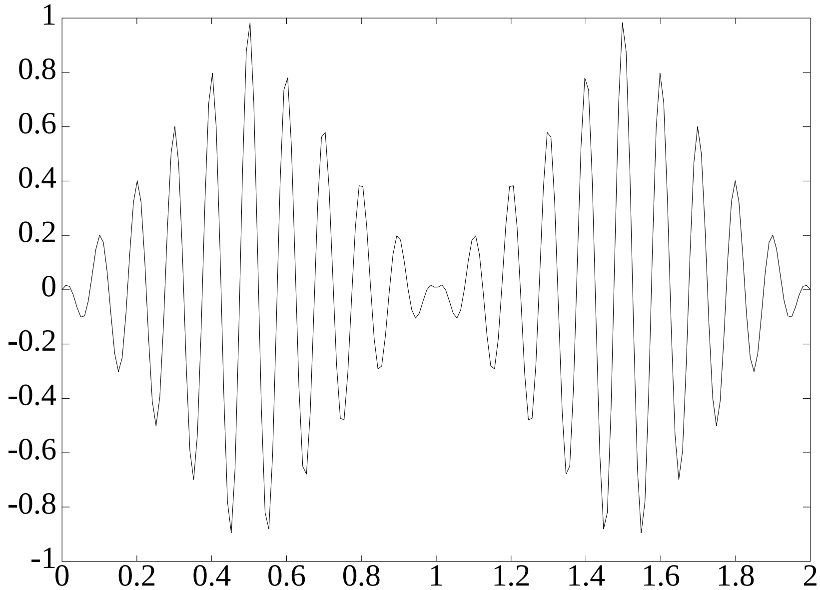
Taking the Fourier transform of both sides, we obtain

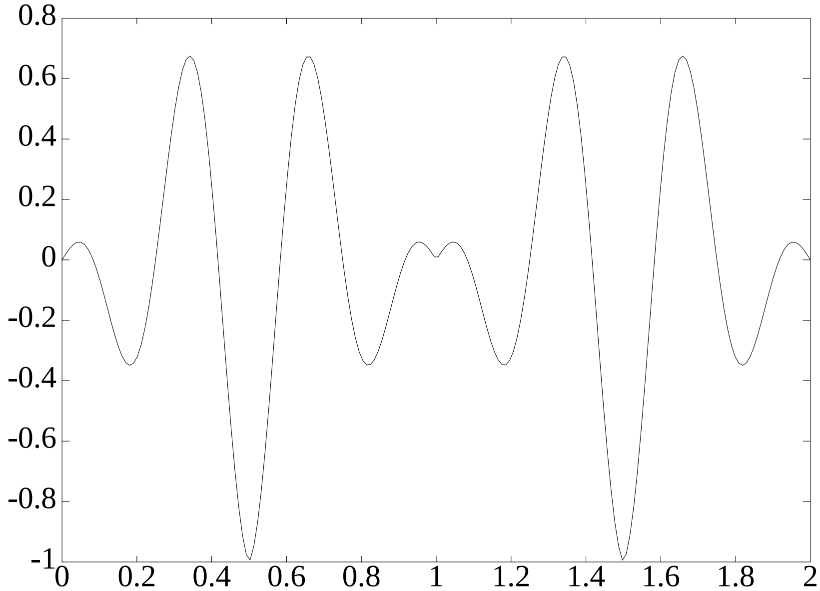
$$\begin{aligned} U(f) &= \frac{A}{2}[\Pi(f) + \Lambda(f)] \star (\delta(f - f_c) + \delta(f + f_c)) \\ &= \frac{A}{2}[\Pi(f - f_c) + \Lambda(f - f_c) + \Pi(f + f_c) + \Lambda(f + f_c)] \end{aligned}$$

$\Pi(f - f_c) \neq 0$ for $|f - f_c| < \frac{1}{2}$, whereas $\Lambda(f - f_c) \neq 0$ for $|f - f_c| < 1$. Hence, the bandwidth of the bandpass filter is 2.

Problem 3.3
The following figure shows the modulated signals for $A = 1$ and $f_0 = 10$. As it is observed both signals have the same envelope but there is a phase reversal at $t = 1$ for the second signal $Am_2(t)\cos(2\pi f_0 t)$ (right plot). This discontinuity is shown clearly in the next figure where we plotted $Am_2(t)\cos(2\pi f_0 t)$ with $f_0 = 3$.







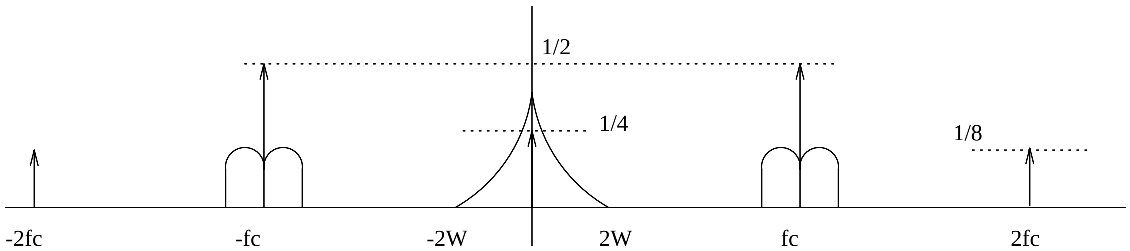
Problem 3.4

$$\begin{aligned}
y(t) &= x(t) + \frac{1}{2}x^2(t) \\
&= m(t) + \cos(2\pi f_c t) + \frac{1}{2}\left(m^2(t) + \cos^2(2\pi f_c t) + 2m(t)\cos(2\pi f_c t)\right) \\
&= m(t) + \cos(2\pi f_c t) + \frac{1}{2}m^2(t) + \frac{1}{4} + \frac{1}{4}\cos(2\pi 2 f_c t) + m(t)\cos(2\pi f_c t)
\end{aligned}$$

Taking the Fourier transform of the previous, we obtain

$$\begin{aligned} Y(f) &= M(f) + \frac{1}{2}M(f) \star M(f) + \frac{1}{2}\left(M(f-f_c) + M(f+f_c)\right) \\ &\quad + \frac{1}{4}\delta(f) + \frac{1}{2}\left(\delta(f-f_c) + \delta(f+f_c)\right) + \frac{1}{8}\left(\delta(f-2f_c) + \delta(f+2f_c)\right) \end{aligned}$$

The next figure depicts the spectrum $Y(f)$



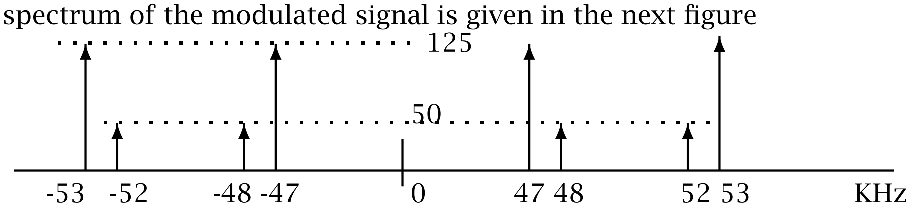
Problem 3.5

$$\begin{aligned} u(t) &= m(t) \cdot c(t) \\ &= 100(2\cos(2\pi 2000t) + 5\cos(2\pi 3000t))\cos(2\pi f_c t) \end{aligned}$$

Thus,

$$\begin{aligned} U(f) &= \frac{100}{2}\left[\delta(f-2000) + \delta(f+2000) + \frac{5}{2}(\delta(f-3000) + \delta(f+3000))\right] \\ &\quad \star \left[\delta(f-50000) + \delta(f+50000)\right] \\ &= 50\left[\delta(f-52000) + \delta(f-48000) + \frac{5}{2}\delta(f-53000) + \frac{5}{2}\delta(f-47000)\right. \\ &\quad \left. + \delta(f+52000) + \delta(f+48000) + \frac{5}{2}\delta(f+53000) + \frac{5}{2}\delta(f+47000)\right] \end{aligned}$$

A plot of the spectrum of the modulated signal is given in the next figure



Problem 3.6

The mixed signal $y(t)$ is given by

$$\begin{aligned} y(t) &= u(t) \cdot x_L(t) = Am(t)\cos(2\pi f_c t)\cos(2\pi f_c t + \theta) \\ &= \frac{A}{2}m(t)\left[\cos(2\pi 2 f_c t + \theta) + \cos(\theta)\right] \end{aligned}$$

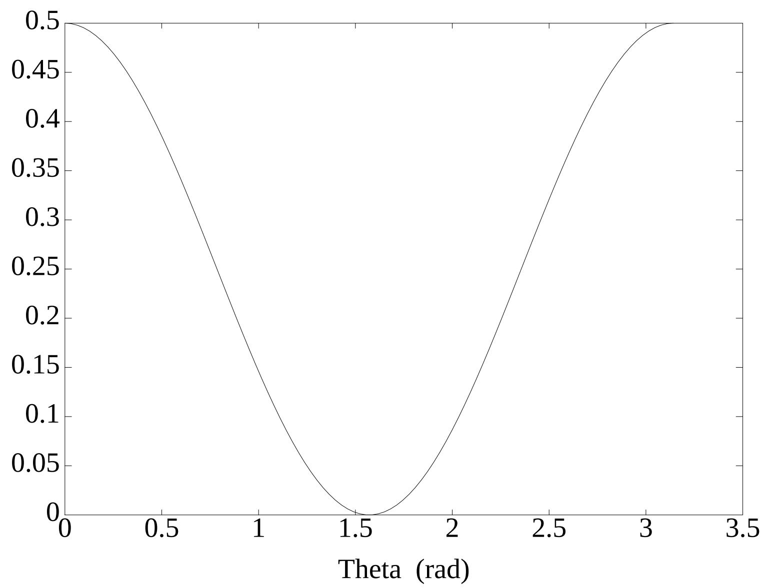
The lowpass filter will cut-off the frequencies above W, where W is the bandwidth of the message signal $m(t)$. Thus, the output of the lowpass filter is

$$z(t) = \frac{A}{2} m(t) \cos(\theta)$$

If the power of $m(t)$ is P_M, then the power of the output signal $z(t)$ is $P_{\text{out}} = P_M \frac{A^2}{4} \cos^2(\theta)$. The power of the modulated signal $u(t) = Am(t)\cos(2\pi f_c t)$ is $P_U = \frac{A^2}{2} P_M$. Hence,

$$\frac{P_{\text{out}}}{P_U} = \frac{1}{2} \cos^2(\theta)$$

A plot of $\frac{P_{\text{out}}}{P_U}$ for $0 \le \theta \le \pi$ is given in the next figure.

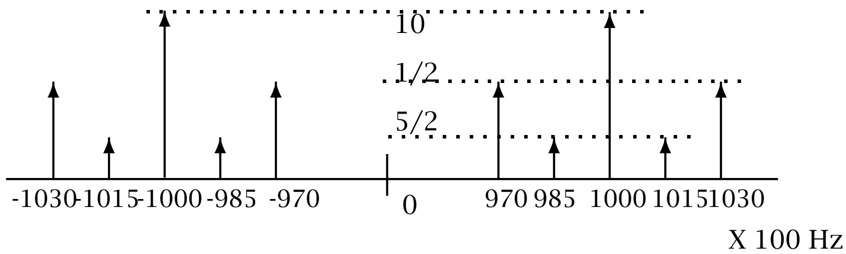


Problem 3.7

1) The spectrum of $u(t)$ is

$$\begin{aligned} U(f) &= \frac{20}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right] \\ &\quad + \frac{2}{4}\left[\delta(f - f_c - 1500) + \delta(f - f_c + 1500)\right. \\ &\quad \left. + \delta(f + f_c - 1500) + \delta(f + f_c + 1500)\right] \\ &\quad + \frac{10}{4}\left[\delta(f - f_c - 3000) + \delta(f - f_c + 3000)\right. \\ &\quad \left. + \delta(f + f_c - 3000) + \delta(f + f_c + 3000)\right] \end{aligned}$$

The next figure depicts the spectrum of $u(t)$.



2) The square of the modulated signal is

$$\begin{aligned} u^2(t) &= 400\cos^2(2\pi f_c t) + \cos^2(2\pi(f_c - 1500)t) + \cos^2(2\pi(f_c + 1500)t) \\ &\quad + 25\cos^2(2\pi(f_c - 3000)t) + 25\cos^2(2\pi(f_c + 3000)t) \\ &\quad + \text{ terms that are multiples of cosines} \end{aligned}$$

If we integrate $u^2(t)$ from $-\frac{T}{2}$ to $\frac{T}{2}$, normalize the integral by $\frac{1}{T}$ and take the limit as $T \to \infty$, then all the terms involving cosines tend to zero, whereas the squares of the cosines give a value of $\frac{1}{2}$. Hence, the power content at the frequency $f_c = 10^5$ Hz is $P_{f_c} = \frac{400}{2} = 200$, the power content at the frequency P_{f_c+1500} is the same as the power content at the frequency P_{f_c-1500} and equal to $\frac{1}{2}$, whereas $P_{f_c+3000} = P_{f_c-3000} = \frac{25}{2}$.

3)

$$\begin{aligned} u(t) &= (20 + 2\cos(2\pi 1500 t) + 10\cos(2\pi 3000 t))\cos(2\pi f_c t) \\ &= 20(1 + \frac{1}{10}\cos(2\pi 1500 t) + \frac{1}{2}\cos(2\pi 3000 t))\cos(2\pi f_c t) \end{aligned}$$

This is the form of a conventional AM signal with message signal

$$\begin{aligned} m(t) &= \frac{1}{10}\cos(2\pi 1500 t) + \frac{1}{2}\cos(2\pi 3000 t) \\ &= \cos^2(2\pi 1500 t) + \frac{1}{10}\cos(2\pi 1500 t) - \frac{1}{2} \end{aligned}$$

The minimum of $g(z) = z^2 + \frac{1}{10}z - \frac{1}{2}$ is achieved for $z = -\frac{1}{20}$ and it is $\min(g(z)) = -\frac{201}{400}$. Since $z = -\frac{1}{20}$ is in the range of $\cos(2\pi 1500 t)$, we conclude that the minimum value of $m(t)$ is $-\frac{201}{400}$. Hence, the modulation index is

$$\alpha = -\frac{201}{400}$$

4)

$$\begin{aligned} u(t) &= 20\cos(2\pi f_c t) + \cos(2\pi(f_c - 1500)t) + \cos(2\pi(f_c - 1500)t) \\ &= 5\cos(2\pi(f_c - 3000)t) + 5\cos(2\pi(f_c + 3000)t) \end{aligned}$$

The power in the sidebands is

$$P_{\text{sidebands}} = \frac{1}{2} + \frac{1}{2} + \frac{25}{2} + \frac{25}{2} = 26$$

The total power is $P_{\text{total}} = P_{\text{carrier}} + P_{\text{sidebands}} = 200 + 26 = 226$. The ratio of the sidebands power to the total power is

$$\frac{P_{\text{sidebands}}}{P_{\text{total}}} = \frac{26}{226}$$

Problem 3.8
1)

$$\begin{aligned}
u(t) &= m(t)c(t) \\
&= 100(\cos(2\pi 1000t) + 2\cos(2\pi 2000t))\cos(2\pi f_c t) \\
&= 100\cos(2\pi 1000t)\cos(2\pi f_c t) + 200\cos(2\pi 2000t)\cos(2\pi f_c t) \\
&= \frac{100}{2}\left[\cos(2\pi(f_c+1000)t) + \cos(2\pi(f_c-1000)t)\right] \\
&\quad \frac{200}{2}\left[\cos(2\pi(f_c+2000)t) + \cos(2\pi(f_c-2000)t)\right]
\end{aligned}$$

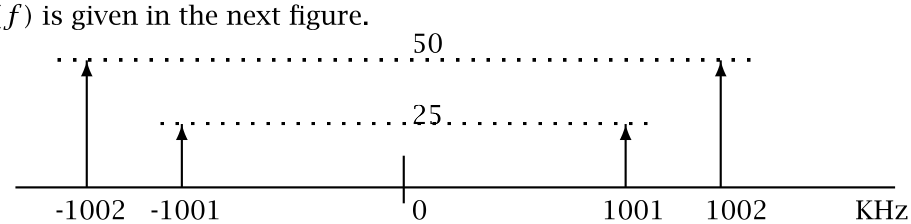
Thus, the upper sideband (USB) signal is

$$u_u(t) = 50\cos(2\pi(f_c+1000)t) + 100\cos(2\pi(f_c+2000)t)$$

2) Taking the Fourier transform of both sides, we obtain

$$\begin{aligned}
U_u(f) &= 25\left(\delta(f-(f_c+1000)) + \delta(f+(f_c+1000))\right) \\
&\quad +50\left(\delta(f-(f_c+2000)) + \delta(f+(f_c+2000))\right)
\end{aligned}$$

A plot of $U_u(f)$ is given in the next figure.



Problem 3.9
If we let

$$x(t) = -\Pi\left(\frac{t+\frac{T_p}{4}}{\frac{T_p}{2}}\right) + \Pi\left(\frac{t-\frac{T_p}{4}}{\frac{T_p}{2}}\right)$$

then using the results of Problem 2.56, we obtain

$$\begin{aligned}
v(t) &= m(t)s(t) = m(t)\sum_{n=-\infty}^{\infty} x(t-nT_p) \\
&= m(t)\frac{1}{T_p}\sum_{n=-\infty}^{\infty} X(\frac{n}{T_p})e^{j2\pi\frac{n}{T_p}t}
\end{aligned}$$

where

$$\begin{aligned} X(\frac{n}{T_p}) &= \mathcal{F}\left[-\Pi\left(\frac{t+\frac{T_p}{4}}{\frac{T_p}{2}}\right)+\Pi\left(\frac{t-\frac{T_p}{4}}{\frac{T_p}{2}}\right)\right]\Big|_{f=\frac{n}{T_p}} \\ &= \frac{T_p}{2}\mathrm{sinc}(f\frac{T_p}{2})\left(e^{-j2\pi f\frac{T_p}{4}}-e^{j2\pi f\frac{T_p}{4}}\right)\Big|_{f=\frac{n}{T_p}} \\ &= \frac{T_p}{2}\mathrm{sinc}(\frac{n}{2})(-2j)\sin(n\frac{\pi}{2}) \end{aligned}$$

Hence, the Fourier transform of $v(t)$ is

$$V(f) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\mathrm{sinc}(\frac{n}{2})(-2j)\sin(n\frac{\pi}{2})M(f-\frac{n}{T_p})$$

The bandpass filter will cut-off all the frequencies except the ones centered at $\frac{1}{T_p}$, that is for $n = \pm 1$. Thus, the output spectrum is

$$\begin{aligned} U(f) &= \mathrm{sinc}(\frac{1}{2})(-j)M(f-\frac{1}{T_p})+\mathrm{sinc}(\frac{1}{2})jM(f+\frac{1}{T_p}) \\ &= -\frac{2}{\pi}jM(f-\frac{1}{T_p})+\frac{2}{\pi}jM(f+\frac{1}{T_p}) \\ &= \frac{4}{\pi}M(f)\star\left[\frac{1}{2j}\delta(f-\frac{1}{T_p})-\frac{1}{2j}\delta(f+\frac{1}{T_p})\right] \end{aligned}$$

Taking the inverse Fourier transform of the previous expression, we obtain

$$u(t) = \frac{4}{\pi}m(t)\sin(2\pi\frac{1}{T_p}t)$$

which has the form of a DSB-SC AM signal, with $c(t) = \frac{4}{\pi}\sin(2\pi\frac{1}{T_p}t)$ being the carrier signal.

Problem 3.10

Assume that $s(t)$ is a periodic signal with period T_p, i.e. $s(t) = \sum_n x(t-nT_p)$. Then

$$\begin{aligned} v(t) &= m(t)s(t) = m(t)\sum_{n=-\infty}^{\infty}x(t-nT_p) \\ &= m(t)\frac{1}{T_p}\sum_{n=-\infty}^{\infty}X(\frac{n}{T_p})e^{j2\pi\frac{n}{T_p}t} \\ &= \frac{1}{T_p}\sum_{n=-\infty}^{\infty}X(\frac{n}{T_p})m(t)e^{j2\pi\frac{n}{T_p}t} \end{aligned}$$

where $X(\frac{n}{T_p}) = \mathcal{F}[x(t)]|_{f=\frac{n}{T_p}}$. The Fourier transform of $v(t)$ is

$$\begin{aligned} V(f) &= \frac{1}{T_p}\mathcal{F}\left[\sum_{n=-\infty}^{\infty}X(\frac{n}{T_p})m(t)e^{j2\pi\frac{n}{T_p}t}\right] \\ &= \frac{1}{T_p}\sum_{n=-\infty}^{\infty}X(\frac{n}{T_p})M(f-\frac{n}{T_p}) \end{aligned}$$

The bandpass filter will cut-off all the frequency components except the ones centered at $f_c = \pm\frac{1}{T_p}$. Hence, the spectrum at the output of the BPF is

$$U(f) = \frac{1}{T_p}X(\frac{1}{T_p})M(f - \frac{1}{T_p}) + \frac{1}{T_p}X(-\frac{1}{T_p})M(f + \frac{1}{T_p})$$

In the time domain the output of the BPF is given by

$$\begin{aligned} u(t) &= \frac{1}{T_p}X(\frac{1}{T_p})m(t)e^{j2\pi\frac{1}{T_p}t} + \frac{1}{T_p}X^*(\frac{1}{T_p})m(t)e^{-j2\pi\frac{1}{T_p}t} \\ &= \frac{1}{T_p}m(t)\left[X(\frac{1}{T_p})e^{j2\pi\frac{1}{T_p}t} + X^*(\frac{1}{T_p})e^{-j2\pi\frac{1}{T_p}t}\right] \\ &= \frac{1}{T_p}2\mathrm{Re}(\mathrm{X}(\frac{1}{\mathrm{T_p}}))\mathrm{m(t)}\cos(2\pi\frac{1}{\mathrm{T_p}}\mathrm{t}) \end{aligned}$$

As it is observed $u(t)$ has the form a modulated DSB-SC signal. The amplitude of the modulating signal is $A_c = \frac{1}{T_p}2\mathrm{Re}(\mathrm{X}(\frac{1}{\mathrm{T_p}}))$ and the carrier frequency $f_c = \frac{1}{T_p}$.

Problem 3.11

1) The spectrum of the modulated signal $Am(t)\cos(2\pi f_c t)$ is

$$V(f) = \frac{A}{2}[M(f - f_c) + M(f + f_c)]$$

The spectrum of the signal at the output of the highpass filter is

$$U(f) = \frac{A}{2}[M(f + f_c)u_{-1}(-f - f_c) + M(f - f_c)u_{-1}(f - f_c)]$$

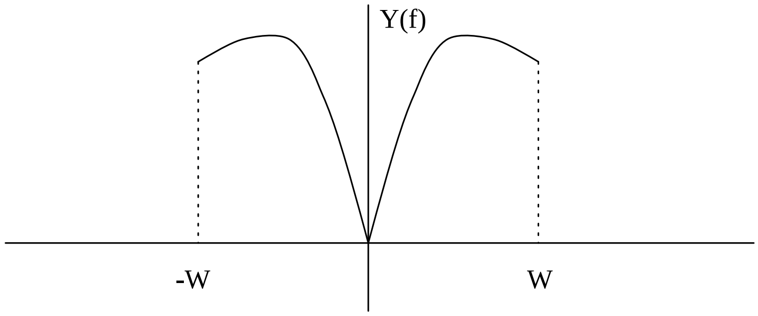
Multiplying the output of the HPF with $A\cos(2\pi(f_c + W)t)$ results in the signal $z(t)$ with spectrum

$$\begin{aligned} Z(f) &= \frac{A}{2}[M(f + f_c)u_{-1}(-f - f_c) + M(f - f_c)u_{-1}(f - f_c)] \\ &\star \frac{A}{2}[\delta(f - (f_c + W)) + \delta(f + f_c + W)] \\ &= \frac{A^2}{4}(M(f + f_c - f_c - W)u_{-1}(-f + f_c + W - f_c) \\ &+M(f + f_c - f_c + W)u_{-1}(f + f_c + W - f_c) \\ &+M(f - 2f_c - W)u_{-1}(f - 2f_c - W) \\ &+M(f + 2f_c + W)u_{-1}(-f - 2f_c - W)) \\ &= \frac{A^2}{4}(M(f - W)u_{-1}(-f + W) + M(f + W)u_{-1}(f + W) \\ &+M(f - 2f_c - W)u_{-1}(f - 2f_c - W) + M(f + 2f_c + W)u_{-1}(-f - 2f_c - W)) \end{aligned}$$

The LPF will cut-off the double frequency components, leaving the spectrum

$$Y(f) = \frac{A^2}{4}[M(f - W)u_{-1}(-f + W) + M(f + W)u_{-1}(f + W)]$$

The next figure depicts $Y(f)$ for $M(f)$ as shown in Fig. P-3.11.



2) As it is observed from the spectrum $Y(f)$, the system shifts the positive frequency components to the negative frequency axis and the negative frequency components to the positive frequency axis. If we transmit the signal $y(t)$ through the system, then we will get a scaled version of the original spectrum $M(f)$.

Problem 3.12
The modulated signal can be written as

$$\begin{aligned} u(t) &= m(t)\cos(2\pi f_c t + \phi) \\ &= m(t)\cos(2\pi f_c t)\cos(\phi) - m(t)\sin(2\pi f_c t)\sin(\phi) \\ &= u_c(t)\cos(2\pi f_c t) - u_s(t)\sin(2\pi f_c t) \end{aligned}$$

where we identify $u_c(t) = m(t)\cos(\phi)$ as the in-phase component and $u_s(t) = m(t)\sin(\phi)$ as the quadrature component. The envelope of the bandpass signal is

$$\begin{aligned} V_u(t) &= \sqrt{u_c^2(t) + u_s^2(t)} = \sqrt{m^2(t)\cos^2(\phi) + m^2(t)\sin^2(\phi)} \\ &= \sqrt{m^2(t)} = |m(t)| \end{aligned}$$

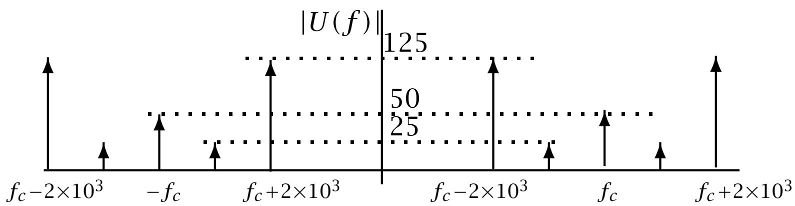
Hence, the envelope is proportional to the absolute value of the message signal.

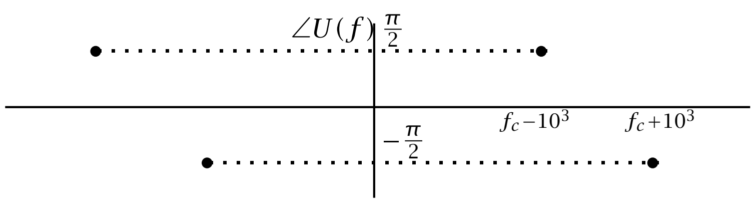
Problem 3.13
1) The modulated signal is

$$\begin{aligned} u(t) &= 100[1 + m(t)]\cos(2\pi 8 \times 10^5 t) \\ &= 100\cos(2\pi 8 \times 10^5 t) + 100\sin(2\pi 10^3 t)\cos(2\pi 8 \times 10^5 t) \\ &\quad +500\cos(2\pi 2 \times 10^3 t)\cos(2\pi 8 \times 10^5 t) \\ &= 100\cos(2\pi 8 \times 10^5 t) + 50[\sin(2\pi(10^3 + 8 \times 10^5)t) - \sin(2\pi(8 \times 10^5 - 10^3)t)] \\ &\quad +250[\cos(2\pi(2 \times 10^3 + 8 \times 10^5)t) + \cos(2\pi(8 \times 10^5 - 2 \times 10^3)t)] \end{aligned}$$

Taking the Fourier transform of the previous expression, we obtain

$$
\begin{aligned}
U(f) &= 50[\delta(f - 8\times 10^5) + \delta(f + 8\times 10^5)] \\
&\quad +25\left[\frac{1}{j}\delta(f - 8\times 10^5 - 10^3) - \frac{1}{j}\delta(f + 8\times 10^5 + 10^3)\right] \\
&\quad -25\left[\frac{1}{j}\delta(f - 8\times 10^5 + 10^3) - \frac{1}{j}\delta(f + 8\times 10^5 - 10^3)\right] \\
&\quad +125\left[\delta(f - 8\times 10^5 - 2\times 10^3) + \delta(f + 8\times 10^5 + 2\times 10^3)\right] \\
&\quad +125\left[\delta(f - 8\times 10^5 - 2\times 10^3) + \delta(f + 8\times 10^5 + 2\times 10^3)\right] \\
&= 50[\delta(f - 8\times 10^5) + \delta(f + 8\times 10^5)] \\
&\quad +25\left[\delta(f - 8\times 10^5 - 10^3)e^{-j\frac{\pi}{2}} + \delta(f + 8\times 10^5 + 10^3)e^{j\frac{\pi}{2}}\right] \\
&\quad +25\left[\delta(f - 8\times 10^5 + 10^3)e^{j\frac{\pi}{2}} + \delta(f + 8\times 10^5 - 10^3)e^{-j\frac{\pi}{2}}\right] \\
&\quad +125\left[\delta(f - 8\times 10^5 - 2\times 10^3) + \delta(f + 8\times 10^5 + 2\times 10^3)\right] \\
&\quad +125\left[\delta(f - 8\times 10^5 - 2\times 10^3) + \delta(f + 8\times 10^5 + 2\times 10^3)\right]
\end{aligned}
$$





2) The average power in the carrier is

$$P_{\text{carrier}} = \frac{A_c^2}{2} = \frac{100^2}{2} = 5000$$

The power in the sidebands is

$$P_{\text{sidebands}} = \frac{50^2}{2} + \frac{50^2}{2} + \frac{250^2}{2} + \frac{250^2}{2} = 65000$$

3) The message signal can be written as

$$
\begin{aligned}
m(t) &= \sin(2\pi 10^3 t) + 5\cos(2\pi 2\times 10^3 t) \\
&= -10\sin(2\pi 10^3 t) + \sin(2\pi 10^3 t) + 5
\end{aligned}
$$

As it is seen the minimum value of $m(t)$ is -6 and is achieved for $\sin(2\pi 10^3 t) = -1$ or $t = \frac{3}{4\times 10^3} + \frac{1}{10^3}k$, with $k \in Z$. Hence, the modulation index is $\alpha = 6$.

4) The power delivered to the load is

$$P_{\text{load}} = \frac{|u(t)|^2}{50} = \frac{100^2(1+m(t))^2\cos^2(2\pi f_c t)}{50}$$

The maximum absolute value of $1+m(t)$ is 6.025 and is achieved for $\sin(2\pi 10^3 t) = \frac{1}{20}$ or $t = \frac{\arcsin(\frac{1}{20})}{2\pi 10^3} + \frac{k}{10^3}$. Since $2\times 10^3 \ll f_c$ the peak power delivered to the load is approximately equal to

$$\max(P_{\text{load}}) = \frac{(100\times 6.025)^2}{50} = 72.6012$$

Problem 3.14
1)

$$\begin{aligned} u(t) &= 5\cos(1800\pi t) + 20\cos(2000\pi t) + 5\cos(2200\pi t) \\ &= 20(1+\frac{1}{2}\cos(200\pi t))\cos(2000\pi t) \end{aligned}$$

The modulating signal is $m(t) = \cos(2\pi 100 t)$ whereas the carrier signal is $c(t) = 20\cos(2\pi 1000 t)$.

2) Since $-1 \le \cos(2\pi 100 t) \le 1$, we immediately have that the modulation index is $\alpha = \frac{1}{2}$.

3) The power of the carrier component is $P_{\text{carrier}} = \frac{400}{2} = 200$, whereas the power in the sidebands is $P_{\text{sidebands}} = \frac{400\alpha^2}{2} = 50$. Hence,

$$\frac{P_{\text{sidebands}}}{P_{\text{carrier}}} = \frac{50}{200} = \frac{1}{4}$$

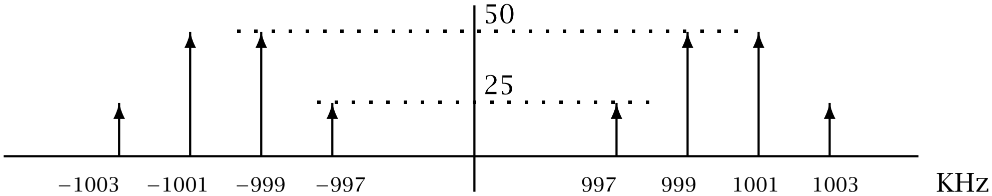
Problem 3.15
1) The modulated signal is written as

$$\begin{aligned} u(t) &= 100(2\cos(2\pi 10^3 t) + \cos(2\pi 3\times 10^3 t))\cos(2\pi f_c t) \\ &= 200\cos(2\pi 10^3 t)\cos(2\pi f_c t) + 100\cos(2\pi 3\times 10^3 t)\cos(2\pi f_c t) \\ &= 100\left[\cos(2\pi(f_c+10^3)t) + \cos(2\pi(f_c-10^3)t)\right] \\ &\quad +50\left[\cos(2\pi(f_c+3\times 10^3)t) + \cos(2\pi(f_c-3\times 10^3)t)\right] \end{aligned}$$

Taking the Fourier transform of the previous expression, we obtain

$$\begin{aligned} U(f) &= 50\Big[\delta(f-(f_c+10^3)) + \delta(f+f_c+10^3) \\ &+ \delta(f-(f_c-10^3)) + \delta(f+f_c-10^3)\Big] \\ &+ 25\Big[\delta(f-(f_c+3\times 10^3)) + \delta(f+f_c+3\times 10^3) \\ &+ \delta(f-(f_c-3\times 10^3)) + \delta(f+f_c-3\times 10^3)\Big] \end{aligned}$$

The spectrum of the signal is depicted in the next figure



2) The average power in the frequencies $f_c + 1000$ and $f_c - 1000$ is

$$P_{f_c+1000} = P_{f_c-1000} = \frac{100^2}{2} = 5000$$

The average power in the frequencies $f_c + 3000$ and $f_c - 3000$ is

$$P_{f_c+3000} = P_{f_c-3000} = \frac{50^2}{2} = 1250$$

Problem 3.16

1) The Hilbert transform of $\cos(2\pi 1000t)$ is $\sin(2\pi 1000t)$, whereas the Hilbert transform of$\sin(2\pi 1000t)$ is $-\cos(2\pi 1000t)$. Thus

$$\hat{m}(t) = \sin(2\pi 1000t) - 2\cos(2\pi 1000t)$$

2) The expression for the LSSB AM signal is

$$u_l(t) = A_c m(t)\cos(2\pi f_c t) + A_c \hat{m}(t)\sin(2\pi f_c t)$$

Substituting $A_c = 100$, $m(t) = \cos(2\pi 1000t) + 2\sin(2\pi 1000t)$ and $\hat{m}(t) = \sin(2\pi 1000t) - 2\cos(2\pi 1000t)$ in the previous, we obtain

$$\begin{aligned} u_l(t) &= 100\left[\cos(2\pi 1000t) + 2\sin(2\pi 1000t)\right]\cos(2\pi f_c t) \\ &+ 100\left[\sin(2\pi 1000t) - 2\cos(2\pi 1000t)\right]\sin(2\pi f_c t) \\ &= 100\left[\cos(2\pi 1000t)\cos(2\pi f_c t) + \sin(2\pi 1000t)\sin(2\pi f_c t)\right] \\ &+ 200\left[\cos(2\pi f_c t)\sin(2\pi 1000t) - \sin(2\pi f_c t)\cos(2\pi 1000t)\right] \\ &= 100\cos(2\pi(f_c - 1000)t) - 200\sin(2\pi(f_c - 1000)t) \end{aligned}$$

3) Taking the Fourier transform of the previous expression we obtain

$$\begin{aligned} U_l(f) &= 50\left(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)\right) \\ &+ 100j\left(\delta(f - f_c + 1000) - \delta(f + f_c - 1000)\right) \\ &= (50 + 100j)\delta(f - f_c + 1000) + (50 - 100j)\delta(f + f_c - 1000) \end{aligned}$$

Hence, the magnitude spectrum is given by

$$\begin{aligned} |U_l(f)| &= \sqrt{50^2 + 100^2}\left(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)\right) \\ &= 10\sqrt{125}\left(\delta(f - f_c + 1000) + \delta(f + f_c - 1000)\right) \end{aligned}$$

Problem 3.17

The input to the upper LPF is

$$\begin{aligned} u_u(t) &= \cos(2\pi f_m t)\cos(2\pi f_1 t) \\ &= \frac{1}{2}[\cos(2\pi(f_1 - f_m)t) + \cos(2\pi(f_1 + f_m)t)] \end{aligned}$$

whereas the input to the lower LPF is

$$\begin{aligned} u_l(t) &= \cos(2\pi f_m t)\sin(2\pi f_1 t) \\ &= \frac{1}{2}[\sin(2\pi(f_1 - f_m)t) + \sin(2\pi(f_1 + f_m)t)] \end{aligned}$$

If we select f_1 such that $|f_1 - f_m| < W$ and $f_1 + f_m > W$, then the two lowpass filters will cut-off the frequency components outside the interval $[-W, W]$, so that the output of the upper and lower LPF is

$$\begin{aligned} y_u(t) &= \cos(2\pi(f_1 - f_m)t) \\ y_l(t) &= \sin(2\pi(f_1 - f_m)t) \end{aligned}$$

The output of the Weaver's modulator is

$$u(t) = \cos(2\pi(f_1 - f_m)t)\cos(2\pi f_2 t) - \sin(2\pi(f_1 - f_m)t)\sin(2\pi f_2 t)$$

which has the form of a SSB signal since $\sin(2\pi(f_1 - f_m)t)$ is the Hilbert transform of $\cos(2\pi(f_1 - f_m)t)$. If we write $u(t)$ as

$$u(t) = \cos(2\pi(f_1 + f_2 - f_m)t)$$

then with $f_1 + f_2 - f_m = f_c + f_m$ we obtain an USSB signal centered at f_c, whereas with $f_1 + f_2 - f_m = f_c - f_m$ we obtain the LSSB signal. In both cases the choice of f_c and f_1 uniquely determine f_2.

Problem 3.18

The signal $x(t)$ is $m(t) + \cos(2\pi f_0 t)$. The spectrum of this signal is $X(f) = M(f) + \frac{1}{2}(\delta(f - f_0) + \delta(f + f_0))$ and its bandwidth equals to $W_x = f_0$. The signal $y_1(t)$ after the Square Law Device is

$$\begin{aligned} y_1(t) &= x^2(t) = (m(t) + \cos(2\pi f_0 t))^2 \\ &= m^2(t) + \cos^2(2\pi f_0 t) + 2m(t)\cos(2\pi f_0 t) \\ &= m^2(t) + \frac{1}{2} + \frac{1}{2}\cos(2\pi 2 f_0 t) + 2m(t)\cos(2\pi f_0 t) \end{aligned}$$

The spectrum of this signal is given by

$$Y_1(f) = M(f) \star M(f) + \frac{1}{2}\delta(f) + \frac{1}{4}(\delta(f - 2f_0) + \delta(f + 2f_0)) + M(f - f_0) + M(f + f_0)$$

and its bandwidth is $W_1 = 2f_0$. The bandpass filter will cut-off the low-frequency components $M(f) \star M(f) + \frac{1}{2}\delta(f)$ and the terms with the double frequency components $\frac{1}{4}(\delta(f-2f_0)+\delta(f+2f_0))$. Thus the spectrum $Y_2(f)$ is given by

$$Y_2(f) = M(f - f_0) + M(f + f_0)$$

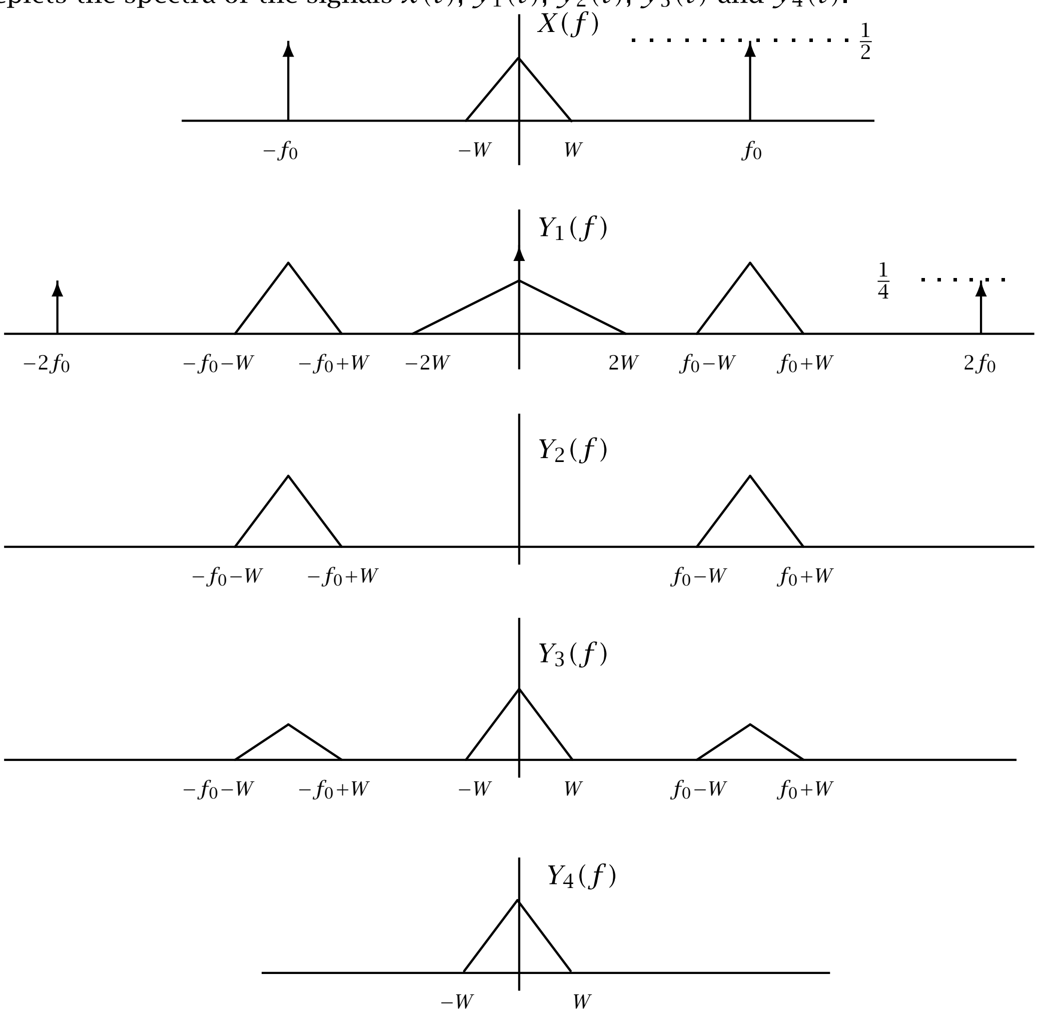
and the bandwidth of $y_2(t)$ is $W_2 = 2W$. The signal $y_3(t)$ is

$$y_3(t) = 2m(t)\cos^2(2\pi f_0 t) = m(t) + m(t)\cos(2\pi f_0 t)$$

with spectrum

$$Y_3(t) = M(f) + \frac{1}{2}(M(f - f_0) + M(f + f_0))$$

and bandwidth $W_3 = f_0 + W$. The lowpass filter will eliminate the spectral components $\frac{1}{2}(M(f - f_0) + M(f + f_0))$, so that $y_4(t) = m(t)$ with spectrum $Y_4 = M(f)$ and bandwidth $W_4 = W$. The next figure depicts the spectra of the signals $x(t)$, $y_1(t)$, $y_2(t)$, $y_3(t)$ and $y_4(t)$.



Problem 3.19

1)

$$\begin{aligned} y(t) &= ax(t) + bx^2(t) \\ &= a(m(t) + \cos(2\pi f_0 t)) + b(m(t) + \cos(2\pi f_0 t))^2 \\ &= am(t) + bm^2(t) + a\cos(2\pi f_0 t) \\ &\quad + b\cos^2(2\pi f_0 t) + 2bm(t)\cos(2\pi f_0 t) \end{aligned}$$

2) The filter should reject the low frequency components, the terms of double frequency and pass only the signal with spectrum centered at f_0. Thus the filter should be a BPF with center frequency f_0 and bandwidth W such that $f_0 - W_M > f_0 - \frac{W}{2} > 2W_M$ where W_M is the bandwidth of the message signal $m(t)$.

3) The AM output signal can be written as

$$u(t) = a(1 + \frac{2b}{a}m(t))\cos(2\pi f_0 t)$$

Since $A_m = \max[|m(t)|]$ we conclude that the modulation index is

$$\alpha = \frac{2bA_m}{a}$$

Problem 3.20

1) When USSB is employed the bandwidth of the modulated signal is the same with the bandwidth of the message signal. Hence,

$$W_{\text{USSB}} = W = 10^4 \text{ Hz}$$

2) When DSB is used, then the bandwidth of the transmitted signal is twice the bandwidth of the message signal. Thus,

$$W_{\text{DSB}} = 2W = 2 \times 10^4 \text{ Hz}$$

3) If conventional AM is employed, then

$$W_{\text{AM}} = 2W = 2 \times 10^4 \text{ Hz}$$

4) Using Carson's rule, the effective bandwidth of the FM modulated signal is

$$B_c = (2\beta + 1)W = 2\left(\frac{k_f \max[|m(t)|]}{W} + 1\right)W = 2(k_f + W) = 140000 \text{ Hz}$$

Problem 3.21

1) The lowpass equivalent transfer function of the system is

$$H_l(f) = 2u_{-1}(f+f_c)H(f+f_c) = 2\begin{cases} \frac{1}{W}f + \frac{1}{2} & |f| \leq \frac{W}{2} \\ 1 & \frac{W}{2} < f \leq W \end{cases}$$

Taking the inverse Fourier transform, we obtain

$$\begin{aligned} h_l(t) &= \mathcal{F}^{-1}[H_l(f)] = \int_{-\frac{W}{2}}^{W} H_l(f)e^{j2\pi ft}df \\ &= 2\int_{-\frac{W}{2}}^{\frac{W}{2}} (\frac{1}{W}f + \frac{1}{2})e^{j2\pi ft}df + 2\int_{\frac{W}{2}}^{W} e^{j2\pi ft}df \\ &= \frac{2}{W}\left(\frac{1}{j2\pi t}fe^{j2\pi ft} + \frac{1}{4\pi^2t^2}e^{j2\pi ft}\right)\Bigg|_{-\frac{W}{2}}^{\frac{W}{2}} + \frac{1}{j2\pi t}e^{j2\pi ft}\Bigg|_{-\frac{W}{2}}^{\frac{W}{2}} + \frac{2}{j2\pi t}e^{j2\pi ft}\Bigg|_{\frac{W}{2}}^{W} \\ &= \frac{1}{j\pi t}e^{j2\pi Wt} + \frac{j}{\pi^2t^2W}\sin(\pi Wt) \\ &= \frac{j}{\pi t}\left[\text{sinc}(Wt) - e^{j2\pi Wt}\right] \end{aligned}$$

2) An expression for the modulated signal is obtained as follows

$$\begin{aligned} u(t) &= \text{Re}[(m(t) \star h_l(t))e^{j2\pi f_c t}] \\ &= \text{Re}\left[(m(t) \star \frac{j}{\pi t}(\text{sinc}(Wt) - e^{j2\pi Wt}))e^{j2\pi f_c t}\right] \\ &= \text{Re}\left[(m(t) \star (\frac{j}{\pi t}\text{sinc}(Wt)))e^{j2\pi f_c t} + (m(t) \star \frac{1}{j\pi t}e^{j2\pi Wt})e^{j2\pi f_c t}\right] \end{aligned}$$

Note that

$$\mathcal{F}[m(t) \star \frac{1}{j\pi t}e^{j2\pi Wt}] = -M(f)\text{sgn}(f-W) = M(f)$$

since $\text{sgn}(f-W) = -1$ for $f < W$. Thus,

$$\begin{aligned} u(t) &= \text{Re}\left[(m(t) \star (\frac{j}{\pi t}\text{sinc}(Wt)))e^{j2\pi f_c t} + m(t)e^{j2\pi f_c t}\right] \\ &= m(t)\cos(2\pi f_c t) - m(t) \star (\frac{1}{\pi t}\text{sinc}(Wt))\sin(2\pi f_c t) \end{aligned}$$

Problem 3.22

a) A DSB modulated signal is written as

$$\begin{aligned} u(t) &= Am(t)\cos(2\pi f_0 t + \phi) \\ &= Am(t)\cos(\phi)\cos(2\pi f_0 t) - Am(t)\sin(\phi)\sin(2\pi f_0 t) \end{aligned}$$

Hence,

$$\begin{aligned}
x_c(t) &= Am(t)\cos(\phi)\\
x_s(t) &= Am(t)\sin(\phi)\\
V(t) &= \sqrt{A^2m^2(t)(\cos^2(\phi)+\sin^2(\phi))} = |Am(t)|\\
\Theta(t) &= \arctan\left(\frac{Am(t)\cos(\phi)}{Am(t)\sin(\phi)}\right) = \arctan(\tan(\phi)) = \phi
\end{aligned}$$

b) A SSB signal has the form

$$u_{\mathrm{SSB}}(t) = Am(t)\cos(2\pi f_0 t) \mp A\hat{m}(t)\sin(2\pi f_0 t)$$

Thus, for the USSB signal (minus sign)

$$\begin{aligned}
x_c(t) &= Am(t)\\
x_s(t) &= A\hat{m}(t)\\
V(t) &= \sqrt{A^2(m^2(t)+\hat{m}^2(t))} = A\sqrt{m^2(t)+\hat{m}^2(t)}\\
\Theta(t) &= \arctan\left(\frac{\hat{m}(t)}{m(t)}\right)
\end{aligned}$$

For the LSSB signal (plus sign)

$$\begin{aligned}
x_c(t) &= Am(t)\\
x_s(t) &= -A\hat{m}(t)\\
V(t) &= \sqrt{A^2(m^2(t)+\hat{m}^2(t))} = A\sqrt{m^2(t)+\hat{m}^2(t)}\\
\Theta(t) &= \arctan\left(-\frac{\hat{m}(t)}{m(t)}\right)
\end{aligned}$$

c) If conventional AM is employed, then

$$\begin{aligned}
u(t) &= A(1+m(t))\cos(2\pi f_0 t + \phi)\\
&= A(1+m(t))\cos(\phi)\cos(2\pi f_0 t) - A(1+m(t))\sin(\phi)\sin(2\pi f_0 t)
\end{aligned}$$

Hence,

$$\begin{aligned}
x_c(t) &= A(1+m(t))\cos(\phi)\\
x_s(t) &= A(1+m(t))\sin(\phi)\\
V(t) &= \sqrt{A^2(1+m(t))^2(\cos^2(\phi)+\sin^2(\phi))} = A|(1+m(t))|\\
\Theta(t) &= \arctan\left(\frac{A(1+m(t))\cos(\phi)}{A(1+m(t))\sin(\phi)}\right) = \arctan(\tan(\phi)) = \phi
\end{aligned}$$

Problem 3.23

1) If SSB is employed, the transmitted signal is

$$u(t) = Am(t)\cos(2\pi f_0 t) \mp A\hat{m}(t)\sin(2\pi f_0 t)$$

Provided that the spectrum of $m(t)$ does not contain any impulses at the origin $P_M = P_{\hat{M}} = \frac{1}{2}$ and

$$P_{\text{SSB}} = \frac{A^2 P_M}{2} + \frac{A^2 P_{\hat{M}}}{2} = A^2 P_M = 400\frac{1}{2} = 200$$

The bandwidth of the modulated signal $u(t)$ is the same with that of the message signal. Hence,

$$W_{\text{SSB}} = 10000 \text{ Hz}$$

2) In the case of DSB-SC modulation $u(t) = Am(t)\cos(2\pi f_0 t)$. The power content of the modulated signal is

$$P_{\text{DSB}} = \frac{A^2 P_M}{2} = 200\frac{1}{2} = 100$$

and the bandwidth $W_{\text{DSB}} = 2W = 20000$ Hz.

3) If conventional AM is employed with modulation index $\alpha = 0.6$, the transmitted signal is

$$u(t) = A[1 + \alpha m(t)]\cos(2\pi f_0 t)$$

The power content is

$$P_{\text{AM}} = \frac{A^2}{2} + \frac{A^2\alpha^2 P_M}{2} = 200 + 200 \cdot 0.6^2 \cdot 0.5 = 236$$

The bandwidth of the signal is $W_{\text{AM}} = 2W = 20000$ Hz.

4) If the modulation is FM with $k_f = 50000$, then

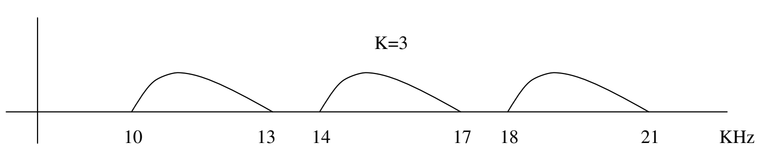
$$P_{\text{FM}} = \frac{A^2}{2} = 200$$

and the effective bandwidth is approximated by Carson's rule as

$$B_c = 2(\beta + 1)W = 2\left(\frac{50000}{W} + 1\right)W = 120000 \text{ Hz}$$

Problem 3.24

1) The next figure illustrates the spectrum of the SSB signal assuming that USSB is employed and $K = 3$. Note, that only the spectrum for the positive frequencies has been plotted.



2) With $LK = 60$ the possible values of the pair (L, K) (or (K, L)) are $\{(1,60), (2,30), (3,20), (4,15), (6,10)\}$. As it is seen the minimum value of $L + K$ is achieved for $L = 6$, $K = 10$ (or $L = 10$, $K = 6$).

3) Assuming that $L = 6$ and $K = 10$ we need 16 carriers with frequencies

$$\begin{array}{ll} f_{k_1} = 10 \text{ KHz} & f_{k_2} = 14 \text{ KHz} \\ f_{k_3} = 18 \text{ KHz} & f_{k_4} = 22 \text{ KHz} \\ f_{k_5} = 26 \text{ KHz} & f_{k_6} = 30 \text{ KHz} \\ f_{k_7} = 34 \text{ KHz} & f_{k_8} = 38 \text{ KHz} \\ f_{k_9} = 42 \text{ KHz} & f_{k_{10}} = 46 \text{ KHz} \end{array}$$

and

$$\begin{array}{ll} f_{l_1} = 290 \text{ KHz} & f_{l_2} = 330 \text{ KHz} \\ f_{l_3} = 370 \text{ KHz} & f_{l_4} = 410 \text{ KHz} \\ f_{l_5} = 450 \text{ KHz} & f_{l_6} = 490 \text{ KHz} \end{array}$$

Computer Problems

Computer Problem 3.1

1) Figures 3.1 and 3.2 present the message and modulated signals, respectively.
2) Spectrum of $m(t)$ and $u(t)$ are given in Figures 3.3 and 3.4, respectively.
3) Figures 3.5, 3.6, 3.7 and 3.8 present the message signal, modulated signal, spectrum of the message signal and the spectrum of the modulated signal for $t_0 = 0.4$.
The MATLAB script for this problem follows.

```
% MATLAB script for Computer Problem 3.1.
% Matlab demonstration script for DSB-AM modulation. The message signal
% is m(t)=sinc(100t).
echo off
t0=.4;                          % signal duration
ts=0.0001;                      % sampling interval
fc=250;                         % carrier frequency
snr=20;                         % SNR in dB (logarithmic)
fs=1/ts;                        % sampling frequency
df=0.3;                         % required freq. resolution
t=[0:ts:t0];                    % time vector
snr_lin=10^(snr/10);            % linear SNR
m=sinc(100*t);                  % the message signal
c=cos(2*pi*fc.*t);              % the carrier signal
u=m.*c;                         % the DSB-AM modulated signal
[M,m,df1]=fftseq(m,ts,df);      % Fourier transform
```

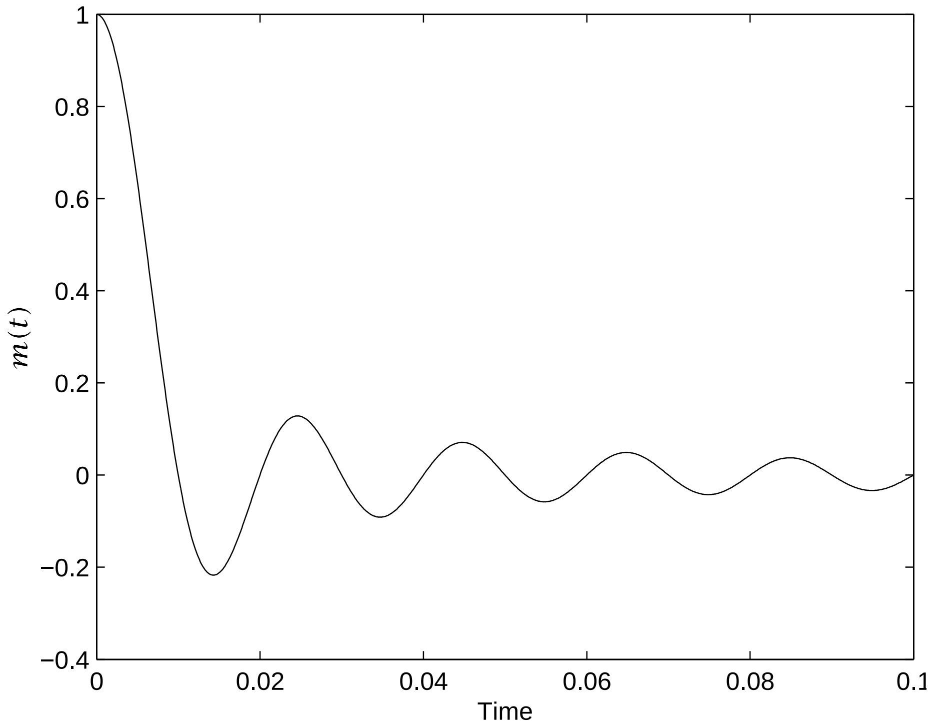


Figure 3.1: The message signal $m(t)$

```
M=M/fs;                                % scaling
[U,u,df1]=fftseq(u,ts,df);             % Fourier transform
U=U/fs;                                % scaling
f=[0:df1:df1*(length(m)−1)]−fs/2;      % frequency vector
% plot the message signal
figure;
plot(t,m(1:length(t)))
xlabel('Time')
% plot the modulated signal.
figure;
plot(t,u(1:length(t)))
xlabel('Time')
% plot the magnitude of the message and the
% modulated signal in the frequency domain.
figure;
plot(f,abs(fftshift(M)))
xlabel('Frequency')
axis([−1000 1000 0 9*10^(−3)]);
figure;
plot(f,abs(fftshift(U)))
xlabel('Frequency')
axis([−1000 1000 0 4.5*10^(−3)]);
```

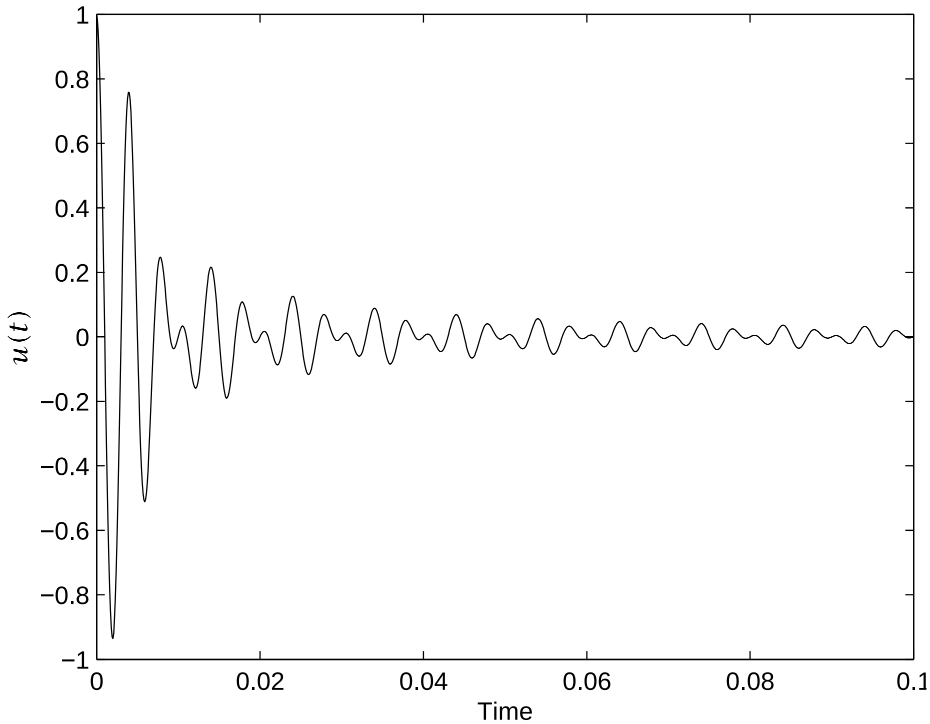


Figure 3.2: The modulated signal $u(t)$

Computer Problem 3.2

The message signal $m(t)$ and the modulated signal $u(t)$ are presented in Figures 3.9 and 3.10, respectively.

2)The spectrum of the message signal is presented in Figure 3.11. Figure 3.12 presents the spectrum of the modulated signal $u(t)$.

3) Figures 3.13, 3.14, 3.15 and 3.16 present the message signal, modulated signal, spectrum of the message signal and the spectrum of the modulated signal for $t_0 = 0.4$.

The MATLAB script for this problem follows.

```
% MATLAB script for Computer Problem 3.2.
t0=.1;                              % signal duration
n=0:1000;
ts=0.0001;                          % sampling interval
df=0.2;                             % frequency resolution
fs=1/ts;                            % sampling frequency
fc=250;                             % carrier frequency
a=0.8;                              % modulation index
t=[0:ts:t0];                        % time vector
m = sinc(100*t);                    % message signal
c=cos(2*pi*fc.*t);                  % carrier signal
m_n=m/max(abs(m));                  % normalized message signal
[M,m,df1]=fftseq(m,ts,df);          % Fourier transform
M=M/fs;                             % scaling
f=[0:df1:df1*(length(m)−1)]−fs/2;   % frequency vector
```

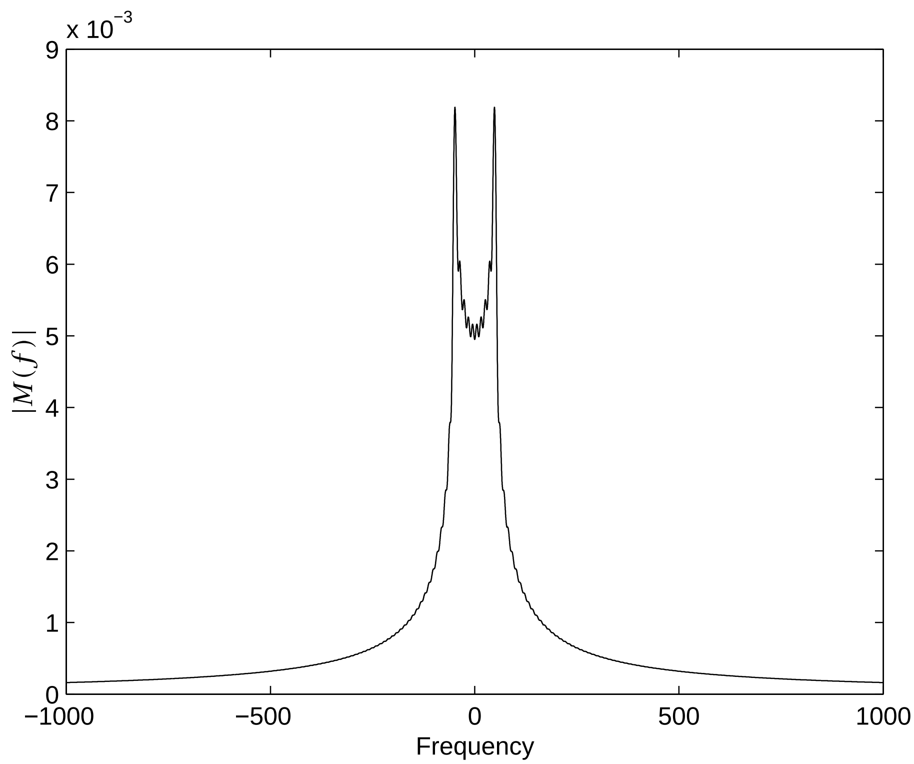


Figure 3.3: The spectrum of the message signal

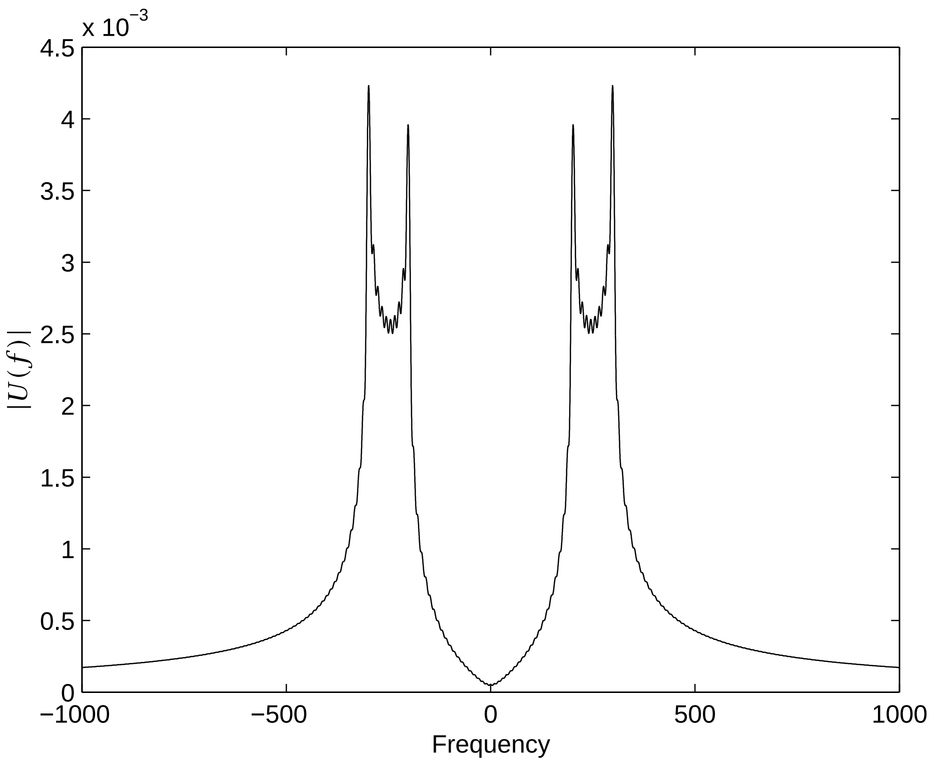


Figure 3.4: The message of the modulated signal

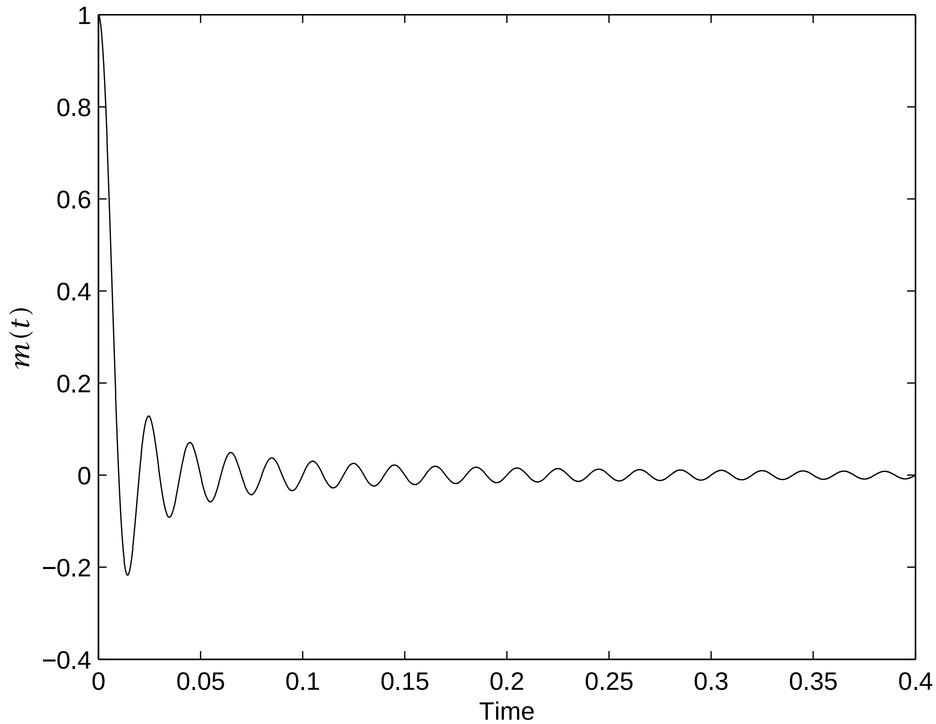


Figure 3.5: The message signal $m(t)$ for $t_0 = 0.4$

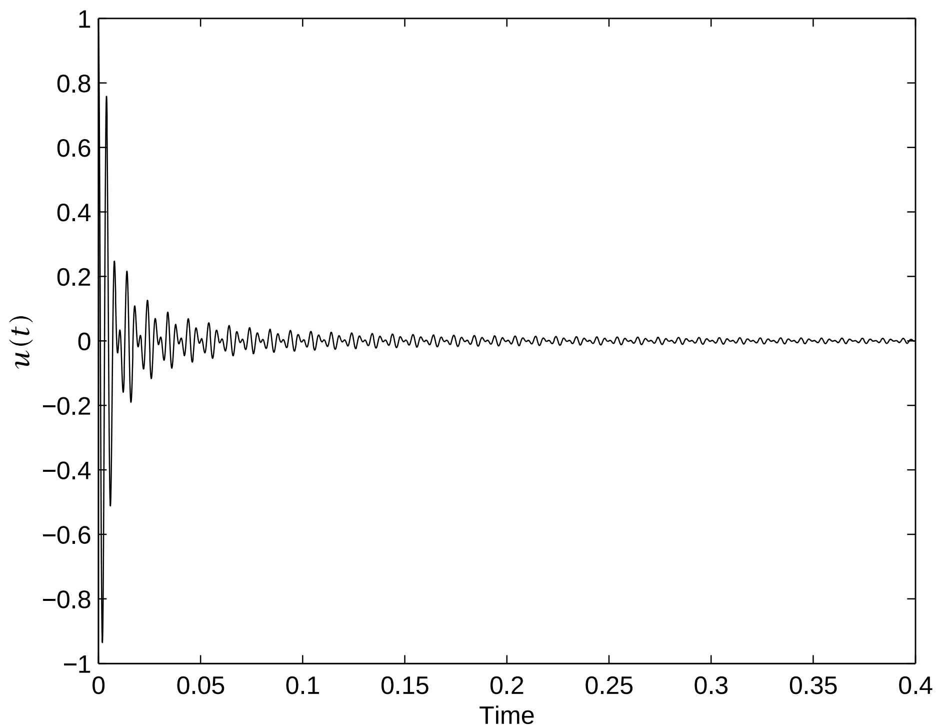


Figure 3.6: The modulated signal $u(t)$ for $t_0 = 0.4$

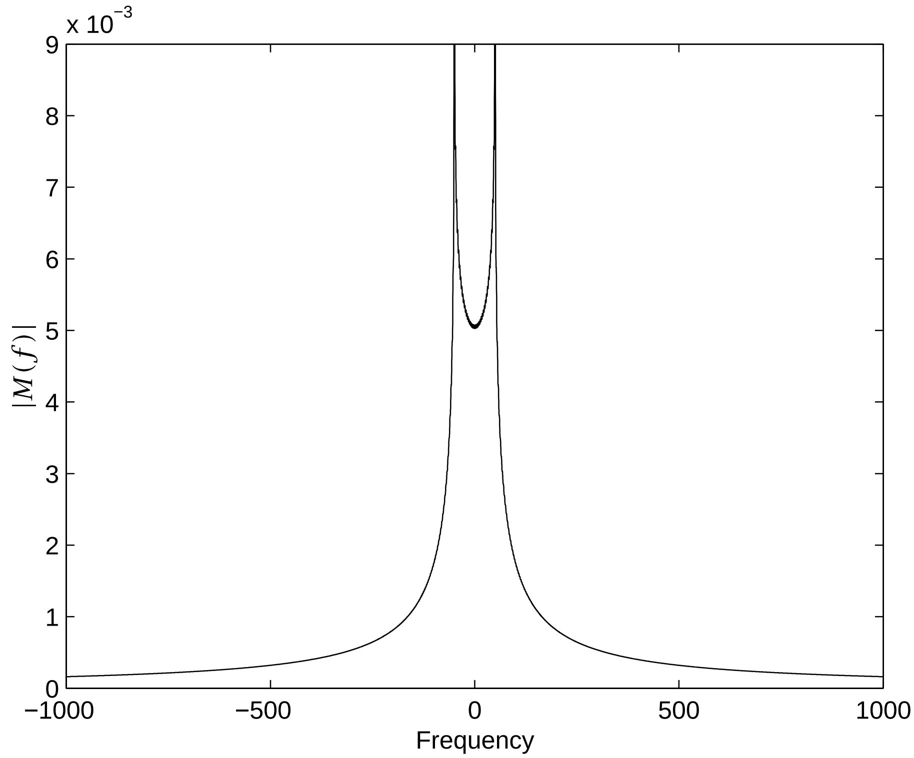


Figure 3.7: The spectrum of the message signal for $t_0 = 0.4$

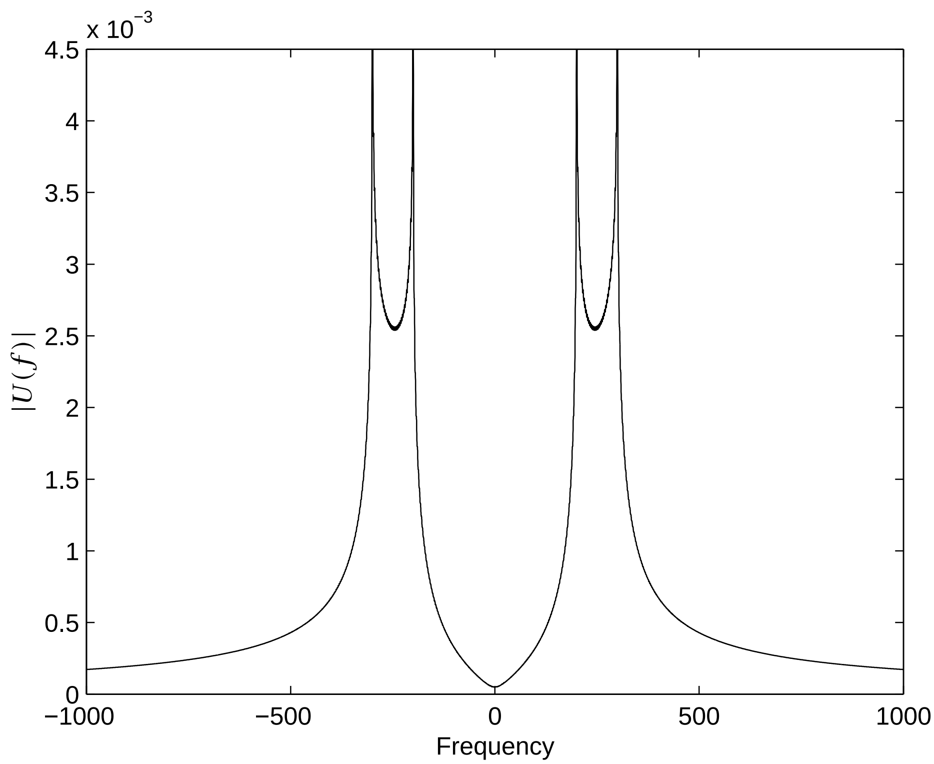


Figure 3.8: The spectrum of the modulated signal for $t_0 = 0.4$

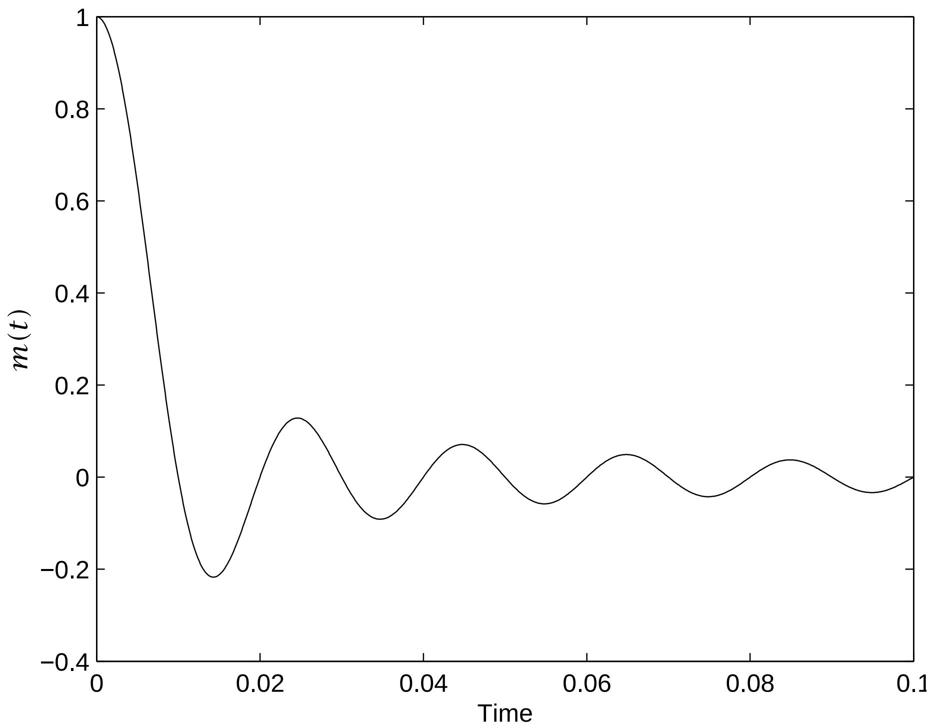


Figure 3.9: The message signal $m(t)$

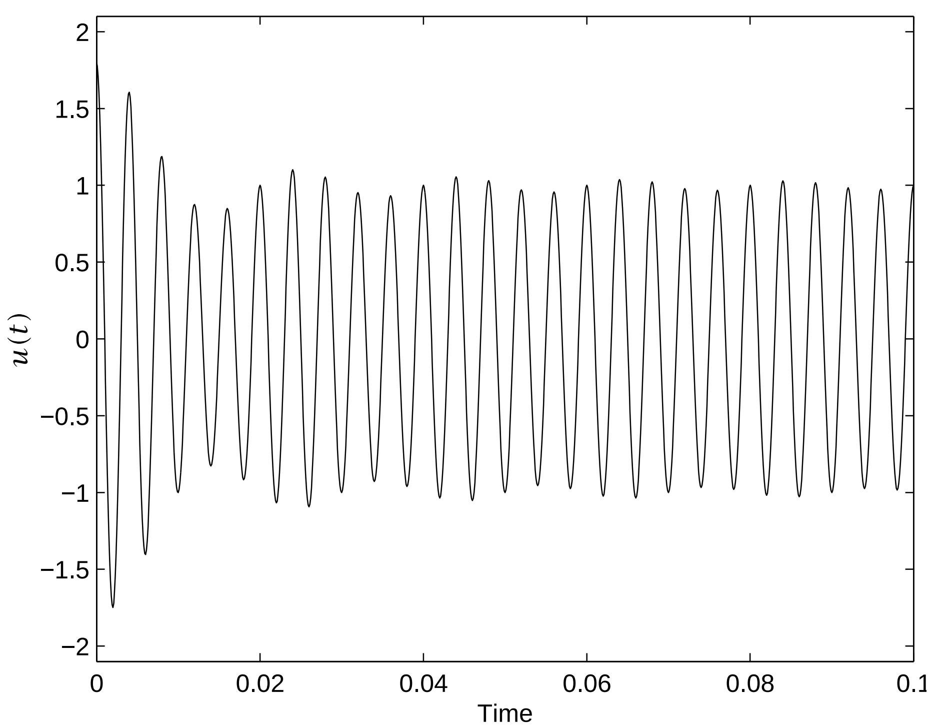


Figure 3.10: The modulated signal $u(t)$

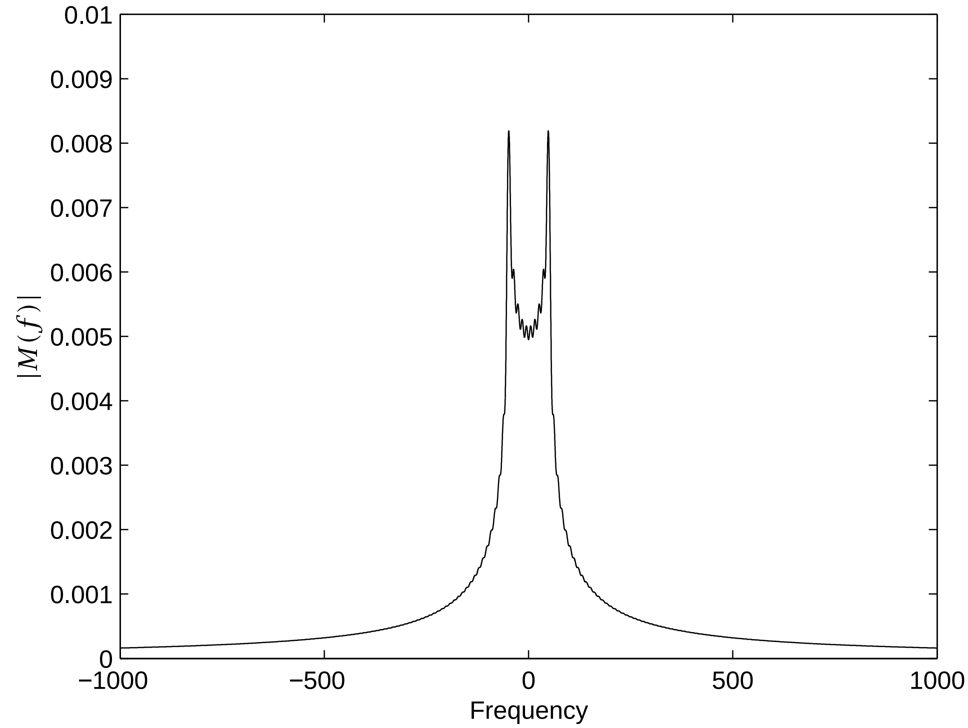


Figure 3.11: The spectrum of the message signal

```
u=(1+a*m_n).*c;                % modulated signal
[U,u,df1]=fftseq(u,ts,df);     % Fourier transform
U=U/fs;                        % scaling
figure;
plot(t,m(1:length(t)))
xlabel('Time')
figure;
plot(t,u(1:length(t)))
axis([0 t0 -2.1 2.1])
xlabel('Time')
figure;
plot(f,abs(fftshift(M)))
xlabel('Frequency')
axis([-1000 1000 0 0.01]);
figure;
plot(f,abs(fftshift(U)))
xlabel('Frequency')
axis([-1000 1000 0 0.06]);
```

Computer Problem 3.3

1) Figures 3.17 and 3.18 present the message signal and its Hilbert transform, respectively. The modulated signal is presented in Figure 3.19.

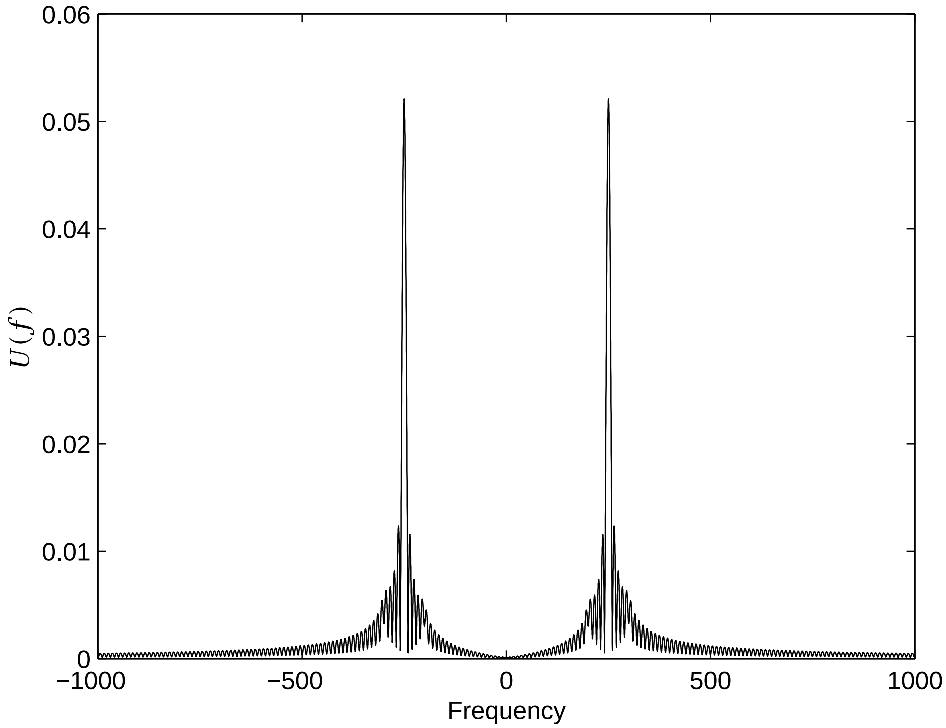


Figure 3.12: The spectrum of the modulated signal

2) The spectrum of the message signal $m(t)$ and the modulated LSSB signal $u(t)$ are presented in Figures 3.20 and 3.21, respectively.
3) Figures 3.22, 3.23, 3.24, 3.25 and 3.26 present the message signal, its Hilbert transform, modulated signal, spectrum of the message signal and the spectrum of the modulated signal for $t_0 = 0.4$, respectively.
The MATLAB script for this problem follows.

```
% MATLAB script for Computer Problem 3.3.
t0=.4;                                  % signal duration
n=0:1000;
ts=0.0001;                              % sampling interval
df=0.2;                                 % frequency resolution
fs=1/ts;                                % sampling frequency
fc=250;                                 % carrier frequency
t=[0:ts:t0];                            % time vector
m = sinc(100*t);                        % message signal
c=cos(2*pi*fc.*t);                      % carrier signal
udsb=m.*c;                                % DSB modulated signal
[UDSB,udssb,df1]=fftseq(udsb,ts,df);      % Fourier transform
UDSB=UDSB/fs;                             % scaling
f=[0:df1:df1*(length(udssb)−1)]−fs/2;     % frequency vector
n2=ceil(fc/df1);                          % location of carrier in freq. vector
% Remove the upper sideband from DSB.
UDSB(n2:length(UDSB)−n2)=zeros(size(UDSB(n2:length(UDSB)−n2)));
```

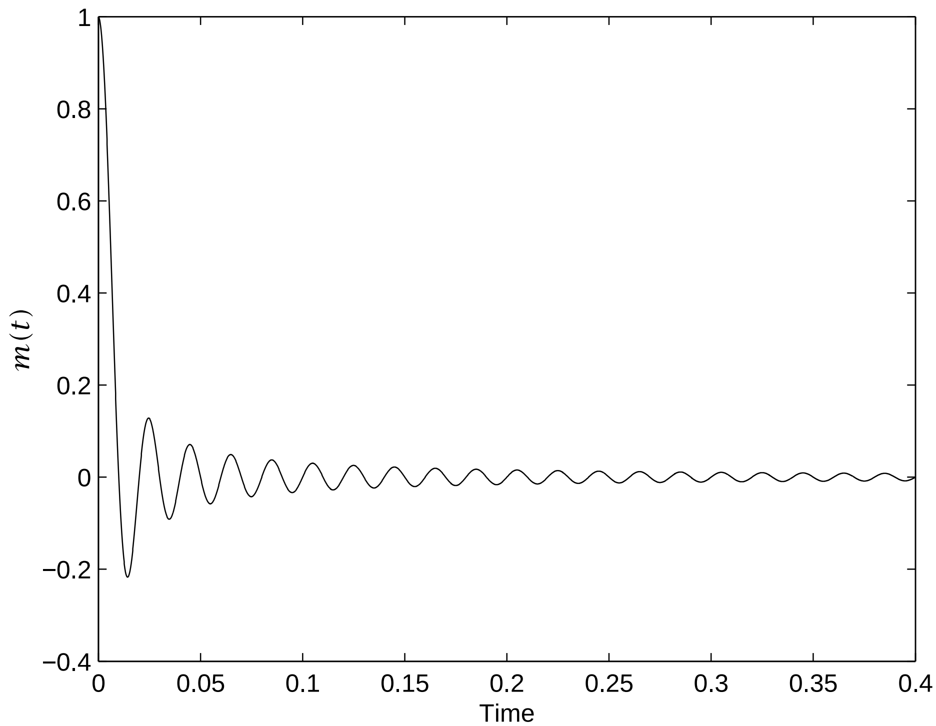


Figure 3.13: The message signal $m(t)$ for $t_0 = 0.4$

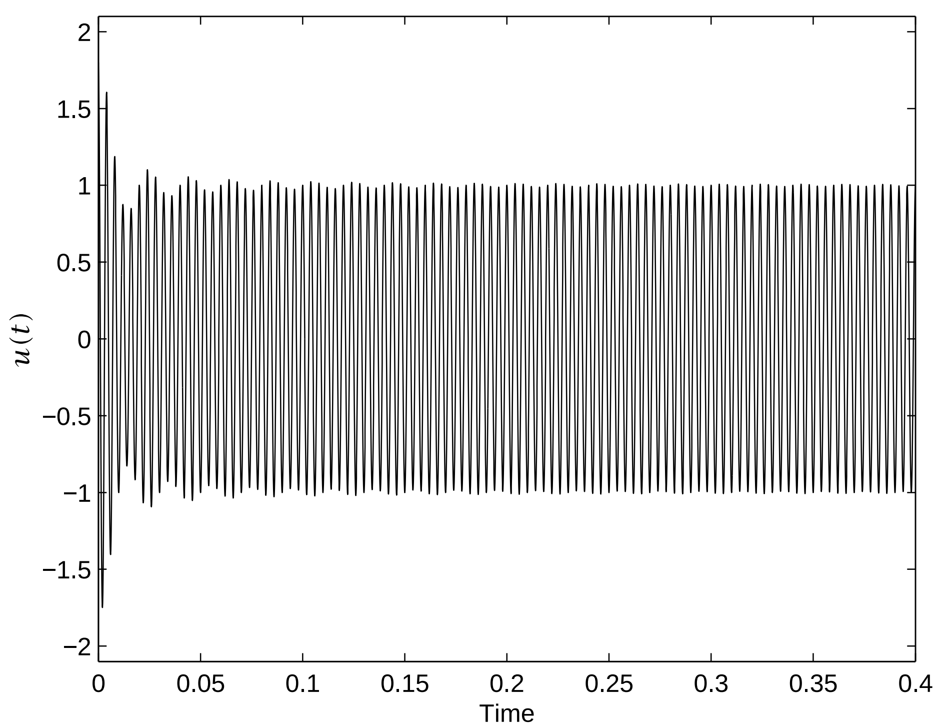


Figure 3.14: The modulated signal $u(t)$ for $t_0 = 0.4$

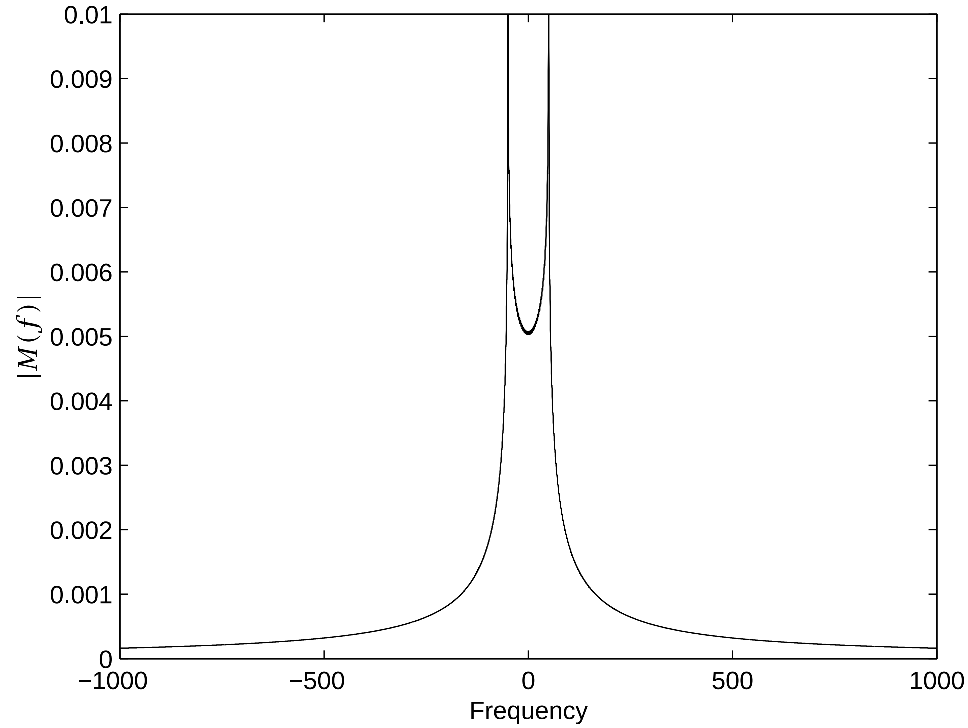


Figure 3.15: The spectrum of the message signal for $t_0 = 0.4$

```
ULSSB=UDSB;                      % Generate LSSB-AM spectrum.
[M,m,df1]=fftseq(m,ts,df);       % Fourier transform
M=M/fs;                          % scaling
u=real(ifft(ULSSB))*fs;          % Generate LSSB signal from spectrum.
%Plot the message signal
figure;
plot(t,m(1:length(t)))
xlabel('Time')
%Plot the Hilbert transform of the message signal
figure;
plot(t, imag(hilbert(m(1:length(t)))))
xlabel('Time');
%plot the LSSB-AM modulated signal
figure;
plot(t, u(1:length(t)))
xlabel('Time')
% Plot the spectrum of the message signal
figure;
plot(f,abs(fftshift(M)))
xlabel('Frequency')
axis([−1000 1000 0 0.009]);

% Plot the spectrum of the LSSB-AM modulated signal
figure;
plot(f,abs(fftshift(ULSSB)))
```

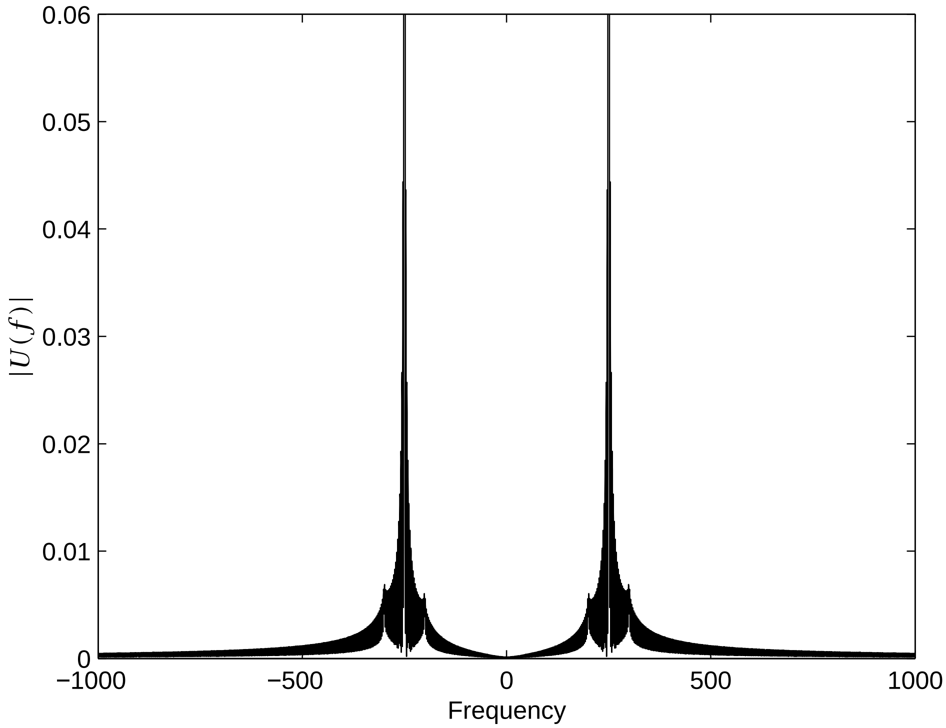


Figure 3.16: The spectrum of the modulated signal for $t_0 = 0.4$

```
xlabel('Frequency')
axis([−1000 1000 0 0.005]);
```

Computer Problem 3.4

1) The message signal $m(t)$ and the modulated signal $u(t)$ are presented in Figures 3.27 and 3.28, respectively.
2) The demodulation output is given in Figure 3.29 for $\phi = 0, \pi/8, \pi/4$, and $\pi/2$.
4) The demodulation output is given in Figure 3.30 for $\phi = 0, \pi/8, \pi/4$, and $\pi/2$.
The MATLAB script for this problem follows.

```
% MATLAB script for Computer Problem 3.4.
t0=.1;                          % signal duration
n=0:1000;
ts=0.001;                       % sampling interval
df=0.2;                         % frequency resolution
fs=1/ts;                        % sampling frequency
```

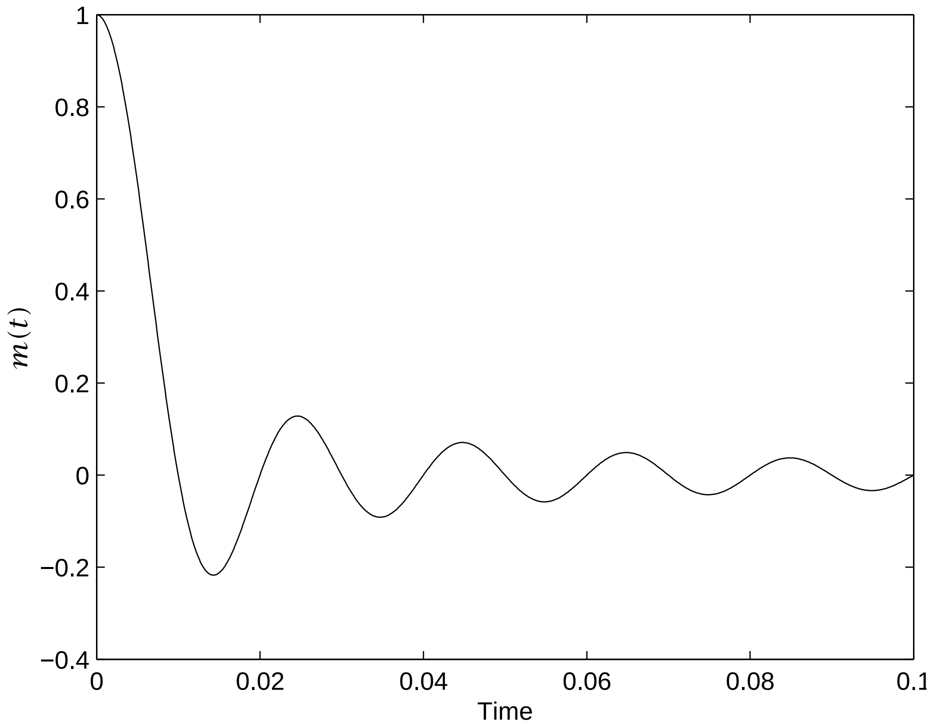


Figure 3.17: Message signal $m(t)$

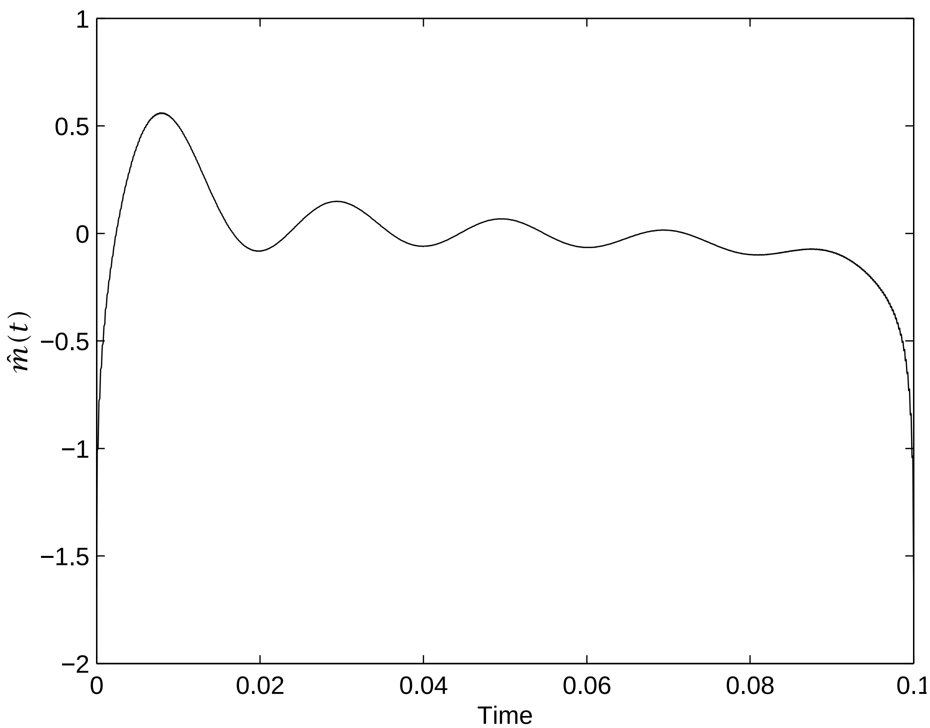


Figure 3.18: Hilbert transform of the message signal $m(t)$

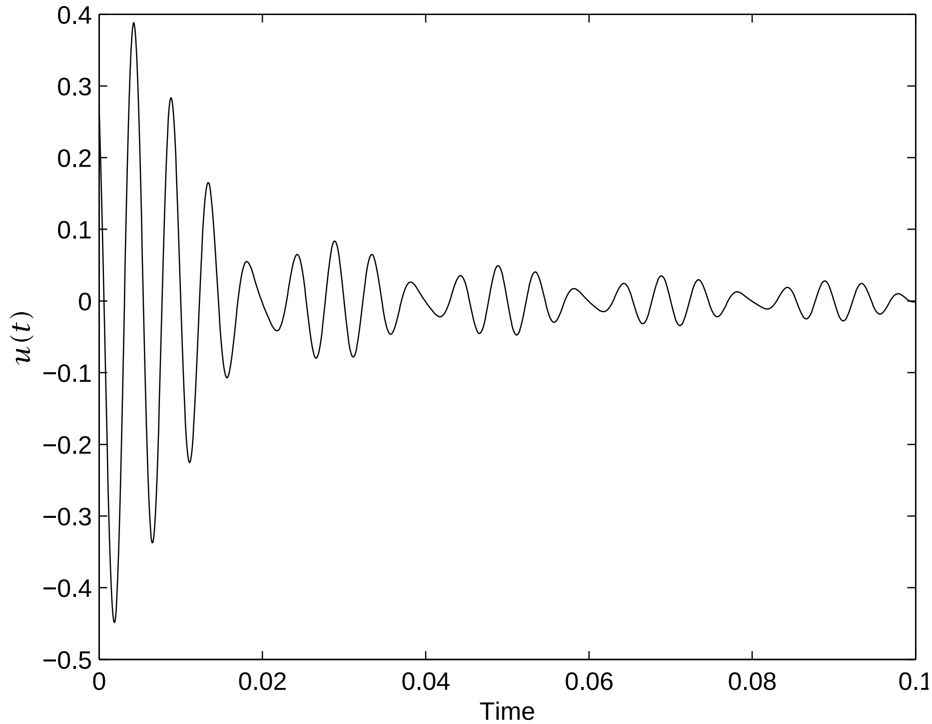


Figure 3.19: Modulated signal

```
fc=250;                          % carrier frequency
t=[0:ts:t0];                     % time vector
m = sinc(100*t);                 % message signal
c = cos(2*pi*fc.*t);
u=m.*c;                          % modulated signal
[M,m,df1]=fftseq(m,ts,df);       % Fourier transform
M=M/fs;                          % scaling
plot(t,m(1:length(t)))
xlabel('t')
figure;
plot(t,u(1:length(t)))
xlabel('t')

% design the filter
fs = 0.16;
fp = 0.0999;
f=[0 fp fs 1];
m=[1 1 0 0];
delta1 = 0.0875;
delta2 = 0.006;
df = fs - fp;
w=[delta2/delta1 1];
h=remez(31,f,m,w);

f_cutoff=100;
```

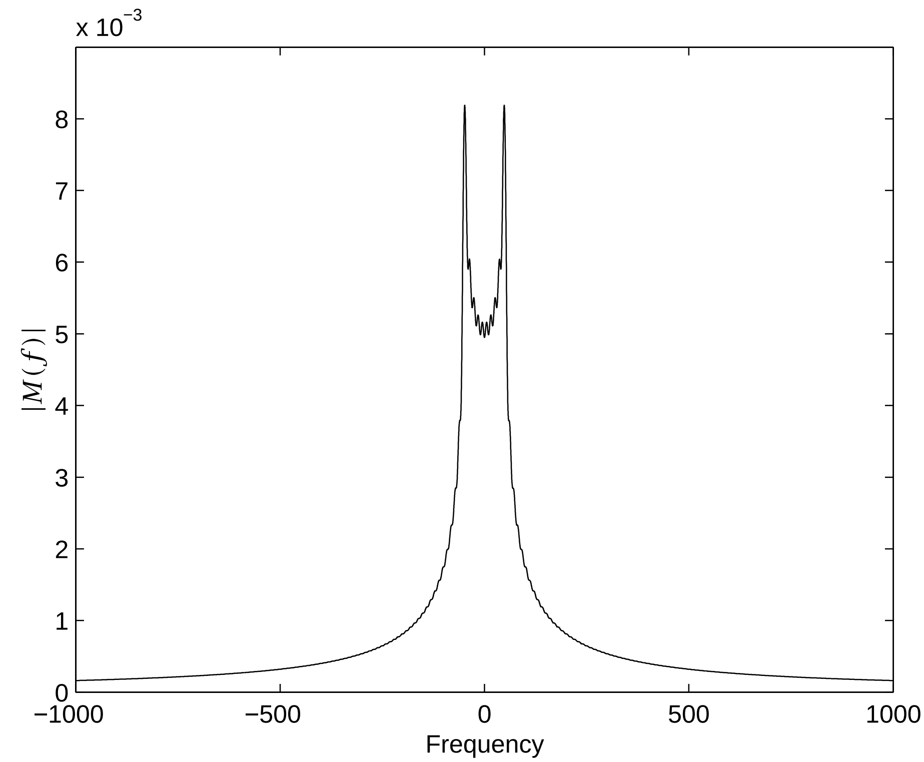


Figure 3.20: Spectrum of the message signal

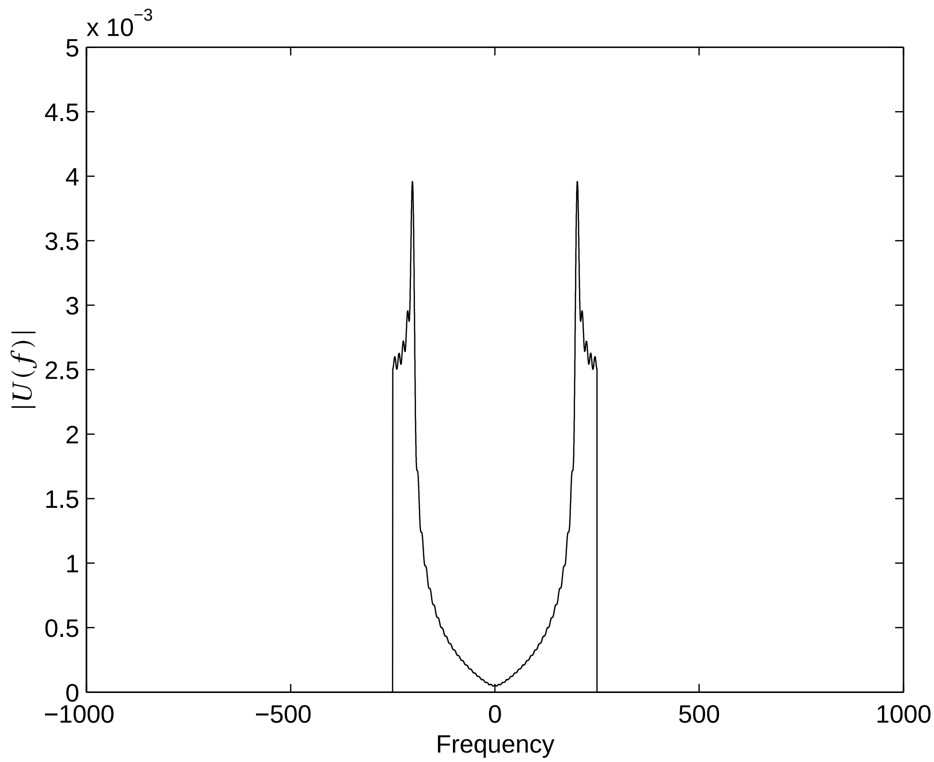


Figure 3.21: The spectrum of the modulated signal

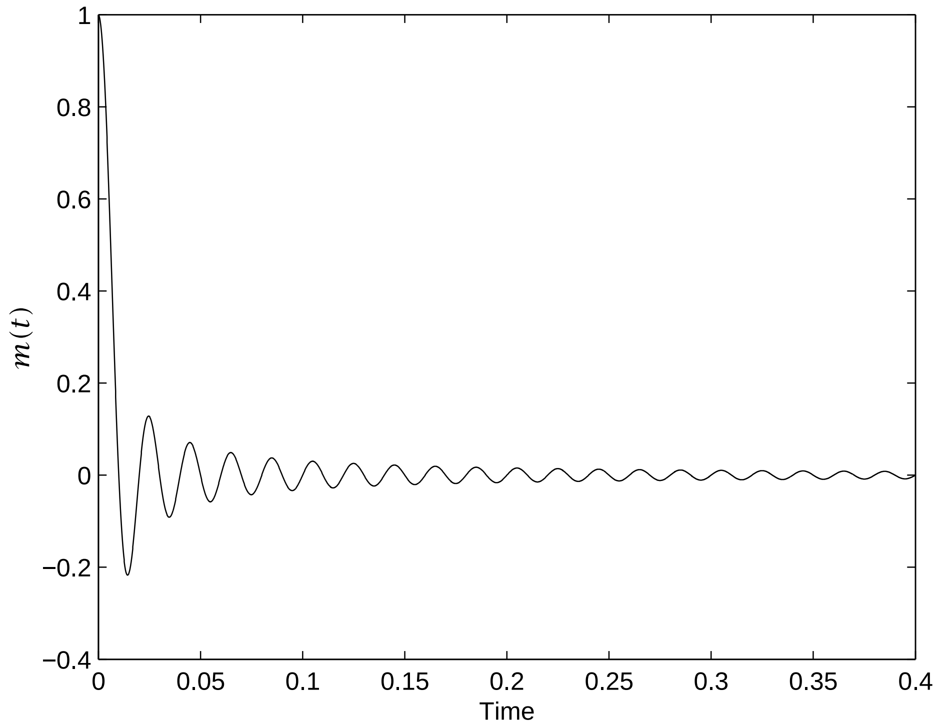


Figure 3.22: Message signal $m(t)$ for $t_0 = 0.4$

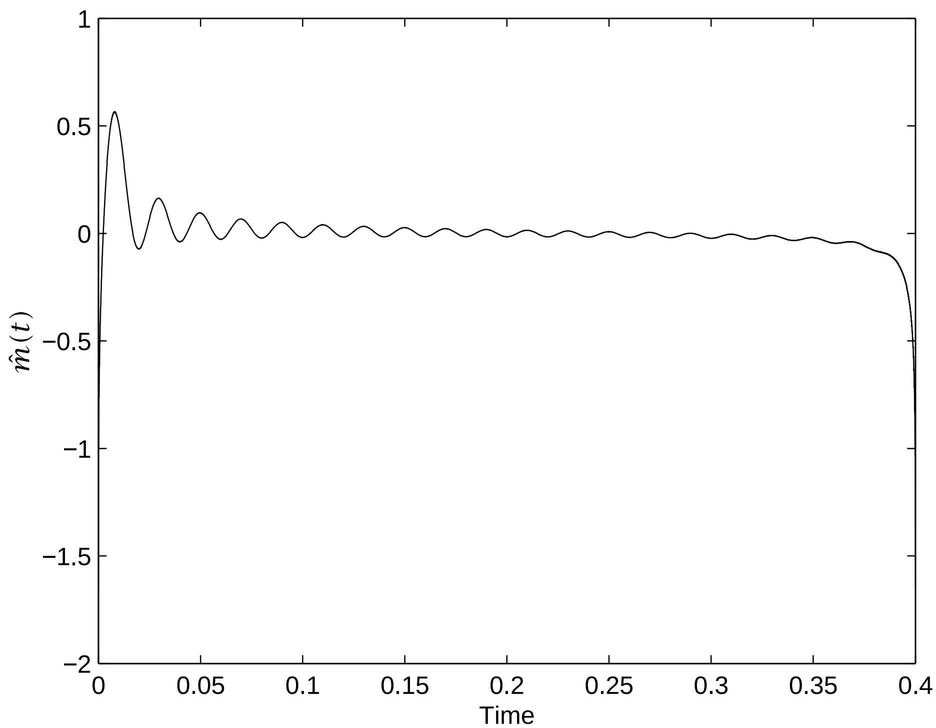


Figure 3.23: Hilbert transform of the message signal for $t_0 = 0.4$

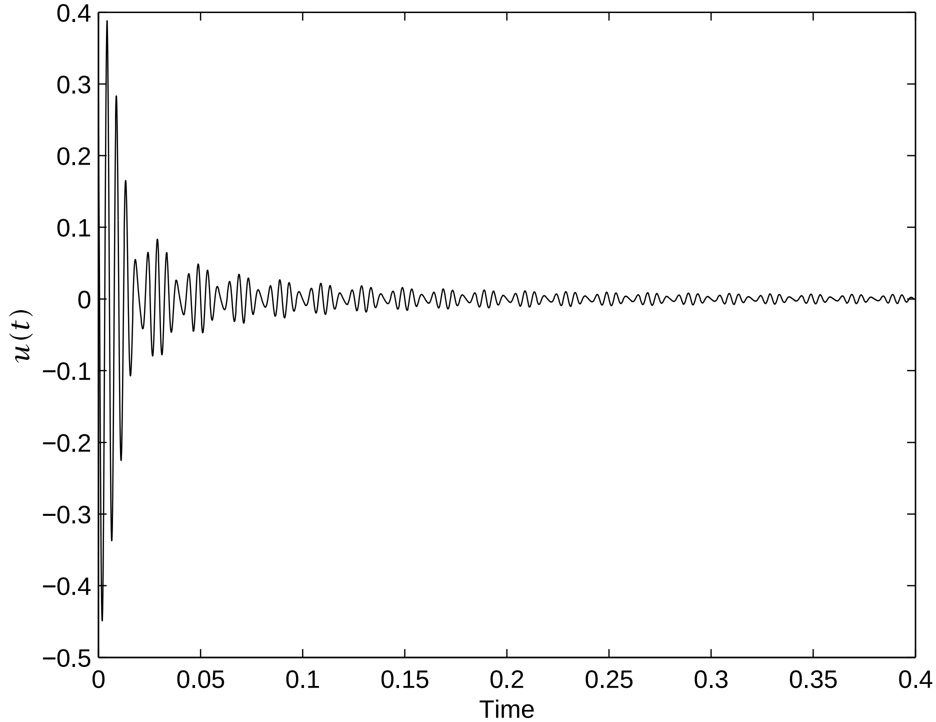


Figure 3.24: Modulated signal $u(t)$ for $t_0 = 0.4$

```
for i = 1:4
    fi = [0 pi/8 pi/4 pi/2];
    tit = ['a', 'b', 'c', 'd'];
    y=u.*cos(2*pi*fc.*t+fi(1));
    dem = filter(h, 1, y);
    figure(3);
    subplot(2,2,i);
    plot(t,dem(1:length(t)))
    xlabel('t')
    title('a');
    [Y,y,df1]=fftseq(y,ts,df);
    n_cutoff=floor(f_cutoff/df1);
    f=[0:df1:df1*(length(y)−1)]−fs/2;
    H=zeros(size(f));
    H(1:n_cutoff)=2*ones(1,n_cutoff);
    H(length(f)−n_cutoff+1:length(f))=2*ones(1,n_cutoff);
    Y=Y/fs;
    DEM=H.*Y;
    dem=real(ifft(DEM))*fs;
    figure(4);
    subplot(2,2,i);
    plot(t,dem(1:length(t)))
    xlabel('t');
    title(tit(i));
end
```

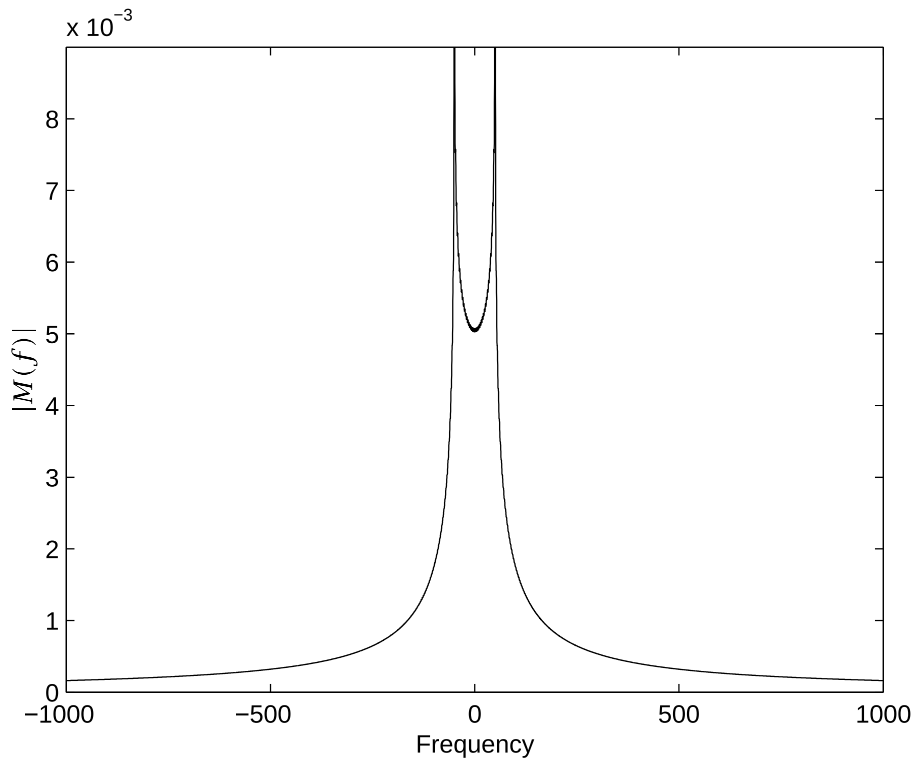


Figure 3.25: Spectrum of the message signal for $t_0 = 0.4$

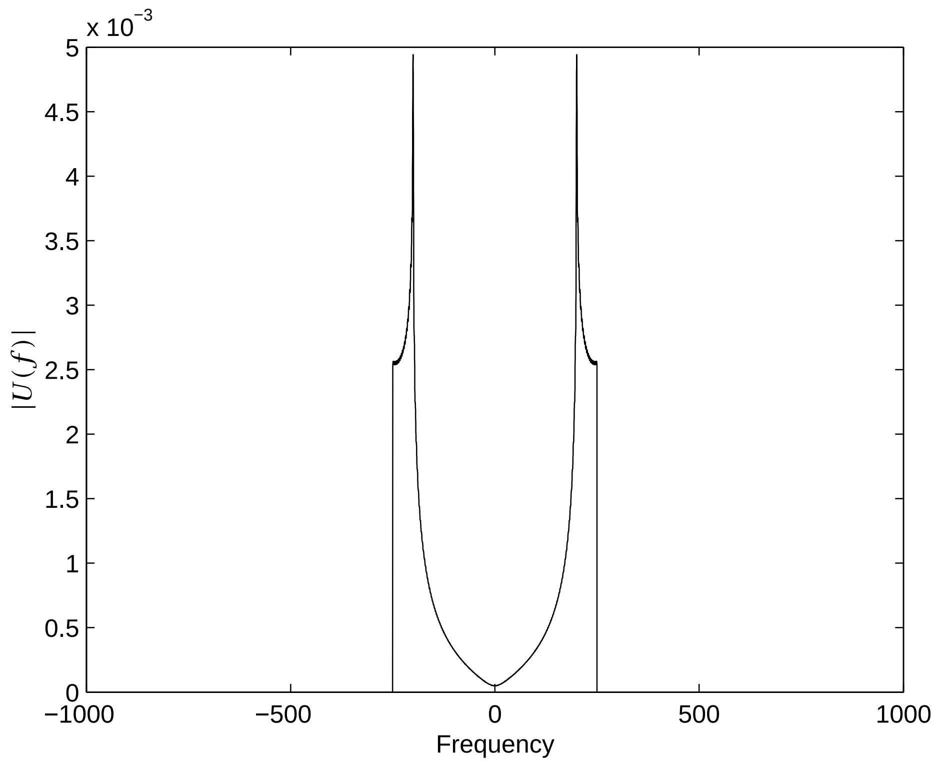


Figure 3.26: Spectrum of the modulated signal for $t_0 = 0.4$

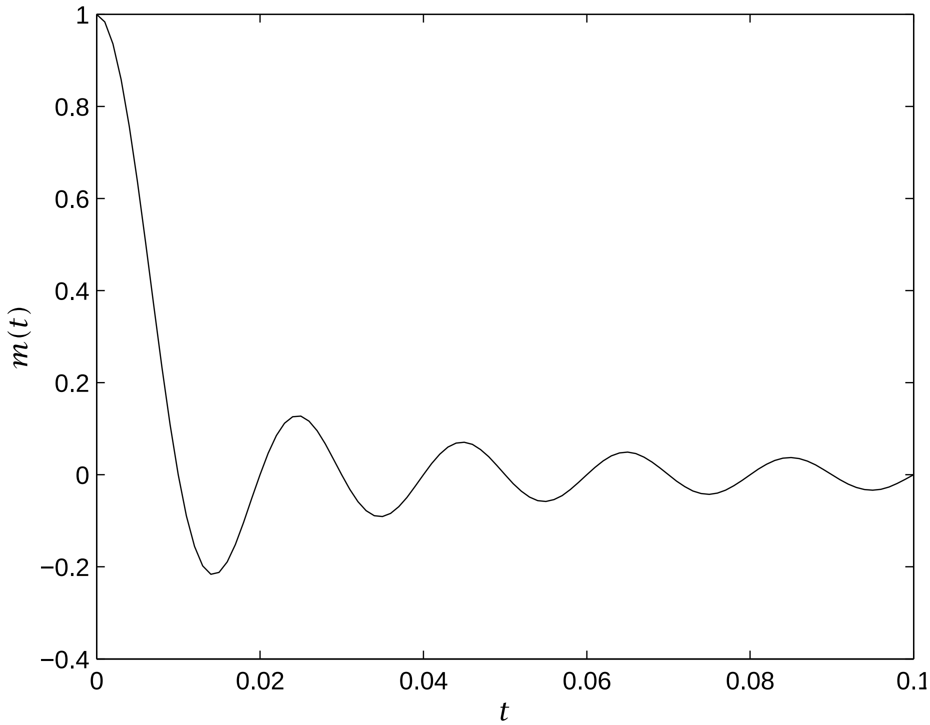


Figure 3.27: The message signal $m(t)$

Computer Problem 3.5

1) The message signal $m(t)$, the modulated signal $u(t)$ and the Hilbert transform of the message signal $\hat{m}(t)$ are presented in Figures 3.31, 3.32 and 3.33, respectively.
2) The demodulation output is given in Figure 3.34 for $\phi = 0, \pi/8, \pi/4$, and $\pi/2$.
4) The demodulation output is given in Figure 3.35 for $\phi = 0, \pi/8, \pi/4$, and $\pi/2$.
The MATLAB script for this problem follows.

```
% MATLAB script for Computer Problem 3.5.
t0=.1;                          % signal duration
ts=0.001;                       % sampling interval
df=0.1;                         % frequency resolution
fs=1/ts;                        % sampling frequency
fc=250;                         % carrier frequency
t=[0:ts:t0];                    % time vector
m = sinc(100*t);                % message signal
m_h = imag(hilbert(m));

u = m.*cos(2*pi*fc*t) + m_h .*sin(2*pi*fc*t);
```

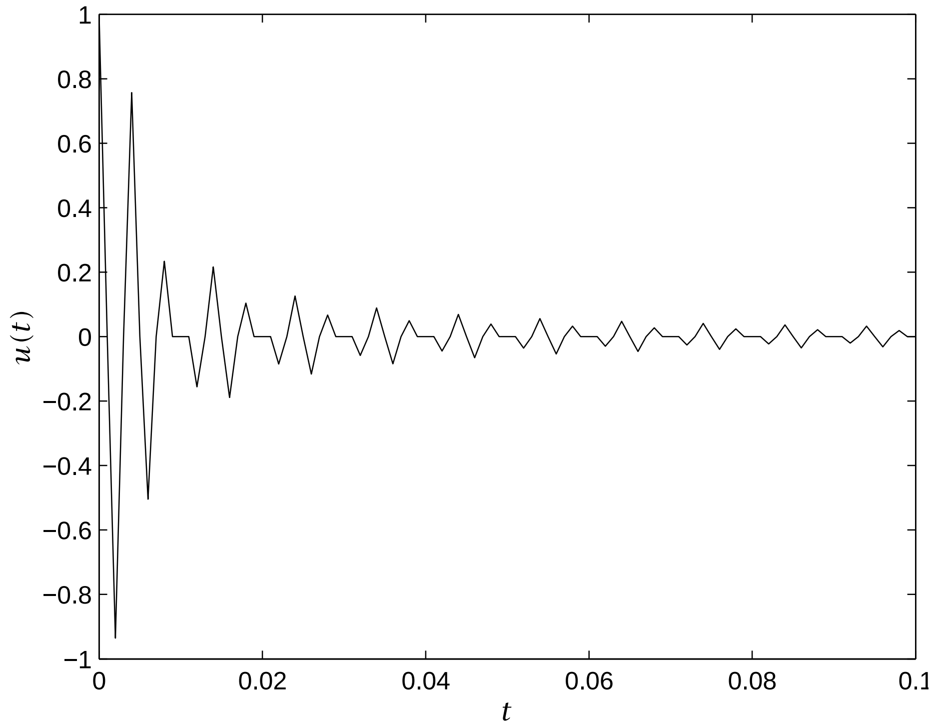


Figure 3.28: The modulated signal $u(t)$

```
plot(t,m(1:length(t)))
xlabel('t')
figure;
plot(t,m_h(1:length(t)))
xlabel('t')
figure;
plot(t,u(1:length(t)))
xlabel('t')
% design the filter
fs = 0.16;
fp = 0.0999;

f=[0 fp fs 1];
m=[1 1 0 0];
delta1 = 0.0875;
delta2 = 0.006;
df = fs - fp;
w=[delta2/delta1 1];
h=remez(31,f,m,w);

f_cutoff=100;
for i = 1:4
    fi = [0 pi/8 pi/4 pi/2];
    tit = ['a', 'b', 'c', 'd'];
    y=u.*cos(2*pi*fc.*t+fi(1));
```

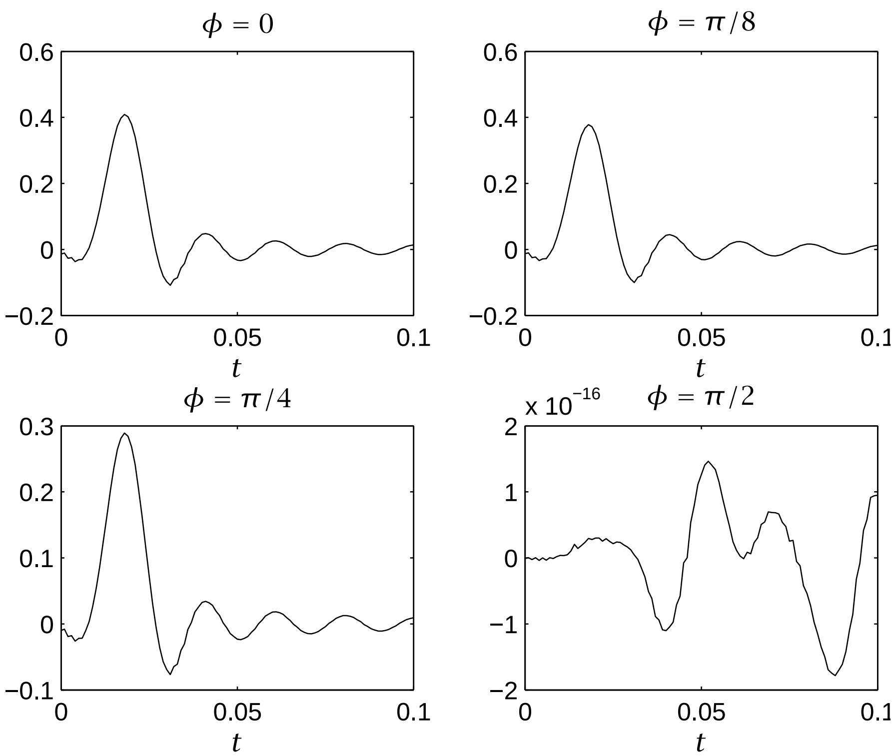


Figure 3.29: The demodulation output

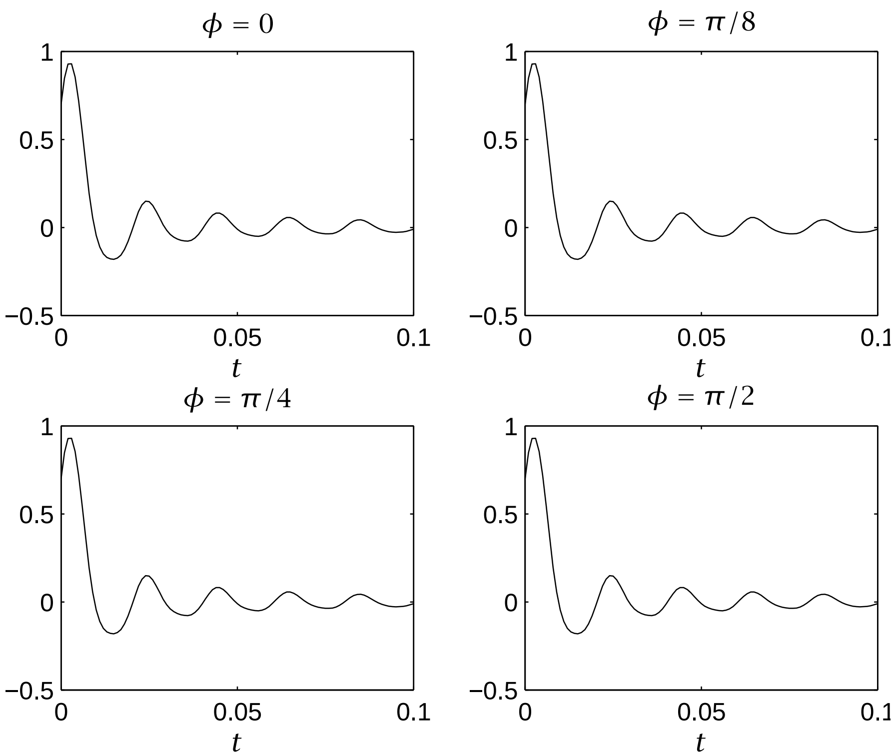


Figure 3.30: The demodulation output

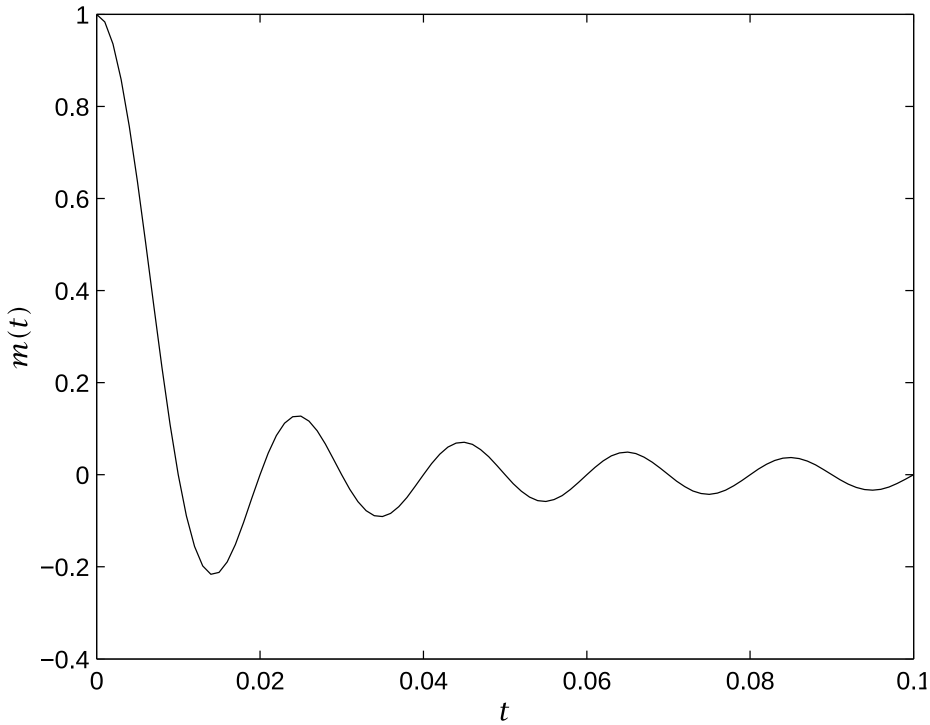


Figure 3.31: The message signal $m(t)$

```
    dem = filter(h, 1, y);
    figure(4);
    subplot(2,2,i);
    plot(t,dem(1:length(t)))
    xlabel('t')
    title('a');
    [Y,y,df1]=fftseq(y,ts,df);
    n_cutoff=floor(f_cutoff/df1);
    f=[0:df1:df1*(length(y)-1)]-fs/2;
    H=zeros(size(f));
    H(1:n_cutoff)=2*ones(1,n_cutoff);
    H(length(f)-n_cutoff+1:length(f))=2*ones(1,n_cutoff);
    Y=Y/fs;
    DEM=H.*Y;
    dem=real(ifft(DEM))*fs;
    figure(5);
    subplot(2,2,i);
    plot(t,dem(1:length(t)))
    xlabel('t');
    title(tit(i));
end
```

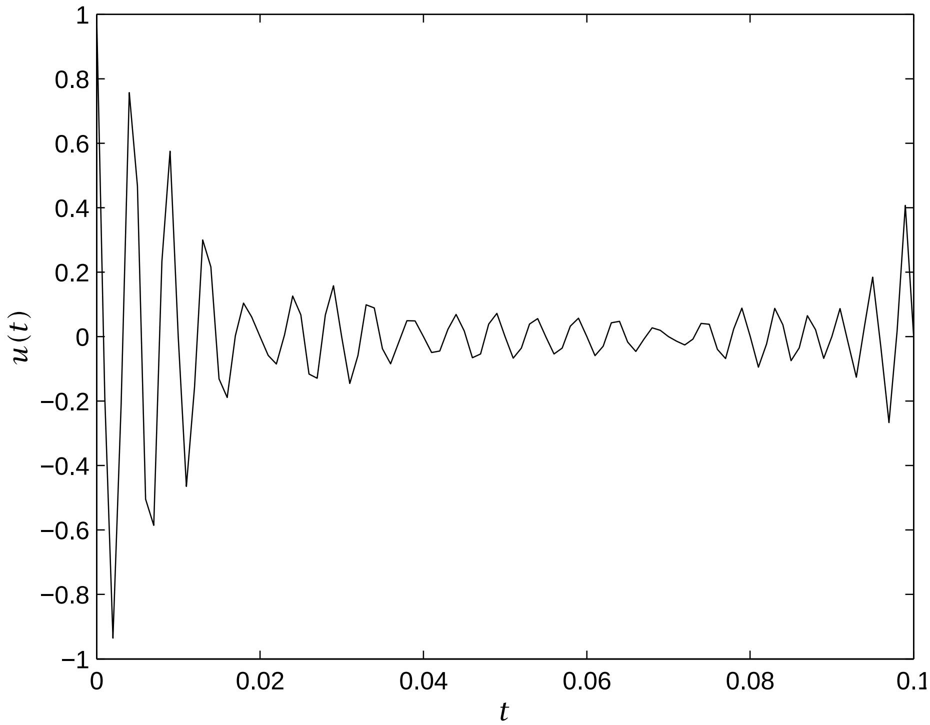


Figure 3.32: The modulated signal $u(t)$

Computer Problem 3.6

1) The message signal and modulated signal are presented in Figures 3.36 and 3.37

2) The demodulated received signal is presented in Figure 3.38

3) In the demodulation process above, we have neglected the effect of the noise-limiting filter, which is a bandpass filter in the first stage of any receiver. In practice, the received signal is passed through the noise-limiting filter and then supplied to the envelope detector. In this example, since the message bandwidth is not finite, passing the received signal through any bandpass filter will cause distortion on the demodulated message, but it will also decrease the amount of noise in the demodulator output.

The MATLAB script for this problem follows.

```
% MATLAB script for Computer Problem 3.6.
t0=.1;                          % signal duration
n=0:1000;
a = 0.8;
ts=0.0001;                      % sampling interval
df=0.2;                         % frequency resolution
fs=1/ts;                        % sampling frequency
fc=250;                         % carrier frequency
t=[0:ts:t0];                    % time vector
m = sinc(100*t);                % message signal
c=cos(2*pi*fc.*t);              % carrier signal
m_n=m/max(abs(m));              % normalized message signal
```

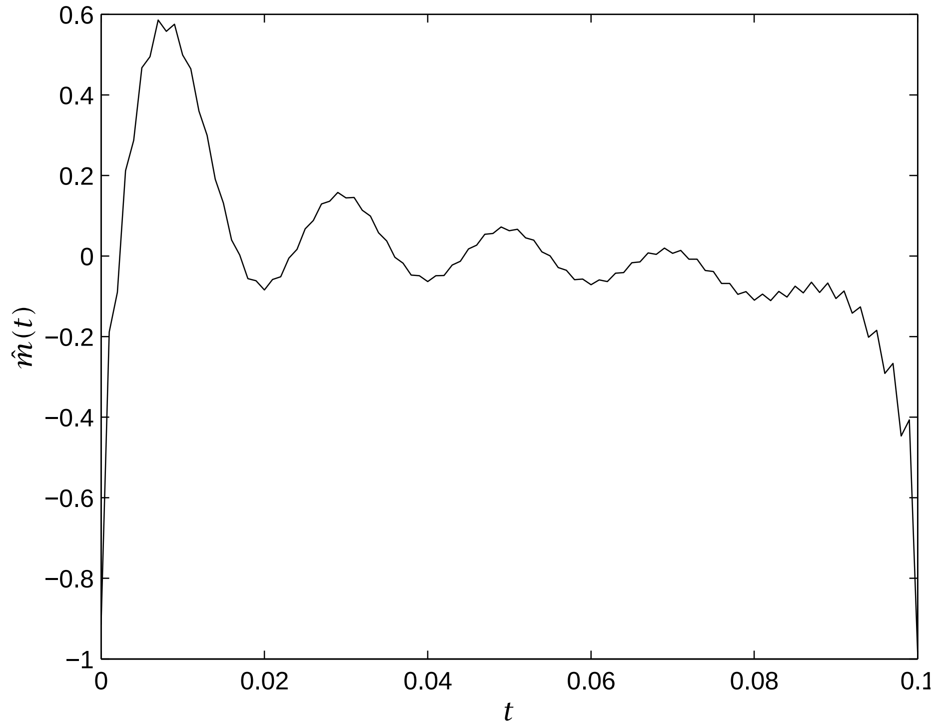


Figure 3.33: The Hilbert transform of the message signal

```
[M,m,df1]=fftseq(m,ts,df);              % Fourier transform
f=[0:df1:df1*(length(m)−1)]−fs/2;       % frequency vector
u=(1+a*m_n).*c;                         % modulated signal
[U,u,df1]=fftseq(u,ts,df);              % Fourier transform
env=env_phas(u, a);                               % Find the envelope.
dem1=2*(env−1)/a;                       % Remove dc and rescale.

% plot the message signal
plot(t,m(1:length(t)))
xlabel('Time')

% plot the modulated signal
figure;
plot(t,u(1:length(t)))
xlabel('Time')

% plot the demodulated signal
figure;
plot(t,dem1(1:length(t)))
xlabel('Time')
```

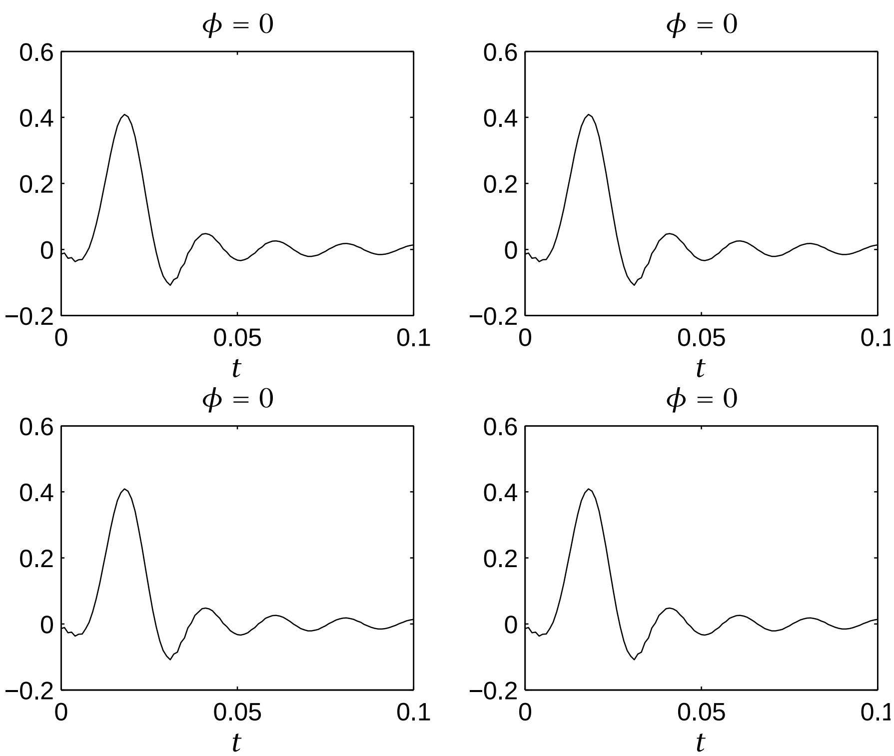


Figure 3.34: The demodulation output

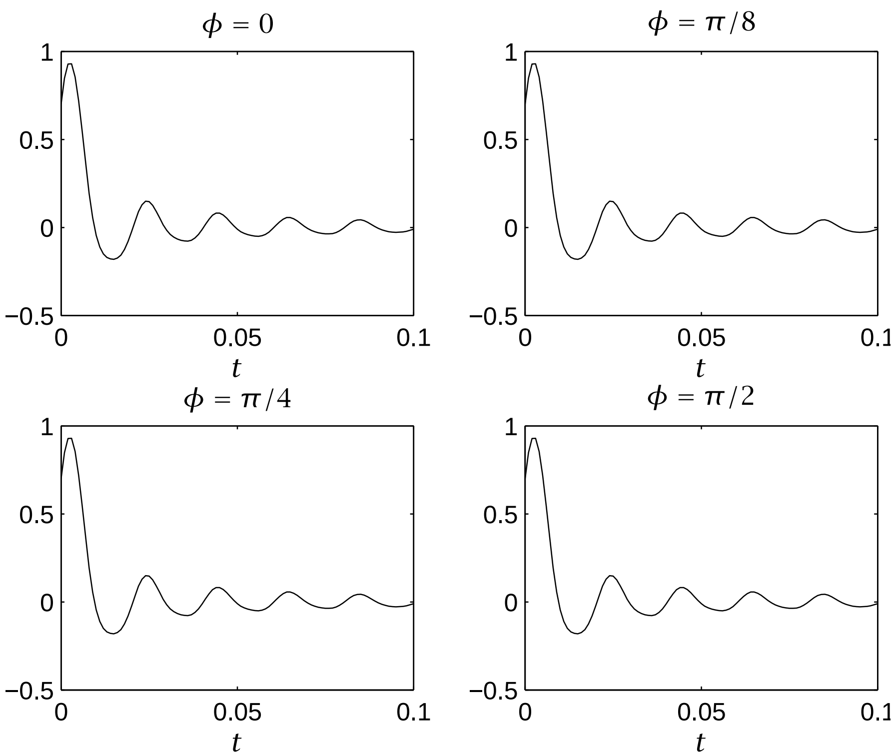


Figure 3.35: The demodulation output

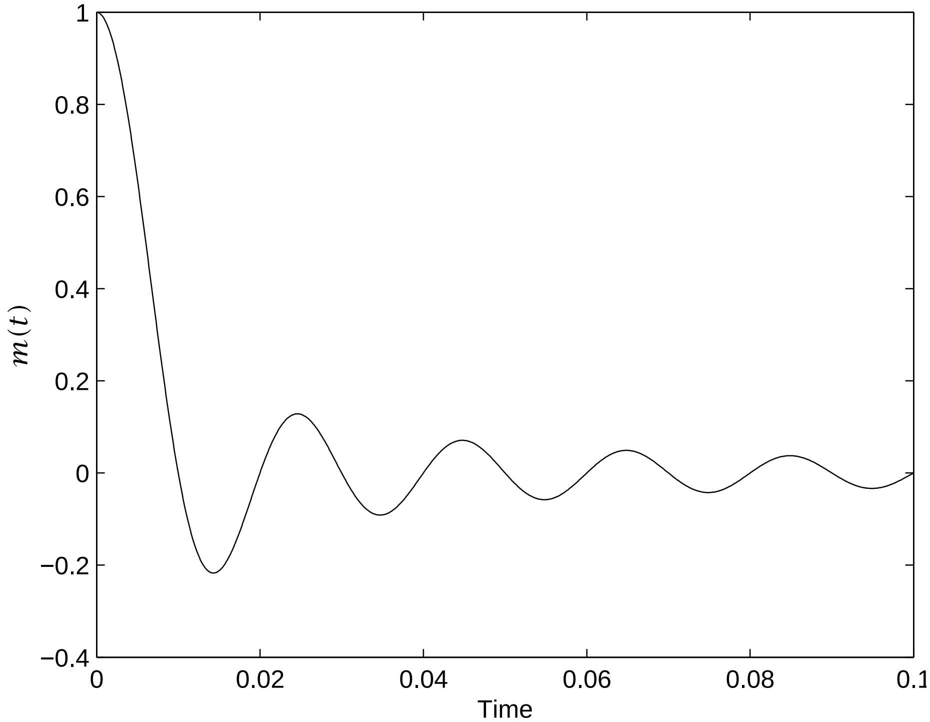


Figure 3.36: Message signal

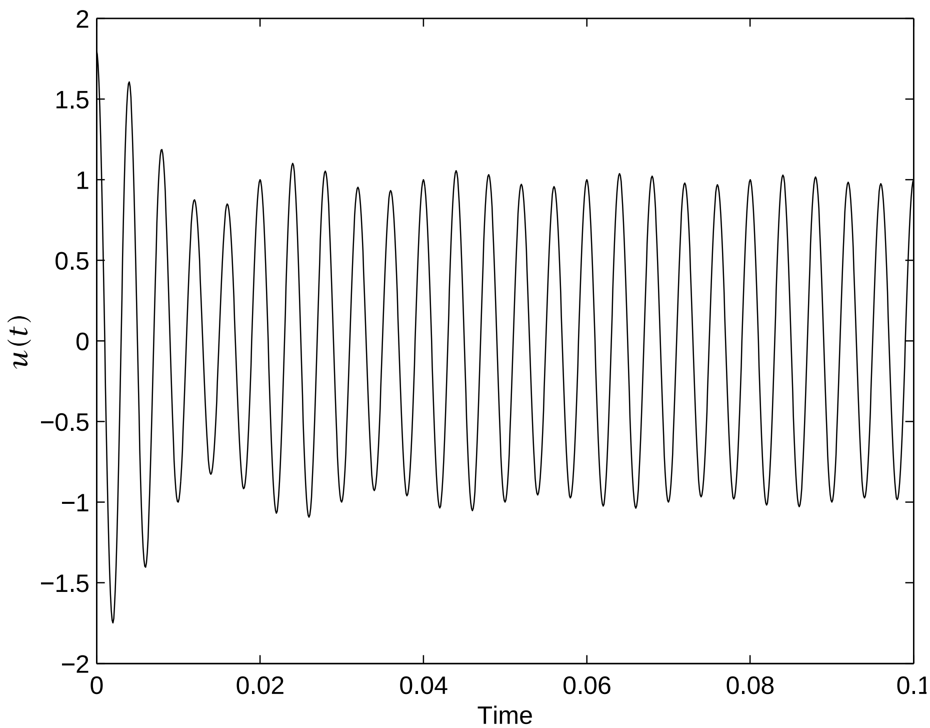


Figure 3.37: Modulated signal

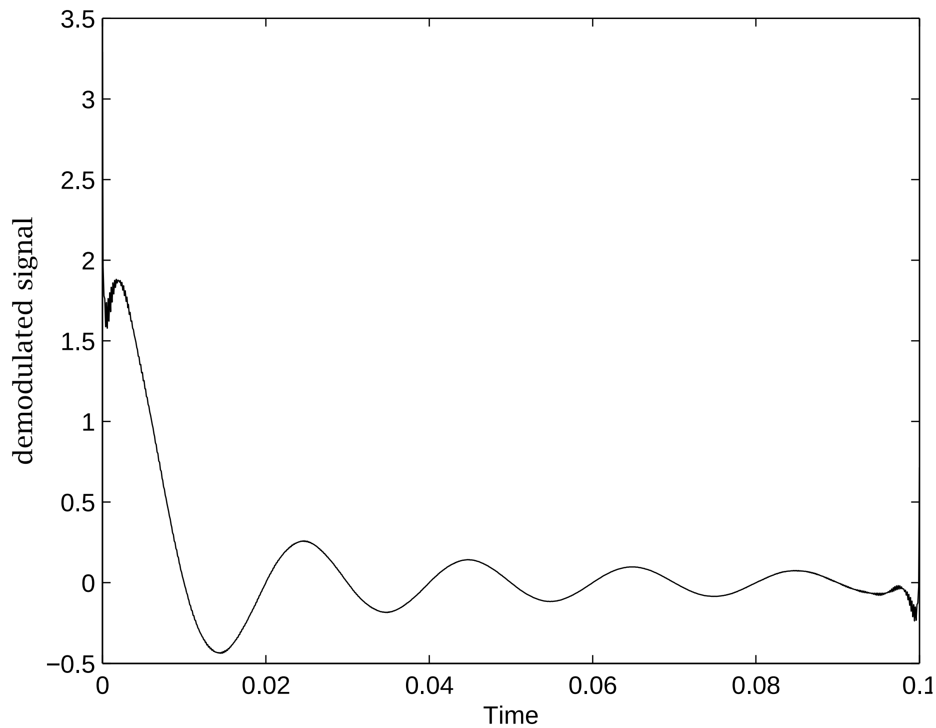


Figure 3.38: Demodulated signal

Chapter 4

Problem 4.1

1) Since $\mathcal{F}[\text{sinc}(400t)] = \frac{1}{400}\Pi(\frac{f}{400})$, the bandwidth of the message signal is $W = 200$ and the resulting modulation index

$$\beta_f = \frac{k_f \max[|m(t)|]}{W} = \frac{k_f 10}{W} = 6 \Longrightarrow k_f = 120$$

Hence, the modulated signal is

$$\begin{aligned} u(t) &= A\cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau) \\ &= 100\cos(2\pi f_c t + +2\pi 1200 \int_{-\infty}^{t} \text{sinc}(400\tau)d\tau) \end{aligned}$$

2) The maximum frequency deviation of the modulated signal is

$$\Delta f_{\max} = \beta_f W = 6 \times 200 = 1200$$

3) Since the modulated signal is essentially a sinusoidal signal with amplitude $A = 100$, we have

$$P = \frac{A^2}{2} = 5000$$

4) Using Carson's rule, the effective bandwidth of the modulated signal can be approximated by

$$B_c = 2(\beta_f + 1)W = 2(6+1)200 = 2800 \text{ Hz}$$

Problem 4.2

1) The maximum phase deviation of the PM signal is

$$\Delta\phi_{\max} = k_p \max[|m(t)|] = k_p$$

The phase of the FM modulated signal is

$$\begin{aligned} \phi(t) &= 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau = 2\pi k_f \int_{0}^{t} m(\tau)d\tau \\ &= \begin{cases} 2\pi k_f \int_0^t \tau d\tau = \pi k_f t^2 & 0 \le t < 1 \\ \pi k_f + 2\pi k_f \int_1^t d\tau = \pi k_f + 2\pi k_f (t-1) & 1 \le t < 2 \\ \pi k_f + 2\pi k_f - 2\pi k_f \int_2^t d\tau = 3\pi k_f - 2\pi k_f (t-2) & 2 \le t < 3 \\ \pi k_f & 3 \le t \end{cases} \end{aligned}$$

The maximum value of $\phi(t)$ is achieved for $t = 2$ and is equal to $3\pi k_f$. Thus, the desired relation between k_p and k_f is

$$k_p = 3\pi k_f$$

2) The instantaneous frequency for the PM modulated signal is

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) = f_c + \frac{1}{2\pi}k_p\frac{d}{dt}m(t)$$

For the $m(t)$ given in Fig. P-4.2, the maximum value of $\frac{d}{dt}m(t)$ is achieved for t in $[0,1]$ and it is equal to one. Hence,

$$\max(f_i(t)) = f_c + \frac{1}{2\pi}$$

For the FM signal $f_i(t) = f_c + k_f m(t)$. Thus, the maximum instantaneous frequency is

$$\max(f_i(t)) = f_c + k_f = f_c + 1$$

Problem 4.3

For an angle modulated signal we have $x(t) = A_c \cos(2\pi f_c t + \phi(t))$, therefore The lowpass equivalent of the signal is $x_l(t) = A_c e^{j\phi(t)}$ with Envelope A_c and phase $\pi(t)$ and in phase an quadrature components $A_c\cos(\phi(t))$ and $A_c\sin(\phi(t))$, respectively. Hence we have the following

$$\text{PM}\begin{cases} A_c & \text{envelope} \\ k_p m(t) & \text{phase} \\ A_c\cos\left(k_p m(t)\right) & \text{in-phase comp.} \\ A_c\sin\left(k_p m(t)\right) & \text{quadrature comp.}\end{cases} \qquad \text{FM}\begin{cases} A_c & \text{envelope} \\ 2\pi k_f \int_{-\infty}^{t} m(\tau)\,d\tau & \text{phase} \\ A_c\cos\left(2\pi k_f \int_{-\infty}^{t} m(\tau)\,d\tau\right) & \text{in-phase comp.} \\ A_c\sin\left(2\pi k_f \int_{-\infty}^{t} m(\tau)\,d\tau\right) & \text{quadrature comp.}\end{cases}$$

Problem 4.4

1) Since an angle modulated signal is essentially a sinusoidal signal with constant amplitude, we have

$$P = \frac{A_c^2}{2} \Longrightarrow P = \frac{100^2}{2} = 5000$$

The same result is obtained if we use the expansion

$$u(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos(2\pi(f_c + nf_m)t)$$

along with the identity

$$J_0^2(\beta) + 2\sum_{n=1}^{\infty} J_n^2(\beta) = 1$$

2) The maximum phase deviation is

$$\Delta\phi_{\max} = \max|4\sin(2000\pi t)| = 4$$

3) The instantaneous frequency is

$$\begin{aligned} f_i &= f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) \\ &= f_c + \frac{4}{2\pi}\cos(2000\pi t)2000\pi = f_c + 4000\cos(2000\pi t) \end{aligned}$$

Hence, the maximum frequency deviation is

$$\Delta f_{\max} = \max|f_i - f_c| = 4000$$

4) The angle modulated signal can be interpreted both as a PM and an FM signal. It is a PM signal with phase deviation constant $k_p = 4$ and message signal $m(t) = \sin(2000\pi t)$ and it is an FM signal with frequency deviation constant $k_f = 4000$ and message signal $m(t) = \cos(2000\pi t)$.

Problem 4.5
The modulated signal can be written as

$$u(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos(2\pi(f_c + nf_m)t)$$

The power in the frequency component $f = f_c + kf_m$ is $P_k = \frac{A_c^2}{2}J_n^2(\beta)$. Hence, the power in the carrier is $P_{\text{carrier}} = \frac{A_c^2}{2}J_0^2(\beta)$ and in order to be zero the modulation index β should be one of the roots of $J_0(x)$. The smallest root of $J_0(x)$ is found from tables to be equal 2.404. Thus,

$$\beta_{\min} = 2.404$$

Problem 4.6
1) If the output of the narrowband FM modulator is,

$$u(t) = A\cos(2\pi f_0 t + \phi(t))$$

then the output of the upper frequency multiplier ($\times n_1$) is

$$u_1(t) = A\cos(2\pi n_1 f_0 t + n_1\phi(t))$$

After mixing with the output of the second frequency multiplier $u_2(t) = A\cos(2\pi n_2 f_0 t)$ we obtain the signal

$$\begin{aligned} y(t) &= A^2\cos(2\pi n_1 f_0 t + n_1\phi(t))\cos(2\pi n_2 f_0 t) \\ &= \frac{A^2}{2}(\cos(2\pi(n_1+n_2)f_0 + n_1\phi(t)) + \cos(2\pi(n_1-n_2)f_0 + n_1\phi(t))) \end{aligned}$$

The bandwidth of the signal is $W = 15$ KHz, so the maximum frequency deviation is $\Delta f = \beta_f W = 0.1 \times 15 = 1.5$ KHz. In order to achieve a frequency deviation of $f = 75$ KHz at the output of the wideband modulator, the frequency multiplier n_1 should be equal to

$$n_1 = \frac{f}{\Delta f} = \frac{75}{1.5} = 50$$

Using an up-converter the frequency modulated signal is given by

$$y(t) = \frac{A^2}{2}\cos(2\pi(n_1+n_2)f_0 + n_1\phi(t))$$

Since the carrier frequency $f_c = (n_1+n_2)f_0$ is 104 MHz, n_2 should be such that

$$(n_1+n_2)100 = 104 \times 10^3 \Longrightarrow n_1 + n_2 = 1040 \text{ or } n_2 = 990$$

2) The maximum allowable drift (d_f) of the 100 kHz oscillator should be such that

$$(n_1+n_2)d_f = 2 \Longrightarrow d_f = \frac{2}{1040} = .0019 \text{ Hz}$$

Problem 4.7

The modulated PM signal is given by

$$\begin{aligned} u(t) &= A_c\cos(2\pi f_c t + k_p m(t)) = A_c\text{Re}\left[e^{j2\pi f_c t}e^{jk_p m(t)}\right] \\ &= A_c\text{Re}\left[e^{j2\pi f_c t}e^{jm(t)}\right] \end{aligned}$$

The signal $e^{jm(t)}$ is periodic with period $T_m = \frac{1}{f_m}$ and Fourier series expansion

$$\begin{aligned} c_n &= \frac{1}{T_m}\int_0^{T_m} e^{jm(t)}e^{-j2\pi n f_m t}dt \\ &= \frac{1}{T_m}\int_0^{\frac{T_m}{2}} e^{j}e^{-j2\pi n f_m t}dt + \frac{1}{T_m}\int_{\frac{T_m}{2}}^{T_m} e^{-j}e^{-j2\pi n f_m t}dt \\ &= -\frac{e^j}{T_m j2\pi n f_m}e^{-j2\pi n f_m t}\Big|_0^{\frac{T_m}{2}} - \frac{e^{-j}}{T_m j2\pi n f_m}e^{-j2\pi n f_m t}\Big|_{\frac{T_m}{2}}^{T_m} \\ &= \frac{(-1)^n - 1}{2\pi n}j(e^j - e^{-j}) = \begin{cases} 0 & n = 2l \\ \frac{2}{\pi(2l+1)}\sin(1) & n = 2l+1 \end{cases} \end{aligned}$$

Hence,

$$e^{jm(t)} = \sum_{l=-\infty}^{\infty} \frac{2}{\pi(2l+1)} \sin(1) e^{j2\pi l f_m t}$$

and

$$\begin{aligned} u(t) &= A_c \mathrm{Re}\left[e^{j2\pi f_c t} e^{jm(t)}\right] = A_c \mathrm{Re}\left[e^{j2\pi f_c t} \sum_{l=-\infty}^{\infty} \frac{2}{\pi(2l+1)} \sin(1) e^{j2\pi l f_m t}\right] \\ &= A_c \sum_{l=-\infty}^{\infty} \left|\frac{2\sin(1)}{\pi(2l+1)}\right| \cos(2\pi(f_c + l f_m)t + \phi_l) \end{aligned}$$

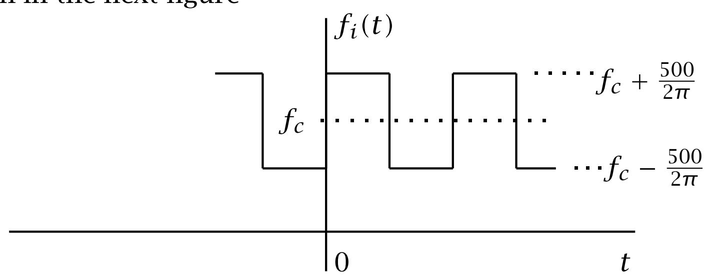
where $\phi_l = 0$ for $l \geq 0$ and $\phi_l = \pi$ for negative values of l.

Problem 4.8

1) The instantaneous frequency is given by

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) = f_c + \frac{1}{2\pi} 100 m(t)$$

A plot of $f_i(t)$ is given in the next figure



2) The peak frequency deviation is given by

$$\Delta f_{\max} = k_f \max[|m(t)|] = \frac{100}{2\pi} 5 = \frac{250}{\pi}$$

Problem 4.9

1) The modulation index is

$$\beta = \frac{k_f \max[|m(t)|]}{f_m} = \frac{\Delta f_{\max}}{f_m} = \frac{20 \times 10^3}{10^4} = 2$$

The modulated signal $u(t)$ has the form

$$\begin{aligned} u(t) &= \sum_{n=-\infty}^{\infty} A_c J_n(\beta) \cos(2\pi(f_c + n f_m)t + \phi_n) \\ &= \sum_{n=-\infty}^{\infty} 100 J_n(2) \cos(2\pi(10^8 + n 10^4)t + \phi_n) \end{aligned}$$

The power of the unmodulated carrier signal is $P = \frac{100^2}{2} = 5000$. The power in the frequency component $f = f_c + k10^4$ is

$$P_{f_c+kf_m} = \frac{100^2 J_k^2(2)}{2}$$

The next table shows the values of $J_k(2)$, the frequency $f_c + kf_m$, the amplitude $100J_k(2)$ and the power $P_{f_c+kf_m}$ for various values of k.

Index k	$J_k(2)$	Frequency Hz	Amplitude $100J_k(2)$	Power $P_{f_c+kf_m}$
0	.2239	10^8	22.39	250.63
1	.5767	$10^8 + 10^4$	57.67	1663.1
2	.3528	$10^8 + 2 \times 10^4$	35.28	622.46
3	.1289	$10^8 + 3 \times 10^4$	12.89	83.13
4	.0340	$10^8 + 4 \times 10^4$	3.40	5.7785

As it is observed from the table the signal components that have a power level greater than 500 (= 10% of the power of the unmodulated signal) are those with frequencies $10^8 + 10^4$ and $10^8 + 2 \times 10^4$. Since $J_n^2(\beta) = J_{-n}^2(\beta)$ it is conceivable that the signal components with frequency $10^8 - 10^4$ and $10^8 - 2 \times 10^4$ will satisfy the condition of minimum power level. Hence, there are four signal components that have a power of at least 10% of the power of the unmodulated signal. The components with frequencies $10^8 + 10^4$, $10^8 - 10^4$ have an amplitude equal to 57.67, whereas the signal components with frequencies $10^8 + 2 \times 10^4$, $10^8 - 2 \times 10^4$ have an amplitude equal to 35.28.

2) Using Carson's rule, the approximate bandwidth of the FM signal is

$$B_c = 2(\beta + 1)f_m = 2(2 + 1)10^4 = 6 \times 10^4 \text{ Hz}$$

Problem 4.10
1)

$$\begin{aligned} \beta_p &= k_p \max[|m(t)|] = 1.5 \times 2 = 3 \\ \beta_f &= \frac{k_f \max[|m(t)|]}{f_m} = \frac{3000 \times 2}{1000} = 6 \end{aligned}$$

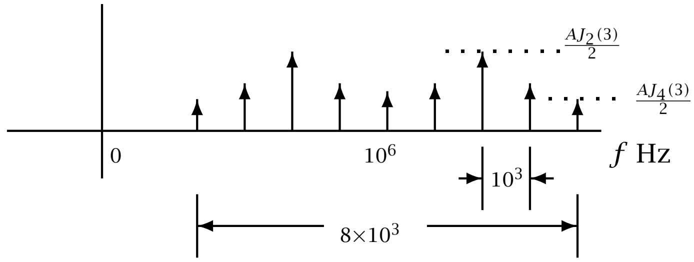
2) Using Carson's rule we obtain

$$\begin{aligned} B_{\text{PM}} &= 2(\beta_p + 1)f_m = 8 \times 1000 = 8000 \\ B_{\text{FM}} &= 2(\beta_f + 1)f_m = 14 \times 1000 = 14000 \end{aligned}$$

3) The PM modulated signal can be written as

$$u(t) = \sum_{n=-\infty}^{\infty} AJ_n(\beta_p)\cos(2\pi(10^6 + n10^3)t)$$

The next figure shows the amplitude of the spectrum for positive frequencies and for these components whose frequencies lie in the interval $[10^6 - 4 \times 10^3, 10^6 + 4 \times 10^3]$. Note that $J_0(3) = -.2601$, $J_1(3) = 0.3391$, $J_2(3) = 0.4861$, $J_3(3) = 0.3091$ and $J_4(3) = 0.1320$.

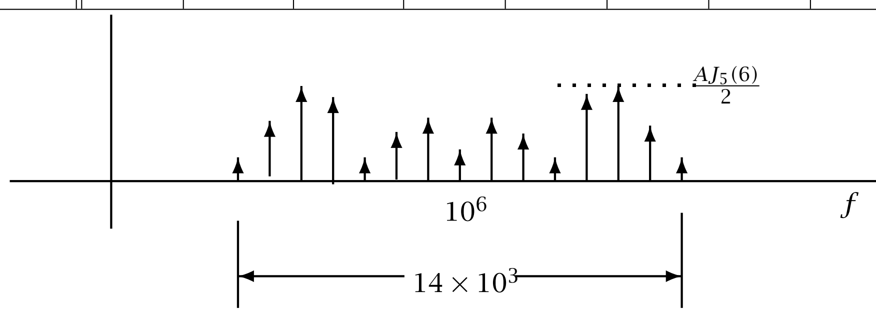


In the case of the FM modulated signal

$$\begin{aligned} u(t) &= A\cos(2\pi f_c t + \beta_f \sin(2000\pi t)) \\ &= \sum_{n=-\infty}^{\infty} AJ_n(6)\cos(2\pi(10^6 + n10^3)t + \phi_n) \end{aligned}$$

The next figure shows the amplitude of the spectrum for positive frequencies and for these components whose frequencies lie in the interval $[10^6 - 7 \times 10^3, 10^6 - 7 \times 10^3]$. The values of $J_n(6)$ for $n = 0, \ldots, 7$ are given in the following table.

n	0	1	2	3	4	5	6	7
$J_n(6)$	.1506	-.2767	-.2429	.1148	.3578	.3621	.2458	.1296



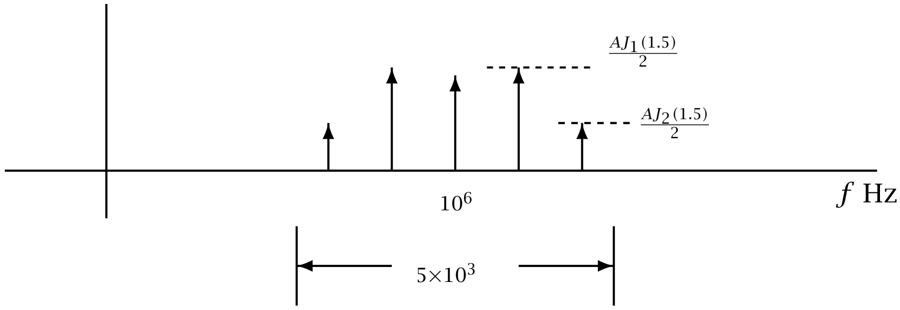
4) If the amplitude of $m(t)$ is decreased by a factor of two, then $m(t) = \cos(2\pi 10^3 t)$ and

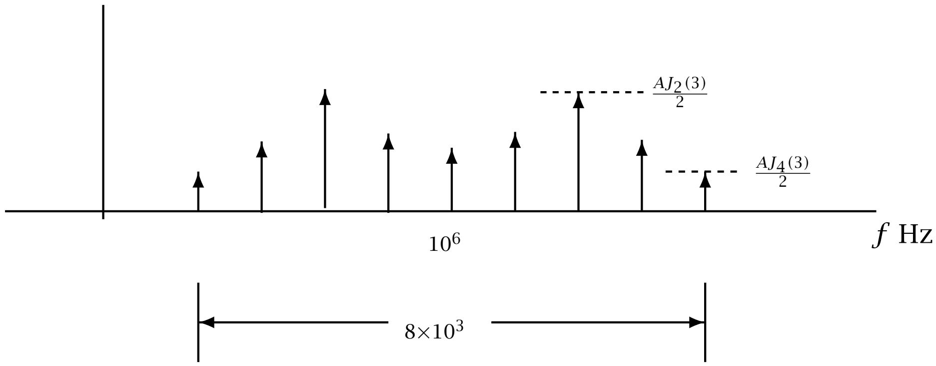
$$\begin{aligned} \beta_p &= k_p \max[|m(t)|] = 1.5 \\ \beta_f &= \frac{k_f \max[|m(t)|]}{f_m} = \frac{3000}{1000} = 3 \end{aligned}$$

The bandwidth is determined using Carson's rule as

$$\begin{aligned} B_{\text{PM}} &= 2(\beta_p + 1)f_m = 5 \times 1000 = 5000 \\ B_{\text{FM}} &= 2(\beta_f + 1)f_m = 8 \times 1000 = 8000 \end{aligned}$$

The amplitude spectrum of the PM and FM modulated signals is plotted in the next figure for positive frequencies. Only those frequency components lying in the previous derived bandwidth are plotted. Note that $J_0(1.5) = .5118$, $J_1(1.5) = .5579$ and $J_2(1.5) = .2321$.





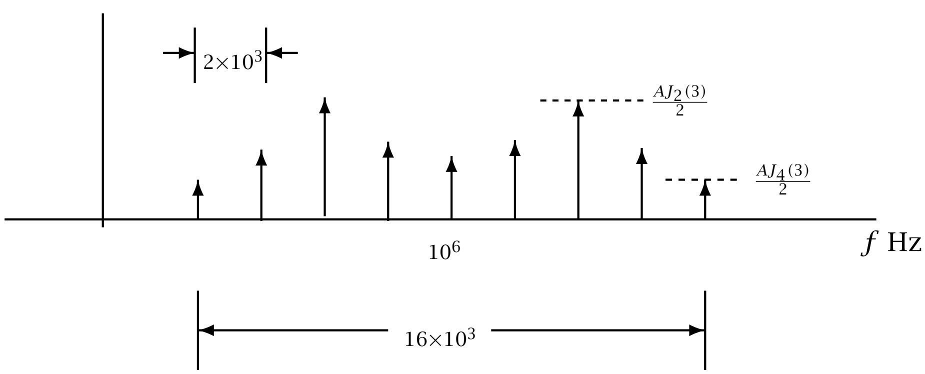
5) If the frequency of $m(t)$ is increased by a factor of two, then $m(t) = 2\cos(2\pi 2\times 10^3 t)$ and

$$\beta_p = k_p \max[|m(t)|] = 1.5 \times 2 = 3$$
$$\beta_f = \frac{k_f \max[|m(t)|]}{f_m} = \frac{3000\times 2}{2000} = 3$$

The bandwidth is determined using Carson's rule as

$$B_{\text{PM}} = 2(\beta_p + 1)f_m = 8 \times 2000 = 16000$$
$$B_{\text{FM}} = 2(\beta_f + 1)f_m = 8 \times 2000 = 16000$$

The amplitude spectrum of the PM and FM modulated signals is plotted in the next figure for positive frequencies. Only those frequency components lying in the previous derived bandwidth are plotted. Note that doubling the frequency has no effect on the number of harmonics in the bandwidth of the PM signal, whereas it decreases the number of harmonics in the bandwidth of the FM signal from 14 to 8.



Problem 4.11

1) The PM modulated signal is

$$\begin{aligned} u(t) &= 100\cos(2\pi f_c t + \frac{\pi}{2}\cos(2\pi 1000 t)) \\ &= \sum_{n=-\infty}^{\infty} 100 J_n\left(\frac{\pi}{2}\right)\cos(2\pi(10^8 + n10^3)t) \end{aligned}$$

The next table tabulates $J_n(\beta)$ for $\beta = \frac{\pi}{2}$ and $n = 0, \ldots, 4$.

n	0	1	2	3	4
$J_n(\beta)$	.4720	.5668	.2497	.0690	.0140

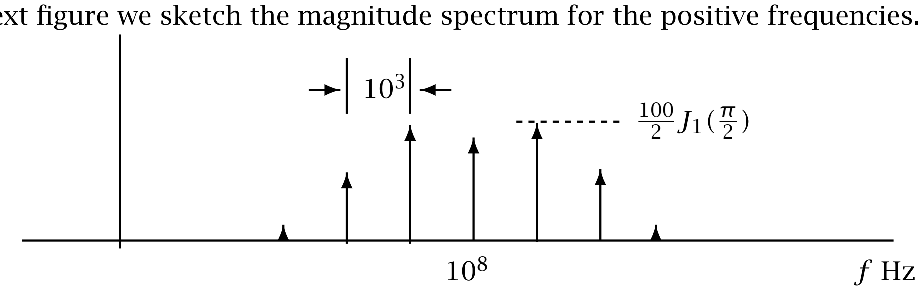
The total power of the modulated signal is $P_{\text{tot}} = \frac{100^2}{2} = 5000$. To find the effective bandwidth of the signal we calculate the index k such that

$$\sum_{n=-k}^{k} \frac{100^2}{2} J_n^2\left(\frac{\pi}{2}\right) \geq 0.99 \times 5000 \Longrightarrow \sum_{n=-k}^{k} J_n^2\left(\frac{\pi}{2}\right) \geq 0.99$$

By trial end error we find that the smallest index k is 2. Hence the effective bandwidth is

$$B_{\text{eff}} = 4 \times 10^3 = 4000$$

In the the next figure we sketch the magnitude spectrum for the positive frequencies.



2) Using Carson's rule, the approximate bandwidth of the PM signal is

$$B_{\text{PM}} = 2(\beta_p + 1)f_m = 2(\frac{\pi}{2} + 1)1000 = 5141.6$$

As it is observed, Carson's rule overestimates the effective bandwidth allowing in this way some margin for the missing harmonics.

Problem 4.12

1) Assuming that $u(t)$ is an FM signal it can be written as

$$\begin{aligned} u(t) &= 100\cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{\infty} \alpha\cos(2\pi f_m \tau)d\tau) \\ &= 100\cos(2\pi f_c t + \frac{k_f \alpha}{f_m}\sin(2\pi f_m t)) \end{aligned}$$

Thus, the modulation index is $\beta_f = \frac{k_f \alpha}{f_m} = 4$ and the bandwidth of the transmitted signal

$$B_{\text{FM}} = 2(\beta_f + 1) f_m = 10 \text{ KHz}$$

2) If we double the frequency, then

$$u(t) = 100 \cos(2\pi f_c t + 4 \sin(2\pi 2 f_m t))$$

Using the same argument as before we find that $\beta_f = 4$ and

$$B_{\text{FM}} = 2(\beta_f + 1) 2 f_m = 20 \text{ KHz}$$

3) If the signal $u(t)$ is PM modulated, then

$$\beta_p = \Delta\phi_{\max} = \max[4 \sin(2\pi f_m t)] = 4$$

The bandwidth of the modulated signal is

$$B_{\text{PM}} = 2(\beta_p + 1) f_m = 10 \text{ KHz}$$

4) If f_m is doubled, then $\beta_p = \Delta\phi_{\max}$ remains unchanged whereas

$$B_{\text{PM}} = 2(\beta_p + 1) 2 f_m = 20 \text{ KHz}$$

Problem 4.13

1) If the signal $m(t) = m_1(t) + m_2(t)$ DSB modulates the carrier $A_c \cos(2\pi f_c t)$ the result is the signal

$$\begin{aligned} u(t) &= A_c m(t) \cos(2\pi f_c t) \\ &= A_c (m_1(t) + m_2(t)) \cos(2\pi f_c t) \\ &= A_c m_1(t) \cos(2\pi f_c t) + A_c m_2(t) \cos(2\pi f_c t) \\ &= u_1(t) + u_2(t) \end{aligned}$$

where $u_1(t)$ and $u_2(t)$ are the DSB modulated signals corresponding to the message signals $m_1(t)$ and $m_2(t)$. Hence, AM modulation satisfies the superposition principle.

2) If $m(t)$ frequency modulates a carrier $A_c \cos(2\pi f_c t)$ the result is

$$\begin{aligned} u(t) &= A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{\infty} (m_1(\tau) + m_2(\tau)) d\tau) \\ &\neq A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{\infty} m_1(\tau) d\tau) \\ &\quad + A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{\infty} m_2(\tau) d\tau) \\ &= u_1(t) + u_2(t) \end{aligned}$$

where the inequality follows from the nonlinearity of the cosine function. Hence, angle modulation is not a linear modulation method.

Problem 4.14

The transfer function of the FM discriminator is

$$H(s) = \frac{R}{R + Ls + \frac{1}{Cs}} = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Thus,

$$|H(f)|^2 = \frac{4\pi^2\left(\frac{R}{L}\right)^2 f^2}{\left(\frac{1}{LC} - 4\pi^2 f^2\right)^2 + 4\pi^2(\frac{R}{L})^2 f^2}$$

As it is observed $|H(f)|^2 \leq 1$ with equality if

$$f = \frac{1}{2\pi\sqrt{LC}}$$

Since this filter is to be used as a slope detector, we require that the frequency content of the signal, which is $[80-6, 80+6]$ MHz, to fall inside the region over which $|H(f)|$ is almost linear. Such a region can be considered the interval $[f_{10}, f_{90}]$, where f_{10} is the frequency such that $|H(f_{10})| = 10\% \max[|H(f)|]$ and f_{90} is the frequency such that $|H(f_{10})| = 90\% \max[|H(f)|]$.

With $\max[|H(f)| = 1$, $f_{10} = 74 \times 10^6$ and $f_{90} = 86 \times 10^6$, we obtain the system of equations

$$4\pi^2 f_{10}^2 + \frac{50 \times 10^3}{L} 2\pi f_{10}[1 - 0.1^2]^{\frac{1}{2}} - \frac{1}{LC} = 0$$
$$4\pi^2 f_{90}^2 + \frac{50 \times 10^3}{L} 2\pi f_{90}[1 - 0.9^2]^{\frac{1}{2}} - \frac{1}{LC} = 0$$

Solving this system, we obtain

$$L = 14.98 \text{ mH} \quad C = 0.018013 \text{ pF}$$

Problem 4.15

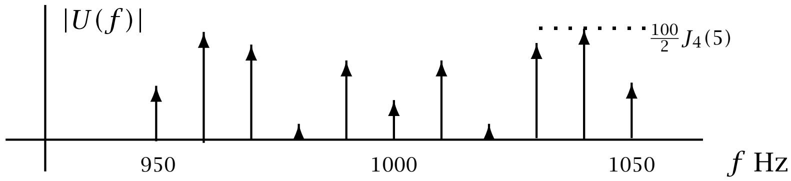
The case of $\phi(t) = \beta\cos(2\pi f_m t)$ has been treated in the text, the modulated signal is

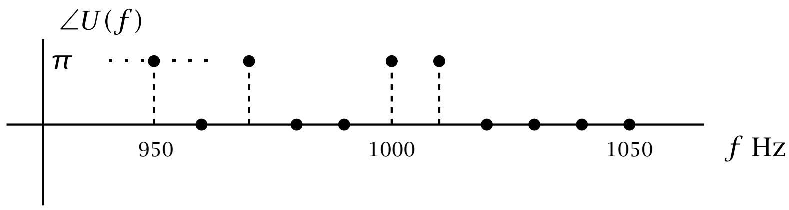
$$\begin{aligned} u(t) &= \sum_{n=-\infty}^{\infty} A_c J_n(\beta)\cos(2\pi(f_c + nf_m)) \\ &= \sum_{n=-\infty}^{\infty} 100 J_n(5)\cos(2\pi(10^3 + n10)) \end{aligned}$$

The following table shows the values of $J_n(5)$ for $n = 0, \ldots, 5$.

n	0	1	2	3	4	5
$J_n(5)$	-.178	-.328	.047	.365	.391	.261

In the next figure we plot the magnitude and the phase spectrum for frequencies in the range $[950, 1050]$ Hz. Note that $J_{-n}(\beta) = J_n(\beta)$ if n is even and $J_{-n}(\beta) = -J_n(\beta)$ if n is odd.





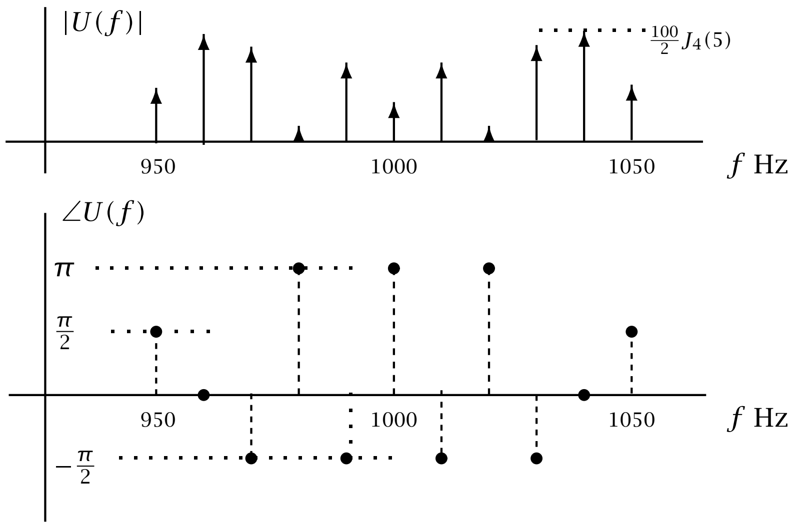
The Fourier Series expansion of $e^{j\beta \sin(2\pi f_m t)}$ is

$$\begin{aligned} c_n &= f_m \int_{\frac{1}{4f_m}}^{\frac{5}{4f_m}} e^{j\beta \sin(2\pi f_m t)} e^{-j2\pi n f_m t} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{j\beta \cos u - jnu} e^{j\frac{n\pi}{2}} du \\ &= e^{j\frac{n\pi}{2}} J_n(\beta) \end{aligned}$$

Hence,

$$\begin{aligned} u(t) &= A_c \text{Re}\left[\sum_{n=-\infty}^{\infty} c_n e^{j2\pi f_c t} e^{j2\pi n f_m t}\right] \\ &= A_c \text{Re}\left[\sum_{n=-\infty}^{\infty} e^{j2\pi(f_c + n f_m)t + \frac{n\pi}{2}}\right] \end{aligned}$$

The magnitude and the phase spectra of $u(t)$ for $\beta = 5$ and frequencies in the interval $[950, 1000]$ Hz are shown in the next figure. Note that the phase spectrum has been plotted modulo 2π in the interval $(-\pi, \pi]$.



Problem 4.16

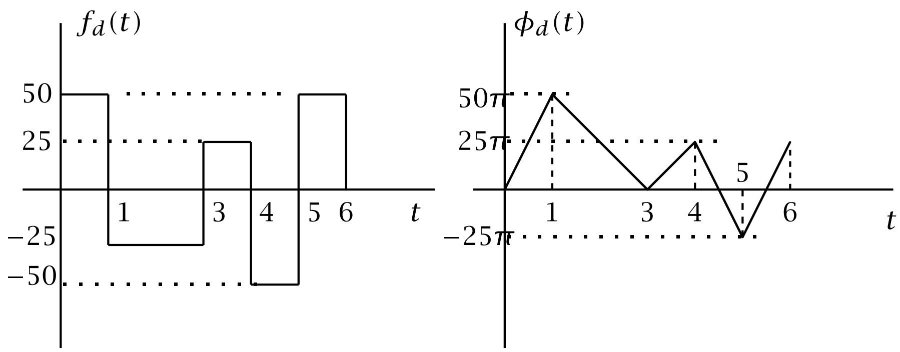
The frequency deviation is given by

$$f_d(t) = f_i(t) - f_c = k_f m(t)$$

whereas the phase deviation is obtained from

$$\phi_d(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau$$

In the next figure we plot the frequency and the phase deviation when $m(t)$ is as in Fig. P-4.16 with $k_f = 25$.



Problem 4.17

Using Carson's rule we obtain

$$B_c = 2(\beta+1)W = 2(\frac{k_f \max[|m(t)|]}{W}+1)W = \begin{cases} 20020 & k_f = 10 \\ 20200 & k_f = 100 \\ 22000 & k_f = 1000 \end{cases}$$

Problem 4.18
The modulation index is

$$\beta = \frac{k_f \max[|m(t)|]}{f_m} = \frac{10\times 10}{8} = 12.5$$

The output of the FM modulator can be written as

$$\begin{aligned} u(t) &= 10\cos(2\pi 2000t + 2\pi k_f \int_{-\infty}^{t} 10\cos(2\pi 8\tau)d\tau) \\ &= \sum_{n=-\infty}^{\infty} 10J_n(12.5)\cos(2\pi(2000+n8)t + \phi_n) \end{aligned}$$

At the output of the BPF only the signal components with frequencies in the interval $[2000-32, 2000+32]$ will be present. These components are the terms of $u(t)$ for which $n = -4,\ldots,4$. The power of the output signal is then

$$\frac{10^2}{2}J_0^2(12.5) + 2\sum_{n=1}^{4}\frac{10^2}{2}J_n^2(12.5) = 50\times 0.2630 = 13.15$$

Since the total transmitted power is $P_{\text{tot}} = \frac{10^2}{2} = 50$, the power at the output of the bandpass filter is only 26.30% of the transmitted power.

Problem 4.19
1) The instantaneous frequency is

$$f_i(t) = f_c + k_f m_1(t)$$

The maximum of $f_i(t)$ is

$$\max[f_i(t)] = \max[f_c + k_f m_1(t)] = 10^6 + 5\times 10^5 = 1.5 \text{ MHz}$$

2) The phase of the PM modulated signal is $\phi(t) = k_p m_1(t)$ and the instantaneous frequency

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{d}{dt}\phi(t) = f_c + \frac{k_p}{2\pi}\frac{d}{dt}m_1(t)$$

The maximum of $f_i(t)$ is achieved for t in $[0,1]$ where $\frac{d}{dt}m_1(t) = 1$. Hence, $\max[f_i(t)] = 10^6 + \frac{3}{2\pi}$.

3) The maximum value of $m_2(t) = \text{sinc}(2 \times 10^4 t)$ is 1 and it is achieved for $t = 0$. Hence,

$$\max[f_i(t)] = \max[f_c + k_f m_2(t)] = 10^6 + 10^3 = 1.001 \text{ MHz}$$

Since, $\mathcal{F}[\text{sinc}(2 \times 10^4 t)] = \frac{1}{2\times10^4}\Pi(\frac{f}{2\times10^4})$ the bandwidth of the message signal is $W = 10^4$. Thus, using Carson's rule, we obtain

$$B = 2(\frac{k_f \max[|m(t)|]}{W} + 1)W = 22 \text{ KHz}$$

Problem 4.20

Since 88 MHz $< f_c <$ 108 MHz and

$$|f_c - f_c'| = 2f_{\text{IF}} \quad \text{if } f_{\text{IF}} < f_{\text{LO}}$$

we conclude that in order for the image frequency f_c' to fall outside the interval [88, 108] MHZ, the minimum frequency f_{IF} is such that

$$2f_{\text{IF}} = 108 - 88 \Longrightarrow f_{\text{IF}} = 10 \text{ MHz}$$

If $f_{\text{IF}} = 10$ MHz, then the range of f_{LO} is $[88 + 10, 108 + 10] = [98, 118]$ MHz.

Computer Problems

Computer Problem 4.1

1) Figures 4.1 and 4.2 present the message signal and its integral, respectively.
2) A plot of $u(t)$ is shown in Figure 4.3.
3) Using MATLAB's Fourier transform routines, we obtain the expression for the spectrum of message and modulated signals shown in Figures 4.4 and 4.5.
4)In this question, the bandwidth of the message signal is not finite, therefore to define the index of modulation, an approximate bandwidth for the message should be used in the expression

$$\beta = \frac{k_f \max |m(t)|}{W} \tag{4.16}$$

defining the bandwidth as the width of the main lobe of the spectrum of $m(t)$ results in

$$W = 20 \text{ Hz}$$

and so

$$\beta = \frac{50 \times 2}{20} = 10$$

The MATLAB script for this problem follows.

```
% MATLAB script for Computer Problem 4.1.
% Demonstration script for frequency modulation. The message signal
% is +1 for 0 < t < t0/3, -2 for t0/3 < t < 2t0/3, and zero otherwise.
echo on
t0=.15;                          % signal duration
ts=0.0001;                       % sampling interval
fc=200;                          % carrier frequency
kf=50;                           % modulation index
fs=1/ts;                         % sampling frequency
t=[0:ts:t0-ts];                  % time vector
df=0.25;                         % required frequency resolution
% message signal
m=[ones(1,t0/(3*ts)),-2*ones(1,t0/(3*ts)),zeros(1,t0/(3*ts)+2)];
int_m(1)=0;
for i=1:length(t)-1              % integral of m
  int_m(i+1)=int_m(i)+m(i)*ts;
  echo off ;
end
echo on ;
[M,m,df1]=fftseq(m,ts,df);       % Fourier transform
M=M/fs;                          % scaling
f=[0:df1:df1*(length(m)-1)]-fs/2; % frequency vector
u=cos(2*pi*fc*t+2*pi*kf*int_m);  % modulated signal.
[U,u,df1]=fftseq(u,ts,df);       % Fourier transform
U=U/fs;                          % scaling
pause  % Press any key to see a plot of the message and the modulated signal.
subplot(2,1,1)
plot(t,m(1:length(t)))
```

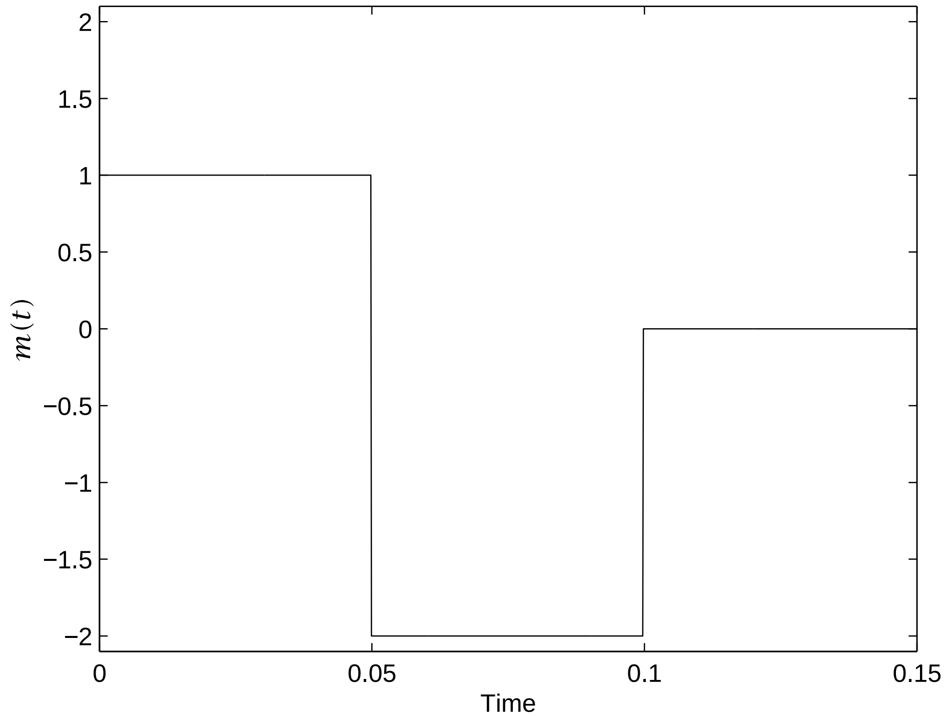


Figure 4.1: The message signal for Computer Problem 4.1

```
axis([0 0.15 −2.1 2.1])
xlabel('Time')
title('The message signal')
subplot(2,1,2)
plot(t,u(1:length(t)))
axis([0 0.15 −2.1 2.1])
xlabel('Time')
title('The modulated signal')
pause   % Press any key to see plots of the magnitude of the message and the
        % modulated signal in the frequency domain.
subplot(2,1,1)
plot(f,abs(fftshift(M)))
xlabel('Frequency')
title('Magnitude spectrum of the message signal')
subplot(2,1,2)
plot(f,abs(fftshift(U)))
title('Magnitude spectrum of the modulated signal')
xlabel('Frequency')
```

Computer Problem 4.2

1) Figures 4.6 and 4.7 present the message signal and its integral, respectively.

2) A plot of $u(t)$ is shown in Figure 4.8.

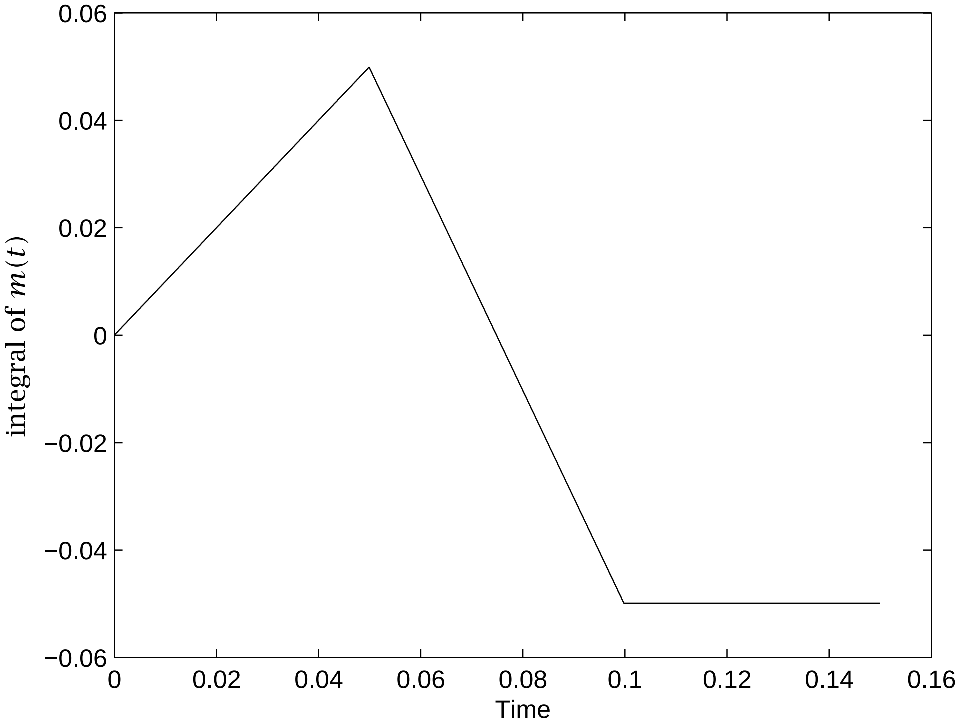


Figure 4.2: The integral of the message signal for Computer Problem 4.1

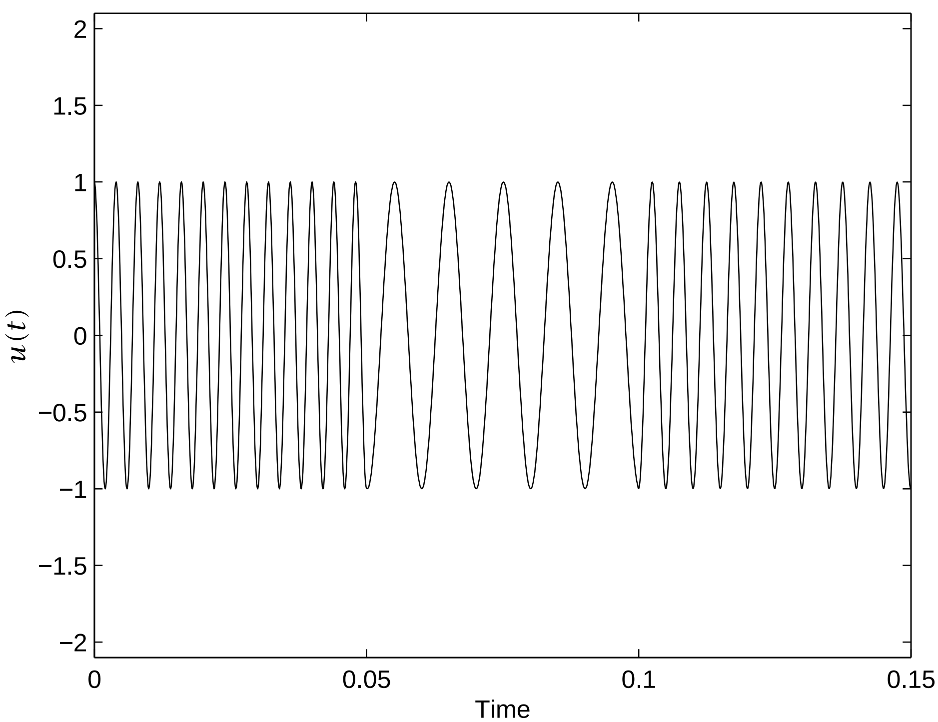


Figure 4.3: The modulated signal for Computer Problem 4.1

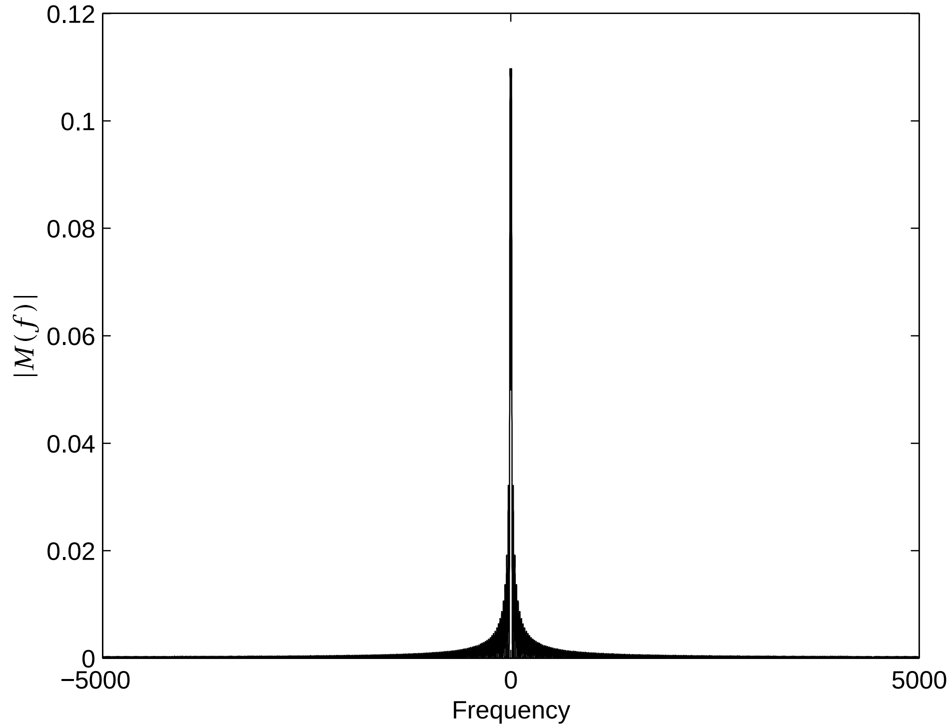


Figure 4.4: The magnitude spectrum of the message signal for Computer Problem 4.1

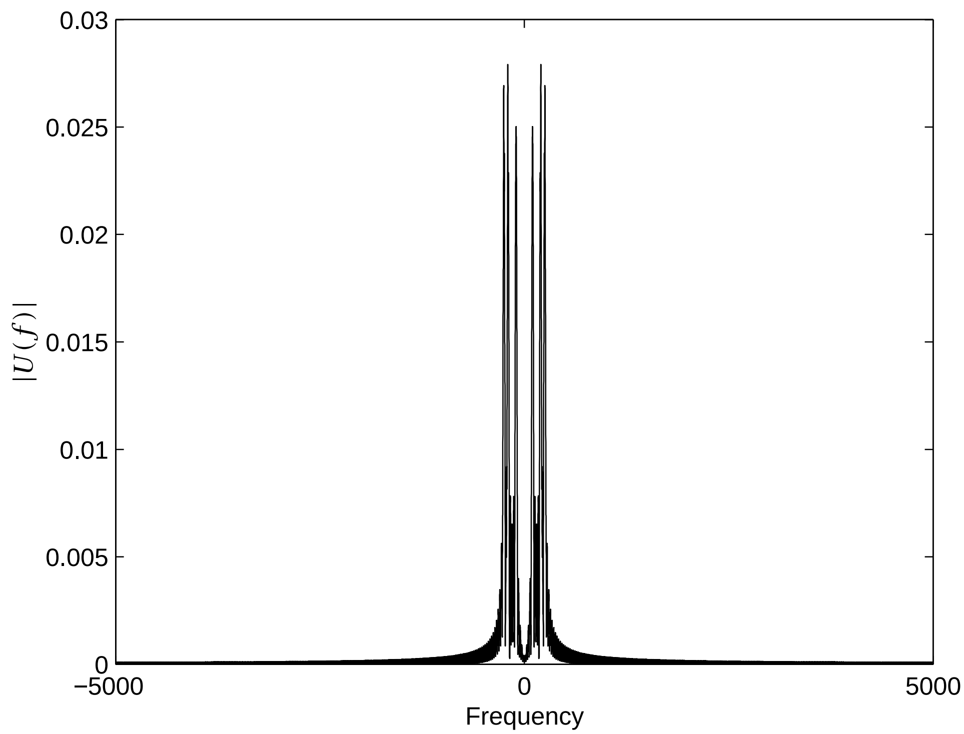


Figure 4.5: The magnitude spectrum of the modulated signal for Computer Problem 4.1

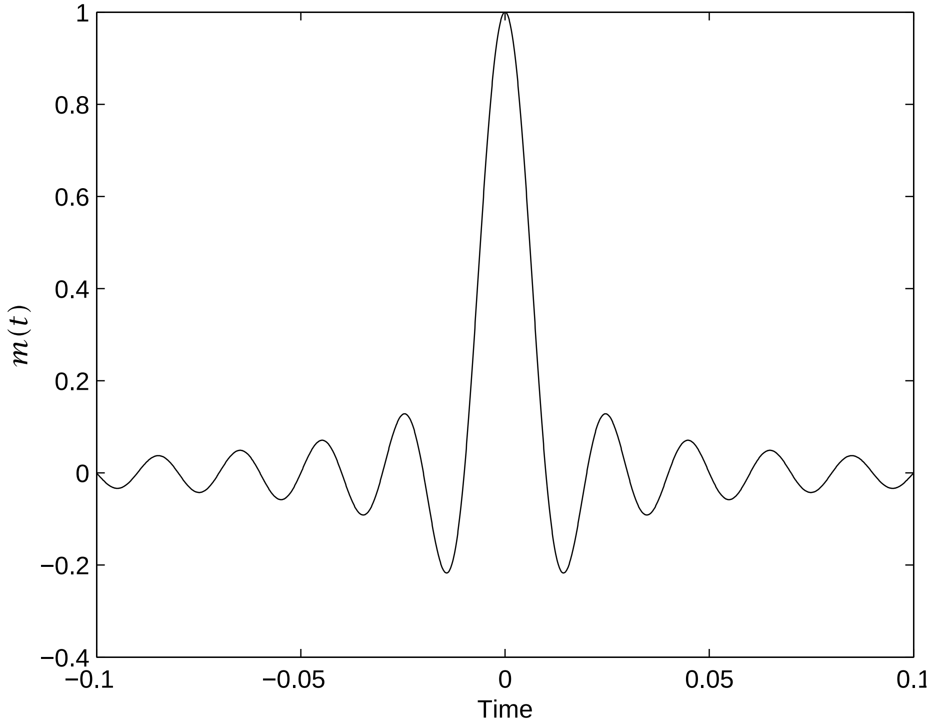


Figure 4.6: The message signal for Computer Problem 4.2

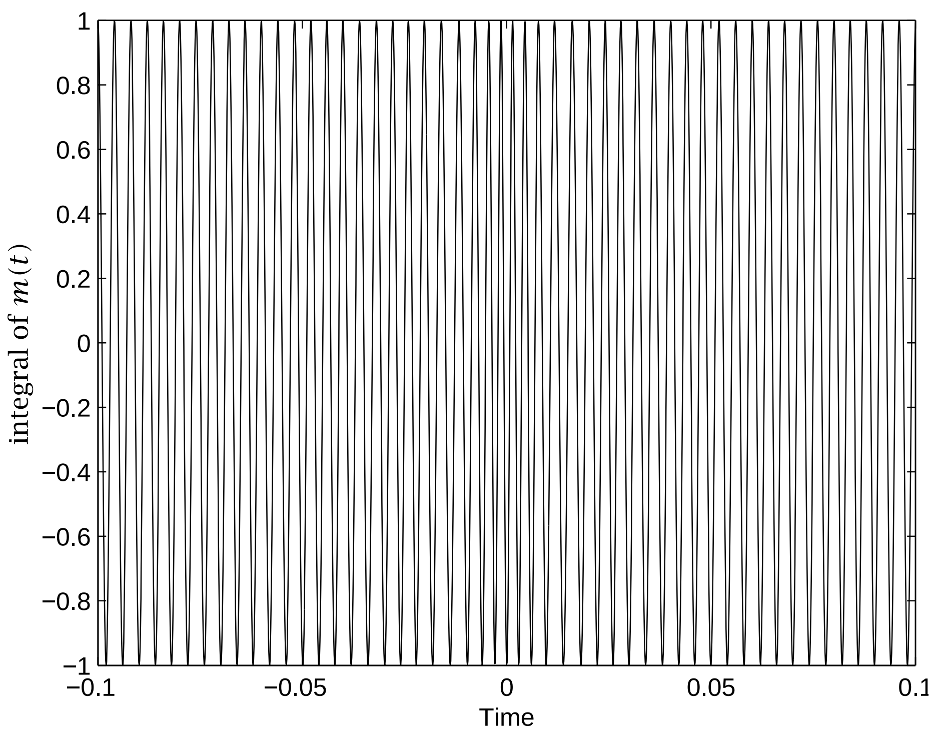


Figure 4.7: The integral of the message signal for Computer Problem 4.2

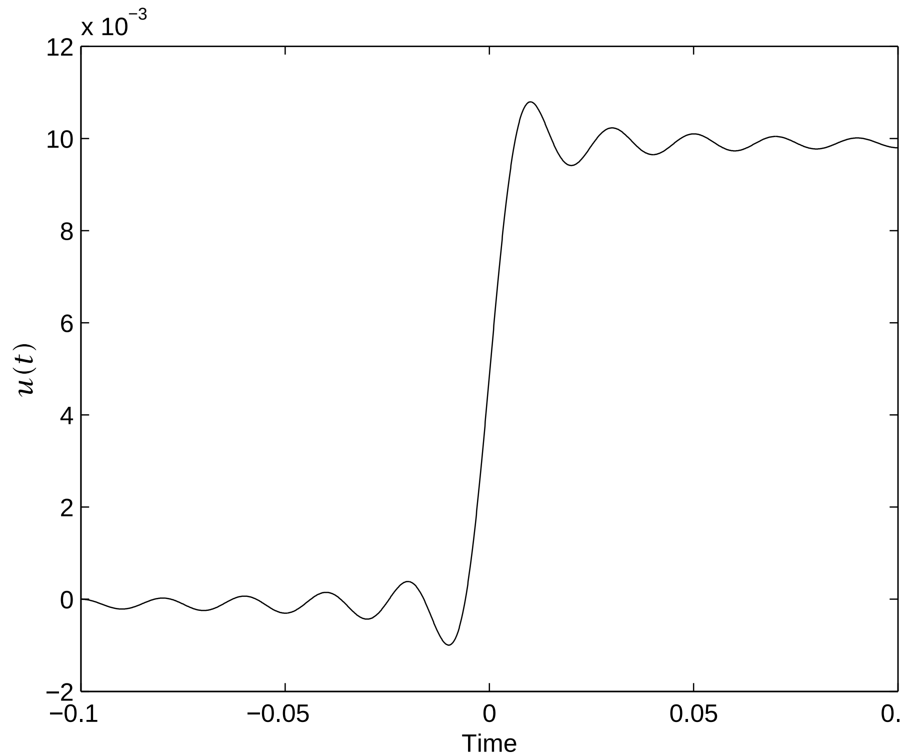


Figure 4.8: The modulated signal for Computer Problem 4.2

3) The spectrum of the message and the modulated signals are shown in Figures 4.9 and 4.11, respectively.
4) The plot of the demodulated signal is shown in Figure 4.11. As you can see, the demodulated signal is quite similar to the message signal.
The MATLAB script for this problem follows.

```
% MATLAB script for Computer Problem 4.2.
% Demonstration script for frequency modulation. The message signal
% is m(t)=sinc(100t).
echo on
t0=.2;                          % signal duration
ts=0.0001;                      % sampling interval
fc=250;                         % carrier frequency
snr=20;                         % SNR in dB (logarithmic)
fs=1/ts;                        % sampling frequency
df=0.3;                         % required freq. resolution
t=[-t0/2:ts:t0/2];              % time vector
kf=100;                         % deviation constant
df=0.25;                        % required frequency resolution
m=sinc(100*t);                  % the message signal
int_m(1)=0;
for i=1:length(t)-1             % integral of m
  int_m(i+1)=int_m(i)+m(i)*ts;
  echo off ;
```

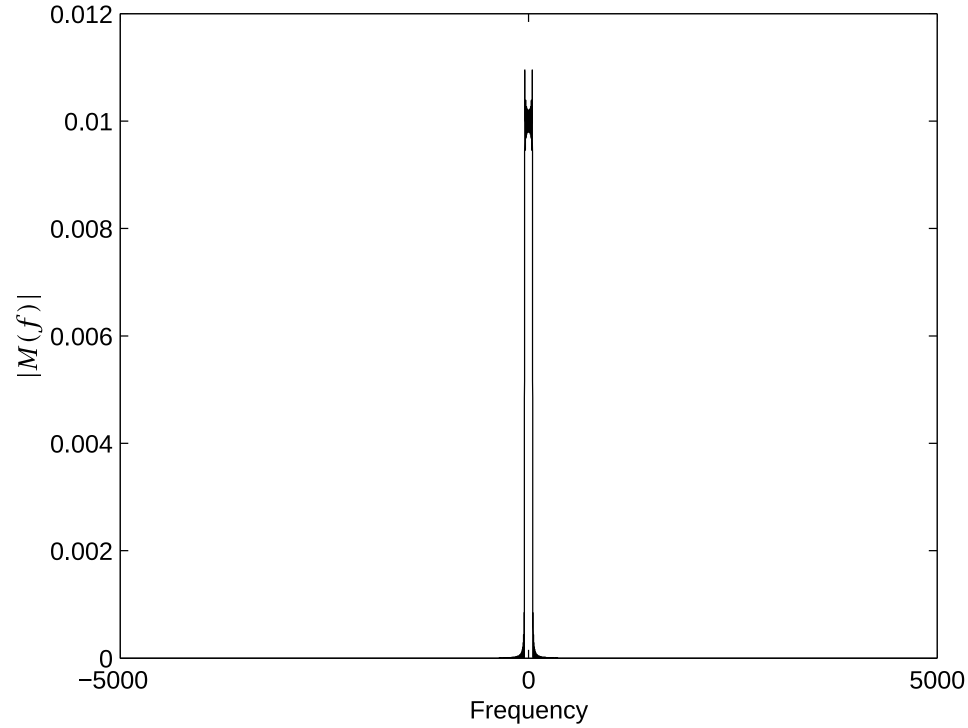


Figure 4.9: The magnitude spectrum of the message signal for Computer Problem 4.2

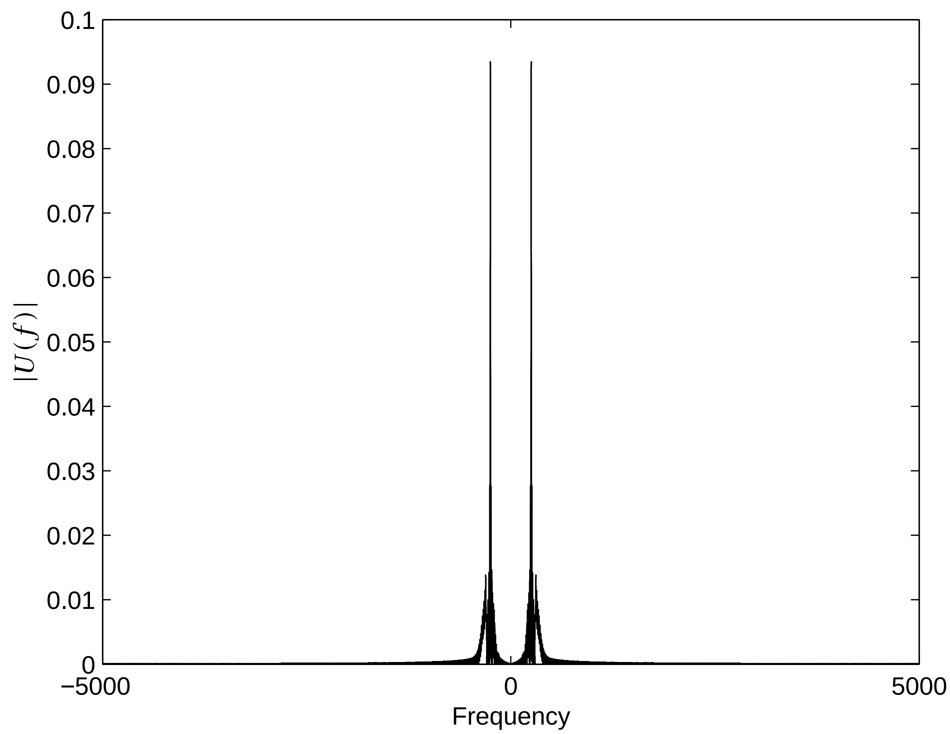


Figure 4.10: The magnitude spectrum of the modulated signal for Computer Problem 4.2

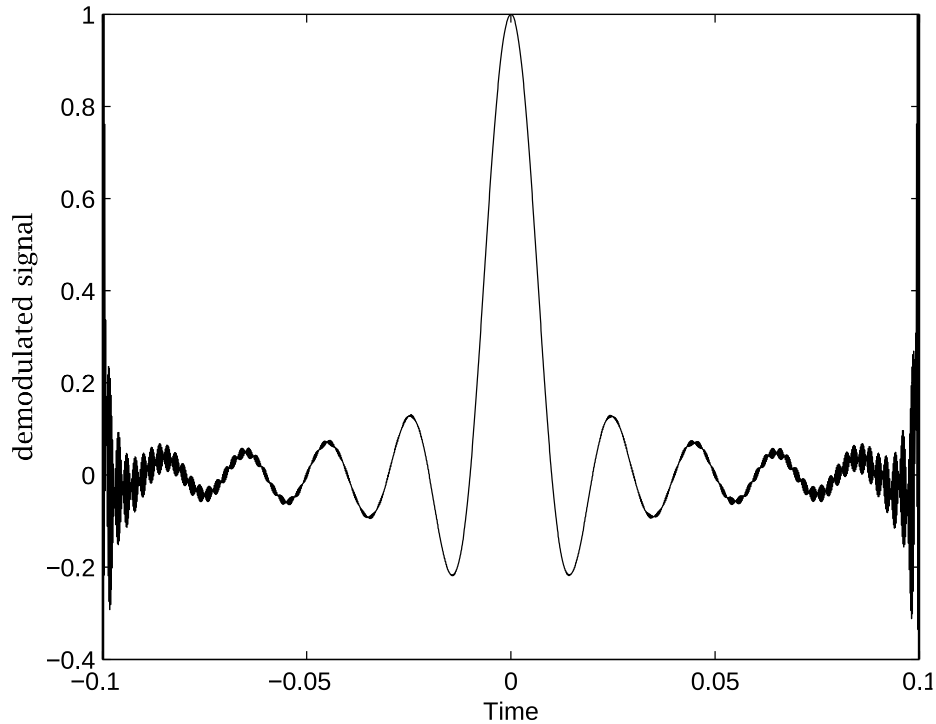


Figure 4.11: The demodulated signal for Computer Problem 4.2

```
end
echo on ;
[M,m,df1]=fftseq(m,ts,df);             % Fourier transform
M=M/fs;                                % scaling
f=[0:df1:df1*(length(m)−1)]−fs/2;      % frequency vector
u=cos(2*pi*fc*t+2*pi*kf*int_m);        % modulated signal
[U,u,df1]=fftseq(u,ts,df);             % Fourier transform
U=U/fs;                                % scaling
[v,phase]=env_phas(u,ts,250);          % demodulation, find phase of u
phi=unwrap(phase);                     % Restore original phase.
dem=(1/(2*pi*kf))*(diff(phi)/ts);      % demodulator output, differentiate and scale phase
pause  % Press any key to see a plot of the message and the modulated signal.
subplot(2,1,1)
plot(t,m(1:length(t)))
xlabel('Time')
title('The message signal')
subplot(2,1,2)
plot(t,u(1:length(t)))
xlabel('Time')
title('The modulated signal')
pause   % Press any key to see plots of the magnitude of the message and the
        % modulated signal in the frequency domain.
subplot(2,1,1)
plot(f,abs(fftshift(M)))
xlabel('Frequency')
title('Magnitude spectrum of the message signal')
```

```
subplot(2,1,2)
plot(f,abs(fftshift(U)))
title('Magnitude-spectrum of the modulated signal')
xlabel('Frequency')
pause  % Press any key to see plots of the message and the demodulator output with no
       % noise.
subplot(2,1,1)
plot(t,m(1:length(t)))
xlabel('Time')
title('The message signal')
subplot(2,1,2)
plot(t,dem(1:length(t)))
xlabel('Time')
title('The demodulated signal')
```

```
function [v,phi]=env_phas(x,ts,f0)
%           [v,phi]=env_phas(x,ts,f0)
%               v=env_phas(x,ts,f0)
%ENV_PHAS   returns the envelope and the phase of the bandpass signal x
%           f0 is the center frequency.
%           ts is the sampling interval.
%
if nargout == 2
  z=loweq(x,ts,f0);
  phi=angle(z);
end
v=abs(hilbert(x));
```

```
function xl=loweq(x,ts,f0)
%           xl=loweq(x,ts,f0)
%LOWEQ      returns the lowpass equivalent of the signal x
%           f0 is the center frequency.
%           ts is the sampling interval.
%
t=[0:ts:ts*(length(x)-1)];
z=hilbert(x);
xl=z.*exp(-j*2*pi*f0*t);
```

Computer Problem 4.3

Similar to Computer Problems 4.2 and 4.3.

Chapter 5

Problem 5.1

Let us denote by r_n (b_n) the event of drawing a red (black) ball with number n. Then

$$\begin{aligned}
1.\ E_1 &= \{r_2, r_4, b_2\} \\
2.\ E_2 &= \{r_2, r_3, r_4\} \\
3.\ E_3 &= \{r_1, r_2, b_1, b_2\} \\
4.\ E_4 &= \{r_1, r_2, r_4, b_1, b_2\} \\
5.\ E_5 &= \{r_2, r_4, b_2\} \cup [\{r_2, r_3, r_4\} \cap \{r_1, r_2, b_1, b_2\}] \\
&= \{r_2, r_4, b_2\} \cup \{r_2\} = \{r_2, r_4, b_2\}
\end{aligned}$$

Problem 5.2

Solution:

Since the seven balls equally likely to be drawn, the probability of each event E_i is proportional to its cardinality.

$$P(E_1) = \frac{3}{7}, \quad P(E_2) = \frac{3}{7}, \quad P(E_3) = \frac{4}{7}, \quad P(E_4) = \frac{5}{7}, \quad P(E_5) = \frac{3}{7}$$

Problem 5.3

Solution:

Let us denote by X the event that a car is of brand X, and by R the event that a car needs repair during its first year of purchase. Then

1)

$$\begin{aligned}
P(R) &= P(A, R) + P(B, R) + P(C, R) \\
&= P(R|A)P(A) + P(R|B)P(B) + P(R|C)P(C) \\
&= \frac{5}{100}\frac{20}{100} + \frac{10}{100}\frac{30}{100} + \frac{15}{100}\frac{50}{100} \\
&= \frac{11.5}{100}
\end{aligned}$$

2)

$$P(A|R) = \frac{P(A, R)}{P(R)} = \frac{P(R|A)P(A)}{P(R)} = \frac{.05.20}{.115} = .087$$

Problem 5.4
Solution:
If two events are mutually exclusive (disjoint) then $P(A \cup B) = P(A) \cup P(B)$ which implies that $P(A \cap B) = 0$. If the events are independent then $P(A \cap B) = P(A) \cap P(B)$. Combining these two conditions we obtain that two disjoint events are independent if

$$P(A \cap B) = P(A)P(B) = 0$$

Thus, at least on of the events should be of zero probability.

Problem 5.5
Let us denote by nS the event that n was produced by the source and sent over the channel, and by nC the event that n was observed at the output of the channel. Then
1)

$$\begin{aligned} P(1C) &= P(1C|1S)P(1S) + P(1C|0C)P(0C) \\ &= .8 \cdot .7 + .2 \cdot .3 = .62 \end{aligned}$$

where we have used the fact that $P(1S) = .7$, $P(0C) = .3$, $P(1C|0C) = .2$ and $P(1C|1S) = 1 - .2 = .8$
2)

$$P(1S|1C) = \frac{P(1C,1S)}{P(1C)} = \frac{P(1C|1S)P(1S)}{P(1C)} = \frac{.8 \cdot .7}{.62} = .9032$$

Problem 5.6
1) X can take four different values. 0, if no head shows up, 1, if only one head shows up in the four flips of the coin, 2, for two heads and 3 if the outcome of each flip is head.
2) X follows the binomial distribution with $n = 3$. Thus

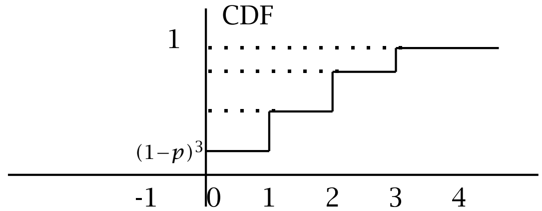
$$P(X = k) = \begin{cases} \begin{pmatrix} 3 \\ k \end{pmatrix} p^k (1-p)^{3-k} & \text{for } 0 \le k \le 3 \\ 0 & \text{otherwise} \end{cases}$$

3)

$$F_X(k) = \sum_{m=0}^{k} \begin{pmatrix} 3 \\ m \end{pmatrix} p^m (1-p)^{3-m}$$

Hence

$$F_X(k) = \begin{cases} 0 & k < 0 \\ (1-p)^3 & k = 0 \\ (1-p)^3 + 3p(1-p)^2 & k = 1 \\ (1-p)^3 + 3p(1-p)^2 + 3p^2(1-p) & k = 2 \\ (1-p)^3 + 3p(1-p)^2 + 3p^2(1-p) + p^3 = 1 & k = 3 \\ 1 & k > 3 \end{cases}$$



4)

$$P(X > 1) = \sum_{k=2}^{3} \binom{3}{k} p^k (1-p)^{3-k} = 3p^2(1-p) + (1-p)^3$$

Problem 5.7

1) The random variables X and Y follow the binomial distribution with $n = 4$ and $p = 1/4$ and $1/2$ respectively. Thus

$$P(X=0) = \binom{4}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^4 = \frac{3^4}{2^8} \qquad P(Y=0) = \binom{4}{0}\left(\frac{1}{2}\right)^4 = \frac{1}{2^4}$$

$$P(X=1) = \binom{4}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^3 = \frac{3^3 4}{2^8} \qquad P(Y=1) = \binom{4}{1}\left(\frac{1}{2}\right)^4 = \frac{4}{2^4}$$

$$P(X=2) = \binom{4}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2 = \frac{3^3 2}{2^8} \qquad P(Y=2) = \binom{4}{2}\left(\frac{1}{2}\right)^4 = \frac{6}{2^4}$$

$$P(X=3) = \binom{4}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^1 = \frac{3 \cdot 4}{2^8} \qquad P(Y=3) = \binom{4}{3}\left(\frac{1}{2}\right)^4 = \frac{4}{2^4}$$

$$P(X=4) = \binom{4}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^0 = \frac{1}{2^8} \qquad P(Y=4) = \binom{4}{4}\left(\frac{1}{2}\right)^4 = \frac{1}{2^4}$$

Since X and Y are independent we have

$$P(X = Y = 2) = P(X=2)P(Y=2) = \frac{3^3 2}{2^8}\frac{6}{2^4} = \frac{81}{1024}$$

2)

$$\begin{aligned} P(X=Y) &= P(X=0)P(Y=0) + P(X=1)P(Y=1) + P(X=2)P(Y=2) \\ &\quad + P(X=3)P(Y=3) + P(X=4)P(Y=4) \\ &= \frac{3^4}{2^12} + \frac{3^3 \cdot 4^2}{2^12} + \frac{3^4 \cdot 2^2}{2^12} + \frac{3 \cdot 4^2}{2^12} + \frac{1}{2^12} = \frac{886}{4096} \end{aligned}$$

3)

$$\begin{aligned} P(X > Y) &= P(Y=0)\,[P(X=1)+P(X=2)+P(X=3)+P(X=4)] + \\ &\quad P(Y=1)\,[P(X=2)+P(X=3)+P(X=4)] + \\ &\quad P(Y=2)\,[P(X=3)+P(X=4)] + \\ &\quad P(Y=3)\,[P(X=4)] \\ &= \frac{535}{4096} \end{aligned}$$

4) In general $P(X+Y \leq 5) = \sum_{l=0}^{5}\sum_{m=0}^{l} P(X=l-m)P(Y=m)$. However it is easier to find $P(X+Y \leq 5)$ through $P(X+Y \leq 5) = 1 - P(X+Y > 5)$ because fewer terms are involved in the calculation of the probability $P(X+Y > 5)$. Note also that $P(X+Y > 5|X=0) = P(X+Y > 5|X=1) = 0$.

$$\begin{aligned} P(X+Y > 5) &= P(X=2)P(Y=4) + P(X=3)[P(Y=3)+P(Y=4)] + \\ &\quad P(X=4)[P(Y=2)+P(Y=3)+P(Y=4)] \\ &= \frac{125}{4096} \end{aligned}$$

Hence, $P(X+Y \leq 5) = 1 - P(X+Y > 5) = 1 - \frac{125}{4096}$

Problem 5.8

1) Since $\lim_{x\to\infty} F_X(x) = 1$ and $F_X(x) = 1$ for all $x \geq 1$ we obtain $K = 1$.

2) The random variable is of the mixed-type since there is a discontinuity at $x = 1$. $\lim_{\epsilon\to 0} F_X(1-\epsilon) = 1/2$ whereas $\lim_{\epsilon\to 0} F_X(1+\epsilon) = 1$

3)

$$P(\frac{1}{2} < X \leq 1) = F_X(1) - F_X(\frac{1}{2}) = 1 - \frac{1}{4} = \frac{3}{4}$$

4)

$$P(\frac{1}{2} < X < 1) = F_X(1^-) - F_X(\frac{1}{2}) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

5)

$$P(X > 2) = 1 - P(X \leq 2) = 1 - F_X(2) = 1 - 1 = 0$$

Problem 5.9

1)

$$\begin{aligned}
x < -1 \quad &\Rightarrow \quad F_X(x) = 0 \\
-1 \le x \le 0 \quad &\Rightarrow \quad F_X(x) = \int_{-1}^{x} (v+1)dv = (\frac{1}{2}v^2 + v)\Big|_{-1}^{x} = \frac{1}{2}x^2 + x + \frac{1}{2} \\
0 \le x \le 1 \quad &\Rightarrow \quad F_X(x) = \int_{-1}^{0} (v+1)dv + \int_{0}^{x} (-v+1)dv = -\frac{1}{2}x^2 + x + \frac{1}{2} \\
1 \le x \quad &\Rightarrow \quad F_X(x) = 1
\end{aligned}$$

2)

$$p(X > \frac{1}{2}) = 1 - F_X(\frac{1}{2}) = 1 - \frac{7}{8} = \frac{1}{8}$$

and

$$p(X > 0 | X < \frac{1}{2}) = \frac{p(X > 0,\ X < \frac{1}{2})}{p(X < \frac{1}{2})} = \frac{F_X(\frac{1}{2}) - F_X(0)}{1 - p(X > \frac{1}{2})} = \frac{3}{7}$$

3) We find first the CDF

$$F_X(x | X > \frac{1}{2}) = p(X \le x | X > \frac{1}{2}) = \frac{p(X \le x,\ X > \frac{1}{2})}{p(X > \frac{1}{2})}$$

If $x \le \frac{1}{2}$ then $p(X \le x | X > \frac{1}{2}) = 0$ since the events $E_1 = \{X \le \frac{1}{2}\}$ and $E_1 = \{X > \frac{1}{2}\}$ are disjoint. If $x > \frac{1}{2}$ then $p(X \le x | X > \frac{1}{2}) = F_X(x) - F_X(\frac{1}{2})$ so that

$$F_X(x | X > \frac{1}{2}) = \frac{F_X(x) - F_X(\frac{1}{2})}{1 - F_X(\frac{1}{2})}$$

Differentiating this equation with respect to x we obtain

$$f_X(x | X > \frac{1}{2}) = \begin{cases} \frac{f_X(x)}{1 - F_X(\frac{1}{2})} & x > \frac{1}{2} \\ 0 & x \le \frac{1}{2} \end{cases}$$

4)

$$\begin{aligned}
E[X | X > 1/2] &= \int_{-\infty}^{\infty} x f_X(x|X > 1/2) dx \\
&= \frac{1}{1 - F_X(1/2)} \int_{\frac{1}{2}}^{\infty} x f_X(x) dx \\
&= 8 \int_{\frac{1}{2}}^{\infty} x(-x+1) dx = 8(-\frac{1}{3}x^3 + \frac{1}{2}x^2)\Big|_{\frac{1}{2}}^{1} \\
&= \frac{2}{3}
\end{aligned}$$

Problem 5.10

In general, if X is a Gaussian RV with mean m and variance σ^2, we have,

$$P(X > \alpha) = Q\left(\frac{\alpha - m}{\sigma}\right)$$

Therefore,

$$P(X > 7) = Q\left(\frac{7-4}{3}\right) = Q(1) = 0.158$$

and using the relation $P(0 < X < 9) = P(X > 0) - P(X > 9)$, we have

$$P(0 < X < 9) = Q\left(\frac{0-4}{3}\right) - Q\left(\frac{9-4}{3}\right) = 1 - Q(1.33) - Q(1.66) \approx 0.858$$

Problem 5.11

1) The random variable X is Gaussian with zero mean and variance $\sigma^2 = 10^{-8}$. Thus $P(X > x) = Q(\frac{x}{\sigma})$ and

$$\begin{aligned} P(X > 10^{-4}) &= Q\left(\frac{10^{-4}}{10^{-4}}\right) = Q(1) = .159 \\ P(X > 4\times 10^{-4}) &= Q\left(\frac{4\times 10^{-4}}{10^{-4}}\right) = Q(4) = 3.17\times 10^{-5} \\ P(-2\times 10^{-4} < X \le 10^{-4}) &= 1 - Q(1) - Q(2) = .8182 \end{aligned}$$

2)

$$P(X > 10^{-4} | X > 0) = \frac{P(X > 10^{-4}, X > 0)}{P(X > 0)} = \frac{P(X > 10^{-4})}{P(X > 0)} = \frac{.159}{.5} = .318$$

3) $y = g(x) = xu(x)$. Clearly $f_Y(y) = 0$ and $F_Y(y) = 0$ for $y < 0$. If $y > 0$, then the equation $y = xu(x)$ has a unique solution $x_1 = y$. Hence, $F_Y(y) = F_X(y)$ and $f_Y(y) = f_X(y)$ for $y > 0$. $F_Y(y)$ is discontinuous at $y = 0$ and the jump of the discontinuity equals $F_X(0)$.

$$F_Y(0^+) - F_Y(0^-) = F_X(0) = \frac{1}{2}$$

In summary the PDF $f_Y(y)$ equals

$$f_Y(y) = f_X(y)u(y) + \frac{1}{2}\delta(y)$$

The general expression for finding $f_Y(y)$ can not be used because $g(x)$ is constant for some interval so that there is an uncountable number of solutions for x in this interval.

4)

$$\begin{aligned} E[Y] &= \int_{-\infty}^{\infty} y f_Y(y) dy \\ &= \int_{-\infty}^{\infty} y \left[f_X(y)u(y) + \frac{1}{2}\delta(y) \right] dy \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy = \frac{\sigma}{\sqrt{2\pi}} \end{aligned}$$

5) $y = g(x) = |x|$. For a given $y > 0$ there are two solutions to the equation $y = g(x) = |x|$, that is $x_{1,2} = \pm y$. Hence for $y > 0$

$$\begin{aligned} f_Y(y) &= \frac{f_X(x_1)}{|\text{sgn}(x_1)|} + \frac{f_X(x_2)}{|\text{sgn}(x_2)|} = f_X(y) + f_X(-y) \\ &= \frac{2}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} \end{aligned}$$

For $y < 0$ there are no solutions to the equation $y = |x|$ and $f_Y(y) = 0$.

$$E[Y] = \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} y e^{-\frac{y^2}{2\sigma^2}} dy = \frac{2\sigma}{\sqrt{2\pi}}$$

Problem 5.12

1) $y = g(x) = ax^2$. Assume without loss of generality that $a > 0$. Then, if $y < 0$ the equation $y = ax^2$ has no real solutions and $f_Y(y) = 0$. If $y > 0$ there are two solutions to the system, namely $x_{1,2} = \sqrt{y/a}$. Hence,

$$\begin{aligned} f_Y(y) &= \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} \\ &= \frac{f_X(\sqrt{y/a})}{2a\sqrt{y/a}} + \frac{f_X(-\sqrt{y/a})}{2a\sqrt{y/a}} \\ &= \frac{1}{\sqrt{ay}\sqrt{2\pi\sigma^2}} e^{-\frac{y}{2a\sigma^2}} \end{aligned}$$

2) The equation $y = g(x)$ has no solutions if $y < -b$. Thus $F_Y(y)$ and $f_Y(y)$ are zero for $y < -b$. If $-b \le y \le b$, then for a fixed y, $g(x) < y$ if $x < y$; hence $F_Y(y) = F_X(y)$. If $y > b$ then $g(x) \le b < y$ for every x; hence $F_Y(y) = 1$. At the points $y = \pm b$, $F_Y(y)$ is discontinuous and the discontinuities equal to

$$F_Y(-b^+) - F_Y(-b^-) = F_X(-b)$$

and

$$F_Y(b^+) - F_Y(b^-) = 1 - F_X(b)$$

The PDF of $y = g(x)$ is

$$\begin{aligned} f_Y(y) &= F_X(-b)\delta(y+b) + (1 - F_X(b))\delta(y-b) + f_X(y)[u_{-1}(y+b) - u_{-1}(y-b)] \\ &= Q\left(\frac{b}{\sigma}\right)(\delta(y+b) + \delta(y-b)) + \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} [u_{-1}(y+b) - u_{-1}(y-b)] \end{aligned}$$

3) In the case of the hard limiter

$$\begin{aligned} P(Y = b) &= P(X < 0) = F_X(0) = \frac{1}{2} \\ P(Y = a) &= P(X > 0) = 1 - F_X(0) = \frac{1}{2} \end{aligned}$$

Thus $F_Y(y)$ is a staircase function and

$$f_Y(y) = F_X(0)\delta(y - b) + (1 - F_X(0))\delta(y - a)$$

4) The random variable $y = g(x)$ takes the values $y_n = x_n$ with probability

$$P(Y = y_n) = P(a_n \leq X \leq a_{n+1}) = F_X(a_{n+1}) - F_X(a_n)$$

Thus, $F_Y(y)$ is a staircase function with $F_Y(y) = 0$ if $y < x_1$ and $F_Y(y) = 1$ if $y > x_N$. The PDF is a sequence of impulse functions, that is

$$\begin{aligned} f_Y(y) &= \sum_{i=1}^{N} \left[F_X(a_{i+1}) - F_X(a_i)\right] \delta(y - x_i) \\ &= \sum_{i=1}^{N} \left[Q\left(\frac{a_i}{\sigma}\right) - Q\left(\frac{a_{i+1}}{\sigma}\right)\right] \delta(y - x_i) \end{aligned}$$

Problem 5.13

The equation $x = \tan\phi$ has a unique solution in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, that is

$$\phi_1 = \arctan x$$

Furthermore

$$x'(\phi) = \left(\frac{\sin\phi}{\cos\phi}\right)' = \frac{1}{\cos^2\phi} = 1 + \frac{\sin^2\phi}{\cos^2\phi} = 1 + x^2$$

Thus,

$$f_X(x) = \frac{f_\Phi(\phi_1)}{|x'(\phi_1)|} = \frac{1}{\pi(1 + x^2)}$$

We observe that $f_X(x)$ is the Cauchy density. Since $f_X(x)$ is even we immediately get $E[X] = 0$. However, the variance is

$$\begin{aligned} \sigma_X^2 &= E[X^2] - (E[X])^2 \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x^2}{1 + x^2} dx = \infty \end{aligned}$$

Problem 5.14

1)

$$\begin{aligned} E[Y] &= \int_0^\infty y f_Y(y)dy \geq \int_\alpha^\infty y f_Y(y)dy \\ &\geq \alpha \int_\alpha^\infty y f_Y(y)dy = \alpha P(Y \geq \alpha) \end{aligned}$$

Thus $P(Y \geq \alpha) \leq E[Y]/\alpha$.

2) Clearly $P(|X - E[X]| > \epsilon) = P((X - E[X])^2 > \epsilon^2)$. Thus using the results of the previous question we obtain

$$P(|X - E[X]| > \epsilon) = P((X - E[X])^2 > \epsilon^2) \leq \frac{E[(X - E[X])^2]}{\epsilon^2} = \frac{\sigma^2}{\epsilon^2}$$

Problem 5.15

The characteristic function of the binomial distribution is

$$\begin{aligned} \psi_X(v) &= \sum_{k=0}^{n} e^{jvk} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} (pe^{jv})^k (1-p)^{n-k} = (pe^{jv} + (1-p))^n \end{aligned}$$

Thus

$$\begin{aligned} E[X] &= m_X^{(1)} = \frac{1}{j}\frac{d}{dv}(pe^{jv} + (1-p))^n \Big|_{v=0} = \frac{1}{j} n (pe^{jv} + (1-p))^{n-1} pje^{jv} \Big|_{v=0} \\ &= n(p + 1 - p)^{n-1} p = np \\ E[X^2] &= m_X^{(2)} = (-1)\frac{d^2}{dv^2}(pe^{jv} + (1-p))^n \Big|_{v=0} \\ &= (-1)\frac{d}{dv}\left[n(pe^{jv} + (1-p)^{n-1} pje^{jv}\right] \Big|_{v=0} \\ &= \left[n(n-1)(pe^{jv} + (1-p))^{n-2} p^2 e^{2jv} + n(pe^{jv} + (1-p))^{n-1} pe^{jv}\right] \Big|_{v=0} \\ &= n(n-1)(p + 1 - p)p^2 + n(p + 1 - p)p \\ &= n(n-1)p^2 + np \end{aligned}$$

Hence the variance of the binomial distribution is

$$\sigma^2 = E[X^2] - (E[X])^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p)$$

Problem 5.16

The characteristic function of the Poisson distribution is

$$\psi_X(v) = \sum_{k=0}^{\infty} e^{jvk}\frac{\lambda^k}{k!}e^{-k} = \sum_{k=0}^{\infty}\frac{(e^{jv-1}\lambda)^k}{k!}$$

But $\sum_{k=0}^{\infty}\frac{a^k}{k!} = e^a$ so that $\psi_X(v) = e^{\lambda(e^{jv-1})}$. Hence

$$\begin{aligned} E[X] &= m_X^{(1)} = \frac{1}{j}\frac{d}{dv}\psi_X(v)\Big|_{v=0} = \frac{1}{j}e^{\lambda(e^{jv-1})}j\lambda e^{jv}\Big|_{v=0} = \lambda \\ E[X^2] &= m_X^{(2)} = (-1)\frac{d^2}{dv^2}\psi_X(v)\Big|_{v=0} = (-1)\frac{d}{dv}\left[\lambda e^{\lambda(e^{jv-1})}e^{jv}j\right]\Big|_{v=0} \\ &= \left[\lambda^2 e^{\lambda(e^{jv-1})}e^{jv} + \lambda e^{\lambda(e^{jv-1})}e^{jv}\right]\Big|_{v=0} = \lambda^2+\lambda \end{aligned}$$

Hence the variance of the Poisson distribution is

$$\sigma^2 = E[X^2] - (E[X])^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Problem 5.17

For n odd, x^n is odd and since the zero-mean Gaussian PDF is even their product is odd. Since the integral of an odd function over the interval $[-\infty,\infty]$ is zero, we obtain $E[X^n] = 0$ for n even. Let $I_n = \int_{-\infty}^{\infty} x^n \exp(-x^2/2\sigma^2)dx$ with n even. Then,

$$\begin{aligned} \frac{d}{dx}I_n &= \int_{-\infty}^{\infty}\left[nx^{n-1}e^{-\frac{x^2}{2\sigma^2}} - \frac{1}{\sigma^2}x^{n+1}e^{-\frac{x^2}{2\sigma^2}}\right]dx = 0 \\ \frac{d^2}{dx^2}I_n &= \int_{-\infty}^{\infty}\left[n(n-1)x^{n-2}e^{-\frac{x^2}{2\sigma^2}} - \frac{2n+1}{\sigma^2}x^n e^{-\frac{x^2}{2\sigma^2}} + \frac{1}{\sigma^4}x^{n+2}e^{-\frac{x^2}{2\sigma^2}}\right]dx \\ &= n(n-1)I_{n-2} - \frac{2n+1}{\sigma^2}I_n + \frac{1}{\sigma^4}I_{n+2} = 0 \end{aligned}$$

Thus,

$$I_{n+2} = \sigma^2(2n+1)I_n - \sigma^4 n(n-1)I_{n-2}$$

with initial conditions $I_0 = \sqrt{2\pi\sigma^2}$, $I_2 = \sigma^2\sqrt{2\pi\sigma^2}$. We prove now that

$$I_n = 1\times 3\times 5\times\cdots\times(n-1)\sigma^n\sqrt{2\pi\sigma^2}$$

The proof is by induction on n. For $n = 2$ it is certainly true since $I_2 = \sigma^2\sqrt{2\pi\sigma^2}$. We assume that the relation holds for n and we will show that it is true for I_{n+2}. Using the previous recursion we have

$$\begin{aligned} I_{n+2} &= 1\times 3\times 5\times\cdots\times(n-1)\sigma^{n+2}(2n+1)\sqrt{2\pi\sigma^2} \\ &\quad -1\times 3\times 5\times\cdots\times(n-3)(n-1)n\sigma^{n-2}\sigma^4\sqrt{2\pi\sigma^2} \\ &= 1\times 3\times 5\times\cdots\times(n-1)(n+1)\sigma^{n+2}\sqrt{2\pi\sigma^2} \end{aligned}$$

Clearly $E[X^n] = \frac{1}{\sqrt{2\pi\sigma^2}}I_n$ and

$$E[X^n] = 1\times 3\times 5\times\cdots\times(n-1)\sigma^n$$

Problem 5.18

1) $f_{X,Y}(x,y)$ is a PDF so that its integral over the support region of x, y should be one.

$$\begin{aligned}
\int_0^1\int_0^1 f_{X,Y}(x,y)dxdy &= K\int_0^1\int_0^1 (x+y)dxdy \\
&= K\left[\int_0^1\int_0^1 xdxdy + \int_0^1\int_0^1 ydxdy\right] \\
&= K\left[\frac{1}{2}x^2\Big|_0^1 y|_0^1 + \frac{1}{2}y^2\Big|_0^1 x|_0^1\right] \\
&= K
\end{aligned}$$

Thus $K = 1$.

2)

$$\begin{aligned}
P(X+Y>1) &= 1 - P(X+Y\leq 1) \\
&= 1 - \int_0^1\int_0^{1-x}(x+y)dxdy \\
&= 1 - \int_0^1 x\int_0^{1-x} dydx - \int_0^1 dx\int_0^{1-x} ydy \\
&= 1 - \int_0^1 x(1-x)dx - \int_0^1 \frac{1}{2}(1-x)^2 dx \\
&= \frac{2}{3}
\end{aligned}$$

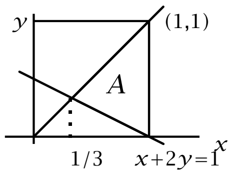
3) By exploiting the symmetry of $f_{X,Y}$ and the fact that it has to integrate to 1, one immediately sees that the answer to this question is 1/2. The "mechanical" solution is:

$$\begin{aligned}
P(X>Y) &= \int_0^1\int_y^1 (x+y)dxdy \\
&= \int_0^1\int_y^1 xdxdy + \int_0^1\int_y^1 ydxdy \\
&= \int_0^1 \frac{1}{2}x^2\Big|_y^1 dy + \int_0^1 yx\Big|_y^1 dy \\
&= \int_0^1 \frac{1}{2}(1-y^2)dy + \int_0^1 y(1-y)dy \\
&= \frac{1}{2}
\end{aligned}$$

4)

$$P(X>Y|X+2Y>1) = P(X>Y, X+2Y>1)/P(X+2Y>1)$$

The region over which we integrate in order to find $P(X > Y, X + 2Y > 1)$ is marked with an A in the following figure.



Thus

$$\begin{aligned}
P(X > Y, X + 2Y > 1) &= \int_{\frac{1}{3}}^{1} \int_{\frac{1-x}{2}}^{x} (x + y) dx dy \\
&= \int_{\frac{1}{3}}^{1} \left[x(x - \frac{1-x}{2}) + \frac{1}{2}(x^2 - (\frac{1-x}{2})^2) \right] dx \\
&= \int_{\frac{1}{3}}^{1} \left(\frac{15}{8}x^2 - \frac{1}{4}x - \frac{1}{8} \right) dx \\
&= \frac{49}{108} \\
P(X + 2Y > 1) &= \int_0^1 \int_{\frac{1-x}{2}}^{1} (x + y) dx dy \\
&= \int_0^1 \left[x(1 - \frac{1-x}{2}) + \frac{1}{2}(1 - (\frac{1-x}{2})^2) \right] dx \\
&= \int_0^1 \left(\frac{3}{8}x^2 + \frac{3}{4}x + \frac{3}{8} \right) dx \\
&= \frac{3}{8} \times \frac{1}{3}x^3 \Big|_0^1 + \frac{3}{4} \times \frac{1}{2}x^2 \Big|_0^1 + \frac{3}{8}x \Big|_0^1 \\
&= \frac{7}{8}
\end{aligned}$$

Hence, $P(X > Y | X + 2Y > 1) = (49/108)/(7/8) = 14/27$

5) When $X = Y$ the volume under integration has measure zero and thus

$$P(X = Y) = 0$$

6) Conditioned on the fact that $X = Y$, the new p.d.f of X is

$$f_{X|X=Y}(x) = \frac{f_{X,Y}(x,x)}{\int_0^1 f_{X,Y}(x,x) dx} = 2x.$$

In words, we re-normalize $f_{X,Y}(x,y)$ so that it integrates to 1 on the region characterized by $X = Y$. The result depends only on x. Then $P(X > \frac{1}{2} | X = Y) = \int_{1/2}^{1} f_{X|X=Y}(x) dx = 3/4$.

7)

$$\begin{aligned} f_X(x) &= \int_0^1 (x+y)dy = x + \int_0^1 y dy = x + \frac{1}{2} \\ f_Y(y) &= \int_0^1 (x+y)dx = y + \int_0^1 x dx = y + \frac{1}{2} \end{aligned}$$

8) $F_X(x|X+2Y>1) = P(X \le x, X+2Y>1)/P(X+2Y>1)$

$$\begin{aligned} P(X \le x, X+2Y>1) &= \int_0^x \int_{\frac{1-v}{2}}^1 (v+y)dv dy \\ &= \int_0^x \left[\frac{3}{8}v^2 + \frac{3}{4}v + \frac{3}{8}\right] dv \\ &= \frac{1}{8}x^3 + \frac{3}{8}x^2 + \frac{3}{8}x \end{aligned}$$

Hence,

$$f_X(x|X+2Y>1) = \frac{\frac{3}{8}x^2 + \frac{6}{8}x + \frac{3}{8}}{P(X+2Y>1)} = \frac{3}{7}x^2 + \frac{6}{7}x + \frac{3}{7}$$

$$\begin{aligned} E[X|X+2Y>1] &= \int_0^1 x f_X(x|X+2Y>1)dx \\ &= \int_0^1 \left(\frac{3}{7}x^3 + \frac{6}{7}x^2 + \frac{3}{7}x\right) \\ &= \frac{3}{7} \times \frac{1}{4}x^4\Big|_0^1 + \frac{6}{7} \times \frac{1}{3}x^3\Big|_0^1 + \frac{3}{7} \times \frac{1}{2}x^2\Big|_0^1 = \frac{17}{28} \end{aligned}$$

Problem 5.19

1)

$$F_Y(y) = P(Y \le y) = P(X_1 \le y \cup X_2 \le y \cup \cdots \cup X_n \le y)$$

Since the previous events are not necessarily disjoint, it is easier to work with the function $1 - [F_Y(y)] = 1 - P(Y \le y)$ in order to take advantage of the independence of X_i's. Clearly

$$\begin{aligned} 1 - P(Y \le y) &= P(Y > y) = P(X_1 > y \cap X_2 > y \cap \cdots \cap X_n > y) \\ &= (1 - F_{X_1}(y))(1 - F_{X_2}(y)) \cdots (1 - F_{X_n}(y)) \end{aligned}$$

Differentiating the previous with respect to y we obtain

$$f_Y(y) = f_{X_1}(y) \prod_{i \ne 1}^n (1 - F_{X_i}(y)) + f_{X_2}(y) \prod_{i \ne 2}^n (1 - F_{X_i}(y)) + \cdots + f_{X_n}(y) \prod_{i \ne n}^n (1 - F_{X_i}(y))$$

2)

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(X_1 \le z, X_2 \le z, \cdots, X_n \le z) \\ &= P(X_1 \le z)P(X_2 \le z) \cdots P(X_n \le z) \end{aligned}$$

Differentiating the previous with respect to z we obtain

$$f_Z(z) = f_{X_1}(z)\prod_{i\neq 1}^{n} F_{X_i}(z) + f_{X_2}(z)\prod_{i\neq 2}^{n} F_{X_i}(z) + \cdots + f_{X_n}(z)\prod_{i\neq n}^{n} F_{X_i}(z)$$

Problem 5.20

$$E[X] = \int_0^\infty x\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}dx = \frac{1}{\sigma^2}\int_0^\infty x^2 e^{-\frac{x^2}{2\sigma^2}}dx$$

However for the Gaussian random variable of zero mean and variance σ^2

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^\infty x^2 e^{-\frac{x^2}{2\sigma^2}}dx = \sigma^2$$

Since the quantity under integration is even, we obtain that

$$\frac{1}{\sqrt{2\pi\sigma^2}}\int_0^\infty x^2 e^{-\frac{x^2}{2\sigma^2}}dx = \frac{1}{2}\sigma^2$$

Thus,

$$E[X] = \frac{1}{\sigma^2}\sqrt{2\pi\sigma^2}\frac{1}{2}\sigma^2 = \sigma\sqrt{\frac{\pi}{2}}$$

In order to find $VAR(X)$ we first calculate $E[X^2]$.

$$\begin{aligned} E[X^2] &= \frac{1}{\sigma^2}\int_0^\infty x^3 e^{-\frac{x^2}{2\sigma^2}}dx = -\int_0^\infty x d[e^{-\frac{x^2}{2\sigma^2}}] \\ &= -x^2 e^{-\frac{x^2}{2\sigma^2}}\Big|_0^\infty + \int_0^\infty 2xe^{-\frac{x^2}{2\sigma^2}}dx \\ &= 0 + 2\sigma^2\int_0^\infty \frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}dx = 2\sigma^2 \end{aligned}$$

Thus,

$$VAR(X) = E[X^2] - (E[X])^2 = 2\sigma^2 - \frac{\pi}{2}\sigma^2 = (2 - \frac{\pi}{2})\sigma^2$$

Problem 5.21

Let $Z = X + Y$. Then,

$$F_Z(z) = P(X + Y \leq z) = \int_{-\infty}^\infty\int_{-\infty}^{z-y} f_{X,Y}(x,y)dxdy$$

Differentiating with respect to z we obtain

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^\infty \frac{d}{dz}\int_{-\infty}^{z-y} f_{X,Y}(x,y)dxdy \\ &= \int_{-\infty}^\infty f_{X,Y}(z-y,y)\frac{d}{dz}(z-y)dy \\ &= \int_{-\infty}^\infty f_{X,Y}(z-y,y)dy \\ &= \int_{-\infty}^\infty f_X(z-y)f_Y(y)dy \end{aligned}$$

where the last line follows from the independence of X and Y. Thus $f_Z(z)$ is the convolution of $f_X(x)$ and $f_Y(y)$. With $f_X(x) = \alpha e^{-\alpha x}u(x)$ and $f_Y(y) = \beta e^{-\beta x}u(x)$ we obtain

$$f_Z(z) = \int_0^z \alpha e^{-\alpha v}\beta e^{-\beta(z-v)}dv$$

If $\alpha = \beta$ then

$$f_Z(z) = \int_0^z \alpha^2 e^{-\alpha z}dv = \alpha^2 z e^{-\alpha z}u_{-1}(z)$$

If $\alpha \neq \beta$ then

$$f_Z(z) = \alpha\beta e^{-\beta z}\int_0^z e^{(\beta-\alpha)v}dv = \frac{\alpha\beta}{\beta-\alpha}\left[e^{-\alpha z} - e^{-\beta z}\right]u_{-1}(z)$$

Problem 5.22

1) $f_{X,Y}(x,y)$ is a PDF, hence its integral over the supporting region of x, and y is 1.

$$\begin{aligned}\int_0^\infty\int_y^\infty f_{X,Y}(x,y)dxdy &= \int_0^\infty\int_y^\infty Ke^{-x-y}dxdy \\ &= K\int_0^\infty e^{-y}\int_y^\infty e^{-x}dxdy \\ &= K\int_0^\infty e^{-2y}dy = K(-\frac{1}{2})e^{-2y}\Big|_0^\infty = K\frac{1}{2}\end{aligned}$$

Thus K should be equal to 2.

2)

$$\begin{aligned}f_X(x) &= \int_0^x 2e^{-x-y}dy = 2e^{-x}(-e^{-y})\Big|_0^x = 2e^{-x}(1-e^{-x}) \\ f_Y(y) &= \int_y^\infty 2e^{-x-y}dy = 2e^{-y}(-e^{-x})\Big|_y^\infty = 2e^{-2y}\end{aligned}$$

3)

$$\begin{aligned}f_X(x)f_Y(y) &= 2e^{-x}(1-e^{-x})2e^{-2y} = 2e^{-x-y}2e^{-y}(1-e^{-x}) \\ &\neq 2e^{-x-y} = f_{X,Y}(x,y)\end{aligned}$$

Thus X and Y are not independent.

4) If $x < y$ then $f_{X|Y}(x|y) = 0$. If $x \geq y$, then with $u = x - y \geq 0$ we obtain

$$f_U(u) = f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2e^{-x-y}}{2e^{-2y}} = e^{-x+y} = e^{-u}$$

5)

$$\begin{aligned} E[X|Y=y] &= \int_y^\infty xe^{-x+y}dx = e^y \int_y^\infty xe^{-x}dx \\ &= e^y \left[-xe^{-x} \Big|_y^\infty + \int_y^\infty e^{-x}dx \right] \\ &= e^y (ye^{-y} + e^{-y}) = y + 1 \end{aligned}$$

6) In this part of the problem we will use extensively the following definite integral

$$\int_0^\infty x^{\nu-1}e^{-\mu x}dx = \frac{1}{\mu^\nu}(\nu-1)!$$

$$\begin{aligned} E[XY] &= \int_0^\infty \int_y^\infty xy2e^{-x-y}dxdy = \int_0^\infty 2ye^{-y} \int_y^\infty xe^{-x}dxdy \\ &= \int_0^\infty 2ye^{-y}(ye^{-y}+e^{-y})dy = 2\int_0^\infty y^2e^{-2y}dy + 2\int_0^\infty ye^{-2y}dy \\ &= 2\frac{1}{2^3}2! + 2\frac{1}{2^2}1! = 1 \end{aligned}$$

$$\begin{aligned} E[X] &= 2\int_0^\infty xe^{-x}(1-e^{-x})dx = 2\int_0^\infty xe^{-x}dx - 2\int_0^\infty xe^{-2x}dx \\ &= 2 - 2\frac{1}{2^2} = \frac{3}{2} \\ E[Y] &= 2\int_0^\infty ye^{-2y}dy = 2\frac{1}{2^2} = \frac{1}{2} \\ E[X^2] &= 2\int_0^\infty x^2e^{-x}(1-e^{-x})dx = 2\int_0^\infty x^2e^{-x}dx - 2\int_0^\infty x^2e^{-2x}dx \\ &= 2\cdot 2! - 2\frac{1}{2^3}2! = \frac{7}{2} \\ E[Y^2] &= 2\int_0^\infty y^2e^{-2y}dy = 2\frac{1}{2^3}2! = \frac{1}{2} \end{aligned}$$

Hence,

$$COV(X,Y) = E[XY] - E[X]E[Y] = 1 - \frac{3}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

and

$$\rho_{X,Y} = \frac{COV(X,Y)}{(E[X^2]-(E[X])^2)^{1/2}(E[Y^2]-(E[Y])^2)^{1/2}} = \frac{1}{\sqrt{5}}$$

Problem 5.23

$$\begin{aligned}
E[X] &= \frac{1}{\pi}\int_0^{\pi}\cos\theta d\theta = \frac{1}{\pi}\sin\theta|_0^{\pi} = 0 \\
E[Y] &= \frac{1}{\pi}\int_0^{\pi}\sin\theta d\theta = \frac{1}{\pi}(-\cos\theta)|_0^{\pi} = \frac{2}{\pi} \\
E[XY] &= \int_0^{\pi}\cos\theta\ \sin\theta\frac{1}{\pi}d\theta \\
&= \frac{1}{2\pi}\int_0^{\pi}\sin 2\theta d\theta = \frac{1}{4\pi}\int_0^{2\pi}\sin x dx = 0 \\
COV(X,Y) &= E[XY] - E[X]E[Y] = 0
\end{aligned}$$

Thus the random variables X and Y are uncorrelated. However they are not independent since $X^2 + Y^2 = 1$. To see this consider the probability $p(|X| < 1/2, Y \geq 1/2)$. Clearly $p(|X| < 1/2)p(Y \geq 1/2)$ is different than zero whereas $p(|X| < 1/2, Y \geq 1/2) = 0$. This is because $|X| < 1/2$ implies that $\pi/3 < \theta < 5\pi/3$ and for these values of θ, $Y = \sin\theta > \sqrt{3}/2 > 1/2$.

Problem 5.24

1) Clearly $X > r$, $Y > r$ implies that $X^2 > r^2$, $Y^2 > r^2$ so that $X^2 + Y^2 > 2r^2$ or $\sqrt{X^2+Y^2} > \sqrt{2}r$. Thus the event $E_1(r) = \{X > r, Y > r\}$ is a subset of the event $E_2(r) = \{\sqrt{X^2+Y^2} > \sqrt{2}r \,|\, X, Y > 0\}$ and $P(E_1(r)) \leq P(E_2(r))$.

2) Since X and Y are independent

$$P(E_1(r)) = P(X > r, Y > r) = P(X > r)P(Y > r) = Q^2(r)$$

3) Using the rectangular to polar transformation $V = \sqrt{X^2+Y^2}$, $\Theta = \arctan\frac{Y}{X}$ it is proved (see text Eq. 4.1.22) that

$$f_{V,\Theta}(v,\theta) = \frac{v}{2\pi\sigma^2}e^{-\frac{v^2}{2\sigma^2}}$$

Hence, with $\sigma^2 = 1$ we obtain

$$\begin{aligned}
P(\sqrt{X^2+Y^2} > \sqrt{2}r \,|\, X, Y > 0) &= \int_{\sqrt{2}r}^{\infty}\int_0^{\frac{\pi}{2}}\frac{v}{2\pi}e^{-\frac{v^2}{2}}dv d\theta \\
&= \frac{1}{4}\int_{\sqrt{2}r}^{\infty} v e^{-\frac{v^2}{2}}dv = \frac{1}{4}(-e^{-\frac{v^2}{2}})\Big|_{\sqrt{2}r}^{\infty} \\
&= \frac{1}{4}e^{-r^2}
\end{aligned}$$

Combining the results of part 1), 2) and 3) we obtain

$$Q^2(r) \leq \frac{1}{4}e^{-r^2} \quad \text{or} \quad Q(r) \leq \frac{1}{2}e^{-\frac{r^2}{2}}$$

Problem 5.25

The following is a program written in Fortran to compute the Q function

```
      REAL*8       x,t,a,q,pi,p,b1,b2,b3,b4,b5
      PARAMETER           (p=.2316419d+00, b1=.31981530d+00,
     +     b2=-.356563782d+00, b3=1.781477937d+00,
     +     b4=-1.821255978d+00, b5=1.330274429d+00)
C-
           pi=4.*atan(1.)
C-INPUT
      PRINT*,      'Enter -x-'
      READ*,       x
C-
           t=1./(1.+p*x)
           a=b1*t + b2*t**2. + b3*t**3. + b4*t**4. + b5*t**5.
           q=(exp(-x**2./2.)/sqrt(2.*pi))*a
C-OUTPUT
      PRINT*,      q
C-
      STOP
      END
```

The results of this approximation along with the actual values of $Q(x)$ (taken from text Table 4.1) are tabulated in the following table. As it is observed a very good approximation is achieved.

x	$Q(x)$	Approximation
1.	1.59×10^{-1}	1.587×10^{-1}
1.5	6.68×10^{-2}	6.685×10^{-2}
2.	2.28×10^{-2}	2.276×10^{-2}
2.5	6.21×10^{-3}	6.214×10^{-3}
3.	1.35×10^{-3}	1.351×10^{-3}
3.5	2.33×10^{-4}	2.328×10^{-4}
4.	3.17×10^{-5}	3.171×10^{-5}
4.5	3.40×10^{-6}	3.404×10^{-6}
5.	2.87×10^{-7}	2.874×10^{-7}

Problem 5.26

The joint distribution of X and Y is given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{1}{2}\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}\right\}$$

The linear transformations $Z = X + Y$ and $W = 2X - Y$ are written in matrix notation as

$$\begin{pmatrix} Z \\ W \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = A\begin{pmatrix} X \\ Y \end{pmatrix}$$

Thus,

$$f_{Z,W}(z,w) = \frac{1}{2\pi \det(M)^{1/2}}\exp\left\{-\frac{1}{2}\begin{pmatrix} Z & W \end{pmatrix}M^{-1}\begin{pmatrix} Z \\ W \end{pmatrix}\right\}$$

where

$$M = A\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}A^t = \begin{pmatrix} 2\sigma^2 & \sigma^2 \\ \sigma^2 & 5\sigma^2 \end{pmatrix} = \begin{pmatrix} \sigma_Z^2 & \rho_{Z,W}\sigma_Z\sigma_W \\ \rho_{Z,W}\sigma_Z\sigma_W & \sigma_W^2 \end{pmatrix}$$

From the last equality we identify $\sigma_Z^2 = 2\sigma^2$, $\sigma_W^2 = 5\sigma^2$ and $\rho_{Z,W} = 1/\sqrt{10}$

Problem 5.27

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\sqrt{2\pi}\sigma_Y}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}}\exp[-A]$$

where

$$\begin{aligned}
A &= \frac{(x-m_X)^2}{2(1-\rho_{X,Y}^2)\sigma_X^2} + \frac{(y-m_Y)^2}{2(1-\rho_{X,Y}^2)\sigma_Y^2} - 2\rho\frac{(x-m_X)(y-m_Y)}{2(1-\rho_{X,Y}^2)\sigma_X\sigma_Y} - \frac{(y-m_Y)^2}{2\sigma_Y^2} \\
&= \frac{1}{2(1-\rho_{X,Y}^2)\sigma_X^2}\left((x-m_X)^2 + \frac{(y-m_Y)^2\sigma_X^2\rho_{X,Y}^2}{\sigma_Y^2} - 2\rho\frac{(x-m_X)(y-m_Y)\sigma_X}{\sigma_Y}\right) \\
&= \frac{1}{2(1-\rho_{X,Y}^2)\sigma_X^2}\left[x - \left(m_X + (y-m_Y)\frac{\rho\sigma_X}{\sigma_Y}\right)\right]^2
\end{aligned}$$

Thus

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{1-\rho_{X,Y}^2}}\exp\left\{-\frac{1}{2(1-\rho_{X,Y}^2)\sigma_X^2}\left[x - \left(m_X + (y-m_Y)\frac{\rho\sigma_X}{\sigma_Y}\right)\right]^2\right\}$$

which is a Gaussian PDF with mean $m_X + (y - m_Y)\rho\sigma_X/\sigma_Y$ and variance $(1 - \rho_{X,Y}^2)\sigma_X^2$. If $\rho = 0$ then $f_{X|Y}(x|y) = f_X(x)$ which implies that Y does not provide any information about X or X, Y are independent. If $\rho = \pm 1$ then the variance of $f_{X|Y}(x|y)$ is zero which means that $X|Y$ is deterministic. This is to be expected since $\rho = \pm 1$ implies a linear relation $X = AY + b$ so that knowledge of Y provides all the information about X.

Problem 5.28
1) Z and W are linear combinations of jointly Gaussian RV's, therefore they are jointly Gaussian too.

2) Since Z and W are jointly Gaussian with zero-mean, they are independent if they are uncorrelated. This implies that they are independent if $E[ZW] = 0$. But $E[ZW] = E[XY](\cos^2\theta - \sin^2\theta)$ where we have used the fact that since X and Y are zero-mean and have the same variance we have $E[X^2] = E[Y^2]$, and therefore, $(E(Y^2) - E(X^2)) \sim \theta\cos\theta = 0$. From above, in order for Z and W to be independent we must have

$$\cos^2\theta - \sin^2\theta = 0 \Longrightarrow \theta = \frac{\pi}{4} + k\frac{\pi}{2}, \quad k \in \mathcal{Z}$$

Note also that if X and Y are independent, then $E[XY] = 0$ and any rotation will produce independent random variables again.

Problem 5.29
1) $f_{X,Y}(x,y)$ is a PDF and its integral over the supporting region of x and y should be one.

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy \\
&= \int_{-\infty}^{0}\int_{-\infty}^{0} \frac{K}{\pi}e^{-\frac{x^2+y^2}{2}}dxdy + \int_{0}^{\infty}\int_{0}^{\infty} \frac{K}{\pi}e^{-\frac{x^2+y^2}{2}}dxdy \\
&= \frac{K}{\pi}\int_{-\infty}^{0} e^{-\frac{x^2}{2}}dx \int_{-\infty}^{0} e^{-\frac{y^2}{2}}dx + \frac{K}{\pi}\int_{0}^{\infty} e^{-\frac{x^2}{2}}dx\int_{0}^{\infty} e^{-\frac{y^2}{2}}dx \\
&= \frac{K}{\pi}\left[2(\frac{1}{2}\sqrt{2\pi})^2\right] = K
\end{aligned}$$

Thus $K = 1$

2) If $x < 0$ then

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{0} \frac{1}{\pi}e^{-\frac{x^2+y^2}{2}}dy = \frac{1}{\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{0} e^{-\frac{y^2}{2}}dy \\
&= \frac{1}{\pi}e^{-\frac{x^2}{2}}\frac{1}{2}\sqrt{2\pi} = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}
\end{aligned}$$

If $x > 0$ then

$$\begin{aligned}
f_X(x) &= \int_{0}^{\infty} \frac{1}{\pi}e^{-\frac{x^2+y^2}{2}}dy = \frac{1}{\pi}e^{-\frac{x^2}{2}}\int_{0}^{\infty} e^{-\frac{y^2}{2}}dy \\
&= \frac{1}{\pi}e^{-\frac{x^2}{2}}\frac{1}{2}\sqrt{2\pi} = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}
\end{aligned}$$

Thus for every x, $f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ which implies that $f_X(x)$ is a zero-mean Gaussian random variable with variance 1. Since $f_{X,Y}(x,y)$ is symmetric to its arguments and the same is true for the region of integration we conclude that $f_Y(y)$ is a zero-mean Gaussian random variable of variance 1.

3) $f_{X,Y}(x,y)$ has not the same form as a binormal distribution. For $xy < 0$, $f_{X,Y}(x,y) = 0$ but a binormal distribution is strictly positive for every x, y.

4) The random variables X and Y are not independent for if $xy < 0$ then $f_X(x)f_Y(y) \neq 0$ whereas $f_{X,Y}(x,y) = 0$.

5)

$$\begin{aligned}
E[XY] &= \frac{1}{\pi}\int_{-\infty}^{0}\int_{-\infty}^{0} XYe^{-\frac{x^2+y^2}{2}}dxdy + \frac{1}{\pi}\int_{0}^{\infty}\int_{0}^{\infty} e^{-\frac{x^2+y^2}{2}}dxdy \\
&= \frac{1}{\pi}\int_{-\infty}^{0} Xe^{-\frac{x^2}{2}}dx \int_{-\infty}^{0} Ye^{-\frac{y^2}{2}}dy + \frac{1}{\pi}\int_{0}^{\infty} Xe^{-\frac{x^2}{2}}dx \int_{0}^{\infty} Ye^{-\frac{y^2}{2}}dy \\
&= \frac{1}{\pi}(-1)(-1) + \frac{1}{\pi} = \frac{2}{\pi}
\end{aligned}$$

Thus the random variables X and Y are correlated since $E[XY] \neq 0$ and $E[X] = E[Y] = 0$, so that $E[XY] - E[X]E[Y] \neq 0$.

6) In general $f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$. If $y > 0$, then

$$f_{X|Y}(x,y) = \begin{cases} 0 & x < 0 \\ \sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}} & x \geq 0 \end{cases}$$

If $y \leq 0$, then

$$f_{X|Y}(x,y) = \begin{cases} 0 & x > 0 \\ \sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}} & x < 0 \end{cases}$$

Thus

$$f_{X|Y}(x,y) = \sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}}u(xy)$$

which is not a Gaussian distribution.

Problem 5.30

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2}\exp\left\{-\frac{(x-m)^2+y^2}{2\sigma^2}\right\}$$

With the transformation

$$V = \sqrt{X^2+Y^2}, \qquad \Theta = \arctan\frac{Y}{X}$$

we obtain

$$\begin{aligned}
f_{V,\Theta}(v,\theta) &= v f_{X,Y}(v\cos\theta, v\sin\theta) \\
&= \frac{v}{2\pi\sigma^2}\exp\left\{-\frac{(v\cos\theta-m)^2+v^2\sin\theta}{2\sigma^2}\right\} \\
&= \frac{v}{2\pi\sigma^2}\exp\left\{-\frac{v^2+m^2-2mv\cos\theta}{2\sigma^2}\right\}
\end{aligned}$$

To obtain the marginal probability density function for the magnitude, we integrate over θ so that

$$\begin{aligned} f_V(v) &= \int_0^{2\pi} \frac{v}{2\pi\sigma^2} e^{-\frac{v^2+m^2}{2\sigma^2}} e^{\frac{mv\cos\theta}{\sigma^2}} d\theta \\ &= \frac{v}{\sigma^2} e^{-\frac{v^2+m^2}{2\sigma^2}} \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{mv\cos\theta}{\sigma^2}} d\theta \\ &= \frac{v}{\sigma^2} e^{-\frac{v^2+m^2}{2\sigma^2}} I_0(\frac{mv}{\sigma^2}) \end{aligned}$$

where

$$I_0(\frac{mv}{\sigma^2}) = \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{mv\cos\theta}{\sigma^2}} d\theta$$

With $m = 0$ we obtain

$$f_V(v) = \begin{cases} \frac{v}{\sigma^2} e^{-\frac{v^2}{2\sigma^2}} & v > 0 \\ 0 & v \le 0 \end{cases}$$

which is the Rayleigh distribution.

Problem 5.31

1) Let X_i be a random variable taking the values 1, 0, with probability $\frac{1}{4}$ and $\frac{3}{4}$ respectively. Then, $m_{X_i} = \frac{1}{4} \cdot 1 + \frac{3}{4} \cdot 0 = \frac{1}{4}$. The weak law of large numbers states that the random variable $Y = \frac{1}{n} \sum_{i=1}^n X_i$ has mean which converges to m_{X_i} with probability one. Using Chebychev's inequality (see Problem 4.13) we have $P(|Y - m_{X_i}| \ge \epsilon) \le \frac{\sigma_Y^2}{\epsilon^2}$ for every $\epsilon > 0$. Hence, with $n = 2000$, $Z = \sum_{i=1}^{2000} X_i$, $m_{X_i} = \frac{1}{4}$ we obtain

$$P(|Z - 500| \ge 2000\epsilon) \le \frac{\sigma_Y^2}{\epsilon^2} \Rightarrow P(500 - 2000\epsilon \le Z \le 500 + 2000\epsilon) \ge 1 - \frac{\sigma_Y^2}{\epsilon^2}$$

The variance σ_Y^2 of $Y = \frac{1}{n} \sum_{i=1}^n X_i$ is $\frac{1}{n}\sigma_{X_i}^2$, where $\sigma_{X_i}^2 = p(1-p) = \frac{3}{16}$ (see Problem 4.13). Thus, with $\epsilon = 0.001$ we obtain

$$P(480 \le Z \le 520) \ge 1 - \frac{3/16}{2 \times 10^{-1}} = .063$$

2) Using the C.L.T. the CDF of the random variable $Y = \frac{1}{n} \sum_{i=1}^n X_i$ converges to the CDF of the random variable $N(m_{X_i}, \frac{\sigma}{\sqrt{n}})$. Hence

$$P = p\left(\frac{480}{n} \le Y \le \frac{520}{n}\right) = Q\left(\frac{\frac{480}{n} - m_{X_i}}{\sigma}\right) - Q\left(\frac{\frac{520}{n} - m_{X_i}}{\sigma}\right)$$

With $n = 2000$, $m_{X_i} = \frac{1}{4}$, $\sigma^2 = \frac{p(1-p)}{n}$ we obtain

$$\begin{aligned} P &= Q\left(\frac{480 - 500}{\sqrt{2000p(1-p)}}\right) - Q\left(\frac{520 - 500}{\sqrt{2000p(1-p)}}\right) \\ &= 1 - 2Q\left(\frac{20}{\sqrt{375}}\right) = .682 \end{aligned}$$

Problem 5.32
The random variable $X(t_0)$ is uniformly distributed over $[-1\ \ 1]$. Hence,

$$m_X(t_0) = E[X(t_0)] = E[X] = 0$$

As it is observed the mean $m_X(t_0)$ is independent of the time instant t_0.

Problem 5.33

$$m_X(t) = E[A + Bt] = E[A] + E[B]t = 0$$

where the last equality follows from the fact that A, B are uniformly distributed over $[-1\ \ 1]$ so that $E[A] = E[B] = 0$.

$$\begin{aligned} R_X(t_1, t_2) &= E[X(t_1)X(t_2)] = E[(A + Bt_1)(A + Bt_2)] \\ &= E[A^2] + E[AB]t_2 + E[BA]t_1 + E[B^2]t_1t_2 \end{aligned}$$

The random variables A, B are independent so that $E[AB] = E[A]E[B] = 0$. Furthermore

$$E[A^2] = E[B^2] = \int_{-1}^{1} x^2 \frac{1}{2} dx = \frac{1}{6}x^3 |_{-1}^{1} = \frac{1}{3}$$

Thus

$$R_X(t_1, t_2) = \frac{1}{3} + \frac{1}{3}t_1t_2$$

Problem 5.34
Since $X(t) = X$ with the random variable uniformly distributed over $[-1\ \ 1]$ we obtain

$$f_{X(t_1),X(t_2),\cdots,X(t_n)}(x_1, x_2, \ldots, x_n) = f_{X,X,\cdots,X}(x_1, x_2, \ldots, x_n)$$

for all $t_1, \ldots, t_n$ and n. Hence, the statistical properties of the process are time independent and by definition we have a stationary process.

Problem 5.35
1) $f(\tau)$ cannot be the autocorrelation function of a random process for $f(0) = 0 < f(1/4f_0) = 1$. Thus the maximum absolute value of $f(\tau)$ is not achieved at the origin $\tau = 0$.

2) $f(\tau)$ cannot be the autocorrelation function of a random process for $f(0) = 0$ whereas $f(\tau) \neq 0$ for $\tau \neq 0$. The maximum absolute value of $f(\tau)$ is not achieved at the origin.

3) $f(0) = 1$ whereas $f(\tau) > f(0)$ for $|\tau| > 1$. Thus $f(\tau)$ cannot be the autocorrelation function of a random process.

4) $f(\tau)$ is even and the maximum is achieved at the origin ($\tau = 0$). We can write $f(\tau)$ as

$$f(\tau) = 1.2\Lambda(\tau) - \Lambda(\tau - 1) - \Lambda(\tau + 1)$$

Taking the Fourier transform of both sides we obtain

$$S(f) = 1.2\text{sinc}^2(f) - \text{sinc}^2(f)\left(e^{-j2\pi f} + e^{j2\pi f}\right) = \text{sinc}^2(f)(1.2 - 2\cos(2\pi f))$$

As we observe the power spectrum $S(f)$ can take negative values, i.e. for $f = 0$. Thus $f(\tau)$ can not be the autocorrelation function of a random process.

Problem 5.36

The random variable ω_i takes the values $\{1, 2, \ldots, 6\}$ with probability $\frac{1}{6}$. Thus

$$\begin{aligned}
E_X &= E\left[\int_{-\infty}^{\infty} X^2(t)dt\right] \\
&= E\left[\int_{-\infty}^{\infty} \omega_i^2 e^{-2t} u_{-1}^2(t)dt\right] = E\left[\int_0^{\infty} \omega_i^2 e^{-2t} dt\right] \\
&= \int_0^{\infty} E[\omega_i^2] e^{-2t} dt = \int_0^{\infty} \frac{1}{6}\sum_{i=1}^{6} i^2 e^{-2t} dt \\
&= \frac{91}{6}\int_0^{\infty} e^{-2t} dt = \frac{91}{6}(-\frac{1}{2}e^{-2t})\Big|_0^{\infty} \\
&= \frac{91}{12}
\end{aligned}$$

Thus the process is an energy-type process. However, this process is not stationary for

$$m_X(t) = E[X(t) = E[\omega_i]e^{-t}u_{-1}(t) = \frac{21}{6}e^{-t}u_{-1}(t)$$

is not constant.

Problem 5.37

1) We find first the probability of an even number of transitions in the interval $(0, \tau]$.

$$\begin{aligned}
p_N(n = even) &= p_N(0) + p_N(2) + p_N(4) + \cdots \\
&= \frac{1}{1 + \alpha\tau}\sum_{l=0}^{\infty}\left(\frac{\alpha\tau}{1 + \alpha\tau}\right)^2 \\
&= \frac{1}{1 + \alpha\tau}\frac{1}{1 - \frac{(\alpha\tau)^2}{(1+\alpha\tau)^2}} \\
&= \frac{1 + \alpha\tau}{1 + 2\alpha\tau}
\end{aligned}$$

The probability $p_N(n = odd)$ is simply $1 - p_N(n = even) = \frac{\alpha\tau}{1+2\alpha\tau}$. The random process $Z(t)$ takes the value of 1 (at time instant t) if an even number of transitions occurred given that $Z(0) = 1$, or if an odd number of transitions occurred given that $Z(0) = 0$. Thus,

$$\begin{aligned} m_Z(t) &= E[Z(t)] = 1 \cdot p(Z(t) = 1) + 0 \cdot p(Z(t) = 0) \\ &= p(Z(t) = 1|Z(0) = 1)p(Z(0) = 1) + p(Z(t) = 1|Z(0) = 0)p(Z(0) = 0) \\ &= p_N(n = even)\frac{1}{2} + p_N(n = odd)\frac{1}{2} \\ &= \frac{1}{2} \end{aligned}$$

2) To determine $R_Z(t_1, t_2)$ note that $Z(t+\tau) = 1$ if $Z(t) = 1$ and an even number of transitions occurred in the interval $(t, t+\tau]$, or if $Z(t) = 0$ and an odd number of transitions have taken place in $(t, t+\tau]$ (we are assuming $\tau > 0$). Hence,

$$\begin{aligned} R_Z(t+\tau, t) &= E[Z(t+\tau)Z(t)] \\ &= 1 \cdot p(Z(t+\tau) = 1, Z(t) = 1) + 0 \cdot p(Z(t+\tau) = 1, Z(t) = 0) \\ &\quad +0 \cdot p(Z(t+\tau) = 0, Z(t) = 1) + 0 \cdot p(Z(t+\tau) = 0, Z(t) = 0) \\ &= p(Z(t+\tau) = 1, Z(t) = 1) = p(Z(t+\tau) = 1|Z(t) = 1)p(Z(t) = 1) \\ &= \frac{1}{2}\frac{1+\alpha\tau}{1+2\alpha\tau} \end{aligned}$$

As it is observed $R_Z(t+\tau, t)$ depends only on τ and thus the process is stationary. The above is for $\tau > 0$, in general we have

$$R_Z(\tau) = \frac{1+\alpha|\tau|}{2(1+2\alpha|\tau|)}$$

Since the process is WSS its PSD is the Fourier transform of its autocorrelation function, finding the Fourier transform of the autocorrelation function is not an easy task. We can use integral tables to show that

$$\begin{aligned} S_Z(f) = \; & \frac{1}{2}\delta(f) + \frac{1}{4\alpha}\mathrm{sgn}(f)\left[\sin\left(\frac{\pi f}{\alpha}\right) - \cos\left(\frac{\pi f}{\alpha}\right)\right] + \frac{\pi}{4\alpha}\cos\left(\frac{\pi f}{\alpha}\right) \\ & - \frac{1}{2\alpha}\sin\left(\frac{\pi f}{\alpha}\right)\mathrm{Si}\left(\frac{\pi f}{\alpha}\right) - \frac{1}{2\alpha}\cos\left(\frac{\pi f}{\alpha}\right)\mathrm{Ci}\left(\frac{\pi f}{\alpha}\right) \end{aligned}$$

where

$$\begin{aligned} \mathrm{Si}(x) &= \int_0^x \frac{\sin(t)}{t}\,dt \\ \mathrm{Ci}(x) &= \gamma + \ln(x) + \int_0^x \frac{\cos(t)-1}{t}\,dt \end{aligned}$$

Finding the power content of the process is much easier and is done by substituting $\tau = 0$ in the autocorrelation function resulting in $P_Z = R_Z(0) = \frac{1}{2}$.

3) Since the process is stationary

$$P_Z = R_Z(0) = \frac{1}{2}$$

Problem 5.38

1)

$$\begin{aligned} m_X(t) &= E[X(t)] = E[X\cos(2\pi f_0 t)] + E[Y\sin(2\pi f_0 t)] \\ &= E[X]\cos(2\pi f_0 t) + E[Y]\sin(2\pi f_0 t) \\ &= 0 \end{aligned}$$

where the last equality follows from the fact that $E[X] = E[Y] = 0$.

2)

$$\begin{aligned} R_X(t+\tau,t) &= E[(X\cos(2\pi f_0(t+\tau)) + Y\sin(2\pi f_0(t+\tau))) \\ &\quad (X\cos(2\pi f_0 t) + Y\sin(2\pi f_0 t))] \\ &= E[X^2\cos(2\pi f_0(t+\tau))\cos(2\pi f_0 t)] + \\ &\quad E[XY\cos(2\pi f_0(t+\tau))\sin(2\pi f_0 t)] + \\ &\quad E[YX\sin(2\pi f_0(t+\tau))\cos(2\pi f_0 t)] + \\ &\quad E[Y^2\sin(2\pi f_0(t+\tau))\sin(2\pi f_0 t)] \\ &= \frac{\sigma^2}{2}[\cos(2\pi f_0(2t+\tau)) + \cos(2\pi f_0\tau)] + \\ &\quad \frac{\sigma^2}{2}[\cos(2\pi f_0\tau) - \cos(2\pi f_0(2t+\tau))] \\ &= \sigma^2\cos(2\pi f_0\tau) \end{aligned}$$

where we have used the fact that $E[XY] = 0$. Thus the process is stationary for $R_X(t+\tau,t)$ depends only on τ.

3) The power spectral density is the Fourier transform of the autocorrelation function, hence

$$S_X(f) = \frac{\sigma^2}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right].$$

4) If $\sigma_X^2 \neq \sigma_Y^2$, then

$$m_X(t) = E[X]\cos(2\pi f_0 t) + E[Y]\sin(2\pi f_0 t) = 0$$

and

$$\begin{aligned} R_X(t+\tau,t) &= E[X^2]\cos(2\pi f_0(t+\tau))\cos(2\pi f_0 t) + \\ &\quad E[Y^2]\sin(2\pi f_0(t+\tau))\sin(2\pi f_0 t) \\ &= \frac{\sigma_X^2}{2}[\cos(2\pi f_0(2t+\tau)) - \cos(2\pi f_0\tau)] + \\ &\quad \frac{\sigma_Y^2}{2}[\cos(2\pi f_0\tau) - \cos(2\pi f_0(2t+\tau))] \\ &= \frac{\sigma_X^2-\sigma_Y^2}{2}\cos(2\pi f_0(2t+\tau) + \\ &\quad \frac{\sigma_X^2+\sigma_Y^2}{2}\cos(2\pi f_0\tau) \end{aligned}$$

The process is not stationary for $R_X(t+\tau,t)$ does not depend only on τ but on t as well. However the process is cyclostationary with period $T_0 = \frac{1}{2f_0}$. Note that if X or Y is not of zero mean then the period of the cyclostationary process is $T_0 = \frac{1}{f_0}$.

Problem 5.39

$$R_{XY}(t_1,t_2) = E[X(t_1)Y(t_2)] = E[Y(t_2)X(t_1)] = R_{YX}(t_2,t_1)$$

If we let $\tau = t_1 - t_2$, then using the previous result and the fact that $X(t)$, $Y(t)$ are jointly stationary, so that $R_{XY}(t_1,t_2)$ depends only on τ, we obtain

$$R_{XY}(t_1,t_2) = R_{XY}(t_1-t_2) = R_{YX}(t_2-t_1) = R_{YX}(-\tau)$$

Taking the Fourier transform of both sides of the previous relation we obtain

$$\begin{aligned} S_{XY}(f) &= \mathcal{F}[R_{XY}(\tau)] = \mathcal{F}[R_{YX}(-\tau)] \\ &= \int_{-\infty}^{\infty} R_{YX}(-\tau)e^{-j2\pi f\tau}d\tau \\ &= \left[\int_{-\infty}^{\infty} R_{YX}(\tau')e^{-j2\pi f\tau'}d\tau'\right]^* = S_{YX}^*(f) \end{aligned}$$

Problem 5.40

1) $S_X(f) = \frac{N_0}{2}$, $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$. The autocorrelation function and the power spectral density of the output are given by

$$R_Y(t) = R_X(\tau) \star h(\tau) \star h(-\tau), \qquad S_Y(f) = S_X(f)|H(f)|^2$$

With $H(f) = \Pi(\frac{f}{2B})$ we have $|H(f)|^2 = \Pi^2(\frac{f}{2B}) = \Pi(\frac{f}{2B})$ so that

$$S_Y(f) = \frac{N_0}{2}\Pi(\frac{f}{2B})$$

Taking the inverse Fourier transform of the previous we obtain the autocorrelation function of the output

$$R_Y(\tau) = 2B\frac{N_0}{2}\text{sinc}(2B\tau) = BN_0\text{sinc}(2B\tau)$$

2) The output random process $Y(t)$ is a zero mean Gaussian process with variance

$$\sigma^2_{Y(t)} = E[Y^2(t)] = E[Y^2(t+\tau)] = R_Y(0) = BN_0$$

The correlation coefficient of the jointly Gaussian processes $Y(t+\tau)$, $Y(t)$ is

$$\rho_{Y(t+\tau)Y(t)} = \frac{COV(Y(t+\tau)Y(t))}{\sigma_{Y(t+\tau)}\sigma_{Y(t)}} = \frac{E[Y(t+\tau)Y(t)]}{BN_0} = \frac{R_Y(\tau)}{BN_0}$$

With $\tau = \frac{1}{2B}$, we have $R_Y(\frac{1}{2B}) = \text{sinc}(1) = 0$ so that $\rho_{Y(t+\tau)Y(t)} = 0$. Hence the joint probability density function of $Y(t)$ and $Y(t+\tau)$ is

$$f_{Y(t+\tau)Y(t)} = \frac{1}{2\pi BN_0}e^{-\frac{Y^2(t+\tau)+Y^2(t)}{2BN_0}}$$

Since the processes are Gaussian and uncorrelated they are also independent.

Problem 5.41

The impulse response of a delay line that introduces a delay equal to Δ is $h(t) = \delta(t-\Delta)$. The output autocorrelation function is

$$R_Y(\tau) = R_X(\tau) \star h(\tau) \star h(-\tau)$$

But,

$$\begin{aligned} h(\tau) \star h(-\tau) &= \int_{-\infty}^{\infty} \delta(-(t-\Delta))\delta(\tau-(t-\Delta))dt \\ &= \int_{-\infty}^{\infty} \delta(t-\Delta)\delta(\tau-(t-\Delta))dt \\ &= \int_{-\infty}^{\infty} \delta(t')\delta(\tau-t')dt' = \delta(\tau) \end{aligned}$$

Hence,

$$R_Y(\tau) = R_X(\tau) \star \delta(\tau) = R_X(\tau)$$

This is to be expected since a delay line does not alter the spectral characteristics of the input process.

Problem 5.42
The converse of the theorem is not true. Consider for example the random process $X(t) = \cos(2\pi f_0 t) + X$ where X is a random variable. Clearly

$$m_X(t) = \cos(2\pi f_0 t) + m_X$$

is a function of time. However, passing this process through the LTI system with transfer function $\Pi(\frac{f}{2W})$ with $W < f_0$ produces the stationary random process $Y(t) = X$.

Problem 5.43
1) $Y(t) = \frac{d}{dt}X(t)$ can be considered as the output process of a differentiator which is known to be a LTI system with impulse response $h(t) = \delta'(t)$. Since $X(t)$ is stationary, its mean is constant so that

$$m_Y(t) = m_{X'}(t) = [m_X(t)]' = 0$$

To prove that $X(t)$ and $\frac{d}{dt}X(t)$ are uncorrelated we have to prove that $R_{XX'}(0) - m_X(t)m_{X'}(t) = 0$ or since $m_{X'}(t) = 0$ it suffices to prove that $R_{XX'}(0) = 0$. But,

$$R_{XX'}(\tau) = R_X(\tau) \star \delta'(-\tau) = -R_X(\tau) \star \delta'(\tau) = -R'_X(\tau)$$

and since $R_X(\tau) = R_X(-\tau)$ we obtain

$$R_{XX'}(\tau) = -R'_X(\tau) = R'_X(-\tau) = -R_{XX'}(-\tau)$$

Thus $R_{XX'}(\tau)$ is an odd function and its value at the origin should be equal to zero

$$R_{XX'}(0) = 0$$

The last proves that $X(t)$ and $\frac{d}{dt}X(t)$ are uncorrelated.

2) The autocorrelation function of the sum $Z(t) = X(t) + \frac{d}{dt}X(t)$ is

$$R_Z(\tau) = R_X(\tau) + R_{X'}(\tau) + R_{XX'}(\tau) + R_{X'X}(\tau)$$

If we take the Fourier transform of both sides we obtain

$$S_Z(f) = S_X(f) + S_{X'}(f) + 2\text{Re}[S_{XX'}(f)]$$

But, $S_{XX'}(f) = \mathcal{F}[-R_X(\tau) \star \delta'(\tau)] = S_X(f)(-j2\pi f)$ so that $\text{Re}[S_{XX'}(f)] = 0$. Thus,

$$S_Z(f) = S_X(f) + S_{X'}(f)$$

3) Since the transfer function of a differentiator is $j2\pi f$, we have $S_{X'}(f) = 4\pi^2 f^2 S_X(f)$, hence

$$S_Z(f) = S_X(f)(1 + 4\pi^2 f^2)$$

Problem 5.44

1) The impulse response of the system is $h(t) = \mathcal{L}[\delta(t)] = \delta'(t) + \delta'(t-T)$. It is a LTI system so that the output process is a stationary. This is true since $Y(t+c) = \mathcal{L}[X(t+c)]$ for all c, so if $X(t)$ and $X(t+c)$ have the same statistical properties, so do the processes $Y(t)$ and $Y(t+c)$.

2) $S_Y(f) = S_X(f)|H(f)|^2$. But, $H(f) = j2\pi f + j2\pi f e^{-j2\pi f T}$ so that

$$\begin{aligned} S_Y(f) &= S_X(f)4\pi^2 f^2 \left|1 + e^{-j2\pi f T}\right|^2 \\ &= S_X(f)4\pi^2 f^2[(1+\cos(2\pi f T))^2 + \sin^2(2\pi f T)] \\ &= S_X(f)8\pi^2 f^2(1+\cos(2\pi f T)) \end{aligned}$$

3) The frequencies for which $|H(f)|^2 = 0$ will not be present at the output. These frequencies are $f = 0$, for which $f^2 = 0$ and $f = \frac{1}{2T} + \frac{k}{T}$, $k \in \mathcal{Z}$, for which $\cos(2\pi f T) = -1$.

Problem 5.45

1) $Y(t) = X(t) \star (\delta(t) - \delta(t-T))$. Hence,

$$\begin{aligned} S_Y(f) &= S_X(f)|H(f)|^2 = S_X(f)|1 - e^{-j2\pi f T}|^2 \\ &= S_X(f)2(1-\cos(2\pi f T)) \end{aligned}$$

2) $Y(t) = X(t) \star (\delta'(t) - \delta(t))$. Hence,

$$\begin{aligned} S_Y(f) &= S_X(f)|H(f)|^2 = S_X(f)|j2\pi f - 1|^2 \\ &= S_X(f)(1+4\pi^2 f^2) \end{aligned}$$

3) $Y(t) = X(t) \star (\delta'(t) - \delta(t-T))$. Hence,

$$\begin{aligned} S_Y(f) &= S_X(f)|H(f)|^2 = S_X(f)|j2\pi f - e^{-j2\pi f T}|^2 \\ &= S_X(f)(1+4\pi^2 f^2 + 4\pi f\sin(2\pi f T)) \end{aligned}$$

Problem 5.46

Using Schwartz's inequality

$$E^2[X(t+\tau)Y(t)] \leq E[X^2(t+\tau)]E[Y^2(t)] = R_X(0)R_Y(0)$$

where equality holds for independent $X(t)$ and $Y(t)$. Thus

$$|R_{XY}(\tau)| = \left(E^2[X(t+\tau)Y(t)]\right)^{\frac{1}{2}} \leq R_X^{1/2}(0)R_Y^{1/2}(0)$$

The second part of the inequality follows from the fact $2ab \leq a^2 + b^2$. Thus, with $a = R_X^{1/2}(0)$ and $b = R_Y^{1/2}(0)$ we obtain

$$R_X^{1/2}(0)R_Y^{1/2}(0) \leq \frac{1}{2}[R_X(0) + R_Y(0)]$$

Problem 5.47

1)

$$\begin{aligned}
R_{XY}(\tau) &= R_X(\tau) \star \delta(-\tau - \Delta) = R_X(\tau) \star \delta(\tau + \Delta) \\
&= e^{-\alpha|\tau|} \star \delta(\tau + \Delta) = e^{-\alpha|\tau+\Delta|} \\
R_Y(\tau) &= R_{XY}(\tau) \star \delta(\tau - \Delta) = e^{-\alpha|\tau+\Delta|} \star \delta(\tau - \Delta) \\
&= e^{-\alpha|\tau|}
\end{aligned}$$

2)

$$\begin{aligned}
R_{XY}(\tau) &= e^{-\alpha|\tau|} \star (-\frac{1}{\tau}) = -\int_{-\infty}^{\infty} \frac{e^{-\alpha|v|}}{t - v} dv \\
R_Y(\tau) &= R_{XY}(\tau) \star \frac{1}{\tau} = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-\alpha|v|}}{s - v}\frac{1}{\tau - s} ds dv
\end{aligned} \tag{5.17}$$

The case of $R_Y(\tau)$ can be simplified as follows. Note that $R_Y(\tau) = \mathcal{F}^{-1}[S_Y(f)]$ where $S_Y(f) = S_X(f)|H(f)|^2$. In our case, $S_X(f) = \frac{2\alpha}{\alpha^2+4\pi^2 f^2}$ and $|H(f)|^2 = \pi^2\mathrm{sgn}^2(f)$. Since $S_X(f)$ does not contain any impulses at the origin ($f = 0$) for which $|H(f)|^2 = 0$, we obtain

$$R_Y(\tau) = \mathcal{F}^{-1}[S_Y(f)] = \pi^2 e^{-\alpha|\tau|}$$

3) The transfer function is $H(f) = \frac{1}{\alpha + j2\pi f}$. Therefore

$$S_Y(f) = S_X(f)|H(f)|^2 = S_X(f)\frac{1}{\alpha^2 + 4\pi^2 f^2} = \frac{2\alpha}{(\alpha^2 + 4\pi^2 f^2)^2}$$

Since $\frac{2\alpha}{\alpha^2+4\pi^2 f^2} \Leftrightarrow e^{-\alpha|\tau|}$, applying the differentiation in the frequency domain result we have

$$\frac{d}{df}\frac{2\alpha}{\alpha^2 + 4\pi^2 f^2} \Leftrightarrow \frac{2\pi}{j}\tau e^{-\alpha|\tau|}$$

resulting in

$$\frac{(j2\pi f)2\alpha}{(\alpha^2 + 4\pi^2 f^2)^2} \Leftrightarrow -\frac{\tau}{2}e^{-\alpha|\tau|}$$

Now we can apply integration in the time domain result to conclude that

$$\frac{2\alpha}{(\alpha^2+4\pi^2 f^2)^2} \Leftrightarrow -\frac{1}{2}\int_{-\infty}^{\tau} u e^{-\alpha|u|}\,du$$

Integration of the right hand side is simple and should be carried out considering $\tau < 0$ and $\tau > 0$ separately. If we do this we will have

$$R_Y(\tau) = \mathcal{F}^{-1}\left[\frac{2\alpha}{(\alpha^2+4\pi^2 f^2)^2}\right] = \frac{1}{2\alpha}|\tau|e^{-\alpha|\tau|} + \frac{1}{2\alpha^2}e^{-\alpha|\tau|}$$

For $S_{XY}(f)$ we have

$$S_{XY}(f) = S_X(f)H^*(f) = \frac{2\alpha}{(\alpha^2+4\pi^2 f^2)(\alpha - j2\pi f)} = \frac{2\alpha(\alpha+j2\pi f)}{(\alpha^2+2\pi^2 f^2)^2}$$

or

$$R_{XY}(\tau) = \mathcal{F}^{-1}\left[\frac{2\alpha^2}{(\alpha^2+4\pi^2 f^2)^2}\right] + \mathcal{F}^{-1}\left[\frac{j2\pi f(2\alpha)}{(\alpha^2+4\pi^2 f^2)^2}\right]$$

The inverse Fourier transform of the first term we have already found, for the second term we apply the differentiation property of the Fourier transform. We have

$$R_{XY}(\tau) = \frac{1}{2\alpha}e^{-\alpha|\tau|} + \frac{1}{2}|\tau|e^{-\alpha|\tau|} + \frac{d}{d\tau}\left(\frac{1}{2\alpha^2}e^{-\alpha|\tau|} + \frac{1}{2\alpha}|\tau|e^{-\alpha|\tau|}\right)$$

This simplifies to

$$R_{XY}(\tau) = \frac{1}{2}e^{-\alpha|\tau|}\left(\frac{1}{\alpha} + |\tau| - \tau\right)$$

4) The system's transfer function is $H(f) = \frac{-1+j2\pi f}{1+j2\pi f}$. Hence,

$$\begin{aligned} S_{XY}(f) &= S_X(f)H^*(f) = \frac{2\alpha}{\alpha^2+4\pi^2 f^2}\frac{-1-j2\pi f}{1-j2\pi f} \\ &= \frac{4\alpha}{1-\alpha^2}\frac{1}{1-j2\pi f} + \frac{\alpha-1}{1+\alpha}\frac{1}{\alpha+j2\pi f} + \frac{1+\alpha}{\alpha-1}\frac{1}{\alpha-j2\pi f} \end{aligned}$$

Thus,

$$\begin{aligned} R_{XY}(\tau) &= \mathcal{F}^{-1}[S_{XY}(f)] \\ &= \frac{4\alpha}{1-\alpha^2}e^{\tau}u_{-1}(-\tau) + \frac{\alpha-1}{1+\alpha}e^{-\alpha\tau}u_{-1}(\tau) + \frac{1+\alpha}{\alpha-1}e^{\alpha\tau}u_{-1}(-\tau) \end{aligned}$$

For the output power spectral density we have $S_Y(f) = S_X(f)|H(f)|^2 = S_X(f)\frac{1+4\pi^2 f^2}{1+4\pi^2 f^2} = S_X(f)$. Hence,

$$R_Y(\tau) = \mathcal{F}^{-1}[S_X(f)] = e^{-\alpha|\tau|}$$

5) The impulse response of the system is $h(t) = \frac{1}{2T}\Pi(\frac{t}{2T})$. Hence,

$$\begin{aligned} R_{XY}(\tau) &= e^{-\alpha|\tau|} \star \frac{1}{2T}\Pi(\frac{-\tau}{2T}) = e^{-\alpha|\tau|} \star \frac{1}{2T}\Pi(\frac{\tau}{2T}) \\ &= \frac{1}{2T}\int_{\tau-T}^{\tau+T} e^{-\alpha|v|}dv \end{aligned}$$

If $\tau \geq T$, then

$$R_{XY}(\tau) = -\frac{1}{2T\alpha}e^{-\alpha v}\Big|_{\tau-T}^{\tau+T} = \frac{1}{2T\alpha}\left(e^{-\alpha(\tau-T)} - e^{-\alpha(\tau+T)}\right)$$

If $0 \leq \tau < T$, then

$$\begin{aligned} R_{XY}(\tau) &= \frac{1}{2T}\int_{\tau-T}^{0} e^{\alpha v}dv + \frac{1}{2T}\int_{0}^{\tau+T} e^{-\alpha v}dv \\ &= \frac{1}{2T\alpha}\left(2 - e^{\alpha(\tau-T)} - e^{-\alpha(\tau+T)}\right) \end{aligned}$$

The autocorrelation of the output is given by

$$\begin{aligned} R_Y(\tau) &= e^{-\alpha|\tau|} \star \frac{1}{2T}\Pi(\frac{\tau}{2T}) \star \frac{1}{2T}\Pi(\frac{\tau}{2T}) \\ &= e^{-\alpha|\tau|} \star \frac{1}{2T}\Lambda(\frac{\tau}{2T}) \\ &= \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|x|}{2T}\right)e^{-\alpha|\tau-x|}dx \end{aligned}$$

If $\tau \geq 2T$, then

$$R_Y(\tau) = \frac{e^{-\alpha\tau}}{2T\alpha^2}\left[e^{2\alpha T} + e^{-2\alpha T} - 2\right]$$

If $0 \leq \tau < 2T$, then

$$R_Y(\tau) = \frac{e^{-2\alpha T}}{4T^2\alpha^2}\left[e^{-\alpha\tau} + e^{\alpha\tau}\right] + \frac{1}{T\alpha} - \frac{\tau}{2T^2\alpha^2} - 2\frac{e^{-\alpha\tau}}{4T^2\alpha^2}$$

Problem 5.48

Consider the random processes $X(t) = Xe^{j2\pi f_0 t}$ and $Y(t) = Ye^{j2\pi f_0 t}$, where X and Y are iid random variables uniformly distributed on $[0,1]$. Clearly

$$R_{XY}(t+\tau, t) = E[X(t+\tau)Y^*(t)] = E[XY]e^{j2\pi f_0 \tau}$$

However, both $X(t)$ and $Y(t)$ are nonstationary for $E[X(t)] = E[X]e^{j2\pi f_0 t} = E[Y(t)] = E[Y]e^{j2\pi f_0 t} = \frac{1}{2}e^{j2\pi f_0 t}$ are not constant, hence $X(t)$ and $Y(t)$ cannot be stationary.

Problem 5.49

1. The power is the area under the power spectral density, which has a triangular shape with a base of 2×10^5 and height of 4×10^{-5}. Therefore

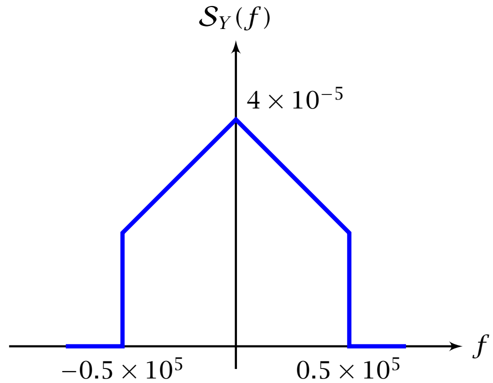
$$P_X = \int_{-\infty}^{\infty} \mathcal{S}_X(f)\,df = \frac{1}{2} \times 2 \times 10^5 \times 4 \times 10^{-5} = 4 \text{ W}$$

2. The range of frequencies are $[-10^5, 10^5]$, hence the bandwidth is 10^5 Hz or 100 kHz.

3. The transfer function of the ideal lowpass filter is $H(f) = \Pi\left(\frac{f}{10^5}\right)$, therefore,

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = 4\times 10^{-5}\Lambda\left(\frac{f}{10^5}\right)\Pi\left(\frac{f}{10^5}\right) = \begin{cases} 4\times 10^{-5}\Lambda\left(\frac{f}{10^5}\right), & |f| < 0.5\times 10^5 \\ 0, & \text{otherwise} \end{cases}$$

and the total power is the area under $\mathcal{S}_Y(f)$. Plot of $\mathcal{S}_Y(f)$ is shown below



Therefore,

$$P_Y = \int_{-\infty}^{\infty} \mathcal{S}_Y(f)\,df = 10^5 \times 2\times 10^{-5} + \frac{1}{2}\times 10^5 \times 2 \times 10^{-5} = 3 \text{ W}$$

4. Since $X(t)$ is Gaussian, $X(0)$ is a Gaussian random variable. Since $X(t)$ is zero-mean, $X(0)$ has mean equal to zero. The variance in $X(0)$ is $\mathsf{E}[X^2(0)]$ which is equal to $R_X(0)$, i.e, the power in $X(t)$ which is 4. Therefore, $X(0)$ is a Gaussian random variable with mean $m = 0$ and variance $\sigma^2 = 4$. The desired PDF is

$$f_{X(0)}(x) = \frac{1}{\sqrt{8\pi}} e^{-x^2/8}$$

5. Since for Gaussian random variables independence means uncorrelated, we need to find the smallest t_0 such that $R_X(t_0) = 0$. But

$$R_X(\tau) = \mathscr{F}^{-1}[\mathcal{S}_X(f)] = 4\,\text{sinc}^2(10^5\tau)$$

and its first zero occurs at $10^5 t_0 = 1$ or $t_0 = 10^{-5}$.

Problem 5.50

1. The power is $P_X = R_X(0) = 4$ Watts.

2. We have

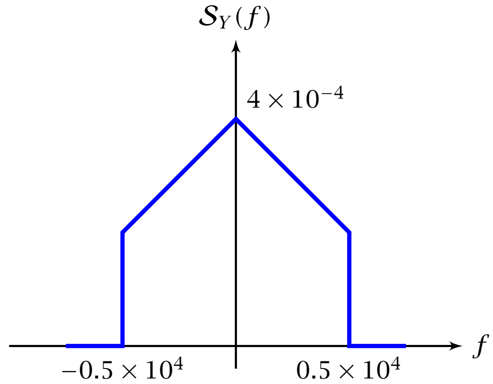
$$\mathcal{S}_X(f) = \mathscr{F}[R_X(\tau)] = 4\times 10^{-4}\Lambda\left(\frac{f}{10^4}\right)$$

3. $\mathcal{S}_X(f)$ occupies the frequency range $[-10^4, 10^4]$, therefore the bandwidth is 10^4 Hz or 10 kHZ.

4. We have

$$\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = 4\times 10^{-4}\Lambda\left(\frac{f}{10^4}\right)\Pi\left(\frac{f}{10^4}\right) = \begin{cases} 4\times 10^{-4}\Lambda\left(\frac{f}{10^4}\right) & |f| < 5\times 10^3 \\ 0 & \text{otherwise} \end{cases}$$

and P_Y is the area under $\mathcal{S}_Y(f)$. The plot of $\mathcal{S}_Y(f)$ is shown below



and $P_Y = \int_{-\infty}^{\infty} \mathcal{S}_Y(f)\,df = 10^4 \times 2\times 10^{-4} + \frac{1}{2}\times 10^4 \times 2 \times 10^{-4} = 3$ W

5. Since $X(t)$ is Gaussian and zero-mean, all random variables are zero-mean Gaussian with variances $\mathsf{E}[X^2(0)]$, $\mathsf{E}[X^2(10^{-4})]$, and $\mathsf{E}[X^2(1.5\times 10^{-4})]$. But all these variances are equal to $R_X(0) = \int_{-\infty}^{\infty} \mathcal{S}_X(f)\,df = 4$, hence all random variables are distributed according a $\mathcal{N}(0,4)$ PDF.

6. The covariance between $X(0)$ and $X(10^{-4})$ is $R_X(10^{-4}) = 0$, therefore these random variables are uncorrelated, and since they are jointly Gaussian, they are also independent. For $X(0)$ and $X(1.5\times 10^{-4})$ the covariance is $R_X(1.5\times 10^{-4}) = 4\,\text{sinc}^2(1.5) \neq 0$, hence the two random variables are correlated and hence dependent.

Problem 5.51

1. The impulse response of the system is obtained by putting $x(t) = \delta(t)$. The output, which is $h(t)$ is $h(t) = \delta(t-1) + \frac{1}{2}\delta'(t-1)$. The transfer function is

$$H(f) = \mathscr{F}[h(t)] = \left(1 + \frac{1}{2}\times j2\pi f\right)e^{-j2\pi f} = (1 + j\pi f)e^{-j2\pi f}$$

We can use $m_Y = m_X H(0) = 2(1+0)\times 1 = 2$.

2. $\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = \mathcal{S}_X(f)(1+\pi^2 f^2) = \begin{cases} 10^{-3}\left(1+\pi^2 f^2\right), & |f| \le 200 \\ 0, & \text{otherwise} \end{cases}$

3. We have

$$P_Y = \int_{-\infty}^{\infty} \mathcal{S}_Y(f)\,df = 2 \times \int_0^{200} 10^{-3} \times (1 + \pi^2 f^2)\,df = 400 + \frac{16\pi^2}{3} \times 10^6 \approx 52638.3 \text{ W}$$

4. $Y(t)$ is the result of passing a WSS process through an LTI system, therefore it is WSS.

5. $Y(t)$ is the result of passing a Gaussian process through an LTI system, therefore it is Gaussian.

6. $Y(1)$ is a zero-mean Gaussian random variable with variance $\mathsf{E}[Y^2(1)] = R_Y(0) = \int_{-\infty}^{\infty} \mathcal{S}_Y(f)\,df \approx$ 52638.3.

Problem 5.52

1. The transfer function from $X(t)$ to $Y(t)$ is the Fourier transform of the impulse response $h(t) = \delta(t) + 2\delta'(t)$, i.e., $H(f) = 1 + j4\pi f$. Hence $m_Y = m_X H(0) = 0$ and $\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|2 = (1 + 16\pi^2 f^2)\frac{N_0}{2}$.

2. $$\mathcal{S}_Z(f) = \mathcal{S}_Y(f)\Pi\left(\frac{f}{2W}\right) = \begin{cases} \frac{N_0}{2}(1 + 16\pi^2 f^2), & |f| \le W \\ 0, \text{otherwise} \end{cases}$$

3. Since $X(t)$ is WSS and system is LTI, $Z(t)$ is also WSS.

4. Since $Z(t)$ is zero mean, its variance is $\mathsf{E}[Z^2(t)] = R_Z(0) = \int_{-\infty}^{\infty} S_Z(f)\,df$. Hence,

$$\sigma_Z^2 = 2\int_0^4 \frac{N_0}{2}(1 + 16\pi^2 f^2)\,df \approx 3372.8 N_0$$

5. The power in $Y(t)$ is the integral of $\mathcal{S}_Y(f)$over all frequencies which is infinite.

Problem 5.53

1)

$$\begin{aligned} E[X(t)] &= \frac{4}{\pi}\int_0^{\frac{\pi}{4}} A\cos(2\pi f_0 t + \theta)d\theta \\ &= \frac{4A}{\pi}\sin(2\pi f_0 t + \theta)\Big|_0^{\frac{\pi}{4}} \\ &= \frac{4A}{\pi}[\sin(2\pi f_0 t + \frac{\pi}{4}) - \sin(2\pi f_0 t)] \end{aligned}$$

Thus, $E[X(t)]$ is periodic with period $T = \frac{1}{f_0}$.

$$\begin{aligned}
R_X(t+\tau,t) &= E[A^2\cos(2\pi f_0(t+\tau)+\Theta)\cos(2\pi f_0 t+\Theta)] \\
&= \frac{A^2}{2}E[\cos(2\pi f_0(2t+\tau)+\Theta)+\cos(2\pi f_0\tau)] \\
&= \frac{A^2}{2}\cos(2\pi f_0\tau)+\frac{A^2}{2}E[\cos(2\pi f_0(2t+\tau)+\Theta)] \\
&= \frac{A^2}{2}\cos(2\pi f_0\tau)+\frac{A^2}{2}\frac{4}{\pi}\int_0^{\frac{\pi}{4}}\cos(2\pi f_0(2t+\tau)+\theta)d\theta \\
&= \frac{A^2}{2}\cos(2\pi f_0\tau)+\frac{A^2}{\pi}(\cos(2\pi f_0(2t+\tau))-\sin(2\pi f_0(2t+\tau)))
\end{aligned}$$

which is periodic with period $T' = \frac{1}{2f_0}$. Thus the process is cyclostationary with period $T = \frac{1}{f_0}$. Using the results of Problem 4.48 we obtain

$$\begin{aligned}
S_X(f) &= \mathcal{F}[\frac{1}{T}\int_0^T R_X(t+\tau,t)dt] \\
&= \mathcal{F}\left[\frac{A^2}{2}\cos(2\pi f_0\tau)+\frac{A^2}{T\pi}\int_0^T(\cos(2\pi f_0(2t+\tau))-\sin(2\pi f_0(2t+\tau))dt\right] \\
&= \mathcal{F}\left[\frac{A^2}{2}\cos(2\pi f_0\tau)\right] \\
&= \frac{A^2}{4}(\delta(f-f_0)+\delta(f+f_0))
\end{aligned}$$

2)

$$\begin{aligned}
R_X(t+\tau,t) &= E[X(t+\tau)X(t)] = E[(X+Y)(X+Y)] \\
&= E[X^2]+E[Y^2]+E[YX]+E[XY] \\
&= E[X^2]+E[Y^2]+2E[X][Y]
\end{aligned}$$

where the last equality follows from the independence of X and Y. But, $E[X] = 0$ since X is uniform on $[-1,\ 1]$ so that

$$R_X(t+\tau,t) = E[X^2]+E[Y^2] = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$$

The Fourier transform of $R_X(t+\tau,t)$ is the power spectral density of $X(t)$. Thus

$$S_X(f) = \mathcal{F}[R_X(t+\tau,t)] = \frac{2}{3}\delta(f)$$

Problem 5.54

$h(t) = e^{-\beta t}u_{-1}(t) \Rightarrow H(f) = \frac{1}{\beta+j2\pi f}$. The power spectral density of the input process is $S_X(f) = \mathcal{F}[e^{-\alpha|\tau|}] = \frac{2\alpha}{\alpha^2+4\pi^2 f^2}$. If $\alpha = \beta$, then

$$S_Y(f) = S_X(f)|H(f)|^2 = \frac{2\alpha}{(\alpha^2+4\pi^2 f^2)^2}$$

If $\alpha \neq \beta$, then

$$S_Y(f) = S_X(f)|H(f)|^2 = \frac{2\alpha}{(\alpha^2 + 4\pi^2 f^2)(\beta^2 + 4\pi^2 f^2)}$$

Problem 5.55

1) Let $Y(t) = X(t) + N(t)$. The process $\hat{X}(t)$ is the response of the system $h(t)$ to the input process $Y(t)$ so that

$$\begin{aligned} R_{Y\hat{X}}(\tau) &= R_Y(\tau) \star h(-\tau) \\ &= [R_X(\tau) + R_N(\tau) + R_{XN}(\tau) + R_{NX}(\tau)] \star h(-\tau) \end{aligned}$$

Also by definition

$$\begin{aligned} R_{Y\hat{X}}(\tau) &= E[(X(t+\tau) + N(t+\tau))\hat{X}(t)] = R_{X\hat{X}}(\tau) + R_{N\hat{X}}(\tau) \\ &= R_{X\hat{X}}(\tau) + R_N(\tau) \star h(-\tau) + R_{NX}(\tau) \star h(-\tau) \end{aligned}$$

Substituting this expression for $R_{Y\hat{X}}(\tau)$ in the previous one, and cancelling common terms we obtain

$$R_{X\hat{X}}(\tau) = R_X(\tau) \star h(-\tau) + R_{XN}(\tau) \star h(-\tau)$$

2)

$$E\left[(X(t) - \hat{X}(t))^2\right] = R_X(0) + R_{\hat{X}}(0) - R_{X\hat{X}}(0) - R_{\hat{X}X}(0)$$

We can write $E\left[(X(t) - \hat{X}(t))^2\right]$ in terms of the spectral densities as

$$\begin{aligned} E\left[(X(t) - \hat{X}(t))^2\right] &= \int_{-\infty}^{\infty} (S_X(f) + S_{\hat{X}}(f) - 2S_{X\hat{X}}(f))df \\ &= \int_{-\infty}^{\infty} \Big[S_X(f) + (S_X(f) + S_N(f) + 2\text{Re}[S_{XN}(f)])|H(f)|^2 \\ &\quad -2(S_X(f) + S_{XN}(f))H^*(f)\Big]df \end{aligned}$$

To find the $H(f)$ that minimizes $E\left[(X(t) - \hat{X}(t))^2\right]$ we set the derivative of the previous expression, with respect to $H(f)$, to zero. By doing so we obtain

$$H(f) = \frac{S_X(f) + S_{XN}(f)}{S_X(f) + S_N(f) + 2\text{Re}[S_{XN}(f)]}$$

3) If $X(t)$ and $N(t)$ are independent, then

$$R_{XN}(\tau) = E[X(t+\tau)N(t)] = E[X(t+\tau)]E[N(t)]$$

Since $E[N(t)] = 0$ we obtain $R_{XN}(\tau) = 0$ and the optimum filter is

$$H(f) = \frac{S_X(f)}{S_X(f) + \frac{N_0}{2}}$$

The corresponding value of $E\left[(X(t)-\hat{X}(t))^2\right]$ is

$$E_{\min}\left[(X(t)-\hat{X}(t))^2\right] = \int_{-\infty}^{\infty} \frac{S_X(f)N_0}{2S_X(f)+N_0}df$$

4) With $S_N(f)=1$, $S_X(f)=\frac{1}{1+f^2}$ and $S_{XN}(f)=0$, then

$$H(f) = \frac{\frac{1}{1+f^2}}{1+\frac{1}{1+f^2}} = \frac{1}{2+f^2}$$

Problem 5.56

1) Let $\hat{X}(t)$ and $\tilde{X}(t)$ be the outputs of the systems $h(t)$ and $g(t)$ when the input $Z(t)$ is applied. Then,

$$\begin{aligned} E[(X(t)-\tilde{X}(t))^2] &= E[(X(t)-\hat{X}(t)+\hat{X}(t)-\tilde{X}(t))^2] \\ &= E[(X(t)-\hat{X}(t))^2] + E[(\hat{X}(t)-\tilde{X}(t))^2] \\ &\quad + E[(X(t)-\hat{X}(t))\cdot(\hat{X}(t)-\tilde{X}(t))] \end{aligned}$$

But,

$$\begin{aligned} &E[(X(t)-\hat{X}(t))\cdot(\hat{X}(t)-\tilde{X}(t))] \\ &= E[(X(t)-\hat{X}(t))\cdot Z(t)\star(h(t)-g(t))] \\ &= E\left[(X(t)-\hat{X}(t))\int_{-\infty}^{\infty}(h(\tau)-g(\tau))Z(t-\tau)d\tau\right] \\ &= \int_{-\infty}^{\infty} E\left[(X(t)-\hat{X}(t))Z(t-\tau)\right](h(\tau)-g(\tau))d\tau = 0 \end{aligned}$$

where the last equality follows from the assumption $E\left[(X(t)-\hat{X}(t))Z(t-\tau)\right]=0$ for all t,τ. Thus,

$$E[(X(t)-\tilde{X}(t))^2] = E[(X(t)-\hat{X}(t))^2] + E[(\hat{X}(t)-\tilde{X}(t))^2]$$

and this proves that

$$E[(X(t)-\hat{X}(t))^2] \le E[(X(t)-\tilde{X}(t))^2]$$

2)

$$E[(X(t)-\hat{X}(t))Z(t-\tau)] = 0 \Rightarrow E[X(t)Z(t-\tau)] = E[\hat{X}(t)Z(t-\tau)]$$

or in terms of crosscorrelation functions $R_{XZ}(\tau)=R_{\hat{X}Z}(\tau)=R_{Z\hat{X}}(-\tau)$. However, $R_{Z\hat{X}}(-\tau)=R_Z(-\tau)\star h(\tau)$ so that

$$R_{XZ}(\tau) = R_Z(-\tau)\star h(\tau) = R_Z(\tau)\star h(\tau)$$

3) Taking the Fourier of both sides of the previous equation we obtain

$$S_{XZ}(f) = S_Z(f)H(f) \quad \text{or} \quad H(f) = \frac{S_{XZ}(f)}{S_Z(f)}$$

4)

$$\begin{aligned}
E[\epsilon^2(t)] &= E\left[(X(t)-\hat{X}(t))((X(t)-\hat{X}(t))\right] \\
&= E[X(t)X(t)] - E[\hat{X}(t)X(t)] \\
&= R_X(0) - E\left[\int_{-\infty}^{\infty} Z(t-v)h(v)X(t)dv\right] \\
&= R_X(0) - \int_{-\infty}^{\infty} R_{ZX}(-v)h(v)dv \\
&= R_X(0) - \int_{-\infty}^{\infty} R_{XZ}(v)h(v)dv
\end{aligned}$$

where we have used the fact that $E[(X(t)-\hat{X}(t))\hat{X}(t)] = E[(X(t)-\hat{X}(t))Z(t) \star h(t)] = 0$

Problem 5.57
the noise equivalent bandwidth of a filter is

$$B_{neq} = \frac{\int_{-\infty}^{\infty} |H(f)|^2 df}{2H_{\max}^2}$$

If we have an ideal bandpass filter of bandwidth W, then $H(f) = 1$ for $|f - f_0| < W$ where f_0 is the central frequency of the filter. Hence,

$$B_{neq} = \frac{1}{2}\left[\int_{-f_0-\frac{W}{2}}^{-f_0+\frac{W}{2}} df + \int_{f_0-\frac{W}{2}}^{f_0+\frac{W}{2}} df\right] = W$$

Problem 5.58
1) The power spectral density of the in-phase and quadrature components is given by

$$S_{n_c}(f) = S_{n_s}(f) = \begin{cases} S_n(f-f_0) + S_n(f+f_0) & |f| < 7 \\ 0 & \text{otherwise} \end{cases}$$

If the passband of the ideal filter extends from 3 to 11 KHz, then f_0 =7 KHz is the mid-band frequency so that

$$S_{n_c}(f) = S_{n_s}(f) = \begin{cases} N_0 & |f| < 7 \\ 0 & \text{otherwise} \end{cases}$$

The cross spectral density is given by

$$S_{n_c n_s}(f) = \begin{cases} j[S_n(f+f_0) - S_n(f-f_0) & |f| < 7 \\ 0 & \text{otherwise} \end{cases}$$

However $S_n(f+f_0) = S_n(f-f_0)$ for $|f| < 7$ and therefore $S_{n_c n_s}(f) = 0$. It turns then that the crosscorrelation $R_{n_c n_s}(\tau)$ is zero.

2) With f_0=6 KHz

$$S_{n_c}(f) = S_{n_s}(f) = \begin{cases} \frac{N_0}{2} & 3 < |f| < 5 \\ N_0 & |f| < 3 \\ 0 & \text{otherwise} \end{cases}$$

The cross spectral density is given by

$$S_{n_c n_s}(f) = \begin{cases} -j\frac{N_0}{2} & -5 < f < 3 \\ j\frac{N_0}{2} & 3 < f < 5 \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$\begin{aligned} R_{n_c n_s}(\tau) &= \mathcal{F}^{-1}\left[-j\frac{N_0}{2}\Pi(\frac{t+4}{2}) + j\frac{N_0}{2}\Pi(\frac{t-4}{2})\right] \\ &= -j\frac{N_0}{2}2\text{sinc}(2\tau)e^{-j2\pi 4\tau} + j\frac{N_0}{2}2\text{sinc}(2\tau)e^{j2\pi 4\tau} \\ &= -2N_0\text{sinc}(2\tau)\sin(2\pi 4\tau) \end{aligned}$$

Problem 5.59

The in-phase component of $X(t)$ is

$$\begin{aligned} X_c(t) &= X(t)\cos(2\pi f_0 t) + \hat{X}(t)\sin(2\pi f_0 t) \\ &= \sum_{n=-\infty}^{\infty} A_n p(t-nT)\cos(2\pi f_0(t-nT)) \\ &\quad + \sum_{n=-\infty}^{\infty} A_n \hat{p}(t-nT)\sin(2\pi f_0(t-nT)) \\ &= \sum_{n=-\infty}^{\infty} A_n \left(p(t-nT)\cos(2\pi f_0(t-nT)) + \hat{p}(t-nT)\sin(2\pi f_0(t-nT))\right) \\ &= \sum_{n=-\infty}^{\infty} A_n p_c(t-nT) \end{aligned}$$

where we have used the fact $p_c(t) = p(t)\cos(2\pi f_0 t) + \hat{p}(t)\sin(2\pi f_0 t)$. Similarly for the quadrature component

$$\begin{aligned} X_s(t) &= \hat{X}(t)\cos(2\pi f_0 t) - X(t)\sin(2\pi f_0 t) \\ &= \sum_{n=-\infty}^{\infty} A_n \hat{p}(t-nT)\cos(2\pi f_0(t-nT)) \\ &\quad - \sum_{n=-\infty}^{\infty} A_n p(t-nT)\sin(2\pi f_0(t-nT)) \\ &= \sum_{n=-\infty}^{\infty} A_n \left(\hat{p}(t-nT)\cos(2\pi f_0(t-nT)) - p(t-nT)\sin(2\pi f_0(t-nT))\right) \\ &= \sum_{n=-\infty}^{\infty} A_n p_s(t-nT) \end{aligned}$$

Problem 5.60

The envelope $V(t)$ of a bandpass process is defined to be

$$V(t) = \sqrt{X_c^2(t) + X_s^2(t)}$$

where $X_c(t)$ and $X_s(t)$ are the in-phase and quadrature components of $X(t)$ respectively. However, both the in-phase and quadrature components are lowpass processes and this makes $V(t)$ a lowpass process independent of the choice of the center frequency f_0.

Problem 5.61

1) The power spectrum of the bandpass signal is

$$S_n(f) = \begin{cases} \frac{N_0}{2} & |f - f_c| < W \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$S_{n_c}(f) = S_{n_s}(f) = \begin{cases} N_0 & |f| < W \\ 0 & \text{otherwise} \end{cases}$$

The power content of the in-phase and quadrature components of $n(t)$ is $P_n = \int_{-W}^{W} N_0 df = 2N_0W$

2) Since $S_{n_cn_s}(f) = 0$, the processes $N_c(t)$, $N_s(t)$ are independent zero-mean Gaussian with variance $\sigma^2 = P_n = 2N_0W$. Hence, $V(t) = \sqrt{N_c^2(t) + N_s^2(t)}$ is Rayleigh distributed and the PDF is given by

$$f_V(v) = \begin{cases} \frac{v^2}{2N_0W} e^{-\frac{v^2}{4N_0W}} & v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

3) $X(t)$ is given by

$$X(t) = (A + N_c(t)) \cos(2\pi f_0 t) - N_S(t) \sin(2\pi f_0 t)$$

The process $A+N_c(t)$ is Gaussian with mean A and variance $2N_0W$. Hence, $V(t) = \sqrt{(A + N_c(t))^2 + N_s^2(t)}$ follows the Rician distribution (see Problem 4.31). The density function of the envelope is given by

$$f_V(v) = \begin{cases} \frac{v}{2N_0W} I_0\left(\frac{Av}{2N_0W}\right) e^{-\frac{v^2+A^2}{4N_0W}} & v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

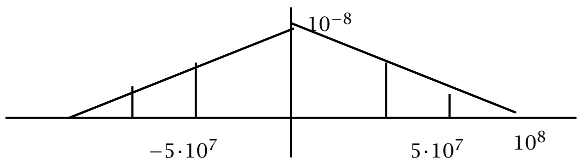
where

$$I_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x \cos u} du$$

Problem 5.62

1) The power spectral density $S_n(f)$ is depicted in the following figure. The output bandpass process has non-zero power content for frequencies in the band $49 \times 10^6 \leq |f| \leq 51 \times 10^6$. The power content is

$$\begin{aligned} P &= \int_{-51\times10^6}^{-49\times10^6} 10^{-8}\left(1+\frac{f}{10^8}\right)df + \int_{49\times10^6}^{51\times10^6} 10^{-8}\left(1-\frac{f}{10^8}\right)df \\ &= 10^{-8}x\Big|_{-51\times10^6}^{-49\times10^6} + 10^{-16}\frac{1}{2}x^2\Big|_{-51\times10^6}^{-49\times10^6} + 10^{-8}x\Big|_{49\times10^6}^{51\times10^6} - 10^{-16}\frac{1}{2}x^2\Big|_{49\times10^6}^{51\times10^6} \\ &= 2\times10^{-2} \end{aligned}$$



2) The output process $N(t)$ can be written as

$$N(t) = N_c(t)\cos(2\pi 50\times10^6 t) - N_s(t)\sin(2\pi 50\times10^6 t)$$

where $N_c(t)$ and $N_s(t)$ are the in-phase and quadrature components respectively, given by

$$\begin{aligned} N_c(t) &= N(t)\cos(2\pi 50\times10^6 t) + \hat{N}(t)\sin(2\pi 50\times10^6 t) \\ N_s(t) &= \hat{N}(t)\cos(2\pi 50\times10^6 t) - N(t)\sin(2\pi 50\times10^6 t) \end{aligned}$$

The power content of the in-phase component is given by

$$\begin{aligned} E[|N_c(t)|^2] &= E[|N(t)|^2]\cos^2(2\pi 50\times10^6 t) + E[|\hat{N}(t)|^2]\sin^2(2\pi 50\times10^6 t) \\ &= E[|N(t)|^2] = 2\times10^{-2} \end{aligned}$$

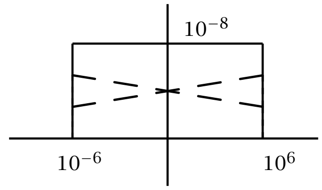
where we have used the fact that $E[|N(t)|^2] = E[|\hat{N}(t)|^2]$. Similarly we find that $E[|N_s(t)|^2] = 2\times10^{-2}$.

3) The power spectral density of $N_c(t)$ and $N_s(t)$ is

$$S_{N_c}(f) = S_{N_s}(f) = \begin{cases} S_N(f-50\times10^6) + S_N(f+50\times10^6) & |f| \leq 50\times10^6 \\ 0 & \text{otherwise} \end{cases}$$

$S_{N_c}(f)$ is depicted in the next figure. The power content of $S_{N_c}(f)$ can now be found easily as

$$P_{N_c} = P_{N_s} = \int_{-10^6}^{10^6} 10^{-8} df = 2\times10^{-2}$$



4) The power spectral density of the output is given by

$$S_Y(f) = S_X(f)|H(f)|^2 = (|f| - 49 \times 10^6)(10^{-8} - 10^{-16}|f|) \quad \text{for } 49 \times 10^6 \le |f| \le 51 \times 10^6$$

Hence, the power content of the output is

$$\begin{aligned} P_Y &= \int_{-51\times10^6}^{-49\times10^6} (-f - 49 \times 10^6)(10^{-8} + 10^{-16} f)df \\ &\quad + \int_{49\times10^6}^{51\times10^6} (f - 49 \times 10^6)(10^{-8} - 10^{-16} f)df \\ &= 2 \times 10^4 - \frac{4}{3}10^2 \end{aligned}$$

The power spectral density of the in-phase and quadrature components of the output process is given by

$$\begin{aligned} S_{Y_c}(f) = S_{Y_s}(f) &= \left((f + 50 \times 10^6) - 49 \times 10^6\right)\left(10^{-8} - 10^{-16}(f + 50 \times 10^6)\right) \\ &\quad + \left(-(f - 50 \times 10^6) - 49 \times 10^6\right)\left(10^{-8} + 10^{-16}(f - 50 \times 10^6)\right) \\ &= -2 \times 10^{-16} f^2 + 10^{-2} \end{aligned}$$

for $|f| \le 10^6$ and zero otherwise. The power content of the in-phase and quadrature component is

$$\begin{aligned} P_{Y_c} = P_{Y_s} &= \int_{-10^6}^{10^6} (-2 \times 10^{-16} f^2 + 10^{-2})df \\ &= -2 \times 10^{-16} \frac{1}{3} f^3 \Big|_{-10^6}^{10^6} + 10^{-2} f \Big|_{-10^6}^{10^6} \\ &= 2 \times 10^4 - \frac{4}{3}10^2 = P_Y \end{aligned}$$

Computer Problems

Computer Problem 5.1

We first generate the uniformly distributed random variable u_i by using matlab function rand, then we generate the random variable X by using

$$x_i = 2\sqrt{(u_i)} \tag{5.18}$$

Figure 5.1 presents the plot of the histogram of the 10000 randomly generated samples. It should be noted that this histogram of the random variables is similar to linear probability density function $f(x)$.
The MATLAB script for this question is given next.

```
% MATLAB script for Computer Problem 5.1

S = 10000; % Number of samples
u = rand(1,S); %Generate uniformly dist. random numbers
x = 2.*sqrt(u);
 N = HIST(x,20);
 x_a = 0:0.1:1.9;
 plot(x_a, N);
```

Computer Problem 5.2
1) The MATLAB function that implements the method given in the question is given as

```
function [gsrv1,gsrv2]=gngauss(m,sgma)
%    [gsrv1,gsrv2]=gngauss(m,sgma)
%    [gsrv1,gsrv2]=gngauss(sgma)
```

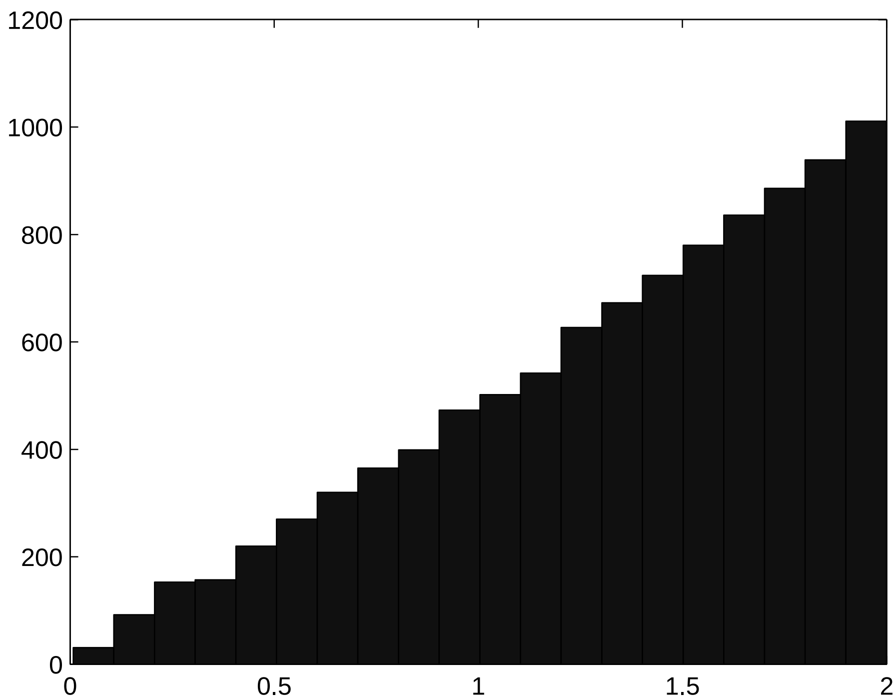


Figure 5.1: Histogram of the random variable X in Computer Problem 5.1

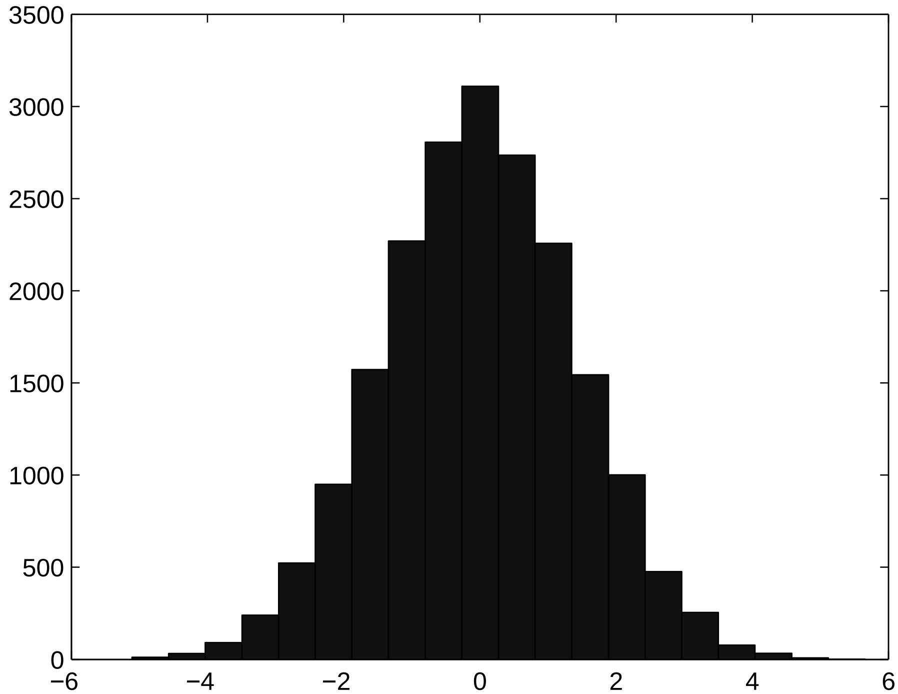


Figure 5.2: Histogram of the random variable X in Computer Problem 5.2

```
%    [gsrv1,gsrv2]=gngauss
%              GNGAUSS  generates two independent Gaussian random variables with mean
%              m and standard deviation sgma. If one of the input arguments is missing,
%              it takes the mean as 0.
%              If neither the mean nor the variance is given, it generates two standard
%              Gaussian random variables.
if nargin == 0,
  m=0; sgma=1;
elseif nargin == 1,
  sgma=m; m=0;
end;
u=rand;                                 % a uniform random variable in (0,1)
z=sgma*(sqrt(2*log(1/(1−u))));          % a Rayleigh distributed random variable
u=rand;                                 % another uniform random variable in (0,1)
gsrv1=m+z*cos(2*pi*u);
gsrv2=m+z*sin(2*pi*u);
```

2) Figure 5.2 presents the plot of the histogram of the 10000 randomly generated samples. It should be noted that this histogram of the random variables is similar to the Gaussian probability density function $f_X(x)$.

Computer Problem 5.3

1) Figure 5.3 presents the plot of $R_X(m)$.

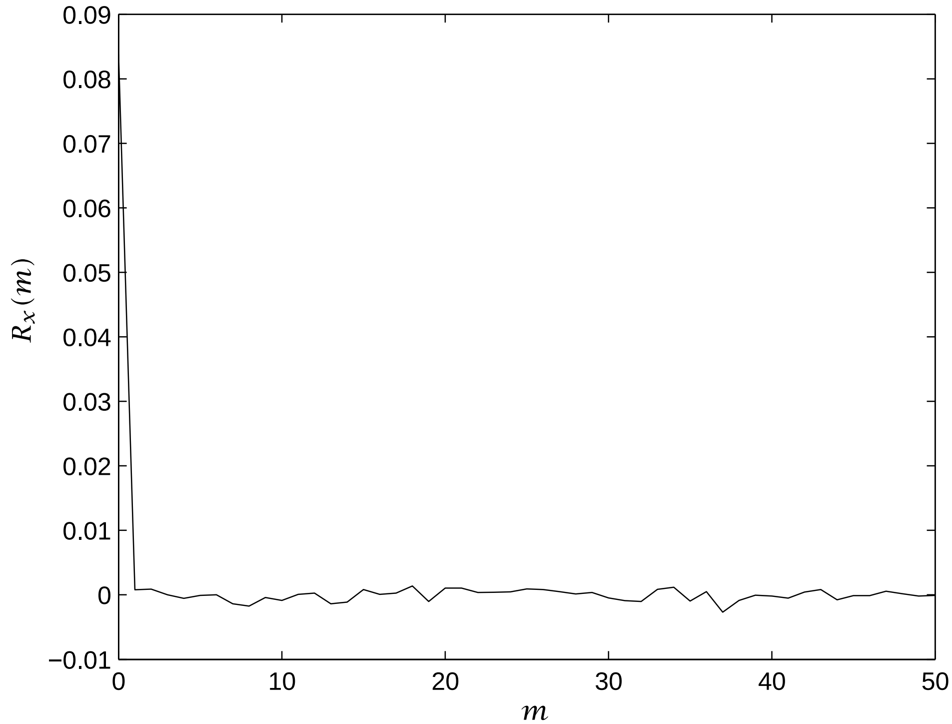


Figure 5.3: The autocorrelation function in Computer Problem 5.3

2) Figure 5.4 presents the plot of $S_X(f)$.
The MATLAB script that implements the generation of the sequence $\{X_n\}$, the computation of the autocorrelation, and the computation of the power spectrum $S_X(f)$ is given next. We should note that the estimates of the autocorrelation function and the power spectrum exhibit a significant variability. Therefore, it is necessary to average the sample autocorrelation over several realizations. $\hat{R}_X(m)$ and $\hat{S}_X(f)$ presented in Figures 5.3 and 5.4 are obtained by running this program using the average autocorrelation over ten realizations of the random process.

```
% MATLAB script for Computer Problem 5.3.
echo on
N=1000;
M=50;
Rx_av=zeros(1,M+1);
Sx_av=zeros(1,M+1);
for j=1:10,                  % Take the ensemble average over ten realizations
    X=rand(1,N)−1/2;         % N i.i.d. uniformly distributed random variables
                             % between -1/2 and 1/2.
    Rx=Rx_est(X,M);              % autocorrelation of the realization
    Sx=fftshift(abs(fft(Rx)));   % power spectrum of the realization
    Rx_av=Rx_av+Rx;              % sum of the autocorrelations
    Sx_av=Sx_av+Sx;              % sum of the spectrums
    echo off ;
end;
echo on ;
```

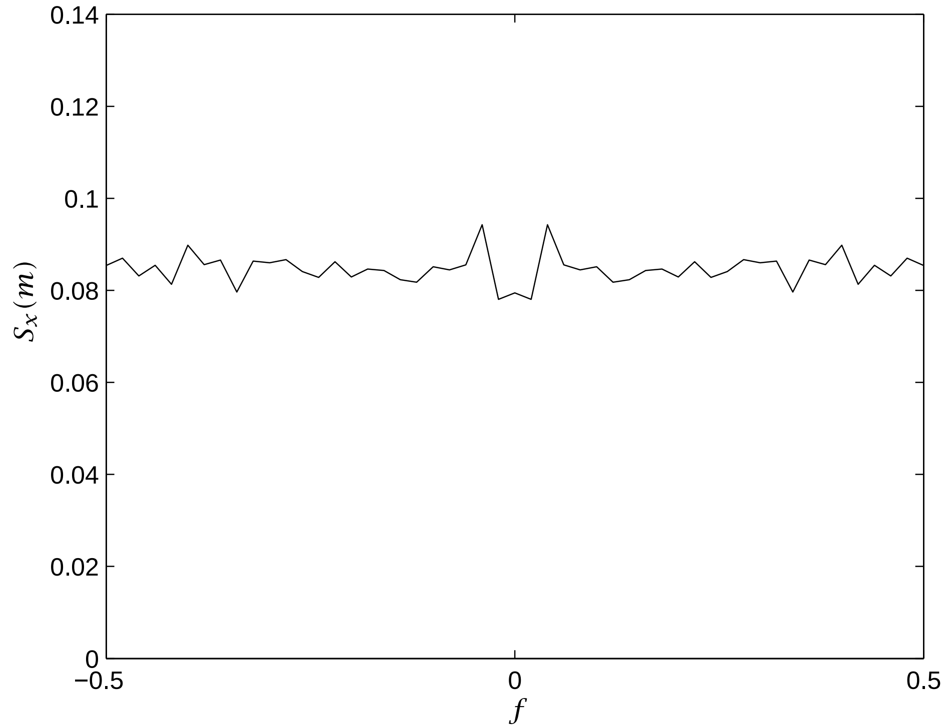


Figure 5.4: The power spectrum in Computer Problem 5.3

```
Rx_av=Rx_av/10;                % ensemble average autocorrelation
Sx_av=Sx_av/10;                % ensemble average spectrum
% Plotting comments follow
```

```
function [Rx]=Rx_est(X,M)
% [Rx]=Rx_est(X,M)
%           RX_EST  estimates the autocorrelation of the sequence of random
%           variables given in X. Only Rx(0), Rx(1), ... , Rx(M) are computed.
%           Note that Rx(m) actually means Rx(m-1).
N=length(X);
Rx=zeros(1,M+1);
for m=1:M+1,
  for n=1:N-m+1,
    Rx(m)=Rx(m)+X(n)*X(n+m-1);
  end;
  Rx(m)=Rx(m)/(N-m+1);
end;
```

Computer Problem 5.4

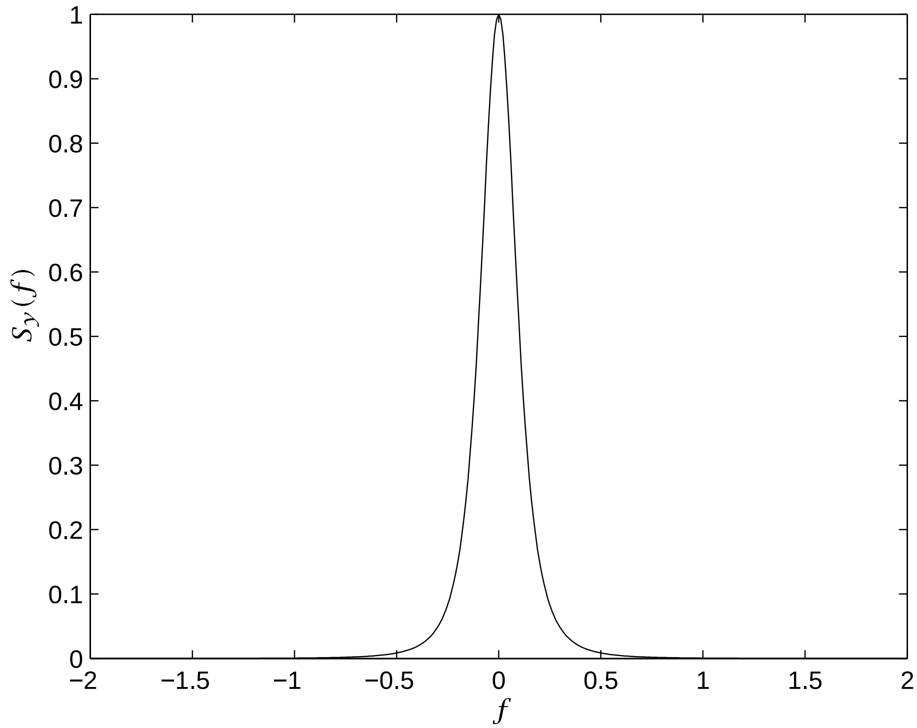


Figure 5.5: Plot of $S_y(f)$ in Computer Problem 5.4

1)The frequency response of the filter is easily shown to be

$$H(f) = \frac{1}{1 + j2\pi f} \tag{5.19}$$

Hence,

$$\begin{aligned} S_y(f) &= |H(f)|^2 \\ &= \frac{1}{1 + (2\pi f)^2} \end{aligned} \tag{5.20}$$

The graph of $S_y(f)$ is illustrated in Figure 5.5.
2) Figure 5.5 presents the plot of the autocorrelation function of the filter output $y(t)$. The MATLAB script for this question is given next.

```
% MATLAB script for Computer Problem 5.4.
echo on
delta=0.01;
F_min=-2;
F_max=2;
f=F_min:delta:F_max;
Sx=ones(1,length(f));
H=1./(1+(2*pi*f).^2);
Sy=Sx.*H.^2;
```

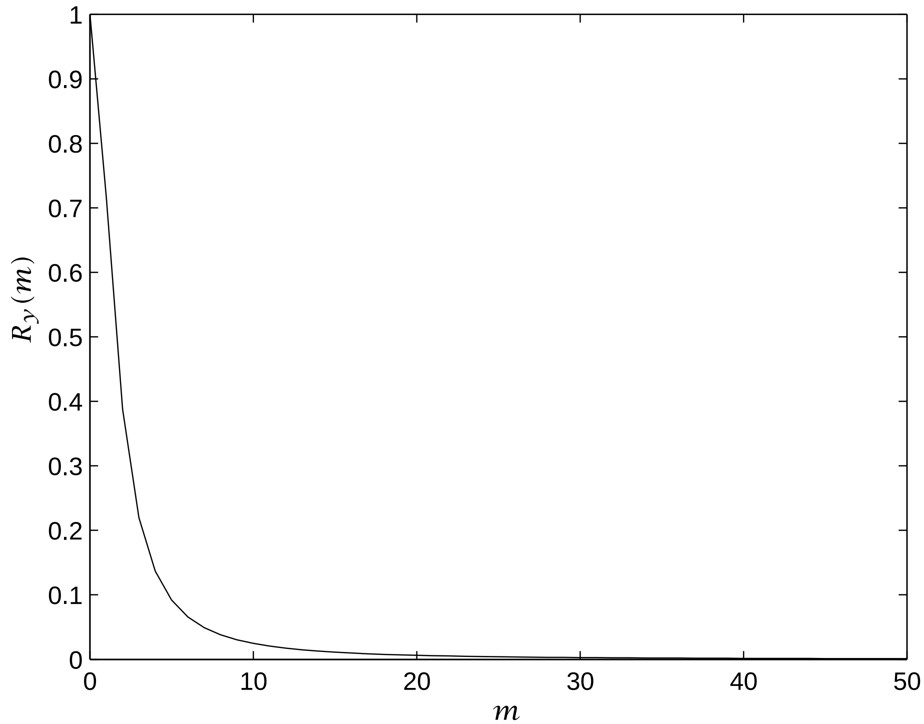


Figure 5.6: Plot of the autocorrelation function of the filter output $y(t)$ in Computer Problem 5.4

```
plot(f, Sy);
N=256;                    % number of samples
deltaf=0.1;               % frequency separation
f=[0:deltaf:(N/2)*deltaf, -(N/2-1)*deltaf:deltaf:-deltaf];
                          % Swap the first half.
Sy=1./(1+(2*pi*f).^2);          % sampled spectrum
Ry=ifft(Sy);                    % autocorrelation of Y
% Plotting command follows.
figure;
plot(fftshift(real(Ry)));
```

Computer Problem 5.5

The MATLAB scripts for all computations are given next. Figures 5.7, 5.8, 5.9 and 5.10 illustrate the estimates of the autocorrelation functions and the power spectra. We note that the plots of the autocorrelation function and the power spectra are averages over ten realizations of the random process.

```
% MATLAB script for Computer Problem 5.5.
N=1000;                              % the maximum value of n
M=50;
Rxav=zeros(1,M+1);
```

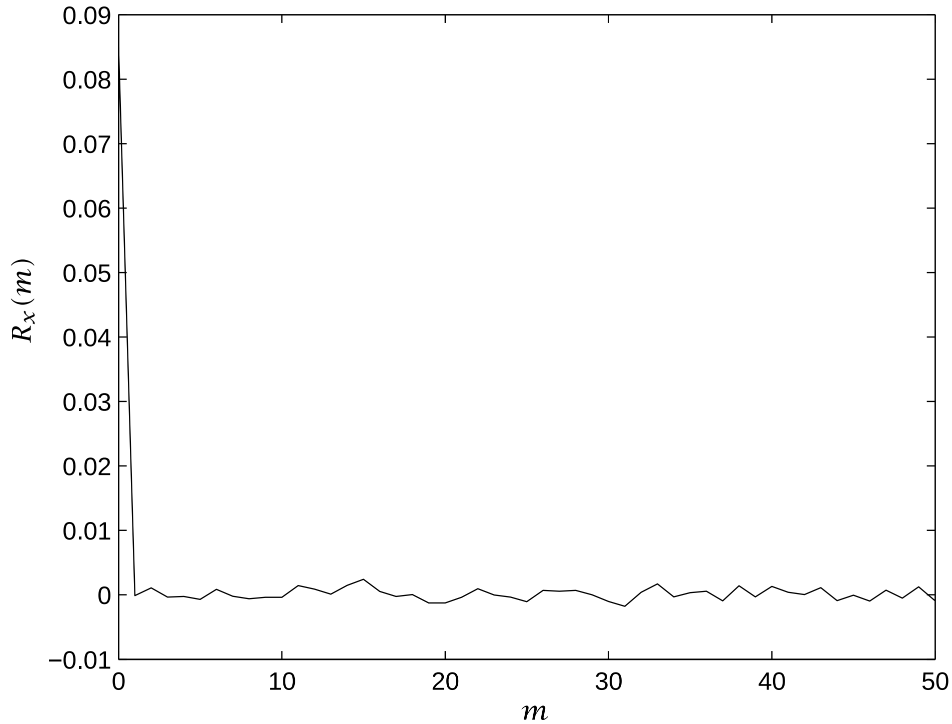


Figure 5.7: The autocorrelation function $R_X(m)$ in Computer Problem 5.5

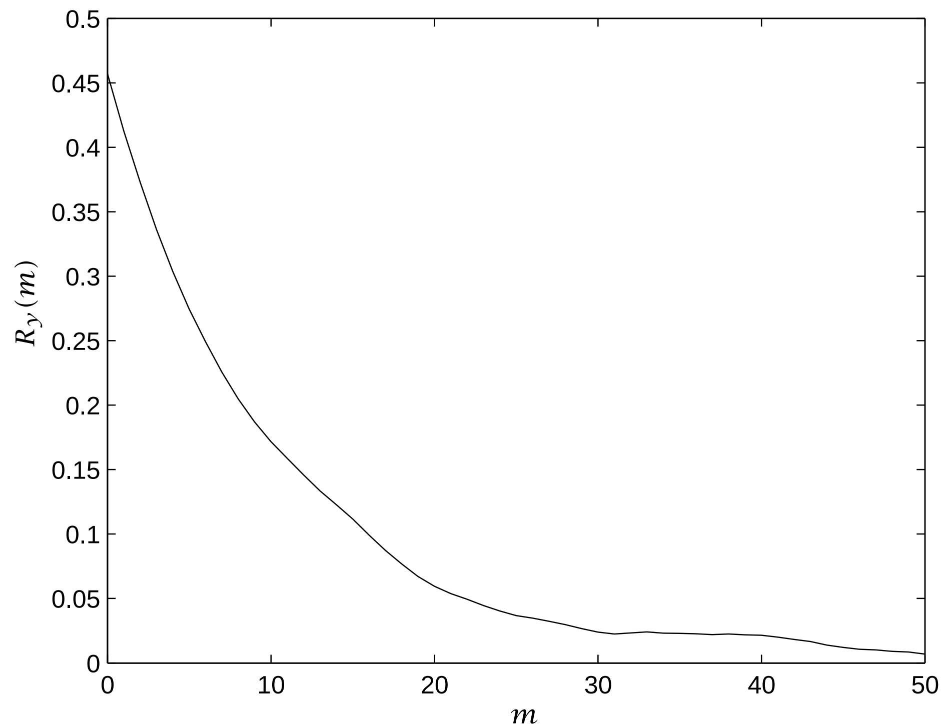


Figure 5.8: The autocorrelation function $R_Y(m)$ in Computer Problem 5.5

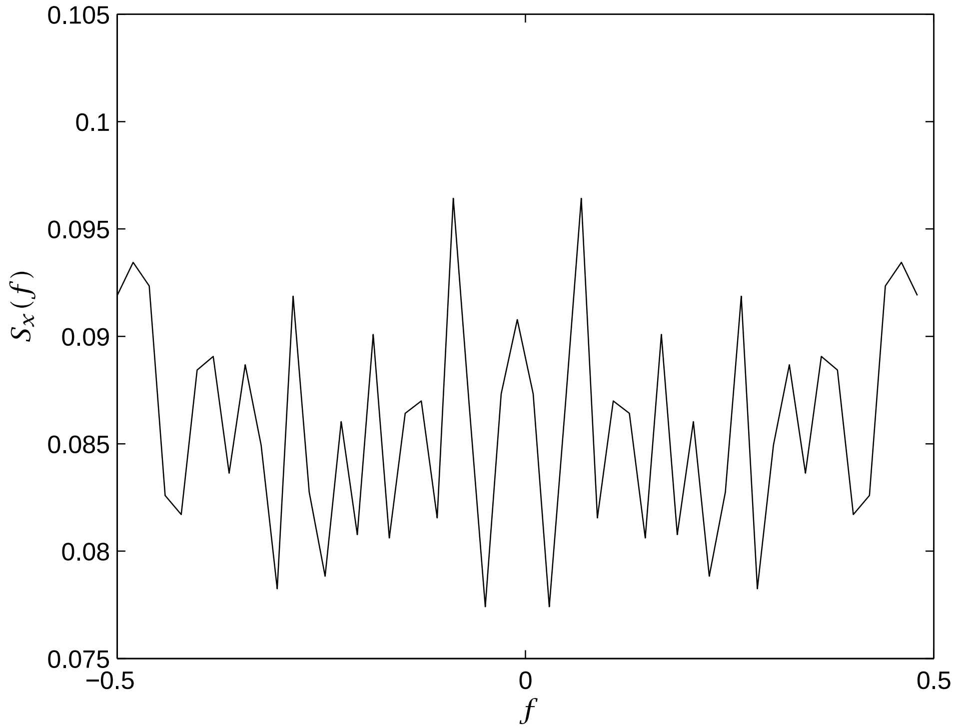


Figure 5.9: The power spectra $S_X(f)$ in Computer Problem 5.5

```
Ryav=zeros(1,M+1);
Sxav=zeros(1,M+1);
Syav=zeros(1,M+1);
for i=1:10,                              % Take the ensemble average over ten realizations.
   X=rand(1,N)−(1/2);                    % Generate a uniform number sequence on (-1/2,1/2).
   Y(1)=0;
   for n=2:N, Y(n)=0.9*Y(n−1)+X(n); end; % Note that Y(n) means Y(n-1).
   Rx=Rx_est(X,M);                       % autocorrelation of {Xn}
   Ry=Rx_est(Y,M);                       % autocorrelation of {Yn}
   Sx=fftshift(abs(fft(Rx)));            % power spectrum of {Xn}
   Sy=fftshift(abs(fft(Ry)));            % power spectrum of {Yn}
   Rxav=Rxav+Rx;
   Ryav=Ryav+Ry;
   Sxav=Sxav+Sx;
   Syav=Syav+Sy;
   echo off ;
end;
echo on ;
Rxav=Rxav/10;
Ryav=Ryav/10;
Sxav=Sxav/10;
Syav=Syav/10;
% Plotting commands follow.
```

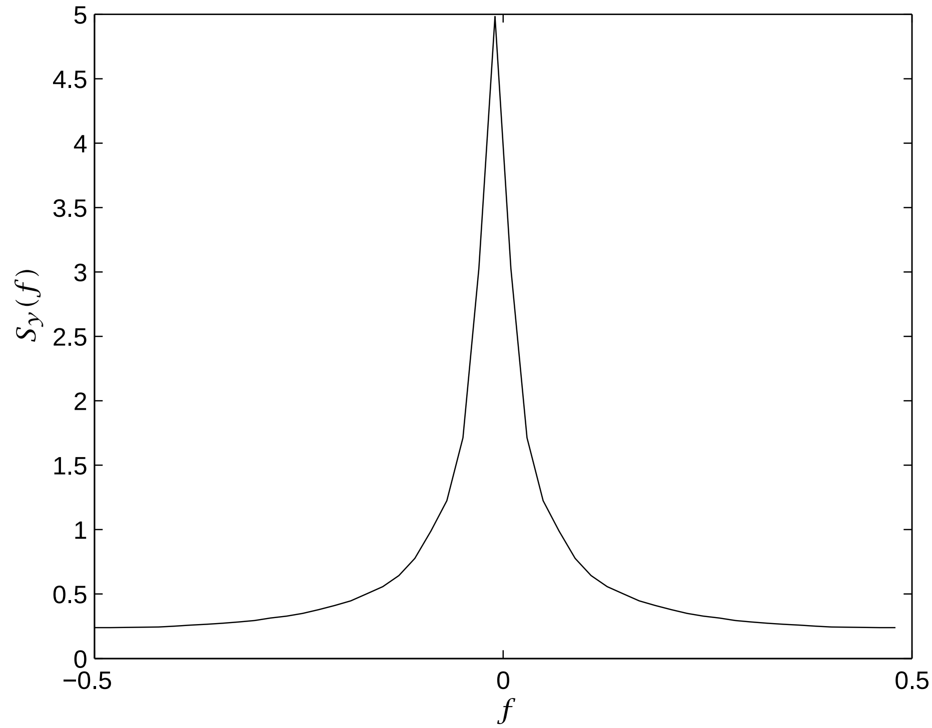


Figure 5.10: The power spectra $S_Y(f)$ in Computer Problem 5.5

Computer Problem 5.6

Figures 5.11, 5.12, and 5.13 present $R_{X_c}(m)$, $R_{X_s}(m)$, and $R_X(m)$, respectively. $S_{X_c}(f)$, $S_{X_s}(f)$, and $S_X(f)$ are presented on Figures 5.14, 5.15, and 5.16, respectively.

The MATLAB script for these computations is given next. For illustrative purposes we have selected the lowpass filer to have transfer function

$$H(z) = \frac{1}{1 - 0.9z^{-1}}$$

```
% MATLAB script for Computer Problem 5.6.
N=1000;                        % number of samples
for i=1:2:N,
   [X1(i) X1(i+1)]=gngauss;
   [X2(i) X2(i+1)]=gngauss;
end;                           % standard Gaussian input noise processes
A=[1 −0.9];                    % lowpass filter parameters
B=1;
Xc=filter(B,A,X1);
Xs=filter(B,A,X2);
fc=1000/pi;                    % carrier frequency
for i=1:N,
```

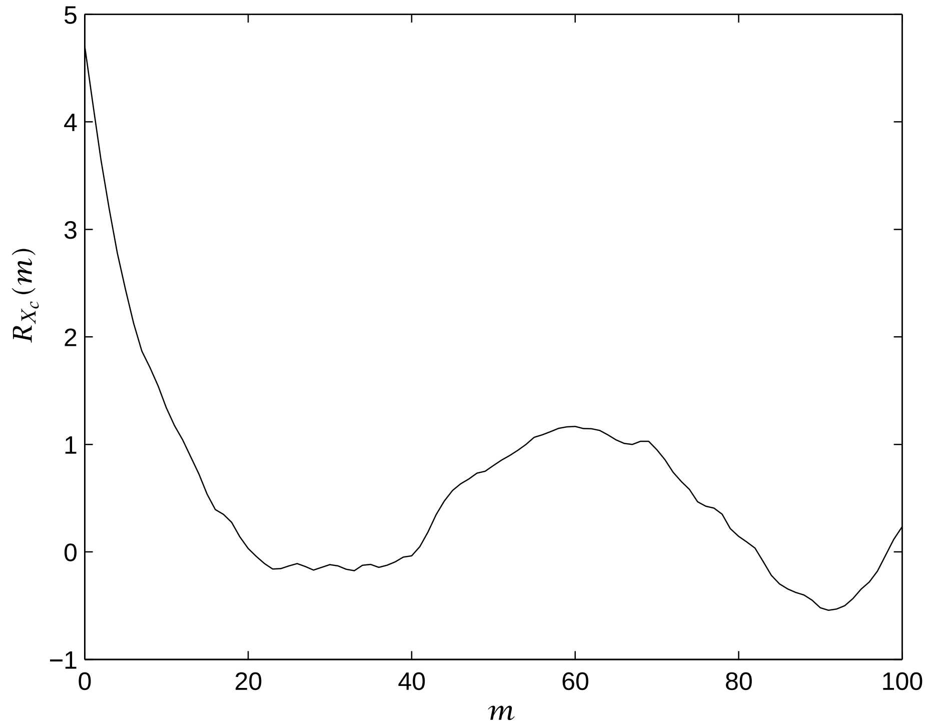


Figure 5.11: The autocorrelation of $X_c(m)$

```
    band_pass_process(i)=Xc(i)*cos(2*pi*fc*i)−Xs(i)*sin(2*pi*fc*i);
end;                                  % T=1 is assumed.
% Determine the autocorrelation and the spectrum of the bandpass process.
M=100;
bpp_autocorr=Rx_est(band_pass_process,M);
bpp_spectrum=fftshift(abs(fft(bpp_autocorr)));

bpp_autocorr=Rx_est(band_pass_process,M);
bpp_spectrum=fftshift(abs(fft(bpp_autocorr)));

Xc_autocorr=Rx_est(Xc,M);
Xc_spectrum=fftshift(abs(fft(Xc)));

Xs_autocorr=Rx_est(Xs,M);
Xs_spectrum=fftshift(abs(fft(Xs)));
% Plotting commands follow.
```

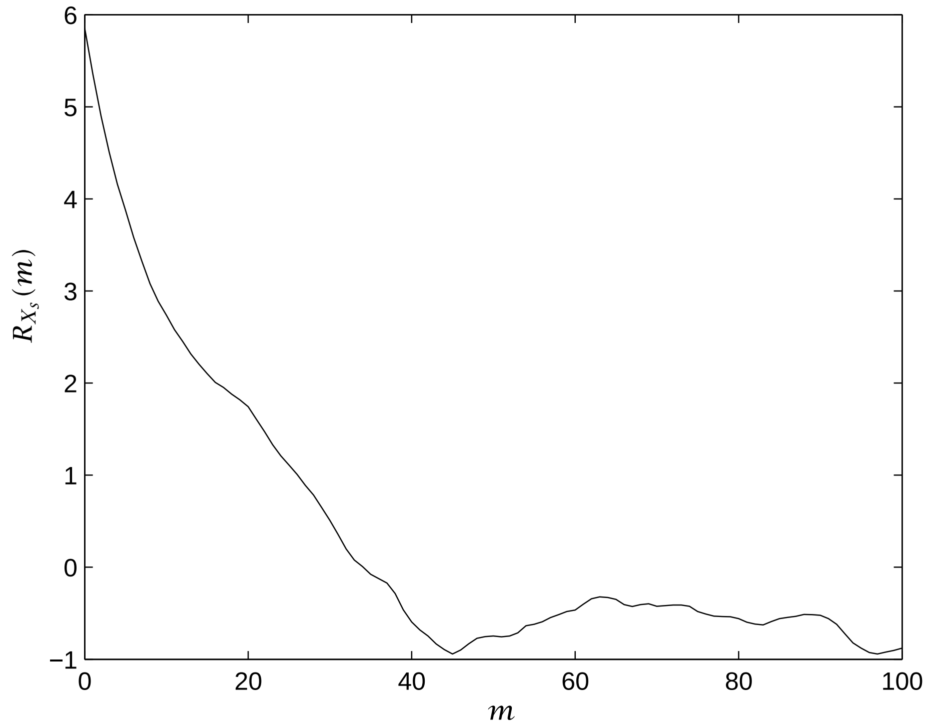


Figure 5.12: The autocorrelation of $X_s(m)$

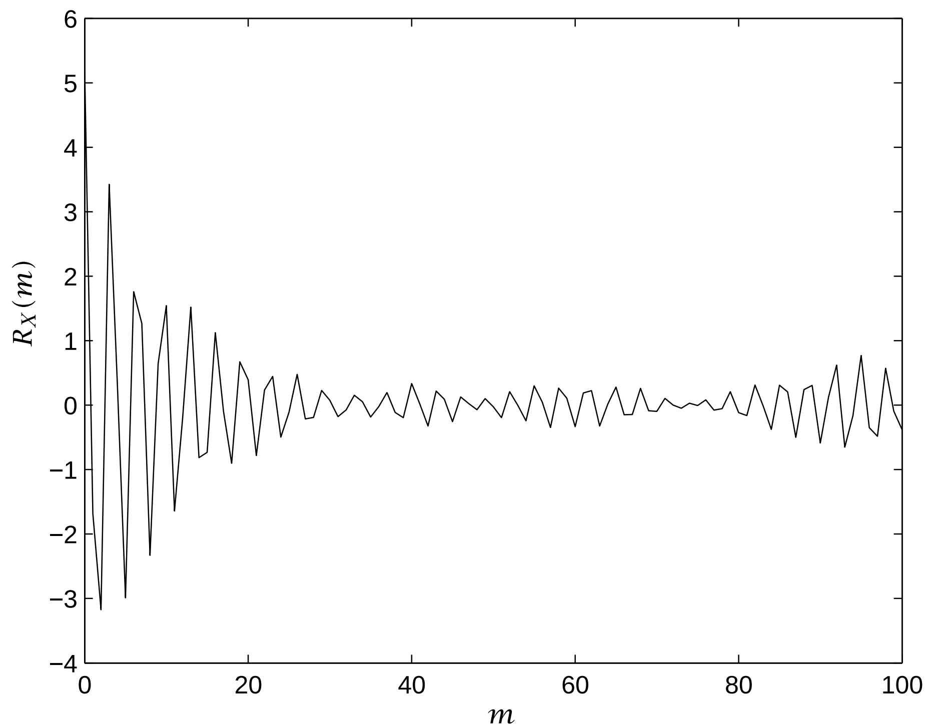


Figure 5.13: The autocorrelation of $X(m)$

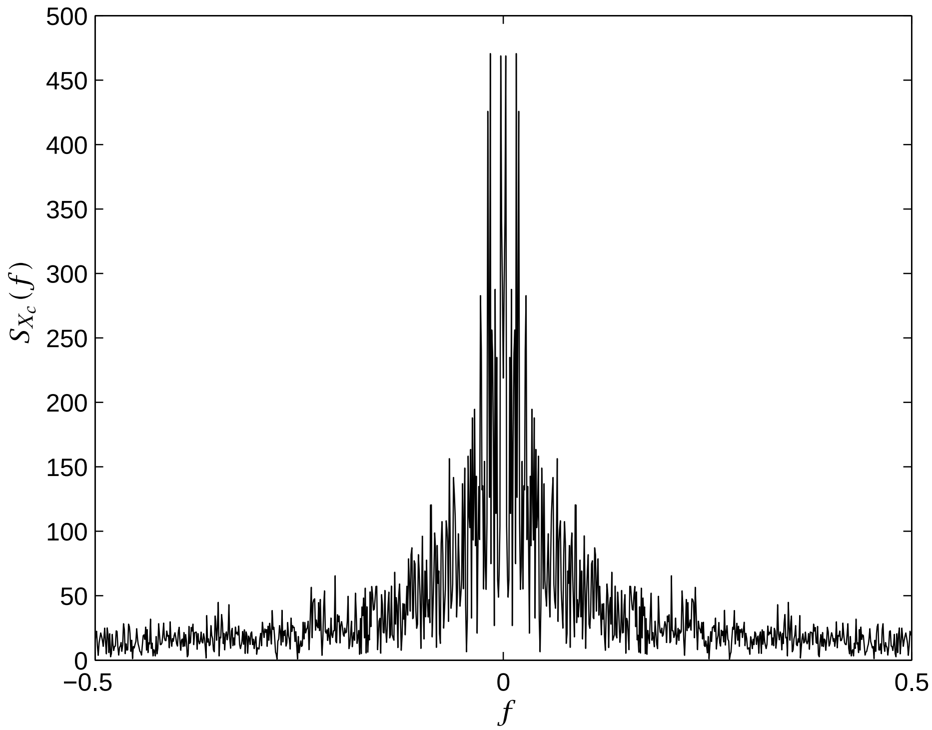


Figure 5.14: The power spectrum of $X_c(m)$

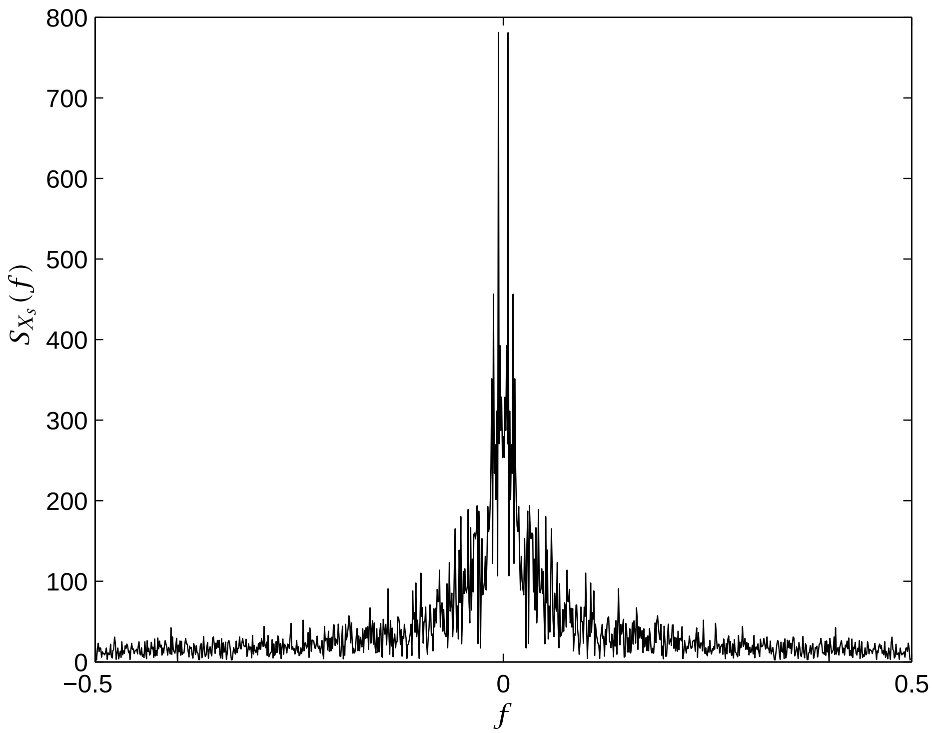


Figure 5.15: The power spectrum of $X_s(m)$

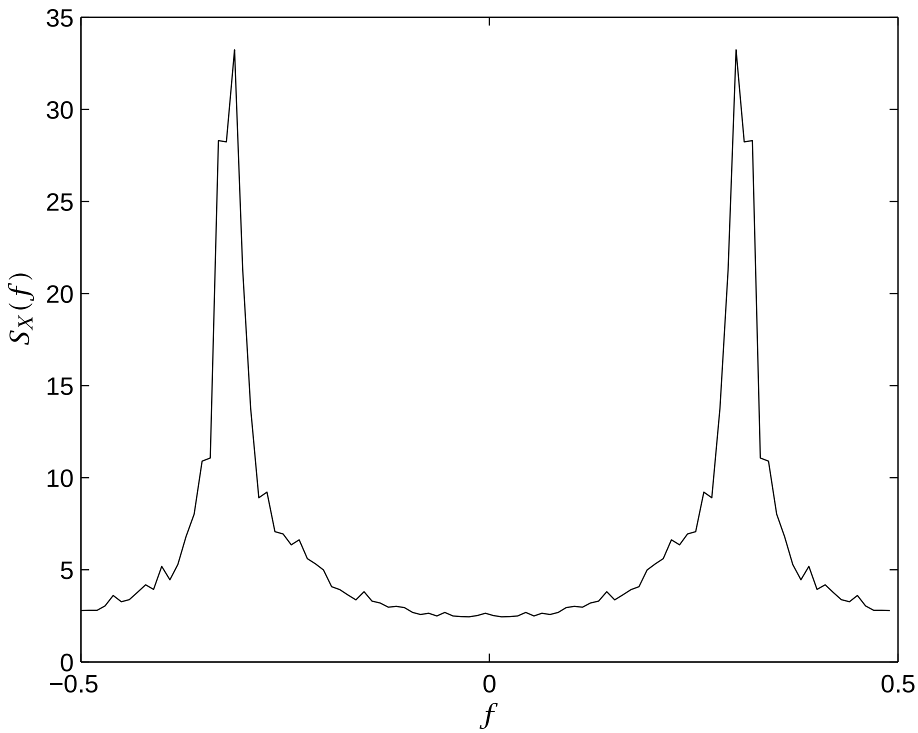


Figure 5.16: The power spectrum of $X(m)$

Chapter 6

Problem 6.1

The spectrum of the signal at the output of the LPF is $S_{s,o}(f) = S_s(f)|\Pi(\frac{f}{2W})|^2$. Hence, the signal power is

$$\begin{aligned} P_{s,o} &= \int_{-\infty}^{\infty} S_{s,o}(f)df = \int_{-W}^{W} \frac{P_0}{1+(f/B)^2}df \\ &= P_0 B \arctan(\frac{f}{B})\Big|_{-W}^{W} = 2P_0 B \arctan(\frac{W}{B}) \end{aligned}$$

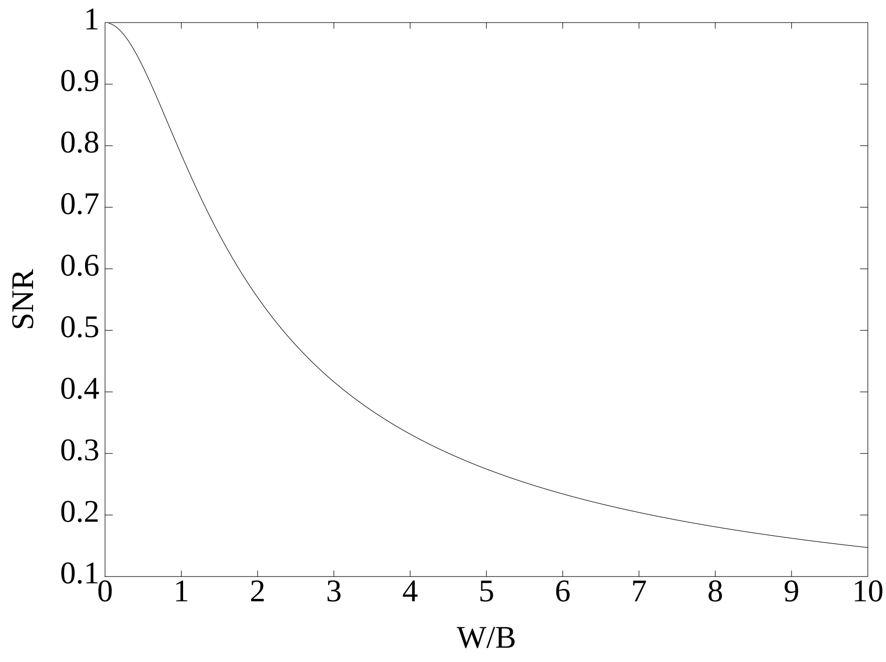
Similarly, noise power at the output of the lowpass filter is

$$P_{n,o} = \int_{-W}^{W} \frac{N_0}{2} df = N_0 W$$

Thus, the SNR is given by

$$\text{SNR} = \frac{2P_0 B \arctan(\frac{W}{B})}{N_0 W} = \frac{2P_0}{N_0} \frac{\arctan(\frac{W}{B})}{\frac{W}{B}}$$

In the next figure we plot SNR as a function of $\frac{W}{B}$ and for $\frac{2P_0}{N_0} = 1$.



Problem 6.2

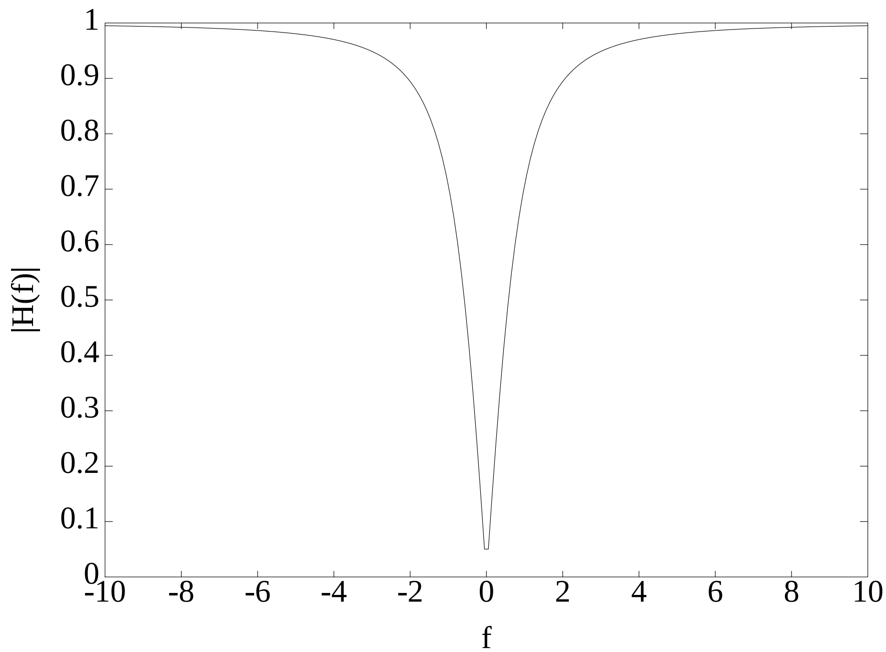
1) The transfer function of the RC filter is

$$H(s) = \frac{R}{\frac{1}{Cs} + R} = \frac{RCs}{1 + RCs}$$

with $s = j2\pi f$. Hence, the magnitude frequency response is

$$|H(f)| = \left(\frac{4\pi^2 (RC)^2 f^2}{1 + 4\pi^2 (RC)^2 f^2} \right)^{\frac{1}{2}}$$

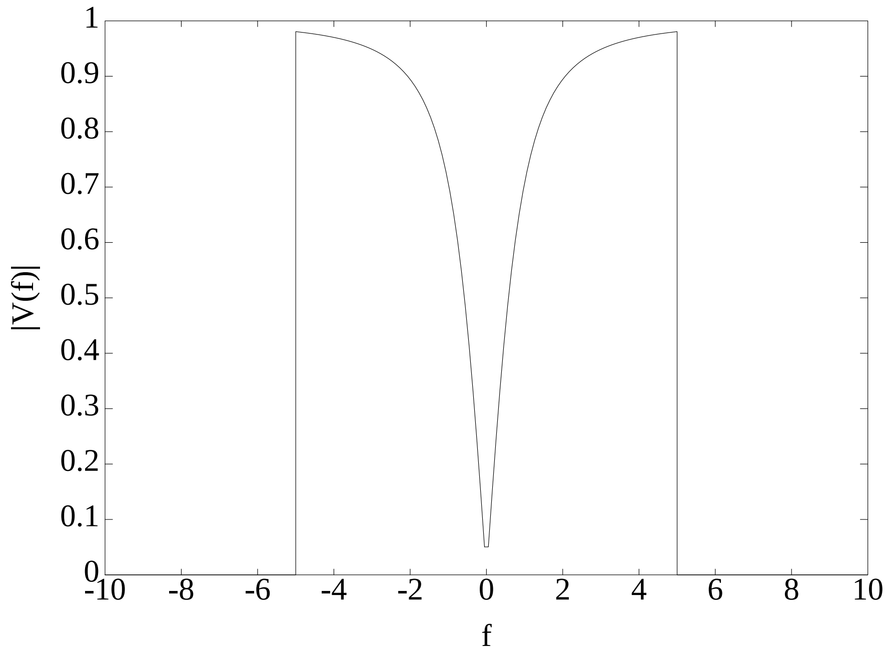
This function is plotted in the next figure for f in $[-10, 10]$ and $4\pi^2 (RC)^2 = 1$.



2) The overall system is the cascade of the RC and the LPF filter. If the bandwidth of the LPF is W, then the transfer function of the system is

$$V(f) = \frac{j2\pi RCf}{1 + j2\pi RCf} \Pi(\frac{f}{2W})$$

The next figure depicts $|V(f)|$ for $W = 5$ and $4\pi^2 (RC)^2 = 1$.



3) The noise output power is

$$\begin{aligned}
P_n &= \int_{-W}^{W} \frac{4\pi^2(RC)^2 f^2}{1+4\pi^2(RC)^2 f^2} \frac{N_0}{2} df \\
&= N_0 W - \frac{N_0}{2} \int_{-W}^{W} \frac{1}{1+4\pi^2(RC)^2 f^2} df \\
&= N_0 W - \frac{N_0}{2} \frac{1}{2\pi RC} \arctan(2\pi RCf) \Big|_{-W}^{W} \\
&= N_0 W - \frac{N_0}{2\pi RC} \arctan(2\pi RCW)
\end{aligned}$$

The output signal is a sinusoidal with frequency f_c and amplitude $A|V(f_c)|$. Since $f_c < W$ we conclude that the amplitude of the sinusoidal ouput signal is

$$A|H(f_c)| = A\sqrt{\frac{4\pi^2(RC)^2 f_c^2}{1+4\pi^2(RC)^2 f_c^2}}$$

and the output signal power

$$P_s = \frac{A^2}{2} \frac{4\pi^2(RC)^2 f_c^2}{1+4\pi^2(RC)^2 f_c^2}$$

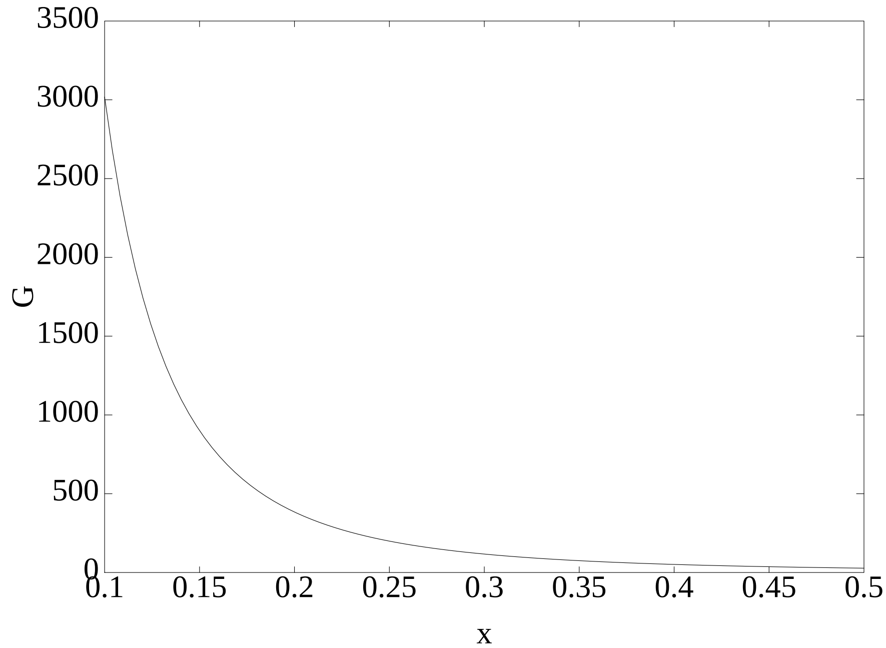
Thus, the SNR at the ouput of the LPF is

$$\text{SNR} = \frac{\frac{A^2}{2} \frac{4\pi^2(RC)^2 f_c^2}{1+4\pi^2(RC)^2 f_c^2}}{N_0 W - \frac{N_0}{2\pi RC} \arctan(2\pi RCW)} = \frac{\frac{A^2}{N_0} \frac{\pi RC f_c^2}{1+4\pi^2(RC)^2 f_c^2}}{2\pi RCW - \arctan(2\pi RCW)}$$

In the next figure we plot

$$G(W) = \frac{1}{2\pi RCW - \arctan(2\pi RCW)}$$

as a function of $x = 2\pi RCW$, when the latter varies from 0.1 to 0.5.



Problem 6.3

The noise power content of the received signal $r(t) = u(t) + n(t)$ is

$$P_n = \int_{-\infty}^{\infty} S_n(f)df = \frac{N_0}{2} \times 4W = 2N_0W$$

If we write $n(t)$ as

$$n(t) = n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

then,

$$\begin{aligned} n(t)\cos(2\pi f_c t) &= n_c(t)\cos^2(2\pi f_c t) - n_s(t)\cos(2\pi f_c t)\sin(2\pi f_c t) \\ &= \frac{1}{2}n_c(t) + \frac{1}{2}n_c(t)\cos(2\pi 2f_c t) - n_s(t)\sin(2\pi 2f_c t) \end{aligned}$$

The noise signal at the output of the LPF is $\frac{1}{2}n_c(t)$ with power content

$$P_{n,o} = \frac{1}{4}P_{n_c} = \frac{1}{4}P_n = \frac{N_0W}{2}$$

If the DSB modulated signal is $u(t) = m(t)\cos(2\pi f_c t)$, then its autocorrelation function is $\bar{R}_u(\tau) = \frac{1}{2}R_M(\tau)\cos(2\pi f_c \tau)$ and its power

$$P_u = \bar{R}_u(0) = \frac{1}{2}R_M(0) = \int_{-\infty}^{\infty} S_u(f)df = 2WP_0$$

From this relation we find $R_M(0) = 4WP_0$. The signal at the output of the LPF is $y(t) = \frac{1}{2}m(t)$ with power content

$$P_{s,o} = \frac{1}{4}E[m^2(t)] = \frac{1}{4}R_M(0) = WP_0$$

Hence, the SNR at the output of the demodulator is

$$\text{SNR} = \frac{P_{s,o}}{P_{n,o}} = \frac{WP_0}{\frac{N_0 W}{2}} = \frac{2P_0}{N_0}$$

Problem 6.4

First we determine the baseband signal to noise ratio $(\frac{S}{N})_b$. With $W = 1.5 \times 10^6$, we obtain

$$\left(\frac{S}{N}\right)_b = \frac{P_R}{N_0 W} = \frac{P_R}{2 \times 0.5 \times 10^{-14} \times 1.5 \times 10^6} = \frac{P_R 10^8}{1.5}$$

Since the channel attenuation is 90 db, then

$$10 \log \frac{P_T}{P_R} = 90 \Longrightarrow P_R = 10^{-9} P_T$$

Hence,

$$\left(\frac{S}{N}\right)_b = \frac{P_R 10^8}{1.5} = \frac{10^8 \times 10^{-9} P_T}{1.5} = \frac{P_T}{15}$$

1) If USSB is employed, then

$$\left(\frac{S}{N}\right)_{o,\text{USSB}} = \left(\frac{S}{N}\right)_b = 10^3 \Longrightarrow P_T = 15 \times 10^3 = 15 \text{ KWatts}$$

2) If conventional AM is used, then

$$\left(\frac{S}{N}\right)_{o,\text{AM}} = \eta \left(\frac{S}{N}\right)_b = \eta \frac{P_T}{15}$$

where, $\eta = \frac{\alpha^2 P_{M_n}}{1+\alpha^2 P_{M_n}}$. Since, $\max[|m(t)| = 1$, we have

$$P_{M_n} = P_M = \int_{-1}^{1} \frac{1}{2} x^2 dx = \frac{1}{3}$$

and, therefore

$$\eta = \frac{0.25 \times \frac{1}{3}}{1 + 0.25 \times \frac{1}{3}} = \frac{1}{13}$$

Hence,

$$\left(\frac{S}{N}\right)_{o,\text{AM}} = \frac{1}{13} \frac{P_T}{15} = 10^3 \Longrightarrow P_T = 195 \text{ KWatts}$$

3) For DSB modulation

$$\left(\frac{S}{N}\right)_{o,\text{DSB}} = \left(\frac{S}{N}\right)_b = \frac{P_T}{15} = 10^3 \Longrightarrow P_T = 15 \text{ KWatts}$$

Problem 6.5

1) Since $|H(f)| = 1$ for $f = |f_c \pm f_m|$, the signal at the output of the noise-limiting filter is

$$r(t) = 10^{-3}[1 + \alpha\cos(2\pi f_m t + \phi)]\cos(2\pi f_c t) + n(t)$$

The signal power is

$$\begin{aligned} P_R &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} 10^{-6}[1 + \alpha\cos(2\pi f_m t + \phi)]^2\cos^2(2\pi f_c t)dt \\ &= \frac{10^{-6}}{2}[1 + \frac{\alpha^2}{2}] = 56.25 \times 10^{-6} \end{aligned}$$

The noise power at the output of the noise-limiting filter is

$$P_{n,o} = \frac{1}{2}P_{n_c} = \frac{1}{2}P_n = \frac{1}{2}\frac{N_0}{2} \times 2 \times 2500 = 25 \times 10^{-10}$$

2) Multiplication of $r(t)$ by $2\cos(2\pi f_c t)$ yields

$$\begin{aligned} y(t) &= \frac{10^{-3}}{2}[1 + \alpha\cos(2\pi f_m t)]2 + \frac{1}{2}n_c(t)2 \\ &\quad + \text{double frequency terms} \end{aligned}$$

The LPF rejects the double frequency components and therefore, the output of the filter is

$$v(t) = 10^{-3}[1 + \alpha\cos(2\pi f_m t)] + n_c(t)$$

If the dc component is blocked, then the signal power at the output of the LPF is

$$P_o = \frac{10^{-6}}{2}0.5^2 = 0.125 \times 10^{-6}$$

whereas, the output noise power is

$$P_{n,o} = P_{n_c} = P_n = 2\frac{N_0}{2}2000 = 40 \times 10^{-10}$$

where we have used the fact that the lowpass filter has a bandwidth of 1000 Hz. Hence, the output SNR is

$$\text{SNR} = \frac{0.125 \times 10^{-6}}{40 \times 10^{-10}} = 31.25 \quad 14.95 \text{ db}$$

Problem 6.6

1) In the case of DSB, the output of the receiver noise-limiting filter is

$$\begin{aligned} r(t) &= u(t) + n(t) \\ &= A_c m(t)\cos(2\pi f_c t + \phi_c(t)) \\ &\quad + n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t) \end{aligned}$$

The power of the received signal is $P_s = \frac{A_c^2}{2}P_m$, whereas the power of the noise

$$P_{n,o} = \frac{1}{2}P_{n_c} + \frac{1}{2}P_{n_s} = P_n$$

Hence, the SNR at the output of the noise-limiting filter is

$$\left(\frac{S}{N}\right)_{o,\text{lim}} = \frac{A_c^2 P_m}{2P_n}$$

Assuming coherent demodulation, the output of the demodulator is

$$y(t) = \frac{1}{2}[A_c m(t) + n_c]$$

The output signal power is $P_o = \frac{1}{4}A_c^2 P_m$ whereas the output noise power

$$P_{n,o} = \frac{1}{4}P_{n_c} = \frac{1}{4}P_n$$

Hence,

$$\left(\frac{S}{N}\right)_{o,\text{dem}} = \frac{A_c^2 P_m}{P_n}$$

and the demodulation gain is given by

$$\text{dem. gain} = \frac{\left(\frac{S}{N}\right)_{o,\text{dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = 2$$

2) In the case of SSB, the output of the receiver noise-limiting filter is

$$r(t) = A_c m(t)\cos(2\pi f_c t) \pm A_c \hat{m}(t)\sin(2\pi f_c t) + n(t)$$

The received signal power is $P_s = A_c^2 P_m$, whereas the received noise power is $P_{n,o} = P_n$. At the output of the demodulator

$$y(t) = \frac{A_c}{2}m(t) + \frac{1}{2}n_c(t)$$

with $P_o = \frac{1}{4}A_c^2 P_m$ and $P_{n,o} = \frac{1}{4}P_{n_c} = \frac{1}{4}P_n$. Therefore,

$$\text{dem. gain} = \frac{\left(\frac{S}{N}\right)_{o,\text{dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = \frac{\frac{A_c^2 P_m}{P_n}}{\frac{A_c^2 P_m}{P_n}} = 1$$

3) In the case of conventional AM modulation, the output of the receiver noise-limiting filter is

$$r(t) = [A_c(1 + \alpha m_n(t)) + n_c(t)]\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

The total pre-detection power in the signal is

$$P_s = \frac{A_c^2}{2}(1 + \alpha^2 P_{M_n})$$

In this case, the demodulation gain is given by

$$\text{dem. gain} = \frac{\left(\frac{S}{N}\right)_{o,\text{dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = \frac{2\alpha^2 P_{M_n}}{1 + \alpha^2 P_{M_n}}$$

The highest gain is achieved for $\alpha = 1$, that is 100% modulation.

4) For an FM system, the output of the receiver front-end (bandwidth B_c) is

$$\begin{aligned} r(t) &= A_c \cos(2\pi f_c t + \phi(t)) + n(t) \\ &= A_c \cos(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau) d\tau) + n(t) \end{aligned}$$

The total signal input power is $P_{s,\text{lim}} = \frac{A_c^2}{2}$, whereas the pre-detection noise power is

$$P_{n,\text{lim}} = \frac{N_0}{2} 2B_c = N_0 B_c = N_0 2(\beta_f + 1)W$$

Hence,

$$\left(\frac{S}{N}\right)_{o,\text{lim}} = \frac{A_c^2}{2N_0 2(\beta_f + 1)W}$$

The output (post-detection) signal to noise ratio is

$$\left(\frac{S}{N}\right)_{o,\text{dem}} = \frac{3k_f^2 A_c^2 P_M}{2N_0 W^3}$$

Thus, the demodulation gain is

$$\text{dem. gain} = \frac{\left(\frac{S}{N}\right)_{o,\text{dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = \frac{3\beta_f^2 P_M 2(\beta_f + 1)}{(\max[|m(t)|])^2} = 6\beta_f^2(\beta_f + 1)P_{M_n}$$

5) Similarly for the PM case we find that

$$\left(\frac{S}{N}\right)_{o,\text{lim}} = \frac{A_c^2}{2N_0 2(\beta_p + 1)W}$$

and

$$\left(\frac{S}{N}\right)_{o,\text{dem}} = \frac{k_p^2 A_c^2 P_M}{2N_0 W}$$

Thus, the demodulation gain for a PM system is

$$\text{dem. gain} = \frac{\left(\frac{S}{N}\right)_{o,\text{dem}}}{\left(\frac{S}{N}\right)_{o,\text{lim}}} = \frac{\beta_p^2 P_M 2(\beta_p + 1)}{(\max[|m(t)|])^2} = 2\beta_p^2(\beta_p + 1)P_{M_n}$$

Problem 6.7

1) Since the channel attenuation is 80 db, then

$$10\log\frac{P_T}{P_R} = 80 \Longrightarrow P_R = 10^{-8}P_T = 10^{-8}\times 40\times 10^3 = 4\times 10^{-4} \text{ Watts}$$

If the noise limiting filter has bandwidth B, then the pre-detection noise power is

$$P_n = 2\int_{f_c-\frac{B}{2}}^{f_c+\frac{B}{2}}\frac{N_0}{2}df = N_0B = 2\times 10^{-10}B \text{ Watts}$$

In the case of DSB or conventional AM modulation, $B = 2W = 2\times 10^4$ Hz, whereas in SSB modulation $B = W = 10^4$. Thus, the pre-detection signal to noise ratio in DSB and conventional AM is

$$\text{SNR}_{\text{DSB,AM}} = \frac{P_R}{P_n} = \frac{4\times 10^{-4}}{2\times 10^{-10}\times 2\times 10^4} = 10^2$$

and for SSB

$$\text{SNR}_{\text{SSB}} = \frac{4\times 10^{-4}}{2\times 10^{-10}\times 10^4} = 2\times 10^2$$

2) For DSB, the demodulation gain (see Problem 5.7) is 2. Hence,

$$\text{SNR}_{\text{DSB},o} = 2\text{SNR}_{\text{DSB},i} = 2\times 10^2$$

3) The demodulation gain of a SSB system is 1. Thus,

$$\text{SNR}_{\text{SSB},o} = \text{SNR}_{\text{SSB},i} = 2\times 10^2$$

4) For conventional AM with $\alpha = 0.8$ and $P_{M_n} = 0.2$, we have

$$\text{SNR}_{\text{AM},o} = \frac{\alpha^2 P_{M_n}}{1+\alpha^2 P_{M_n}}\text{SNR}_{\text{AM},i} = 0.1135\times 2\times 10^2$$

Problem 6.8

1) For an FM system that utilizes the whole bandwidth $B_c = 2(\beta_f+1)W$, therefore

$$2(\beta_f+1) = \frac{100}{4} \Longrightarrow b_f = 11.5$$

Hence,

$$\left(\frac{S}{N}\right)_{o,\text{FM}} = 3\frac{A_c^2}{2}\left(\frac{\beta_f}{\max[|m(t)|]}\right)^2\frac{P_M}{N_0W} = 3\frac{A_c^2}{2}\beta_f^2\frac{P_{M_n}}{N_0W}$$

For an AM system

$$\left(\frac{S}{N}\right)_{o,\text{AM}} = \frac{A_c^2\alpha^2 P_{M_n}}{2N_0W}$$

Hence,

$$\frac{\left(\frac{S}{N}\right)_{o,\text{FM}}}{\left(\frac{S}{N}\right)_{o,\text{AM}}} = \frac{3\beta_f^2}{\alpha^2} = 549.139 \sim 27.3967 \text{ dB}$$

2) Since the PM and FM systems provide the same SNR

$$\left(\frac{S}{N}\right)_{o,\text{PM}} = \frac{k_p^2 A_c^2}{2} \frac{P_M}{N_0 W} = \frac{3k_f^2 A_c^2}{2W^2} \frac{P_M}{N_0 W} = \left(\frac{S}{N}\right)_{o,\text{FM}}$$

or

$$\frac{k_p^2}{3k_f^2} = \frac{1}{W^2} \Longrightarrow \frac{\beta_p^2}{3\beta_f^2 W^2} = \frac{1}{W^2}$$

Hence,

$$\frac{\text{BW}_{\text{PM}}}{\text{BW}_{\text{FM}}} = \frac{2(\beta_p + 1)W}{2(\beta_f + 1)W} = \frac{\sqrt{3}\beta_f + 1}{\beta_f + 1}$$

Problem 6.9

1) The received signal power can be found from

$$10 \log \frac{P_T}{P_R} = 80 \Longrightarrow P_R = 10^{-8} P_T = 10^{-4} \text{ Watts}$$

$$\left(\frac{S}{N}\right)_o = \frac{\alpha^2 P_{M_n}}{1 + \alpha^2 P_{M_n}} \left(\frac{S}{N}\right)_b = \frac{\alpha^2 P_{M_n}}{1 + \alpha^2 P_{M_n}} \frac{P_R}{N_0 W}$$

Thus, with $P_R = 10^{-4}$, $P_{M_n} = 0.1$, $\alpha = 0.8$ and

$$N_0 W = 2 \times 0.5 \times 10^{-12} \times 5 \times 10^3 = 5 \times 10^{-9}$$

we find that

$$\left(\frac{S}{N}\right)_o = 1204 \quad 30.806 \text{ db}$$

2) Using Carson's rule, we obtain

$$B_c = 2(\beta + 1)W \Longrightarrow 100 \times 10^3 = 2(\beta + 1) 5 \times 10^3 \Longrightarrow \beta = 9$$

We check now if the threshold imposes any restrictions.

$$\left(\frac{S}{N}\right)_{b,th} = \frac{P_R}{N_0 W} = 20(\beta + 1) = \frac{10^{-4}}{10^{-12} \times 5 \times 10^3} \Longrightarrow \beta = 999$$

Since we are limited in bandwidth we choose $\beta = 9$. The output signal to noise ratio is

$$\left(\frac{S}{N}\right)_o = 3\beta^2 0.1 \left(\frac{S}{N}\right)_b = 3 \times 9^2 \times 0.1 \times \frac{10^5}{5} = 486000 \quad 56.866 \text{ db}$$

Problem 6.10

1) First we check whether the threshold or the bandwidth impose a restrictive bound on the modulation index. By Carson's rule

$$B_c = 2(\beta+1)W \Longrightarrow 60\times 10^3 = 2(\beta+1)\times 8\times 10^3 \Longrightarrow \beta = 2.75$$

Using the relation

$$\left(\frac{S}{N}\right)_o = 60\beta^2(\beta+1)P_{M_n}$$

with $\left(\frac{S}{N}\right)_o = 10^4$ and $P_{M_n} = \frac{1}{2}$ we find

$$10^4 = 30\beta^2(\beta+1) \Longrightarrow \beta = 6.6158$$

Since we are limited in bandwidth we choose $\beta = 2.75$. Then,

$$\left(\frac{S}{N}\right)_o = 3\beta^2 P_{M_n}\left(\frac{S}{N}\right)_b \Longrightarrow \left(\frac{S}{N}\right)_b = \frac{2\times 10^4}{3\times 2.75^2} = 881.542$$

Thus,

$$\left(\frac{S}{N}\right)_b = \frac{P_R}{N_0 W} = 881.542 \Longrightarrow P_R = 881.542\times 2\times 10^{-12}\times 8\times 10^3 = 1.41\times 10^{-5}$$

Since the channel attenuation is 40 db, we find

$$P_T = 10^4 P_R = 0.141 \text{ Watts}$$

2) If the minimum required SNR is increased to 60 db, then the β from Carson's rule remains the same, whereas from the relation

$$\left(\frac{S}{N}\right)_o = 60\beta^2(\beta+1)P_{M_n} = 10^6$$

we find $\beta = 31.8531$. As in part 1) we choose $\beta = 2.75$, and therefore

$$\left(\frac{S}{N}\right)_b = \frac{1}{3\beta^2 P_{M_n}}\left(\frac{S}{N}\right)_o = 8.8154\times 10^4$$

Thus,

$$P_R = N_0 W 8.8154\times 10^4 = 2\times 10^{-12}\times 8\times 10^3\times 8.8154\times 10^4 = 0.0014$$

and

$$P_T = 10^4 P_R = 14 \text{ Watts}$$

3) The frequency response of the receiver (de-emphasis) filter is given by

$$H_d(f) = \frac{1}{1+j\frac{f}{f_0}}$$

with $f_0 = \frac{1}{2\pi\times 75\times 10^{-6}} = 2100$ Hz. In this case,

$$\left(\frac{S}{N}\right)_{o,\text{PD}} = \frac{(\frac{W}{f_0})^3}{3\left(\frac{W}{f_0} - \arctan\frac{W}{f_0}\right)}\left(\frac{S}{N}\right)_o = 10^6$$

From this relation we find

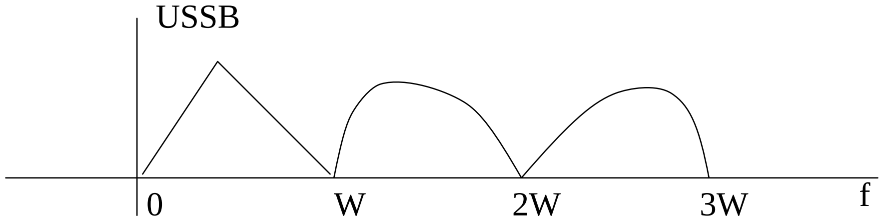
$$\left(\frac{S}{N}\right)_o = 1.3541\times 10^5 \Longrightarrow P_R = 9.55\times 10^{-5}$$

and therefore,

$$P_T = 10^4 P_R = 0.955 \text{ Watts}$$

Problem 6.11

1) In the next figure we plot a typical USSB spectrum for $K = 3$. Note that only the positive frequency axis is shown.



2) The bandwidth of the signal $m(t)$ is $W_m = KW$.

3) The noise power at the output of the LPF of the FM demodulator is

$$P_{n,o} = \int_{-W_m}^{W_m} S_{n,o}(f)df = \frac{2N_0W_m^3}{3A_c^2} = \frac{2N_0W^3}{3A_c^2}K^3$$

where A_c is the amplitude of the FM signal. As it is observed the power of the noise that enters the USSB demodulators is proportional to the cube of the number of multiplexed signals.

The i^{th} message USSB signal occupies the frequency band $[(i-1)W, iW]$. Since the power spectral density of the noise at the output of the FM demodulator is $S_{n,o}(f) = \frac{N_0}{A_c^2}f^2$, we conclude that the noise power at the output of the i^{th} USSB demodulator is

$$P_{n,o_i} = \frac{1}{4}P_{n_i} = \frac{1}{4}2\int_{-(i-1)W}^{iW}\frac{N_0}{A_c^2}f^2df = \frac{N_0}{2A_c^2}\frac{1}{3}f^3\Big|_{-(i-1)W}^{iW} = \frac{N_0W^3}{6A_c^2}(3i^2-3i+1)$$

Hence, the noise power at the output of the i^{th} USSB demodulator depends on i.

4) Using the results of the previous part, we obtain

$$\frac{P_{n,o_i}}{P_{n,o_j}} = \frac{3i^2-3i+1}{3j^2-3j+1}$$

5) The output signal power of the i^{th} USSB demodulator is $P_{s_i} = \frac{A_i^2}{4} P_{M_i}$. Hence, the SNR at the output of the i^{th} demodulator is

$$\text{SNR}_i = \frac{\frac{A_i^2}{4} P_{M_i}}{\frac{N_0 W^3}{6A_c^2}(3i^2 - 3i + 1)}$$

Assuming that P_{M_i} is the same for all i, then in order to guarantee a constant SNR_i we have to select A_i^2 proportional to $3i^2 - 3i + 1$.

Problem 6.12

1) The power is given by

$$P = \frac{V^2}{R}$$

Hence, with $R = 50$, $P = 20$, we obtain

$$V^2 = PR = 20 \times 50 = 1000 \Longrightarrow V = 1000^{\frac{1}{2}} = 31.6228 \text{ Volts}$$

2) The current through the load resistance is

$$i = \frac{V}{R} = \frac{31.6228}{50} = 0.6325 \text{ Amp}$$

3) The dBm unit is defined as

$$\text{dBm} = 10 \log\left(\frac{\text{actual power in Watts}}{10^{-3}}\right) = 30 + 10 \log(\text{actual power in Watts})$$

Hence,

$$P = 30 + 10 \log(50) = 46.9897 \text{ dBm}$$

Problem 6.13

1) The overall loss in 200 Km is $200 \times 20 = 400$ dB. Since the line is loaded with the characteristic impedance, the delivered power to the line is twice the power delivered to the load in absence of line loss. Hence, the required power is $20 + 400 = 420$ dBm.

2) Each repeater provides a gain of 20 dB, therefore the spacing between two adjacent receivers can be up to $20/2 = 10$ Km to attain a constant signal level at the input of all repeaters. This means that a total of $200/10 = 20$ repeaters are required.

Problem 6.14

1) Since the noise figure is 2 dB, we have

$$10 \log \left(1 + \frac{\mathcal{T}_e}{290}\right) = 2$$

and therefore $\mathcal{T}_e = 169.62°$ K.

2) To determine the output power we have

$$P_{no} = \mathcal{G} k B_{\text{neq}} (\mathcal{T} + \mathcal{T}_e)$$

where $10 \log \mathcal{G} = 35$, and therefore, $\mathcal{G} = 10^3.5 = 3162$. From this we obtain

$$P_{no} = 3162 \times 1.38 \times 10^{-23} \times 10 \times 10^6 (169.62 + 50) = 9.58 \times 10^{-11} \quad \text{Watts} \sim -161.6 \quad \text{dBm}$$

Problem 6.15

Using the relation $P_{n_o} = \mathcal{G} k B_{neq} (\mathcal{T} + \mathcal{T}_e)$ with $P_{n_o} = 10^8 k T_0$, $B_{neq} = 25 \times 10^3$, $\mathcal{G} = 10^3$ and $\mathcal{T} = T_0$, we obtain

$$(10^8 - 25 \times 10^6) T_0 = 25 \times 10^6 \mathcal{T}_e \Longrightarrow \mathcal{T}_e = 3T_0$$

The noise figure of the amplifier is

$$F = \left(1 + \frac{\mathcal{T}_e}{\mathcal{T}}\right) = 1 + 3 = 4$$

Problem 6.16

The proof is by induction on m, the number of the amplifiers. We assume that the physical temperature $\mathcal{T}$ is the same for all the amplifiers. For $m = 2$, the overall gain of the cascade of the two amplifiers is $\mathcal{G} = \mathcal{G}_1 \mathcal{G}_2$, whereas the total noise at the output of the second amplifier is due to the source noise amplified by two stages, the first stage noise excess noise amplified by the second stage, and the second stage excess noise. Hence,

$$\begin{aligned} P_{n_2} &= \mathcal{G}_1 \mathcal{G}_2 P_{n_s} + \mathcal{G}_2 P_{n_i,1} + P_{n_i,2} \\ &= \mathcal{G}_1 \mathcal{G}_2 k \mathcal{T} B_{neq} + \mathcal{G}_2 (\mathcal{G}_1 k B_{neq} \mathcal{T}_{e_1}) + \mathcal{G}_2 k B_{neq} \mathcal{T}_{e_2} \end{aligned}$$

The noise of a single stage model with effective noise temperature $\mathcal{T}_e$, and gain $\mathcal{G}_1 \mathcal{G}_2$ is

$$P_n = \mathcal{G}_1 \mathcal{G}_2 k B_{neq} (\mathcal{T} + \mathcal{T}_e)$$

Equating the two expressions for the output noise we obtain

$$\mathcal{G}_1 \mathcal{G}_2 (\mathcal{T} + \mathcal{T}_e) = \mathcal{G}_1 \mathcal{G}_2 \mathcal{T} + \mathcal{G}_1 \mathcal{G}_2 \mathcal{T}_{e_1} + \mathcal{G}_2 \mathcal{T}_{e_2}$$

or

$$\mathcal{T}_e = \mathcal{T}_{e_1} + \frac{\mathcal{T}_{e_2}}{\mathcal{G}_1}$$

Assume now that if the number of the amplifiers is $m-1$, then

$$\mathcal{T}'_e = \mathcal{T}_{e_1} + \frac{\mathcal{T}_{e_2}}{\mathcal{G}_1} + \cdots \frac{\mathcal{T}_{e_{m-1}}}{\mathcal{G}_1 \cdots \mathcal{G}_{m-2}}$$

Then for the cascade of m amplifiers

$$\mathcal{T}_e = \mathcal{T}'_e + \frac{\mathcal{T}_{e_m}}{\mathcal{G}'}$$

where $\mathcal{G}' = \mathcal{G}_1 \cdots \mathcal{G}_{m-1}$ is the gain of the $m-1$ amplifiers and we have used the results for $m = 2$. Thus,

$$\mathcal{T}_e = \mathcal{T}_{e_1} + \frac{\mathcal{T}_{e_2}}{\mathcal{G}_1} + \cdots \frac{\mathcal{T}_{e_{m-1}}}{\mathcal{G}_1 \cdots \mathcal{G}_{m-2}} + \frac{\mathcal{T}_{e_m}}{\mathcal{G}_1 \cdots \mathcal{G}_{m-1}}$$

Proof of Fries formula follows easily with the substitution $F_k = \left(1 + \frac{\mathcal{T}_{ek}}{\mathcal{T}_0}\right)$ into the above equation.

Computer Problems

Computer Problem 6.1

The plot of the message signal $m(t)$ and the modulated signal $u(t)$ are shown in Figures 6.1 and 6.2. Also Figures 6.3, 6.4 and 6.5 illustrate the modulated signal $\{r(n)\}$with various channel noise values of σ: $\sigma = 0.1$, $\sigma = 1$, and $\sigma = 2$, respectively.

We design a linear lowpass filter with 31 taps, cutoff frequency (-3 dB) of 100 Hz and a stopband attenuation of at least 30 dB. The frequency responce of the filter is given in Figure 6.6. The demodulated signals for different values of noise are shown in Figures 6.7, 6.8 and 6.9.

The FIR filter introduces a short delay on demodulated signal. Therefore, and in order to determine the signal to noise ratio at the output of the demodulator, one must consider this delay. The signal to noise ratio for different values of the σ: $\sigma = 0.1$, $\sigma = 1$, and $\sigma = 2$ are SNR=-3.8027 dB, -7.6224 dB and -11.842 dB, respectively.

The MATLAB script for this question is given next.

```
% MATLAB script for Computer Problem 6.1.
% Matlab demonstration script for DSB-AM modulation. The message signal
% is m(t)=sinc(100t).
echo on
t0=.1;                     % signal duration
ts=0.0001;                 % sampling interval
fc=250;                    % carrier frequency
fs=1/ts;                   % sampling frequency

t=[0:ts:t0-ts];            % time vector
m=sinc(100*t);             % the message signal
```

```
c=cos(2*pi*fc.*t);              % the carrier signal
u=m.*c;                         % the DSB-AM modulated signal

Wc=randn(1, 1000);
Ws=randn(1, 1000);
sgma = 0.1;
r_01 = u+sgma*(Wc.*cos(2*pi*fc*t)- Ws.*sin(3*pi*fc*t));
sgma = 1;
r_1 = u+sgma*(Wc.*cos(2*pi*fc*t)- Ws.*sin(3*pi*fc*t));
sgma = 2;
r_2 = u+sgma*(Wc.*cos(2*pi*fc*t)- Ws.*sin(3*pi*fc*t));

f_cutoff=100;                   % the desired cutoff frequency
f_stopband=555;                 % the actual stopband frequency
fs=10000;                       % the sampling frequency
f1=2*f_cutoff/fs;               % the normalized passband frequency
f2=2*f_stopband/fs;             % the normalized stopband frequency
N=32;                           % This number is found by experiment.
F=[0 f1 f2 1];
M=[1 1 0 0];                    % describes the lowpass filter
B=remez(N-1,F,M);               % returns the FIR tap coefficients
[H,W]=freqz(B);
```

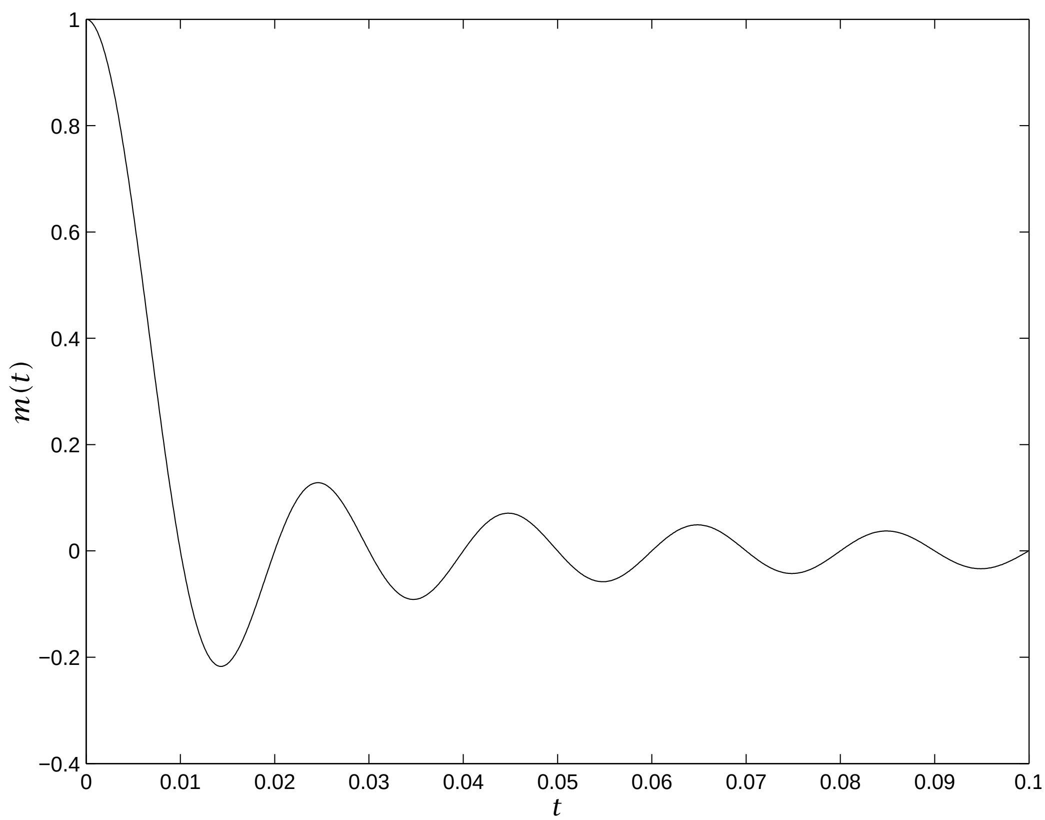


Figure 6.1: The message signal

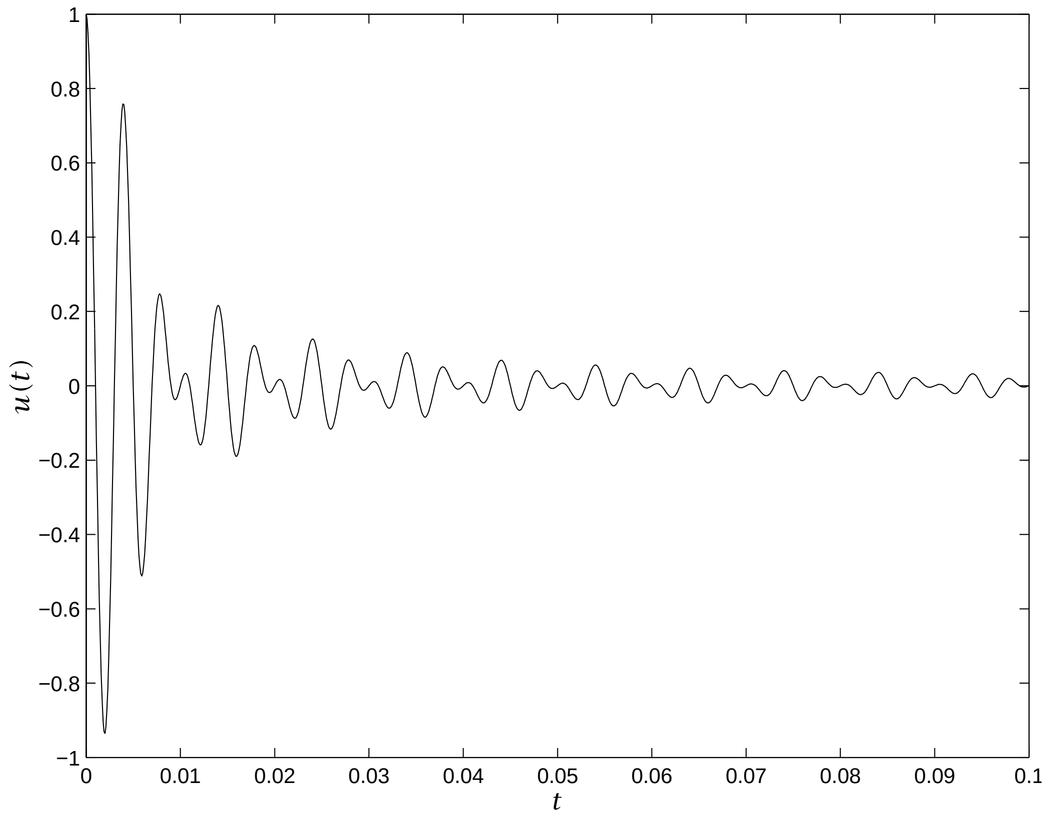


Figure 6.2: The modulated signal $u(t)$

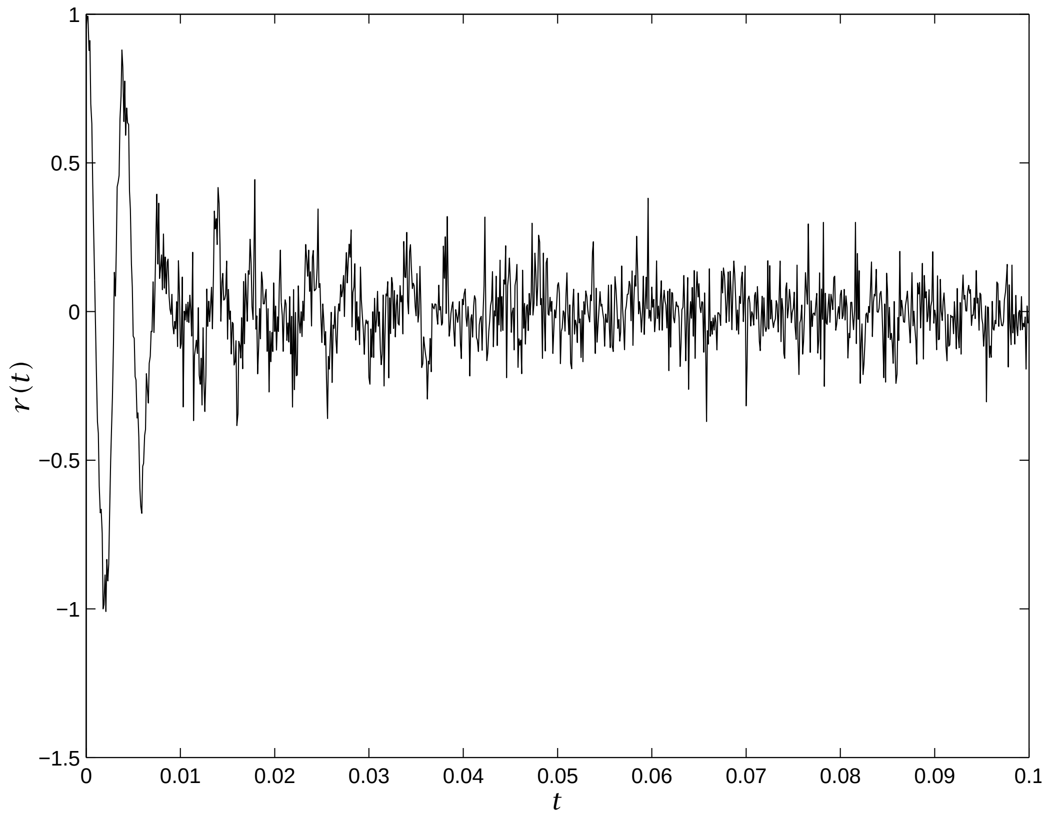


Figure 6.3: The modulated signal with noise $\sigma = 0.1$

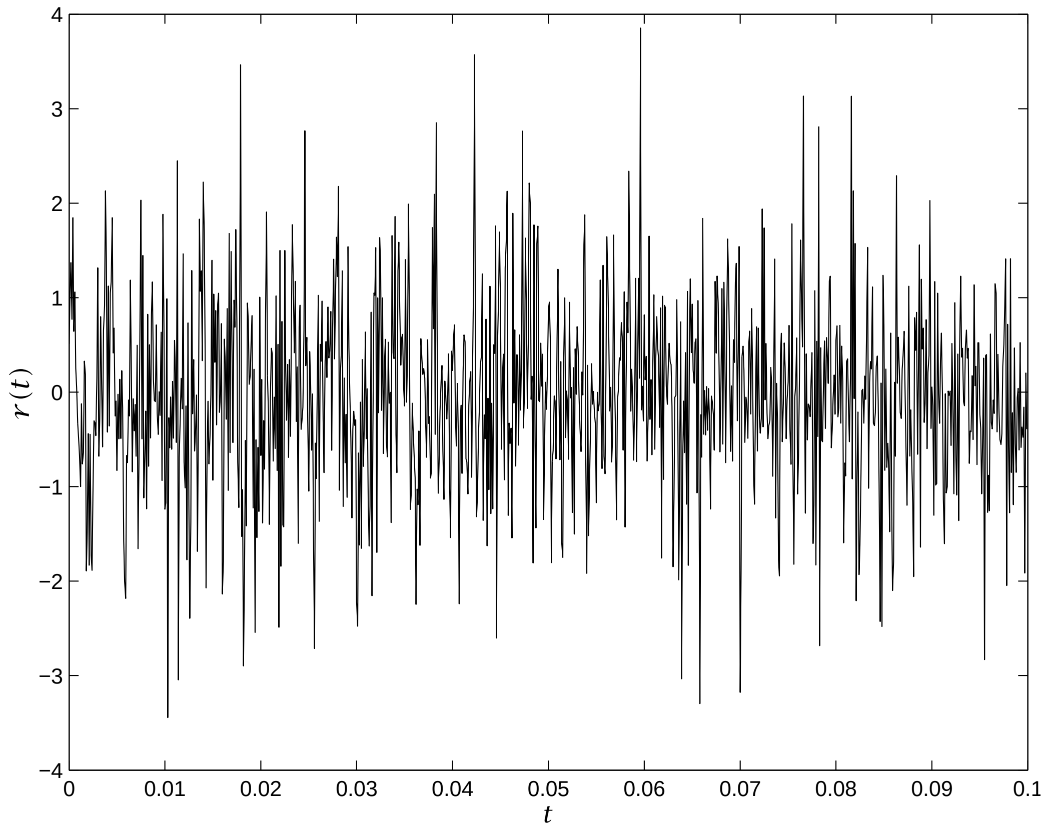


Figure 6.4: The modulated signal with noise $\sigma = 1$

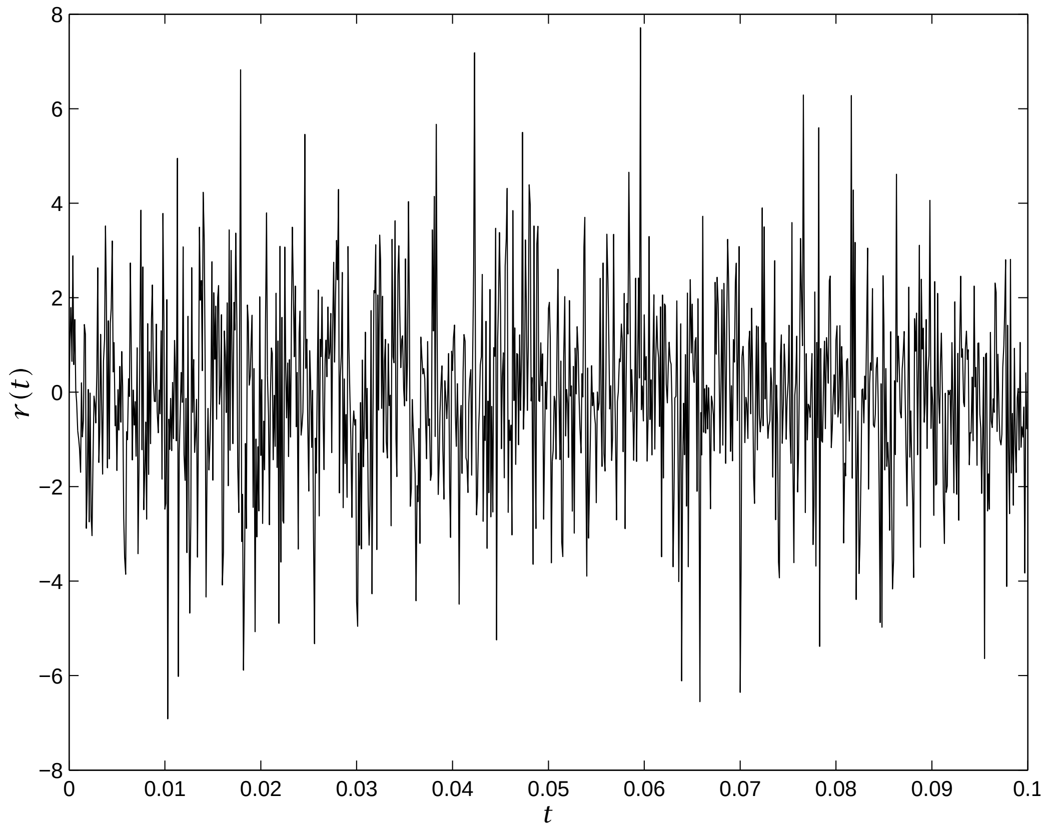


Figure 6.5: The modulated signal with noise $\sigma = 2$

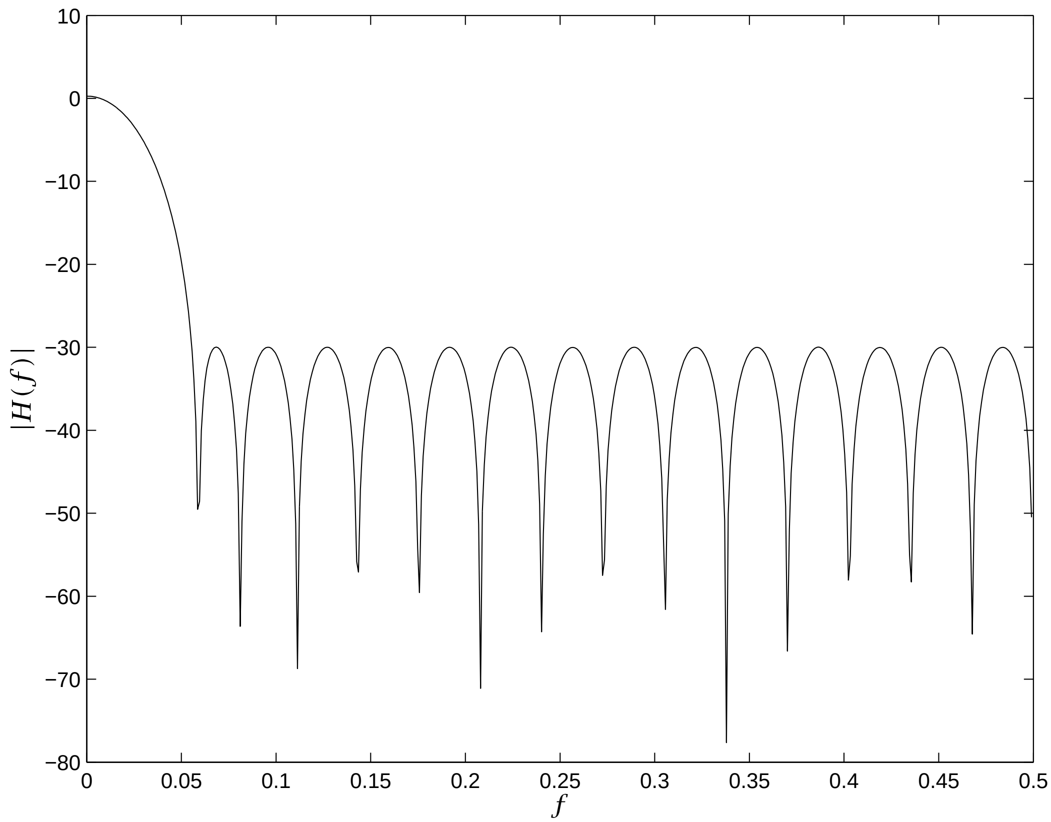


Figure 6.6: Frequency response of the linear lowpass filter

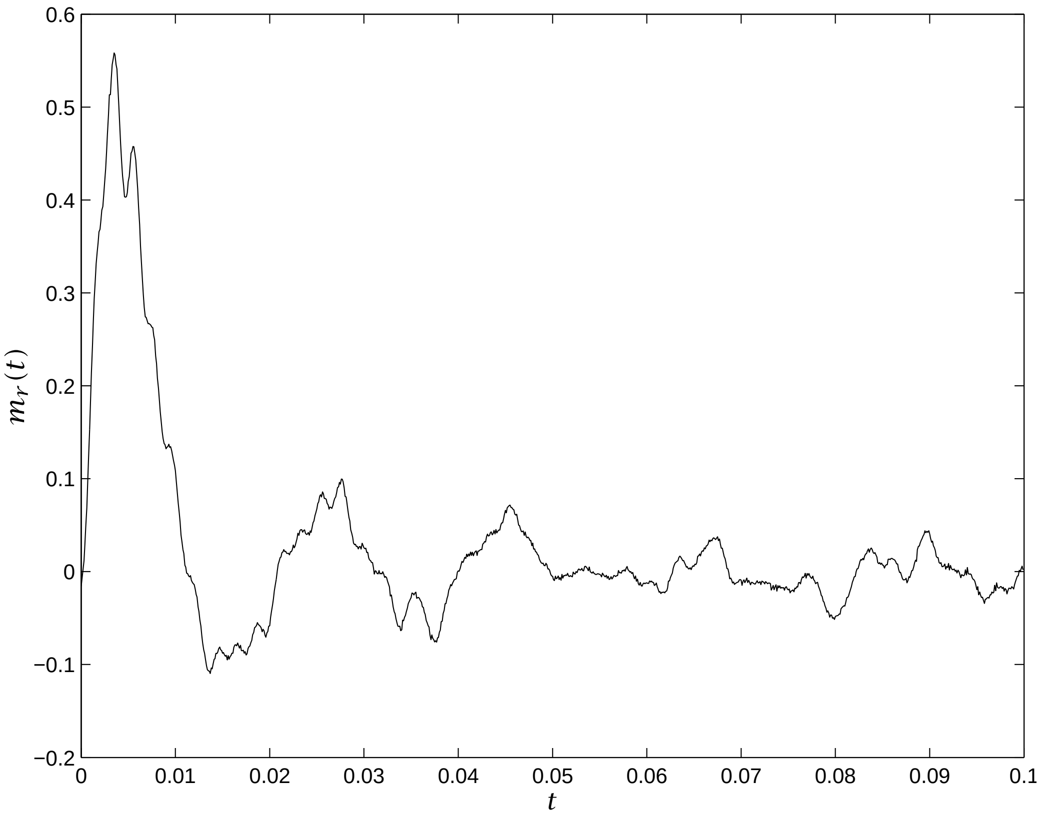


Figure 6.7: The demodulated signal with noise $\sigma = 0.1$

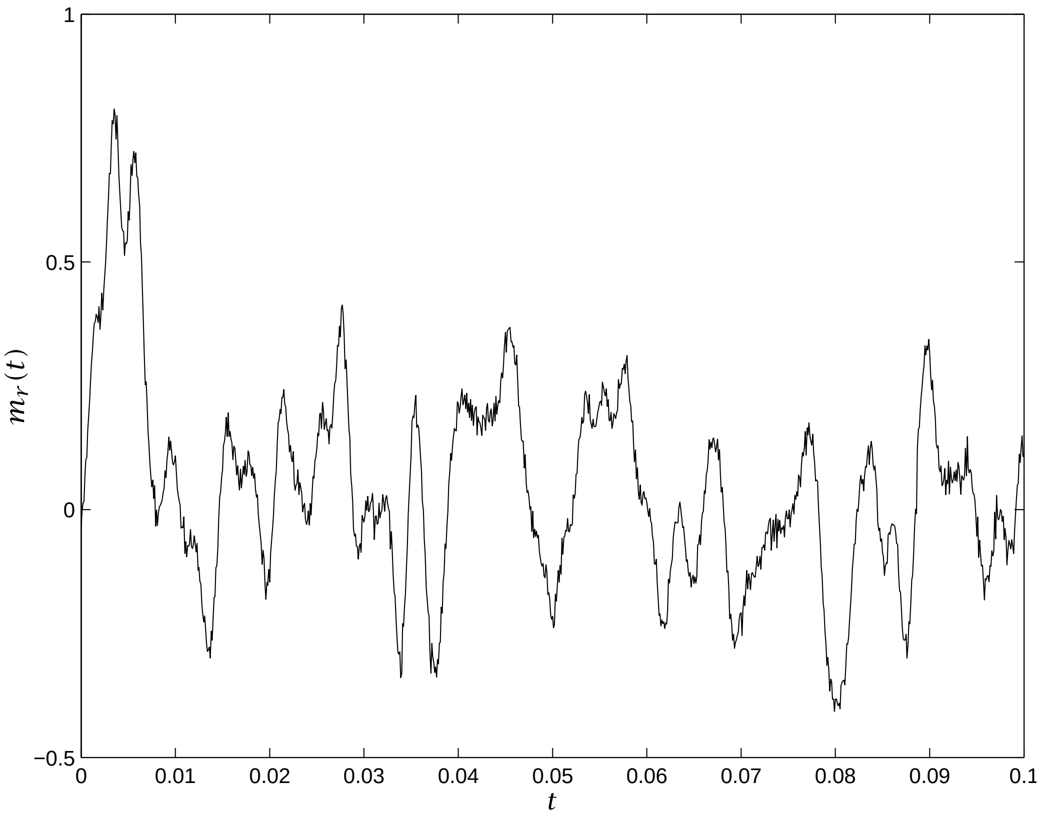


Figure 6.8: The demodulated signal with noise $\sigma = 1$

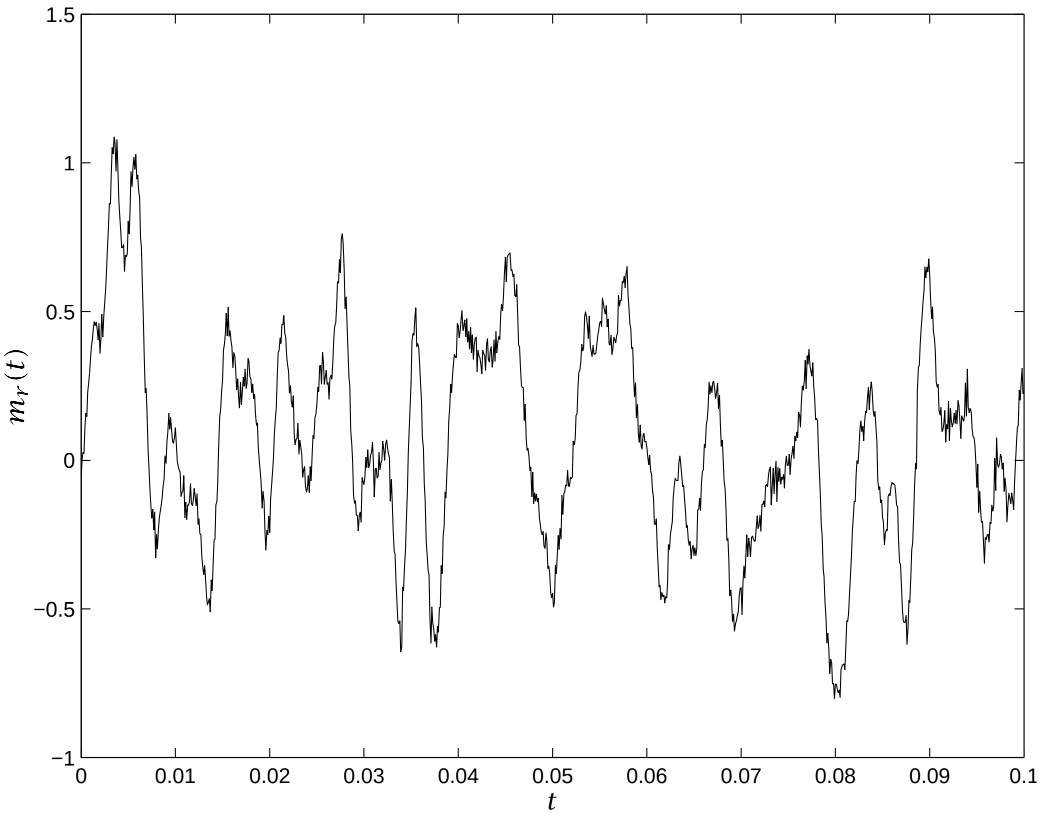


Figure 6.9: The demodulated signal with noise $\sigma = 2$

```
H_in_dB=20*log10(abs(H));

r_m_01 = filter(B, 1, r_01.*c);
r_m_1 = filter(B, 1, r_1.*c);
r_m_2 = filter(B, 1, r_2.*c);

delayed_u = filter(B, 1, u);
p01=10*log10(spower(delayed_u)/spower(r_m_01−delayed_u));
p1 =10*log10(spower(delayed_u)/spower(r_m_1−delayed_u));
p2 =10*log10(spower(delayed_u)/spower(r_m_2−delayed_u));
% Plotting command follows.
```

Computer Problem 6.2

The message signal $m(t)$ is similar to the message signal in previous problem, which is shown in Figure 6.1. The modulated signal $u(t)$ is shown in Figure 6.10. Also Figures 6.11, 6.12 and 6.13 illustrate the modulated signal $\{r(n)\}$with various channel noise values of σ: $\sigma = 0.1$, $\sigma = 1$, and $\sigma = 2$, respectively.

We design a linear lowpass filter with 31 taps, cutoff frequency (-3 dB) of 100 Hz and a stopband attenuation of at least 30 dB. The frequency responce of the filter is given in Figure 6.6. The demodulated signals for different values of noise are shown in Figure 6.14.

The FIR filter introduces a short delay on demodulated signal. Therefore, and in order to determine the signal to noise ratio at the output of the demodulator, one must consider this delay. The signal to noise ratio for different values of the σ: $\sigma = 0.1$, $\sigma = 1$, and $\sigma = 2$ are SNR=-3.8027 dB, -7.6224 dB and -11.842 dB, respectively.

The MATLAB script for this question is given next.

```
% MATLAB script for Computer Problem 6.2.
% Matlab demonstration script for SSB-AM modulation. The message signal
% is m(t)=sinc(100t).
echo on
t0=.1;                          % signal duration
ts=0.0001;                      % sampling interval
fc=250;                         % carrier frequency
fs=1/ts;                        % sampling frequency
df=1;
t=[0:ts:t0−ts];                 % time vector
m=sinc(100*t);                  % the message signal
c=cos(2*pi*fc.*t);              % the carrier signal
udsb=m.*c;                                % DSB modulated signal
[UDSB,udssb,df1]=fftseq(udsb,ts,df);      % Fourier transform
UDSB=UDSB/fs;                             % scaling
f=[0:df1:df1*(length(udssb)−1)]−fs/2;     % frequency vector
n2=ceil(fc/df1);                          % location of carrier in freq. vector
% Remove the upper sideband from DSB.
UDSB(n2:length(UDSB)−n2)=zeros(size(UDSB(n2:length(UDSB)−n2)));
ULSSB=UDSB;                               % Generate LSSB-AM spectrum.
[M,m,df1]=fftseq(m,ts,df);                % Fourier transform
M=M/fs;                                   % scaling
```

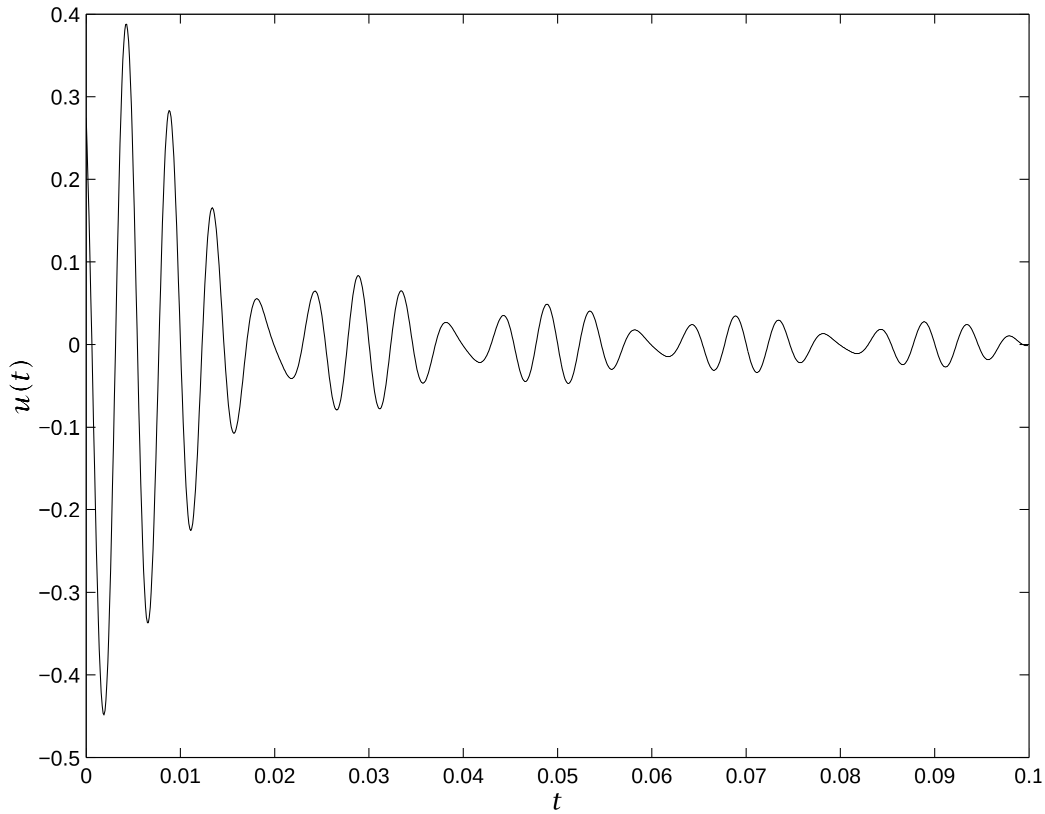


Figure 6.10: The modulated signal $u(t)$ for SSB-AM signal

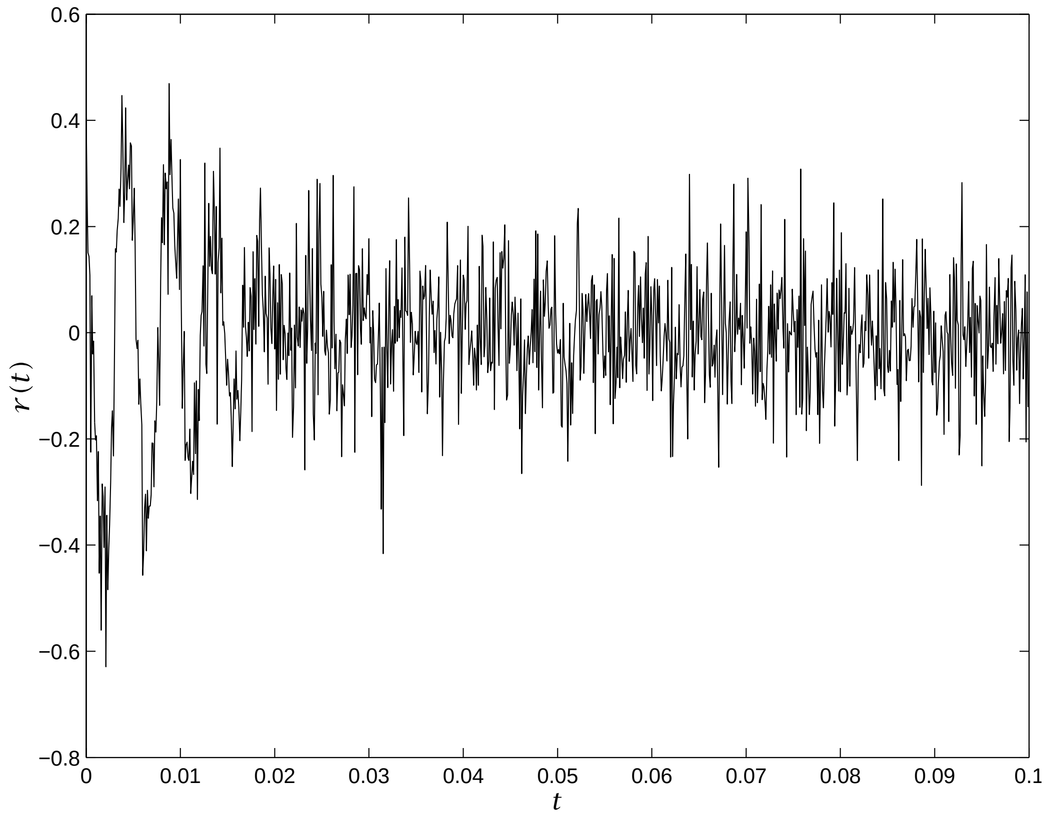


Figure 6.11: The SSB-AM modulated signal with noise $\sigma = 0.1$

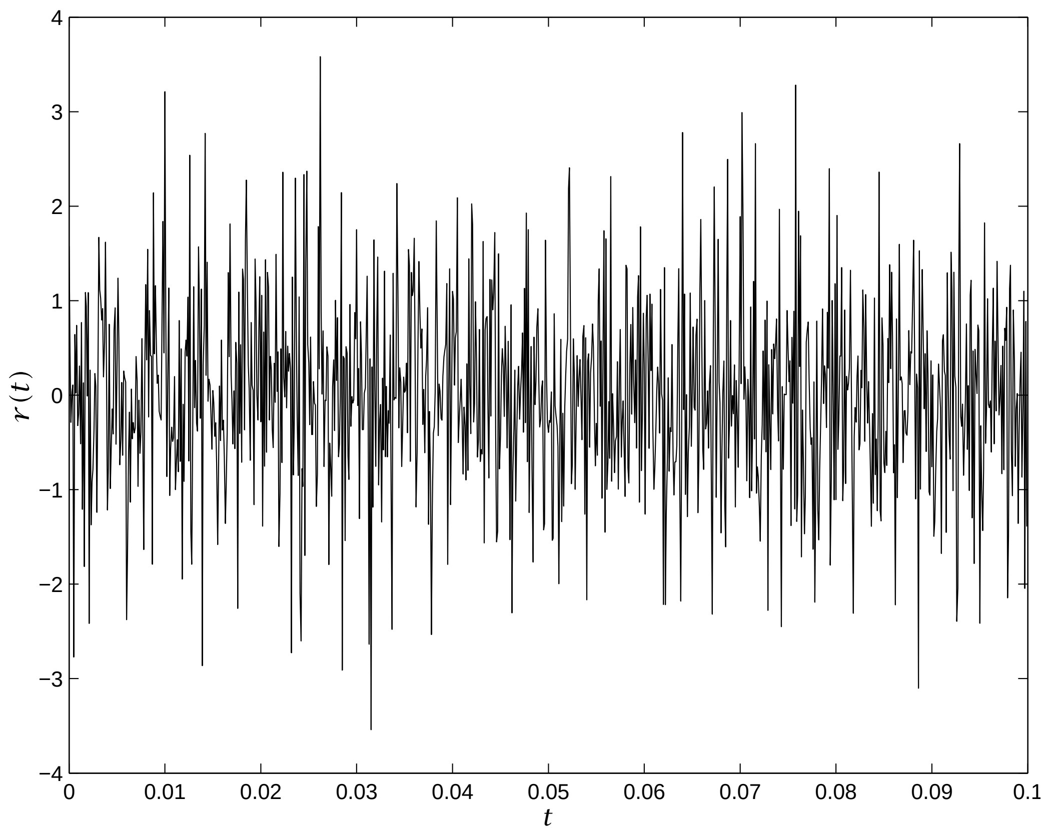


Figure 6.12: The SSB-AM modulated signal with noise $\sigma = 1$

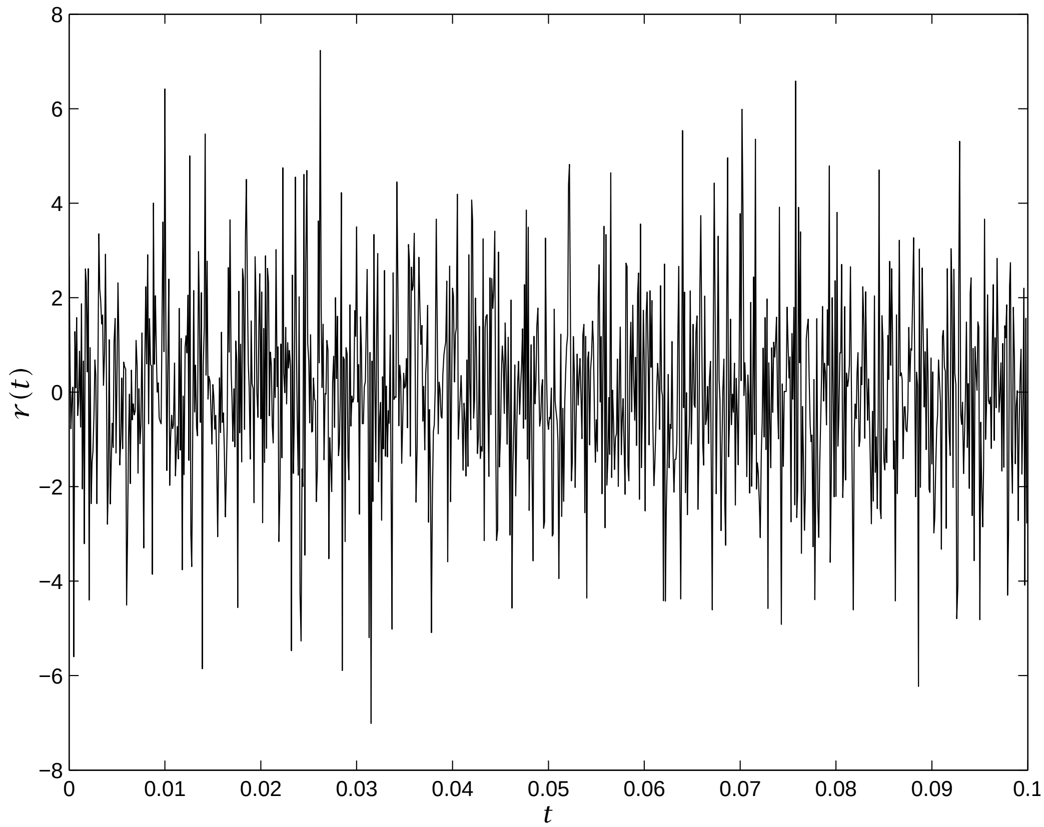


Figure 6.13: The SSB-AM modulated signal with noise $\sigma = 2$

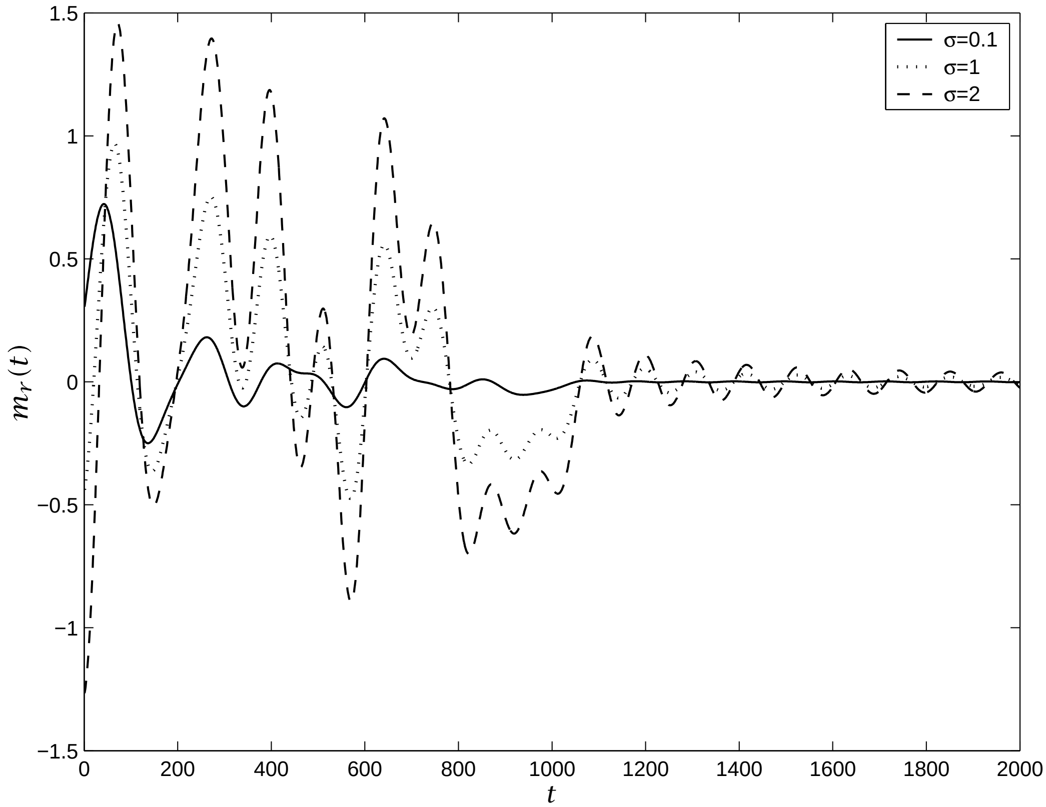


Figure 6.14: The SSB-AM demodulated signals with noise $\sigma = 0.1$, $\sigma = 1$ and $\sigma = 2$

```
u1=real(ifft(ULSSB))*fs;                % Generate LSSB signal from spectrum.
u=u1(1:length(t));
signal_power=spower(udsb(1:length(t)))/2;
Wc=randn(1, 1000);
Ws=randn(1, 1000);
sgma = 1;
r_1 = u+sgma*(Wc.*cos(2*pi*fc*t)- Ws.*sin(3*pi*fc*t));
y1=r_1.*cos(2*pi*fc*[0:ts:ts*(length(u)-1)]);
[Y1,y1,df1]=fftseq(y1,ts,df);           % spectrum of the output of the mixer
Y1=Y1/fs;                                         % scaling
f_cutoff=150;                               % Choose the cutoff freq. of the filter.
n_cutoff=floor(150/df);                 % Design the filter.
H=zeros(size(f));
H(1:n_cutoff)=4*ones(1,n_cutoff);
% spectrum of the filter output
H(length(f)-n_cutoff+1:length(f))=4*ones(1,n_cutoff);
DEM1=H.*Y1;                                    % spectrum of the filter output
dem1=real(ifft(DEM1))*fs;                 % filter output
```

Computer Problem 6.3

Figures 6.15 and 6.16 show the message signal $m(t)$ and the modulated signal $u(t)$. Also Figures 6.17, 6.18 and 6.19 illustrate the modulated signals $\{r(n)\}$with various channel noise values of σ: $\sigma = 0.1$, $\sigma = 1$, and $\sigma = 2$, respectively.

The demodulated signals for different values of noise are shown in Figures 6.20, 6.21 and 6.22. The MATLAB script for this question is given next.

```
% MATLAB script for Computer Problem 6.3.
% Demonstration script for DSB-AM modulation. The message signal
% is sinc(100t) for 0 < t < t0 and zero otherwise.
echo on
t0=.1;                          % signal duration
ts=0.0001;                      % sampling interval
fc=250;                         % carrier frequency
a=0.80;                         % modulation index
fs=1/ts;                        % sampling frequency
t=[0:ts:t0-ts];                   % time vector
% message signal
m=sinc(100*t);
m_n=m/max(abs(m));              % normalized message signal
c=cos(2*pi*fc.*t);              % carrier signal
u=(1+a*m_n).*c;                 % modulated signal
figure;plot(t, m);xlabel('Time');
figure;plot(t, u);xlabel('Time');

Wc=randn(1, 1000);
Ws=randn(1, 1000);
sgma = 1;
r_1 = u+sgma*(Wc.*cos(2*pi*fc*t)- Ws.*sin(3*pi*fc*t));
e_1 = sqrt((1+ a.*m+ sgma.*Wc).^2 + (sgma.*Ws).^2);
```

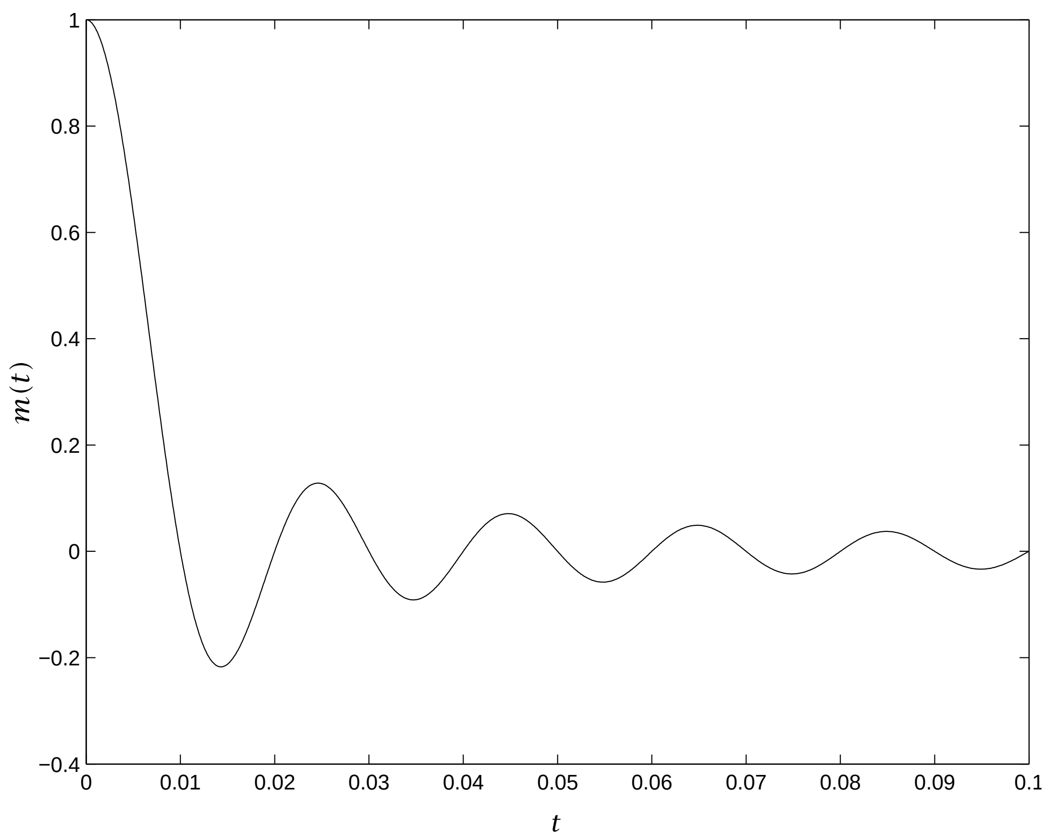


Figure 6.15: The message signal $m(t)$

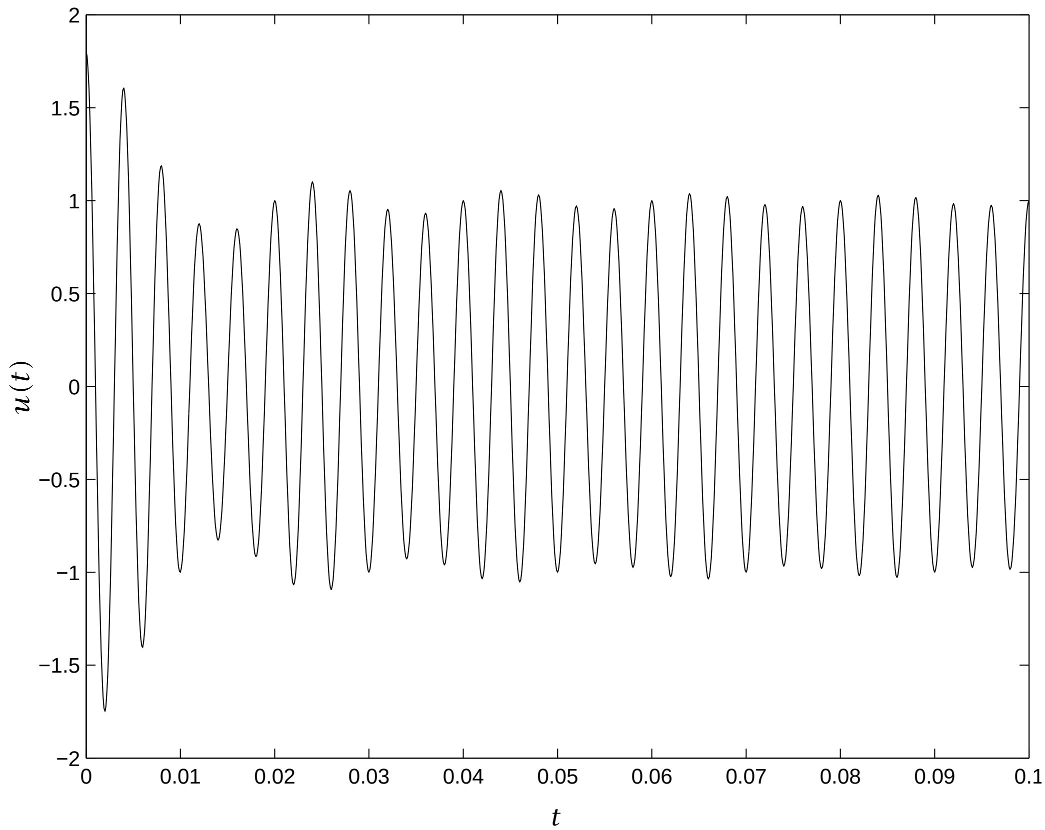


Figure 6.16: The AM modulated signal $u(t)$

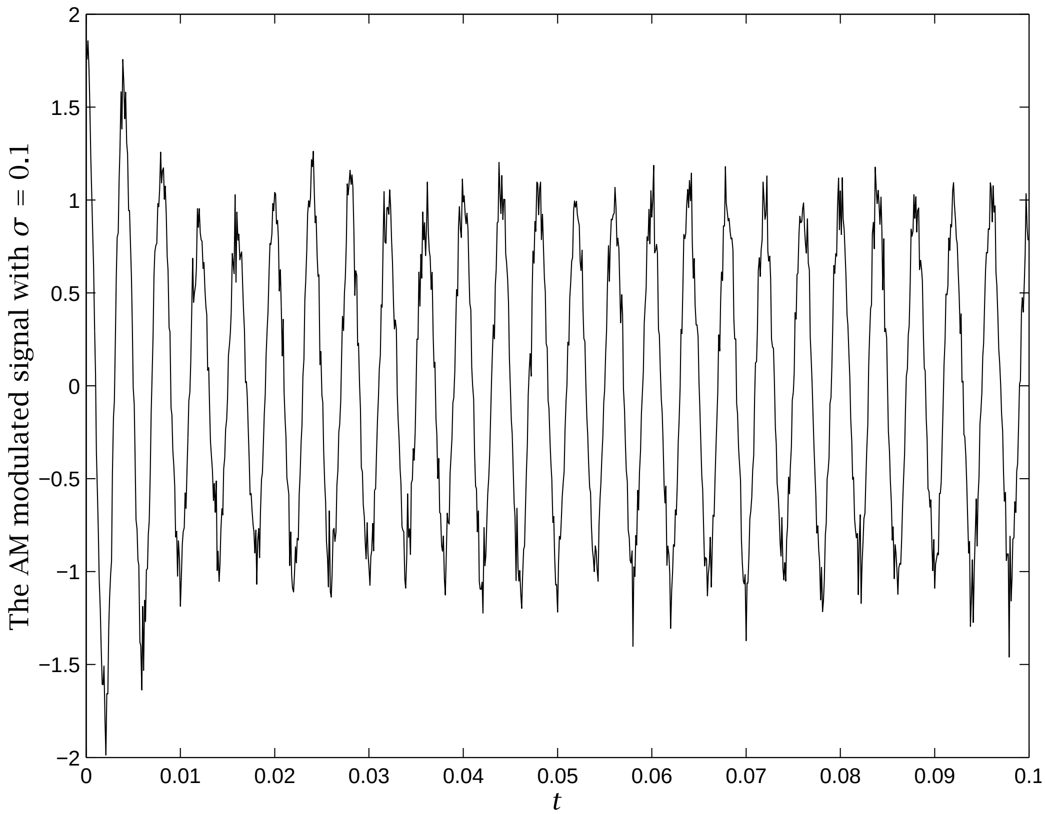


Figure 6.17: The AM modulated signal with noise $\sigma = 0.1$

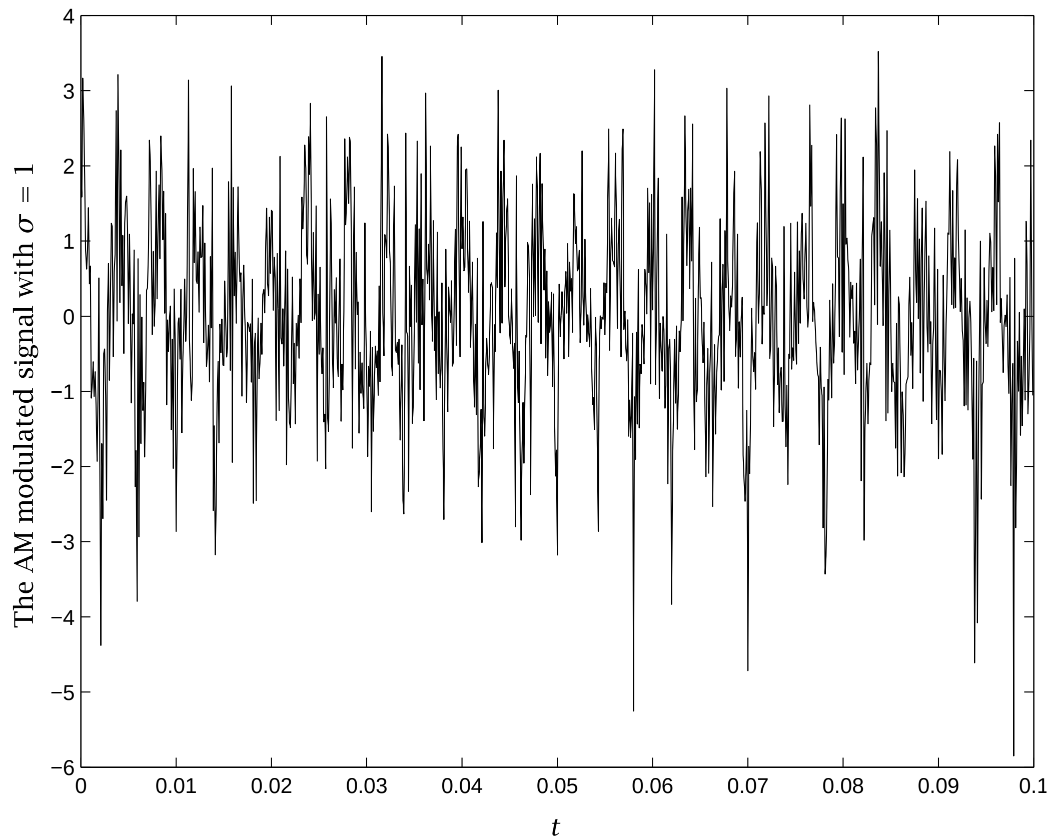


Figure 6.18: The AM modulated signal with noise $\sigma = 1$

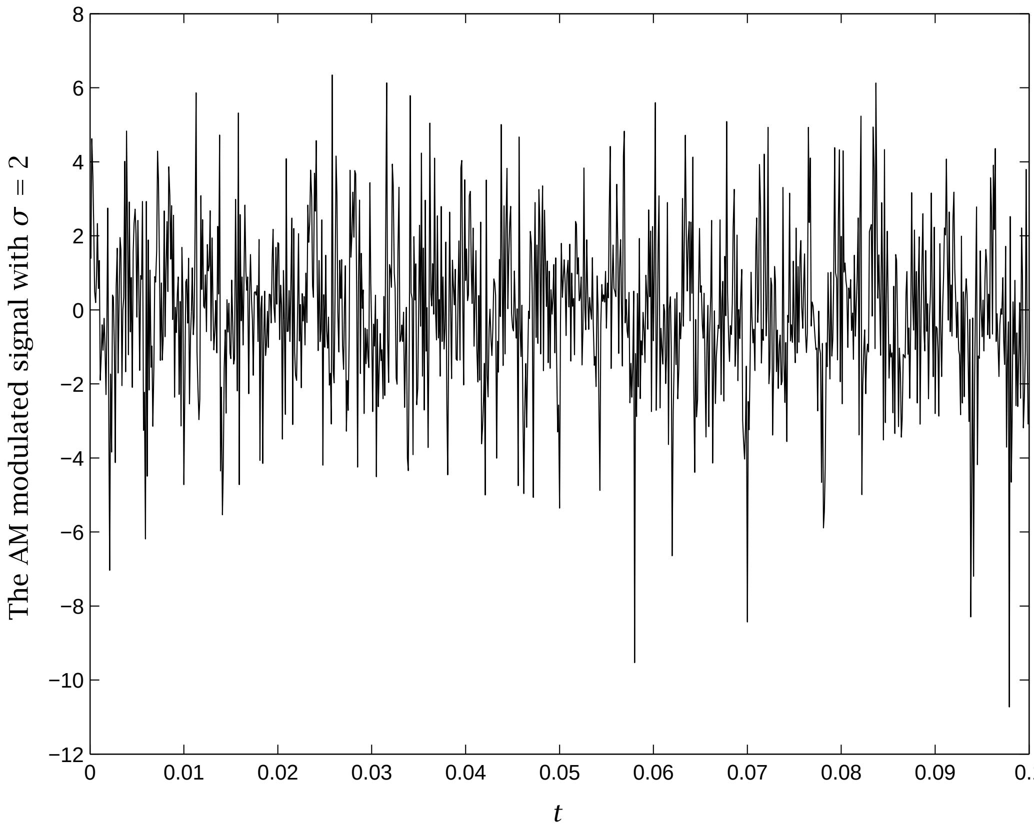


Figure 6.19: The AM modulated signal with noise $\sigma = 2$

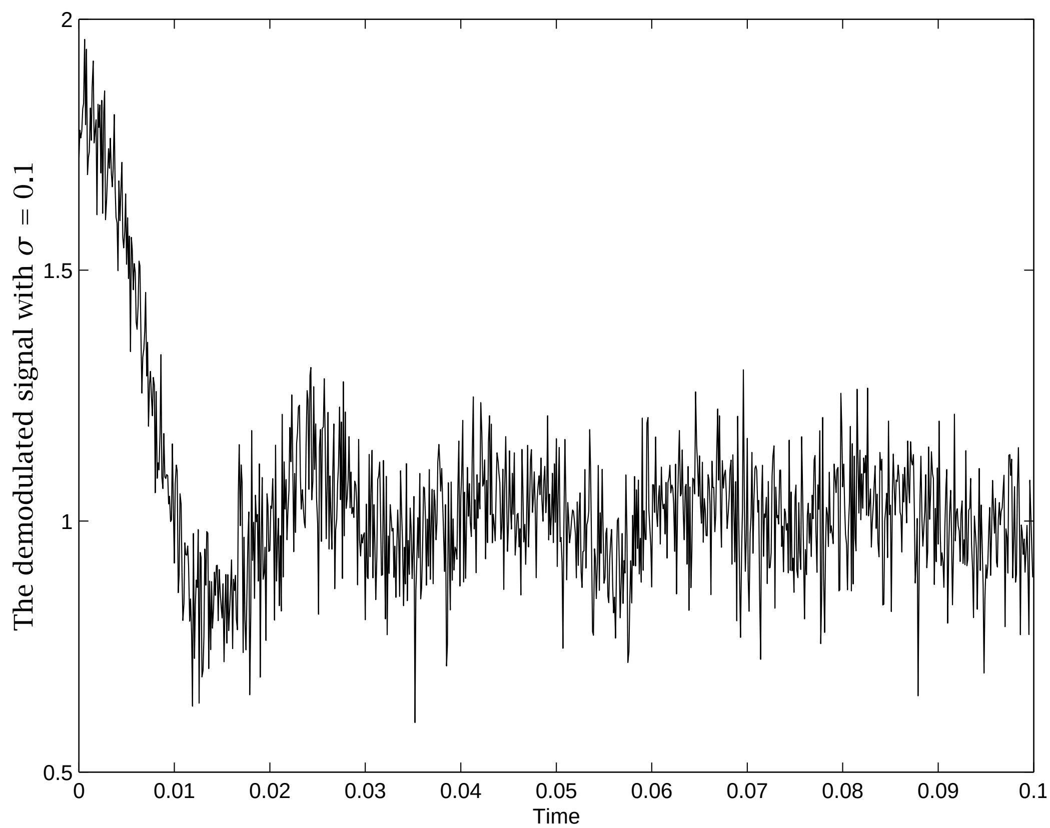


Figure 6.20: The demodulated signal with noise $\sigma = 0.1$

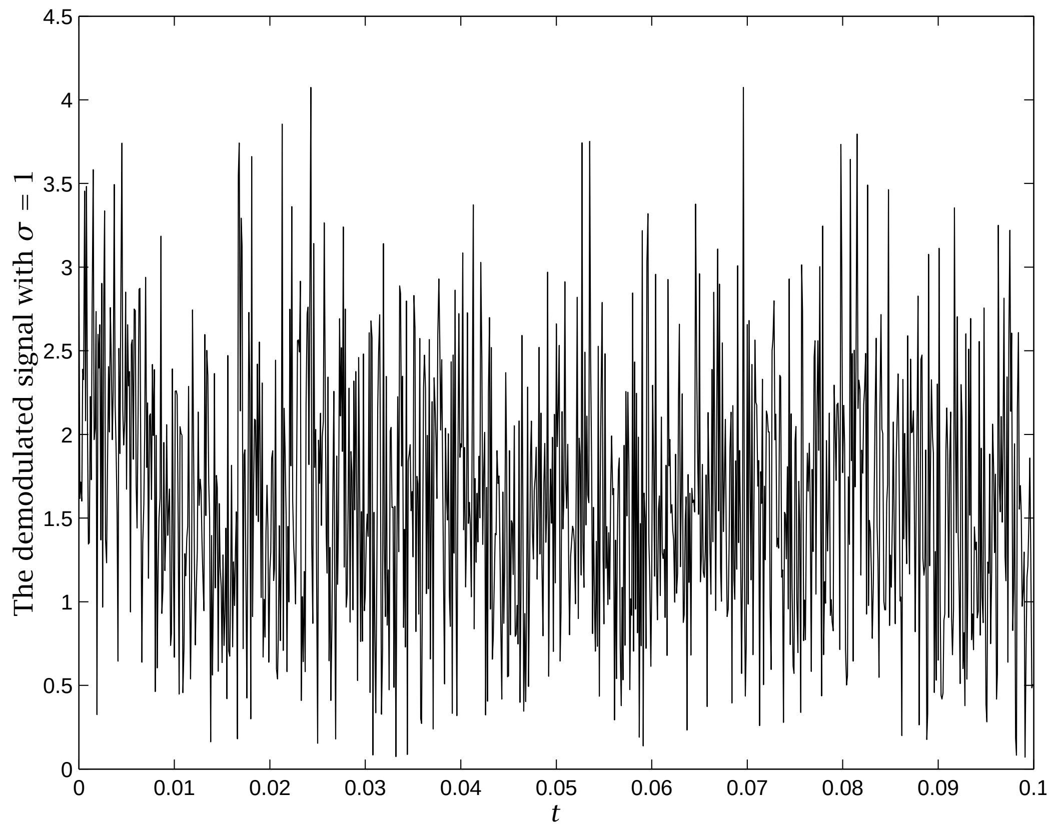


Figure 6.21: The demodulated signal with noise $\sigma = 1$

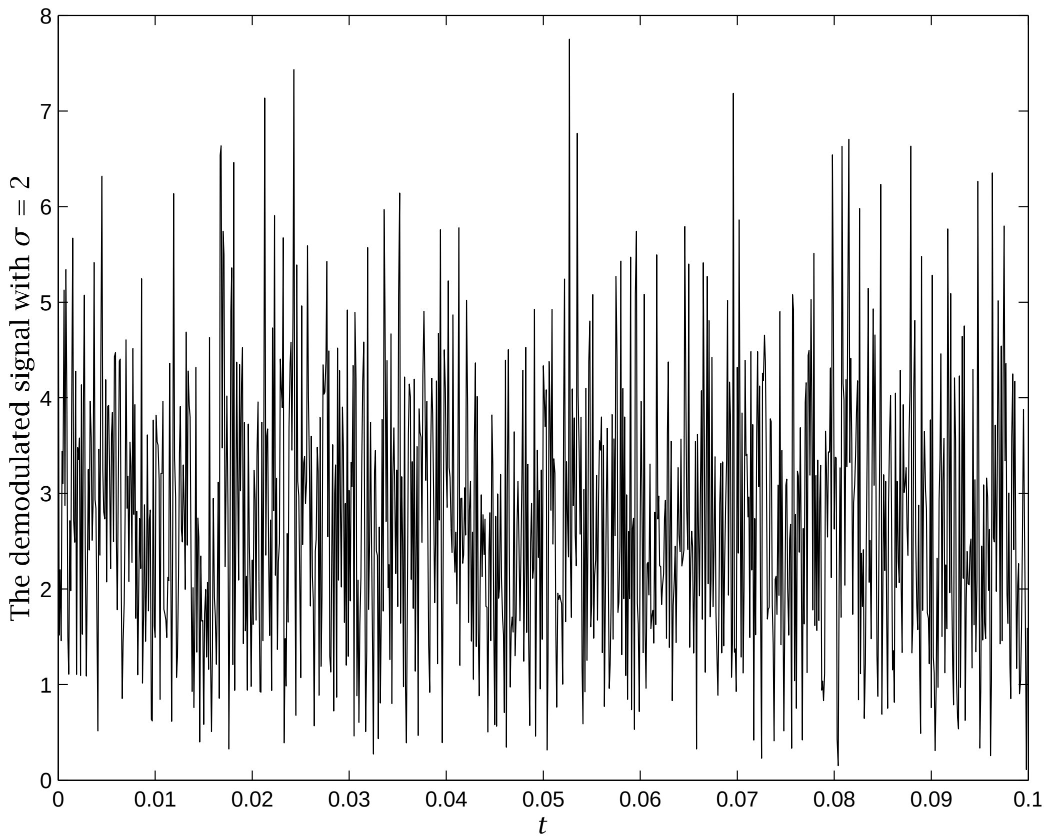


Figure 6.22: The demodulated signal with noise $\sigma = 2$

```
[M,m,df1]=fftseq(m,ts,df);                 % Fourier transform
M=M/fs;                                    % scaling
f=[0:df1:df1*(length(m)−1)]−fs/2;          % frequency vector
[U,u,df1]=fftseq(u,ts,df);                 % Fourier transform
U=U/fs;                                    % scaling
signal_power=spower(u(1:length(t)));       % power in modulated signal
% power in normalized message
pmn=spower(m(1:length(t)))/(max(abs(m)))^2;
eta=(a^2*pmn)/(1+a^2*pmn);                 % modulation efficiency
noise_power=eta*signal_power/snr_lin;      % noise power
noise_std=sqrt(noise_power);               % noise standard deviation
noise=noise_std*randn(1,length(u));        % Generate noise.
r=u+noise;                                 % Add noise to the modulated signal
[R,r,df1]=fftseq(r,ts,df);                 % Fourier transform.
R=R/fs;                                    % scaling
pause % Press a key to show the modulated signal power.
signal_power
pause % Press a key to show the modulation efficiency.
eta
pause % Press any key to see a plot of the message.
subplot(2,2,1)
plot(t,m(1:length(t)))
axis([0 0.15 −2.1 2.1])
xlabel('Time')
title('The message signal')
pause
pause % Press any key to see a plot of the carrier.
subplot(2,2,2)
plot(t,c(1:length(t)))
axis([0 0.15 −2.1 2.1])
xlabel('Time')
title('The carrier')
pause % Press any key to see a plot of the modulated signal.
subplot(2,2,3)
plot(t,u(1:length(t)))
axis([0 0.15 −2.1 2.1])
xlabel('Time')
title('The modulated signal')
pause  % Press any key to see plots of the magnitude of the message and the
% modulated signal in the frequency domain.
subplot(2,1,1)
plot(f,abs(fftshift(M)))
xlabel('Frequency')
title('Spectrum of the message signal')
subplot(2,1,2)
plot(f,abs(fftshift(U)))
title('Spectrum of the modulated signal')
xlabel('Frequency')
pause % Press a key to see a noise sample.
subplot(2,1,1)
plot(t,noise(1:length(t)))
title('Noise sample')
xlabel('Time')
pause % Press a key to see the modulated signal and noise.
```

```
subplot(2,1,2)
plot(t,r(1:length(t)))
title('Signal and noise')
xlabel('Time')
pause % Press a key to see the modulated signal and noise in freq. domain.
subplot(2,1,1)
plot(f,abs(fftshift(U)))
title('Signal spectrum')
xlabel('Frequency')
subplot(2,1,2)
plot(f,abs(fftshift(R)))
title('Signal and noise spectrum')
xlabel('Frequency')
```

Computer Problem 6.4

Figures 6.23 and 6.24 show the message signal $m(t)$ and it's integral signal and Figure 6.25 illustrates the FM modulated signal. Using MATLABŠs Fourier transform routine, we plot the spectra of $m(t)$ and $u(t)$ on Figures 6.26 and 6.27.

The impact of the channel noise on the modulated signals $\{r(n)\}$ for two channel noise values of $\sigma = 0.1$ and $\sigma = 1$ are shown in Figure 6.28. The demodulated signal are also shown in Figure 6.29. The MATLAB script for this question is given next.

```
% MATLAB script for Computer Problem 6.4.
% Demonstration script for frequency modulation. The message signal
% is m(t)=sinc(100t).
echo on
t0=.1;                          % signal duration
ts=0.0001;                      % sampling interval
fc=250;                         % carrier frequency
fs=1/ts;                        % sampling frequency
t=[0:ts:t0-ts];                    % time vector
kf=10000;                       % deviation constant
df=0.25;                        % required frequency resolution
m=sinc(100*t);                  % the message signal

int_m(1)=0;
for i=1:length(t)-1             % integral of m
  int_m(i+1)=int_m(i)+m(i)*ts;
  echo off ;
end
echo on ;

figure;
plot(t, m);
xlabel('Time');ylabel('y');
figure;
plot(t, int_m);
xlabel('Time');ylabel('y');
```

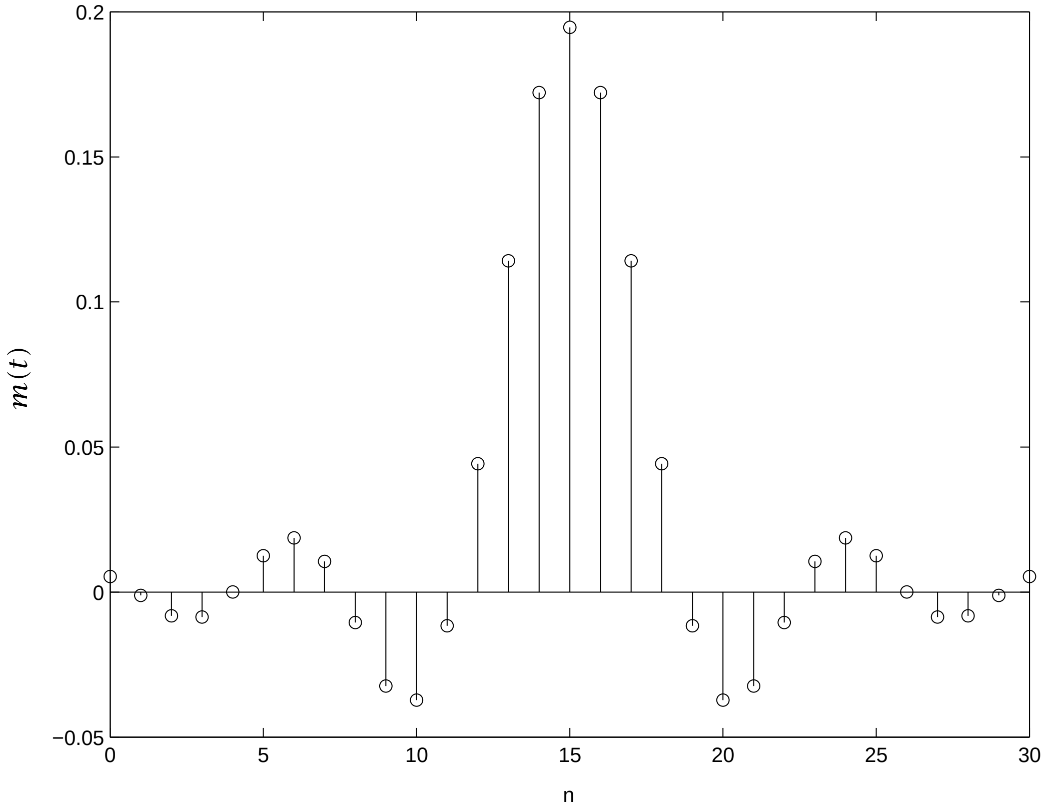


Figure 6.23: The message signal $m(t)$

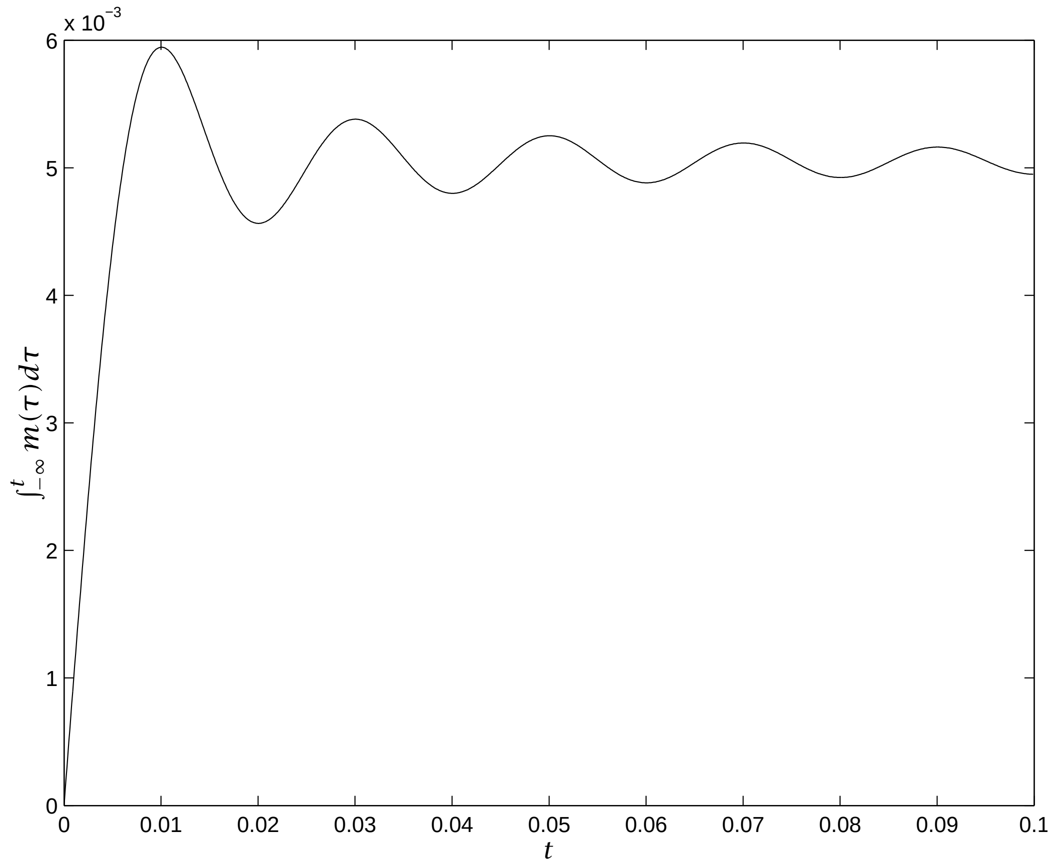


Figure 6.24: The integral of the message signal

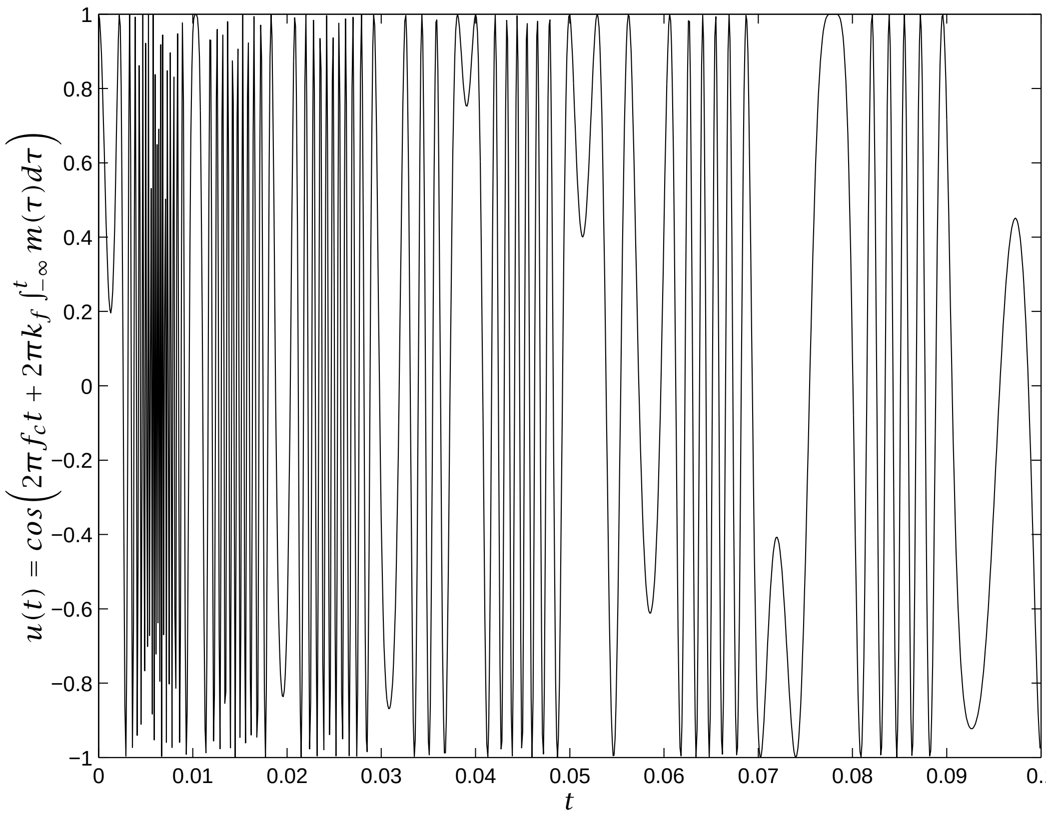


Figure 6.25: The FM modulated signal

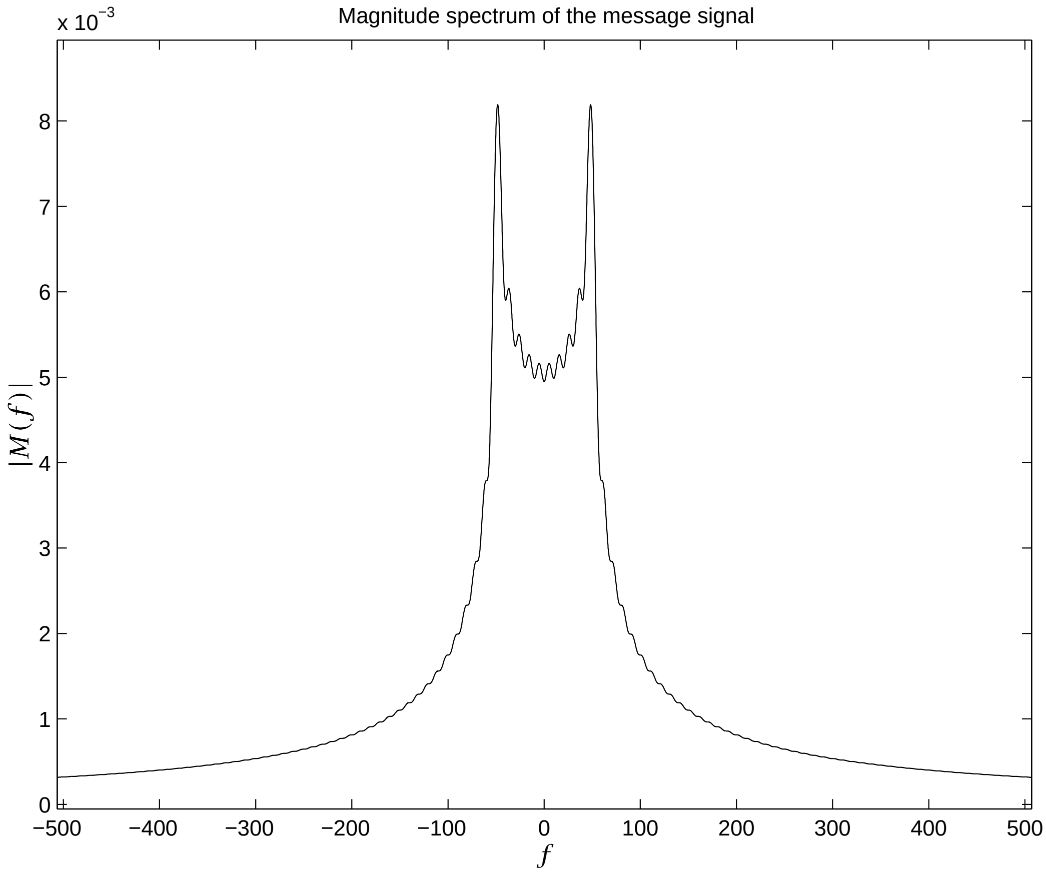


Figure 6.26: The Fourier transform of $M(f)$

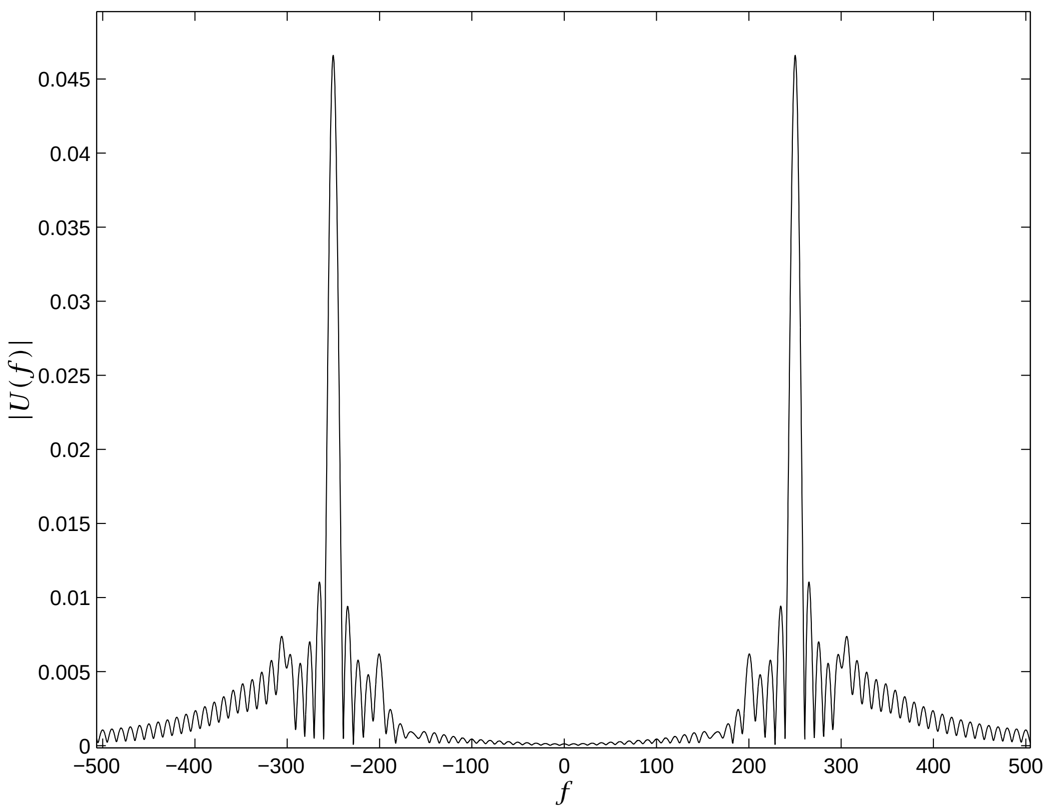


Figure 6.27: The Fourier transform of $U(f)$

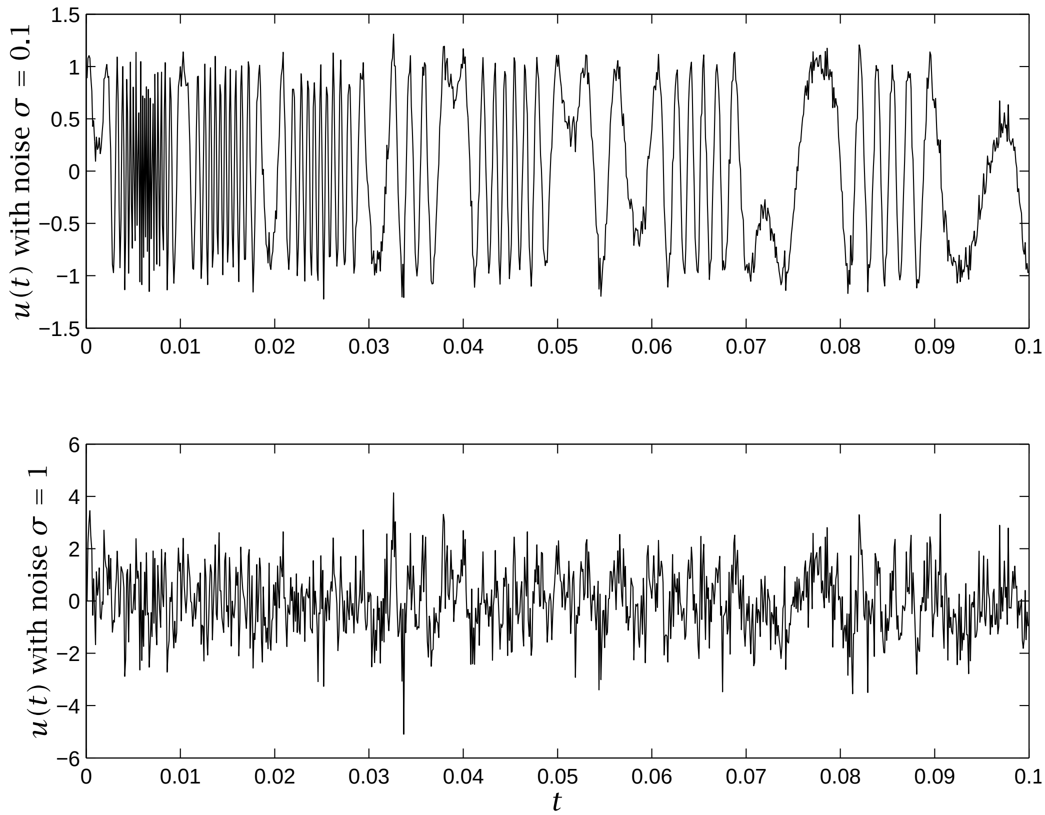


Figure 6.28: The FM modulated signals with noise of $\sigma = 0.1$ and $\sigma = 1$

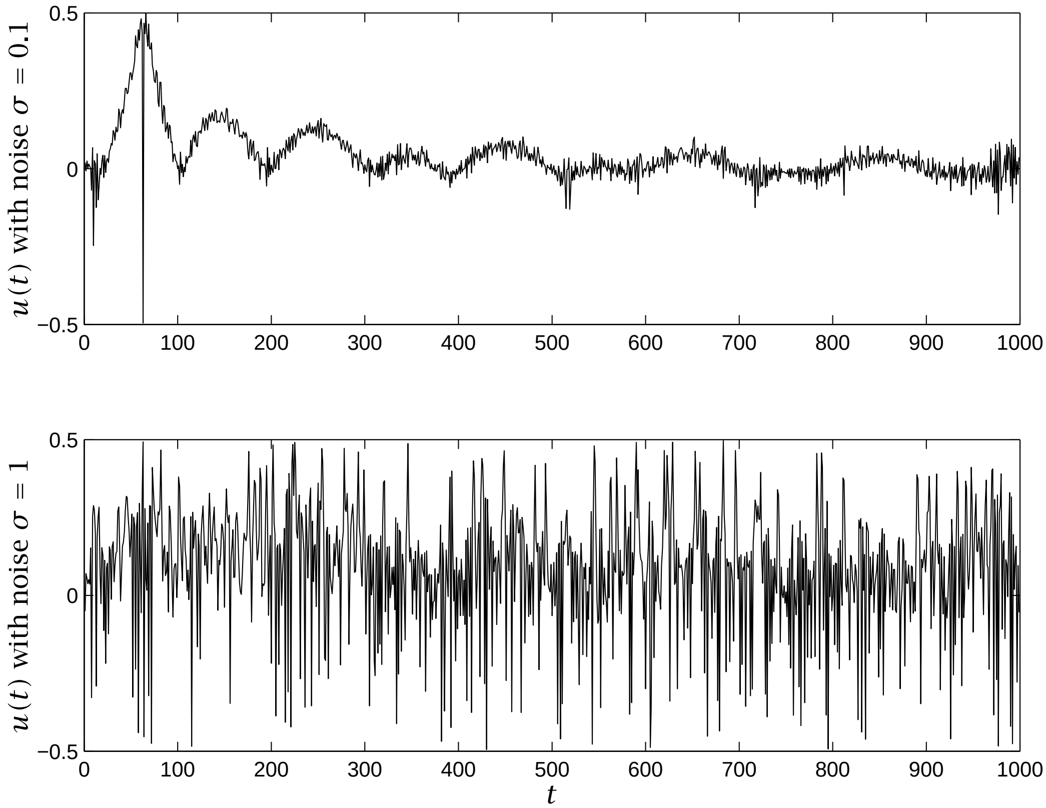


Figure 6.29: The demodulated signals with noise of $\sigma = 0.1$ and $\sigma = 1$

```
[M,m,df1]=fftseq(m,ts,df);              % Fourier transform
M=M/fs;                                 % scaling
f=[0:df1:df1*(length(m)-1)]-fs/2;       % frequency vector
u=cos(2*pi*fc*t+2*pi*kf*int_m);         % modulated signal
figure;
plot(u);
xlabel('Time');ylabel('y');

Wc=randn(1, 1000);
Ws=randn(1, 1000);
sgma = 1;
r_1 = u+sgma*(Wc.*cos(2*pi*fc*t)- Ws.*sin(2*pi*fc*t));
[U,u_pad,df1]=fftseq(u,ts,df);          % Fourier transform
U=U/fs;                                 % scaling

[v,phase]=env_phas(u,ts,250);           % demodulation, find phase of u
phi=unwrap(phase);                      % Restore original phase.
dem=(1/(2*pi*kf))*(diff(phi)/ts);       % demodulator output, differentiate and scale phase

[v_1,phase_1]=env_phas(r_1,ts,250);           % demodulation, find phase of u
phi_1=unwrap(phase_1);                        % Restore original phase.
dem_1=(1/(2*pi*kf))*(diff(phi_1)/ts);         % demodulator output, differentiate and scale phase
figure; plot(dem_1);
% pause % Press any key to see a plot of the message and the modulated signal.
subplot(2,1,1)
plot(t,m(1:length(t)))
xlabel('Time')
title('The message signal')
subplot(2,1,2)
plot(t,u(1:length(t)))
xlabel('Time')
title('The modulated signal')
% pause  % Press any key to see plots of the magnitude of the message and the
       % modulated signal in the frequency domain.
subplot(2,1,1)
plot(f,abs(fftshift(M)))
xlabel('Frequency')
title('Magnitude spectrum of the message signal')
subplot(2,1,2)
plot(f,abs(fftshift(U)))
title('Magnitude-spectrum of the modulated signal')
xlabel('Frequency')
pause  % Press any key to see plots of the message and the demodulator output with no
       % noise.
subplot(2,1,1)
plot(t,m(1:length(t)))
xlabel('Time')
title('The message signal')
subplot(2,1,2)
plot(t,dem(1:length(t)))
xlabel('Time')
title('The demodulated signal');
```

Chapter 7

Problem 7.1

1. Since the maximum frequency in $X(f)$ is is 40 KHz, the minimum sampling rate is $f_s = 2W = 80$ KHz.
2. Here $f_s = 2W + W_G = 2 \times 40 + 10 = 90$ KHz.
3. $X_1(f) = \frac{1}{2}X(f-40000)+\frac{1}{2}X(f+40000)$, the maximum frequency in $X_1(f)$ is 40000+40000=80000 Hz, and the minimum sampling rate is $f_s = 2 \times 80000 = 160000$. From this the maximum sampling interval is $T_s = 1/f_s = 1/160000 = 6.25\mu$ sec.

Problem 7.2

For no aliasing to occur we must sample at the Nyquist rate

$$f_s = 2 \cdot 6000 \text{ samples/sec } = 12000 \text{ samples/sec}$$

With a guard band of 2000

$$f_s - 2W = 2000 \Longrightarrow f_s = 14000$$

The reconstruction filter should not pick-up frequencies of the images of the spectrum $X(f)$. The nearest image spectrum is centered at f_s and occupies the frequency band $[f_s - W, f_s + W]$. Thus the highest frequency of the reconstruction filter ($= 10000$) should satisfy

$$10000 \leq f_s - W \Longrightarrow f_s \geq 16000$$

For the value $f_s = 16000$, K should be such that

$$K \cdot f_s = 1 \Longrightarrow K = (16000)^{-1}$$

Problem 7.3

$$x(t) = A\text{sinc}(1000\pi t) \Longrightarrow X(f) = \frac{A}{1000}\Pi(\frac{f}{1000})$$

Thus the bandwidth W of $x(t)$ is $1000/2 = 500$. Since we sample at $f_s = 2000$ there is a gap between the image spectra equal to

$$2000 - 500 - W = 1000$$

The reconstruction filter should have a bandwidth W' such that $500 < W' < 1500$. A filter that satisfy these conditions is

$$H(f) = T_s \Pi\left(\frac{f}{2W'}\right) = \frac{1}{2000}\Pi\left(\frac{f}{2W'}\right)$$

and the more general reconstruction filters have the form

$$H(f) = \begin{cases} \frac{1}{2000} & |f| < 500 \\ \text{arbitrary} & 500 < |f| < 1500 \\ 0 & |f| > 1500 \end{cases}$$

Problem 7.4

1)

$$\begin{aligned} x_p(t) &= \sum_{n=-\infty}^{\infty} x(nT_s)p(t - nT_s) \\ &= p(t) \star \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \\ &= p(t) \star x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \end{aligned}$$

Thus

$$\begin{aligned} X_p(f) &= P(f) \cdot \mathcal{F}\left[x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right] \\ &= P(f)X(f) \star \mathcal{F}\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right] \\ &= P(f)X(f) \star \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T_s}) \\ &= \frac{1}{T_s} P(f) \sum_{n=-\infty}^{\infty} X(f - \frac{n}{T_s}) \end{aligned}$$

2) In order to avoid aliasing $\frac{1}{T_s} > 2W$. Furthermore the spectrum $P(f)$ should be invertible for $|f| < W$.

3) $X(f)$ can be recovered using the reconstruction filter $\Pi(\frac{f}{2W_\Pi})$ with $W < W_\Pi < \frac{1}{T_s} - W$. In this case

$$X(f) = X_p(f) T_s P^{-1}(f) \Pi(\frac{f}{2W_\Pi})$$

Problem 7.5

1)

$$\begin{aligned} x_1(t) &= \sum_{n=-\infty}^{\infty} (-1)^n x(nT_s)\delta(t-nT_s) = x(t)\sum_{n=-\infty}^{\infty}(-1)^n\delta(t-nT_s) \\ &= x(t)\left[\sum_{l=-\infty}^{\infty}\delta(t-2lT_s) - \sum_{l=-\infty}^{\infty}\delta(t-T_s-2lT_s)\right] \end{aligned}$$

Thus

$$\begin{aligned} X_1(f) &= X(f)\star\left[\frac{1}{2T_s}\sum_{l=-\infty}^{\infty}\delta(f-\frac{l}{2T_s}) - \frac{1}{2T_s}\sum_{l=-\infty}^{\infty}\delta(f-\frac{l}{2T_s})e^{-j2\pi fT_s}\right] \\ &= \frac{1}{2T_s}\sum_{l=-\infty}^{\infty}X(f-\frac{l}{2T_s}) - \frac{1}{2T_s}\sum_{l=-\infty}^{\infty}X(f-\frac{l}{2T_s})e^{-j2\pi\frac{l}{2T_s}T_s} \\ &= \frac{1}{2T_s}\sum_{l=-\infty}^{\infty}X(f-\frac{l}{2T_s}) - \frac{1}{2T_s}\sum_{l=-\infty}^{\infty}X(f-\frac{l}{2T_s})(-1)^l \\ &= \frac{1}{T_s}\sum_{l=-\infty}^{\infty}X(f-\frac{1}{2T_s}-\frac{l}{T_s}) \end{aligned}$$

2) The spectrum of $x(t)$ occupies the frequency band $[-W,W]$. Suppose that from the periodic spectrum $X_1(f)$ we isolate $X_k(f) = \frac{1}{T_s}X(f-\frac{1}{2T_s}-\frac{k}{T_s})$, with a bandpass filter, and we use it to reconstruct $x(t)$. Since $X_k(f)$ occupies the frequency band $[2kW, 2(k+1)W]$, then for all k, $X_k(f)$ cannot cover the whole interval $[-W,W]$. Thus at the output of the reconstruction filter there will exist frequency components which are not present in the input spectrum. Hence, the reconstruction filter has to be a time-varying filter. To see this in the time domain, note that the original spectrum has been shifted by $f' = \frac{1}{2T_s}$. In order to bring the spectrum back to the origin and reconstruct $x(t)$ the sampled signal $x_1(t)$ has to be multiplied by $e^{-j2\pi\frac{1}{2T_s}t} = e^{-j2\pi Wt}$. However the system described by

$$y(t) = e^{j2\pi Wt}x(t)$$

is a time-varying system.

3) Using a time-varying system we can reconstruct $x(t)$ as follows. Use the bandpass filter $T_s\Pi(\frac{f-W}{2W})$ to extract the component $X(f-\frac{1}{2T_s})$. Invert $X(f-\frac{1}{2T_s})$ and multiply the resultant signal by $e^{-j2\pi Wt}$. Thus

$$x(t) = e^{-j2\pi Wt}\mathcal{F}^{-1}\left[T_s\Pi(\frac{f-W}{2W})X_1(f)\right]$$

Problem 7.6

1) The linear interpolation system can be viewed as a linear filter where the sampled signal

$x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$ is passed through the filter with impulse response

$$h(t) = \begin{cases} 1+\frac{t}{T_s} & -T_s \le f \le 0 \\ 1-\frac{t}{T_s} & 0 \le f \le T_s \\ 0 & \text{otherwise} \end{cases}$$

To see this write

$$x_1(t) = \left[x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right] \star h(t) = \sum_{n=-\infty}^{\infty} x(nT_s)h(t-nT_s)$$

Comparing this with the interpolation formula in the interval $[nT_s, (n+1)T_s]$

$$\begin{aligned} x_1(t) &= x(nT_s) + \frac{t-nT_s}{T_s}\left(x((n+1)T_s) - x(nT_s)\right) \\ &= x(nT_s)\left[1-\frac{t-nT_s}{T_s}\right] + x((n+1)T_s)\left[1+\frac{t-(n+1)T_s}{T_s}\right] \\ &= x(nT_s)h(t-nT_s) + x((n+1)T_s)h(t-(n+1)T_s) \end{aligned}$$

we observe that $h(t)$ does not extend beyond $[-T_s, T_s]$ and in this interval its form should be the one described above. The power spectrum of $x_1(t)$ is $S_{X_1}(f) = |X_1(f)|^2$ where

$$\begin{aligned} X_1(f) &= \mathcal{F}[x_1(t)] = \mathcal{F}\left[h(t) \star x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\right] \\ &= H(f)\left[X(f) \star \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta(f-\frac{n}{T_s})\right] \\ &= \text{sinc}^2(fT_s)\sum_{n=-\infty}^{\infty} X(f-\frac{n}{T_s}) \end{aligned}$$

2) The system function $\text{sinc}^2(fT_s)$ has zeros at the frequencies f such that

$$fT_s = k, \quad k \in \mathcal{Z} - \{0\}$$

In order to recover $X(f)$, the bandwidth W of $x(t)$ should be smaller than $1/T_s$, so that the whole $X(f)$ lies inside the main lobe of $\text{sinc}^2(fT_s)$. This condition is automatically satisfied if we choose T_s such that to avoid aliasing ($2W < 1/T_s$). In this case we can recover $X(f)$ from $X_1(f)$ using the lowpass filter $\Pi(\frac{f}{2W})$.

$$\Pi(\frac{f}{2W})X_1(f) = \text{sinc}^2(fT_s)X(f)$$

or

$$X(f) = (\text{sinc}^2(fT_s))^{-1}\Pi(\frac{f}{2W})X_1(f)$$

If $T_s \ll 1/W$, then $\text{sinc}^2(fT_s) \approx 1$ for $|f| < W$ and $X(f)$ is available using $X(f) = \Pi(\frac{f}{2W})X_1(f)$.

Problem 7.7

1) $W = 50$Hz so that $T_s = 1/2W = 10^{-2}$sec. The reconstructed signal is

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} x(nT_s)\text{sinc}(\frac{t}{T_s} - n) \\ &= -\sum_{n=-4}^{-1} \text{sinc}(\frac{t}{T_s} - n) + \sum_{n=1}^{4} \text{sinc}(\frac{t}{T_s} - n) \end{aligned}$$

With $T_s = 10^{-2}$ and $t = 5 \cdot 10^{-3}$ we obtain

$$\begin{aligned} x(.005) &= -\sum_{n=1}^{4} \text{sinc}(\frac{1}{2} + n) + \sum_{n=1}^{4} \text{sinc}(\frac{1}{2} - n) \\ &= -[\text{sinc}(\frac{3}{2}) + \text{sinc}(\frac{5}{2}) + \text{sinc}(\frac{7}{2}) + \text{sinc}(\frac{9}{2})] \\ &\quad +[\text{sinc}(-\frac{1}{2}) + \text{sinc}(-\frac{3}{2}) + \text{sinc}(-\frac{5}{2}) + \text{sinc}(-\frac{7}{2})] \\ &= \text{sinc}(\frac{1}{2}) - \text{sinc}(\frac{9}{2}) = \frac{2}{\pi}\sin(\frac{\pi}{2}) - \frac{2}{9\pi}\sin(\frac{9\pi}{2}) \\ &= \frac{16}{9\pi} \end{aligned}$$

where we have used the fact that sinc(t) is an even function.

2) Note that (see Problem 7.8)

$$\int_{-\infty}^{\infty} \text{sinc}(2Wt - m)\text{sinc}^*(2Wt - n)dt = \frac{1}{2W}\delta_{mn}$$

with δ_{mn} the Kronecker delta. Thus,

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)x^*(t)dt \\ &= \sum_{n=-\infty}^{\infty} x(nT_s)x^*(mT_s)\int_{-\infty}^{\infty} \text{sinc}(2Wt - m)\text{sinc}^*(2Wt - n)dt \\ &= \sum_{n=-\infty}^{\infty} |x(nT_s)|^2 \frac{1}{2W} \end{aligned}$$

Hence

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2W}\left[\sum_{n=-4}^{-1} 1 + \sum_{n=1}^{4} 1\right] = \frac{4}{W} = 8 \cdot 10^{-2}$$

Problem 7.8

1) Using Parseval's theorem we obtain

$$\begin{aligned}
A &= \int_{-\infty}^{\infty} \text{sinc}(2Wt-m)\text{sinc}(2Wt-n)dt \\
&= \int_{-\infty}^{\infty} \mathcal{F}[\text{sinc}(2Wt-m)]\mathcal{F}[\text{sinc}(2Wt-n)]dt \\
&= \int_{-\infty}^{\infty} (\frac{1}{2W})^2\Pi^2(\frac{f}{2W})e^{-j2\pi f\frac{m-n}{2W}}df \\
&= \frac{1}{4W^2}\int_{-W}^{W} e^{-j2\pi f\frac{m-n}{2W}}df = \frac{1}{2W}\delta_{mn}
\end{aligned}$$

where δ_{mn} is the Kronecker's delta. The latter implies that $\{\text{sinc}(2Wt-m)\}$ form an orthogonal set of signals. In order to generate an orthonormal set of signals we have to weight each function by $1/\sqrt{2W}$.

2) The bandlimited signal $x(t)$ can be written as

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\text{sinc}(2Wt-n)$$

where $x(nT_s)$ are the samples taken at the Nyquist rate. This is an orthogonal expansion relation where the basis functions $\{\text{sinc}(2Wt-m)\}$ are weighted by $x(mT_s)$.

3)

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t)\text{sinc}(2Wt-n)dt &= \int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x(mT_s)\text{sinc}(2Wt-m)\text{sinc}(2Wt-n)dt \\
&= \sum_{m=-\infty}^{\infty} x(mT_s)\int_{-\infty}^{\infty}\text{sinc}(2Wt-m)\text{sinc}(2Wt-n)dt \\
&= \sum_{m=-\infty}^{\infty} x(mT_s)\frac{1}{2W}\delta_{mn} = \frac{1}{2W}x(nT_s)
\end{aligned}$$

Problem 7.9

1) From Table 7.1 we find that for a unit variance Gaussian process, the optimal level spacing for a 16-level uniform quantizer is .3352. This number has to be multiplied by σ to provide the optimal level spacing when the variance of the process is σ^2. In our case $\sigma^2 = 10$ and $\Delta = \sqrt{10}\cdot 0.3352 = 1.060$.

The quantization levels are

$$\begin{aligned}
\hat{x}_1 = -\hat{x}_{16} &= -7 \cdot 1.060 - \frac{1}{2} \cdot 1.060 = -7.950 \\
\hat{x}_2 = -\hat{x}_{15} &= -6 \cdot 1.060 - \frac{1}{2} \cdot 1.060 = -6.890 \\
\hat{x}_3 = -\hat{x}_{14} &= -5 \cdot 1.060 - \frac{1}{2} \cdot 1.060 = -5.830 \\
\hat{x}_4 = -\hat{x}_{13} &= -4 \cdot 1.060 - \frac{1}{2} \cdot 1.060 = -4.770 \\
\hat{x}_5 = -\hat{x}_{12} &= -3 \cdot 1.060 - \frac{1}{2} \cdot 1.060 = -3.710 \\
\hat{x}_6 = -\hat{x}_{11} &= -2 \cdot 1.060 - \frac{1}{2} \cdot 1.060 = -2.650 \\
\hat{x}_7 = -\hat{x}_{10} &= -1 \cdot 1.060 - \frac{1}{2} \cdot 1.060 = -1.590 \\
\hat{x}_8 = -\hat{x}_9 &= -\frac{1}{2} \cdot 1.060 = -0.530
\end{aligned}$$

The boundaries of the quantization regions are given by

$$\begin{aligned}
a_1 = a_{15} &= -7 \cdot 1.060 = -7.420 \\
a_2 = a_{14} &= -6 \cdot 1.060 = -6.360 \\
a_3 = a_{13} &= -5 \cdot 1.060 = -5.300 \\
a_4 = a_{12} &= -4 \cdot 1.060 = -4.240 \\
a_5 = a_{11} &= -3 \cdot 1.060 = -3.180 \\
a_6 = a_{10} &= -2 \cdot 1.060 = -2.120 \\
a_7 = a_9 &= -1 \cdot 1.060 = -1.060 \\
a_8 &= 0
\end{aligned}$$

2) The resulting distortion is $D = \sigma^2 \cdot 0.01154 = 0.1154$.

3) Substituting $\sigma^2 = 10$ and $D = 0.1154$ in the rate-distortion bound, we obtain

$$R = \frac{1}{2} \log_2 \frac{\sigma^2}{D} = 3.2186$$

5) The distortion of the 16-level optimal quantizer is $D_{16} = \sigma^2 \cdot 0.01154$ whereas that of the 8-level optimal quantizer is $D_8 = \sigma^2 \cdot 0.03744$. Hence, the amount of increase in SQNR (db) is

$$10 \log_{10} \frac{\text{SQNR}_{16}}{\text{SQNR}_8} = 10 \cdot \log_{10} \frac{0.03744}{0.01154} = 5.111 \text{ db}$$

Problem 7.10

With 8 quantization levels and $\sigma^2 = 400$ we obtain

$$\Delta = \sigma .5860 = 20 \cdot 0.5860 = 11.72$$

Hence, the quantization levels are

$$\begin{aligned}
\hat{x}_1 = -\hat{x}_8 &= -3 \cdot 11.72 - \frac{1}{2} 11.72 = -41.020 \\
\hat{x}_2 = -\hat{x}_7 &= -2 \cdot 11.72 - \frac{1}{2} 11.72 = -29.300 \\
\hat{x}_3 = -\hat{x}_6 &= -1 \cdot 11.72 - \frac{1}{2} 11.72 = -17.580 \\
\hat{x}_4 = -\hat{x}_5 &= -\frac{1}{2} 11.72 = -5.860
\end{aligned}$$

The distortion of the optimum quantizer is

$$D = \sigma^2 \cdot 0.03744 = 14.976$$

As it is observed the distortion of the optimum quantizer is significantly less than that of Example 7.2.1.

Problem 7.11

Using Table 7.2 we find the quantization regions and the quantized values for $N = 16$. These values should be multiplied by $\sigma = P_X^{1/2} = \sqrt{10}$, since Table 6.3 provides the optimum values for a unit variance Gaussian source.

$$\begin{aligned}
a_1 = -a_{15} &= -\sqrt{10} \cdot 2.401 = -7.5926 \\
a_2 = -a_{14} &= -\sqrt{10} \cdot 1.844 = -5.8312 \\
a_3 = -a_{13} &= -\sqrt{10} \cdot 1.437 = -4.5442 \\
a_4 = -a_{12} &= -\sqrt{10} \cdot 1.099 = -3.4753 \\
a_5 = -a_{11} &= -\sqrt{10} \cdot 0.7996 = -2.5286 \\
a_6 = -a_{10} &= -\sqrt{10} \cdot 0.5224 = -1.6520 \\
a_7 = -a_9 &= -\sqrt{10} \cdot 0.2582 = -0.8165 \\
a_8 &= 0
\end{aligned}$$

The quantized values are

$$\begin{aligned}
\hat{x}_1 = -\hat{x}_{16} &= -\sqrt{10}\cdot 2.733 = -8.6425 \\
\hat{x}_2 = -\hat{x}_{15} &= -\sqrt{10}\cdot 2.069 = -6.5428 \\
\hat{x}_3 = -\hat{x}_{14} &= -\sqrt{10}\cdot 1.618 = -5.1166 \\
\hat{x}_4 = -\hat{x}_{13} &= -\sqrt{10}\cdot 1.256 = -3.9718 \\
\hat{x}_5 = -\hat{x}_{12} &= -\sqrt{10}\cdot 0.9424 = -2.9801 \\
\hat{x}_6 = -\hat{x}_{11} &= -\sqrt{10}\cdot 0.6568 = -2.0770 \\
\hat{x}_7 = -\hat{x}_{10} &= -\sqrt{10}\cdot 0.3881 = -1.2273 \\
\hat{x}_8 = -\hat{x}_9 &= -\sqrt{10}\cdot 0.1284 = -0.4060
\end{aligned}$$

The resulting distortion is $D = 10\cdot 0.009494 = 0.09494$. From Table 7.2 we find that the minimum number of bits per source symbol is $H(\hat{X}) = 3.765$.

Problem 7.12

1) The area between the two squares is $4\times 4 - 2\times 2 = 12$. Hence, $f_{X,Y}(x,y) = \frac{1}{12}$. The marginal probability $f_X(x)$ is given by $f_X(x) = \int_{-2}^{2} f_{X,Y}(x,y)dy$. If $-2 \le X < -1$, then

$$f_X(x) = \int_{-2}^{2} f_{X,Y}(x,y)dy = \frac{1}{12}y\Big|_{-2}^{2} = \frac{1}{3}$$

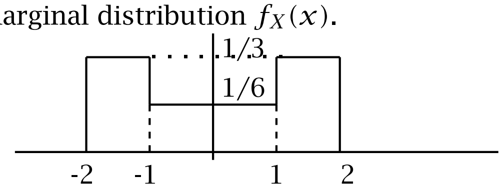
If $-1 \le X < 1$, then

$$f_X(x) = \int_{-2}^{-1}\frac{1}{12}dy + \int_{1}^{2}\frac{1}{12}dy = \frac{1}{6}$$

Finally, if $1 \le X \le 2$, then

$$f_X(x) = \int_{-2}^{2} f_{X,Y}(x,y)dy = \frac{1}{12}y\Big|_{-2}^{2} = \frac{1}{3}$$

The next figure depicts the marginal distribution $f_X(x)$.



Similarly we find that

$$f_Y(y) = \begin{cases} \frac{1}{3} & -2 \le y < -1 \\ \frac{1}{6} & -1 \le y < -1 \\ \frac{1}{3} & 1 \le y \le 2 \end{cases}$$

2) The quantization levels $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ and $\hat{x}_4$ are set to $-\frac{3}{2}$, $-\frac{1}{2}$, $\frac{1}{2}$ and $\frac{3}{2}$ respectively. The resulting distortion is

$$\begin{aligned}
D_X &= 2\int_{-2}^{-1}(x+\frac{3}{2})^2 f_X(x)dx + 2\int_{-1}^{0}(x+\frac{1}{2})^2 f_X(x)dx \\
&= \frac{2}{3}\int_{-2}^{-1}(x^2+3x+\frac{9}{4})dx + \frac{2}{6}\int_{-1}^{0}(x^2+x+\frac{1}{4})dx \\
&= \frac{2}{3}\left(\frac{1}{3}x^3+\frac{3}{2}x^2+\frac{9}{4}x\right)\Big|_{-2}^{-1} + \frac{2}{6}\left(\frac{1}{3}x^3+\frac{1}{2}x^2+\frac{1}{4}x\right)\Big|_{-1}^{0} \\
&= \frac{1}{12}
\end{aligned}$$

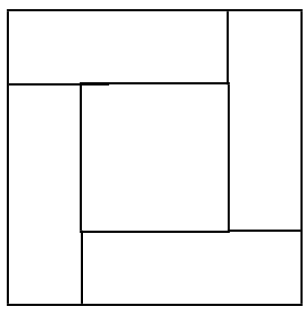
The total distortion is

$$D_{\text{total}} = D_X + D_Y = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

whereas the resulting number of bits per (X, Y) pair

$$R = R_X + R_Y = \log_2 4 + \log_2 4 = 4$$

3) Suppose that we divide the region over which $p(x, y) \neq 0$ into L equal subregions. The case of $L = 4$ is depicted in the next figure.



For each subregion the quantization output vector $(\hat{x}, \hat{y})$ is the centroid of the corresponding rectangle. Since, each subregion has the same shape (uniform quantization), a rectangle with width equal to one and length $12/L$, the distortion of the vector quantizer is

$$\begin{aligned}
D &= \int_0^1\int_0^{\frac{12}{L}}[(x,y)-(\frac{1}{2},\frac{12}{2L})]^2\frac{L}{12}dxdy \\
&= \frac{L}{12}\int_0^1\int_0^{\frac{12}{L}}\left[(x-\frac{1}{2})^2+(y-\frac{12}{2L})^2\right]dxdy \\
&= \frac{L}{12}\left[\frac{12}{L}\frac{1}{12}+\frac{12^3}{L^3}\frac{1}{12}\right] = \frac{1}{12}+\frac{12}{L^2}
\end{aligned}$$

If we set $D = \frac{1}{6}$, we obtain

$$\frac{12}{L^2} = \frac{1}{12} \Longrightarrow L = \sqrt{144} = 12$$

Thus, we have to divide the area over which $p(x, y) \neq 0$, into 12 equal subregions in order to achieve the same distortion. In this case the resulting number of bits per source output pair (X, Y) is $R = \log_2 12 = 3.585$.

Problem 7.13

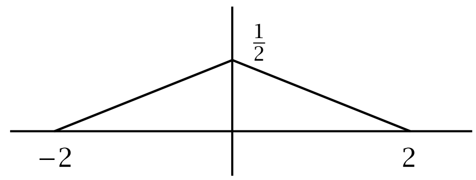
1) The joint probability density function is $f_{XY}(x,y) = \frac{1}{(2\sqrt{2})^2} = \frac{1}{8}$. The marginal distribution $f_X(x)$ is $f_X(x) = \int_y f_{XY}(x,y)dy$. If $-2 \le x \le 0$,then

$$f_X(x) = \int_{-x-2}^{x+2} f_{X,Y}(x,y)dy = \frac{1}{8}y|_{-x-2}^{x+2} = \frac{x+2}{4}$$

If $0 \le x \le 2$,then

$$f_X(x) = \int_{x-2}^{-x+2} f_{X,Y}(x,y)dy = \frac{1}{8}y|_{x-2}^{-x+2} = \frac{-x+2}{4}$$

The next figure depicts $f_X(x)$.



From the symmetry of the problem we have

$$f_Y(y) = \begin{cases} \frac{y+2}{4} & -2 \le y < 0 \\ \frac{-y+2}{4} & 0 \le y \le 2 \end{cases}$$

2)

$$\begin{aligned} D_X &= 2\int_{-2}^{-1}(x+\frac{3}{2})^2 f_X(x)dx + 2\int_{-1}^{0}(x+\frac{1}{2})^2 f_X(x)dx \\ &= \frac{1}{2}\int_{-2}^{-1}(x+\frac{3}{2})^2(x+2)dx + \frac{1}{2}\int_{-1}^{0}(x+\frac{1}{2})^2(-x+2)dx \\ &= \frac{1}{2}\left(\frac{1}{4}x^4 + \frac{5}{3}x^3 + \frac{33}{8}x^2 + \frac{9}{2}x\right)\Big|_{-2}^{-1} + \frac{1}{2}\left(\frac{1}{4}x^4 + x^3 + \frac{9}{8}x^2 + \frac{1}{2}x\right)\Big|_{-1}^{0} \\ &= \frac{1}{12} \end{aligned}$$

The total distortion is

$$D_{\text{total}} = D_X + D_Y = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

whereas the required number of bits per source output pair

$$R = R_X + R_Y = \log_2 4 + \log_2 4 = 4$$

3) We divide the square over which $p(x,y) \ne 0$ into $2^4 = 16$ equal square regions. The area of each square is $\frac{1}{2}$ and the resulting distortion

$$\begin{aligned}
D &= \frac{16}{8}\int_0^{\frac{1}{\sqrt{2}}}\int_0^{\frac{1}{\sqrt{2}}}\left[(x-\frac{1}{2\sqrt{2}})^2+(y-\frac{1}{2\sqrt{2}})^2\right]dxdy \\
&= 4\int_0^{\frac{1}{\sqrt{2}}}\int_0^{\frac{1}{\sqrt{2}}}(x-\frac{1}{2\sqrt{2}})^2 dxdy \\
&= \frac{4}{\sqrt{2}}\int_0^{\frac{1}{\sqrt{2}}}(x^2+\frac{1}{8}-\frac{x}{\sqrt{2}})dx \\
&= \frac{4}{\sqrt{2}}\left(\frac{1}{3}x^3+\frac{1}{8}x-\frac{1}{2\sqrt{2}}x^2\right)\Bigg|_0^{\frac{1}{\sqrt{2}}} \\
&= \frac{1}{12}
\end{aligned}$$

Hence, using vector quantization and the same rate we obtain half the distortion.

Problem 7.14

$\breve{X} = \frac{X}{x_{\max}} = X/2$. Hence,

$$E[\breve{X}^2] = \frac{1}{4}\int_{-2}^{2}\frac{X^2}{4}dx = \frac{1}{16\cdot 3}x^3\Bigg|_{-2}^{2} = \frac{1}{3}$$

With $\nu = 8$ and $\overline{\breve{X}^2} = \frac{1}{3}$, we obtain

$$\text{SQNR} = 3\cdot 4^8\cdot\frac{1}{3} = 4^8 = 48.165(\text{db})$$

Problem 7.15

1)

$$\sigma^2 = E[X^2(t)] = R_X(\tau)|_{\tau=0} = \frac{A^2}{2}$$

Hence,

$$\text{SQNR} = 3\cdot 4^\nu\overline{\breve{X}^2} = 3\cdot 4^\nu\frac{\overline{X^2}}{x_{\max}^2} = 3\cdot 4^\nu\frac{A^2}{2A^2}$$

With SQNR = 60 db, we obtain

$$10\log_{10}\left(\frac{3\cdot 4^q}{2}\right) = 60 \Longrightarrow q = 9.6733$$

The smallest integer larger that q is 10. Hence, the required number of quantization levels is $\nu = 10$.

2) The minimum bandwidth requirement for transmission of a binary PCM signal is BW $= \nu W$. Since $\nu = 10$, we have BW $= 10W$.

Problem 7.16

1)

$$
\begin{aligned}
E[X^2(t)] &= \int_{-2}^{0} x^2\left(\frac{x+2}{4}\right)dx + \int_{0}^{2} x^2\left(\frac{-x+2}{4}\right)dx \\
&= \frac{1}{4}\left(\frac{1}{4}x^4 + \frac{2}{3}x^3\right)\Big|_{-2}^{0} + \frac{1}{4}\left(-\frac{1}{4}x^4 + \frac{2}{3}x^3\right)\Big|_{0}^{2} \\
&= \frac{2}{3}
\end{aligned}
$$

Hence,

$$
\text{SQNR} = \frac{3 \times 4^{\nu} \times \frac{2}{3}}{x_{\max}^2} = \frac{3 \times 4^5 \times \frac{2}{3}}{2^2} = 512 = 27.093(\text{db})
$$

2) If the available bandwidth of the channel is 40 KHz, then the maximum rate of transmission is $\nu = 40/5 = 8$. In this case the highest achievable SQNR is

$$
\text{SQNR} = \frac{3 \times 4^8 \times \frac{2}{3}}{2^2} = 32768 = 45.154(\text{db})
$$

3) In the case of a guard band of 2 KHz the sampling rate is $f_s = 2W + 2000 = 12$ KHz. The highest achievable rate is $\nu = \frac{2\text{BW}}{f_s} = 6.6667$ and since ν should be an integer we set $\nu = 6$. Thus, the achievable SQNR is

$$
\text{SQNR} = \frac{3 \times 4^6 \times \frac{2}{3}}{2^2} = 2048 = 33.11(\text{db})
$$

Problem 7.17

Let $\tilde{X} = X - Q(X)$. Clearly if $|\tilde{X}| > 0.5$, then $p(\tilde{X}) = 0$. If $|\tilde{X}| \leq 0.5$, then there are four solutions to the equation $\tilde{X} = X - Q(X)$, which are denoted by x_1, x_2, x_3 and x_4. The solution x_1 corresponds to the case $-2 \leq X \leq -1$, x_2 is the solution for $-1 \leq X \leq 0$ and so on. Hence,

$$
\begin{aligned}
f_X(x_1) &= \frac{x_1+2}{4} = \frac{(\tilde{x}-1.5)+2}{4} & \qquad f_X(x_3) &= \frac{-x_3+2}{4} = \frac{-(\tilde{x}+0.5)+2}{4} \\
f_X(x_2) &= \frac{x_2+2}{4} = \frac{(\tilde{x}-0.5)+2}{4} & \qquad f_X(x_4) &= \frac{-x_4+2}{4} = \frac{-(\tilde{x}+1.5)+2}{4}
\end{aligned}
$$

The absolute value of $(X - Q(X))'$ is one for $X = x_1, \ldots, x_4$. Thus, for $|\tilde{X}| \leq 0.5$

$$
\begin{aligned}
f_{\tilde{X}}(\tilde{x}) &= \sum_{i=1}^{4} \frac{f_X(x_i)}{|(x_i - Q(x_i))'|} \\
&= \frac{(\tilde{x}-1.5)+2}{4} + \frac{(\tilde{x}-0.5)+2}{4} + \frac{-(\tilde{x}+0.5)+2}{4} + \frac{-(\tilde{x}+1.5)+2}{4} \\
&= 1
\end{aligned}
$$

Problem 7.18

1)

$$\begin{aligned} R_X(t+\tau,t) &= E[X(t+\tau)X(t)] \\ &= E[Y^2\cos(2\pi f_0(t+\tau)+\Theta)\cos(2\pi f_0 t+\Theta)] \\ &= \frac{1}{2}E[Y^2]E[\cos(2\pi f_0\tau)+\cos(2\pi f_0(2t+\tau)+2\Theta)] \end{aligned}$$

and since

$$E[\cos(2\pi f_0(2t+\tau)+2\Theta)] = \frac{1}{2\pi}\int_0^{2\pi}\cos(2\pi f_0(2t+\tau)+2\theta)d\theta = 0$$

we conclude that

$$R_X(t+\tau,t) = \frac{1}{2}E[Y^2]\cos(2\pi f_0\tau) = \frac{3}{2}\cos(2\pi f_0\tau)$$

2)

$$10\log_{10}\text{SQNR} = 10\log_{10}\left(\frac{3\times 4^\nu \times R_X(0)}{x_{\max}^2}\right) = 40$$

Thus,

$$\log_{10}\left(\frac{4^\nu}{2}\right) = 4 \text{ or } \nu = 8$$

The bandwidth of the process is $W = f_0$, so that the minimum bandwidth requirement of the PCM system is $\text{BW} = 8f_0$.

3) If SQNR = 64 db, then

$$\nu' = \log_4(2\cdot 10^{6.4}) = 12$$

Thus, $\nu' - \nu = 4$ more bits are needed to increase SQNR by 24 db. The new minimum bandwidth requirement is $\text{BW}' = 12f_0$.

Problem 7.19

1. The power spectral density ofthe process is $\mathcal{S}_X(f) = \mathscr{F}[R_X(\tau)] = 2\times 10^{-4}\Lambda\left(\frac{f}{10^4}\right)$, from wich the bandwidth of the process is $W = 10^4$ Hz. Therefore, $f_s = 2W + W_G = 2\times 10^4 + 2500 = 22500$ samples/sec. and the rate is $R = \nu f_s = 22500\times\log_2 128 = 22500\times 7 = 157500$ bits/sc. We also observe that $P_X = R_X(0) = 2$ and $x_{\max} = 10$, hence

$$\text{SQNR(dB)} = 4.8 + 6\nu + 10\log_{10}\frac{P_X}{x_{\max}^2} = 4.8 + 6\times 7 + 10\log_{10}\frac{2}{100} \approx 29.8\,\text{dB}$$

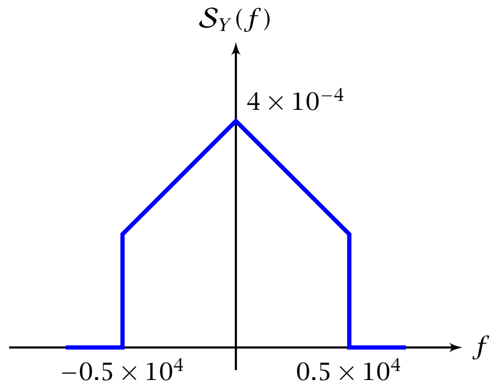
and $B_T = R/2 = 78750$ Hz.

2. Here we need to improve the SQNR by $56 - 29.8 = 26.2$ dB. Since each doubling of N improves SQNR by 6 dB, we have to double N at least $26.2/6$ or 5 times, i.e., the new N is $2^5 = 32$ times the old N, i.e., $N = 128\times 32 = 4096$. The resulting SQNR is $5\times 6 = 30$ dB more than the original SQNR, i.e., $29.8 + 30 = 59.8$ dB.

3. Using $B_T = R/2 = \nu f_s/2 = \nu(W + W_G/2)$ we want to find the largest integer ν that satisfies $93000 = \nu(10000 + W_G/2)$ Clearly $\nu = 9$ is the answer which gives $W_G = 6000/9 \approx 667$ Hz. Since $\nu = 9$ is an increase of 2 bits/sample compared to $\nu = 7$ the resulting SQNR is 12 dB higher or $12 + 29.8 = 41.8$ dB. For this system $N = 2^9 = 512$ and $f_s = 2W + W_G \approx 20667$ samples/sec.

Problem 7.20

1. $P_X = R_X(0) = 4$ Watts.
2. $\mathcal{S}_X(f) = \mathscr{F}[R_X(\tau)] = 4 \times 10^{-4}\Lambda\left(\frac{f}{10^4}\right)$.
3. From $\mathcal{S}_X(f)$ the range of frequencies are $[-10^4, 10^4]$, hence, $W = 10^4$.
4. $\mathcal{S}_Y(f) = \mathcal{S}_X(f)|H(f)|^2 = \mathcal{S}_X(f)\Pi\left(\frac{f}{10^4}\right) = \begin{cases} 10^{-4}\Lambda\left(\frac{f}{10^4}\right), & |f| \leq 5000 \\ 0, & \text{otherwise} \end{cases}$. The plot of $\mathcal{S}_Y(f)$ is shown below



and the power is the area under the power spectral density

$$P_Y = \int_{-\infty}^{\infty} \mathscr{S}_Y(f)\,df = 10^4 \times 2 \times 10^{-4} + \frac{1}{2} \times 10^4 \times 2 \times 10^{-4} = 3 \text{ W}$$

5. Since $X(t)$ is Gaussian and zero-mean, all random variables are zero-mean Gaussian with variances $\mathsf{E}[X^2(0)]$, $\mathsf{E}[X^2(10^{-4})]$, and $\mathsf{E}[X^2(1.5 \times 10^{-4})]$. But all these variances are equal to $R_X(0) = \int_{-\infty}^{\infty} \mathcal{S}_X(f)\,df = 4$, hence all random variables are distributed according a $\mathcal{N}(0, 4)$ PDF.
6. The covariance between $X(0)$ and $X(10^{-4})$ is $R_X(10^{-4}) = 0$, therefore these random variables are uncorrelated, and since they are jointly Gaussian, they are also independent. For $X(0)$ and $X(1.5 \times 10^{-4})$, the covariance is $R_X(1.5 \times 10^{-4}) = 4\,\text{sinc}^2(1.5) \neq 0$, hence the two random variables are correlated and hence dependent.

Problem 7.21

1. $f_s = 2W + W_G = 2\times 5000 + 2000 = 12000$ samples/sec. and $R = \nu f_s = 12000 \times \log_2 128 = 84000$ bps.

2. From $f_X(d)$ we find the power as

$$P_X = \mathsf{E}\left[X^2(t)\right] = \int_{-2}^{0} \frac{1}{3}x^2\,dx + \int_0^2 \frac{1}{6}x^2\,dx = \frac{4}{3}$$

   and SQNR(dB) $= 4.8 + 6\nu + 10\log_{10}\frac{P_X}{x_{\max}^2} = 4.8 + 42 + 10\log_{10}\frac{4/3}{4} \approx 42.$

3. $B_T = R/2 = 84000/2 = 42000$ Hz.

4. Using $B_T = R/2 = \nu f_s/2 = \nu(W + W_G/2)$ we want to find the largest integer ν that satisfies $70000 = \nu(5000 + W_G/2)$ Clearly $\nu = 14$ is the answer which gives $W_G = 0$ Hz. Since $\nu = 14$ is an increase of 7 bits/sample compared to $\nu = 7$ the resulting SQNR is 42 dB higher or $42 + 42 = 84$ dB.

Problem 7.22

1. The power is the integral of the power spectral density or the area below it, i.e., $P_X = \frac{1}{2} \times 2 \times (4000 + 6000) = 10000$ W.

2. $\nu = \log_2 512 = 9$, hence, SQNR$= 4.8 + 6\nu + 10\log_{10}\frac{P_X}{x_{\max}^2} = 4.8 + 6\times 9 + 10\log_{10}\frac{10000}{200^2} = 4.8 + 54 - 6 = 52.2$ dB. We also have $f_s = 2W + W_G = 2\times 3000 + 1000 = 7000$ Hz, then $R = \nu f_s = 9 \times 7000 = 63000$ and $B_T = R/2 = 31500$ Hz.

3. Using $B_T = R/2 = \nu f_s/2 = \nu(W + W_G/2)$ we want to find the largest integer ν that satisfies $47000 = \nu(3000 + W_G/2)$ Clearly $\nu = 15$ is the answer which gives $W_G = 4000/15 \approx 267$ Hz. Since $\nu = 15$ is an increase of 6 bits/sample compared to $\nu = 9$ the resulting SQNR is 36 dB higher or $36 + 52.2 = 88.2$ dB.

Problem 7.23

1. $P_X = \frac{1}{\pi}\int_{-200}^{200}\frac{1}{1+f^2}\,df = \frac{1}{\pi}\tan^{-1} f\Big]_{-200}^{200} \approx 1$ where we have used the approximations $\tan^{-1} 200 = \frac{\pi}{2}$ and $\tan^{-1}(-200) = -\frac{\pi}{2}$. To find the SQNR, we have $\text{SQNR}_{\text{dB}} = 6\times 8 + 4.8 + 10\log\frac{1}{100} = 48 + 4.8 - 20 = 32.8$ dB.

2. To increase the SQNR by 20 dB, we need at least 4 more bits per sample (each bit improved the SQNR by 6 dB). The new number of bits per sample is 8+4=12 bits and the new number of levels is $2^{12} = 4096$.

3. The minimum transmission bandwidth is obtained from $\text{BW} = \nu W = 12 \times 200 = 2400$ Hz.

Problem 7.24

1. Any probability density function satisfies $\int_{-\infty}^{\infty} f(x)\,dx = 1$ here the area under the density function has to be one. This is the area of the left triangle plus the area of the right rectangle in the plot of $f(x)$. Therefore, we should have

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{2a}{2} + 2a = 3a = 1 \Rightarrow a = \frac{1}{3}$$

2. The equation for $f(x)$ is

$$f(x) = \begin{cases} \frac{x}{6} + \frac{1}{3}, & -2 \le x < 0 \\ \frac{1}{3}, & 0 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

Here the density function is given therefore the power can be obtained from $E[X^2(t)] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$. We have

$$\begin{aligned} P_X &= \int_{-2}^{0} x^2\left(\frac{x}{6} + \frac{1}{3}\right)dx + \int_{0}^{2} x^2 \frac{1}{3}\,dx \\ &= \left[\frac{1}{24}x^4 + \frac{1}{9}x^3\right]_{-2}^{0} + \left[\frac{1}{9}x^3\right]_{0}^{2} \\ &= \frac{2}{9} + \frac{8}{9} = \frac{10}{9} \end{aligned}$$

3. Here $x_{\max} = 2$, and $N = 2^{\nu} = 32$, therefore $\nu = 5$, and

$$\begin{aligned} \text{SQNR}|_{\text{dB}} &= 4.8 + 6\nu + 10\log_{10} \frac{P_X}{x_{\max}^2} \\ &= 4.8 + 30 + 10\log_{10} \frac{10/9}{4} \\ &= 34.8 + 10\log_{10} \frac{5}{18} \\ &= 34.8 - 5.56 = 29.24 \ \text{ dB} \end{aligned}$$

4. $B_T = \nu W = 5 \times 12000 = 60000$ Hz.

5. Each extra bit improves SQNR by 6 dB, since we need an extra 20 dB, we need at least 4 more bits (3 more bits can improve the performance by only 18 dB), therefore the new ν will be 5+4=9 and the new bandwidth will be $B_T = 9 \times 12000 = 108000$ Hz . Compared to the previous required bandwidth of 60000 Hz, this is an increase of eighty per cent.

Problem 7.25

1. $\mathcal{S}_X(f)$ is a triangle of height one extending from -5000 to 5000. Therefore the area under it which is the power in the process is $P_X = 5000$ Watts.
2. $f_s = 2W + W_G = 2 \times 5000 + 2000 = 12000$ samples [er second.
3. $256 = 2^\nu$, hence $\nu = 8$. We have
$$\text{SQNR}_{\text{dB}} = 4.8 + 6\nu + 10\log_{10}\frac{P_X}{x_{\max}^2} = 4.8 + 48 + 10\log_{10}\frac{5000}{360000} = 34.23 \text{ dB}$$
4. $R = \nu f_s = 8 \times 12000 = 96000$ bits/sec.
5. $\text{BW}_\text{T} = \frac{R}{2} = 48000$ Hz.
6. Since each extra bit increases the SNR by 6 dB's, for at least 25 dB's we need at least 5 extra bits. The new value of nu is 8+5=13, and we have $N = 2^\nu = 2^{13} = 8192$, $\text{SQNR}_{\text{dB}} = 34.23 + 6 \times 5 = 64.23$, and $\text{BW}_\text{T} = \frac{\nu f_s}{2} = 78000$.

Problem 7.26

Suppose that the transmitted sequence is $\mathbf{x}$. If an error occurs at the i^{th} bit of the sequence, then the received sequence $\mathbf{x}'$ is
$$\mathbf{x}' = \mathbf{x} + [0\ldots010\ldots0]$$
where addition is modulo 2. Thus the error sequence is $e_i = [0\ldots010\ldots0]$, which in natural binary coding has the value 2^{i-1}. If the spacing between levels is Δ, then the error introduced by the channel is $2^{i-1}\Delta$.

2)

$$\begin{aligned}
D_{\text{channel}} &= \sum_{i=1}^{\nu} p(\text{error in } i \text{ bit}) \cdot (2^{i-1}\Delta)^2 \\
&= \sum_{i=1}^{\nu} p_b\Delta^2 4^{i-1} = p_b\Delta^2\frac{1-4^\nu}{1-4} \\
&= p_b\Delta^2\frac{4^\nu-1}{3}
\end{aligned}$$

3) The total distortion is

$$\begin{aligned}
D_{\text{total}} &= D_{\text{channel}} + D_{\text{quantiz.}} = p_b\Delta^2\frac{4^\nu-1}{3} + \frac{x_{\max}^2}{3\cdot N^2} \\
&= p_b\frac{4\cdot x_{\max}^2}{N^2}\frac{4^\nu-1}{3} + \frac{x_{\max}^2}{3\cdot N^2}
\end{aligned}$$

or since $N = 2^\nu$

$$D_{\text{total}} = \frac{x_{\max}^2}{3 \cdot 4^\nu}(1 + 4p_b(4^\nu - 1)) = \frac{x_{\max}^2}{3N^2}(1 + 4p_b(N^2 - 1))$$

4)

$$\text{SNR} = \frac{E[X^2]}{D_{\text{total}}} = \frac{E[X^2]3N^2}{x_{\max}^2(1 + 4p_b(N^2 - 1))}$$

If we let $\breve{X} = \frac{X}{x_{\max}}$, then $\frac{E[X^2]}{x_{\max}^2} = E[\breve{X}^2] = \overline{\breve{X}^2}$. Hence,

$$\text{SNR} = \frac{3N^2\overline{\breve{X}^2}}{1 + 4p_b(N^2 - 1)} = \frac{3 \cdot 4^\nu \overline{\breve{X}^2}}{1 + 4p_b(4^\nu - 1)}$$

Problem 7.27

The sampling rate is $f_s = 44100$ meaning that we take 44100 samples per second. Each sample is quantized using 16 bits so the total number of bits per second is 44100×16. For a music piece of duration 50 min = 3000 sec the resulting number of bits per channel (left and right) is

$$44100 \times 16 \times 3000 = 2.1168 \times 10^9$$

and the overall number of bits is

$$2.1168 \times 10^9 \times 2 = 4.2336 \times 10^9$$

Computer Problems

Computer Problem 7.1

The following MATLAB script finds the quantization levels as $(-5.1865, -4.2168, -2.3706, 0.7228, -0.4599, 1.5101, 3.2827, 5.1865)$.

```
% MATLAB script for Computer Problem 7.1.

echo on ;
a=[−10,−5,−4,−2,0,1,3,5,10];
for i=1:length(a)−1
  y(i)=centroid('normal',a(i),a(i+1),0.001,0,1);
  echo off ;
end
```

In this MATLAB script the MATLAB function centroid.m given next finds the centroid of a region.

```
function y=centroid(funfcn,a,b,tol,p1,p2,p3)
% CENTROID   Finds the centroid of a function over a region.
%            Y=CENTROID('F',A,B,TOL,P1,P2,P3) finds the centroid of the
%            function F defined in an m-file on the [A,B] region. The
%            function can contain up to three parameters, P1, P2, P3.
%            tol=the relative error.

args=[];
for n=1:nargin-4
  args=[args,',p',int2str(n)];
end
args=[args,')'];
funfcn1='x_fnct';
y1=eval(['quad(funfcn1,a,b,tol,[],funfcn',args]);
y2=eval(['quad(funfcn,a,b,tol,[]',args]);
y=y1/y2;
```

MATLAB functions xfunct.m and normal.m that arse used in centroid.m are given next

```
function y=x_fnct(x,funfcn,p1,p2,p3)
% y=x_fnct(x,funfcn,p1,p2,p3)
%  Returns the function funfcn(x) times x

args=[];
for nn=1:nargin-2
  args=[args,',p',int2str(nn)];
end
args=[args,')'] ;
y=eval([funfcn,'(x',args,'.*x' ]);
```

```
function y=normal(x,m,s)
% FUNCTION y=NORMAL(x,m,s)
% Gaussian distribution
% m=mean
% s=standard deviation
y=(1/sqrt(2*pi*s^2))*exp(-((x-m).^2)/(2*s^2));
```

Computer Problem 7.2

1) By the symmetry assumption the boundaries of the quantization regions are $0, \pm 1, \pm 2, \pm 3, \pm 4$, and ± 5.

2) The quantization regions are $(-\infty, -5]$, $(-5, -4]$, $(-4, -3]$, $(-3, -2]$, $(-2, -1]$, $(-1, 0]$, $(0, 1]$, $(1, 2]$, $(2, 3]$, $(3, 4]$, $(4, 5]$, and $(5, +\infty)$.

3) The MATLAB function uq_dist.m is used to find the distortion of a uniform quantizer (it is assumed that the quantization levels are set to the centroids of the quantization regions). uq_dist.m and the function mse_dist.m called by uq_dist.m are given next

```
function [y,dist]=uq_dist(funfcn,b,c,n,delta,s,tol,p1,p2,p3)
%UQ_DIST     returns the distortion of a uniform quantizer
%            with quantization points set to the centroids
%            [Y,DIST]=UQ_DIST(FUNFCN,B,C,N,DELTA,S,TOL,P1,P2,P3)
%            funfcn=source density function given in an m-file
%            with at most three parameters, p1,p2,p3.
%            [b,c]=The support of the source density function.
%            n=number of levels.
%            delta=level size.
%            s=the leftmost quantization region boundary.
%            p1,p2,p3=parameters of the input function.
%            y=quantization levels.
%            dist=distortion.
%            tol=the relative error.

if (c-b<delta*(n-2))
  error('Too many levels for this range.'); return
end
if (s<b)
  error('The leftmost boundary too small.'); return
end
if (s+(n-2)*delta>c)
  error('The leftmost boundary too large.'); return
end
args=[];
for j=1:nargin-7
  args=[args,',p',int2str(j)];
end
args=[args,')'];
a(1)=b;
for i=2:n
  a(i)=s+(i-2)*delta;
end
a(n+1)=c;
[y,dist]=eval(['mse_dist(funfcn,a,tol',args]);
```

```
function [y,dist]=mse_dist(funfcn,a,tol,p1,p2,p3)
%MSE_DIST    returns the mean-squared quantization error.
%            [Y,DIST]=MSE_DIST(FUNFCN,A,TOL,P1,P2,P3)
%            funfcn=The distribution function given
%            in an m-file. It can depend on up to three
%            parameters, p1,p2,p3.
%            a=the vector defining the boundaries of the
%            quantization regions. (Note: [a(1),a(length(a))]
%            is the support of funfcn.)
%            p1,p2,p3=parameters of funfcn.
%            tol=the relative error.

args=[];
for n=1:nargin-3
  args=[args,',p',int2str(n)];
end
```

```
args=[args,')'];
for i=1:length(a)−1
  y(i)=eval(['centroid(funfcn,a(i),a(i+1),tol',args]);
end
dist=0;
for i=1:length(a)−1
  newfun = 'x_a2_fnct' ;
  dist=dist+eval(['quad(newfun,a(i),a(i+1),tol,[],funfcn,', num2str(y(i)), args]);
end
```

In uq_dist.m function we can substitute $b = -20$, $c = 20$, $\Delta = 1$, $n = 12$, $s = -5$, tol = 0.001, $p_1 = 0$, and $p_2 = 2$. Substituting these values into uq_dist.m, we obtain a squared error distortion of 0.0851 and quantization values of ±0.4897, ±1.4691, ±2.4487, ±3.4286, ±4.4089, and ±5.6455.

Computer Problem 7.3

In order to design a a Lloyd-Max quantizer, the m-file lloydmax.m given next is used

```
function [a,y,dist]=lloydmax(funfcn,b,n,tol,p1,p2,p3)
%LLOYDMAX    returns the the Lloyd-Max quantizer and the mean-squared
%            quantization error for a symmetric distribution
%            [A,Y,DIST]=LLOYDMAX(FUNFCN,B,N,TOL,P1,P2,P3).
%            funfcn=the density function given
%            in an m-file. It can depend on up to three
%            parameters, p1,p2,p3.
%            a=the vector giving the boundaries of the
%            quantization regions.
%            [-b,b] approximates support of the density function.
%            n=the number of quantization regions.
%            y=the quantization levels.
%            p1,p2,p3=parameters of funfcn.
%            tol=the relative error.

args=[];
for j=1:nargin−4
  args=[args,',p',int2str(j)];
end
args=[args,')'];
v=eval(['variance(funfcn,-b,b,tol',args]);
a(1)=−b;
d=2*b/n;
for i=2:n
  a(i)=a(i−1)+d;
end
a(n+1)=b;
dist=v;
[y,newdist]=eval(['mse_dist(funfcn,a,tol',args]);
while(newdist<0.99*dist),
  for i=2:n
    a(i)=(y(i−1)+y(i))/2;
  end
```

```
  dist=newdist;
  [y,newdist]=eval(['mse_dist(funfcn,a,tol',args]);
end
```

1) Using $b = 10$, $n = 10$, tol = 0.01, $p_1 = 0$, and $p_2 = 1$ in lloydmax.m, we obtain the quantization boundaries and quantization levels vectors $\boldsymbol{a}$ and $\boldsymbol{y}$ as

$$\boldsymbol{a} = \pm 10, \pm 2.16, \pm 1.51, \pm 0.98, \pm 0.48, 0$$
$$\boldsymbol{y} = \pm 2.52, \pm 1.78, \pm 1.22, \pm 0.72, \pm 0.24$$

2) The mean squared distortion is found (using lloydmax.m) to be 0.02.

Computer Problem 7.4

The m-file u_pcm.m given next takes as its input a sequence of sampled values and the number of desired quantization levels and finds the quantized sequence, the encoded sequence, and the resulting SQNR (in decibels).

```
function [sqnr,a_quan,code]=u_pcm(a,n)
%U_PCM        uniform PCM encoding of a sequence
%             [SQNR,A_QUAN,CODE]=U_PCM(A,N)
%             a=input sequence.
%             n=number of quantization levels (even).
%             sqnr=output SQNR (in dB).
%             a_quan=quantized output before encoding.
%             code=the encoded output.

amax=max(abs(a));
a_quan=a/amax;
b_quan=a_quan;
d=2/n;
q=d.*[0:n-1];
q=q-((n-1)/2)*d;
for i=1:n
  a_quan(find((q(i)-d/2 <= a_quan) & (a_quan <= q(i)+d/2)))=...
  q(i).*ones(1,length(find((q(i)-d/2 <= a_quan) & (a_quan <= q(i)+d/2))));
  b_quan(find( a_quan==q(i) ))=(i-1).*ones(1,length(find( a_quan==q(i) )));
end
a_quan=a_quan*amax;
nu=ceil(log2(n));
code=zeros(length(a),nu);
for i=1:length(a)
  for j=nu:-1:0
    if ( fix(b_quan(i)/(2^j)) == 1)
        code(i,(nu-j)) = 1;
        b_quan(i) = b_quan(i) - 2^j;
    end
  end
end
sqnr=20*log10(norm(a)/norm(a-a_quan));
```

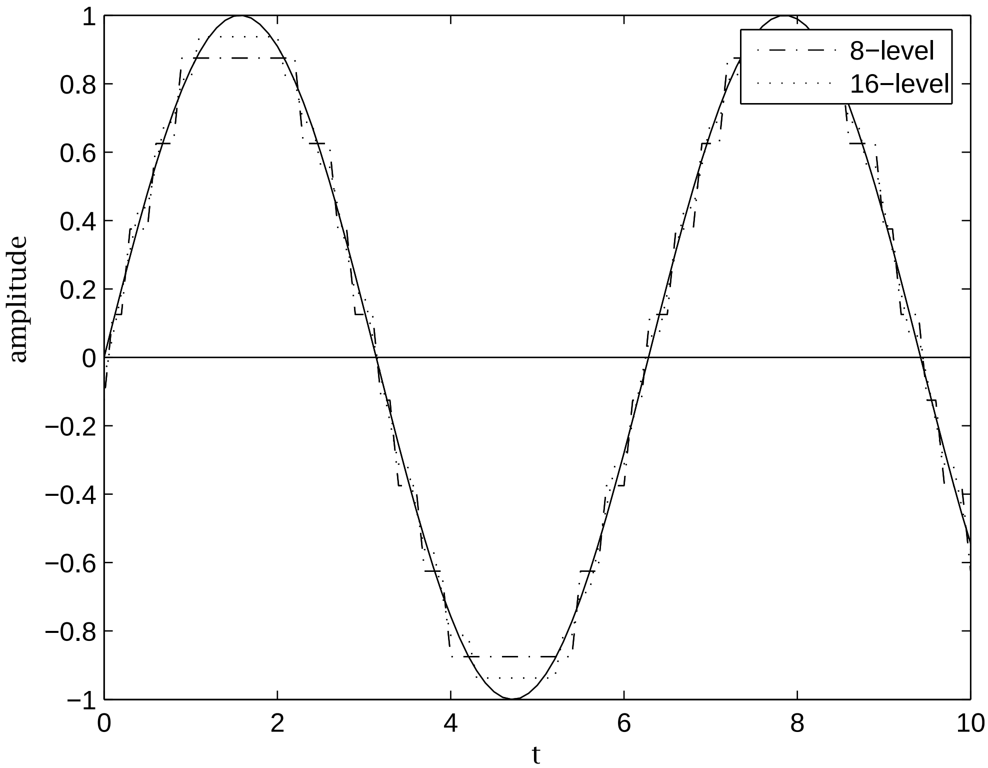


Figure 7.1: Uniform PCM for a sinusoidal signal using 8 and 16 levels

---

1) We arbitrarily choose the duration of the signal to be 10 s. Then using the u_pcm.m m-file, we generate the quantized signals for the two cases of 8 and 16 quantization levels. The plots are shown in Figure 7.1.

2) The resulting SQNRs are 18.8532 dB for the 8-level PCM and 25.1153 dB for the 16-level uniform PCM.

A MATLAB script for this problem is shown next.

```
% MATLAB script for Computer Problem 7.4.
echo on
t=[0:0.1:10];
a=sin(t);
[sqnr8,aquan8,code8]=u_pcm(a,8);
[sqnr16,aquan16,code16]=u_pcm(a,16);
pause % Press a key to see the SQNR for N = 8.
sqnr8
pause % Press a key to see the SQNR for N = 16.
sqnr16
```

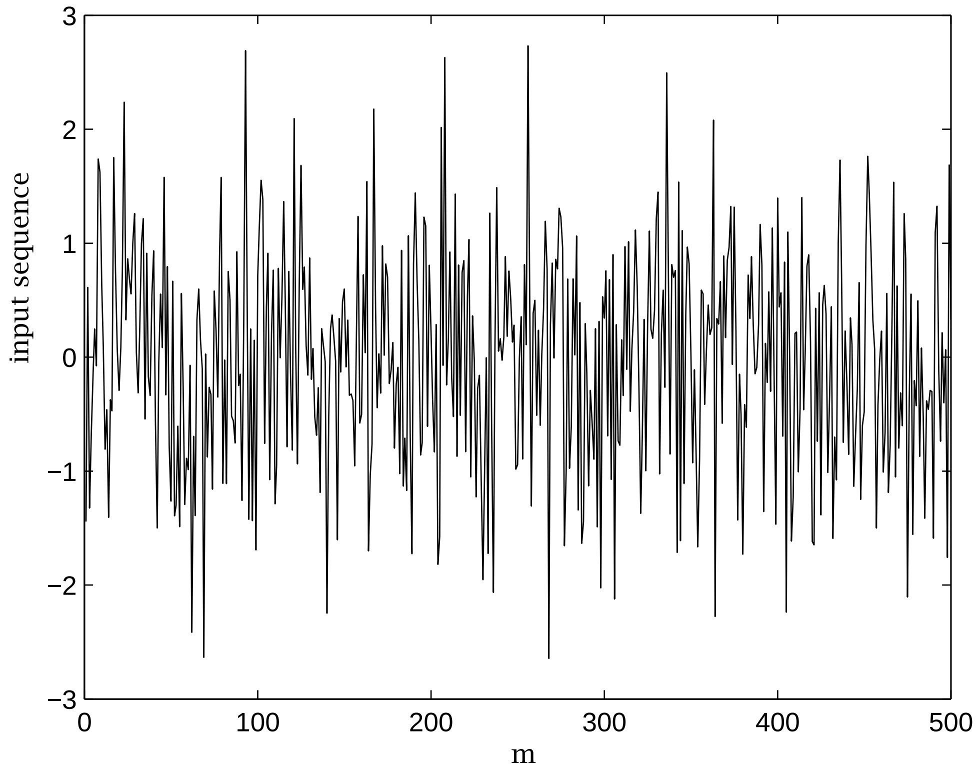


Figure 7.2: The plot of input sequence for 64 quantization levels

```
pause  % Press a key to see the plot of the signal and its quantized versions.
plot(t,a,'-',t,aquan8,'-.',t,aquan16,'-',t,zeros(1,length(t)))
```

Computer Problem 7.5

1) The plot of 500 point sequence is given in Figure 7.2

2) Using the MATLAB function u_pcm.m given in Computer Problem 7.4, we find the SQNR for the 64-level quantizer to be 31.66 dB.

3) Again by using the MATLAB function u_pcm.m, the first five values of the sequence, the corresponding quantized values, and the corresponding PCM codewords are given as

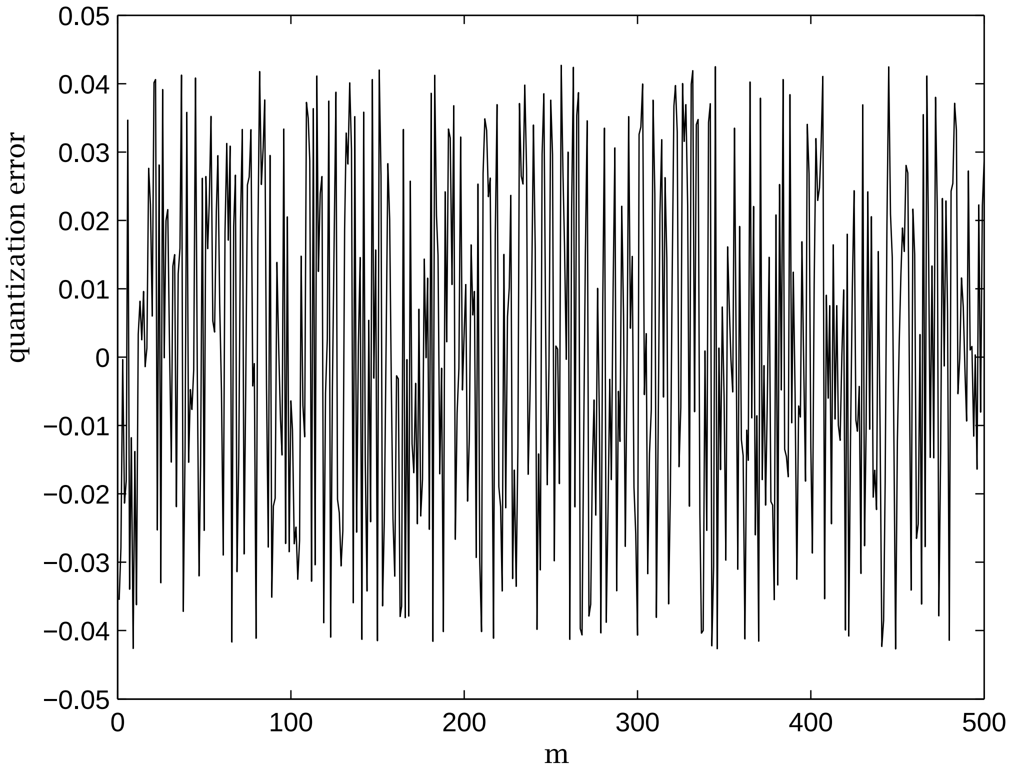


Figure 7.3: Quantization error in uniform PCM for 64 quantization levels

$$\text{Input} = [-0.4326, -1.6656, 0.1253, 0.2877, -1.1465] \tag{7.21}$$

$$\text{Quantized values} = [-0.4331, -1.6931, 0.1181, 0.2756, -1.1419] \tag{7.22}$$

$$\text{Codewords} = \left\{ \begin{array}{cccccc} 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right. \tag{7.23}$$

4) Plot of the quantization error is shown in Figure 7.3

Computer Problem 7.6

1) This question is solved by using the m-file mula_pcm.m, which is the equivalent of the m-file u_pcm.m when using a μlaw PCM scheme. This file is given next

```
function [sqnr,a_quan,code]=mula_pcm(a,n,mu)
```

```
%MULA_PCM     mu-law PCM encoding of a sequence
%             [SQNR,A_QUAN,CODE]=MULA_PCM(A,N,MU).
%             a=input sequence.
%             n=number of quantization levels (even).
%             sqnr=output SQNR (in dB).
%             a_quan=quantized output before encoding.
%             code=the encoded output.

[y,maximum]=mulaw(a,mu);
[sqnr,y_q,code]=u_pcm(y,n);
a_quan=invmulaw(y_q,mu);
a_quan=maximum*a_quan;
sqnr=20*log10(norm(a)/norm(a-a_quan));
```

The two m-files mulaw.m and invmulaw.m given below implement μ-law nonlinearity and its inverse. signum.m function that finds the signum of a vector is also given next.

```
function [y,a]=mulaw(x,mu)
%MULAW         mu-law nonlinearity for nonuniform PCM
%              Y=MULAW(X,MU).
%              X=input vector.

a=max(abs(x));
y=(log(1+mu*abs(x/a))./log(1+mu)).*signum(x);
```

```
function x=invmulaw(y,mu)
%INVMULAW              the inverse of mu-law nonlinearity
%X=INVMULAW(Y,MU)  Y=normalized output of the mu-law nonlinearity.

x=(((1+mu).^(abs(y))-1)./mu).*signum(y);
```

```
function y=signum(x)
%SIGNUM finds the signum of a vector.
%        Y=SIGNUM(X)
%        X=input vector

y=x;
y(find(x>0))=ones(size(find(x>0)));
y(find(x<0))=-ones(size(find(x<0)));
y(find(x==0))=zeros(size(find(x==0)));
```

Let the vector $\boldsymbol{a}$ be the vector of length 500 generated according to $\mathcal{N}(0,1)$; that is, let

$$a = \text{randn}(1, 500)$$

Then by using

$$[\text{dist,a\_quan,code}] = \text{mula\_pcm}(a, 16, 255)$$

we can obtain the quantized sequence and the SQNR for a 16-level quantization. Plots of the input–output relation for the quantizer and the quantization error are given in Figures 7.4, 7.5, and 7.6.

Using mula_perm.m, the SQNR is found to be 13.96 dB. For the case of 64 levels we obtain SQNR = 26.30 dB, and for 128 levels we have SQNR = 31.49 dB.

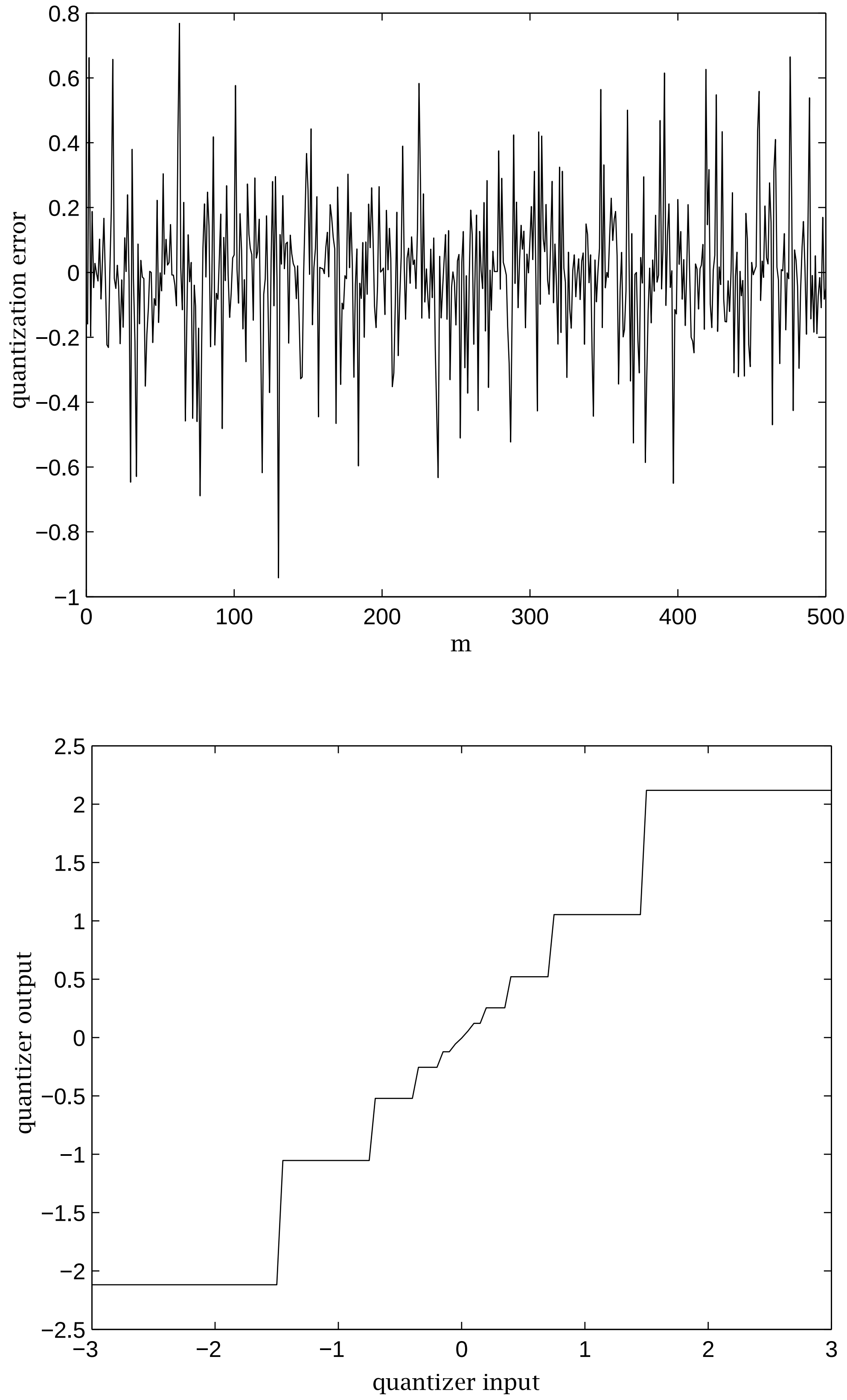


Figure 7.4: Quantization error and quantizer input–output relation for a 16-level μ-law PCM

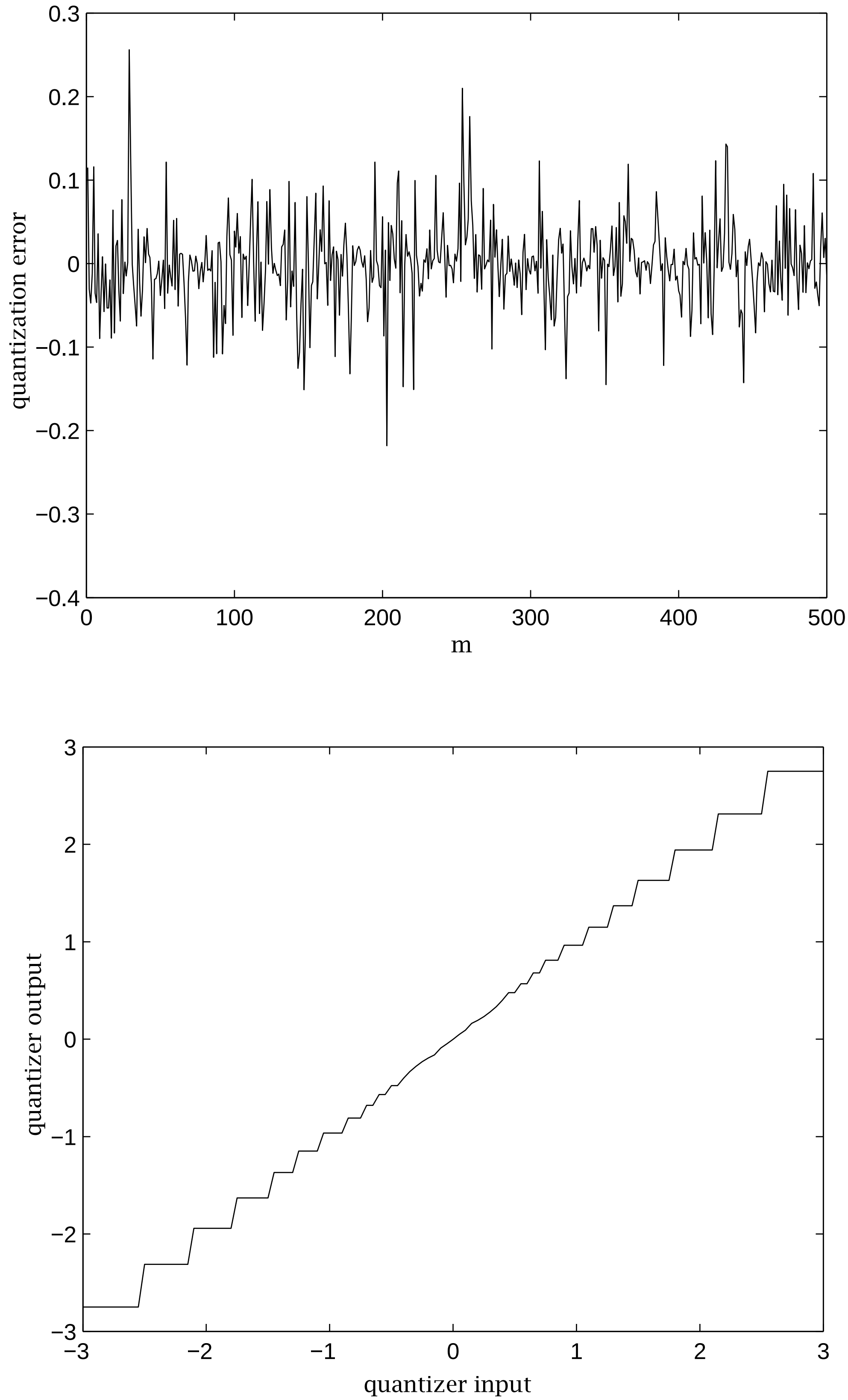


Figure 7.5: Quantization error and quantizer input–output relation for a 64-level μ-law PCM

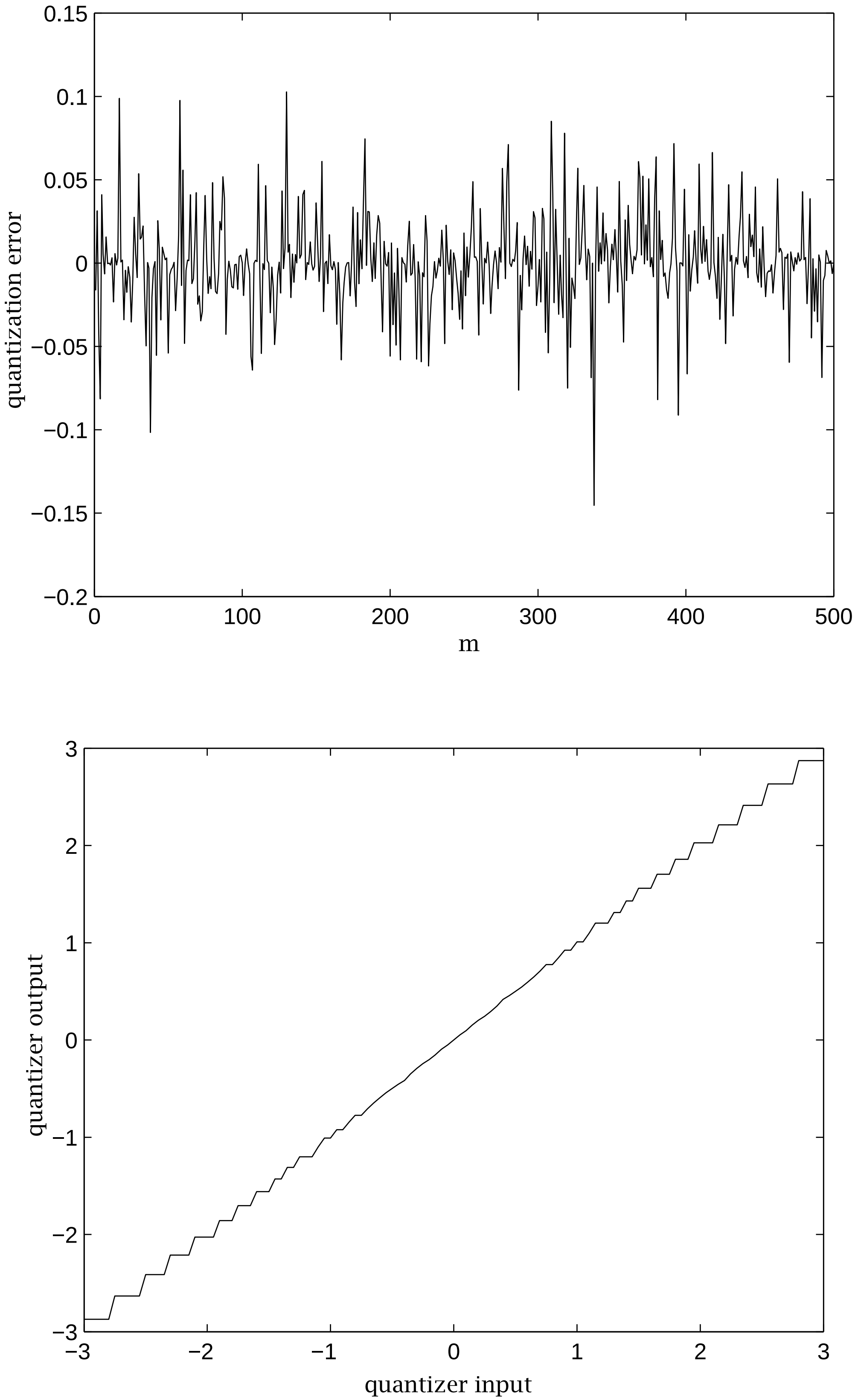


Figure 7.6: Quantization error and quantizer input–output relation for a 128-level μ-law PCM

Chapter 8

Problem 8.1

1) To show that the waveforms $\psi_n(t)$, $n = 1, 2, 3$ are orthogonal we have to prove that

$$\int_{-\infty}^{\infty} \psi_m(t)\psi_n(t)dt = 0, \qquad m \neq n$$

Clearly,

$$\begin{aligned} c_{12} &= \int_{-\infty}^{\infty} \psi_1(t)\psi_2(t)dt = \int_0^4 \psi_1(t)\psi_2(t)dt \\ &= \int_0^2 \psi_1(t)\psi_2(t)dt + \int_2^4 \psi_1(t)\psi_2(t)dt \\ &= \frac{1}{4}\int_0^2 dt - \frac{1}{4}\int_2^4 dt = \frac{1}{4}\times 2 - \frac{1}{4}\times(4-2) \\ &= 0 \end{aligned}$$

Similarly,

$$\begin{aligned} c_{13} &= \int_{-\infty}^{\infty} \psi_1(t)\psi_3(t)dt = \int_0^4 \psi_1(t)\psi_3(t)dt \\ &= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt - \frac{1}{4}\int_2^3 dt + \frac{1}{4}\int_3^4 dt \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} c_{23} &= \int_{-\infty}^{\infty} \psi_2(t)\psi_3(t)dt = \int_0^4 \psi_2(t)\psi_3(t)dt \\ &= \frac{1}{4}\int_0^1 dt - \frac{1}{4}\int_1^2 dt + \frac{1}{4}\int_2^3 dt - \frac{1}{4}\int_3^4 dt \\ &= 0 \end{aligned}$$

Thus, the signals $\psi_n(t)$ are orthogonal.

2) We first determine the weighting coefficients

$$x_n = \int_{-\infty}^{\infty} x(t)\psi_n(t)dt, \qquad n = 1, 2, 3$$

$$\begin{aligned} x_1 &= \int_0^4 x(t)\psi_1(t)dt = -\frac{1}{2}\int_0^1 dt + \frac{1}{2}\int_1^2 dt - \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0 \\ x_2 &= \int_0^4 x(t)\psi_2(t)dt = \frac{1}{2}\int_0^4 x(t)dt = 0 \\ x_3 &= \int_0^4 x(t)\psi_3(t)dt = -\frac{1}{2}\int_0^1 dt - \frac{1}{2}\int_1^2 dt + \frac{1}{2}\int_2^3 dt + \frac{1}{2}\int_3^4 dt = 0 \end{aligned}$$

As it is observed, $x(t)$ is orthogonal to the signal waveforms $\psi_n(t)$, $n = 1, 2, 3$ and thus it can not represented as a linear combination of these functions.

Problem 8.2

1) The expansion coefficients $\{c_n\}$, that minimize the mean square error, satisfy

$$c_n = \int_{-\infty}^{\infty} x(t)\psi_n(t)dt = \int_0^4 \sin\frac{\pi t}{4}\psi_n(t)dt$$

Hence,

$$\begin{aligned} c_1 &= \int_0^4 \sin\frac{\pi t}{4}\psi_1(t)dt = \frac{1}{2}\int_0^2 \sin\frac{\pi t}{4}dt - \frac{1}{2}\int_2^4 \sin\frac{\pi t}{4}dt \\ &= -\frac{2}{\pi}\cos\frac{\pi t}{4}\bigg|_0^2 + \frac{2}{\pi}\cos\frac{\pi t}{4}\bigg|_2^4 \\ &= -\frac{2}{\pi}(0-1) + \frac{2}{\pi}(-1-0) = 0 \end{aligned}$$

Similarly,

$$\begin{aligned} c_2 &= \int_0^4 \sin\frac{\pi t}{4}\psi_2(t)dt = \frac{1}{2}\int_0^4 \sin\frac{\pi t}{4}dt \\ &= -\frac{2}{\pi}\cos\frac{\pi t}{4}\bigg|_0^4 = -\frac{2}{\pi}(-1-1) = \frac{4}{\pi} \end{aligned}$$

and

$$\begin{aligned} c_3 &= \int_0^4 \sin\frac{\pi t}{4}\psi_3(t)dt \\ &= \frac{1}{2}\int_0^1 \sin\frac{\pi t}{4}dt - \frac{1}{2}\int_1^2 \sin\frac{\pi t}{4}dt + \frac{1}{2}\int_2^3 \sin\frac{\pi t}{4}dt - \frac{1}{2}\int_3^4 \sin\frac{\pi t}{4}dt \\ &= 0 \end{aligned}$$

Note that c_1, c_2 can be found by inspection since $\sin\frac{\pi t}{4}$ is even with respect to the $x = 2$ axis and $\psi_1(t)$, $\psi_3(t)$ are odd with respect to the same axis.

2) The residual mean square error $E_{\min}$ can be found from

$$E_{\min} = \int_{-\infty}^{\infty} |x(t)|^2 dt - \sum_{i=1}^{3} |c_i|^2$$

Thus,

$$\begin{aligned} E_{\min} &= \int_0^4 \left(\sin\frac{\pi t}{4}\right)^2 dt - \left(\frac{4}{\pi}\right)^2 = \frac{1}{2}\int_0^4 \left(1-\cos\frac{\pi t}{2}\right)dt - \frac{16}{\pi^2} \\ &= 2 - \frac{1}{\pi}\sin\frac{\pi t}{2}\bigg|_0^4 - \frac{16}{\pi^2} = 2 - \frac{16}{\pi^2} \end{aligned}$$

Problem 8.3

1) As an orthonormal set of basis functions we consider the set

$$\psi_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad \psi_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases}$$

$$\psi_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases} \qquad \psi_4(t) = \begin{cases} 1 & 3 \le t < 4 \\ 0 & \text{o.w} \end{cases}$$

In matrix notation, the four waveforms can be represented as

$$\begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \\ s_4(t) \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \\ \psi_4(t) \end{pmatrix}$$

Note that the rank of the transformation matrix is 4 and therefore, the dimensionality of the waveforms is 4

2) The representation vectors are

$$\begin{aligned} \mathbf{s}_1 &= \begin{bmatrix} 2 & -1 & -1 & -1 \end{bmatrix} \\ \mathbf{s}_2 &= \begin{bmatrix} -2 & 1 & 1 & 0 \end{bmatrix} \\ \mathbf{s}_3 &= \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \\ \mathbf{s}_4 &= \begin{bmatrix} 1 & -2 & -2 & 2 \end{bmatrix} \end{aligned}$$

3) The distance between the first and the second vector is

$$d_{1,2} = \sqrt{|\mathbf{s}_1 - \mathbf{s}_2|^2} = \sqrt{\left|\begin{bmatrix} 4 & -2 & -2 & -1 \end{bmatrix}\right|^2} = \sqrt{25}$$

Similarly we find that

$$\begin{aligned}
d_{1,3} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_3|^2} = \sqrt{\left|\left[\begin{array}{cccc} 1 & 0 & -2 & 0 \end{array}\right]\right|^2} = \sqrt{5} \\
d_{1,4} &= \sqrt{|\mathbf{s}_1 - \mathbf{s}_4|^2} = \sqrt{\left|\left[\begin{array}{cccc} 1 & 1 & 1 & -3 \end{array}\right]\right|^2} = \sqrt{12} \\
d_{2,3} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_3|^2} = \sqrt{\left|\left[\begin{array}{cccc} -3 & 2 & 0 & 1 \end{array}\right]\right|^2} = \sqrt{14} \\
d_{2,4} &= \sqrt{|\mathbf{s}_2 - \mathbf{s}_4|^2} = \sqrt{\left|\left[\begin{array}{cccc} -3 & 3 & 3 & -2 \end{array}\right]\right|^2} = \sqrt{31} \\
d_{3,4} &= \sqrt{|\mathbf{s}_3 - \mathbf{s}_4|^2} = \sqrt{\left|\left[\begin{array}{cccc} 0 & 1 & 3 & -3 \end{array}\right]\right|^2} = \sqrt{19}
\end{aligned}$$

Thus, the minimum distance between any pair of vectors is $d_{\min} = \sqrt{5}$.

Problem 8.4

As a set of orthonormal functions we consider the waveforms

$$\psi_1(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{o.w} \end{cases} \qquad \psi_2(t) = \begin{cases} 1 & 1 \le t < 2 \\ 0 & \text{o.w} \end{cases} \qquad \psi_3(t) = \begin{cases} 1 & 2 \le t < 3 \\ 0 & \text{o.w} \end{cases}$$

The vector representation of the signals is

$$\begin{aligned}
\mathbf{s}_1 &= \left[\begin{array}{ccc} 2 & 2 & 2 \end{array}\right] \\
\mathbf{s}_2 &= \left[\begin{array}{ccc} 2 & 0 & 0 \end{array}\right] \\
\mathbf{s}_3 &= \left[\begin{array}{ccc} 0 & -2 & -2 \end{array}\right] \\
\mathbf{s}_4 &= \left[\begin{array}{ccc} 2 & 2 & 0 \end{array}\right]
\end{aligned}$$

Note that $s_3(t) = s_2(t) - s_1(t)$ and that the dimensionality of the waveforms is 3.

Problem 8.5

Case 1: $f_c = \frac{k}{2T_b}$, where k is a positive integer:

$$\begin{aligned}
E_1 = E_2 &= \int_0^{T_b} s_1^2(t)\,dt = \frac{2\mathcal{E}_b}{T_b}\int_0^{T_b} \cos^2(2\pi f_c t)\,dt \\
&= \frac{\mathcal{E}_b}{T_b}\int_0^{T_b}\left[1 + \cos\left(4\pi\frac{k}{2T_b}t\right)\right]dt \\
&= \frac{\mathcal{E}_b}{T_b}\left[T_b + \left[\frac{T_b}{2\pi k}\sin\left(4\pi\frac{k}{2T_b}t\right)\right]_0^{T_b}\right] \\
&= \mathcal{E}_b
\end{aligned}$$

Case 2: $f_c \neq \frac{k}{2T_b}$, but $f_c T_b \gg 1$:

$$\begin{aligned} E_1 = E_2 = \int_0^{T_b} s_1^2(t)\,dt &= \frac{2\mathcal{E}_b}{T_b}\int_0^{T_b} \cos^2(2\pi f_c t)\,dt \\ &= \frac{\mathcal{E}_b}{T_b}\int_0^{T_b} [1+\cos(4\pi f_c t)]\,dt \\ &= \frac{\mathcal{E}_b}{T_b}\left[T_b + \left[\frac{1}{4\pi f_c}\sin(4\pi f_c t)\right]_0^{T_b}\right] \\ &= \mathcal{E}_b + \frac{\mathcal{E}_b}{2\pi f_c T_b}\sin(4\pi f_c T_b) \end{aligned}$$

Noting that $|\sin(4\pi f_c T_b)| \leq 1$ and $f_c T_b \gg 1$, we conclude that teh seconfd term is negligible compared to the first term; hence $E_1 = E_2 \approx \mathcal{E}_b$.

Problem 8.6

Proof of the energy part in this problem is the same as the solution of problem 8.5. For the orthogonality, we have

$$\begin{aligned} \int_0^{T_b} s_1(t)s_2(t)\,dt &= \frac{2\mathcal{E}_b}{T_b}\int_0^{T_b} \cos(2\pi f_1 t)\cos(2\pi f_2 t)\,dt \\ &= \frac{2\mathcal{E}_b}{2T_b}\int_0^{T_b} [\cos 2\pi(f_1+f_2)t + \cos 2\pi(f_1-f_2)t]\,dt \\ &= \frac{2\mathcal{E}_b}{2T_b}\left[\frac{1}{2\pi(f_1+f_2)}\sin 2\pi\left(\frac{k_1+k_2}{2T_b}\right)t + \frac{1}{2\pi(f_1-f_2)}\sin 2\pi\left(\frac{k_1-k_2}{2T_b}\right)t\right]_0^{T_b} \\ &= 0 \end{aligned}$$

The proof for the case where $f_1 T_b \gg 1$ and $f_2 T_b \gg 1$ is similar to the proof of case 2 in the solution of Problem 8.5.

Problem 8.7

1) The impulse response of the filter matched to $s(t)$ is

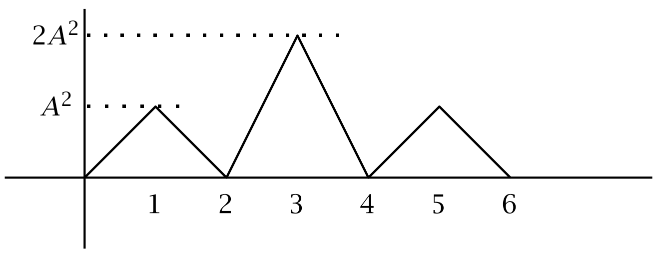
$$h(t) = s(T-t) = s(3-t) = s(t)$$

where we have used the fact that $s(t)$ is even with respect to the $t = \frac{T}{2} = \frac{3}{2}$ axis.

2) The output of the matched filter is

$$
\begin{aligned}
y(t) &= s(t) \star s(t) = \int_0^t s(\tau)s(t-\tau)d\tau \\
&= \begin{cases} 0 & t < 0 \\ A^2 t & 0 \le t < 1 \\ A^2(2-t) & 1 \le t < 2 \\ 2A^2(t-2) & 2 \le t < 3 \\ 2A^2(4-t) & 3 \le t < 4 \\ A^2(t-4) & 4 \le t < 5 \\ A^2(6-t) & 5 \le t < 6 \\ 0 & 6 \le t \end{cases}
\end{aligned}
$$

A sketch of $y(t)$ is depicted in the next figure



3) At the output of the matched filter and for $t = T = 3$ the noise is

$$
\begin{aligned}
n_T &= \int_0^T n(\tau)h(T-\tau)d\tau \\
&= \int_0^T n(\tau)s(T-(T-\tau))d\tau = \int_0^T n(\tau)s(\tau)d\tau
\end{aligned}
$$

The variance of the noise is

$$
\begin{aligned}
\sigma_{n_T}^2 &= E\left[\int_0^T \int_0^T n(\tau)n(v)s(\tau)s(v)d\tau dv\right] \\
&= \int_0^T \int_0^T s(\tau)s(v)E[n(\tau)n(v)]d\tau dv \\
&= \frac{N_0}{2}\int_0^T \int_0^T s(\tau)s(v)\delta(\tau - v)d\tau dv \\
&= \frac{N_0}{2}\int_0^T s^2(\tau)d\tau = N_0 A^2
\end{aligned}
$$

4) For antipodal equiprobable signals the probability of error is

$$P(e) = Q\left[\sqrt{\left(\frac{S}{N}\right)_o}\right]$$

where $\left(\frac{S}{N}\right)_o$ is the output SNR from the matched filter. Since

$$\left(\frac{S}{N}\right)_o = \frac{y^2(T)}{E[n_T^2]} = \frac{4A^4}{N_0 A^2}$$

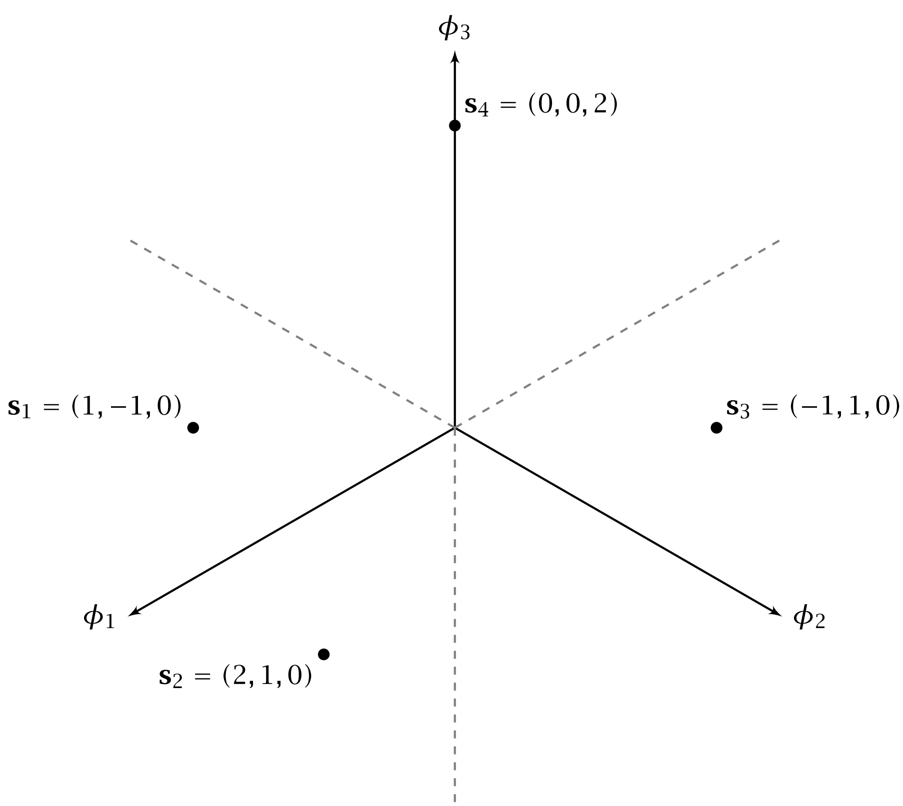
we obtain

$$P(e) = Q\left[\sqrt{\frac{4A^2}{N_0}}\right]$$

Problem 8.8

1. Since $s_3(t) = -s_1(t)$, it is sufficient to consider just $s_1(t)$, $s_2(t)$ and $s_4(t)$. By inspection, we can choose $\phi_1(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}$, $\phi_2(t) = \phi_1(t-1)$, and $\phi_3(t) = \phi_1(t-2)$. With this selection $\boldsymbol{s}_1 = (1, -1, 0)$, $\boldsymbol{s}_2 = (2, 1, 1)$, $\boldsymbol{s}_3 = (-1, 1, 0)$, and $\boldsymbol{s}_4 = (0, 0, 2)$.

2. The constellation is shown below



3. The matrix representation of the four vectors is

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The three columns are clearly linearly independent, hence the rank of the matrix is 3. Therefore the dimensionality of the signal space is 3.

4. We know that in general $E_m = \|\boldsymbol{s}_m\|^2$, hence, $E_1 = \|\boldsymbol{s}_1\|^2 = 2$, $E_2 = \|\boldsymbol{s}_2\|^2 = 5$, $E_3 = \|\boldsymbol{s}_3\|^2 = 2$, and $E_4 = \|\boldsymbol{s}_4\|^2 = 4$. Therefore, $E_{\text{avg}} = \frac{1}{4}(2+5+2+4) = \frac{13}{4}$ and $E_{\text{bavg}} = \frac{E_{\text{avg}}}{\log_2 M} = \frac{13}{8}$.

Problem 8.9

1) Taking the inverse Fourier transform of $H(f)$, we obtain

$$\begin{aligned} h(t) &= \mathcal{F}^{-1}[H(f)] = \mathcal{F}^{-1}\left[\frac{1}{j2\pi f}\right] - \mathcal{F}^{-1}\left[\frac{e^{-j2\pi fT}}{j2\pi f}\right] \\ &= \text{sgn}(t) - \text{sgn}(t-T) = 2\Pi\left(\frac{t-\frac{T}{2}}{T}\right) \end{aligned}$$

2) The signal waveform, to which $h(t)$ is matched, is

$$s(t) = h(T-t) = 2\Pi\left(\frac{T-t-\frac{T}{2}}{T}\right) = 2\Pi\left(\frac{\frac{T}{2}-t}{T}\right) = h(t)$$

where we have used the symmetry of $\Pi\left(\frac{t-\frac{T}{2}}{T}\right)$ with respect to the $t = \frac{T}{2}$ axis.

Problem 8.10

If $g_T(t) = \text{sinc}(t)$, then its matched waveform is $h(t) = \text{sinc}(-t) = \text{sinc}(t)$. Since, (see Problem 2.17)

$$\text{sinc}(t) \star \text{sinc}(t) = \text{sinc}(t)$$

the output of the matched filter is the same sinc pulse. If

$$g_T(t) = \text{sinc}\left(\frac{2}{T}\left(t - \frac{T}{2}\right)\right)$$

then the matched waveform is

$$h(t) = g_T(T-t) = \text{sinc}\left(\frac{2}{T}\left(\frac{T}{2}-t\right)\right) = g_T(t)$$

where the last equality follows from the fact that $g_T(t)$ is even with respect to the $t = \frac{T}{2}$ axis. The output of the matched filter is

$$\begin{aligned}
y(t) &= \mathcal{F}^{-1}[g_T(t) \star g_T(t)] \\
&= \mathcal{F}^{-1}\left[\frac{T^2}{4}\Pi\left(\frac{T}{2}f\right)e^{-j2\pi fT}\right] \\
&= \frac{T}{2}\text{sinc}\left(\frac{2}{T}(t-T)\right) = \frac{T}{2}g_T\left(t-\frac{T}{2}\right)
\end{aligned}$$

Thus the output of the matched filter is the same sinc function, scaled by $\frac{T}{2}$ and centered at $t = T$.

Problem 8.11

1) The output of the integrator is

$$\begin{aligned}
y(t) &= \int_0^t r(\tau)d\tau = \int_0^t [s_i(\tau) + n(\tau)]d\tau \\
&= \int_0^t s_i(\tau)d\tau + \int_0^t n(\tau)d\tau
\end{aligned}$$

At time $t = T$ we have

$$y(T) = \int_0^T s_i(\tau)d\tau + \int_0^T n(\tau)d\tau = \pm\sqrt{\frac{\mathcal{E}_b}{T}}T + \int_0^T n(\tau)d\tau$$

The signal energy at the output of the integrator at $t = T$ is

$$\mathcal{E}_s = \left(\pm\sqrt{\frac{\mathcal{E}_b}{T}}T\right)^2 = \mathcal{E}_b T$$

whereas the noise power

$$\begin{aligned}
P_n &= E\left[\int_0^T\int_0^T n(\tau)n(v)d\tau dv\right] \\
&= \int_0^T\int_0^T E[n(\tau)n(v)]d\tau dv \\
&= \frac{N_0}{2}\int_0^T\int_0^T \delta(\tau - v)d\tau dv = \frac{N_0}{2}T
\end{aligned}$$

Hence, the output SNR is

$$\text{SNR} = \frac{\mathcal{E}_s}{P_n} = \frac{2\mathcal{E}_b}{N_0}$$

2) The transfer function of the RC filter is

$$H(f) = \frac{1}{1 + j2\pi RCf}$$

Thus, the impulse response of the filter is

$$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u_{-1}(t)$$

and the output signal is given by

$$\begin{aligned} y(t) &= \frac{1}{RC}\int_{-\infty}^{t} r(\tau) e^{-\frac{t-\tau}{RC}} d\tau \\ &= \frac{1}{RC}\int_{-\infty}^{t} (s_i(\tau) + n(\tau)) e^{-\frac{t-\tau}{RC}} d\tau \\ &= \frac{1}{RC} e^{-\frac{t}{RC}} \int_{0}^{t} s_i(\tau) e^{\frac{\tau}{RC}} d\tau + \frac{1}{RC} e^{-\frac{t}{RC}} \int_{-\infty}^{t} n(\tau) e^{\frac{\tau}{RC}} d\tau \end{aligned}$$

At time $t = T$ we obtain

$$y(T) = \frac{1}{RC} e^{-\frac{T}{RC}} \int_{0}^{T} s_i(\tau) e^{\frac{\tau}{RC}} d\tau + \frac{1}{RC} e^{-\frac{T}{RC}} \int_{-\infty}^{T} n(\tau) e^{\frac{\tau}{RC}} d\tau$$

The signal energy at the output of the filter is

$$\begin{aligned} \mathcal{E}_s &= \frac{1}{(RC)^2} e^{-\frac{2T}{RC}} \int_{0}^{T}\int_{0}^{T} s_i(\tau) s_i(v) e^{\frac{\tau}{RC}} e^{\frac{v}{RC}} d\tau dv \\ &= \frac{1}{(RC)^2} e^{-\frac{2T}{RC}} \frac{\mathcal{E}_b}{T} \left(\int_{0}^{T} e^{\frac{\tau}{RC}} d\tau\right)^2 \\ &= e^{-\frac{2T}{RC}} \frac{\mathcal{E}_b}{T} \left(e^{\frac{T}{RC}} - 1\right)^2 \\ &= \frac{\mathcal{E}_b}{T} \left(1 - e^{-\frac{T}{RC}}\right)^2 \end{aligned}$$

The noise power at the output of the filter is

$$\begin{aligned} P_n &= \frac{1}{(RC)^2} e^{-\frac{2T}{RC}} \int_{-\infty}^{T}\int_{-\infty}^{T} E[n(\tau)n(v)] d\tau dv \\ &= \frac{1}{(RC)^2} e^{-\frac{2T}{RC}} \int_{-\infty}^{T}\int_{-\infty}^{T} \frac{N_0}{2} \delta(\tau - v) e^{\frac{\tau+v}{RC}} d\tau dv \\ &= \frac{1}{(RC)^2} e^{-\frac{2T}{RC}} \int_{-\infty}^{T} \frac{N_0}{2} e^{\frac{2\tau}{RC}} d\tau \\ &= \frac{1}{2RC} e^{-\frac{2T}{RC}} \frac{N_0}{2} e^{\frac{2T}{RC}} = \frac{1}{2RC}\frac{N_0}{2} \end{aligned}$$

Hence,

$$\text{SNR} = \frac{\mathcal{E}_s}{P_n} = \frac{4\mathcal{E}_b RC}{TN_0} \left(1 - e^{-\frac{T}{RC}}\right)^2$$

3) The value of RC that maximizes SNR, can be found by setting the partial derivative of SNR with respect to RC equal to zero. Thus, if $a = RC$, then

$$\frac{\vartheta \text{SNR}}{\vartheta a} = 0 = (1 - e^{-\frac{T}{a}}) - \frac{T}{a} e^{-\frac{T}{a}} = -e^{-\frac{T}{a}} \left(1 + \frac{T}{a}\right) + 1$$

Solving this transcendental equation numerically for a, we obtain

$$\frac{T}{a} = 1.26 \Longrightarrow RC = a = \frac{T}{1.26}$$

Problem 8.12

1) The matched filter is

$$h_1(t) = s_1(T-t) = \begin{cases} -\frac{1}{T}t + 1, & 0 \leq t < T \\ 0 & \text{otherwise} \end{cases}$$

The output of the matched filter is

$$y_1(t) = \int_{-\infty}^{\infty} s_1(\tau)h_1(t-\tau)d\tau$$

If $t \leq 0$, then $y_1(t) = 0$, If $0 < t \leq T$, then

$$\begin{aligned} y_1(t) &= \int_0^{\infty} \frac{\tau}{T}\left(-\frac{1}{T}(t-\tau)+1\right)d\tau \\ &= \int_0^t \tau\left(\frac{1}{T} - \frac{t}{T^2}\right)d\tau + \frac{1}{T^2}\int_0^t \tau^2 d\tau \\ &= -\frac{t^3}{6T^2} + \frac{t^2}{2T} \end{aligned}$$

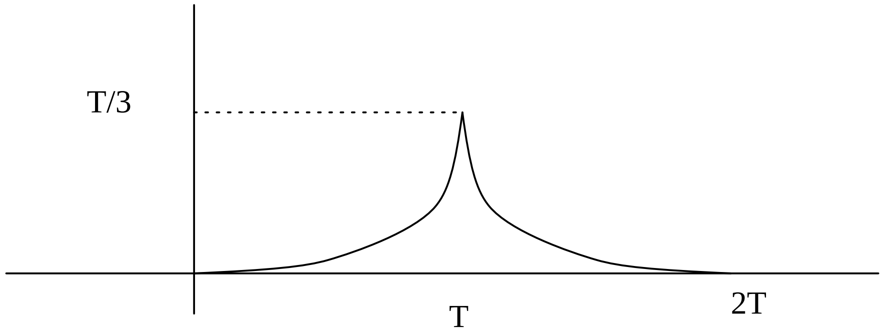
If $T \leq t \leq 2T$, then

$$\begin{aligned} y_1(t) &= \int_{t-\tau}^{T} \frac{\tau}{T}\left(-\frac{1}{T}(t-\tau)+1\right)d\tau \\ &= \int_{t-\tau}^{T} \tau\left(\frac{1}{T} - \frac{t}{T^2}\right)d\tau + \frac{1}{T^2}\int_{t-\tau}^{T} \tau^2 d\tau \\ &= \frac{(t-T)^3}{6T^2} - \frac{t-T}{2} + \frac{T}{3} \end{aligned}$$

For $2T < 0$, we obtain $y_1(t) = 0$. In summary

$$y_1(t) = \begin{cases} 0 & t \leq 0 \\ -\frac{t^3}{6T^2} + \frac{t^2}{2T} & 0 < t \leq T \\ \frac{(t-T)^3}{6T^2} - \frac{t-T}{2} + \frac{T}{3} & T < t \leq 2T \\ 0 & 2T < t \end{cases}$$

A sketch of $y_1(t)$ is given in the next figure. As it is observed the maximum of $y_1(t)$, which is $\frac{T}{3}$, is achieved for $t = T$.



2) The signal waveform matched to $s_2(t)$ is

$$h_2(t) = \begin{cases} -1, & 0 \leq t \leq \frac{T}{2} \\ 2, & \frac{T}{2} < t \leq T \end{cases}$$

The output of the matched filter is

$$y_2(t) = \int_{-\infty}^{\infty} s_2(\tau) h_2(t-\tau) d\tau$$

If $t \leq 0$ or $t \geq 2T$, then $y_2(t) = 0$. If $0 < t \leq \frac{T}{2}$, then $y_2(t) = \int_0^t (-2) d\tau = -2t$. If $\frac{T}{2} < t \leq T$, then

$$y_2(t) = \int_0^{t-\frac{T}{2}} 4 d\tau + \int_{t-\frac{T}{2}}^{\frac{T}{2}} (-2) d\tau + \int_{-\frac{T}{2}}^{t} d\tau = 7t - \frac{9}{2}T$$

If $T < t \leq \frac{3T}{2}$, then

$$y_2(t) = \int_{t-T}^{\frac{T}{2}} 4 d\tau + \int_{\frac{T}{2}}^{t-\frac{T}{2}} (-2) d\tau + \int_{t-\frac{T}{2}}^{T} d\tau = \frac{19T}{2} - 7t$$

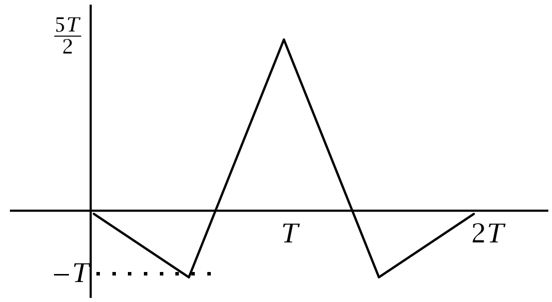
For, $\frac{3T}{2} < t \leq 2T$, we obtain

$$y_2(t) = \int_{t-T}^{T} (-2) d\tau = 2t - 4T$$

In summary

$$y_2(t) = \begin{cases} 0 & t \leq 0 \\ -2t & 0 < t \leq \frac{T}{2} \\ 7t - \frac{9}{2}T & \frac{T}{2} < t \leq T \\ \frac{19T}{2} - 7t & T < t \leq \frac{3T}{2} \\ 2t - 4T & \frac{3T}{2} < t \leq 2T \\ 0 & 2T < t \end{cases}$$

A plot of $y_2(t)$ is shown in the next figure



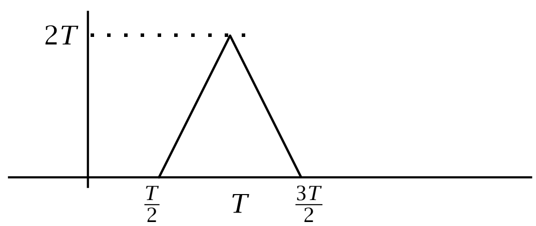
3) The signal waveform matched to $s_3(t)$ is

$$h_3(t) = \begin{cases} 2 & 0 \le t \le \frac{T}{2} \\ 0 & \frac{T}{2} < t \le T \end{cases}$$

The output of the matched filter is

$$y_3(t) = h_3(t) \star s_3(t) = \begin{cases} 4t - 2T & \frac{T}{2} \le t < T \\ -4t + 6T & T \le t \le \frac{3T}{2} \end{cases}$$

In the next figure we have plotted $y_3(t)$.



Problem 8.13

Since the rate of transmission is $R = 10^5$ bits/sec, the bit interval T_b is 10^{-5} sec. The probability of error in a binary PAM system is

$$P(e) = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]$$

where the bit energy is $\mathcal{E}_b = A^2 T_b$. With $P(e) = P_2 = 10^{-6}$, we obtain

$$\sqrt{\frac{2\mathcal{E}_b}{N_0}} = 4.75 \Longrightarrow \mathcal{E}_b = \frac{4.75^2 N_0}{2} = 0.112813$$

Thus

$$A^2 T_b = 0.112813 \Longrightarrow A = \sqrt{0.112813 \times 10^5} = 106.21$$

Problem 8.14

1) For a binary PAM system for which the two signals have unequal probability, the optimum detector is

$$r \underset{s_2}{\overset{s_1}{\gtrless}} \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln \frac{1-p}{p} = \alpha^*$$

Here $\sqrt{\mathcal{E}_b}/N_0 = 10$ and $p = 0.3$. Substituting in the above gives $\alpha^* = 0.025 \times \ln \frac{7}{3} \approx 0.02118$.

2) The average probability of error is

$$\begin{aligned}
P(e) &= P(e|s_1)P(s_1) + P(e|s_2)P(s_2) \\
&= pP(e|s_1) + (1-p)P(e|s_2) \\
&= p\int_{-\infty}^{\alpha^*} f(r|s_1)dr + (1-p)\int_{\alpha^*}^{\infty} f(r|s_1)dr \\
&= p\int_{-\infty}^{\alpha^*} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r-\sqrt{\mathcal{E}_b})^2}{N_0}} dr + (1-p)\int_{\alpha^*}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r+\sqrt{\mathcal{E}_b})^2}{N_0}} dr \\
&= p\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha_1^*} e^{-\frac{x^2}{2}} dx + (1-p)\frac{1}{\sqrt{2\pi}}\int_{\alpha_2^*}^{\infty} e^{-\frac{x^2}{2}} dx
\end{aligned}$$

where

$$\alpha_1^* = -\sqrt{\frac{2\mathcal{E}_b}{N_0}} + \alpha^*\sqrt{\frac{2}{N_0}} \qquad \alpha_2^* = \sqrt{\frac{2\mathcal{E}_b}{N_0}} + \alpha^*\sqrt{\frac{2}{N_0}}$$

Thus,

$$P(e) = pQ\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} - \alpha^*\sqrt{\frac{2}{N_0}}\right] + (1-p)Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} + \alpha^*\sqrt{\frac{2}{N_0}}\right]$$

If $p = 0.3$ and $\frac{\mathcal{E}_b}{N_0} = 10$, then

$$\begin{aligned}
P(e) &= 0.3Q[4.3774] + 0.7Q[4.5668] \\
&= 3.5348 \times 10^{-6}
\end{aligned}$$

If the symbols are equiprobable, then we have

$$P(e) = Q[\sqrt{\frac{2\mathcal{E}_b}{N_0}}] = Q[\sqrt{20}] = 3.8721 \times 10^{-6}$$

Problem 8.15

Assuming that $E[n^2(t)] = \sigma_n^2$, we obtain

$$\begin{aligned} E[n_1 n_2] &= E\left[\left(\int_0^T s_1(t)n(t)dt\right)\left(\int_0^T s_2(v)n(v)dv\right)\right] \\ &= \int_0^T \int_0^T s_1(t)s_2(v)E[n(t)n(v)]dtdv \\ &= \sigma_n^2 \int_0^T s_1(t)s_2(t)dt \\ &= 0 \end{aligned}$$

where the last equality follows from the orthogonality of the signal waveforms $s_1(t)$ and $s_2(t)$.

Problem 8.16

1) The optimum threshold is given by

$$\alpha^* = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln \frac{1-p}{p} = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2$$

2) The average probability of error is ($\alpha^* = \frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln 2$)

$$\begin{aligned} P(e) &= p(a_m = -1) \int_{\alpha^*}^{\infty} \frac{1}{\sqrt{\pi N_0}} e^{-(r+\sqrt{\mathcal{E}_b})^2/N_0} dr \\ &\quad + p(a_m = 1) \int_{-\infty}^{\alpha^*} \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{\mathcal{E}_b})^2/N_0} dr \\ &= \frac{2}{3} Q\left[\frac{\alpha^* + \sqrt{\mathcal{E}_b}}{\sqrt{N_0/2}}\right] + \frac{1}{3} Q\left[\frac{\sqrt{\mathcal{E}_b} - \alpha^*}{\sqrt{N_0/2}}\right] \\ &= \frac{2}{3} Q\left[\frac{\sqrt{2N_0/\mathcal{E}_b}\ln 2}{4} + \sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] + \frac{1}{3} Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} - \frac{\sqrt{2N_0/\mathcal{E}_b}\ln 2}{4}\right] \end{aligned}$$

3) Here we have $P_e = \frac{2}{3} Q\left[\frac{\sqrt{2N_0/\mathcal{E}_b}\ln 2}{4} + \sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] + \frac{1}{3} Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}} - \frac{\sqrt{2N_0/\mathcal{E}_b}\ln 2}{4}\right]$, substituting $\mathcal{E}_b = 1$ and $N_0 = 0.1$ we obtain

$$P_e = \frac{2}{3} Q\left[\frac{\sqrt{0.2}\times\ln 2}{4} + \sqrt{20}\right] + \frac{1}{3}\left[\sqrt{20} + \frac{\sqrt{0.2}\times\ln 2}{4}\right] = \frac{2}{3} Q(4.5496) - \frac{1}{3} Q(4.3946)$$

The result is $P_e = 3.64 \times 10^{-6}$.

Problem 8.17

1) The optimal receiver (see Problem 8.11) computes the metrics

$$C(\mathbf{r},\mathbf{s}_m) = \int_{-\infty}^{\infty} r(t)s_m(t)dt - \frac{1}{2}\int_{-\infty}^{\infty} |s_m(t)|^2 dt + \frac{N_0}{2}\ln P(\mathbf{s}_m)$$

and decides in favor of the signal with the largest $C(\mathbf{r},\mathbf{s}_m)$. Since $s_1(t) = -s_2(t)$, the energy of the two message signals is the same, and therefore the detection rule is written as

$$\int_{-\infty}^{\infty} r(t)s_1(t)dt \overset{s_1}{\underset{s_2}{\gtrless}} \frac{N_0}{4}\ln\frac{P(\mathbf{s}_2)}{P(\mathbf{s}_1)} = \frac{N_0}{4}\ln\frac{p_2}{p_1}$$

2) If $s_1(t)$ is transmitted, then the output of the correlator is

$$\begin{aligned}
\int_{-\infty}^{\infty} r(t)s_1(t)dt &= \int_0^T (s_1(t))^2 dt + \int_0^T n(t)s_1(t)dt \\
&= \mathcal{E}_s + n
\end{aligned}$$

where $\mathcal{E}_s$ is the energy of the signal and n is a zero-mean Gaussian random variable with variance

$$\begin{aligned}
\sigma_n^2 &= E\left[\int_0^T\int_0^T n(\tau)n(v)s_1(\tau)s_1(v)d\tau dv\right] \\
&= \int_0^T\int_0^T s_1(\tau)s_1(v)E[n(\tau)n(v)]d\tau dv \\
&= \frac{N_0}{2}\int_0^T\int_0^T s_1(\tau)s_1(v)\delta(\tau - v)d\tau dv \\
&= \frac{N_0}{2}\int_0^T |s_1(\tau)|^2 d\tau = \frac{N_0}{2}\mathcal{E}_s
\end{aligned}$$

Hence, the probability of error $P(e|\mathbf{s}_1)$ is

$$\begin{aligned}
P(e|\mathbf{s}_1) &= \int_{-\infty}^{\frac{N_0}{4}\ln\frac{p_2}{p_1} - \mathcal{E}_s} \frac{1}{\sqrt{\pi N_0 \mathcal{E}_s}} e^{-\frac{x^2}{N_0\mathcal{E}_s}} dx \\
&= Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} - \frac{1}{4}\sqrt{\frac{2N_0}{\mathcal{E}_s}}\ln\frac{p_2}{p_1}\right]
\end{aligned}$$

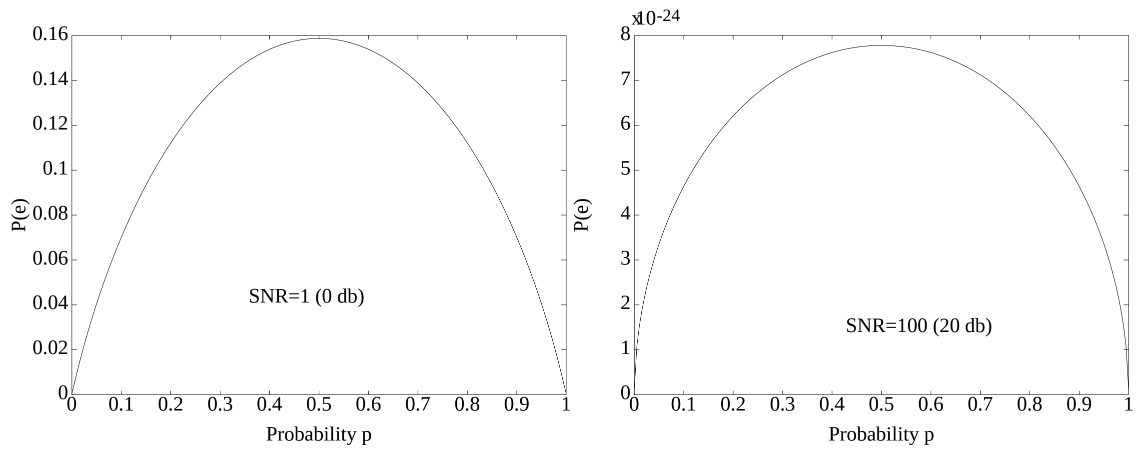
Similarly we find that

$$P(e|\mathbf{s}_2) = Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} + \frac{1}{4}\sqrt{\frac{2N_0}{\mathcal{E}_s}}\ln\frac{p_2}{p_1}\right]$$

The average probability of error is

$$\begin{aligned}
P(e) &= p_1 P(e|\mathbf{s}_1) + p_2 P(e|\mathbf{s}_2) \\
&= p_1 Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} - \frac{1}{4}\sqrt{\frac{2N_0}{\mathcal{E}_s}}\ln\frac{1-p_1}{p_1}\right] + (1-p_1)Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}} + \frac{1}{4}\sqrt{\frac{2N_0}{\mathcal{E}_s}}\ln\frac{1-p_1}{p_1}\right]
\end{aligned}$$

3) In the next figure we plot the probability of error as a function of p_1, for two values of the SNR $= \frac{2\mathcal{E}_s}{N_0}$. As it is observed the probability of error attains its maximum for equiprobable signals.



Problem 8.18

1) The two equiprobable signals have the same energy and therefore the optimal receiver bases its decisions on the rule

$$\int_{-\infty}^{\infty} r(t)s_1(t)dt \ \mathop{\gtrless}_{s_2}^{s_1} \ \int_{-\infty}^{\infty} r(t)s_2(t)dt$$

2) If the message signal $s_1(t)$ is transmitted, then $r(t) = s_1(t) + n(t)$ and the decision rule becomes

$$\begin{aligned}
&\int_{-\infty}^{\infty} (s_1(t) + n(t))(s_1(t) - s_2(t))dt \\
&= \int_{-\infty}^{\infty} s_1(t)(s_1(t) - s_2(t))dt + \int_{-\infty}^{\infty} n(t)(s_1(t) - s_2(t))dt \\
&= \int_{-\infty}^{\infty} s_1(t)(s_1(t) - s_2(t))dt + n \ \mathop{\gtrless}_{s_2}^{s_1} \ 0
\end{aligned}$$

where n is a zero mean Gaussian random variable with variance

$$\begin{aligned}
\sigma_n^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(s_1(\tau)-s_2(\tau))(s_1(v)-s_2(v))E[n(\tau)n(v)]d\tau dv \\
&= \int_0^T\int_0^T(s_1(\tau)-s_2(\tau))(s_1(v)-s_2(v))\frac{N_0}{2}\delta(\tau-v)d\tau dv \\
&= \frac{N_0}{2}\int_0^T(s_1(\tau)-s_2(\tau))^2 d\tau \\
&= \frac{N_0}{2}\int_0^T\int_0^T\left(\frac{2A\tau}{T}-A\right)^2 d\tau \\
&= \frac{N_0}{2}\frac{A^2T}{3}
\end{aligned}$$

Since

$$\begin{aligned}
\int_{-\infty}^{\infty}s_1(t)(s_1(t)-s_2(t))dt &= \int_0^T\frac{At}{T}\left(\frac{2At}{T}-A\right)dt \\
&= \frac{A^2T}{6}
\end{aligned}$$

the probability of error $P(e|\mathbf{s}_1)$ is given by

$$\begin{aligned}
P(e|\mathbf{s}_1) &= P(\frac{A^2T}{6}+n<0) \\
&= \frac{1}{\sqrt{2\pi\frac{A^2TN_0}{6}}}\int_{-\infty}^{-\frac{A^2T}{6}}\exp\left(-\frac{x^2}{2\frac{A^2TN_0}{6}}\right)dx \\
&= Q\left[\sqrt{\frac{A^2T}{6N_0}}\right]
\end{aligned}$$

Similarly we find that

$$P(e|\mathbf{s}_2) = Q\left[\sqrt{\frac{A^2T}{6N_0}}\right]$$

and since the two signals are equiprobable, the average probability of error is given by

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|\mathbf{s}_1)+\frac{1}{2}P(e|\mathbf{s}_2) \\
&= Q\left[\sqrt{\frac{A^2T}{6N_0}}\right] = Q\left[\sqrt{\frac{\mathcal{E}_s}{2N_0}}\right]
\end{aligned}$$

where $\mathcal{E}_s$ is the energy of the transmitted signals.

Problem 8.19

For binary phase modulation, the error probability is

$$P_2 = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{\frac{A^2T}{N_0}}\right]$$

With $P_2 = 10^{-6}$ we find from tables that

$$\sqrt{\frac{A^2T}{N_0}} = 4.74 \Longrightarrow A^2T = 44.9352 \times 10^{-10}$$

If the data rate is 10 Kbps, then the bit interval is $T = 10^{-4}$ and therefore, the signal amplitude is

$$A = \sqrt{44.9352 \times 10^{-10} \times 10^4} = 6.7034 \times 10^{-3}$$

Similarly we find that when the rate is 10^5 bps and 10^6 bps, the required amplitude of the signal is $A = 2.12 \times 10^{-2}$ and $A = 6.703 \times 10^{-2}$ respectively.

Problem 8.20

The energy of the two signals $s_1(t)$ and $s_2(t)$ is

$$\mathcal{E}_b = A^2T$$

The dimensionality of the signal space is one, and by choosing the basis function as

$$\psi(t) = \begin{cases} \frac{1}{\sqrt{T}} & 0 \le t < \frac{T}{2} \\ -\frac{1}{\sqrt{T}} & \frac{T}{2} \le t \le T \end{cases}$$

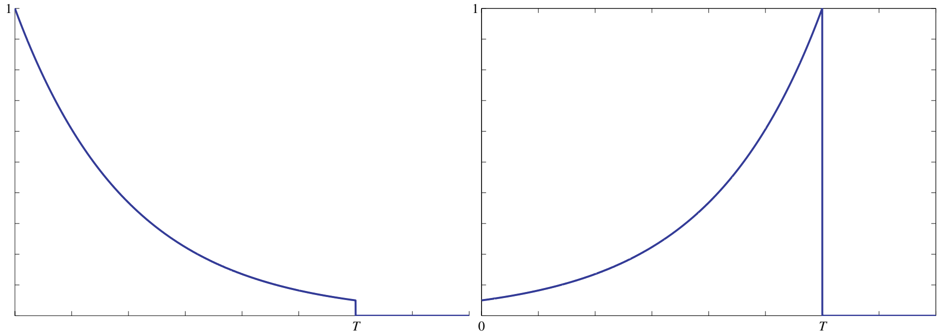
we find the vector representation of the signals as

$$s_{1,2} = \pm A\sqrt{T} + n$$

with n a zero-mean Gaussian random variable of variance $\frac{N_0}{2}$. The probability of error for antipodal signals is given by, where $\mathcal{E}_b = A^2T$. Hence,

$$P(e) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left[\sqrt{\frac{2A^2T}{N_0}}\right]$$

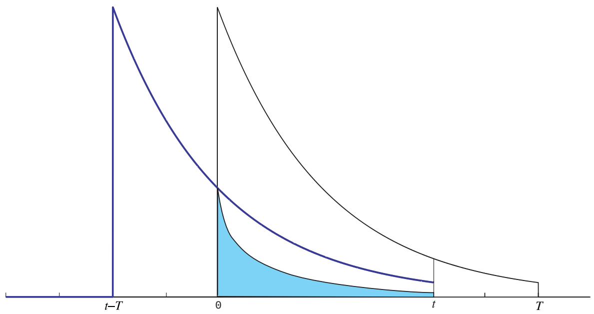
Problem 8.21
Plots of $s(t)$ and $h(t)$ are shown on left and right, respectively.



The output of the matched filter is

$$y(t) = \int_{-\infty}^{\infty} s(\tau)h(t-\tau)\,d\tau$$

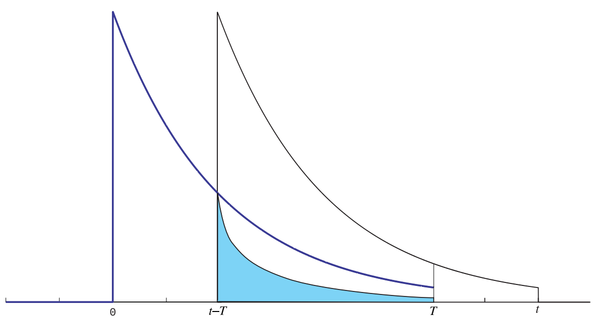
For $t < 0$, there is no overlap and the integral is zero. For $0 < t \leq T$ we have the following figure, where the the product of the two signals in the overlapping region is $s(\tau)h(t-\tau) = e^{-\tau} \times e^{-(T-t+\tau)} = e^{t-T-2\tau}$ and the integral is the area of the shaded region.



For this case we have

$$\begin{aligned} y(t) &= e^{t-T}\int_0^t e^{-2\tau}\,d\tau \\ &= e^{t-T}\left[-\frac{1}{2}e^{-2\tau}\right]_0^t \\ &= \frac{1}{2}e^{-T}\left(e^t - e^{-t}\right) \end{aligned}$$

For $T < t \leq 2T$ we have the following figure



and

$$\begin{aligned} y(t) &= e^{t-T}\int_{t-T}^{T} e^{-2\tau}\,d\tau \\ &= e^{t-T}\left[-\frac{1}{2}e^{-2\tau}\right]_{t-T}^{T} \\ &= \frac{1}{2}e^{-t+T} - \frac{1}{2}e^{t-3T} \end{aligned}$$

Therefore

$$y(t) = \begin{cases} \frac{1}{2}e^{-T}\left(e^t - e^{-t}\right) & 0 < t \leq T \\ \frac{1}{2}e^{-t+T} - \frac{1}{2}e^{t-3T} & T < t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Problem 8.22

We have $P_{av} = R\mathcal{E}_b = 2 \times 10^6 \mathcal{E}_b$, hence

$$P_b = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) = Q\left(\sqrt{\frac{2P_{av}}{RN_0}}\right) = 10^{-6}$$

Using the Q-function table (page 220) we have $Q(4.77) \approx 10^{-6}$, therefore

$$\sqrt{\frac{2P_{av}}{RN_0}} = \sqrt{\frac{2P_{av}}{2 \times 10^6 N_0}} = Q(4.77) \Rightarrow \frac{P_{av}}{10^6 N_0} = 4.77^2$$

From this we have $\frac{P_{av}}{N_0} = 22.753 \times 10^6$.

Problem 8.23

a) The received signal may be expressed as

$$r(t) = \begin{cases} n(t) & \text{if } s_0(t) \text{ was transmitted} \\ A + n(t) & \text{if } s_1(t) \text{ was transmitted} \end{cases}$$

Assuming that $s(t)$ has unit energy, then the sampled outputs of the crosscorrelators are

$$r = s_m + n, \qquad m = 0, 1$$

where $s_0 = 0$, $s_1 = A\sqrt{T}$ and the noise term n is a zero-mean Gaussian random variable with variance

$$\begin{aligned}
\sigma_n^2 &= E\left[\frac{1}{\sqrt{T}}\int_0^T n(t)dt \frac{1}{\sqrt{T}}\int_0^T n(\tau)d\tau\right] \\
&= \frac{1}{T}\int_0^T\int_0^T E\left[n(t)n(\tau)\right]dtd\tau \\
&= \frac{N_0}{2T}\int_0^T\int_0^T \delta(t-\tau)dtd\tau = \frac{N_0}{2}
\end{aligned}$$

The probability density function for the sampled output is

$$\begin{aligned}
f(r|s_0) &= \frac{1}{\sqrt{\pi N_0}}e^{-\frac{r^2}{N_0}} \\
f(r|s_1) &= \frac{1}{\sqrt{\pi N_0}}e^{-\frac{(r-A\sqrt{T})^2}{N_0}}
\end{aligned}$$

Since the signals are equally probable, the optimal detector decides in favor of s_0 if

$$\text{PM}(\mathbf{r}, \mathbf{s}_0) = f(r|s_0) > f(r|s_1) = \text{PM}(\mathbf{r}, \mathbf{s}_1)$$

otherwise it decides in favor of s_1. The decision rule may be expressed as

$$\frac{\mathrm{PM}(\mathbf{r},\mathbf{s}_0)}{\mathrm{PM}(\mathbf{r},\mathbf{s}_1)} = e^{\frac{(r-A\sqrt{T})^2-r^2}{N_0}} = e^{-\frac{(2r-A\sqrt{T})A\sqrt{T}}{N_0}} \underset{s_1}{\overset{s_0}{\gtrless}} 1$$

or equivalently

$$r \underset{s_0}{\overset{s_1}{\gtrless}} \frac{1}{2}A\sqrt{T}$$

The optimum threshold is $\frac{1}{2}A\sqrt{T}$.
b) The average probability of error is

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|s_0) + \frac{1}{2}P(e|s_1) \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} f(r|s_0)dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} f(r|s_1)dr \\
&= \frac{1}{2}\int_{\frac{1}{2}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{r^2}{N_0}}dr + \frac{1}{2}\int_{-\infty}^{\frac{1}{2}A\sqrt{T}} \frac{1}{\sqrt{\pi N_0}}e^{-\frac{(r-A\sqrt{T})^2}{N_0}}dr \\
&= \frac{1}{2}\int_{\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx + \frac{1}{2}\int_{-\infty}^{-\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
&= Q\left[\frac{1}{2}\sqrt{\frac{2}{N_0}}A\sqrt{T}\right] = Q\left[\sqrt{\mathrm{SNR}}\right]
\end{aligned}$$

where

$$\mathrm{SNR} = \frac{\frac{1}{2}A^2T}{N_0}$$

Thus, the on-off signaling requires a factor of two more energy to achieve the same probability of error as the antipodal signaling.

Problem 8.24
1) The impulse response of the matched filter is

$$s(t) = u(T-t) = \begin{cases} \frac{A}{T}(T-t)\cos(2\pi f_c(T-t)) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

2) The output of the matched filter at $t = T$ is

$$\begin{aligned}
g(T) &= u(t) \star s(t)\big|_{t=T} = \int_0^T u(T-\tau)s(\tau)d\tau \\
&= \frac{A^2}{T^2}\int_0^T (T-\tau)^2\cos^2(2\pi f_c(T-\tau))d\tau \\
&\overset{v=T-\tau}{=} \frac{A^2}{T^2}\int_0^T v^2\cos^2(2\pi f_c v)dv \\
&= \frac{A^2}{T^2}\left[\frac{v^3}{6} + \left(\frac{v^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c v) + \frac{v\cos(4\pi f_c v)}{4(2\pi f_c)^2}\right]\Bigg|_0^T \\
&= \frac{A^2}{T^2}\left[\frac{T^3}{6} + \left(\frac{T^2}{4\times 2\pi f_c} - \frac{1}{8\times(2\pi f_c)^3}\right)\sin(4\pi f_c T) + \frac{T\cos(4\pi f_c T)}{4(2\pi f_c)^2}\right]
\end{aligned}$$

3) The output of the correlator at $t = T$ is

$$\begin{aligned}
q(T) &= \int_0^T u^2(\tau)d\tau \\
&= \frac{A^2}{T^2}\int_0^T \tau^2\cos^2(2\pi f_c\tau)d\tau
\end{aligned}$$

However, this is the same expression with the case of the output of the matched filter sampled at $t = T$. Thus, the correlator can substitute the matched filter in a demodulation system and vise versa.

Problem 8.25

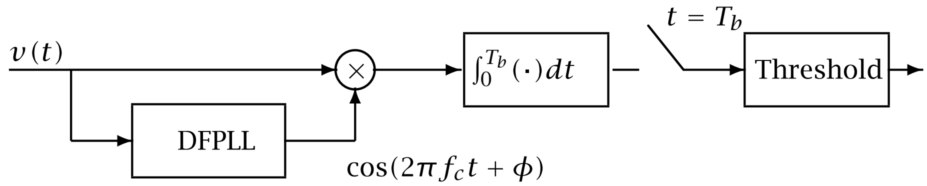
1) The signal $r(t)$ can be written as

$$\begin{aligned}
r(t) &= \pm\sqrt{2P_s}\cos(2\pi f_c t + \phi) + \sqrt{2P_c}\sin(2\pi f_c t + \phi) \\
&= \sqrt{2(P_c + P_s)}\sin\left(2\pi f_c t + \phi + a_n\tan^{-1}\left(\sqrt{\frac{P_s}{P_c}}\right)\right) \\
&= \sqrt{2P_T}\sin\left(2\pi f_c t + \phi + a_n\cos^{-1}\left(\sqrt{\frac{P_c}{P_T}}\right)\right)
\end{aligned}$$

where $a_n = \pm 1$ are the information symbols and P_T is the total transmitted power. As it is observed the signal has the form of a PM signal where

$$\theta_n = a_n\cos^{-1}\left(\sqrt{\frac{P_c}{P_T}}\right)$$

Any method used to extract the carrier phase from the received signal can be employed at the receiver. The following figure shows the structure of a receiver that employs a decision-feedback PLL. The operation of the PLL is described in the next part.



2) At the receiver the signal is demodulated by crosscorrelating the received signal

$$r(t) = \sqrt{2P_T}\sin\left(2\pi f_c t + \phi + a_n \cos^{-1}\left(\sqrt{\frac{P_c}{P_T}}\right)\right) + n(t)$$

with $\cos(2\pi f_c t + \hat{\phi})$ and $\sin(2\pi f_c t + \hat{\phi})$. The sampled values at the output of the correlators are

$$\begin{aligned} r_1 &= \frac{1}{2}\left[\sqrt{2P_T} - n_s(t)\right]\sin(\phi - \hat{\phi} + \theta_n) + \frac{1}{2}n_c(t)\cos(\phi - \hat{\phi} + \theta_n) \\ r_2 &= \frac{1}{2}\left[\sqrt{2P_T} - n_s(t)\right]\cos(\phi - \hat{\phi} + \theta_n) + \frac{1}{2}n_c(t)\sin(\hat{\phi} - \phi - \theta_n) \end{aligned}$$

where $n_c(t)$, $n_s(t)$ are the in-phase and quadrature components of the noise $n(t)$. If the detector has made the correct decision on the transmitted point, then by multiplying r_1 by $\cos(\theta_n)$ and r_2 by $\sin(\theta_n)$ and subtracting the results, we obtain (after ignoring the noise)

$$\begin{aligned} r_1\cos(\theta_n) &= \frac{1}{2}\sqrt{2P_T}\left[\sin(\phi - \hat{\phi})\cos^2(\theta_n) + \cos(\phi - \hat{\phi})\sin(\theta_n)\cos(\theta_n)\right] \\ r_2\sin(\theta_n) &= \frac{1}{2}\sqrt{2P_T}\left[\cos(\phi - \hat{\phi})\cos(\theta_n)\sin(\theta_n) - \sin(\phi - \hat{\phi})\sin^2(\theta_n)\right] \\ e(t) &= r_1\cos(\theta_n) - r_2\sin(\theta_n) = \frac{1}{2}\sqrt{2P_T}\sin(\phi - \hat{\phi}) \end{aligned}$$

The error $e(t)$ is passed to the loop filter of the DFPLL that drives the VCO. As it is seen only the phase θ_n is used to estimate the carrier phase.

3) Having a correct carrier phase estimate, the output of the lowpass filter sampled at $t = T_b$ is

$$\begin{aligned} r &= \pm\frac{1}{2}\sqrt{2P_T}\sin\cos^{-1}\left(\sqrt{\frac{P_c}{P_T}}\right) + n \\ &= \pm\frac{1}{2}\sqrt{2P_T}\sqrt{1 - \frac{P_c}{P_T}} + n \\ &= \pm\frac{1}{2}\sqrt{2P_T\left(1 - \frac{P_c}{P_T}\right)} + n \end{aligned}$$

where n is a zero-mean Gaussian random variable with variance

$$\begin{aligned} \sigma_n^2 &= E\left[\int_0^{T_b}\int_0^{T_b} n(t)n(\tau)\cos(2\pi f_c t + \phi)\cos(2\pi f_c \tau + \phi)dt d\tau\right] \\ &= \frac{N_0}{2}\int_0^{T_b}\cos^2(2\pi f_c t + \phi)dt \\ &= \frac{N_0}{4} \end{aligned}$$

Note that T_b has been normalized to 1 since the problem has been stated in terms of the power of the involved signals. The probability of error is given by

$$P(\text{error}) = Q\left[\sqrt{\frac{2P_T}{N_0}\left(1-\frac{P_c}{P_T}\right)}\right]$$

The loss due to the allocation of power to the pilot signal is

$$\text{SNR}_{\text{loss}} = 10\log_{10}\left(1-\frac{P_c}{P_T}\right)$$

When $P_c/P_T = 0.1$, then

$$\text{SNR}_{\text{loss}} = 10\log_{10}(0.9) = -0.4576 \text{ dB}$$

The negative sign indicates that the SNR is decreased by 0.4576 dB.

Problem 8.26

1) If the received signal is

$$r(t) = \pm g_T(t)\cos(2\pi f_c t + \phi) + n(t)$$

then by crosscorrelating with the signal at the output of the PLL

$$\psi(t) = \sqrt{\frac{2}{\mathcal{E}_g}} g_t(t)\cos(2\pi f_c t + \hat{\phi})$$

we obtain

$$\begin{aligned}
\int_0^T r(t)\psi(t)dt &= \pm\sqrt{\frac{2}{\mathcal{E}_g}}\int_0^T g_T^2(t)\cos(2\pi f_c t+\phi)\cos(2\pi f_c t+\hat{\phi})dt \\
&\quad + \int_0^T n(t)\sqrt{\frac{2}{\mathcal{E}_g}} g_t(t)\cos(2\pi f_c t+\hat{\phi})dt \\
&= \pm\sqrt{\frac{2}{\mathcal{E}_g}}\int_0^T \frac{g_T^2(t)}{2}\left(\cos(2\pi 2 f_c t+\phi+\hat{\phi})+\cos(\phi-\hat{\phi})\right)dt + n \\
&= \pm\sqrt{\frac{\mathcal{E}_g}{2}}\cos(\phi-\hat{\phi}) + n
\end{aligned}$$

where n is a zero-mean Gaussian random variable with variance $\frac{N_0}{2}$. If we assume that the signal $s_1(t) = g_T(t)\cos(2\pi f_c t + \phi)$ was transmitted, then the probability of error is

$$\begin{aligned}
P(\text{error}|s_1(t)) &= P\left(\sqrt{\frac{\mathcal{E}_g}{2}}\cos(\phi-\hat{\phi}) + n < 0\right) \\
&= Q\left[\sqrt{\frac{\mathcal{E}_g\cos^2(\phi-\hat{\phi})}{N_0}}\right] = Q\left[\sqrt{\frac{2\mathcal{E}_s\cos^2(\phi-\hat{\phi})}{N_0}}\right]
\end{aligned}$$

where $\mathcal{E}_s = \mathcal{E}_g/2$ is the energy of the transmitted signal. As it is observed the phase error $\phi - \hat{\phi}$ reduces the SNR by a factor

$$\text{SNR}_{\text{loss}} = -10\log_{10}\cos^2(\phi-\hat{\phi})$$

2) When $\phi - \hat{\phi} = 45^o$, then the loss due to the phase error is

$$\text{SNR}_{\text{loss}} = -10\log_{10}\cos^2(45^o) = -10\log_{10}\frac{1}{2} = 3.01 \text{ dB}$$

Problem 8.27

1) The bandwidth of the bandpass channel is

$$W = 3000 - 600 = 2400 \text{ Hz}$$

Since each symbol of the QPSK constellation conveys 2 bits of information, the symbol rate of transmission is

$$R = \frac{2400}{2} = 1200 \text{ symbols/sec}$$

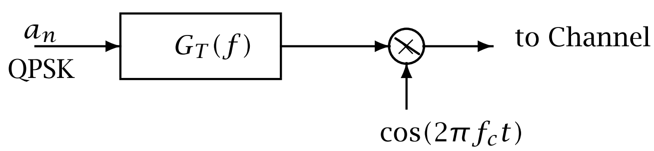
Thus, for spectral shaping we can use a signal pulse with a raised cosine spectrum and roll-off factor $\alpha = 1$, that is

$$X_{rc}(f) = \frac{T}{2}[1 + \cos(\pi T|f|)] = \frac{1}{2400}\cos^2\left(\frac{\pi|f|}{2400}\right)$$

If the desired spectral characteristic is split evenly between the transmitting filter $G_T(f)$ and the receiving filter $G_R(f)$, then

$$G_T(f) = G_R(f) = \sqrt{\frac{1}{1200}}\cos\left(\frac{\pi|f|}{2400}\right), \qquad |f| < \frac{1}{T} = 1200$$

A block diagram of the transmitter is shown in the next figure.



2) If the bit rate is 4800 bps, then the symbol rate is

$$R = \frac{4800}{2} = 2400 \text{ symbols/sec}$$

In order to satisfy the Nyquist criterion, the the signal pulse used for spectral shaping, should have the spectrum

$$X(f) = T\Pi\left(\frac{f}{W}\right)$$

Thus, the frequency response of the transmitting filter is $G_T(f) = \sqrt{T}\Pi\left(\frac{f}{W}\right)$.

Problem 8.28

The constellation of Fig. P-10.9(a) has four points at a distance $2A$ from the origin and four points at a distance $2\sqrt{2}A$. Thus, the average transmitted power of the constellation is

$$P_a = \frac{1}{8}\left[4 \times (2A)^2 + 4 \times (2\sqrt{2}A)^2\right] = 6A^2$$

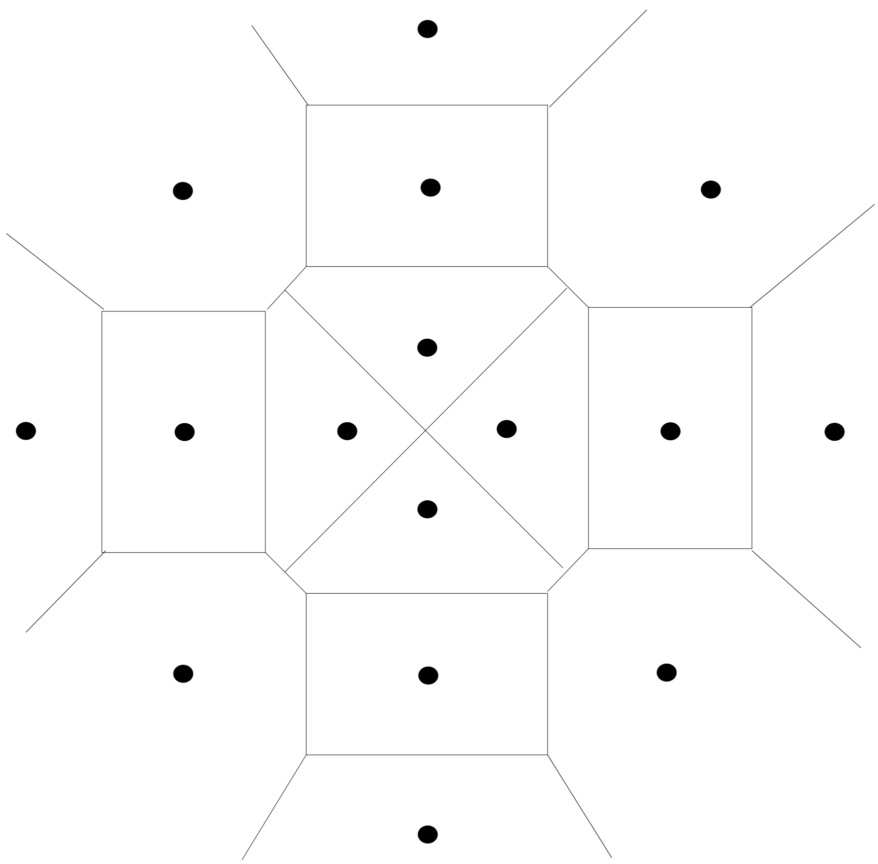
The second constellation has four points at a distance $\sqrt{7}A$ from the origin, two points at a distance $\sqrt{3}A$ and two points at a distance A. Thus, the average transmitted power of the second constellation is

$$P_b = \frac{1}{8}\left[4 \times (\sqrt{7}A)^2 + 2 \times (\sqrt{3}A)^2 + 2A^2\right] = \frac{9}{2}A^2$$

Since $P_b < P_a$ the second constellation is more power efficient.

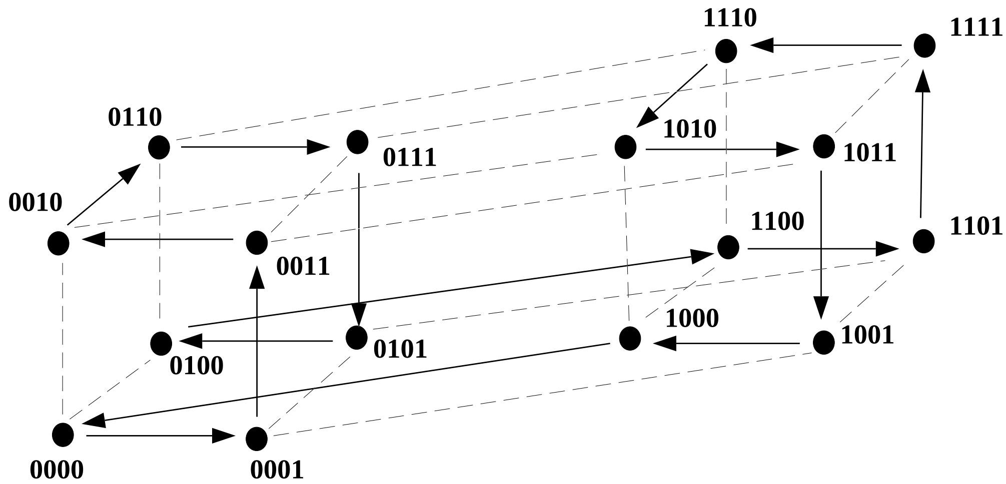
Problem 8.29

The optimum decision boundary of a point is determined by the perpendicular bisectors of each line segment connecting the point with its neighbors. The decision regions for the V.29 constellation are depicted in the next figure.

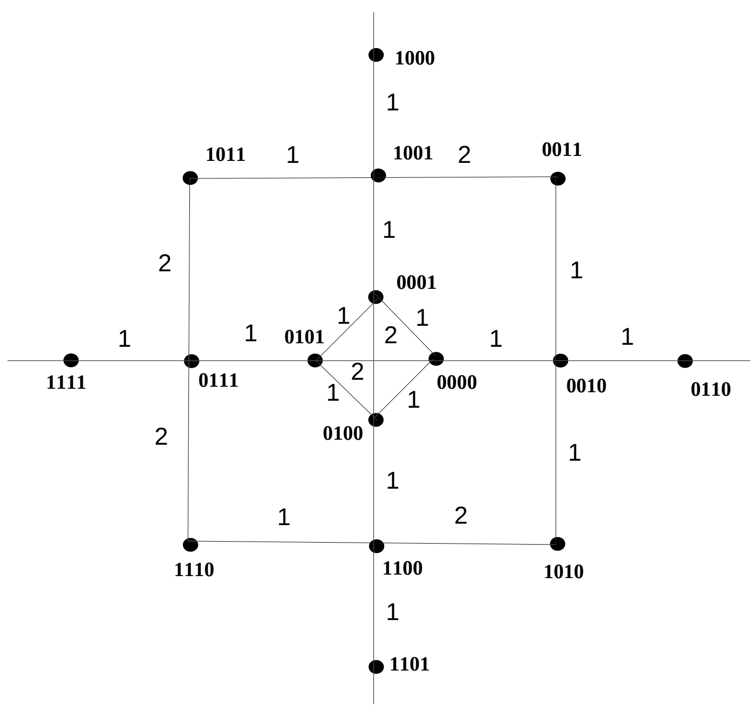


Problem 8.30

The following figure depicts a 4-cube and the way that one can traverse it in Gray-code order (see John F. Wakerly, Digital Design Principles and Practices, Prentice Hall, 1990). Adjacent points are connected with solid or dashed lines.



One way to label the points of the V.29 constellation using the Gray-code is depicted in the next figure. Note that the maximum Hamming distance between points with distance between them as large as 3 is only 2. Having labeled the innermost points, all the adjacent nodes can be found using the previous figure.



Problem 8.31

1) Consider the QAM constellation of Fig. P-10.12. Using the Pythagorean theorem we can find the radius of the inner circle as

$$a^2 + a^2 = A^2 \Longrightarrow a = \frac{1}{\sqrt{2}}A$$

The radius of the outer circle can be found using the cosine rule. Since b is the third side of a triangle with a and A the two other sides and angle between then equal to $\theta = 75^o$, we obtain

$$b^2 = a^2 + A^2 - 2aA\cos 75^o \Longrightarrow b = \frac{1+\sqrt{3}}{2}A$$

2) If we denote by r the radius of the circle, then using the cosine theorem we obtain

$$A^2 = r^2 + r^2 - 2r\cos 45^o \Longrightarrow r = \frac{A}{\sqrt{2-\sqrt{2}}}$$

3) The average transmitted power of the PSK constellation is

$$P_{\text{PSK}} = 8 \times \frac{1}{8} \times \left(\frac{A}{\sqrt{2-\sqrt{2}}}\right)^2 \Longrightarrow P_{\text{PSK}} = \frac{A^2}{2-\sqrt{2}}$$

whereas the average transmitted power of the QAM constellation

$$P_{\text{QAM}} = \frac{1}{8}\left(4\frac{A^2}{2} + 4\frac{(1+\sqrt{3})^2}{4}A^2\right) \Longrightarrow P_{\text{QAM}} = \left[\frac{2+(1+\sqrt{3})^2}{8}\right]A^2$$

The relative power advantage of the PSK constellation over the QAM constellation is

$$\text{gain} = \frac{P_{\text{PSK}}}{P_{\text{QAM}}} = \frac{8}{(2+(1+\sqrt{3})^2)(2-\sqrt{2})} = 1.5927 \text{ dB}$$

Problem 8.32

1) Although it is possible to assign three bits to each point of the 8-PSK signal constellation so that adjacent points differ in only one bit, this is not the case for the 8-QAM constellation of Figure P-10.12. This is because there are fully connected graphs consisted of three points. To see this consider an equilateral triangle with vertices A, B and C. If, without loss of generality, we assign the all zero sequence $\{0,0,\ldots,0\}$ to point A, then point B and C should have the form

$$\text{B} = \{0,\ldots,0,1,0,\ldots,0\} \qquad \text{C} = \{0,\ldots,0,1,0,\ldots,0\}$$

where the position of the 1 in the sequences is not the same, otherwise B=C. Thus, the sequences of B and C differ in two bits.

2) Since each symbol conveys 3 bits of information, the resulted symbol rate is

$$R_s = \frac{90\times10^6}{3} = 30\times10^6 \text{ symbols/sec}$$

3) The probability of error for an M-ary PSK signal is

$$P_M = 2Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\sin\frac{\pi}{M}\right]$$

whereas the probability of error for an M-ary QAM signal is upper bounded by

$$P_M = 4Q\left[\sqrt{\frac{3\mathcal{E}_{av}}{(M-1)N_0}}\right]$$

Since, the probability of error is dominated by the argument of the Q function, the two signals will achieve the same probability of error if

$$\sqrt{2\mathrm{SNR}_{\mathrm{PSK}}}\sin\frac{\pi}{M} = \sqrt{\frac{3\mathrm{SNR}_{\mathrm{QAM}}}{M-1}}$$

With $M = 8$ we obtain

$$\sqrt{2\mathrm{SNR}_{\mathrm{PSK}}}\sin\frac{\pi}{8} = \sqrt{\frac{3\mathrm{SNR}_{\mathrm{QAM}}}{7}} \Longrightarrow \frac{\mathrm{SNR}_{\mathrm{PSK}}}{\mathrm{SNR}_{\mathrm{QAM}}} = \frac{3}{7\times 2\times 0.3827^2} = 1.4627$$

4) Assuming that the magnitude of the signal points is detected correctly, then the detector for the 8-PSK signal will make an error if the phase error (magnitude) is greater than 22.5^o. In the case of the 8-QAM constellation an error will be made if the magnitude phase error exceeds 45^o. Hence, the QAM constellation is more immune to phase errors.

Problem 8.33

The channel bandwidth is $W = 4000$ Hz.
(1) Binary PSK with a pulse shape that has $\alpha = \frac{1}{2}$. Hence

$$\frac{1}{2T}(1+\alpha) = 2000$$

and $\frac{1}{T} = 2667$, the bit rate is 2667 bps.
(2) Four-phase PSK with a pulse shape that has $\alpha = \frac{1}{2}$. From (a) the symbol rate is $\frac{1}{T} = 2667$ and the bit rate is 5334 bps.
(3) $M = 8$ QAM with a pulse shape that has $\alpha = \frac{1}{2}$. From (a), the symbol rate is $\frac{1}{T} = 2667$ and hence the bit rate $\frac{3}{T} = 8001$ bps.
(4) Binary FSK with noncoherent detection. Assuming that the frequency separation between the two frequencies is $\Delta f = \frac{1}{T}$, where $\frac{1}{T}$ is the bit rate, the two frequencies are $f_c + \frac{1}{2T}$ and $f_c - \frac{1}{2T}$. Since $W = 4000$ Hz, we may select $\frac{1}{2T} = 1000$, or, equivalently, $\frac{1}{T} = 2000$. Hence, the bit rate is 2000 bps, and the two FSK signals are orthogonal.
(5) Four FSK with noncoherent detection. In this case we need four frequencies with separation of $\frac{1}{T}$ between adjacent frequencies. We select $f_1 = f_c - \frac{1.5}{T}$, $f_2 = f_c - \frac{1}{2T}$, $f_3 = f_c + \frac{1}{2T}$, and $f_4 = f_c + \frac{1.5}{T}$, where $\frac{1}{2T} = 500$ Hz. Hence, the symbol rate is $\frac{1}{T} = 1000$ symbols per second and since each symbol

carries two bits of information, the bit rate is 2000 bps.
(6) $M = 8$ FSK with noncoherent detection. In this case we require eight frequencies with frequency separation of $\frac{1}{T} = 500$ Hz for orthogonality. Since each symbol carries 3 bits of information, the bit rate is 1500 bps.

Problem 8.34

The three symbols A, 0 and $-A$ are used with equal probability. Hence, the optimal detector uses two thresholds, which are $\frac{A}{2}$ and $-\frac{A}{2}$, and it bases its decisions on the criterion

$$\begin{array}{ll} A: & r > \frac{A}{2} \\ 0: & -\frac{A}{2} < r < \frac{A}{2} \\ -A: & r < -\frac{A}{2} \end{array}$$

If the variance of the AWG noise is σ_n^2, then the average probability of error is

$$\begin{aligned} P(e) &= \frac{1}{3}\int_{-\infty}^{\frac{A}{2}} \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(r-A)^2}{2\sigma_n^2}} dr + \frac{1}{3}\left(1 - \int_{-\frac{A}{2}}^{\frac{A}{2}} \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{r^2}{2\sigma_n^2}} dr\right) \\ &\quad + \frac{1}{3}\int_{-\frac{A}{2}}^{\infty} \frac{1}{\sqrt{2\pi\sigma_n^2}} e^{-\frac{(r+A)^2}{2\sigma_n^2}} dr \\ &= \frac{1}{3}Q\left[\frac{A}{2\sigma_n}\right] + \frac{1}{3}2Q\left[\frac{A}{2\sigma_n}\right] + \frac{1}{3}Q\left[\frac{A}{2\sigma_n}\right] \\ &= \frac{4}{3}Q\left[\frac{A}{2\sigma_n}\right] \end{aligned}$$

Problem 8.35

1) The PDF of the noise n is

$$f(n) = \frac{\lambda}{2} e^{-\lambda|n|}$$

The optimal receiver uses the criterion

$$\frac{f(r|A)}{f(r|-A)} = e^{-\lambda[|r-A|-|r+A|]} \underset{-A}{\overset{A}{\gtrless}} 1 \Longrightarrow r \underset{-A}{\overset{A}{\gtrless}} 0$$

The average probability of error is

$$\begin{aligned}
P(e) &= \frac{1}{2}P(e|A) + \frac{1}{2}P(e|-A) \\
&= \frac{1}{2}\int_{-\infty}^{0} f(r|A)dr + \frac{1}{2}\int_{0}^{\infty} f(r|-A)dr \\
&= \frac{1}{2}\int_{-\infty}^{0} \lambda_2 e^{-\lambda|r-A|}dr + \frac{1}{2}\int_{0}^{\infty} \lambda_2 e^{-\lambda|r+A|}dr \\
&= \frac{\lambda}{4}\int_{-\infty}^{-A} e^{-\lambda|x|}dx + \frac{\lambda}{4}\int_{A}^{\infty} e^{-\lambda|x|}dx \\
&= \frac{\lambda}{4}\frac{1}{\lambda}e^{\lambda x}\Big|_{-\infty}^{-A} + \frac{\lambda}{4}\left(-\frac{1}{\lambda}\right)e^{-\lambda x}\Big|_{A}^{\infty} \\
&= \frac{1}{2}e^{-\lambda A}
\end{aligned}$$

2) The variance of the noise is

$$\begin{aligned}
\sigma_n^2 &= \frac{\lambda}{2}\int_{-\infty}^{\infty} e^{-\lambda|x|}x^2 dx \\
&= \lambda\int_{0}^{\infty} e^{-\lambda x}x^2 dx = \lambda\frac{2!}{\lambda^3} = \frac{2}{\lambda^2}
\end{aligned}$$

Hence, the SNR is

$$\text{SNR} = \frac{A^2}{\frac{2}{\lambda^2}} = \frac{A^2\lambda^2}{2}$$

and the probability of error is given by

$$P(e) = \frac{1}{2}e^{-\sqrt{\lambda^2 A^2}} = \frac{1}{2}e^{-\sqrt{2\text{SNR}}}$$

For $P(e) = 10^{-5}$ we obtain

$$\ln(2\times 10^{-5}) = -\sqrt{2\text{SNR}} \Longrightarrow \text{SNR} = 58.534 = 17.6741 \text{ dB}$$

If the noise was Gaussian, then

$$P(e) = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{\text{SNR}}\right]$$

where SNR is the signal to noise ratio at the output of the matched filter. With $P(e) = 10^{-5}$ we find $\sqrt{\text{SNR}} = 4.26$ and therefore SNR = 18.1476 = 12.594 dB. Thus the required signal to noise ratio is 5 dB less when the additive noise is Gaussian.

Problem 8.36

The points in the constellation are at distance $\pm d, \pm 3d, \pm 5d, \ldots, \pm(M-1)d$ from the origin. Since the square of the distance of a point in the constellation from the origin is equal to the energy of the signal corresponding to that point, we have two signals with energy d^2, two signals with energy $9d^2$, two signals with energy $25d^2$,...., and two signals with energy $(M-1)^2d^2$. The average energy is

$$E_{av} = \frac{1}{M}\sum_i E_i = \frac{2d^2}{M}\left(1+9+25+\cdots+(M-1)^2\right)$$

Using the well known relation

$$1^2+2^2+3^2+\cdots+M^2 = \frac{M(M+1)(2M+1)}{6}$$

we have (note that $M = 2^k$ is even)

$$2^2+4^2+\cdots+(M)^2 = 4\left(1^2+2^2+\cdots+\left(\frac{M}{2}\right)^2\right) = \frac{M(M+1)(M+2)}{6}$$

subtracting the latter two series gives

$$1^2+3^2+\cdots+(M-1)^2 = \frac{M(M+1)(2M+1)}{6} - \frac{M(M+1)(M+2)}{6} = \frac{M\,(M^2-1)}{6}$$

Therefore,

$$E_{av} = \frac{2d^2}{M}\times\frac{M\,(M^2-1)}{6} = \frac{d^2(M^2-1)}{3}$$

Problem 8.37

The optimal receiver bases its decisions on the metrics

$$\text{PM}(\mathbf{r},\mathbf{s}_m) = f(\mathbf{r}|\mathbf{s}_m)P(\mathbf{s}_m)$$

For an additive noise channel $\mathbf{r} = \mathbf{s}_m + \mathbf{n}$, so

$$\text{PM}(\mathbf{r},\mathbf{s}_m) = f(\mathbf{n})P(\mathbf{s}_m)$$

where $f(\mathbf{n})$ is the N-dimensional PDF for the noise channel vector. If the noise is AWG, then

$$f\mathbf{n}) = \frac{1}{(\pi N_0)^{\frac{N}{2}}}e^{-\frac{|\mathbf{r}-\mathbf{s}_m|^2}{N_0}}$$

Maximizing $f(\mathbf{r}|\mathbf{s}_m)P(\mathbf{s}_m)$ is the same as minimizing the reciprocal $e^{\frac{|\mathbf{r}-\mathbf{s}_m|^2}{N_0}}/P(\mathbf{s}_m)$, or by taking the natural logarithm, minimizing the cost

$$D(\mathbf{r},\mathbf{s}_m) = |\mathbf{r}-\mathbf{s}_m|^2 - N_0P(\mathbf{s}_m)$$

This is equivalent to the maximization of the quantity

$$C(\mathbf{r},\mathbf{s}_m) = \mathbf{r}\cdot\mathbf{s}_m - \frac{1}{2}|\mathbf{s}_m|^2 + \frac{N_0}{2}\ln P(\mathbf{s}_m)$$

If the vectors $\mathbf{r}$, $\mathbf{s}_m$ correspond to the waveforms $r(t)$ and $s_m(t)$, where

$$\begin{aligned} r(t) &= \sum_{i=1}^{N} r_i \psi_i(t) \\ s_m(t) &= \sum_{i=1}^{N} s_{m,i} \psi_i(t) \end{aligned}$$

then,

$$\begin{aligned} \int_{-\infty}^{\infty} r(t) s_m(t) dt &= \int_{-\infty}^{\infty} \sum_{i=1}^{N} r_i \psi_i(t) \sum_{j=1}^{N} s_{m,j} \psi_j(t) dt \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} r_i s_{m,j} \int_{-\infty}^{\infty} \psi_i(t) \psi_j(t) dt \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} r_i s_{m,j} \delta_{i,j} = \sum_{i=1}^{N} r_i s_{m,i} \\ &= \mathbf{r} \cdot \mathbf{s}_m \end{aligned}$$

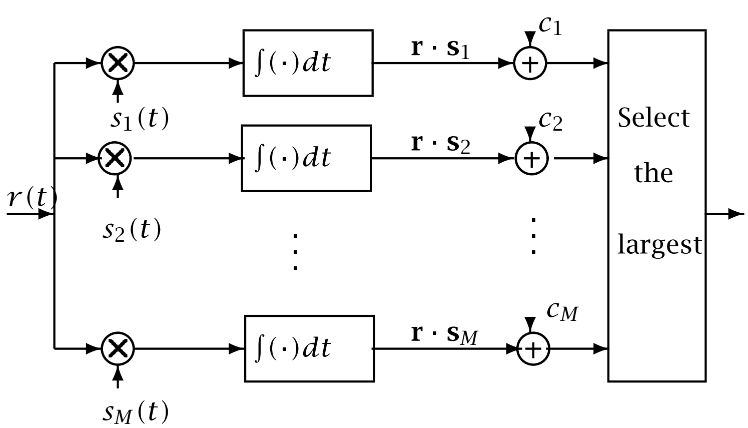
Similarly we obtain

$$\int_{-\infty}^{\infty} |s_m(t)|^2 dt = |\mathbf{s}_m|^2 = \mathcal{E}_{s_m}$$

Therefore, the optimal receiver can use the costs

$$\begin{aligned} C(\mathbf{r}, \mathbf{s}_m) &= \int_{-\infty}^{\infty} r(t) s_m(t) dt - \frac{1}{2} \int_{-\infty}^{\infty} |s_m(t)|^2 dt + \frac{N_0}{2} \ln P(\mathbf{s}_m) \\ &= \int_{-\infty}^{\infty} r(t) s_m(t) dt + c_m \end{aligned}$$

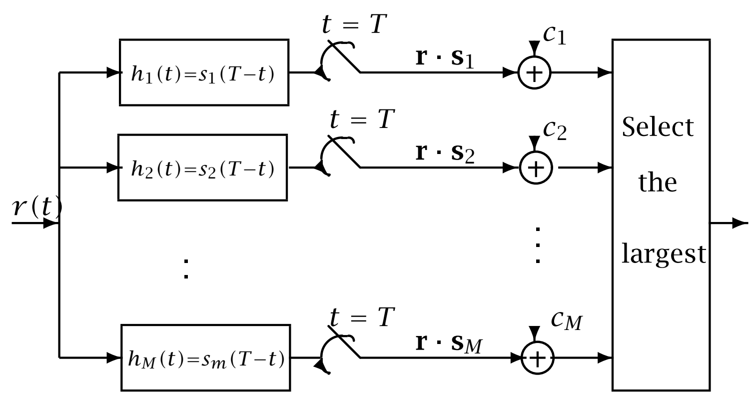
to base its decisions. This receiver can be implemented using M correlators to evaluate $\int_{-\infty}^{\infty} r(t) s_m(t) dt$. The bias constants c_m can be precomputed and added to the output of the correlators. The structure of the receiver is shown in the next figure.



Parallel to the development of the optimal receiver using N filters matched to the orthonormal functions $\psi_i(t)$, $i = 1, \ldots, N$, the M correlators can be replaced by M equivalent filters matched to the signal waveforms $s_m(t)$. The output of the m^{th} matched filter $h_m(t)$, at the time instant T is

$$\begin{aligned}\int_0^T r(\tau)h_m(T-\tau)d\tau &= \int_0^T r(\tau)s_m(T-(T-\tau))d\tau \\ &= \int_0^T r(\tau)s_m(\tau)d\tau \\ &= \mathbf{r}\cdot\mathbf{s}_m\end{aligned}$$

The structure of this optimal receiver is shown in the next figure. The optimal receivers, derived in this problem, are more costly than those derived in the text, since N is usually less than M, the number of signal waveforms. For example, in an M-ary PAM system, $N = 1$ always less than M.



Problem 8.38

The bandwidth required for transmission of an M-ary PAM signal is

$$W = \frac{R_b}{2\log_2 M}\ \text{Hz}$$

Since,

$$R_b = 8\times 10^3\,\frac{\text{samples}}{\text{sec}}\times 8\,\frac{\text{bits}}{\text{sample}} = 64\times 10^3\,\frac{\text{bits}}{\text{sec}}$$

we obtain

$$W = \begin{cases} 16\ \text{KHz} & M=4 \\ 10.667\ \text{KHz} & M=8 \\ 8\ \text{KHz} & M=16 \end{cases}$$

Problem 8.39

The vector $\mathbf{r} = [r_1, r_2]$ at the output of the integrators is

$$\mathbf{r} = [r_1, r_2] = [\int_0^{1.5} r(t)dt, \quad \int_1^2 r(t)dt]$$

If $s_1(t)$ is transmitted, then

$$\begin{aligned}
\int_0^{1.5} r(t)dt &= \int_0^{1.5} [s_1(t) + n(t)]dt = 1 + \int_0^{1.5} n(t)dt \\
&= 1 + n_1 \\
\int_1^2 r(t)dt &= \int_1^2 [s_1(t) + n(t)]dt = \int_1^2 n(t)dt \\
&= n_2
\end{aligned}$$

where n_1 is a zero-mean Gaussian random variable with variance

$$\sigma_{n_1}^2 = E\left[\int_0^{1.5}\int_0^{1.5} n(\tau)n(v)d\tau dv\right] = \frac{N_0}{2}\int_0^{1.5} d\tau = 1.5$$

and n_2 is is a zero-mean Gaussian random variable with variance

$$\sigma_{n_2}^2 = E\left[\int_1^2\int_1^2 n(\tau)n(v)d\tau dv\right] = \frac{N_0}{2}\int_1^2 d\tau = 1$$

Thus, the vector representation of the received signal (at the output of the integrators) is

$$\mathbf{r} = [1 + n_1, \; n_2]$$

Similarly we find that if $s_2(t)$ is transmitted, then

$$\mathbf{r} = [0.5 + n_1, \; 1 + n_2]$$

Suppose now that the detector bases its decisions on the rule

$$r_1 - r_2 \underset{s_2}{\overset{s_1}{\gtrless}} T$$

The probability of error $P(e|s_1)$ is obtained as

$$\begin{aligned}
P(e|s_1) &= P(r1 - r_2 < T|s_1) \\
&= P(1 + n_1 - n_2 < T) = P(n_1 - n_2 < T - 1) \\
&= P(n < T)
\end{aligned}$$

where the random variable $n = n_1 - n_2$ is zero-mean Gaussian with variance

$$\begin{aligned}
\sigma_n^2 &= \sigma_{n_1}^2 + \sigma_{n_2}^2 - 2E[n_1 n_2] \\
&= \sigma_{n_1}^2 + \sigma_{n_2}^2 - 2\int_1^{1.5} \frac{N_0}{2} d\tau \\
&= 1.5 + 1 - 2 \times 0.5 = 1.5
\end{aligned}$$

Hence,

$$P(e|s_1) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \int_{-\infty}^{T-1} e^{-\frac{x^2}{2\sigma_n^2}} dx$$

Similarly we find that

$$\begin{aligned} P(e|s_2) &= P(0.5 + n_1 - 1 - n_2 > T) \\ &= P(n_1 - n_2 > T + 0.5) \\ &= \frac{1}{\sqrt{2\pi\sigma_n^2}} \int_{T+0.5}^{\infty} e^{-\frac{x^2}{2\sigma_n^2}} dx \end{aligned}$$

The average probability of error is

$$\begin{aligned} P(e) &= \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) \\ &= \frac{1}{2\sqrt{2\pi\sigma_n^2}} \int_{-\infty}^{T-1} e^{-\frac{x^2}{2\sigma_n^2}} dx + \frac{1}{2\sqrt{2\pi\sigma_n^2}} \int_{T+0.5}^{\infty} e^{-\frac{x^2}{2\sigma_n^2}} dx \end{aligned}$$

To find the value of T that minimizes the probability of error, we set the derivative of $P(e)$ with respect to T equal to zero. Using the Leibnitz rule for the differentiation of definite integrals, we obtain

$$\frac{\vartheta P(e)}{\vartheta T} = \frac{1}{2\sqrt{2\pi\sigma_n^2}} \left[e^{-\frac{(T-1)^2}{2\sigma_n^2}} - e^{-\frac{(T+0.5)^2}{2\sigma_n^2}} \right] = 0$$

or

$$(T-1)^2 = (T+0.5)^2 \Longrightarrow T = 0.25$$

Thus, the optimal decision rule is

$$r_1 - r_2 \underset{s_2}{\overset{s_1}{\gtrless}} 0.25$$

Problem 8.40

1) For n repeaters in cascade, the probability of i out of n repeaters to produce an error is given by the binomial distribution

$$P_i = \binom{n}{i} p^i (1-p)^{n-i}$$

However, there is a bit error at the output of the terminal receiver only when an odd number of repeaters produces an error. Hence, the overall probability of error is

$$P_n = P_{\text{odd}} = \sum_{i=\text{odd}} \binom{n}{i} p^i (1-p)^{n-i}$$

Let P_{even} be the probability that an even number of repeaters produces an error. Then

$$P_{\text{even}} = \sum_{i=\text{even}} \binom{n}{i} p^i (1-p)^{n-i}$$

and therefore,

$$P_{\text{even}} + P_{\text{odd}} = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = (p+1-p)^n = 1$$

One more relation between P_{even} and P_{odd} can be provided if we consider the difference $P_{\text{even}} - P_{\text{odd}}$. Clearly,

$$\begin{aligned} P_{\text{even}} - P_{\text{odd}} &= \sum_{i=\text{even}} \binom{n}{i} p^i (1-p)^{n-i} - \sum_{i=\text{odd}} \binom{n}{i} p^i (1-p)^{n-i} \\ &\stackrel{a}{=} \sum_{i=\text{even}} \binom{n}{i} (-p)^i (1-p)^{n-i} + \sum_{i=\text{odd}} \binom{n}{i} (-p)^i (1-p)^{n-i} \\ &= (1-p-p)^n = (1-2p)^n \end{aligned}$$

where the equality (a) follows from the fact that $(-1)^i$ is 1 for i even and -1 when i is odd. Solving the system

$$\begin{aligned} P_{\text{even}} + P_{\text{odd}} &= 1 \\ P_{\text{even}} - P_{\text{odd}} &= (1-2p)^n \end{aligned}$$

we obtain

$$P_n = P_{\text{odd}} = \frac{1}{2}(1-(1-2p)^n)$$

2) Expanding the quantity $(1-2p)^n$, we obtain

$$(1-2p)^n = 1 - n2p + \frac{n(n-1)}{2}(2p)^2 + \cdots$$

Since, $p \ll 1$ we can ignore all the powers of p which are greater than one. Hence,

$$P_n \approx \frac{1}{2}(1-1+n2p) = np = 100 \times 10^{-6} = 10^{-4}$$

Problem 8.41

The overall probability of error is approximated by

$$P(e) = KQ\left[\sqrt{\frac{\mathcal{E}_b}{N_0}}\right]$$

Thus, with $P(e) = 10^{-6}$ and $K = 100$, we obtain the probability of each repeater $P_r = Q\left[\sqrt{\frac{\mathcal{E}_b}{N_0}}\right] = 10^{-8}$. The argument of the function $Q[\cdot]$ that provides a value of 10^{-8} is found from tables to be

$$\sqrt{\frac{\mathcal{E}_b}{N_0}} = 5.61$$

Hence, the required $\frac{\mathcal{E}_b}{N_0}$ is $5.61^2 = 31.47$

Problem 8.42

The one-sided noise equivalent bandwidth is defined as

$$B_{eq} = \frac{\int_0^\infty |H(f)|^2 df}{|H(f)|^2_{\max}}$$

It is usually convenient to substitute $|H(f)|^2_{f=0}$ for $|H(f)|^2_{\max}$ in the denominator, since the peaking of the magnitude transfer function may be high (especially for small ζ) creating in this way anomalies. On the other hand if ζ is less, but close, to one, $|H(f)|^2_{\max}$ can be very well approximated by $|H(f)|^2_{f=0}$. Hence,

$$B_{eq} = \frac{\int_0^\infty |H(f)|^2 df}{|H(f)|^2_{f=0}}$$

and since

$$|H(f)|^2 = \frac{\omega_n^2 + j2\pi f\left(2\zeta\omega_n - \frac{\omega_n^2}{K}\right)}{\omega_n^2 - 4\pi^2 f^2 + j2\pi f 2\zeta\omega_n}$$

we find that $|H(0)| = 1$. Therefore,

$$B_{eq} = \int_0^\infty |H(f)|^2 df$$

For the passive second order filter

$$H(s) = \frac{s(2\zeta\omega_n - \frac{\omega_n^2}{K}) + \omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

$\tau_1 \gg 1$, so that $\frac{\omega_n^2}{K} = \frac{1}{\tau_1} \approx 0$ and

$$H(s) = \frac{s2\zeta\omega_n + \omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

The B_{eq} can be written as

$$B_{eq} = \frac{1}{4\pi j}\int_{-j\infty}^{j\infty} H(s)H(-s)ds$$

Since, $H(s) = \frac{KG(s)/s}{1+KG(s)/s}$ we obtain $\lim_{|s|\to\infty} H(s)H(-s) = 0$. Hence, the integral for B_{eq} can be taken along a contour, which contains the imaginary axis and the left half plane. Furthermore, since $G(s)$

is a rational function of s, the integral is equal to half the sum of the residues of the left half plane poles of $H(s)H(-s)$. Hence,

$$\begin{aligned}
B_{eq} &= \frac{1}{2}\left[(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})H(s)H(-s)\Big|_{s=-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}}\right.\\
&\quad \left.+(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})H(s)H(-s)\Big|_{s=-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}}\right]\\
&= \frac{\omega_n}{8}(4\zeta+\frac{1}{\zeta}) = \frac{1+4\zeta^2}{8\zeta/\omega_n}\\
&= \frac{1+\omega_n^2\tau_2^2+(\frac{\omega_n}{K})^2+2\frac{\omega_n^2\tau_2}{K}}{8\zeta/\omega_n}\\
&\approx \frac{1+\omega_n^2\tau_2^2}{8\zeta/\omega_n}
\end{aligned}$$

where we have used the approximation $\frac{\omega_n}{K}\approx 0$.

Problem 8.43

1) The closed loop transfer function is

$$H(s) = \frac{G(s)/s}{1+G(s)/s} = \frac{G(s)}{s+G(s)} = \frac{1}{s^2+\sqrt{2}s+1}$$

The poles of the system are the roots of the denominator, that is

$$\rho_{1,2} = \frac{-\sqrt{2}\pm\sqrt{2-4}}{2} = -\frac{1}{\sqrt{2}} \pm j\frac{1}{\sqrt{2}}$$

Since the real part of the roots is negative, the poles lie in the left half plane and therefore, the system is stable.

2) Writing the denominator in the form

$$D = s^2+2\zeta\omega_n s+\omega_n^2$$

we identify the natural frequency of the loop as $\omega_n = 1$ and the damping factor as $\zeta = \frac{1}{\sqrt{2}}$.

Problem 8.44

1) The closed loop transfer function is

$$H(s) = \frac{G(s)/s}{1+G(s)/s} = \frac{G(s)}{s+G(s)} = \frac{K}{\tau_1 s^2+s+K} = \frac{\frac{K}{\tau_1}}{s^2+\frac{1}{\tau_1}s+\frac{K}{\tau_1}}$$

The gain of the system at $f=0$ is

$$|H(0)| = |H(s)|_{s=0} = 1$$

2) The poles of the system are the roots of the denominator, that is

$$\rho_{1,2} = \frac{-1 \pm \sqrt{1-4K\tau_1}}{2\tau_1} =$$

In order for the system to be stable the real part of the poles must be negative. Since K is greater than zero, the latter implies that τ_1 is positive. If in addition we require that the damping factor $\zeta = \frac{1}{2\sqrt{\tau_1 K}}$ is less than 1, then the gain K should satisfy the condition

$$K > \frac{1}{4\tau_1}$$

Problem 8.45

The transfer function of the RC circuit is

$$G(s) = \frac{R_2 + \frac{1}{Cs}}{R_1 + R_2 + \frac{1}{Cs}} = \frac{1 + R_2 Cs}{1 + (R_1 + R_2)Cs} = \frac{1 + \tau_2 s}{1 + \tau_1 s}$$

From the last equality we identify the time constants as

$$\tau_2 = R_2 C, \qquad \tau_1 = (R_1 + R_2)C$$

Problem 8.46

Assuming that the input resistance of the operational amplifier is high so that no current flows through it, then the voltage-current equations of the circuit are

$$\begin{aligned} V_2 &= -AV_1 \\ V_1 - V_2 &= \left(R_1 + \frac{1}{Cs}\right) i \\ V_1 - V_0 &= iR \end{aligned}$$

where, V_1, V_2 is the input and output voltage of the amplifier respectively, and V_0 is the signal at the input of the filter. Eliminating i and V_1, we obtain

$$\frac{V_2}{V_1} = \frac{\frac{R_1 + \frac{1}{Cs}}{R}}{1 + \frac{1}{A} - \frac{R_1 + \frac{1}{Cs}}{AR}}$$

If we let $A \to \infty$ (ideal amplifier), then

$$\frac{V_2}{V_1} = \frac{1 + R_1 Cs}{RCs} = \frac{1 + \tau_2 s}{\tau_1 s}$$

Hence, the constants τ_1, τ_2 of the active filter are given by

$$\tau_1 = RC, \qquad \tau_2 = R_1 C$$

Problem 8.47

In the non decision-directed timing recovery method we maximize the function

$$\Lambda_2(\tau) = \sum_m y_m^2(\tau)$$

with respect to τ. Thus, we obtain the condition

$$\frac{d\Lambda_2(\tau)}{d\tau} = 2\sum_m y_m(\tau)\frac{dy_m(\tau)}{d\tau} = 0$$

Suppose now that we approximate the derivative of the log-likelihood $\Lambda_2(\tau)$ by the finite difference

$$\frac{d\Lambda_2(\tau)}{d\tau} \approx \frac{\Lambda_2(\tau+\delta) - \Lambda_2(\tau-\delta)}{2\delta}$$

Then, if we substitute the expression of $\Lambda_2(\tau)$ in the previous approximation, we obtain

$$\begin{aligned}\frac{d\Lambda_2(\tau)}{d\tau} &= \frac{\sum_m y_m^2(\tau+\delta) - \sum_m y_m^2(\tau-\delta)}{2\delta} \\ &= \frac{1}{2\delta}\sum_m \left[\left(\int r(t)u(t-mT-\tau-\delta)dt\right)^2 - \left(\int r(t)u(t-mT-\tau+\delta)dt\right)^2\right]\end{aligned}$$

where $u(-t) = g_R(t)$ is the impulse response of the matched filter in the receiver. However, this is the expression of the early-late gate synchronizer, where the lowpass filter has been substituted by the summation operator. Thus, the early-late gate synchronizer is a close approximation to the timing recovery system.

Problem 8.48

1) r is a Gaussian random variable. If $\sqrt{\mathcal{E}_b}$ is the transmitted signal point, then

$$E(r) = E(r_1) + E(r_2) = (1+k)\sqrt{\mathcal{E}_b} \equiv m_r$$

and the variance is

$$\sigma_r^2 = \sigma_1^2 + k^2\sigma_2^2$$

The probability density function of r is

$$f(r) = \frac{1}{\sqrt{2\pi}\sigma_r}e^{-\frac{(r-m_r)^2}{2\sigma_r^2}}$$

and the probability of error is

$$\begin{aligned} P_2 &= \int_{-\infty}^{0} f(r)\,dr \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{m_r}{\sigma_r}} e^{-\frac{x^2}{2}}\,dx \\ &= Q\left(\sqrt{\frac{m_r^2}{\sigma_r^2}}\right) \end{aligned}$$

where

$$\frac{m_r^2}{\sigma_r^2} = \frac{(1+k)^2 \mathcal{E}_b}{\sigma_1^2 + k^2 \sigma_2^2}$$

The value of k that maximizes this ratio is obtained by differentiating this expression and solving for the value of k that forces the derivative to zero. Thus, we obtain

$$k = \frac{\sigma_1^2}{\sigma_2^2}$$

Note that if $\sigma_1 > \sigma_2$, then $k > 1$ and r_2 is given greater weight than r_1. On the other hand, if $\sigma_2 > \sigma_1$, then $k < 1$ and r_1 is given greater weight than r_2. When $\sigma_1 = \sigma_2$, $k = 1$. In this case

$$\frac{m_r^2}{\sigma_r^2} = \frac{2\mathcal{E}_b}{\sigma_1^2}$$

2) When $\sigma_2^2 = 3\sigma_1^2$, $k = \frac{1}{3}$, and

$$\frac{m_r^2}{\sigma_r^2} = \frac{(1+\frac{1}{3})^2 \mathcal{E}_b}{\sigma_1^2 + \frac{1}{9}(3\sigma_1^2)} = \frac{4}{3}\left(\frac{\mathcal{E}_b}{\sigma_1^2}\right)$$

On the other hand, if k is set to unity we have

$$\frac{m_r^2}{\sigma_r^2} = \frac{4\mathcal{E}_b}{\sigma_1^2 + 3\sigma_1^2} = \frac{\mathcal{E}_b}{\sigma_1^2}$$

Therefore, the optimum weighting provides a gain of

$$10\log\frac{4}{3} = 1.25 \text{ dB}$$

Computer Problems

Computer Problem 8.1

Figure 8.1 illustrates the results of this simulation for the transmission of $N = 10000$ bits at several different values of SNR. Note the agreement between the simulation results and the theoretical value of P_2. We should also note that a simulation of $N = 10000$ data bits allows us to estimate the error probability reliably down to about $P_2 = 10^{-3}$. In other words, with $N = 10000$ data bits, we should have at least ten errors for a reliable estimate of P_e. MATLAB scripts for this problem are given next.

```
% MATLAB script for Computer Problem 8.1.
echo on
SNRindB1=0:1:12;
SNRindB2=0:0.1:12;
for i=1:length(SNRindB1),
  % simulated error rate
  smld_err_prb(i)=smldPe81(SNRindB1(i));
  echo off ;
end;
echo on ;
for i=1:length(SNRindB2),
  SNR=exp(SNRindB2(i)*log(10)/10);
  % theoretical error rate
```

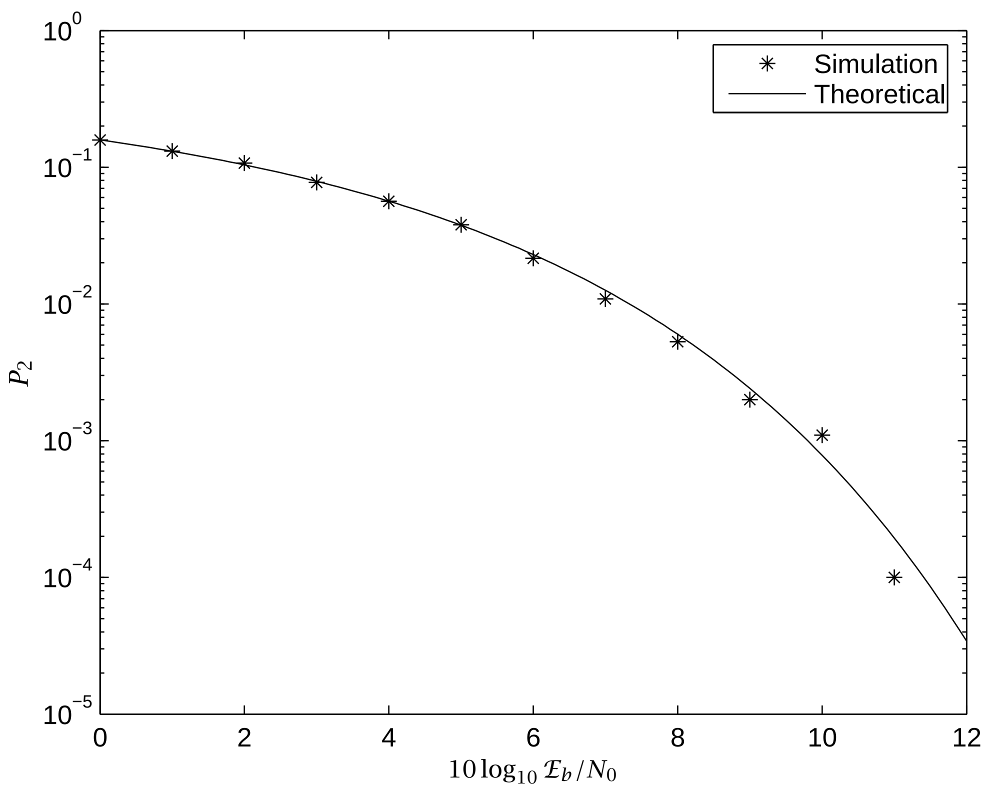


Figure 8.1: Error probability from Monte Carlo simulation compared with theoretical error probability for orthogonal signaling

```
  theo_err_prb(i)=Qfunct(sqrt(SNR));
  echo off ;
end;
echo on;
% Plotting commands follow.
semilogy(SNRindB1,smld_err_prb,'*');
hold
semilogy(SNRindB2,theo_err_prb);
```

```
function [p]=smldPe81(snr_in_dB)
% [p]=smldPe54(snr_in_dB)
%              SMLDPE81  finds the probability of error for the given
%              snr_in_dB, signal-to-noise ratio in dB.
E=1;
SNR=exp(snr_in_dB*log(10)/10);       % signal-to-noise ratio
sgma=E/sqrt(2*SNR);                  % sigma, standard deviation of noise
N=10000;
% generation of the binary data source
for i=1:N,
  temp=rand;                         % a uniform random variable over (0,1)
  if (temp<0.5),
    dsource(i)=0;                    % With probability 1/2, source output is 0.
  else
    dsource(i)=1;                    % With probability 1/2, source output is 1.
  end
end;
% detection, and probability of error calculation
numoferr=0;
for i=1:N,
  % matched filter outputs
  if (dsource(i)==0),
    r0=E+gngauss(sgma);
    r1=gngauss(sgma);                % if the source output is "0"
  else
    r0=gngauss(sgma);
    r1=E+gngauss(sgma);              % if the source output is "1"
  end;
  % Detector follows.
  if (r0>r1),
    decis=0;                         % Decision is "0".
  else
    decis=1;                         % Decision is "1".
  end;
  if (decis~=dsource(i)),            % If it is an error, increase the error counter.
    numoferr=numoferr+1;
  end;
end;
p=numoferr/N;                        % probability of error estimate
```

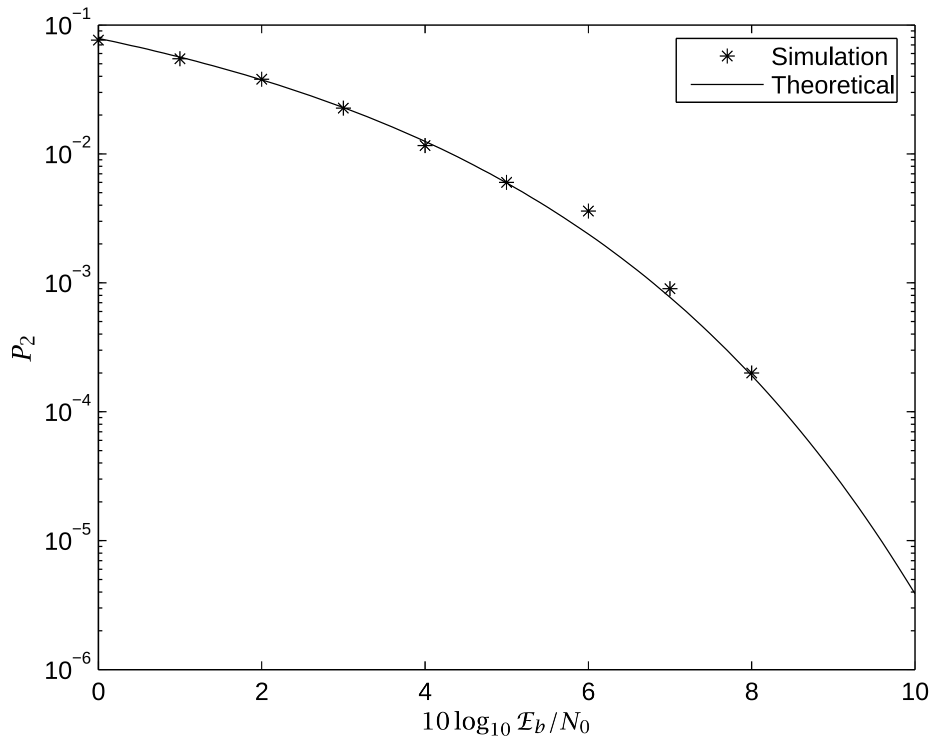


Figure 8.2: Error probability from Monte Carlo simulation compared with theoretical error probability for antipodal signals

Computer Problem 8.2

Figure 8.2 illustrates the results of this simulation for the transmission of 10000 bits at several different values of SNR. The theoretical value for P_2 is also plotted in Figure 8.2 for comparison. The MATLAB scripts for this problem are given next.

```
% MATLAB script for Computer Problem 8.2.
echo on
SNRindB1=0:1:10;
SNRindB2=0:0.1:10;
for i=1:length(SNRindB1),
  % simulated error rate
  smld_err_prb(i)=smldPe82(SNRindB1(i));
  echo off;
end;
echo on;
for i=1:length(SNRindB2),
  SNR=exp(SNRindB2(i)*log(10)/10);
  % theoretical error rate
  theo_err_prb(i)=Qfunct(sqrt(2*SNR));
  echo off;
end;
echo on;
% Plotting commands follow.
```

```
semilogy(SNRindB1,smld_err_prb,'*');
hold
semilogy(SNRindB2,theo_err_prb);
```

```
function [p]=smldPe82(snr_in_dB)
% [p]=smldPe82(snr_in_dB)
%              SMLDPE82  simulates the probability of error for the particular
%              value of snr_in_dB, signal-to-noise ratio in dB.
E=1;
SNR=exp(snr_in_dB*log(10)/10);       % signal-to-noise ratio
sgma=E/sqrt(2*SNR);                  % sigma, standard deviation of noise
N=10000;
% Generation of the binary data source follows.
for i=1:N,
  temp=rand;                         % a uniform random variable over (0,1)
  if (temp<0.5),
    dsource(i)=0;                    % With probability 1/2, source output is 0.
  else
    dsource(i)=1;                    % With probability 1/2, source output is 1.
  end
end;
% The detection, and probability of error calculation follows.
numoferr=0;
for i=1:N,
  % the matched filter outputs
  if (dsource(i)==0),
    r=-E+gngauss(sgma);              % if the source output is "0"
  else
    r=E+gngauss(sgma);               % if the source output is "1"
  end;
  % Detector follows.
  if (r<0),
    decis=0;                         % Decision is "0".
  else
    decis=1;                         % Decision is "1".
  end;
  if (decis~=dsource(i)),            % If it is an error, increase the error counter.
    numoferr=numoferr+1;
  end;
end;
p=numoferr/N;                        % probability of error estimate
```

Computer Problem 8.3

Figure 8.3 illustrates the estimated error probability based on 10000 binary digits. The theoretical error rate P_2 is also illustrated in this figure.

The MATLAB scripts for this problem are given next.

```
% MATLAB script for Computer Problem 8.3.
```

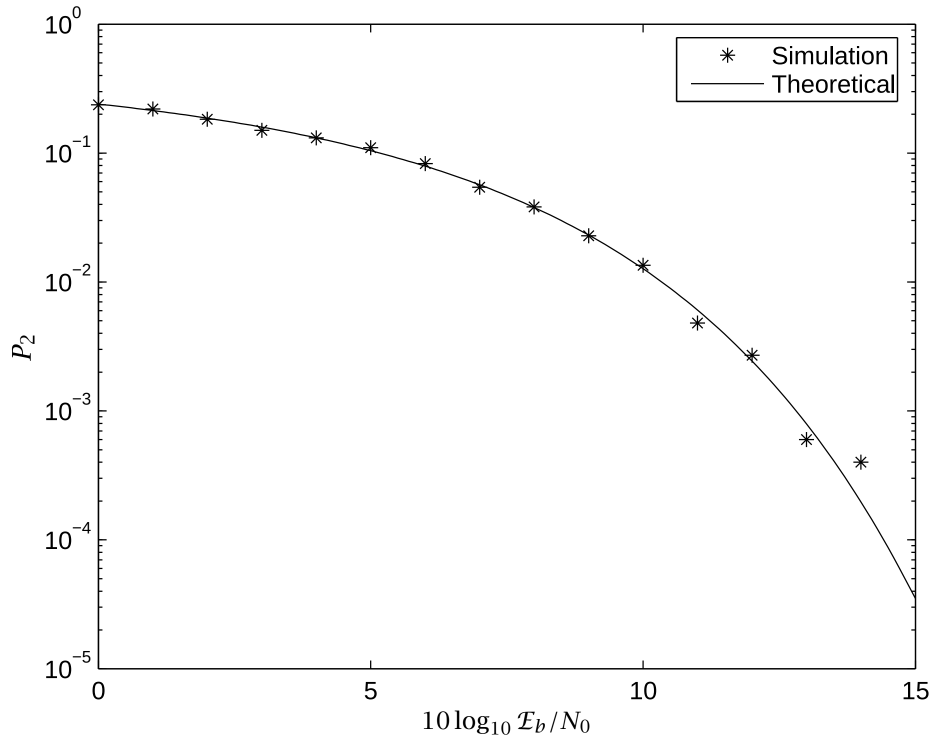


Figure 8.3: Error probability from Monte Carlo simulation compared with theoretical error probability for on-off signals

```
echo on
SNRindB1=0:1:15;
SNRindB2=0:0.1:15;
for i=1:length(SNRindB1),
    smld_err_prb(i)=smldPe83(SNRindB1(i));      % simulated error rate
    echo off;
end;
echo on;
for i=1:length(SNRindB2),
    SNR=exp(SNRindB2(i)*log(10)/10);            % signal-to-noise ratio
    theo_err_prb(i)=Qfunct(sqrt(SNR/2));        % theoretical error rate
    echo off;
end;
echo on;
% Plotting commands follow.
semilogy(SNRindB1,smld_err_prb,'*');
hold
semilogy(SNRindB2,theo_err_prb);
```

```
function [p]=smldPe83(snr_in_dB)
% [p]=smldPe83(snr_in_dB)
%               SMLDPE83 simulates the probability of error for a given
%               snr_in_dB, signal-to-noise ratio in dB.

E=1;
alpha_opt=1/2;
SNR=exp(snr_in_dB*log(10)/10);          % signal-to-noise ratio
sgma=E/sqrt(2*SNR);                     % sigma, standard deviation of noise
N=10000;
% Generation of the binary data source follows.
for i=1:N,
    temp=rand;                          % a uniform random variable over (0,1)
    if (temp<0.5),
        dsource(i)=0;                   % With probability 1/2, source output is 0.
    else
        dsource(i)=1;                   % With probability 1/2, source output is 1.
    end
end;
% detection, and probability of error calculation
numoferr=0;
for i=1:N,
    % the matched filter outputs
    if (dsource(i)==0),
        r=gngauss(sgma);                % if the source output is "0"
    else
        r=E+gngauss(sgma);              % if the source output is "1"
    end;
    % Detector follows.
    if (r<alpha_opt),
        decis=0;                        % Decision is "0".
    else
        decis=1;                        % Decision is "1".
```

```
    end;
    if (decis~=dsource(i)),          % If it is an error, increase the error counter.
        numoferr=numoferr+1;
    end;
end;
p=numoferr/N;                        % probability of error estimate
```

Computer Problem 8.4

The results of the Monte Carlo simulation are shown in Figure 8.4. Note that at a low noise power level (σ small) the effect of the noise on performance (error rate) of the communication system is small. As the noise power level increases, the noise components increase in size and cause more errors.

The MATLAB script for this problem for $\sigma = 0.5$ is given next.

```
% MATLAB script for Computer Problem 8.4.
echo on
n0=.5*randn(100,1);
n1=.5*randn(100,1);
n2=.5*randn(100,1);
n3=.5*randn(100,1);
x1=1.+n0;
y1=n1;
x2=n2;
y2=1.+n3;
plot(x1,y1,'o',x2,y2,'*')
axis('square')
```

Computer Problem 8.5

Figure 8.5 illustrates the results of the simulation for the transmissions of $N = 10000$ symbols at different values of the average bit SNR. Note the agreement between the simulation results and the theoretical values of P_4 computed from

$$P_4 = \frac{3}{2} Q\left(\sqrt{\frac{2\mathcal{E}_{\text{av}}}{5N_0}}\right) \tag{8.24}$$

The MATLAB scripts for this problem are given next.

```
% MATLAB script for Computer Problem 8.5.

echo on
SNRindB1=0:1:12;
SNRindB2=0:0.1:12;
for i=1:length(SNRindB1),
  % simulated error rate
  smld_err_prb(i)=smldPe85(SNRindB1(i));
```

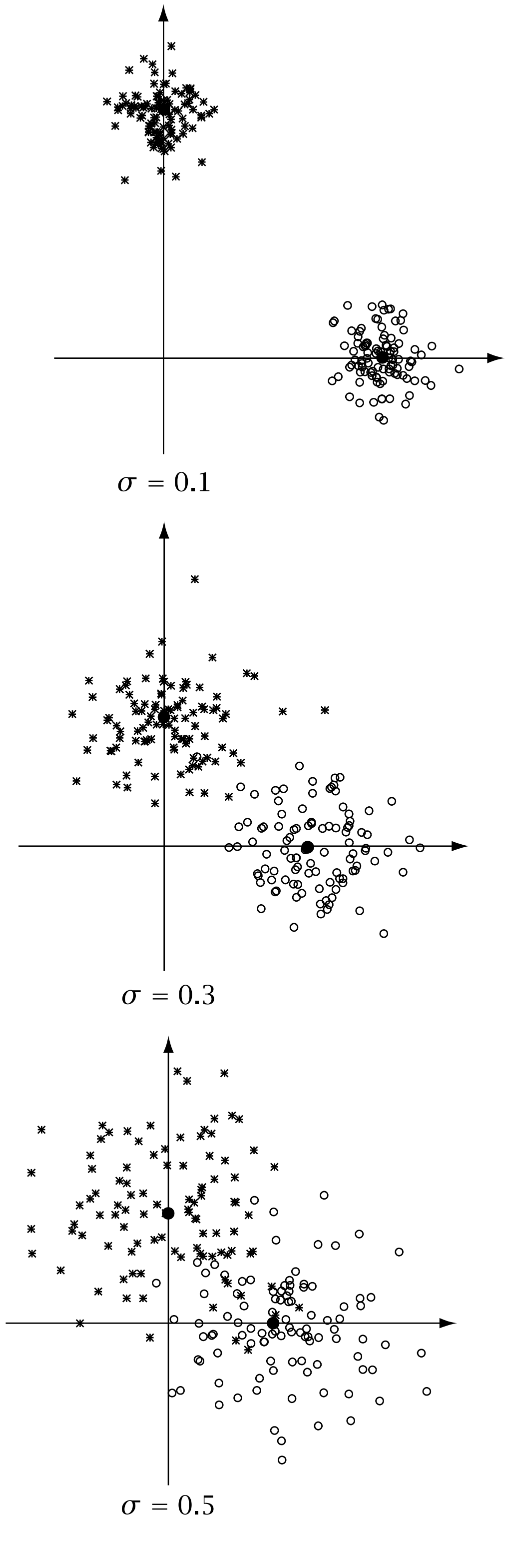


Figure 8.4: Received signal points at input to the detector for orthogonal signals (Monte Carlo simulation)

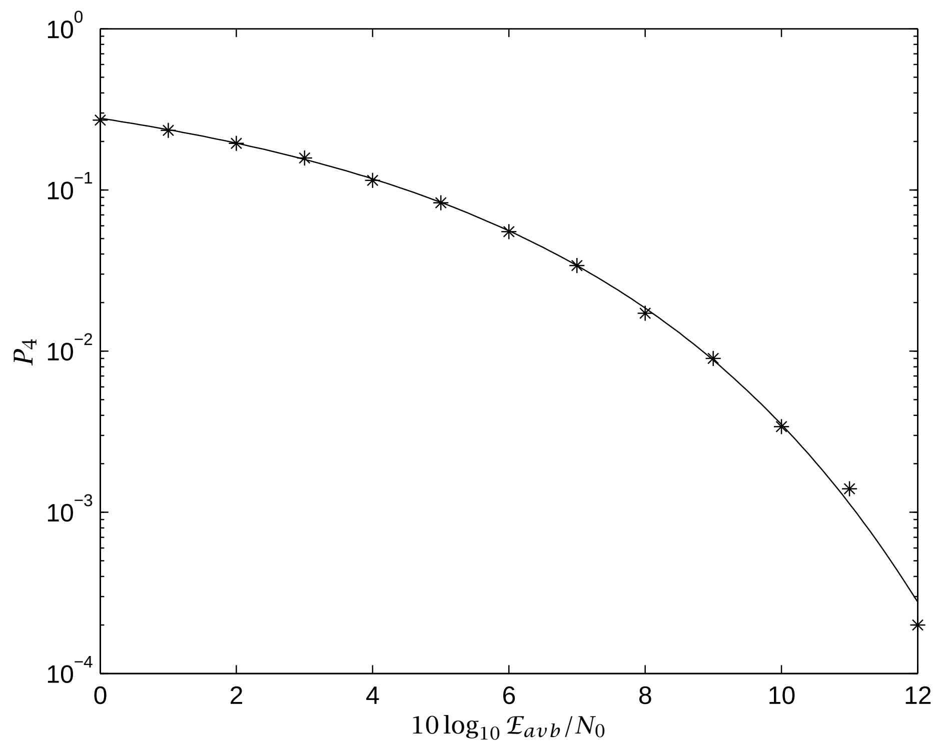


Figure 8.5: Probability of symbol error for four-level PAM

```
  echo off;
end;
echo on;
for i=1:length(SNRindB2),
  % signal-to-noise ratio
  SNR_per_bit=exp(SNRindB2(i)*log(10)/10);
  % theoretical error rate
  theo_err_prb(i)=(3/2)*Qfunct(sqrt((4/5)*SNR_per_bit));
  echo off;
end;
echo on;
% Plotting commands follow.
semilogy(SNRindB1,smld_err_prb,'*');
hold
semilogy(SNRindB2,theo_err_prb);
```

```
function [p]=smldPe85(snr_in_dB)
% [p]=smldPe85(snr_in_dB)
%          SMLDPE85  simulates the probability of error for the given
%          snr_in_dB, signal to noise ratio in dB.
d=1;
SNR=exp(snr_in_dB*log(10)/10);      % signal to noise ratio per bit
sgma=sqrt((5*d^2)/(4*SNR));         % sigma, standard deviation of noise
N=10000;                            % number of symbols being simulated
% Generation of the quaternary data source follows.
for i=1:N,
  temp=rand;                        % a uniform random variable over (0,1)
  if (temp<0.25),
    dsource(i)=0;                   % With probability 1/4, source output is "00."
  elseif (temp<0.5),
    dsource(i)=1;                   % With probability 1/4, source output is "01."
  elseif (temp<0.75),
    dsource(i)=2;                   % With probability 1/4, source output is "10."
  else
    dsource(i)=3;                   % With probability 1/4, source output is "11."
  end
end;
% detection, and probability of error calculation
numoferr=0;
for i=1:N,
  % the matched filter outputs
  if (dsource(i)==0),
    r=-3*d+gngauss(sgma);           % if the source output is "00"
  elseif (dsource(i)==1),
    r=-d+gngauss(sgma);             % if the source output is "01"
  elseif (dsource(i)==2)
    r=d+gngauss(sgma);              % if the source output is "10"
  else
    r=3*d+gngauss(sgma);            % if the source output is "11"
  end;
  % Detector follows.
  if (r<-2*d),
```

```
    decis=0;                    % Decision is "00."
  elseif (r<0),
    decis=1;                    % Decision is "01."
  elseif (r<2*d),
    decis=2;                    % Decision is "10."
  else
    decis=3;                    % Decision is "11."
  end;
  if (decis~=dsource(i)),       % If it is an error, increase the error counter.
    numoferr=numoferr+1;
  end;
end;
p=numoferr/N;                   % probability of error estimate
```

Computer Problem 8.6

Figure 8.6 illustrates the measured symbol error rate for 10000 transmitted symbols and the theoretical symbol error rate given by

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6(\log_2 M)\mathcal{E}_{avb}}{(M^2-1)N_0}}\right) \tag{8.25}$$

where $M = 16$.

The MATLAB scripts for this problem are given next.

```
% MATLAB script for Computer Problem 8.6.
echo on
SNRindB1=5:1:25;
SNRindB2=5:0.1:25;
M=16;
for i=1:length(SNRindB1),
  % simulated error rate
  smld_err_prb(i)=smldPe86(SNRindB1(i));
  echo off;
end;
echo on ;
for i=1:length(SNRindB2),
  SNR_per_bit=exp(SNRindB2(i)*log(10)/10);
  % theoretical error rate
  theo_err_prb(i)=(2*(M-1)/M)*Qfunct(sqrt((6*log2(M)/(M^2-1))*SNR_per_bit));
  echo off;
end;
echo on;
% Plotting commands follow.
semilogy(SNRindB1,smld_err_prb,'*');
hold
semilogy(SNRindB2,theo_err_prb);
```

```
function [p]=smldPe86(snr_in_dB)
```

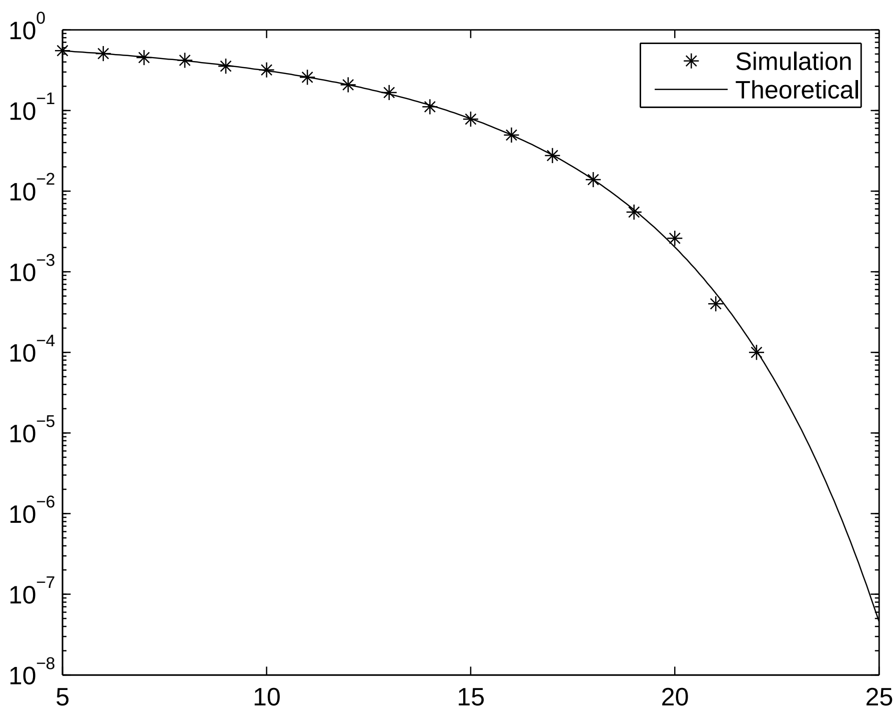


Figure 8.6: Error rate from Monte Carlo simulation compared with the theoretical error probability for $M = 16$ PAM

```
% [p]=smldPe86(snr_in_dB)
%               SMLDPE86  simulates the error probability for the given
%               snr_in_dB, signal-to-noise ratio in dB.
M=16;                                % 16-ary PAM
d=1;
SNR=exp(snr_in_dB*log(10)/10);       % signal-to-noise ratio per bit
sgma=sqrt((85*d^2)/(8*SNR));         % sigma, standard deviation of noise
N=10000;                             % number of symbols being simulated
% generation of the data source
for i=1:N,
   temp=rand;                        % a uniform random variable over (0,1)
   index=floor(M*temp);              % The index is an integer from 0 to M-1, where
                                     % all the possible values are equally likely.
   dsource(i)=index;
end;
% detection, and probability of error calculation
numoferr=0;
for i=1:N,
  % matched filter outputs
  % (2*dsource(i)-M+1)*d is the mapping to the 16-ary constellation.
  r=(2*dsource(i)-M+1)*d+gngauss(sgma);
  % the detector
  if (r>(M-2)*d),
    decis=15;
  elseif (r>(M-4)*d),
    decis=14;
  elseif (r>(M-6)*d),
    decis=13;
  elseif (r>(M-8)*d),
    decis=12;
  elseif (r>(M-10)*d),
    decis=11;
  elseif (r>(M-12)*d),
    decis=10;
  elseif (r>(M-14)*d),
    decis=9;
  elseif (r>(M-16)*d),
    decis=8;
  elseif (r>(M-18)*d),
    decis=7;
  elseif (r>(M-20)*d),
    decis=6;
  elseif (r>(M-22)*d),
    decis=5;
  elseif (r>(M-24)*d),
    decis=4;
  elseif (r>(M-26)*d),
    decis=3;
  elseif (r>(M-28)*d),
    decis=2;
  elseif (r>(M-30)*d),
    decis=1;
  else
    decis=0;
```

```
  end;
  if (decis~=dsource(i)),    % If it is an error, increase the error counter.
    numoferr=numoferr+1;
  end;
end;
p=numoferr/N;                  % probability of error estimate
```

Computer Problem 8.7

Figure 8.7 illustrates the eight waveforms for the case in which $f_c = 6/T$. The MATLAB script for this computation is given next.

```
% MATLAB script for Computer Problem 8.7.
echo on
T=1;
M=8;
Es=T/2;
fc=6/T;                        % carrier frequency
N=100;                         % number of samples
delta_T=T/(N-1);
t=0:delta_T:T;
u0=sqrt(2*Es/T)*cos(2*pi*fc*t);
u1=sqrt(2*Es/T)*cos(2*pi*fc*t+2*pi/M);
u2=sqrt(2*Es/T)*cos(2*pi*fc*t+4*pi/M);
u3=sqrt(2*Es/T)*cos(2*pi*fc*t+6*pi/M);
u4=sqrt(2*Es/T)*cos(2*pi*fc*t+8*pi/M);
u5=sqrt(2*Es/T)*cos(2*pi*fc*t+10*pi/M);
u6=sqrt(2*Es/T)*cos(2*pi*fc*t+12*pi/M);
u7=sqrt(2*Es/T)*cos(2*pi*fc*t+14*pi/M);
% plotting commands follow
subplot(8,1,1);
plot(t,u0);
subplot(8,1,2);
plot(t,u1);
subplot(8,1,3);
plot(t,u2);
subplot(8,1,4);
plot(t,u3);
subplot(8,1,5);
plot(t,u4);
subplot(8,1,6);
plot(t,u5);
subplot(8,1,7);
plot(t,u6);
subplot(8,1,8);
plot(t,u7);
```

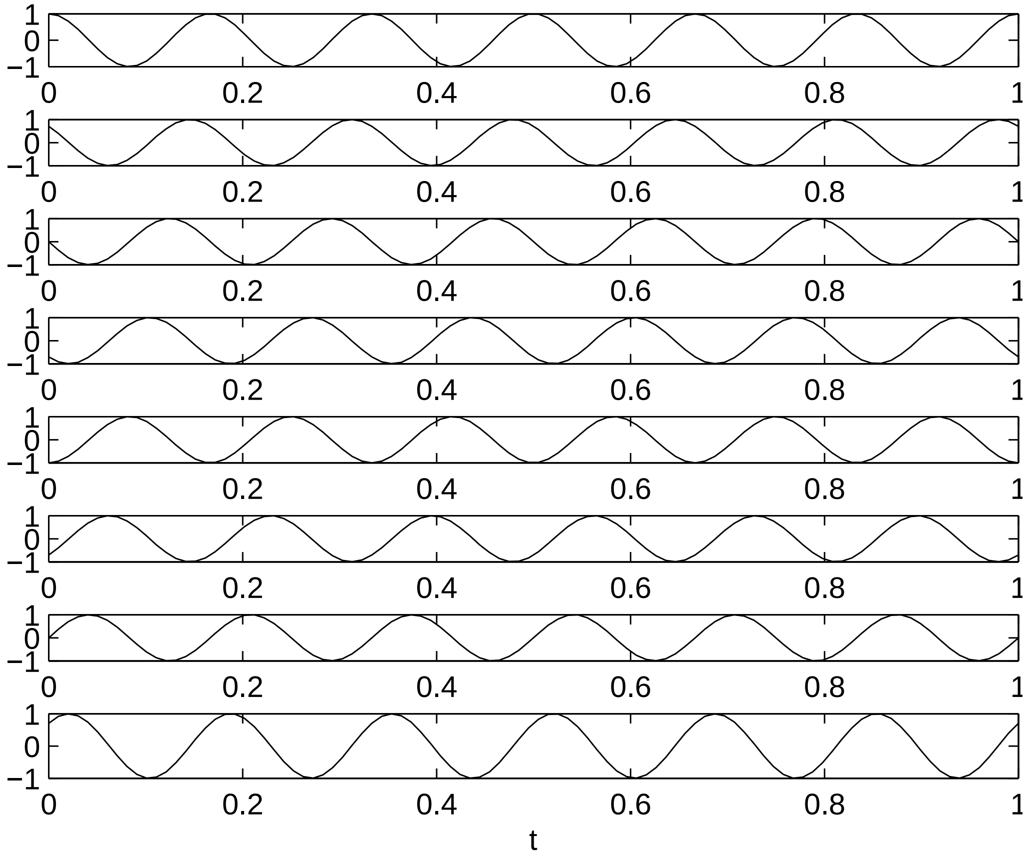


Figure 8.7: $M = 8$ constant-amplitude PSK waveforms

Computer Problem 8.8

For convenience we set $T = 1$. The following figure illustrates the correlator outputs over the entire signal interval for the four possible transmitted phases. Note that the double frequency terms average out to zero, as best observed in the case where $\sigma^2 = 0$. Secondly, we observe the effect of the additive noise on the correlator outputs as σ^2 increases.

The MATLAB script for the problem is given below.

```
% MATLAB script for Computer Problem 8.8

M = 4;
Es = 1;                              % Energy per symbol
T = 1;
Ts = 100/T;
fc = 30/T;
t = 0:T/100:T;
l_t = length(t);
g_T = sqrt(2/T)*ones(1,l_t);
si_1 = g_T.*cos(2*pi*fc*t);
si_2 = -g_T.*sin(2*pi*fc*t);
for m = 0 : 3
    % Generation of the transmitted signal:
    s_mc = sqrt(Es) * cos(2*pi*m/M);
    s_ms = sqrt(Es) * sin(2*pi*m/M);
    u_m = s_mc.*si_1 + s_ms.*si_2;
    var = [ 0 0.05 0.5];             % Noise variance vector
    if (m == 2)
        figure
    end
    for k = 1 : length(var)
        % Generation of the noise components:
        n_c = sqrt(var(k))*randn(1,l_t);
        n_s = sqrt(var(k))*randn(1,l_t);
        % The received signal:
        r = u_m + n_c.*cos(2*pi*fc*t) - n_s.*sin(2*pi*fc*t);
        % The correlator outputs:
        y_c = zeros(1,l_t);
        y_s = zeros(1,l_t);
        for i = 1:l_t
            y_c(i) = sum(r(1:i).*si_1(1:i));
            y_s(i) = sum(r(1:i).*si_2(1:i));
        end
        % Plotting the results:
        subplot(3,2,2*k-1+mod(m,2))
        plot([0 1:length(y_c)-1],y_c,'.-')
        hold
        plot([0 1:length(y_s)-1],y_s)
        title(['\sigma^2 = ',num2str(var(k))])
        xlabel(['n (m=',num2str(m),')'])
        axis auto
    end
end
```

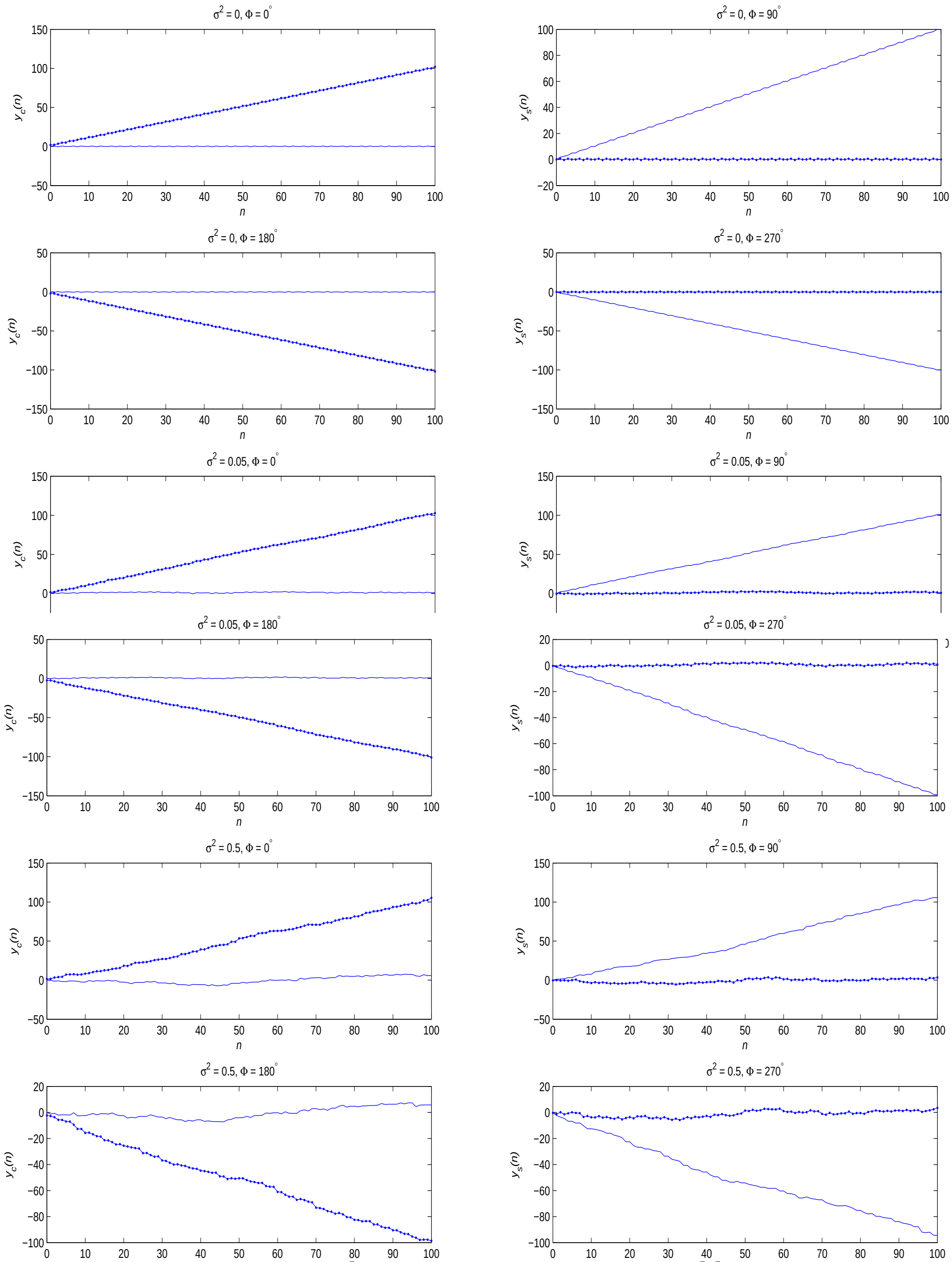


Figure 8.8: Correlator outputs in Computer Problem 8.8

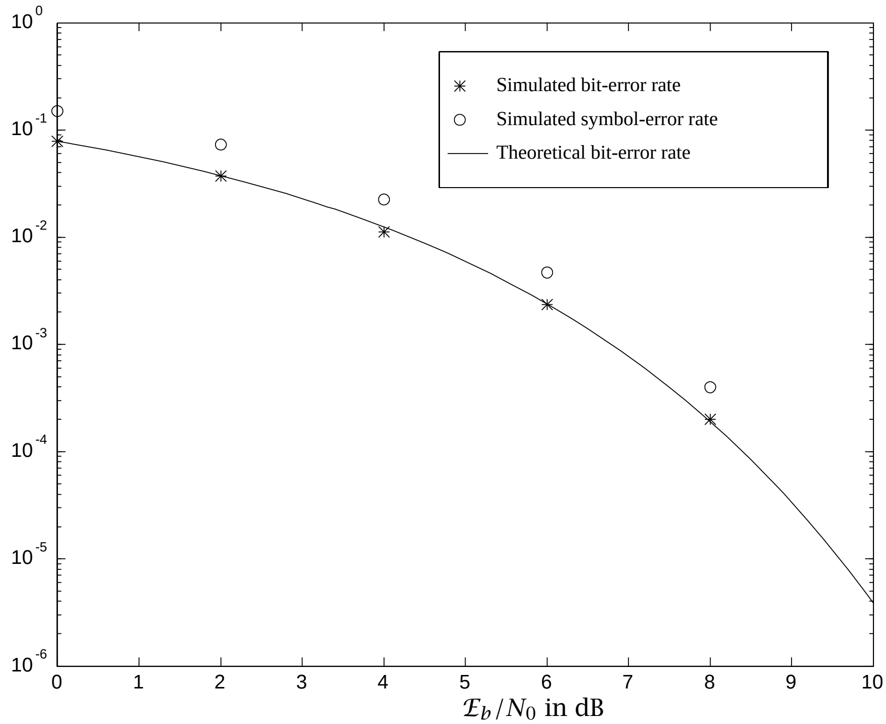


Figure 8.9: Performance of a four-phase PSK system from the Monte Carlo simulation

---

Computer Problem 8.9

Figure 8.9 illustrates the results of the Monte Carlo simulation for the transmission of N=10000 symbols at different values of the SNR parameter $\mathcal{E}_b/N_0$, where $\mathcal{E}_b = \mathcal{E}_s/2$ is the bit energy. Also shown in Figure 8.9 is the bit-error rate, which is defined as $P_b \approx P_M/2$, and the corresponding theoretical error probability, given by

$$\begin{aligned} P_M &\approx 2Q\left(\sqrt{\frac{2\mathcal{E}_s}{N_0}}\sin\frac{\pi}{M}\right) \\ &\approx 2Q\left(\sqrt{\frac{2k\mathcal{E}_b}{N_0}}\sin\frac{\pi}{M}\right) \end{aligned} \tag{8.26}$$

The MATLAB scripts for this Monte Carlo simulation are given next.

```
% MATLAB script for Computer Problem 8.9.
echo on
SNRindB1=0:2:10;
SNRindB2=0:0.1:10;
for i=1:length(SNRindB1),
  [pb,ps]=cm_sm32(SNRindB1(i));         % simulated bit and symbol error rates
```

```
  smld_bit_err_prb(i)=pb;
  smld_symbol_err_prb(i)=ps;
  echo off ;
end;
echo on;
for i=1:length(SNRindB2),
  SNR=exp(SNRindB2(i)*log(10)/10);        % signal-to-noise ratio
  theo_err_prb(i)=Qfunct(sqrt(2*SNR));   % theoretical bit-error rate
  echo off ;
end;
echo on ;
% Plotting commands follow.
semilogy(SNRindB1,smld_bit_err_prb,'*');
hold
semilogy(SNRindB1,smld_symbol_err_prb,'o');
semilogy(SNRindB2,theo_err_prb);
```

```
function [pb,ps]=cm_sm32(snr_in_dB)
% [pb,ps]=cm_sm32(snr_in_dB)
%               CM_SM32  finds the probability of bit error and symbol error for the
%               given value of snr_in_dB, signal-to-noise ratio in dB.
N=10000;
E=1;                                   % energy per symbol
snr=10^(snr_in_dB/10);                 % signal-to-noise ratio
sgma=sqrt(E/snr)/2;                    % noise variance
% the signal mapping
s00=[1 0];
s01=[0 1];
s11=[−1 0];
s10=[0 −1];
% generation of the data source
for i=1:N,
  temp=rand;                           % a uniform random variable between 0 and 1
  if (temp<0.25),                      % With probability 1/4, source output is "00."
    dsource1(i)=0;
    dsource2(i)=0;
  elseif (temp<0.5),                   % With probability 1/4, source output is "01."
    dsource1(i)=0;
    dsource2(i)=1;
  elseif (temp<0.75),                  % With probability 1/4, source output is "10."
    dsource1(i)=1;
    dsource2(i)=0;
  else                                 % With probability 1/4, source output is "11."
    dsource1(i)=1;
    dsource2(i)=1;
  end;
end;
% detection and the probability of error calculation
numofsymbolerror=0;
numofbiterror=0;
for i=1:N,
  % The received signal at the detector, for the ith symbol, is:
```

```
   n(1)=gngauss(sgma);
   n(2)=gngauss(sgma);
   if ((dsource1(i)==0) & (dsource2(i)==0)),
      r=s00+n;
   elseif ((dsource1(i)==0) & (dsource2(i)==1)),
      r=s01+n;
   elseif ((dsource1(i)==1) & (dsource2(i)==0)),
      r=s10+n;
   else
      r=s11+n;
   end;
   % The correlation metrics are computed below.
   c00=dot(r,s00);
   c01=dot(r,s01);
   c10=dot(r,s10);
   c11=dot(r,s11);
   % The decision on the ith symbol is made next.
   c_max=max([c00 c01 c10 c11]);
   if (c00==c_max),
      decis1=0; decis2=0;
   elseif (c01==c_max),
      decis1=0; decis2=1;
   elseif (c10==c_max),
      decis1=1; decis2=0;
   else
      decis1=1; decis2=1;
   end;
   % Increment the error counter, if the decision is not correct.
   symbolerror=0;
   if (decis1~=dsource1(i)),
      numofbiterror=numofbiterror+1;
      symbolerror=1;
   end;
   if (decis2~=dsource2(i)),
      numofbiterror=numofbiterror+1;
      symbolerror=1;
   end;
   if (symbolerror==1),
      numofsymbolerror = numofsymbolerror+1;
   end;
end;
ps=numofsymbolerror/N;          % since there are totally N symbols
pb=numofbiterror/(2*N);         % since 2N bits are transmitted
```

Computer Problem 8.10

Figure 8.10 illustrates the results of the Monte Carlo simulation for the transmission of N=10000 symbols at different values of the SNR parameter $\mathcal{E}_b/N_0$, where $\mathcal{E}_b = \mathcal{E}_s/2$ is the bit energy. Also shown in Figure 8.10 is the theoretical value of the symbol error rate based on the approximation that the term $n_k n_{k-1}^*$ is negligible. We observe from Figure 8.10 that the approximation results in an upper bound to the error probability.

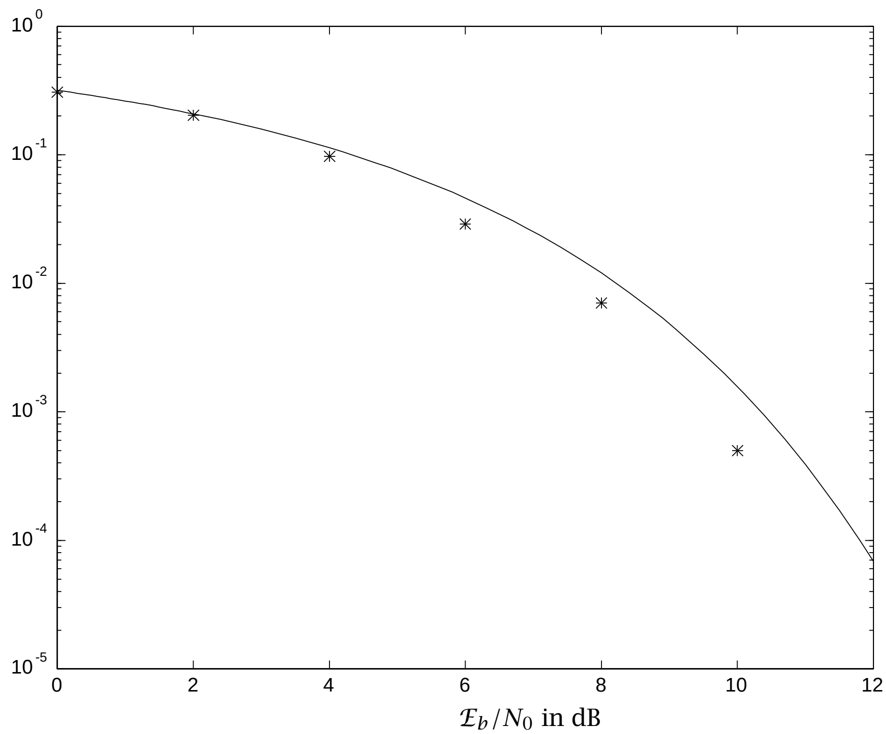


Figure 8.10: Performance of four-phase DPSK system from Monte Carlo simulation (the solid curve is an upper bound based on approximation that neglects the noise term $n_l n_{k-1}^*$)

The MATLAB scripts for this Monte Carlo simulation are given next.

```
% MATLAB script for Computer Problem 8.10.
echo on
SNRindB1=0:2:12;
SNRindB2=0:0.1:12;
for i=1:length(SNRindB1),
  smld_err_prb(i)=cm_sm34(SNRindB1(i)); % simulated error rate
  echo off ;
end;
echo on ;
for i=1:length(SNRindB2),
  SNR=exp(SNRindB2(i)*log(10)/10);      % signal-to-noise ratio
  theo_err_prb(i)=2*Qfunct(sqrt(SNR));   % theoretical symbol error rate
  echo off ;
end;
echo on ;
% Plotting commands follow
semilogy(SNRindB1,smld_err_prb,'*');
hold
semilogy(SNRindB2,theo_err_prb);
```

```
function [p]=cm_sm34(snr_in_dB)
% [p]=cm_sm34(snr_in_dB)
%               CM_SM34  finds the probability of error for the given
%               value of snr_in_dB, signal-to-noise ratio in dB.
N=10000;
E=1;                              % energy per symbol
snr=10^(snr_in_dB/10);            % signal-to-noise ratio
sgma=sqrt(E/(4*snr));             % noise variance
% Generation of the data source follows.
for i=1:2*N,
  temp=rand;                      % a uniform random variable between 0 and 1
  if (temp<0.5),
    dsource(i)=0;                 % With probability 1/2, source output is "0."
  else.
    dsource(i)=1;                 % With probability 1/2, source output is "1."
  end;
end;
% Differential encoding of the data source follows.
mapping=[0 1 3 2];
M=4;
[diff_enc_output] = cm_dpske(E,M,mapping,dsource);
% Received signal is then
for i=1:N,
  [n(1) n(2)]=gngauss(sgma);
  r(i,:)=diff_enc_output(i,:)+n;
end;
% detection and the probability of error calculation
numoferr=0;
prev_theta=0;
for i=1:N,
```

```
  theta=angle(r(i,1)+j*r(i,2));
  delta_theta=mod(theta-prev_theta,2*pi);
  if ((delta_theta<pi/4) | (delta_theta>7*pi/4)),
    decis=[0 0];
  elseif (delta_theta<3*pi/4),
    decis=[0 1];
  elseif (delta_theta<5*pi/4)
    decis=[1 1];
  else
    decis=[1 0];
  end;
  prev_theta=theta;
  % Increase the error counter, if the decision is not correct.
  if ((decis(1)~=dsource(2*i-1)) | (decis(2)~=dsource(2*i))),
    numoferr=numoferr+1;
  end;
end;
p=numoferr/N;
```

```
function [enc_comp] = cm_dpske(E,M,mapping,sequence);
% [enc_comp] = cm_dpske(E,M,mapping,sequence)
%          CM_DPSKE differentially encodes a sequence.
%          E is the average energy, M is the number of constellation points,
%          and mapping is the vector defining how the constellation points are
%          allocated. Finally, "sequence" is the uncoded binary data sequence.
k=log2(M);
N=length(sequence);
% If N is not divisible by k, append zeros, so that it is...
remainder=rem(N,k);
if (remainder~=0),
  for i=N+1:N+k-remainder,
    sequence(i)=0;
  end;
  N=N+k-remainder;
end;
theta=0;                          % Initially, assume that theta=0.
for i=1:k:N,
  index=0;
  for j=i:i+k-1,
    index=2*index+sequence(j);
  end;
  index=index+1;
  theta=mod(2*pi*mapping(index)/M+theta,2*pi);
  enc_comp((i+k-1)/k,1)=sqrt(E)*cos(theta);
  enc_comp((i+k-1)/k,2)=sqrt(E)*sin(theta);
end;
```

Computer Problem 8.11

The position of the eight signal points are $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$, $(1+\sqrt{3},0)$, $(-1,-\sqrt{3},0)$, $(0,1+\sqrt{3})$, and $(0,-1-\sqrt{3})$. For convenience, we set $T=1$. Figure 8.11 illustrates the correlator outputs over the signal interval when the transmitted symbol is $(1,-1)$. Note that the double frequency terms average out to zero, as best observed in the case where $\sigma^2=0$. Furthermore, we observe the effect of the additive noise on the correlator output as σ^2 increases.

The MATLAB script for the problem is given below.

```
% MATLAB script for Computer Problem 8.11

M = 8;
Es = 1;                         % Energy oer symbol
T = 1;
Ts = 100/T;
fc = 30/T;
t = 0:T/100:T;
l_t = length(t);
A_mc = 1/sqrt(Es);              % Signal Amplitude
A_ms = -1/sqrt(Es);             % Signal Amplitude
g_T = sqrt(2/T)*ones(1,l_t);
phi = 2*pi*rand;
si_1 = g_T.*cos(2*pi*fc*t + phi);
si_2 = g_T.*sin(2*pi*fc*t + phi);
var = [ 0 0.05 0.5];            % Noise variance vector
for k = 1 : length(var)
    % Generation of the noise components:
    n_c = sqrt(var(k))*randn(1,l_t);
    n_s = sqrt(var(k))*randn(1,l_t);
    noise = n_c.*cos(2*pi*fc+t) - n_s.*sin(2*pi*fc+t);
    % The received signal
    r = A_mc*g_T.*cos(2*pi*fc*t+phi) + A_ms*g_T.*sin(2*pi*fc*t+phi) + noise;
    % The correlator outputs:
    y_c = zeros(1,l_t);
    y_s = zeros(1,l_t);
    for i = 1:l_t
        y_c(i) = sum(r(1:i).*si_1(1:i));
        y_s(i) = sum(r(1:i).*si_2(1:i));
    end
    % Plotting the results:
    subplot(3,1,k)
    plot([0 1:length(y_c)-1],y_c,'.-')
    hold
    plot([0 1:length(y_s)-1],y_s)
    title(['\sigma^2 = ',num2str(var(k))])
    xlabel('n')
    axis auto
end
```

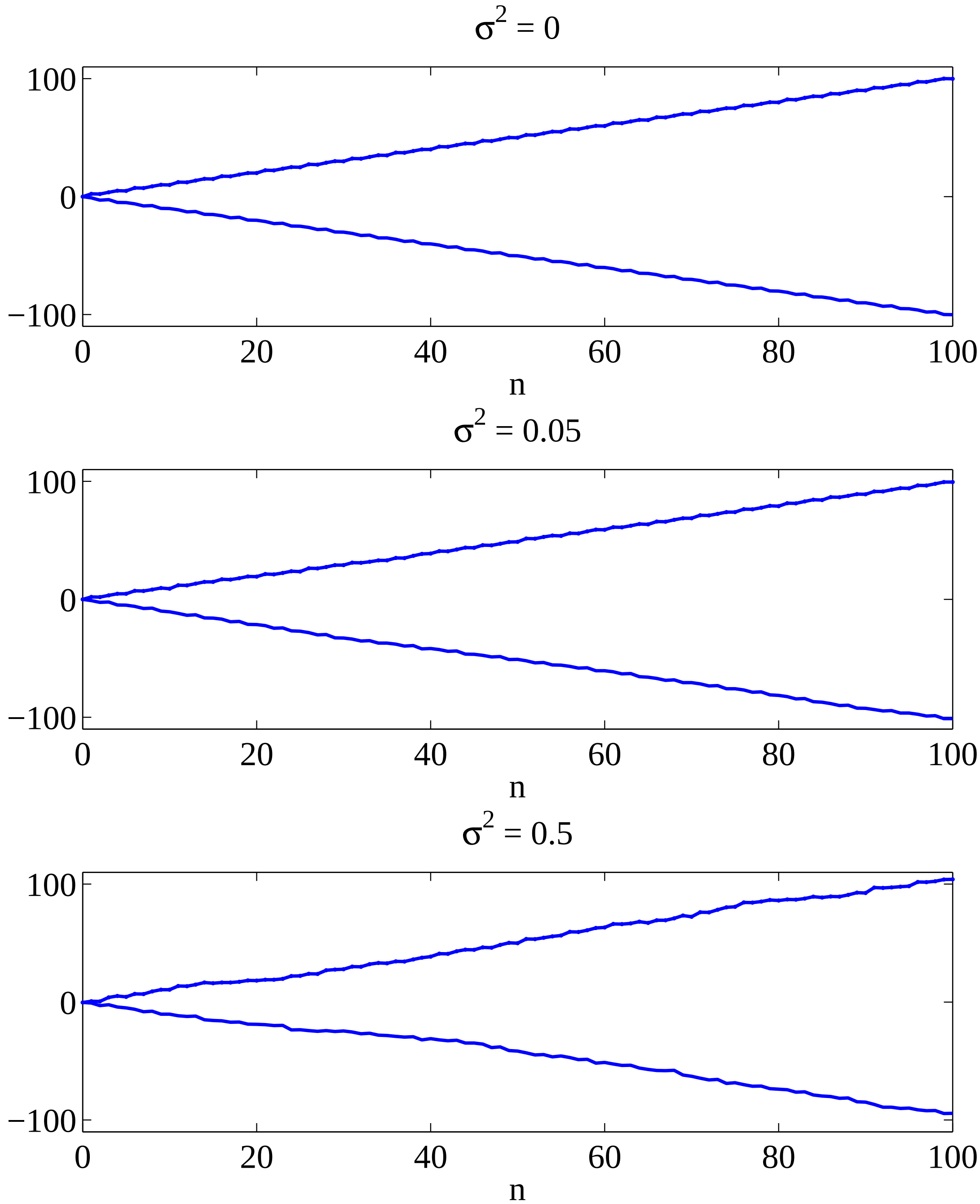


Figure 8.11: Correlator outputs in Computer Problem 8.11.

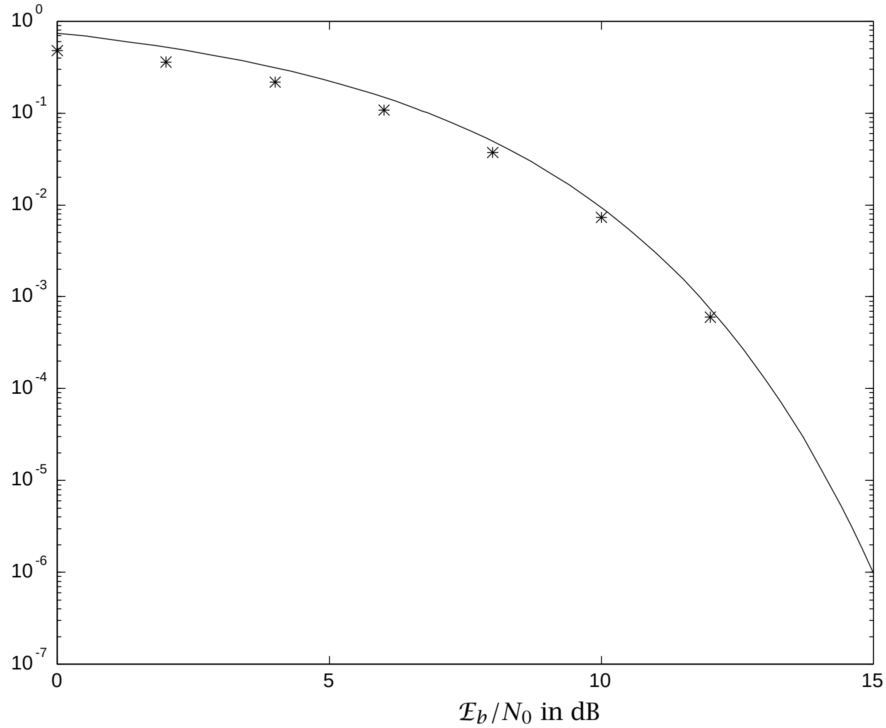


Figure 8.12: Performance of $M = 16$-QAM system from the Monte Carlo simulation.

Computer Problem 8.12

Figure 8.12 illustrates the results of the Monte Carlo simulation for the transmission of N=10000 symbols at different values of the SNR parameter $\mathcal{E}_b/N_0$, where $\mathcal{E}_b = \mathcal{E}_s/4$ is the bit energy. Also shown in Figure 8.12 is the theoretical value of the symbol-error probability given by (8.27) and (8.28).

$$P_{\sqrt{M}} = 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left(\sqrt{\frac{3}{M-1}\frac{\mathcal{E}_{\text{av}}}{N_0}}\right) \tag{8.27}$$

where $\mathcal{E}_{\text{av}}/N_0$ is the average SNR per symbol. Therefore, the probability of a symbol error for the M-ary QAM is

$$P_M = 1 - \left(1 - P_{\sqrt{M}}\right)^2 \tag{8.28}$$

The MATLAB scripts for this problem are given next.

```
% MATLAB script for Computer Problem 8.12.
echo on
SNRindB1=0:2:15;
SNRindB2=0:0.1:15;
M=16;
k=log2(M);
for i=1:length(SNRindB1),
  smld_err_prb(i)=cm_sm41(SNRindB1(i)); % simulated error rate
  echo off;
```

```
end;
echo on ;
for i=1:length(SNRindB2),
  SNR=exp(SNRindB2(i)*log(10)/10);       % signal-to-noise ratio
  % theoretical symbol error rate
  theo_err_prb(i)=4*Qfunct(sqrt(3*k*SNR/(M−1)));
  echo off ;
end;
echo on ;
% Plotting commands follow.
semilogy(SNRindB1,smld_err_prb,'*');
hold
semilogy(SNRindB2,theo_err_prb);
```

```
function [p]=cm_sm41(snr_in_dB)
% [p]=cm_sm41(snr_in_dB)
%               CM_SM41  finds the probability of error for the given
%               value of snr_in_dB, SNR in dB.
N=10000;
d=1;                                  % min. distance between symbols
Eav=10*d^2;                           % energy per symbol
snr=10^(snr_in_dB/10);                % SNR per bit (given)
sgma=sqrt(Eav/(8*snr));               % noise variance
M=16;
% Generation of the data source follows.
for i=1:N,
  temp=rand;                          % a uniform R.V. between 0 and 1
  dsource(i)=1+floor(M*temp);         % a number between 1 and 16, uniform
end;
% Mapping to the signal constellation follows.
mapping=[−3*d 3*d;
           −d  3*d;
            d  3*d;
          3*d  3*d;
         −3*d   d;
           −d   d;
            d   d;
          3*d   d;
         −3*d  −d;
           −d  −d;
            d  −d;
          3*d  −d;
         −3*d  −3*d;
           −d  −3*d;
            d  −3*d;
          3*d  −3*d];
for i=1:N,
  qam_sig(i,:)=mapping(dsource(i),:);
end;
% received signal
for i=1:N,
  [n(1) n(2)]=gngauss(sgma);
```

```
  r(i,:)=qam_sig(i,:)+n;
end;
% detection and error probability calculation
numoferr=0;
for i=1:N,
  % Metric computation follows.
  for j=1:M,
    metrics(j)=(r(i,1)−mapping(j,1))^2+(r(i,2)−mapping(j,2))^2;
  end;
  [min_metric decis] = min(metrics);
  if (decis~=dsource(i)),
    numoferr=numoferr+1;
  end;
end;
p=numoferr/(N);
```

Computer Problem 8.13

Figure 8.13 illustrates the correlator outputs for different noise variances. The MATLAB script for the computation is given next

```
% MATLAB script for Computer Problem 8.13.

% Initialization:
K=20;            % Number of samples
A=1;             % Signal amplitude
l=0:K;
s_0=A*ones(1,K);% Signal waveform
r_0=zeros(1,K); % Output signal

% Case 1: noise~N(0,0)
    noise=random('Normal',0,0,1,K);
    % Sub-case s = s_0:
    s=s_0;
    r=s+noise;  % received signal
    for n=1:K
        r_0(n)=sum(r(1:n).*s_0(1:n));
    end
    % Plotting the results:
    subplot(3,2,1)
    plot(l,[0 r_0])
    set(gca,'XTickLabel',{'0','5Tb','10Tb','15Tb','20Tb'})
    axis([0 20 0 25])
    xlabel('(a) \sigma^2= 0 & S_{0} is transmitted ','fontsize',10)
%      text(15,3,'\fontsize{10} r_{0}: - & r_{1}: -','hor','left')
    % Sub-case s = s_1:
    s=−s_0;
    r=s+noise;  % received signal
    for n=1:K
        r_0(n)=sum(r(1:n).*s_0(1:n));
```

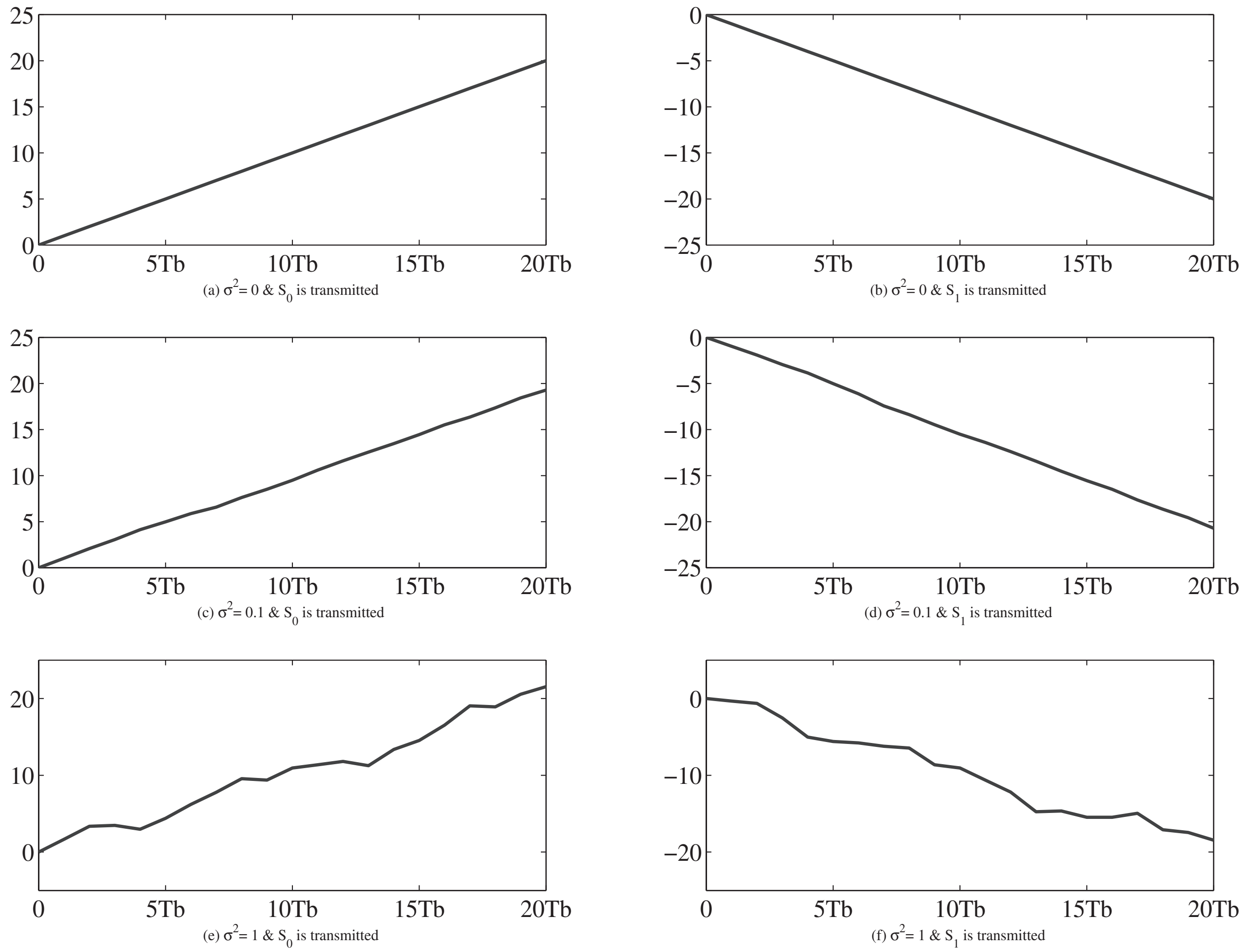


Figure 8.13: Correlator outputs in Computer Problem8.13.

```
    end
    % Plotting the results:
    subplot(3,2,2)
    plot(l,[0 r_0])
    set(gca,'XTickLabel',{'0','5Tb','10Tb','15Tb','20Tb'})
    axis([0 20 -25 0])
    xlabel('(b) \sigma^2= 0 & S_{1} is transmitted ','fontsize',10)
% Case 2: noise~N(0,0.1)
    noise=random('Normal',0,0.1,1,K);
    % Sub-case s = s_0:
    s=s_0;
    r=s+noise;  % received signal
    for n=1:K
        r_0(n)=sum(r(1:n).*s_0(1:n));
    end
    % Plotting the results:
    subplot(3,2,3)
    plot(l,[0 r_0])
    set(gca,'XTickLabel',{'0','5Tb','10Tb','15Tb','20Tb'})
    axis([0 20 0 25])
    xlabel('(c) \sigma^2= 0.1 & S_{0} is transmitted ','fontsize',10)
    % Sub-case s = s_1:
    s=-s_0;
    r=s+noise;  % received signal
    for n=1:K
        r_0(n)=sum(r(1:n).*s_0(1:n));
    end
    % Plotting the results:
    subplot(3,2,4)
    plot(l,[0 r_0])
    set(gca,'XTickLabel',{'0','5Tb','10Tb','15Tb','20Tb'})
    axis([0 20 -25 0])
    xlabel('(d) \sigma^2= 0.1 & S_{1} is transmitted ','fontsize',10)
% Case 3: noise~N(0,1)
    noise=random('Normal',0,1,1,K);
    % Sub-case s = s_0:
    s=s_0;
    r=s+noise;  % received signal
    for n=1:K
        r_0(n)=sum(r(1:n).*s_0(1:n));
    end
    % Plotting the results:
    subplot(3,2,5)
    plot(l,[0 r_0])
    set(gca,'XTickLabel',{'0','5Tb','10Tb','15Tb','20Tb'})
    axis([0 20 -5 25])
    xlabel('(e) \sigma^2= 1 & S_{0} is transmitted ','fontsize',10)
    % Sub-case s = s_1:
    s=-s_0;
    r=s+noise;  % received signal
    for n=1:K
        r_0(n)=sum(r(1:n).*s_0(1:n));
    end
    % Plotting the results:
```

```
subplot(3,2,6)
plot(l,[0 r_0])
set(gca,'XTickLabel',{'0','5Tb','10Tb','15Tb','20Tb'})
axis([0 20 -25 5])
xlabel('(f) \sigma^2= 1 & S_{1} is transmitted ','fontsize',10)
```

Computer Problem 8.14

Figure 8.14 illustrates the correlator outputs for different noise variances when $s_1(t)$ is sent. The MATLAB script for the computation is given next

```
T=20;
s1=zeros(1,T);
s1(1:T/2)=1;
s2=zeros(1,T); >> s2(T/2+1:T)=1;
n1=sqrt(0.1)*randn(1,20);
n2=sqrt(0.5)*randn(1,20);
n3=randn(1,20);
r1=s1+n1;
r2=s1+n2;
r3=s1+n3;
for k=1:20
  y11(k)=0;y12(k)=0;y13(k)=0;
  y21(k)=0;y22(k)=0;y23(k)=0;
  for n=1:k
    y11(k)=y11(n)+0.05*r1(n)*s1(n);
    y12(k)=y12(n)+0.05*r2(n)*s1(n);
    y13(k)=y13(n)+0.05*r3(n)*s1(n);
    y21(k)=y21(n)+0.05*r1(n)*s2(n);
    y22(k)=y22(n)+0.05*r2(n)*s2(n);
    y23(k)=y23(n)+0.05*r3(n)*s2(n);
  end
end
subplot(3,2,1), stem(y11)
subplot(3,2,2), stem(y21)
subplot(3,2,3), stem(y12)
subplot(3,2,4), stem(y22)
subplot(3,2,5), stem(y13)
subplot(3,2,6), stem(y23)
```

A similar script gives the outputs when $s_2(t)$ is transmitted. The resulting plots are shown in Figure 8.15.

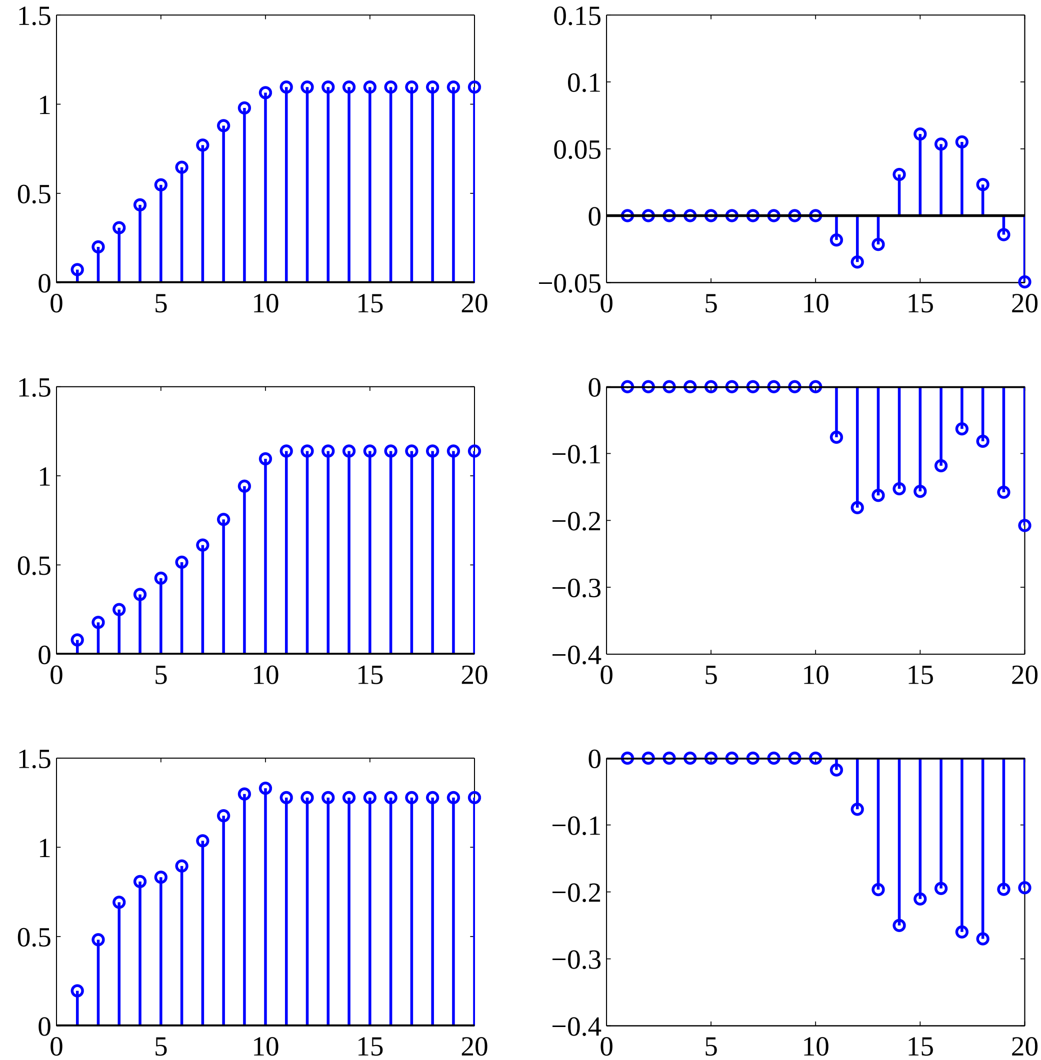


Figure 8.14: Correlator outputs in Computer Problem8.14 when $s_1(t)$ is transmitted. Left column is the result of correlation with $s_1(t)$ and right column is the output of correlator with $s_2(t)$. Rows one to three correspond to $\sigma^2 = 0.1, 0.5, 1$, respectively. Note that vertical scales in left and right columns are different.

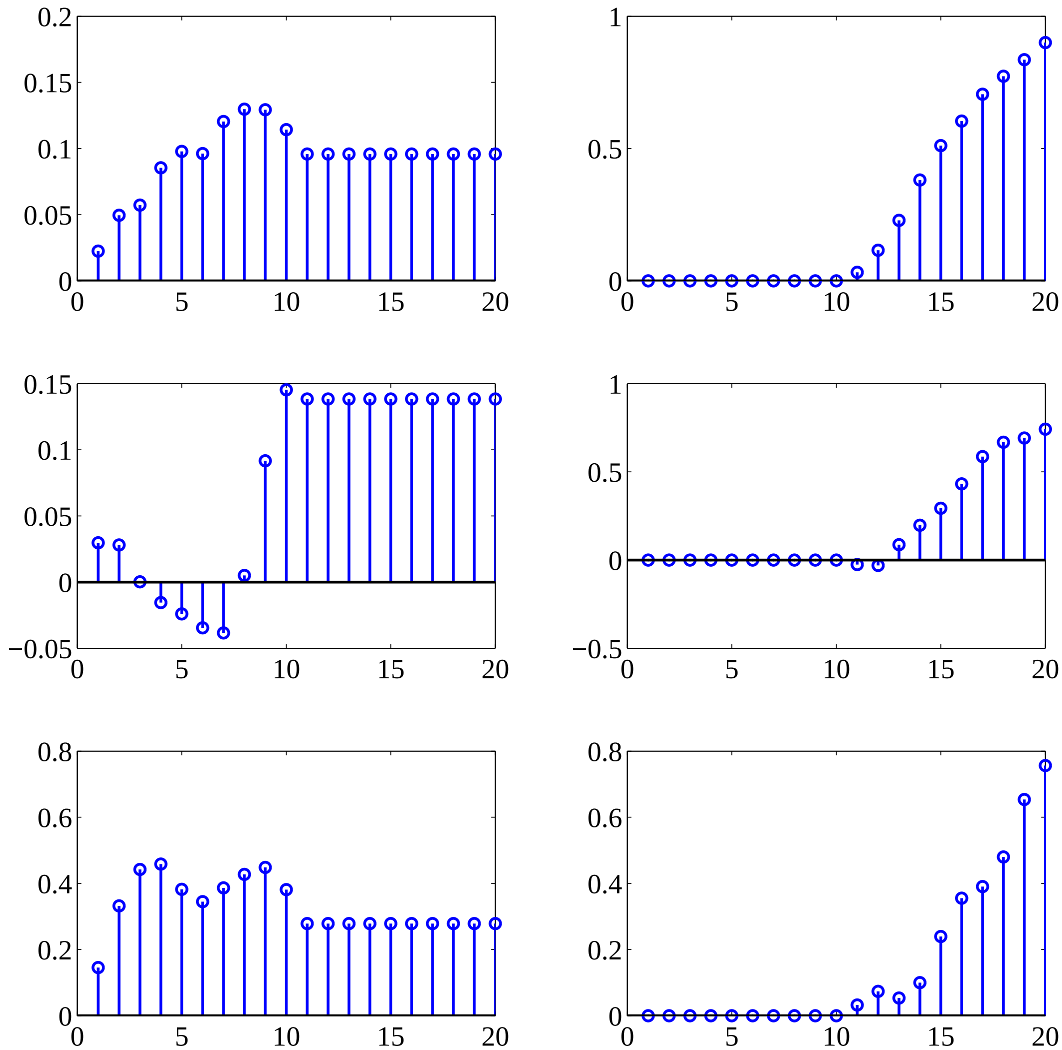


Figure 8.15: Correlator outputs in Computer Problem8.14 when $s_2(t)$ is transmitted. Left column is the result of correlation with $s_1(t)$ and right column is the output of correlator with $s_2(t)$. Rows one to three correspond to $\sigma^2 = 0.1, 0.5, 1$, respectively. Note that vertical scales in left and right columns are different.

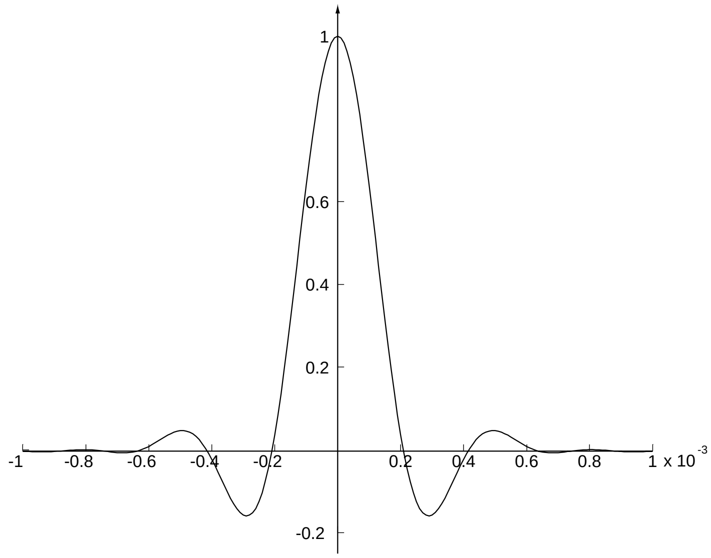


Figure 8.16: The raised-cosine signal

Computer Problem 8.15

The plot of $x(t)$ is given in Figure 8.16.

From Figure 8.16 it is clear that, for all practical purposes, it is sufficient to consider only the interval $|t| \leq 0.6 \times 10^{-3}$, which is roughly $[-3T, 3T]$. Truncating the raised-cosine pulse to this interval and computing the autocorrelation function result in the waveform shown in Figure 8.17.

In the MATLAB script given next, the raised-cosine signal and the autocorrelation function are first computed and plotted. In this particular example the length of the autocorrelation function is 1201 and the maximum (i.e., the optimum sampling time) occurs at the 600th component. Two cases are examined: one when the incorrect sampling time is 700 and one when it is 500. In both cases the early-late gate corrects the sampling time to the optimum 600.

```
% MATLAB script for Computer Problem 8.15
echo on
num=[0.01 1];
den=[1 1.01 1];
[a,b,c,d]=tf2ss(num,den);
dt=0.01;
u=ones(1,2000);
x=zeros(2,2001);
for i=1:2000
  x(:,i+1)=x(:,i)+dt.*a*x(:,i)+dt.*b*u(i);
  y(i)=c*x(:,i);
  echo off;
end
echo on;
t=[0:dt:20];
plot(t(1:2000),y)
```

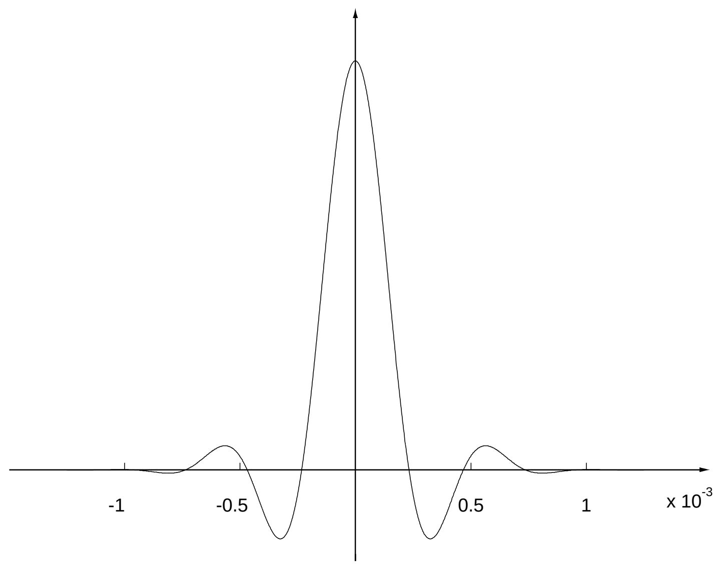


Figure 8.17: The autocorrelation function of the raised-cosine signal

Chapter 9

Problem 9.1

1) The first set represents a 4-PAM signal constellation. The points of the constellation are $\{\pm A, \pm 3A\}$. The second set consists of four orthogonal signals. The geometric representation of the signals is

$$\mathbf{s}_1 = [\ A \quad 0 \quad 0 \quad 0\ ] \qquad \mathbf{s}_3 = [\ 0 \quad 0 \quad A \quad 0\ ]$$

$$\mathbf{s}_2 = [\ 0 \quad A \quad 0 \quad 0\ ] \qquad \mathbf{s}_4 = [\ 0 \quad 0 \quad 0 \quad A\ ]$$

This set can be classified as a 4-FSK signal. The third set can be classified as a 4-QAM signal constellation. The geometric representation of the signals is

$$\mathbf{s}_1 = [\ \tfrac{A}{\sqrt{2}} \quad \tfrac{A}{\sqrt{2}}\ ] \qquad \mathbf{s}_3 = [\ -\tfrac{A}{\sqrt{2}} \quad -\tfrac{A}{\sqrt{2}}\ ]$$
$$\mathbf{s}_2 = [\ \tfrac{A}{\sqrt{2}} \quad -\tfrac{A}{\sqrt{2}}\ ] \qquad \mathbf{s}_4 = [\ -\tfrac{A}{\sqrt{2}} \quad \tfrac{A}{\sqrt{2}}\ ]$$

2) The average transmitted energy for sets I, II and III is

$$\begin{aligned}
\mathcal{E}_{av,I} &= \frac{1}{4}\sum_{i=1}^{4}\|\mathbf{s}_i\|^2 = \frac{1}{4}(A^2 + 9A^2 + 9A^2 + A^2) = 5A^2 \\
\mathcal{E}_{av,II} &= \frac{1}{4}\sum_{i=1}^{4}\|\mathbf{s}_i\|^2 = \frac{1}{4}(4A^2) = A^2 \\
\mathcal{E}_{av,III} &= \frac{1}{4}\sum_{i=1}^{4}\|\mathbf{s}_i\|^2 = \frac{1}{4}(4 \times (\frac{A^2}{2} + \frac{A^2}{2})) = A^2
\end{aligned}$$

3) The probability of error for the 4-PAM signal is given by

$$P_{4,I} = \frac{2(M-1)}{M} Q\left[\sqrt{\frac{6\mathcal{E}_{av,I}}{(M^2-1)N_0}}\right] = \frac{3}{2}Q\left[\sqrt{\frac{6 \times 5 \times A^2}{15N_0}}\right] = \frac{3}{2}Q\left[\sqrt{\frac{2A^2}{N_0}}\right]$$

4) When coherent detection is employed, then an upper bound on the probability of error is given by

$$P_{4,II,\text{coherent}} \le (M-1)Q\left[\sqrt{\frac{\mathcal{E}_s}{N_0}}\right] = 3Q\left[\sqrt{\frac{A^2}{N_0}}\right]$$

If the detection is performed noncoherently, then the probability of error is given by

$$\begin{aligned}
P_{4,II,\text{noncoherent}} &= \sum_{n=1}^{M-1}(-1)^{n+1}\begin{pmatrix} M-1 \\ n \end{pmatrix}\frac{1}{n+1}e^{-n\rho_s/(n=1)} \\
&= \frac{3}{2}e^{-\frac{\rho_s}{2}} - e^{-\frac{2\rho_s}{3}} + \frac{1}{4}e^{-\frac{3\rho_s}{4}} \\
&= \frac{3}{2}e^{-\frac{A^2}{2N_0}} - e^{-\frac{2A^2}{3N_0}} + \frac{1}{4}e^{-\frac{3A^2}{4N_0}}
\end{aligned}$$

5) It is not possible to use noncoherent detection for the signal set III. This is because all signals have the same square amplitude for every $t \in [0, 2T]$.

6) The following table shows the bit rate to bandwidth ratio for the different types of signaling and the results for $M = 4$.

Type	R/W	$M = 4$
PAM	$2\log_2 M$	4
QAM	$\log_2 M$	2
FSK (coherent)	$\frac{2\log_2 M}{M}$	1
FSK (noncoherent)	$\frac{\log_2 M}{M}$	0.5

To achieve a ratio $\frac{R}{W}$ of at least 2, we have to select either the first signal set (PAM) or the second signal set (QAM).

Problem 9.2

The correlation coefficient between the m^{th} and the n^{th} signal points is

$$\gamma_{mn} = \frac{\mathbf{s}_m \cdot \mathbf{s}_n}{|\mathbf{s}_m||\mathbf{s}_n|}$$

where $\mathbf{s}_m = (s_{m1}, s_{m2}, \ldots, s_{mN})$ and $s_{mj} = \pm\sqrt{\frac{\mathcal{E}_s}{N}}$. Two adjacent signal points differ in only one coordinate, for which s_{mk} and s_{nk} have opposite signs. Hence,

$$\begin{aligned}\mathbf{s}_m \cdot \mathbf{s}_n &= \sum_{j=1}^{N} s_{mj}s_{nj} = \sum_{j\neq k} s_{mj}s_{nj} + s_{mk}s_{nk} \\ &= (N-1)\frac{\mathcal{E}_s}{N} - \frac{\mathcal{E}_s}{N} = \frac{N-2}{N}\mathcal{E}_s\end{aligned}$$

Furthermore, $|\mathbf{s}_m| = |\mathbf{s}_n| = (\mathcal{E}_s)^{\frac{1}{2}}$ so that

$$\gamma_{mn} = \frac{N-2}{N}$$

The Euclidean distance between the two adjacent signal points is

$$d = \sqrt{|\mathbf{s}_m - \mathbf{s}_n|^2} = \sqrt{\left|\pm 2\sqrt{\mathcal{E}_s/N}\right|^2} = \sqrt{4\frac{\mathcal{E}_s}{N}} = 2\sqrt{\frac{\mathcal{E}_s}{N}}$$

Problem 9.3

The energy of the signal waveform $s'_m(t)$ is

$$\begin{aligned}
\mathcal{E}' &= \int_{-\infty}^{\infty} |s'_m(t)|^2 \, dt = \int_{-\infty}^{\infty} \left| s_m(t) - \frac{1}{M}\sum_{k=1}^{M} s_k(t) \right|^2 dt \\
&= \int_{-\infty}^{\infty} s_m^2(t)dt + \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_k(t)s_l(t)dt \\
&\quad -\frac{1}{M}\sum_{k=1}^{M}\int_{-\infty}^{\infty} s_m(t)s_k(t)dt - \frac{1}{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty} s_m(t)s_l(t)dt \\
&= \mathcal{E} + \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}\mathcal{E}\delta_{kl} - \frac{2}{M}\mathcal{E} \\
&= \mathcal{E} + \frac{1}{M}\mathcal{E} - \frac{2}{M}\mathcal{E} = \left(\frac{M-1}{M}\right)\mathcal{E}
\end{aligned}$$

The correlation coefficient is given by

$$\begin{aligned}
\gamma_{mn} &= \frac{\int_{-\infty}^{\infty} s'_m(t)s'_n(t)dt}{\left[\int_{-\infty}^{\infty}|s'_m(t)|^2 dt\right]^{\frac{1}{2}}\left[\int_{-\infty}^{\infty}|s'_n(t)|^2 dt\right]^{\frac{1}{2}}} \\
&= \frac{1}{\mathcal{E}'}\int_{-\infty}^{\infty}\left(s_m(t) - \frac{1}{M}\sum_{k=1}^{M}s_k(t)\right)\left(s_n(t) - \frac{1}{M}\sum_{l=1}^{M}s_l(t)\right)dt \\
&= \frac{1}{\mathcal{E}'}\left(\int_{-\infty}^{\infty}s_m(t)s_n(t)dt + \frac{1}{M^2}\sum_{k=1}^{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty}s_k(t)s_l(t)dt\right) \\
&\quad -\frac{1}{\mathcal{E}'}\left(\frac{1}{M}\sum_{k=1}^{M}\int_{-\infty}^{\infty}s_n(t)s_k(t)dt + \frac{1}{M}\sum_{l=1}^{M}\int_{-\infty}^{\infty}s_m(t)s_l(t)dt\right) \\
&= \frac{\frac{1}{M^2}M\mathcal{E} - \frac{1}{M}\mathcal{E} - \frac{1}{M}\mathcal{E}}{\frac{M-1}{M}\mathcal{E}} = -\frac{1}{M-1}
\end{aligned}$$

Problem 9.4

The biorthogonal signal set has the form

$$\begin{aligned}
\mathbf{s}_1 &= [\sqrt{\mathcal{E}_s}, 0, 0, 0] & \mathbf{s}_5 &= [-\sqrt{\mathcal{E}_s}, 0, 0, 0] \\
\mathbf{s}_2 &= [0, \sqrt{\mathcal{E}_s}, 0, 0] & \mathbf{s}_6 &= [0, -\sqrt{\mathcal{E}_s}, 0, 0] \\
\mathbf{s}_3 &= [0, 0, \sqrt{\mathcal{E}_s}, 0] & \mathbf{s}_7 &= [0, 0, -\sqrt{\mathcal{E}_s}, 0] \\
\mathbf{s}_4 &= [0, 0, 0, \sqrt{\mathcal{E}_s}] & \mathbf{s}_8 &= [0, 0, 0, -\sqrt{\mathcal{E}_s}]
\end{aligned}$$

For each point $\mathbf{s}_i$, there are $M - 2 = 6$ points at a distance

$$d_{i,k} = |\mathbf{s}_i - \mathbf{s}_k| = \sqrt{2\mathcal{E}_s}$$

and one vector $(-\mathbf{s}_i)$ at a distance $d_{i,m} = 2\sqrt{\mathcal{E}_s}$. Hence, the union bound on the probability of error $P(e|\mathbf{s}_i)$ is given by

$$P_{\mathrm{UB}}(e|\mathbf{s}_i) = \sum_{k=1,k\neq i}^{M} Q\left[\frac{d_{i,k}}{\sqrt{2N_0}}\right] = 6Q\left[\sqrt{\frac{\mathcal{E}_s}{N_0}}\right] + Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]$$

Since all the signals are equiprobable, we find that

$$P_{\mathrm{UB}}(e) = 6Q\left[\sqrt{\frac{\mathcal{E}_s}{N_0}}\right] + Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]$$

With $M = 8 = 2^3$, $\mathcal{E}_s = 3\mathcal{E}_b$ and therefore,

$$P_{\mathrm{UB}}(e) = 6Q\left[\sqrt{\frac{3\mathcal{E}_b}{N_0}}\right] + Q\left[\sqrt{\frac{6\mathcal{E}_b}{N_0}}\right]$$

Problem 9.5

It is convenient to find first the probability of a correct decision. Since all signals are equiprobable

$$P(C) = \sum_{i=1}^{M} \frac{1}{M} P(C|\mathbf{s}_i)$$

All the $P(C|\mathbf{s}_i)$, $i = 1, \ldots, M$ are identical because of the symmetry of the constellation. By translating the vector $\mathbf{s}_i$ to the origin we can find the probability of a correct decision, given that $\mathbf{s}_i$ was transmitted, as

$$P(C|\mathbf{s}_i) = \int_{-\frac{d}{2}}^{\infty} f(n_1)dn_1 \int_{-\frac{d}{2}}^{\infty} f(n_2)dn_2 \ldots \int_{-\frac{d}{2}}^{\infty} f(n_N)dn_N$$

where the number of the integrals on the right side of the equation is N, d is the minimum distance between the points and

$$f(n_i) = \frac{1}{\pi N_0} e^{-\frac{n_i^2}{N_0}}$$

Hence,

$$\begin{aligned} P(C|\mathbf{s}_i) &= \left(\int_{-\frac{d}{2}}^{\infty} f(n)dn\right)^N = \left(1 - \int_{-\infty}^{-\frac{d}{2}} f(n)dn\right)^N \\ &= \left(1 - Q\left[\frac{d}{\sqrt{2N_0}}\right]\right)^N \end{aligned}$$

and therefore, the probability of error is given by

$$\begin{aligned} P(e) &= 1 - P(C) = 1 - \sum_{i=1}^{M} \frac{1}{M}\left(1 - Q\left[\frac{d}{\sqrt{2N_0}}\right]\right)^N \\ &= 1 - \left(1 - Q\left[\frac{d}{\sqrt{2N_0}}\right]\right)^N \end{aligned}$$

Note that since

$$\mathcal{E}_s = \sum_{i=1}^{N} s_{m,i}^2 = \sum_{i=1}^{N} (\frac{d}{2})^2 = N\frac{d^2}{4}$$

the probability of error can be written as

$$P(e) = 1 - \left(1 - Q\left[\sqrt{\frac{2\mathcal{E}_s}{NN_0}}\right]\right)^N$$

Problem 9.6

Consider first the signal

$$y(t) = \sum_{k=1}^{n} c_k \delta(t - kT_c)$$

The signal $y(t)$ has duration $T = nT_c$ and its matched filter is

$$\begin{aligned} g(t) &= y(T-t) = y(nT_c - t) = \sum_{k=1}^{n} c_k \delta(nT_c - kT_c - t) \\ &= \sum_{i=1}^{n} c_{n-i+1}\delta((i-1)T_c - t) = \sum_{i=1}^{n} c_{n-i+1}\delta(t - (i-1)T_c) \end{aligned}$$

that is, a sequence of impulses starting at $t = 0$ and weighted by the mirror image sequence of $\{c_i\}$. Since,

$$s(t) = \sum_{k=1}^{n} c_k p(t - kT_c) = p(t) \star \sum_{k=1}^{n} c_k \delta(t - kT_c)$$

the Fourier transform of the signal $s(t)$ is

$$S(f) = P(f) \sum_{k=1}^{n} c_k e^{-j2\pi f k T_c}$$

and therefore, the Fourier transform of the signal matched to $s(t)$ is

$$\begin{aligned} H(f) &= S^*(f)e^{-j2\pi fT} = S^*(f)e^{-j2\pi f nT_c} \\ &= P^*(f)\sum_{k=1}^{n} c_k e^{j2\pi f kT_c} e^{-j2\pi f nT_c} \\ &= P^*(f)\sum_{i=1}^{n} c_{n-i+1} e^{-j2\pi f(i-1)T-c} \\ &= P^*(f)\mathcal{F}[g(t)] \end{aligned}$$

Thus, the matched filter $H(f)$ can be considered as the cascade of a filter,with impulse response $p(-t)$, matched to the pulse $p(t)$ and a filter, with impulse response $g(t)$, matched to the signal

$y(t) = \sum_{k=1}^{n} c_k \delta(t - kT_c)$. The output of the matched filter at $t = nT_c$ is

$$\begin{aligned}\int_{-\infty}^{\infty} |s(t)|^2 &= \sum_{k=1}^{n} c_k^2 \int_{-\infty}^{\infty} p^2(t - kT_c)dt \\ &= T_c \sum_{k=1}^{n} c_k^2\end{aligned}$$

where we have used the fact that $p(t)$ is a rectangular pulse of unit amplitude and duration T_c.

Problem 9.7

1) The inner product of $s_i(t)$ and $s_j(t)$ is

$$\begin{aligned}\int_{-\infty}^{\infty} s_i(t)s_j(t)dt &= \int_{-\infty}^{\infty} \sum_{k=1}^{n} c_{ik}p(t - kT_c) \sum_{l=1}^{n} c_{jl}p(t - lT_c)dt \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} c_{ik}c_{jl} \int_{-\infty}^{\infty} p(t - kT_c)p(t - lT_c)dt \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} c_{ik}c_{jl}\mathcal{E}_p\delta_{kl} \\ &= \mathcal{E}_p \sum_{k=1}^{n} c_{ik}c_{jk}\end{aligned}$$

The quantity $\sum_{k=1}^{n} c_{ik}c_{jk}$ is the inner product of the row vectors $\underline{C}_i$ and $\underline{C}_j$. Since the rows of the matrix H_n are orthogonal by construction, we obtain

$$\int_{-\infty}^{\infty} s_i(t)s_j(t)dt = \mathcal{E}_p \sum_{k=1}^{n} c_{ik}^2\delta_{ij} = n\mathcal{E}_p\delta_{ij}$$

Thus, the waveforms $s_i(t)$ and $s_j(t)$ are orthogonal.

2) Using the results of Problem 8.30, we obtain that the filter matched to the waveform

$$s_i(t) = \sum_{k=1}^{n} c_{ik}p(t - kT_c)$$

can be realized as the cascade of a filter matched to $p(t)$ followed by a discrete-time filter matched to the vector $\underline{C}_i = [c_{i1}, \ldots, c_{in}]$. Since the pulse $p(t)$ is common to all the signal waveforms $s_i(t)$, we conclude that the n matched filters can be realized by a filter matched to $p(t)$ followed by n discrete-time filters matched to the vectors $\underline{C}_i$, $i = 1, \ldots, n$.

Problem 9.8

1) The optimal ML detector selects the sequence $\underline{C}_i$ that minimizes the quantity

$$D(\mathbf{r}, \underline{C}_i) = \sum_{k=1}^{n} (r_k - \sqrt{\mathcal{E}_b} C_{ik})^2$$

The metrics of the two possible transmitted sequences are

$$D(\mathbf{r}, \underline{C}_1) = \sum_{k=1}^{w} (r_k - \sqrt{\mathcal{E}_b})^2 + \sum_{k=w+1}^{n} (r_k - \sqrt{\mathcal{E}_b})^2$$

and

$$D(\mathbf{r}, \underline{C}_2) = \sum_{k=1}^{w} (r_k - \sqrt{\mathcal{E}_b})^2 + \sum_{k=w+1}^{n} (r_k + \sqrt{\mathcal{E}_b})^2$$

Since the first term of the right side is common for the two equations, we conclude that the optimal ML detector can base its decisions only on the last $n - w$ received elements of $\mathbf{r}$. That is

$$\sum_{k=w+1}^{n} (r_k - \sqrt{\mathcal{E}_b})^2 - \sum_{k=w+1}^{n} (r_k + \sqrt{\mathcal{E}_b})^2 \underset{\underline{C}_1}{\overset{\underline{C}_2}{\gtrless}} 0$$

or equivalently

$$\sum_{k=w+1}^{n} r_k \underset{\underline{C}_2}{\overset{\underline{C}_1}{\gtrless}} 0$$

2) Since $r_k = \sqrt{\mathcal{E}_b} C_{ik} + n_k$, the probability of error $P(e|\underline{C}_1)$ is

$$\begin{aligned} P(e|\underline{C}_1) &= P\left(\sqrt{\mathcal{E}_b}(n-w) + \sum_{k=w+1}^{n} n_k < 0\right) \\ &= P\left(\sum_{k=w+1}^{n} n_k < -(n-w)\sqrt{\mathcal{E}_b}\right) \end{aligned}$$

The random variable $u = \sum_{k=w+1}^{n} n_k$ is zero-mean Gaussian with variance $\sigma_u^2 = (n-w)\sigma^2$. Hence

$$P(e|\underline{C}_1) = \frac{1}{\sqrt{2\pi(n-w)\sigma^2}} \int_{-\infty}^{-\sqrt{\mathcal{E}_b}(n-w)} \exp\left(-\frac{x^2}{2\pi(n-w)\sigma^2}\right) dx = Q\left[\sqrt{\frac{\mathcal{E}_b(n-w)}{\sigma^2}}\right]$$

Similarly we find that $P(e|\underline{C}_2) = P(e|\underline{C}_1)$ and since the two sequences are equiprobable

$$P(e) = Q\left[\sqrt{\frac{\mathcal{E}_b(n-w)}{\sigma^2}}\right]$$

3) The probability of error $P(e)$ is minimized when $\frac{\mathcal{E}_b(n-w)}{\sigma^2}$ is maximized, that is for $w = 0$. This implies that $\underline{C}_1 = -\underline{C}_2$ and thus the distance between the two sequences is the maximum possible.

Problem 9.9

Consider the following waveforms of the binary FSK signaling:

$$\begin{aligned} u_1(t) &= \sqrt{\frac{2\mathcal{E}_b}{T}}\cos(2\pi f_c t) \\ u_2(t) &= \sqrt{\frac{2\mathcal{E}_b}{T}}\cos(2\pi f_c t + 2\pi\Delta f t) \end{aligned}$$

The correlation of the two signals is

$$\begin{aligned} \gamma_{12} &= \frac{1}{\mathcal{E}_b}\int_0^T u_1(t)u_2(t)dt \\ &= \frac{1}{\mathcal{E}_b}\int_0^T \frac{2\mathcal{E}_b}{T}\cos(2\pi f_c t)\cos(2\pi f_c t + 2\pi\Delta f t)dt \\ &= \frac{1}{T}\int_0^T \cos(2\pi\Delta f t)dt + \frac{1}{T}\int_0^T \cos(2\pi 2 f_c t + 2\pi\Delta f t)dt \end{aligned}$$

If $f_c \gg \frac{1}{T}$, then

$$\gamma_{12} = \frac{1}{T}\int_0^T \cos(2\pi\Delta f t)dt = \frac{\sin(2\pi\Delta f T)}{2\pi\Delta f T}$$

To find the minimum value of the correlation, we set the derivative of γ_{12} with respect to Δf equal to zero. Thus,

$$\frac{\vartheta\gamma_{12}}{\vartheta\Delta f} = 0 = \frac{\cos(2\pi\Delta f T)2\pi T}{2\pi\Delta f T} - \frac{\sin(2\pi\Delta f T)}{(2\pi\Delta f T)^2}2\pi T$$

and therefore,

$$2\pi\Delta f T = \tan(2\pi\Delta f T)$$

Solving numerically the equation $x = \tan(x)$, we obtain $x = 4.4934$. Thus,

$$2\pi\Delta f T = 4.4934 \Longrightarrow \Delta f = \frac{0.7151}{T}$$

and the value of γ_{12} is -0.2172. Note that when a gradient method like the Gauss-Newton is used to solve the equation $f(x) = x - \tan(x) = 0$, then in order to find the smallest nonzero root, the initial value of the algorithm x_0 should be selected in the range $(\frac{\pi}{2}, \frac{3\pi}{2})$.

The probability of error can be expressed in terms of the distance d_{12} between the signal points, as

$$p_b = Q\left[\sqrt{\frac{d_{12}^2}{2N_0}}\right]$$

The two signal vectors $\mathbf{u}_1$, $\mathbf{u}_2$ are of equal energy

$$\|\mathbf{u}_1\|^2 = \|\mathbf{u}_2\|^2 = \mathcal{E}_b$$

and the angle θ_{12} between them is such that

$$\cos(\theta_{12}) = \gamma_{12}$$

Hence,

$$d_{12}^2 = \|\mathbf{u}_1\|^2 + \|\mathbf{u}_2\|^2 - 2\|\mathbf{u}_1\|\|\mathbf{u}_2\|\cos(\theta_{12}) = 2\mathcal{E}_s(1-\gamma_{12})$$

and therefore,

$$p_b = Q\left[\sqrt{\frac{2\mathcal{E}_s(1-\gamma_{12})}{2N_0}}\right] = Q\left[\sqrt{\frac{\mathcal{E}_s(1+0.2172)}{N_0}}\right]$$

Problem 9.10

1) If the transmitted signal is

$$u_0(t) = \sqrt{\frac{2\mathcal{E}_s}{T}}\cos(2\pi f_c t), \qquad 0 \le t \le T$$

then the received signal is

$$r(t) = \sqrt{\frac{2\mathcal{E}_s}{T}}\cos(2\pi f_c t + \phi) + n(t)$$

In the phase-coherent demodulation of M-ary FSK signals, the received signal is correlated with each of the M-possible received signals $\cos(2\pi f_c t + 2\pi m\Delta f t + \hat{\phi}_m)$, where $\hat{\phi}_m$ are the carrier phase estimates. The output of the m^{th} correlator is

$$\begin{aligned}
r_m &= \int_0^T r(t)\cos(2\pi f_c t + 2\pi m\Delta f t + \hat{\phi}_m)dt \\
&= \int_0^T \sqrt{\frac{2\mathcal{E}_s}{T}}\cos(2\pi f_c t + \phi)\cos(2\pi f_c t + 2\pi m\Delta f t + \hat{\phi}_m)dt \\
&\quad + \int_0^T n(t)\cos(2\pi f_c t + 2\pi m\Delta f t + \hat{\phi}_m)dt \\
&= \sqrt{\frac{2\mathcal{E}_s}{T}}\int_0^T \frac{1}{2}\Big(\cos(2\pi 2 f_c t + 2\pi m\Delta f t + \hat{\phi}_m + \phi) + \cos(2\pi m\Delta f t + \hat{\phi}_m - \phi)\Big) + n \\
&= \sqrt{\frac{2\mathcal{E}_s}{T}}\frac{1}{2}\int_0^T \cos(2\pi m\Delta f t + \hat{\phi}_m - \phi)dt + n
\end{aligned}$$

where n is a zero-mean Gaussian random variable with variance $\frac{N_0}{2}$.

2) In order to obtain orthogonal signals at the demodulator, the expected value of r_m, $E[r_m]$, should be equal to zero for every $m \neq 0$. Since $E[n] = 0$, the latter implies that

$$\int_0^T \cos(2\pi m\Delta f t + \hat{\phi}_m - \phi)dt = 0, \qquad \forall m \neq 0$$

The equality is true when $m\Delta f$ is a multiple of $\frac{1}{T}$. Since the smallest value of m is 1, the necessary condition for orthogonality is

$$\Delta f = \frac{1}{T}$$

Problem 9.11

The noise components in the sampled output of the two correlators for the m^{th} FSK signal, are given by

$$\begin{aligned}
n_{mc} &= \int_0^T n(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t + 2\pi m\Delta f t)dt \\
n_{ms} &= \int_0^T n(t)\sqrt{\frac{2}{T}}\sin(2\pi f_c t + 2\pi m\Delta f t)dt
\end{aligned}$$

Clearly, n_{mc}, n_{ms} are zero-mean random variables since

$$\begin{aligned}
E[n_{mc}] &= E\left[\int_0^T n(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t + 2\pi m\Delta f t)dt\right] \\
&= \int_0^T E[n(t)]\sqrt{\frac{2}{T}}\cos(2\pi f_c t + 2\pi m\Delta f t)dt = 0 \\
E[n_{ms}] &= E\left[\int_0^T n(t)\sqrt{\frac{2}{T}}\sin(2\pi f_c t + 2\pi m\Delta f t)dt\right] \\
&= \int_0^T E[n(t)]\sqrt{\frac{2}{T}}\sin(2\pi f_c t + 2\pi m\Delta f t)dt = 0
\end{aligned}$$

Furthermore,

$$\begin{aligned}
E[n_{mc}n_{kc}] &= E\left[\int_0^T\int_0^T \frac{2}{T} n(t)n(\tau)\cos(2\pi f_c t + 2\pi m\Delta f t)\cos(2\pi f_c t + 2\pi k\Delta f \tau)dtd\tau\right] \\
&= \frac{2}{T}\int_0^T\int_0^T E[n(t)n(\tau)]\cos(2\pi f_c t + 2\pi m\Delta f t)\cos(2\pi f_c t + 2\pi k\Delta f \tau)dtd\tau \\
&= \frac{2}{T}\frac{N_0}{2}\int_0^T \cos(2\pi f_c t + 2\pi m\Delta f t)\cos(2\pi f_c t + 2\pi k\Delta f t)dt \\
&= \frac{2}{T}\frac{N_0}{2}\int_0^T \frac{1}{2}\left(\cos(2\pi 2 f_c t + 2\pi(m+k)\Delta f t) + \cos(2\pi(m-k)\Delta f t)\right)dt \\
&= \frac{2}{T}\frac{N_0}{2}\int_0^T \frac{1}{2}\delta_{mk}dt = \frac{N_0}{2}\delta_{mk}
\end{aligned}$$

where we have used the fact that for $f_c \gg \frac{1}{T}$

$$\int_0^T \cos(2\pi 2 f_c t + 2\pi(m+k)\Delta f t)dt \approx 0$$

and for $\Delta f = \frac{1}{T}$

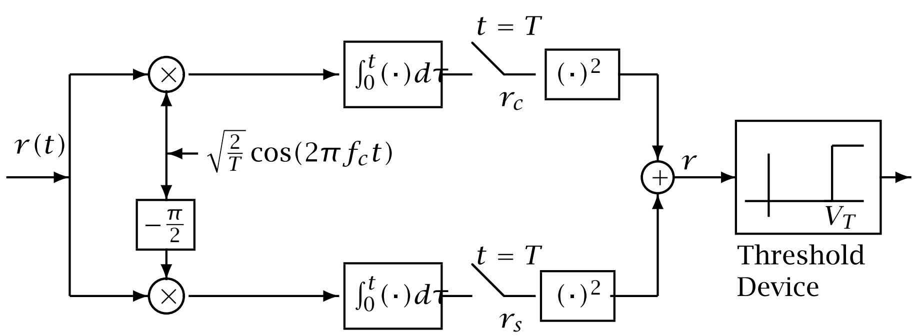
$$\int_0^T \cos(2\pi(m-k)\Delta f t)dt = 0, \qquad m \neq k$$

Thus, n_{mc}, n_{kc} are uncorrelated for $m \neq k$ and since they are zero-mean Gaussian they are independent. Similarly we obtain

$$\begin{aligned}
E[n_{mc}n_{ks}] &= E\left[\int_0^T \int_0^T \frac{2}{T} n(t)n(\tau)\cos(2\pi f_c t + 2\pi m\Delta f t)\sin(2\pi f_c t + 2\pi k \Delta f \tau)dt d\tau\right] \\
&= \frac{2}{T}\int_0^T \int_0^T E[n(t)n(\tau)]\cos(2\pi f_c t + 2\pi m\Delta f t)\sin(2\pi f_c t + 2\pi k \Delta f \tau)dt d\tau \\
&= \frac{2}{T}\frac{N_0}{2}\int_0^T \cos(2\pi f_c t + 2\pi m\Delta f t)\sin(2\pi f_c t + 2\pi k\Delta f t)dt \\
&= \frac{2}{T}\frac{N_0}{2}\int_0^T \frac{1}{2}\left(\sin(2\pi 2 f_c t + 2\pi(m+k)\Delta f t) - \sin(2\pi(m-k)\Delta f t)\right)dt \\
&= 0 \\
E[n_{ms}n_{ks}] &= \frac{N_0}{2}\delta_{mk}
\end{aligned}$$

Problem 9.12

1) The noncoherent envelope detector for the on-off keying signal is depicted in the next figure.



2) If $s_0(t)$ is sent, then the received signal is $r(t) = n(t)$ and therefore the sampled outputs r_c, r_s are zero-mean independent Gaussian random variables with variance $\frac{N_0}{2}$. Hence, the random variable $r = \sqrt{r_c^2 + r_s^2}$ is Rayleigh distributed and the PDF is given by

$$p(r|s_0(t)) = \frac{r}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}} = \frac{2r}{N_0}e^{-\frac{r^2}{N_0}}$$

If $s_1(t)$ is transmitted, then the received signal is

$$r(t) = \sqrt{\frac{2\mathcal{E}_b}{T_b}}\cos(2\pi f_c t + \phi) + n(t)$$

Crosscorrelating $r(t)$ by $\sqrt{\frac{2}{T}}\cos(2\pi f_c t)$ and sampling the output at $t = T$, results in

$$\begin{aligned} r_c &= \int_0^T r(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t)dt \\ &= \int_0^T \frac{2\sqrt{\mathcal{E}_b}}{T_b}\cos(2\pi f_c t+\phi)\cos(2\pi f_c t)dt + \int_0^T n(t)\sqrt{\frac{2}{T}}\cos(2\pi f_c t)dt \\ &= \frac{2\sqrt{\mathcal{E}_b}}{T_b}\int_0^T \frac{1}{2}\left(\cos(2\pi 2 f_c t+\phi)+\cos(\phi)\right)dt + n_c \\ &= \sqrt{\mathcal{E}_b}\cos(\phi) + n_c \end{aligned}$$

where n_c is zero-mean Gaussian random variable with variance $\frac{N_0}{2}$. Similarly, for the quadrature component we have

$$r_s = \sqrt{\mathcal{E}_b}\sin(\phi) + n_s$$

The PDF of the random variable $r = \sqrt{r_c^2 + r_s^2} = \sqrt{\mathcal{E}_b + n_c^2 + n_s^2}$ is (see Problem 4.31)

$$p(r|s_1(t)) = \frac{r}{\sigma^2}e^{-\frac{r^2+\mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right) = \frac{2r}{N_0}e^{-\frac{r^2+\mathcal{E}_b}{N_0}} I_0\left(\frac{2r\sqrt{\mathcal{E}_b}}{N_0}\right)$$

that is a Rician PDF.

3) For equiprobable signals the probability of error is given by

$$P(\text{error}) = \frac{1}{2}\int_{-\infty}^{V_T} p(r|s_1(t))dr + \frac{1}{2}\int_{V_T}^{\infty} p(r|s_0(t))dr$$

Since $r > 0$ the expression for the probability of error takes the form

$$\begin{aligned} P(\text{error}) &= \frac{1}{2}\int_0^{V_T} p(r|s_1(t))dr + \frac{1}{2}\int_{V_T}^{\infty} p(r|s_0(t))dr \\ &= \frac{1}{2}\int_0^{V_T} \frac{r}{\sigma^2}e^{-\frac{r^2+\mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right)dr + \frac{1}{2}\int_{V_T}^{\infty}\frac{r}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}}dr \end{aligned}$$

The optimum threshold level is the value of V_T that minimizes the probability of error. However, when $\frac{\mathcal{E}_b}{N_0} \gg 1$ the optimum value is close to $\frac{\sqrt{\mathcal{E}_b}}{2}$ and we will use this threshold to simplify the analysis. The integral involving the Bessel function cannot be evaluated in closed form. Instead of $I_0(x)$ we will use the approximation

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}$$

which is valid for large x, that is for high SNR. In this case

$$\frac{1}{2}\int_0^{V_T} \frac{r}{\sigma^2}e^{-\frac{r^2+\mathcal{E}_b}{2\sigma^2}} I_0\left(\frac{r\sqrt{\mathcal{E}_b}}{\sigma^2}\right)dr \approx \frac{1}{2}\int_0^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}}e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2}dr$$

This integral is further simplified if we observe that for high SNR, the integrand is dominant in the vicinity of $\sqrt{\mathcal{E}_b}$ and therefore, the lower limit can be substituted by $-\infty$. Also

$$\sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} \approx \sqrt{\frac{1}{2\pi\sigma^2}}$$

and therefore,

$$\begin{aligned}\frac{1}{2}\int_0^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{r}{2\pi\sigma^2\sqrt{\mathcal{E}_b}}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr &\approx \frac{1}{2}\int_{-\infty}^{\frac{\sqrt{\mathcal{E}_b}}{2}} \sqrt{\frac{1}{2\pi\sigma^2}} e^{-(r-\sqrt{\mathcal{E}_b})^2/2\sigma^2} dr \\ &= \frac{1}{2} Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right]\end{aligned}$$

Finally

$$\begin{aligned}P(\text{error}) &= \frac{1}{2} Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] + \frac{1}{2}\int_{\frac{\sqrt{\mathcal{E}_b}}{2}}^{\infty} \frac{2r}{N_0} e^{-\frac{r^2}{N_0}} dr \\ &\leq \frac{1}{2} Q\left[\sqrt{\frac{\mathcal{E}_b}{2N_0}}\right] + \frac{1}{2} e^{-\frac{\mathcal{E}_b}{4N_0}}\end{aligned}$$

Problem 9.13

(a) Four phase PSK

If we use a pulse shape having a raised cosine spectrum with a rolloff α, the symbol rate is determined from the relation

$$\frac{1}{2T}(1+\alpha) = 50000$$

Hence,

$$\frac{1}{T} = \frac{10^5}{1+\alpha}$$

where $W = 10^5$ Hz is the channel bandwidth. The bit rate is

$$\frac{2}{T} = \frac{2\times 10^5}{1+\alpha} \quad \text{bps}$$

(b) Binary FSK with noncoherent detection

In this case we select the two frequencies to have a frequency separation of $\frac{1}{T}$, where $\frac{1}{T}$ is the symbol rate. Hence

$$f_1 = f_c - \frac{1}{2T}$$

$$f_2 = f + c + \frac{1}{2T}$$

where f_c is the carrier in the center of the channel band. Thus, we have

$$\frac{1}{2T} = 50000$$

or equivalently

$$\frac{1}{T} = 10^5$$

Hence, the bit rate is 10^5 bps.

(c) $M = 4$ FSK with noncoherent detection

In this case we require four frequencies with adjacent frequencies separation of $\frac{1}{T}$. Hence, we select

$$f_1 = f_c - |\frac{1.5}{T}, \quad f_2 = f_c - \frac{1}{2T}, \quad f_3 = f_c + \frac{1}{2T}, \quad f_4 = f_c + \frac{1.5}{T}$$

where f_c is the carrier frequency and $\frac{1}{2T} = 25000$, or, equivalently,

$$\frac{1}{T} = 50000$$

Since the symbol rate is 50000 symbols per second and each symbol conveys 2 bits, the bit rate is 10^5 bps.

Problem 9.14

We assume that the input bits 0, 1 are mapped to the symbols -1 and 1 respectively. The terminal phase of an MSK signal at time instant n is given by

$$\theta(n;\mathbf{a}) = \frac{\pi}{2}\sum_{k=0}^{k} a_k + \theta_0$$

where θ_0 is the initial phase and a_k is ± 1 depending on the input bit at the time instant k. The following table shows $\theta(n;\mathbf{a})$ for two different values of θ_0 $(0,\pi)$, and the four input pairs of data: $\{00, 01, 10, 11\}$.

θ_0	b_0	b_1	a_0	a_1	$\theta(n;\mathbf{a})$
0	0	0	-1	-1	$-\pi$
0	0	1	-1	1	0
0	1	0	1	-1	0
0	1	1	1	1	π
π	0	0	-1	-1	0
π	0	1	-1	1	π
π	1	0	1	-1	π
π	1	1	1	1	2π

Problem 9.15

1) The envelope of the signal is

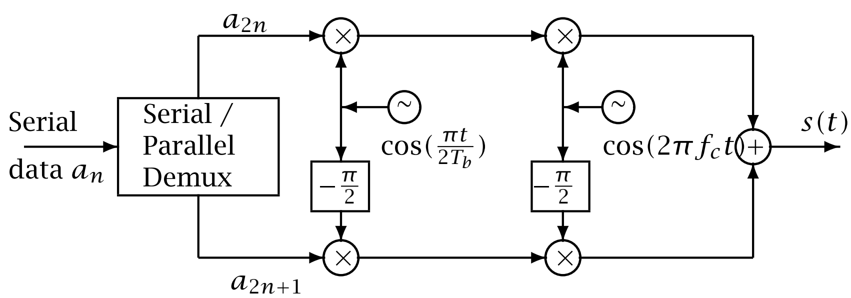
$$\begin{aligned} |s(t)| &= \sqrt{|s_c(t)|^2 + |s_s(t)|^2} \\ &= \sqrt{\frac{2\mathcal{E}_b}{T_b}\cos^2\left(\frac{\pi t}{2T_b}\right) + \frac{2\mathcal{E}_b}{T_b}\sin^2\left(\frac{\pi t}{2T_b}\right)} \\ &= \sqrt{\frac{2\mathcal{E}_b}{T_b}} \end{aligned}$$

Thus, the signal has constant amplitude.

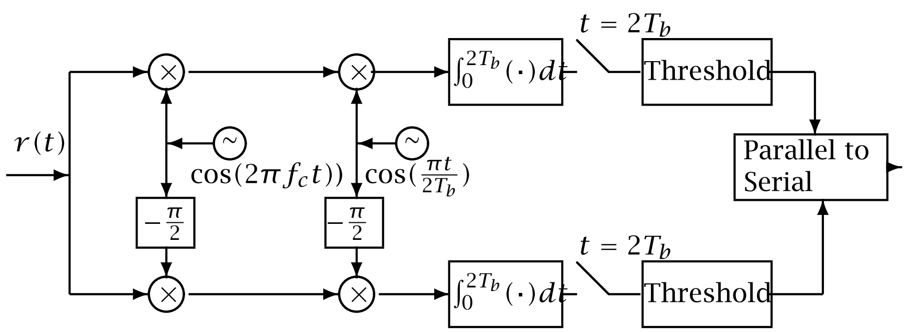
2) The signal $s(t)$ has the form of the four-phase PSK signal with

$$g_T(t) = \cos\left(\frac{\pi t}{2T_b}\right), \qquad 0 \le t \le 2T_b$$

Hence, it is an MSK signal. A block diagram of the modulator for synthesizing the signal is given in the next figure.



3) A sketch of the demodulator is shown in the next figure.



Problem 9.16

Since $p = 2$, m is odd ($m = 1$) and $M = 2$, there are

$$N_s = 2pM = 8$$

phase states, which we denote as $S_n = (\theta_n, a_{n-1})$. The $2p = 4$ phase states corresponding to θ_n are

$$\Theta_s = \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$$

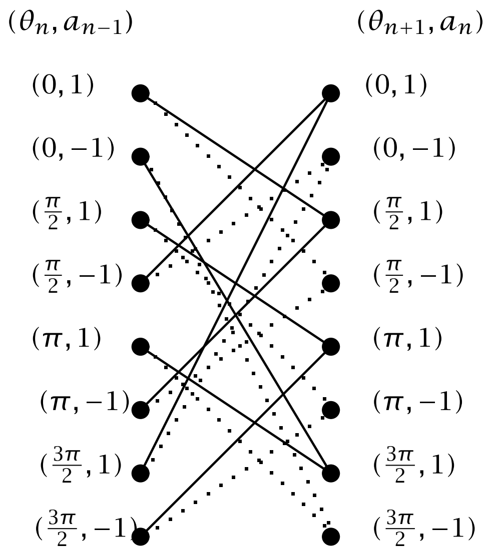
and therefore, the 8 states S_n are

$$\left\{(0,1),\ (0,-1),\ \left(\frac{\pi}{2},1\right),\ \left(\frac{\pi}{2},-1\right),\ (\pi,1),\ (\pi,-1),\ \left(\frac{3\pi}{2},1\right),\ \left(\frac{3\pi}{2},-1\right)\right\}$$

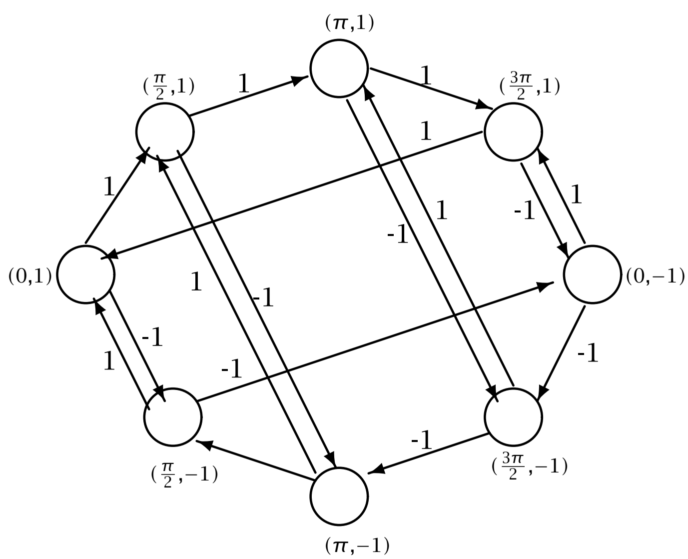
Having at our disposal the state (θ_n, a_{n-1}) and the transmitted symbol a_n, we can find the new phase state as

$$(\theta_n, a_{n-1}) \xrightarrow{a_n} (\theta_n + \frac{\pi}{2}a_{n-1}, a_n) = (\theta_{n+1}, a_n)$$

The following figure shows one frame of the phase-trellis of the partial response CPM signal.



The following is a sketch of the state diagram of the partial response CPM signal.



Problem 9.17

1) For a full response CPFSK signal, L is equal to 1. If $h = \frac{2}{3}$, then since m is even, there are p terminal phase states. If $h = \frac{3}{4}$, the number of states is $N_s = 2p$.

2) With $L = 3$ and $h = \frac{2}{3}$, the number of states is $N_s = p2^2 = 12$. When $L = 3$ and $h = \frac{3}{4}$, the number of states is $N_s = 2p2^2 = 32$.

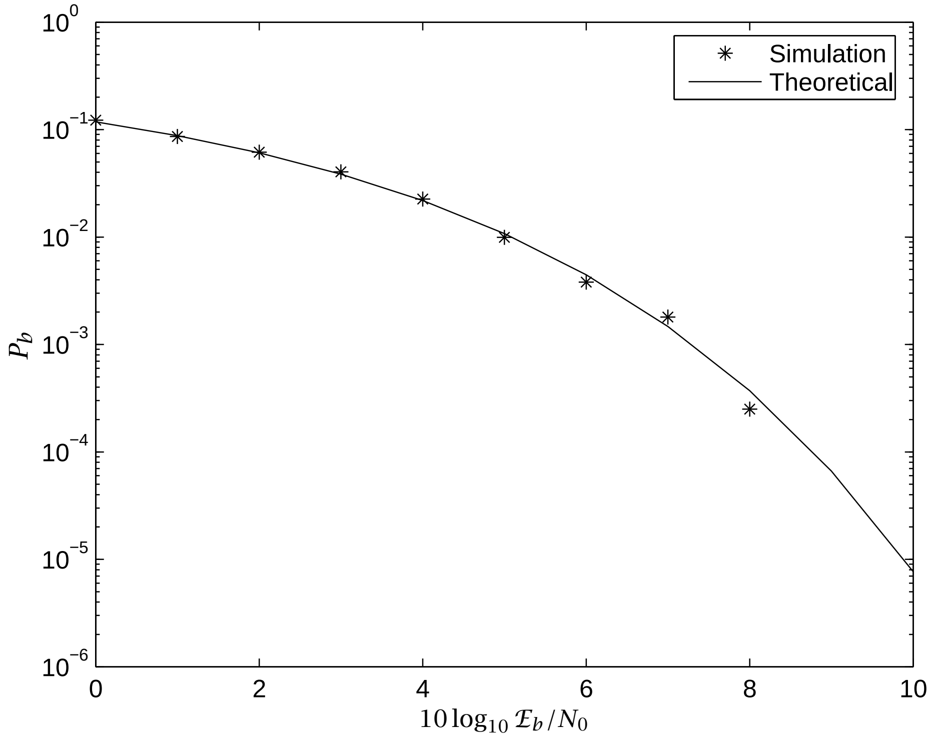


Figure 9.1: Bit-error probability for $M = 4$ orthogonal signals from a Monte Carlo simulation compared with theoretical error probability

Computer Problems

Computer Problem 9.1

Figure 9.1 illustrates the results of the simulation for the transmission of 20000 bits at several different values of the SNR $\mathcal{E}_b/N_0$. Note the agreement between the simulation results and the theoretical value of P_b given by

$$P_b = \frac{2^{k-1}}{2^k - 1} P_M \tag{9.29}$$

The MATLAB scripts for this problem are given next.

```
% MATLAB script for Computer Problem 9.1.
echo on
clear all;
tolerance=1e-8;                    % Tolerance used for the integration
minus_inf=-100000000;                    % This is practically -infinity
plus_inf=100000000;                      % This is practically infinity
SNRindB=0:1:10;
for i=1:length(SNRindB),
```

```
    % simulated error rate
    smld_err_prb(i)=smldP87(SNRindB(i));
      echo off;
end;
for i=1:length(SNRindB),
   snr=10^(SNRindB(i)/10);
  % theoretical error rate
  theo_err_prb(i)=(2/3)*quad8('bdt_int',minus_inf,plus_inf,tolerance,[],snr,4);
  echo off;
end;

echo on;
% Plotting commands follow
semilogy(SNRindB,smld_err_prb,'*');
hold
semilogy(SNRindB,theo_err_prb);
legend('Simulation','Theoretical');
```

```
function [p]=smldP87(snr_in_dB)
% [p]=smldP87(snr_in_dB)
%            SMLDP87 simulates the probability of error for the given
%            snr_in_dB, signal-to-noise ratio in dB.
M=4;                              % quaternary orthogonal signaling
E=1;
SNR=exp(snr_in_dB*log(10)/10);    % signal-to-noise ratio per bit
sgma=sqrt(E^2/(4*SNR));           % sigma, standard deviation of noise
N=10000;                          % number of symbols being simulated
% generation of the quaternary data source
for i=1:N,
  temp=rand;                      % a uniform random variable over (0,1)
  if (temp<0.25),
    dsource1(i)=0;
    dsource2(i)=0;
  elseif (temp<0.5),
    dsource1(i)=0;
    dsource2(i)=1;
  elseif (temp<0.75),
    dsource1(i)=1;
    dsource2(i)=0;
  else
    dsource1(i)=1;
    dsource2(i)=1;
  end
end;
% detection, and probability of error calculation
numoferr=0;
for i=1:N,
  % matched filter outputs
  if ((dsource1(i)==0) & (dsource2(i)==0)),
    r0=sqrt(E)+gngauss(sgma);
    r1=gngauss(sgma);
    r2=gngauss(sgma);
```

```
      r3=gngauss(sgma);
    elseif ((dsource1(i)==0) & (dsource2(i)==1)),
      r0=gngauss(sgma);
      r1=sqrt(E)+gngauss(sgma);
      r2=gngauss(sgma);
      r3=gngauss(sgma);
    elseif ((dsource1(i)==1) & (dsource2(i)==0)),
      r0=gngauss(sgma);
      r1=gngauss(sgma);
      r2=sqrt(E)+gngauss(sgma);
      r3=gngauss(sgma);
    else
      r0=gngauss(sgma);
      r1=gngauss(sgma);
      r2=gngauss(sgma);
      r3=sqrt(E)+gngauss(sgma);
    end;
    % the detector
    max_r=max([r0 r1 r2 r3]);
    if (r0==max_r),
      decis1=0;
      decis2=0;
    elseif (r1==max_r),
      decis1=0;
      decis2=1;
    elseif (r2==max_r),
      decis1=1;
      decis2=0;
    else
      decis1=1;
      decis2=1;
    end;
    % Count the number of bit errors made in this decision.
    if (decis1~=dsource1(i)),          % If it is an error, increase the error counter.
      numoferr=numoferr+1;
    end;
    if (decis2~=dsource2(i)),          % If it is an error, increase the error counter.
      numoferr=numoferr+1;
    end;
  end;
  p=numoferr/(2*N);                    % bit error probability estimate
```

Computer Problem 9.2

Figure 9.2 illustrates the results of the simulation for the transmission of 20000 bits at several different values of the SNR $\mathcal{E}_b/N_0$. Note the agreement between the simulation results and the theoretical value of P_b given by
The MATLAB scripts for this problem are given next.

```
% MATLAB script for Computer Problem 9.2.
```

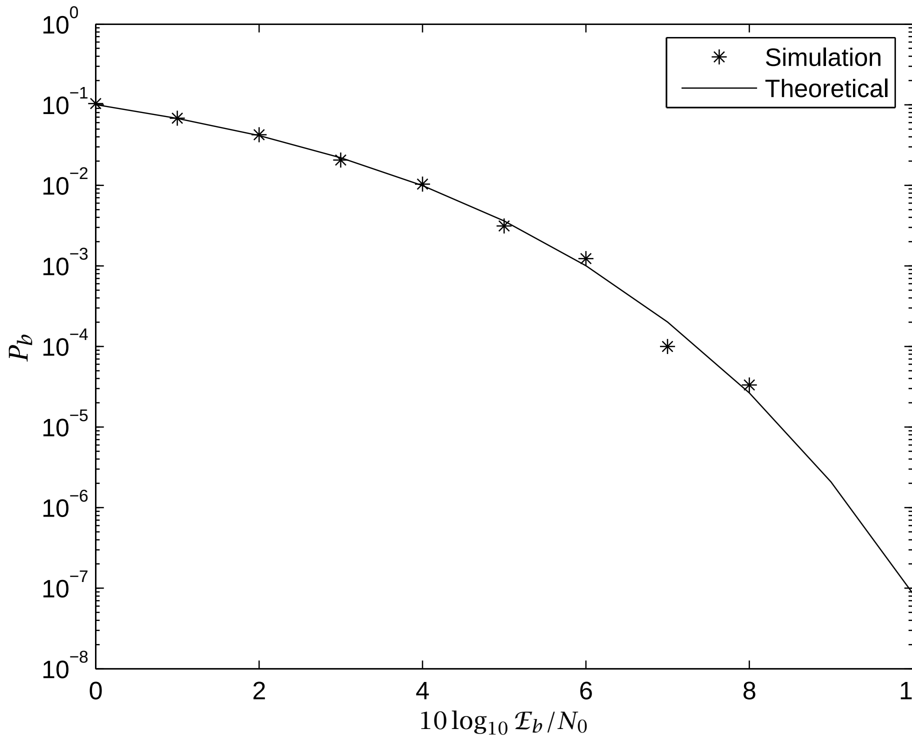


Figure 9.2: Bit-error probability for $M = 8$ orthogonal signals from a Monte Carlo simulation compared with theoretical error probability

```
clear all;
tolerance=1e-15;                    % Tolerance used for the integration
minus_inf=-60;                  % This is practically -infinity
plus_inf=60;                    % This is practically infinity
SNRindB=0:1:10;
for i=1:length(SNRindB),
    % simulated error rate
    smld_err_prb(i)=smldP88(SNRindB(i));
end;
for i=1:length(SNRindB),
   snr=10^(SNRindB(i)/10);
  % theoretical error rate
  theo_err_prb(i)=(4/7)*quad8('bdt_int',minus_inf,plus_inf,tolerance,[],snr,8);
end;
%Plotting commands follow
semilogy(SNRindB,smld_err_prb,'*');
hold on;
semilogy(SNRindB,theo_err_prb);
legend('Simulation','Theoretical');
hold on;
```

```
function [p]=smldP87(snr_in_dB)
% [p]=smldP87(snr_in_dB)
%               SMLDP87 simulates the probability of error for the given
%               snr_in_dB, signal-to-noise ratio in dB.
M=8;                            % quaternary orthogonal signaling
E=1;
numoferr = 0;
SNR=exp(snr_in_dB*log(10)/10);  % signal-to-noise ratio per bit
sgma=sqrt(E^2/(6*SNR));         % sigma, standard deviation of noise
N=10000;                        % number of symbols being simulated
% generation of the quaternary data source
for i=1:N,
    temp=rand;                      % a uniform random variable over (0,1)
    if (temp<1/8),
        dsource1(i)=0;
        dsource2(i)=0;
        dsource3(i)=0;
    elseif (temp<2/8),
        dsource1(i)=0;
        dsource2(i)=0;
        dsource3(i)=1;
    elseif (temp<3/8),
        dsource1(i)=0;
        dsource2(i)=1;
        dsource3(i)=0;
    elseif (temp<4/8),
        dsource1(i)=0;
        dsource2(i)=1;
        dsource3(i)=1;
    elseif (temp<5/8),
        dsource1(i)=1;
```

```
            dsource2(i)=0;
            dsource3(i)=0;
        elseif (temp<6/8),
            dsource1(i)=1;
            dsource2(i)=0;
            dsource3(i)=1;
        elseif (temp<7/8),
            dsource1(i)=1;
            dsource2(i)=1;
            dsource3(i)=0;
        else
            dsource1(i)=1;
            dsource2(i)=1;
            dsource3(i)=1;
        end
    end;
    % detection, and probability of error calculation
    numoferr=0;
    for i=1:N,
        % matched filter outputs
        if ((dsource1(i)==0) & (dsource2(i)==0) & (dsource3(i)==0)),
            r0=sqrt(E)+gngauss(sgma);
            r1=gngauss(sgma);
            r2=gngauss(sgma);
            r3=gngauss(sgma);
            r4=gngauss(sgma);
            r5=gngauss(sgma);
            r6=gngauss(sgma);
            r7=gngauss(sgma);
        elseif ((dsource1(i)==0) & (dsource2(i)==0) & (dsource3(i)==1)),
            r0=gngauss(sgma);
            r1=sqrt(E)+gngauss(sgma);
            r2=gngauss(sgma);
            r3=gngauss(sgma);
            r4=gngauss(sgma);
            r5=gngauss(sgma);
            r6=gngauss(sgma);
            r7=gngauss(sgma);
        elseif ((dsource1(i)==0) & (dsource2(i)==1) & (dsource3(i)==0)),
            r0=gngauss(sgma);
            r1=gngauss(sgma);
            r2=sqrt(E)+gngauss(sgma);
            r3=gngauss(sgma);
            r4=gngauss(sgma);
            r5=gngauss(sgma);
            r6=gngauss(sgma);
            r7=gngauss(sgma);
        elseif ((dsource1(i)==0) & (dsource2(i)==1) & (dsource3(i)==1)),
            r0=gngauss(sgma);
            r1=gngauss(sgma);
            r2=gngauss(sgma);
            r3=sqrt(E)+gngauss(sgma);
            r4=gngauss(sgma);
            r5=gngauss(sgma);
```

```
      r6=gngauss(sgma);
      r7=gngauss(sgma);
   elseif ((dsource1(i)==1) & (dsource2(i)==0) & (dsource3(i)==0)),
      r0=gngauss(sgma);
      r1=gngauss(sgma);
      r2=gngauss(sgma);
      r3=gngauss(sgma);
      r4=sqrt(E)+gngauss(sgma);
      r5=gngauss(sgma);
      r6=gngauss(sgma);
      r7=gngauss(sgma);
   elseif ((dsource1(i)==1) & (dsource2(i)==0) & (dsource3(i)==1)),
      r0=gngauss(sgma);
      r1=gngauss(sgma);
      r2=gngauss(sgma);
      r3=gngauss(sgma);
      r4=gngauss(sgma);
      r5=sqrt(E)+gngauss(sgma);
      r6=gngauss(sgma);
      r7=gngauss(sgma);
   elseif ((dsource1(i)==1) & (dsource2(i)==1) & (dsource3(i)==0)),
      r0=gngauss(sgma);
      r1=gngauss(sgma);
      r2=gngauss(sgma);
      r3=gngauss(sgma);
      r4=gngauss(sgma);
      r5=gngauss(sgma);
      r6=sqrt(E)+gngauss(sgma);
      r7=gngauss(sgma);
   elseif ((dsource1(i)==1) & (dsource2(i)==1) & (dsource3(i)==1)),
      r0=gngauss(sgma);
      r1=gngauss(sgma);
      r2=gngauss(sgma);
      r3=gngauss(sgma);
      r4=gngauss(sgma);
      r5=gngauss(sgma);
      r6=gngauss(sgma);
      r7=sqrt(E)+gngauss(sgma);
   else
      r0=gngauss(sgma);
      r1=gngauss(sgma);
      r2=gngauss(sgma);
      r3=gngauss(sgma);
      r4=gngauss(sgma);
      r5=gngauss(sgma);
      r6=gngauss(sgma);
      r7=sqrt(E)+gngauss(sgma);
  end;
   % the detector
   max_r=max([r0 r1 r2 r3 r4 r5 r6 r7]);
   if (r0==max_r),
      decis1=0;
      decis2=0;
      decis3=0;
```

```
    elseif (r1==max_r),
        decis1=0;
        decis2=0;
        decis3=1;
    elseif (r2==max_r),
        decis1=0;
        decis2=1;
        decis3=0;
    elseif(r3==max_r)
        decis1=0;
        decis2=1;
        decis3=1;
    elseif(r4==max_r)
        decis1=1;
        decis2=0;
        decis3=0;
    elseif(r5==max_r)
        decis1=1;
        decis2=0;
        decis3=1;
    elseif(r6==max_r)
        decis1=1;
        decis2=1;
        decis3=0;
    else
        decis1=1;
        decis2=1;
        decis3=1;
    end;
    % Count the number of bit errors made in this decision.
    if (decis1~=dsource1(i)),             % If it is an error, increase the error counter.
        numoferr=numoferr+1;
    end;
    if (decis2~=dsource2(i)),             % If it is an error, increase the error counter.
        numoferr=numoferr+1;
    end;
    if (decis3~=dsource3(i)),             % If it is an error, increase the error counter.
        numoferr=numoferr+1;
    end;
end;
p=numoferr/(3*N);                         % bit error probability estimate
```

Computer Problem 9.3

Figure 9.3 illustrates the outputs of the two correlators for different noise variances and transmitted signals. The MATLAB script for the computation is given next.

```
% MATLAB script for Computer Problem 9.3.

% Initialization:
```

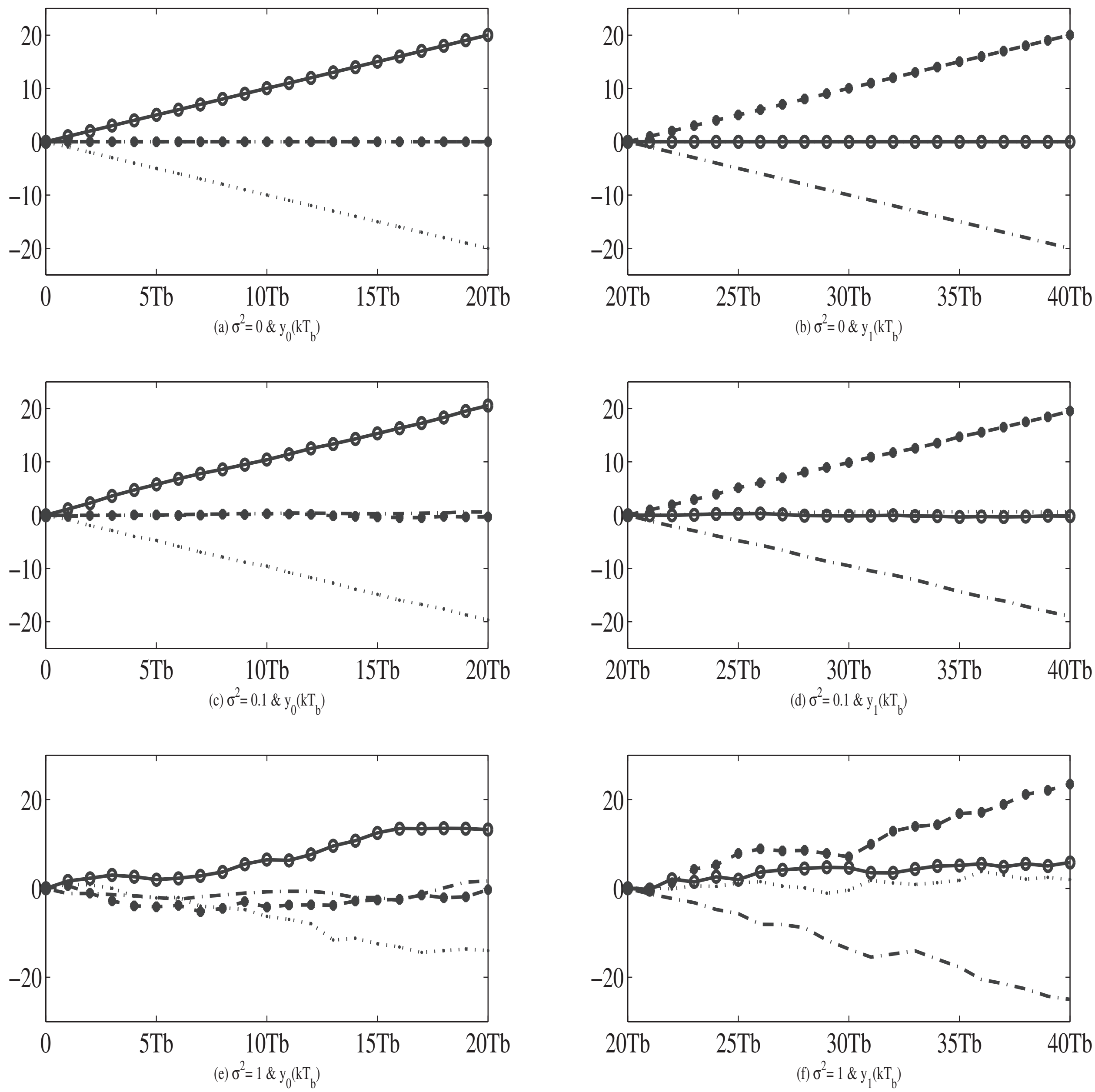


Figure 9.3: Correlator outputs in Computer Problem 9.3. Solid, dashed, dotted, and dash-dotted plots correspond to transmission of $s_0(t)$, $s_1(t)$, $s_2(t)$, and $s_3(t)$, respectively.

```
K=40;        % Number of samples
A=1;         % Signal amplitude
m=0:K/2;
n=K/2:K;
% Defining signal waveforms:
s_0=[A*ones(1,K/2) zeros(1,K/2)];
s_1=[zeros(1,K/2) A*ones(1,K/2)];
s_2=[-A*ones(1,K/2) zeros(1,K/2)];
s_3=[zeros(1,K/2) -A*ones(1,K/2)];
% Initializing Outputs:
y_0_0=zeros(1,K);
y_0_1=zeros(1,K);
y_0_2=zeros(1,K);
y_0_3=zeros(1,K);
y_1_0=zeros(1,K);
y_1_1=zeros(1,K);
y_1_2=zeros(1,K);
y_1_3=zeros(1,K);

% Case 1: noise~N(0,0)
  noise=random('Normal',0,0,1,K);
  r_0=s_0+noise; r_1=s_1+noise; % received signals
  r_2=s_2+noise; r_3=s_3+noise; % received signals
  for k=1:K/2
    y_0_0(k)=sum(r_0(1:k).*s_0(1:k));
    y_0_1(k)=sum(r_1(1:k).*s_0(1:k));
    y_0_2(k)=sum(r_2(1:k).*s_0(1:k));
    y_0_3(k)=sum(r_3(1:k).*s_0(1:k));
    l=K/2+k;
    y_1_0(l)=sum(r_0(21:l).*s_1(21:l));
    y_1_1(l)=sum(r_1(21:l).*s_1(21:l));
    y_1_2(l)=sum(r_2(21:l).*s_1(21:l));
    y_1_3(l)=sum(r_3(21:l).*s_1(21:l));
  end
  % Plotting the results:
  subplot(3,2,1)
  plot(m,[0 y_0_0(1:K/2)],'-bo',m,[0 y_0_1(1:K/2)],'--b*',...
    m,[0 y_0_2(1:K/2)],':b.',m,[0 y_0_3(1:K/2)],'-.')
  set(gca,'XTickLabel',{'0','5Tb','10Tb','15Tb','20Tb'})
  axis([0 20 -25 25])
  xlabel('(a) \sigma^2= 0 & y_{0}(kT_{b})' ,'fontsize',10)
  subplot(3,2,2)
  plot(n,[0 y_1_0(K/2+1:K)],'-bo',n,[0 y_1_1(K/2+1:K)],'--b*',...
    n,[0 y_1_2(K/2+1:K)],':b.',n,[0 y_1_3(K/2+1:K)],'-.')
  set(gca,'XTickLabel',{'20Tb','25Tb','30Tb','35Tb','40Tb'})
  axis([20 40 -25 25])
  xlabel('(b) \sigma^2= 0 & y_{1}(kT_{b})' ,'fontsize',10)
  % Case 2: noise~N(0,0.1)
  noise=random('Normal',0,0.1,4,K);
  r_0=s_0+noise(1,:); r_1=s_1+noise(2,:); % received signals
  r_2=s_2+noise(3,:); r_3=s_3+noise(4,:); % received signals
  for k=1:K/2
    y_0_0(k)=sum(r_0(1:k).*s_0(1:k));
    y_0_1(k)=sum(r_1(1:k).*s_0(1:k));
```

```
    y_0_2(k)=sum(r_2(1:k).*s_0(1:k));
    y_0_3(k)=sum(r_3(1:k).*s_0(1:k));
    l=K/2+k;
    y_1_0(l)=sum(r_0(21:l).*s_1(21:l));
    y_1_1(l)=sum(r_1(21:l).*s_1(21:l));
    y_1_2(l)=sum(r_2(21:l).*s_1(21:l));
    y_1_3(l)=sum(r_3(21:l).*s_1(21:l));
end
% Plotting the results:
subplot(3,2,3)
plot(m,[0 y_0_0(1:K/2)],'-bo',m,[0 y_0_1(1:K/2)],'--b*'...
    ,m,[0 y_0_2(1:K/2)],':b.',m,[0 y_0_3(1:K/2)],'-.')
set(gca,'XTickLabel',{'0','5Tb','10Tb','15Tb','20Tb'})
axis([0 20 -25 25])
xlabel('(c) \sigma^2= 0.1 & y_{0}(kT_{b})' ,'fontsize',10)
subplot(3,2,4)
plot(n,[0 y_1_0(K/2+1:K)],'-bo',n,[0 y_1_1(K/2+1:K)],'--b*',...
    n,[0 y_1_2(K/2+1:K)],':b.',n,[0 y_1_3(K/2+1:K)],'-.')
set(gca,'XTickLabel',{'20Tb','25Tb','30Tb','35Tb','40Tb'})
axis([20 40 -25 25])
xlabel('(d) \sigma^2= 0.1 & y_{1}(kT_{b})' ,'fontsize',10)

% Case 3: noise~N(0,1)
noise=random('Normal',0,1,4,K);
r_0=s_0+noise(1,:); r_1=s_1+noise(2,:); % received signals
r_2=s_2+noise(3,:); r_3=s_3+noise(4,:); % received signals
for k=1:K/2
    y_0_0(k)=sum(r_0(1:k).*s_0(1:k));
    y_0_1(k)=sum(r_1(1:k).*s_0(1:k));
    y_0_2(k)=sum(r_2(1:k).*s_0(1:k));
    y_0_3(k)=sum(r_3(1:k).*s_0(1:k));
    l=K/2+k;
    y_1_0(l)=sum(r_0(21:l).*s_1(21:l));
    y_1_1(l)=sum(r_1(21:l).*s_1(21:l));
    y_1_2(l)=sum(r_2(21:l).*s_1(21:l));
    y_1_3(l)=sum(r_3(21:l).*s_1(21:l));
end
% Plotting the results:
subplot(3,2,5)
plot(m,[0 y_0_0(1:K/2)],'-bo',m,[0 y_0_1(1:K/2)],'--b*',...
    m,[0 y_0_2(1:K/2)],':b.',m,[0 y_0_3(1:K/2)],'-.')
set(gca,'XTickLabel',{'0','5Tb','10Tb','15Tb','20Tb'})
axis([0 20 -30 30])
xlabel('(e) \sigma^2= 1 & y_{0}(kT_{b})' ,'fontsize',10)
subplot(3,2,6)
plot(n,[0 y_1_0(K/2+1:K)],'-bo',n,[0 y_1_1(K/2+1:K)],'--b*',...
    n,[0 y_1_2(K/2+1:K)],':b.',n,[0 y_1_3(K/2+1:K)],'-.')
set(gca,'XTickLabel',{'20Tb','25Tb','30Tb','35Tb','40Tb'})
axis([20 40 -30 30])
xlabel('(f) \sigma^2= 1 & y_{1}(kT_{b})' ,'fontsize',10)
```

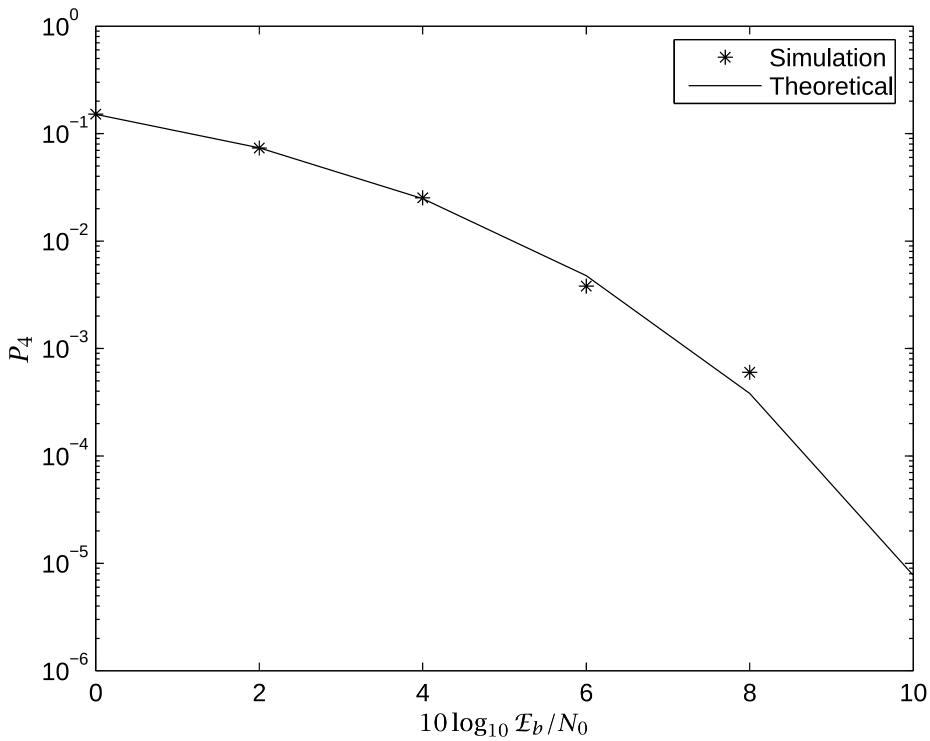


Figure 9.4: Symbol-error probability for $M = 4$ biorthogonal signals from Monte Carlo simulation compared with theoretical error probability

Computer Problem 9.4

Figure 9.4 illustrates the results of the simulation for the transmission of 20000 bits at several different values of the SNR $\mathcal{E}_b/N_0$. Note the agreement between the simulation results and the theoretical value of P_4 given by (9.30) and (9.31).

$$P_M = 1 - P_c \tag{9.30}$$

where P_c is given by

$$P_c = \int_0^\infty \left[\frac{1}{\sqrt{2\pi}} \int_{-r_0/\sqrt{\mathcal{E}N_0/2}}^{r_0/\sqrt{\mathcal{E}N_0/2}} e^{-x^2/2}\,dx \right]^{M-1} p(r_0)\,dr_0 \tag{9.31}$$

The MATLAB scripts for this problem are given next.

```
% MATLAB script for Computer Problem 9.4.
echo on
SNRindB=0:2:10;
for i=1:length(SNRindB),
  % simulated error rate
  smld_err_prb(i)=smldP89(SNRindB(i));
  echo off;
end;
```

```
echo on ;
% Plotting commands follow.
```

```
function [p]=smldP89(snr_in_dB)
% [p]=smldP89(snr_in_dB)
%              SMLDP89  simulates the probability of error for the given
%              snr_in_dB, signal-to-noise ratio in dB, for the system
%              described in Computer Problem 9.4.
M=4;                              % quaternary biorthogonal signaling
E=1;
SNR=exp(snr_in_dB*log(10)/10);    % signal-to-noise ratio per bit
sgma=sqrt(E^2/(4*SNR));           % sigma, standard deviation of noise
N=10000;                          % number of symbols being simulated
% generation of the quaternary data source
for i=1:N,
  temp=rand;                      % uniform random variable over (0,1)
  if (temp<0.25),
    dsource(i)=0;
  elseif (temp<0.5),
    dsource(i)=1;
  elseif (temp<0.75),
    dsource(i)=2;
  else
    dsource(i)=3;
  end
end;
% detection, and error probability computation
numoferr=0;
for i=1:N,
  % the matched filter outputs
  if (dsource(i)==0)
    r0=sqrt(E)+gngauss(sgma);
    r1=gngauss(sgma);
  elseif (dsource(i)==1)
    r0=gngauss(sgma);
    r1=sqrt(E)+gngauss(sgma);
  elseif (dsource(i)==2)
    r0=-sqrt(E)+gngauss(sgma);
    r1=gngauss(sgma);
  else
    r0=gngauss(sgma);
    r1=-sqrt(E)+gngauss(sgma);
  end;
  % detector follows
  if (r0>abs(r1)),
    decis=0;
  elseif (r1>abs(r0)),
    decis=1;
  elseif (r0<-abs(r1)),
    decis=2;
  else
    decis=3;
```

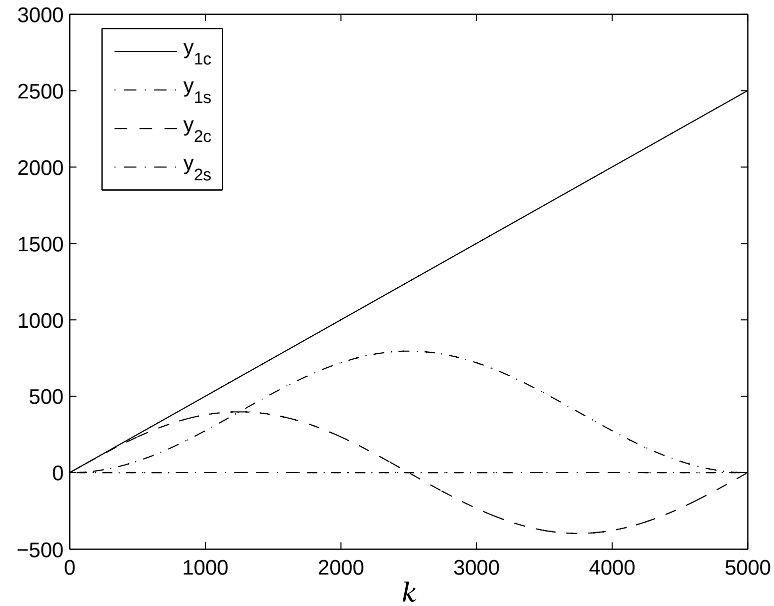


Figure 9.5: Correlator outputs for $\cos(2\pi f_1 t)$

```
    end;
    if (decis~=dsource(i)),            % If it is an error, increase the error counter.
        numoferr=numoferr+1;
    end;
end;
p=numoferr/N;                          % bit error probability estimate
```

Computer Problem 9.5

Figures 9.5 and 9.6 present the correlator outputs for $\cos(2\pi f_1 t)$ and $\cos(2\pi f_2 t)$, respectively. $y_1 = 6.2525e + 006$ and $y_2 = 1$ for $\cos(2\pi f_1 t)$. $y_1 = 1$ and $y_2 = 6.2525e + 006$ for $\cos(2\pi f_2 t)$. The MATLAB script for this problem is given next.

```
% MATLAB script for Computer Problem 9.5.
Tb=1;
f1=1000/Tb;
f2=f1+1/Tb;
phi=pi/4;
N=5000;                                % number of samples
t=0:Tb/(N−1):Tb;
u1=cos(2*pi*f1*t);
u2=cos(2*pi*f2*t);
% Assuming that the received signa is r = cos(2*pi*f1*t)
sgma=1;
for i=1:N,
  r(i)=cos(2*pi*f1*t(i));
```

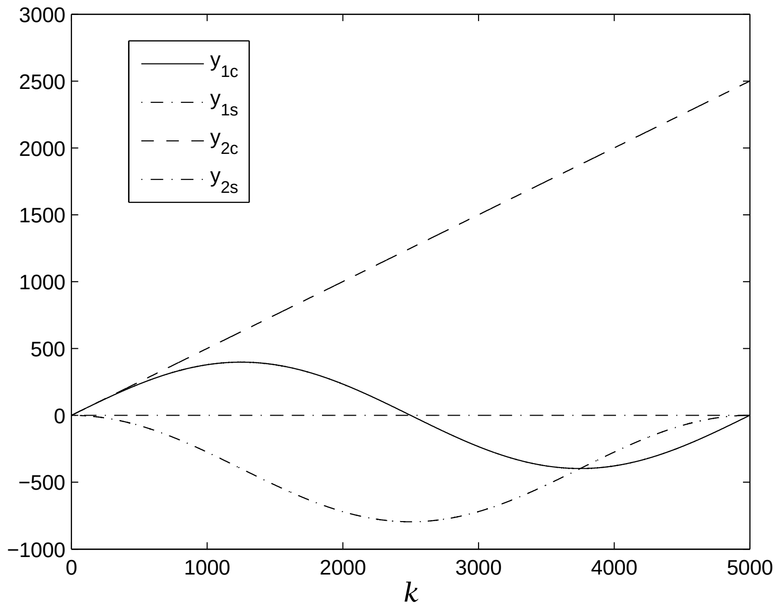


Figure 9.6: Correlator outputs for $\cos(2\pi f_2 t)$

```
end;
% The correlator outputs are computed next.
v1=sin(2*pi*f1*t);
v2=sin(2*pi*f2*t);
y1c(1)=r(1)*u1(1);
y1s(1)=r(1)*v1(1);
y2c(1)=r(1)*u2(1);
y2s(1)=r(1)*v2(1);
for k=2:N,
  y1c(k)=y1c(k−1)+r(k)*u1(k);
  y1s(k)=y1s(k−1)+r(k)*v1(k);
  y2c(k)=y2c(k−1)+r(k)*u2(k);
  y2s(k)=y2s(k−1)+r(k)*v2(k);
end;
% decision variables
y1=y1c(5000)^2+y1s(5000)^2
y2=y2c(5000)^2+y2s(5000)^2
% Plotting commands follow.
plot(y1c, '-');
hold on;
plot(y1s, '-.');
hold on;
plot(y2c,'--');
hold on;
plot(y2s,'-.');
legend('y_{1c}', 'y_{1s}', 'y_{2c}', 'y_{2s}');
xlabel('k');
```

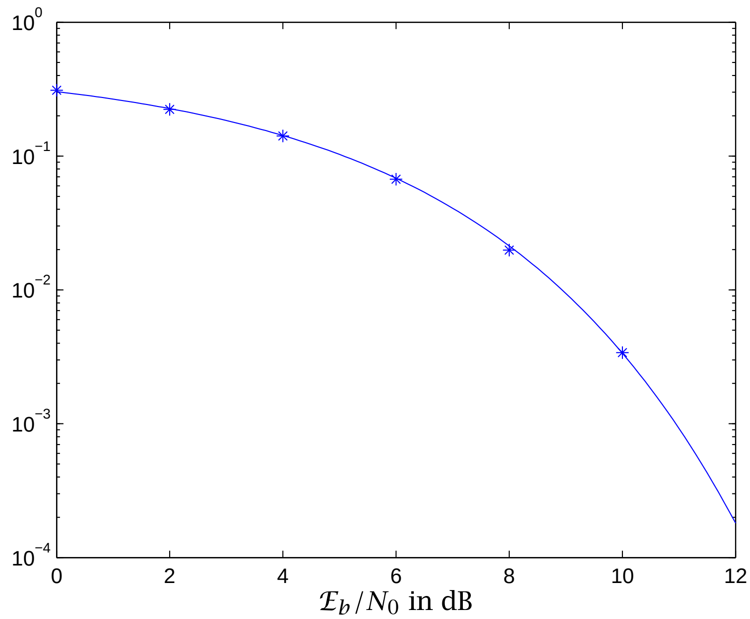


Figure 9.7: Theoretical error probability and Monte Carlo simulation results for a binary FSK system

---

Computer Problem 9.6

Figure 9.7 presents the measured error rate and compares it with the theoretical error probability. The MATLAB scripts for this problem are given next.

```
% MATLAB script for Computer Problem 9.6.
echo on
SNRindB1=0:2:12;
SNRindB2=0:0.1:12;
for i=1:length(SNRindB1),
  smld_err_prb(i)=cm_sm52(SNRindB1(i)); % simulated error rate
  echo off ;
end;
echo on ;
for i=1:length(SNRindB2),
  SNR=exp(SNRindB2(i)*log(10)/10);     % signal-to-noise ratio
  theo_err_prb(i)=(1/2)*exp(-SNR/2);   % theoretical symbol error rate
  echo off;
end;
echo on;
% Plotting commands follow.
semilogy(SNRindB1,smld_err_prb,'*');
hold on;
semilogy(SNRindB2,theo_err_prb);
```

```
function [p]=cm_sm52(snr_in_dB)
% [p]=cm_sm52(snr_in_dB)
%           CM_SM52 Returns the probability of error for the given
%           value of snr_in_dB, signal-to-noise ratio in dB.
N=10000;
Eb=1;
d=1;
snr=10^(snr_in_dB/10);              % signal-to-noise ratio per bit
sgma=sqrt(Eb/(2*snr));              % noise variance
phi=0;
% Generation of the data source follows.
for i=1:N,
  temp=rand;                        % a uniform random variable between 0 and 1
  if (temp<0.5),
    dsource(i)=0;
  else
    dsource(i)=1;
  end;
end;
% detection and the probability of error calculation
numoferr=0;
for i=1:N,
  % demodulator output
  if (dsource(i)==0),
    y0c=sqrt(Eb)*cos(phi)+gngauss(sgma);
    y0s=sqrt(Eb)*sin(phi)+gngauss(sgma);
    y1c=gngauss(sgma);
    y1s=gngauss(sgma);
  else
    y0c=gngauss(sgma);
    y0s=gngauss(sgma);
    y1c=sqrt(Eb)*cos(phi)+gngauss(sgma);
    y1s=sqrt(Eb)*sin(phi)+gngauss(sgma);
  end;
  % square-law detector outputs
  y0=y0c^2+y0s^2;
  y1=y1c^2+y1s^2;
  % Decision is made next.
  if (y0>y1),
    decis=0;
  else
    decis=1;
  end;
  % If the decision is not correct the error counter is increased.
  if (decis~=dsource(i)),
    numoferr=numoferr+1;
  end;
end;
p=numoferr/(N);
```

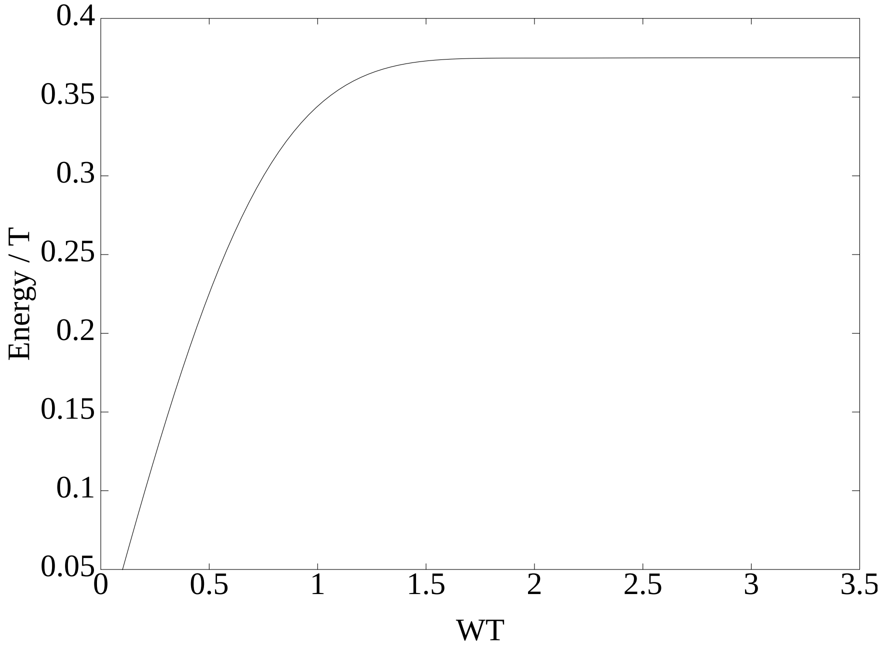
Chapter 10

Problem 10.1

1) The following table shows the values of $\mathcal{E}_h(W)/T$ obtained using an adaptive recursive Newton-Cotes numerical integration rule.

WT	0.5	1.0	1.5	2.0	2.5	3.0
$\mathcal{E}_h(W)/T$	0.2253	0.3442	0.3730	0.3748	0.3479	0.3750

A plot of $\mathcal{E}_h(W)/T$ as a function of WT is given in the next figure



2) The value of $\mathcal{E}_h(W)$ as $W \to \infty$ is

$$\begin{aligned}
\lim_{W\to\infty} \mathcal{E}_h(W) &= \int_{-\infty}^{\infty} g_T^2(t)dt = \int_0^T g_T^2(t)dt \\
&= \frac{1}{4}\int_0^T \left(1+\cos\frac{2\pi}{T}\left(t-\frac{T}{2}\right)\right)^2 dt \\
&= \frac{T}{4} + \frac{1}{2}\int_0^T \cos\frac{2\pi}{T}\left(t-\frac{T}{2}\right)dt \\
&\quad + \frac{1}{8}\int_0^T \left[1+\cos\frac{2\pi}{T}2\left(t-\frac{T}{2}\right)\right]dt \\
&= \frac{T}{4}+\frac{T}{8} = \frac{3T}{8} = 0.3750T
\end{aligned}$$

Problem 10.2

We have

$$y = \begin{cases} a + n - \frac{1}{2} & \text{with Prob. } \frac{1}{4} \\ a + n + \frac{1}{2} & \text{with Prob. } \frac{1}{4} \\ a + n & \text{with Prob. } \frac{1}{2} \end{cases}$$

By symmetry, $P_e = P(e|a = 1) = P(e|a = -1)$, hence,

$$\begin{aligned} P_e = P(e|a = -1) &= \frac{1}{2}P(n - 1 > 0) + \frac{1}{4}P\left(n - \frac{3}{2} > 0\right) + \frac{1}{4}P\left(n - \frac{1}{2} > 0\right) \\ &= \frac{1}{2}Q\left(\frac{1}{\sigma_n}\right) + \frac{1}{4}Q\left(\frac{3}{2\sigma_n}\right) + \frac{1}{4}Q\left(\frac{1}{2\sigma_n}\right) \end{aligned}$$

Problem 10.3

1) If the transmitted signal is

$$r(t) = \sum_{n=-\infty}^{\infty} a_n h(t - nT) + n(t)$$

then the output of the receiving filter is

$$y(t) = \sum_{n=-\infty}^{\infty} a_n x(t - nT) + \nu(t)$$

where $x(t) = h(t) \star h(t)$ and $\nu(t) = n(t) \star h(t)$. If the sampling time is off by 10%, then the samples at the output of the correlator are taken at $t = (m \pm \frac{1}{10})T$. Assuming that $t = (m - \frac{1}{10})T$ without loss of generality, then the sampled sequence is

$$y_m = \sum_{n=-\infty}^{\infty} a_n x((m - \frac{1}{10}T - nT) + \nu((m - \frac{1}{10})T)$$

If the signal pulse is rectangular with amplitude A and duration T, then $\sum_{n=-\infty}^{\infty} a_n x((m - \frac{1}{10}T - nT)$ is nonzero only for $n = m$ and $n = m - 1$ and therefore, the sampled sequence is given by

$$\begin{aligned} y_m &= a_m x(-\frac{1}{10}T) + a_{m-1}x(T - \frac{1}{10}T) + \nu((m - \frac{1}{10})T) \\ &= \frac{9}{10}a_m A^2 T + a_{m-1}\frac{1}{10}A^2 T + \nu((m - \frac{1}{10})T) \end{aligned}$$

The power spectral density of the noise at the output of the correlator is

$$S_\nu(f) = S_n(f)|H(f)|^2 = \frac{N_0}{2}A^2T^2\text{sinc}^2(fT)$$

Thus, the variance of the noise is

$$\sigma_n u^2 = \int_{-\infty}^{\infty} \frac{N_0}{2} A^2 T^2 \text{sinc}^2(fT) df = \frac{N_0}{2} A^2 T^2 \frac{1}{T} = \frac{N_0}{2} A^2 T$$

and therefore, the SNR is

$$\text{SNR} = \left(\frac{9}{10}\right)^2 \frac{2(A^2T)^2}{N_0 A^2 T} = \frac{81}{100} \frac{2A^2T}{N_0}$$

As it is observed, there is a loss of $10 \log_{10} \frac{81}{100} = -0.9151$ dB due to the mistiming.

2) Recall from part a) that the sampled sequence is

$$y_m = \frac{9}{10} a_m A^2 T + a_{m-1} \frac{1}{10} A^2 T + \nu_m$$

The term $a_{m-1} \frac{A^2T}{10}$ expresses the ISI introduced to the system. If $a_m = 1$ is transmitted, then the probability of error is

$$\begin{aligned} P(e|a_m = 1) &= \frac{1}{2} P(e|a_m = 1, a_{m-1} = 1) + \frac{1}{2} P(e|a_m = 1, a_{m-1} = -1) \\ &= \frac{1}{2\sqrt{\pi N_0 A^2 T}} \int_{-\infty}^{-A^2T} e^{-\frac{\nu^2}{N_0 A^2 T}} d\nu + \frac{1}{2\sqrt{\pi N_0 A^2 T}} \int_{-\infty}^{-\frac{8}{10}A^2T} e^{-\frac{\nu^2}{N_0 A^2 T}} d\nu \\ &= \frac{1}{2} Q\left[\sqrt{\frac{2A^2T}{N_0}}\right] + \frac{1}{2} Q\left[\sqrt{\left(\frac{8}{10}\right)^2 \frac{2A^2T}{N_0}}\right] \end{aligned}$$

Since the symbols of the binary PAM system are equiprobable the previous derived expression is the probability of error when a symbol by symbol detector is employed. Comparing this with the probability of error of a system with no ISI, we observe that there is an increase of the probability of error by

$$P_{\text{diff}}(e) = \frac{1}{2} Q\left[\sqrt{\left(\frac{8}{10}\right)^2 \frac{2A^2T}{N_0}}\right] - \frac{1}{2} Q\left[\sqrt{\frac{2A^2T}{N_0}}\right]$$

Problem 10.4

1) Taking the inverse Fourier transform of $H(f)$, we obtain

$$h(t) = \mathcal{F}^{-1}[H(f)] = \delta(t) + \frac{\alpha}{2}\delta(t - t_0) + \frac{\alpha}{2}\delta(t + t_0)$$

Hence,

$$y(t) = s(t) \star h(t) = s(t) + \frac{\alpha}{2} s(t - t_0) + \frac{\alpha}{2} s(t + t_0)$$

2) If the signal $s(t)$ is used to modulate the sequence $\{a_n\}$, then the transmitted signal is

$$u(t) = \sum_{n=-\infty}^{\infty} a_n s(t - nT)$$

The received signal is the convolution of $u(t)$ with $h(t)$. Hence,

$$\begin{aligned} y(t) &= u(t) \star h(t) = \left(\sum_{n=-\infty}^{\infty} a_n s(t-nT) \right) \star \left(\delta(t) + \frac{\alpha}{2}\delta(t-t_0) + \frac{\alpha}{2}\delta(t+t_0) \right) \\ &= \sum_{n=-\infty}^{\infty} a_n s(t-nT) + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} a_n s(t-t_0-nT) + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} a_n s(t+t_0-nT) \end{aligned}$$

Thus, the output of the matched filter $s(-t)$ at the time instant t_1 is

$$\begin{aligned} w(t_1) &= \sum_{n=-\infty}^{\infty} a_n \int_{-\infty}^{\infty} s(\tau-nT)s(\tau-t_1)d\tau \\ &\quad + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} a_n \int_{-\infty}^{\infty} s(\tau-t_0-nT)s(\tau-t_1)d\tau \\ &\quad + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} a_n \int_{-\infty}^{\infty} s(\tau+t_0-nT)s(\tau-t_1)d\tau \end{aligned}$$

If we denote the signal $s(t) \star s(t)$ by $x(t)$, then the output of the matched filter at $t_1 = kT$ is

$$\begin{aligned} w(kT) &= \sum_{n=-\infty}^{\infty} a_n x(kT-nT) \\ &\quad + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} a_n x(kT-t_0-nT) + \frac{\alpha}{2} \sum_{n=-\infty}^{\infty} a_n x(kT+t_0-nT) \end{aligned}$$

3) With $t_0 = T$ and $k = n$ in the previous equation, we obtain

$$\begin{aligned} w_k &= a_k x_0 + \sum_{n \neq k} a_n x_{k-n} \\ &\quad + \frac{\alpha}{2} a_k x_{-1} + \frac{\alpha}{2} \sum_{n \neq k} a_n x_{k-n-1} + \frac{\alpha}{2} a_k x_1 + \frac{\alpha}{2} \sum_{n \neq k} a_n x_{k-n+1} \\ &= a_k \left(x_0 + \frac{\alpha}{2} x_{-1} + \frac{\alpha}{2} x_1 \right) + \sum_{n \neq k} a_n \left[x_{k-n} + \frac{\alpha}{2} x_{k-n-1} + \frac{\alpha}{2} x_{k-n+1} \right] \end{aligned}$$

The terms under the summation is the ISI introduced by the channel.

Problem 10.5

The pulse $x(t)$ having the raised cosine spectrum is

$$x(t) = \text{sinc}(t/T) \frac{\cos(\pi \alpha t/T)}{1 - 4\alpha^2 t^2/T^2}$$

The function $\text{sinc}(t/T)$ is 1 when $t = 0$ and 0 when $t = nT$. On the other hand

$$g(t) = \frac{\cos(\pi \alpha t/T)}{1 - 4\alpha^2 t^2/T^2} = \begin{cases} 1 & t = 0 \\ \text{bounded} & t \neq 0 \end{cases}$$

The function $g(t)$ needs to be checked only for those values of t such that $4\alpha^2 t^2/T^2 = 1$ or $\alpha t = \frac{T}{2}$. However,

$$\lim_{\alpha t \to \frac{T}{2}} \frac{\cos(\pi\alpha t/T)}{1-4\alpha^2 t^2/T^2} = \lim_{x\to 1} \frac{\cos(\frac{\pi}{2}x)}{1-x}$$

and by using L'Hospital's rule

$$\lim_{x\to 1} \frac{\cos(\frac{\pi}{2}x)}{1-x} = \lim_{x\to 1} \frac{\pi}{2}\sin(\frac{\pi}{2}x) = \frac{\pi}{2} < \infty$$

Hence,

$$x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

meaning that the pulse $x(t)$ satisfies the Nyquist criterion.

Problem 10.6

Substituting the expression of $X_{rc}(f)$ in the desired integral, we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} X_{rc}(f)df &= \int_{-\frac{1+\alpha}{2T}}^{-\frac{1-\alpha}{2T}} \frac{T}{2}\left[1+\cos\frac{\pi T}{\alpha}(-f-\frac{1-\alpha}{2T})\right]df + \int_{-\frac{1-\alpha}{2T}}^{\frac{1-\alpha}{2T}} T df \\
&\quad + \int_{\frac{1-\alpha}{2T}}^{\frac{1+\alpha}{2T}} \frac{T}{2}\left[1+\cos\frac{\pi T}{\alpha}(f-\frac{1-\alpha}{2T})\right]df \\
&= \int_{-\frac{1+\alpha}{2T}}^{-\frac{1-\alpha}{2T}} \frac{T}{2}df + T\left(\frac{1-\alpha}{T}\right) + \int_{\frac{1-\alpha}{2T}}^{\frac{1+\alpha}{2T}} \frac{T}{2}df \\
&\quad + \int_{-\frac{1+\alpha}{2T}}^{-\frac{1-\alpha}{2T}} \cos\frac{\pi T}{\alpha}(f+\frac{1-\alpha}{2T})df + \int_{\frac{1-\alpha}{2T}}^{\frac{1+\alpha}{2T}} \cos\frac{\pi T}{\alpha}(f-\frac{1-\alpha}{2T})df \\
&= 1 + \int_{-\frac{\alpha}{T}}^{0} \cos\frac{\pi T}{\alpha}x dx + \int_{0}^{\frac{\alpha}{T}} \cos\frac{\pi T}{\alpha}x dx \\
&= 1 + \int_{-\frac{\alpha}{T}}^{\frac{\alpha}{T}} \cos\frac{\pi T}{\alpha}x dx = 1 + 0 = 1
\end{aligned}$$

Problem 10.7

Let $X(f)$ be such that

$$\mathrm{Re}[X(f)] = \begin{cases} T\Pi(fT) + U(f) & |f| < \frac{1}{T} \\ 0 & \text{otherwise} \end{cases} \qquad \mathrm{Im}[X(f)] = \begin{cases} V(f) & |f| < \frac{1}{T} \\ 0 & \text{otherwise} \end{cases}$$

with $U(f)$ even with respect to 0 and odd with respect to $f = \frac{1}{2T}$ Since $x(t)$ is real, $V(f)$ is odd with respect to 0 and by assumption it is even with respect to $f = \frac{1}{2T}$. Then,

$$\begin{aligned}
x(t) &= \mathcal{F}^{-1}[X(f)] \\
&= \int_{-\frac{1}{T}}^{\frac{1}{2T}} X(f)e^{j2\pi ft}df + \int_{-\frac{1}{2T}}^{\frac{1}{2T}} X(f)e^{j2\pi ft}df + \int_{\frac{1}{2T}}^{\frac{1}{T}} X(f)e^{j2\pi ft}df \\
&= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} Te^{j2\pi ft}df + \int_{-\frac{1}{T}}^{\frac{1}{T}} [U(f) + jV(f)]e^{j2\pi ft}df \\
&= \text{sinc}(t/T) + \int_{-\frac{1}{T}}^{\frac{1}{T}} [U(f) + jV(f)]e^{j2\pi ft}df
\end{aligned}$$

Consider first the integral $\int_{-\frac{1}{T}}^{\frac{1}{T}} U(f)e^{j2\pi ft}df$. Clearly,

$$\int_{-\frac{1}{T}}^{\frac{1}{T}} U(f)e^{j2\pi ft}df = \int_{-\frac{1}{T}}^{0} U(f)e^{j2\pi ft}df + \int_{0}^{\frac{1}{T}} U(f)e^{j2\pi ft}df$$

and by using the change of variables $f' = f + \frac{1}{2T}$ and $f' = f - \frac{1}{2T}$ for the two integrals on the right hand side respectively, we obtain

$$\begin{aligned}
&\int_{-\frac{1}{T}}^{\frac{1}{T}} U(f)e^{j2\pi ft}df \\
&= e^{-j\frac{\pi}{T}t} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' - \frac{1}{2T})e^{j2\pi f't}df' + e^{j\frac{\pi}{T}t} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T})e^{j2\pi f't}df' \\
&\stackrel{a}{=} \left(e^{j\frac{\pi}{T}t} - e^{-j\frac{\pi}{T}t}\right) \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T})e^{j2\pi f't}df' \\
&= 2j\sin(\frac{\pi}{T}t) \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T})e^{j2\pi f't}df'
\end{aligned}$$

where for step (a) we used the odd symmetry of $U(f')$ with respect to $f' = \frac{1}{2T}$, that is

$$U(f' - \frac{1}{2T}) = -U(f' + \frac{1}{2T})$$

For the integral $\int_{-\frac{1}{T}}^{\frac{1}{T}} V(f)e^{j2\pi ft}df$ we have

$$\begin{aligned}
&\int_{-\frac{1}{T}}^{\frac{1}{T}} V(f)e^{j2\pi ft}df \\
&= \int_{-\frac{1}{T}}^{0} V(f)e^{j2\pi ft}df + \int_{0}^{\frac{1}{T}} V(f)e^{j2\pi ft}df \\
&= e^{-j\frac{\pi}{T}t} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f' - \frac{1}{2T})e^{j2\pi f't}df' + e^{j\frac{\pi}{T}t} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f' + \frac{1}{2T})e^{j2\pi f't}df'
\end{aligned}$$

However, $V(f)$ is odd with respect to 0 and since $V(f' + \frac{1}{2T})$ and $V(f' - \frac{1}{2T})$ are even, the translated spectra satisfy

$$\int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f' - \frac{1}{2T}) e^{j2\pi f' t} df' = -\int_{-\frac{1}{2T}}^{\frac{1}{2T}} V(f' + \frac{1}{2T}) e^{j2\pi f' t} df'$$

Hence,

$$\begin{aligned} x(t) &= \text{sinc}(t/T) + 2j\sin(\frac{\pi}{T}t) \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T}) e^{j2\pi f' t} df' \\ &\quad -2\sin(\frac{\pi}{T}t) \int_{-\frac{1}{2T}}^{\frac{1}{2T}} U(f' + \frac{1}{2T}) e^{j2\pi f' t} df' \end{aligned}$$

and therefore,

$$x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Thus, the signal $x(t)$ satisfies the Nyquist criterion.

Problem 10.8

The bandwidth of the channel is 1400 Hz. Since the minimum transmission bandwidth required for baseband signaling is $R/2$, where R is the symbol rate, we conclude that the maximum value of the symbol rate for the given channel is $R_{\text{max}} = 2800$. If an M-ary PAM modulation is used for transmission, then in order to achieve a bit-rate of 9600 bps, with maximum symbol rate of R_{max}, the minimum size of the constellation is $M = 2^k = 16$. In this case, the symbol rate is

$$R = \frac{9600}{k} = 2400 \text{ symbols/sec}$$

and the symbol interval $T = \frac{1}{R} = \frac{1}{2400}$ sec. The roll-off factor α of the raised cosine pulse used for transmission is determined by noting that $1200(1 + \alpha) = 1400$, and hence, $\alpha = 0.166$. Therefore, the squared root raised cosine pulse can have a roll-off of $\alpha = 0.166$.

Problem 10.9

Since the bandwidth of the ideal lowpass channel is $W = 2400$ Hz, the rate of transmission is

$$R = 2 \times 2400 = 4800 \text{ symbols/sec}$$

The number of bits per symbol is

$$k = \frac{14400}{4800} = 3$$

Hence, the number of transmitted symbols is $2^3 = 8$. If a duobinary pulse is used for transmission, then the number of possible transmitted symbols is $2M - 1 = 15$. These symbols have the form

$$b_n = 0, \pm 2d, \pm 4d, \ldots, \pm 12d$$

where $2d$ is the minimum distance between the points of the 8-PAM constellation. The probability mass function of the received symbols is

$$P(b = 2md) = \frac{8 - |m|}{64}, \qquad m = 0, \pm 1, \ldots, \pm 7$$

An upper bound of the probability of error is given by (see (9.3.33))

$$P_M < 2\left(1 - \frac{1}{M^2}\right) Q\left[\sqrt{\left(\frac{\pi}{4}\right)^2 \frac{6}{M^2 - 1} \frac{k\mathcal{E}_{b,av}}{N_0}}\right]$$

With $P_M = 10^{-6}$ and $M = 8$ we obtain

$$\frac{k\mathcal{E}_{b,av}}{N_0} = 1.3193 \times 10^3 \Longrightarrow \mathcal{E}_{b,av} = 0.088$$

Problem 10.10

1) If the power spectral density of the additive noise is $S_n(f)$, then the PSD of the noise at the output of the prewhitening filter is

$$S_\nu(f) = S_n(f)|H_p(f)|^2$$

In order for $S_\nu(f)$ to be flat (white noise), $H_p(f)$ should be such that

$$H_p(f) = \frac{1}{\sqrt{S_n(f)}}$$

2) Let $h_p(t)$ be the impulse response of the prewhitening filter $H_p(f)$. That is, $h_p(t) = \mathcal{F}^{-1}[H_p(f)]$. Then, the input to the matched filter is the signal $\tilde{s}(t) = s(t) \star h_p(t)$. The frequency response of the filter matched to $\tilde{s}(t)$ is

$$\tilde{S}_m(f) = \tilde{S}^*(f)e^{-j2\pi f t_0} == S^*(f)H_p^*(f)e^{-j2\pi f t_0}$$

where t_0 is some nominal time-delay at which we sample the filter output.

3) The frequency response of the overall system, prewhitening filter followed by the matched filter, is

$$G(f) = \tilde{S}_m(f)H_p(f) = S^*(f)|H_p(f)|^2 e^{-j2\pi f t_0} = \frac{S^*(f)}{S_n(f)} e^{-j2\pi f t_0}$$

4) The variance of the noise at the output of the generalized matched filter is

$$\sigma^2 = \int_{-\infty}^{\infty} S_n(f)|G(f)|^2 df = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{S_n(f)} df$$

At the sampling instant $t = t_0 = T$, the signal component at the output of the matched filter is

$$\begin{aligned} y(T) &= \int_{-\infty}^{\infty} Y(f)e^{j2\pi fT} df = \int_{-\infty}^{\infty} s(\tau)g(T-\tau)d\tau \\ &= \int_{-\infty}^{\infty} S(f)\frac{S^*(f)}{S_n(f)} df = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{S_n(f)} df \end{aligned}$$

Hence, the output SNR is

$$\text{SNR} = \frac{y^2(T)}{\sigma^2} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{S_n(f)} df$$

Problem 10.11

The roll-off factor α is related to the bandwidth by the expression $\frac{1+\alpha}{T} = 2W$, or equivalently $R(1+\alpha) = 2W$. The following table shows the symbol rate for the various values of the excess bandwidth and for $W = 1500$ Hz.

α	.25	.33	.50	.67	.75	1.00
R	2400	2256	2000	1796	1714	1500

Problem 10.12

The following table shows the precoded sequence, the transmitted amplitude levels, the received signal levels and the decoded sequence, when the data sequence 10010110010 modulates a duobinary transmitting filter.

Data seq. d_n:		1	0	0	1	0	1	1	0	0	1	0
Precoded seq. p_n:	0	1	1	1	0	0	1	0	0	0	1	1
Transmitted seq. a_n:	-1	1	1	1	-1	-1	1	-1	-1	-1	1	1
Received seq. b_n:		0	2	2	0	-2	0	0	-2	-2	0	2
Decoded seq. d_n:		1	0	0	1	0	1	1	0	0	1	0

Problem 10.13

The following table shows the precoded sequence, the transmitted amplitude levels, the received signal levels and the decoded sequence, when the data sequence 10010110010 modulates a modified duobinary transmitting filter.

Data seq. d_n:			1	0	0	1	0	1	1	0	0	1	0
Precoded seq. p_n:	0	0	1	0	1	1	1	0	0	0	0	1	0
Transmitted seq. a_n:	-1	-1	1	-1	1	1	1	-1	-1	-1	-1	1	-1
Received seq. b_n:			2	0	0	2	0	-2	-2	0	0	2	0
Decoded seq. d_n:			1	0	0	1	0	1	1	0	0	1	0

Problem 10.14

Let $X(z)$ denote the $\mathcal{Z}$-transform of the sequence x_n, that is

$$X(z) = \sum_n x_n z^{-n}$$

Then the precoding operation can be described as

$$P(z) = \frac{D(z)}{X(z)} \quad \text{mod} - M$$

where $D(z)$ and $P(z)$ are the $\mathcal{Z}$-transforms of the data and precoded sequences respectively. For example, if $M = 2$ and $X(z) = 1 + z^{-1}$ (duobinary signaling), then

$$P(z) = \frac{D(z)}{1 + z^{-1}} \Longrightarrow P(z) = D(z) - z^{-1}P(z)$$

which in the time domain is written as

$$p_n = d_n - p_{n-1}$$

and the subtraction is mod-2.

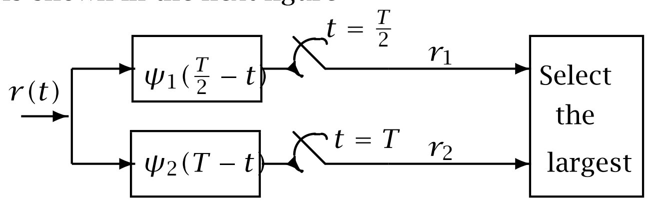
However, the inverse filter $\frac{1}{X(z)}$ exists only if x_0, the first coefficient of $X(z)$ is relatively prime with M. If this is not the case, then the precoded symbols p_n cannot be determined uniquely from the data sequence d_n.

Problem 10.15

1) The dimensionality of the signal space is two. An orthonormal basis set for the signal space is formed by the signals

$$\psi_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \le t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases} \qquad \psi_2(t) = \begin{cases} \sqrt{\frac{2}{T}}, & \frac{T}{2} \le t < T \\ 0, & \text{otherwise} \end{cases}$$

2) The optimal receiver is shown in the next figure



3) Assuming that the signal $s_1(t)$ is transmitted, the received vector at the output of the samplers is

$$\mathbf{r} = [\sqrt{\frac{A^2T}{2}} + n_1, n_2]$$

where n_1, n_2 are zero mean Gaussian random variables with variance $\frac{N_0}{2}$. The probability of error $P(e|s_1)$ is

$$\begin{aligned} P(e|s_1) &= P(n-2-n_1 > \sqrt{\frac{A^2T}{2}}) \\ &= \frac{1}{\sqrt{2\pi N_0}} \int_{\frac{A^2T}{2}}^{\infty} e^{-\frac{x^2}{2N_0}} dx = Q\left[\sqrt{\frac{A^2T}{2N_0}}\right] \end{aligned}$$

where we have used the fact the $n = n_2 - n_1$ is a zero-mean Gaussian random variable with variance N_0. Similarly we find that $P(e|s_1) = Q\left[\sqrt{\frac{A^2T}{2N_0}}\right]$, so that

$$P(e) = \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) = Q\left[\sqrt{\frac{A^2T}{2N_0}}\right]$$

4) The signal waveform $\psi_1(\frac{T}{2} - t)$ matched to $\psi_1(t)$ is exactly the same with the signal waveform $\psi_2(T - t)$ matched to $\psi_2(t)$. That is,

$$\psi_1(\frac{T}{2} - t) = \psi_2(T - t) = \psi_1(t) = \begin{cases} \sqrt{\frac{2}{T}}, & 0 \le t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

Thus, the optimal receiver can be implemented by using just one filter followed by a sampler which samples the output of the matched filter at $t = \frac{T}{2}$ and $t = T$ to produce the random variables r_1 and r_2 respectively.

5) If the signal $s_1(t)$ is transmitted, then the received signal $r(t)$ is

$$r(t) = s_1(t) + \frac{1}{2}s_1(t - \frac{T}{2}) + n(t)$$

The output of the sampler at $t = \frac{T}{2}$ and $t = T$ is given by

$$\begin{aligned} r_1 &= A\sqrt{\frac{2}{T}\frac{T}{4}} + \frac{3A}{2}\sqrt{\frac{2}{T}\frac{T}{4}} + n_1 = \frac{5}{2}\sqrt{\frac{A^2T}{8}} + n_1 \\ r_2 &= \frac{A}{2}\sqrt{\frac{2}{T}\frac{T}{4}} + n_2 = \frac{1}{2}\sqrt{\frac{A^2T}{8}} + n_2 \end{aligned}$$

If the optimal receiver uses a threshold V to base its decisions, that is

$$r_1 - r_2 \underset{s_2}{\overset{s_1}{\gtrless}} V$$

then the probability of error $P(e|s_1)$ is

$$P(e|s_1) = P(n_2 - n_1 > 2\sqrt{\frac{A^2T}{8}} - V) = Q\left[2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right]$$

If $s_2(t)$ is transmitted, then

$$r(t) = s_2(t) + \frac{1}{2}s_2(t - \frac{T}{2}) + n(t)$$

The output of the sampler at $t = \frac{T}{2}$ and $t = T$ is given by

$$\begin{aligned} r_1 &= n_1 \\ r_2 &= A\sqrt{\frac{2}{T}\frac{T}{4}} + \frac{3A}{2}\sqrt{\frac{2}{T}\frac{T}{4}} + n_2 \\ &= \frac{5}{2}\sqrt{\frac{A^2T}{8}} + n_2 \end{aligned}$$

The probability of error $P(e|s_2)$ is

$$P(e|s_2) = P(n_1 - n_2 > \frac{5}{2}\sqrt{\frac{A^2T}{8}} + V) = Q\left[\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right]$$

Thus, the average probability of error is given by

$$\begin{aligned} P(e) &= \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) \\ &= \frac{1}{2}Q\left[2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right] + \frac{1}{2}Q\left[\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right] \end{aligned}$$

The optimal value of V can be found by setting $\frac{\vartheta P(e)}{\vartheta V}$ equal to zero. Using Leibnitz rule to differentiate definite integrals, we obtain

$$\frac{\vartheta P(e)}{\vartheta V} = 0 = \left(2\sqrt{\frac{A^2T}{8N_0}} - \frac{V}{\sqrt{N_0}}\right)^2 - \left(\frac{5}{2}\sqrt{\frac{A^2T}{8N_0}} + \frac{V}{\sqrt{N_0}}\right)^2$$

or by solving in terms of V

$$V = -\frac{1}{8}\sqrt{\frac{A^2T}{2}}$$

6) Let a be fixed to some value between 0 and 1. Then, if we argue as in part 5) we obtain

$$\begin{aligned} P(e|s_1,a) &= P(n_2 - n_1 > 2\sqrt{\frac{A^2T}{8}} - V(a)) \\ P(e|s_2,a) &= P(n_1 - n_2 > (a+2)\sqrt{\frac{A^2T}{8}} + V(a)) \end{aligned}$$

and the probability of error is

$$P(e|a) = \frac{1}{2}P(e|s_1,a) + \frac{1}{2}P(e|s_2,a)$$

For a given a, the optimal value of $V(a)$ is found by setting $\frac{\vartheta P(e|a)}{\vartheta V(a)}$ equal to zero. By doing so we find that

$$V(a) = -\frac{a}{4}\sqrt{\frac{A^2T}{2}}$$

The mean square estimation of $V(a)$ is

$$V = \int_0^1 V(a)f(a)da = -\frac{1}{4}\sqrt{\frac{A^2T}{2}}\int_0^1 ada = -\frac{1}{8}\sqrt{\frac{A^2T}{2}}$$

Problem 10.16

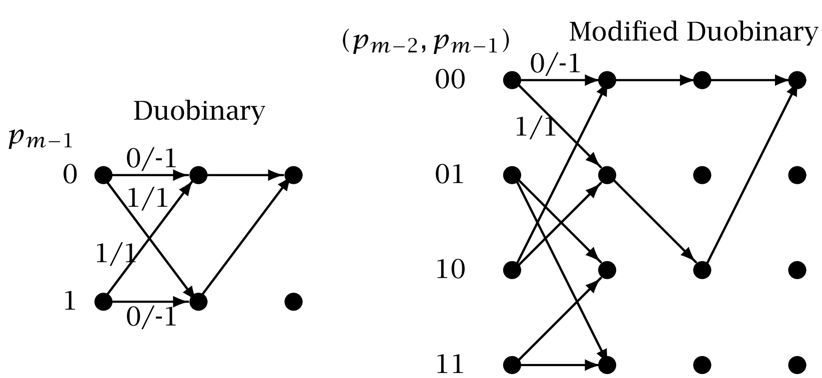
The precoding for the duobinary signaling is given by

$$p_m = d_m \ominus p_{m-1}$$

The corresponding trellis has two states associated with the binary values of the history p_{m-1}. For the modified duobinary signaling the precoding is

$$p_m = d_m \oplus p_{m-2}$$

Hence, the corresponding trellis has four states depending on the values of the pair (p_{m-2}, p_{m-1}). The two trellises are depicted in the next figure. The branches have been labelled as x/y, where x is the binary input data d_m and y is the actual transmitted symbol. Note that the trellis for the modified duobinary signal has more states, but the minimum free distance between the paths is $d_{\text{free}} = 3$, whereas the minimum free distance between paths for the duobinary signal is 2.



Problem 10.17

1) The spectrum of the baseband signal is

$$S_V(f) = \frac{1}{T} S_a(f) |X_{rc}(f)|^2 = \frac{1}{T} |X_{rc}(f)|^2$$

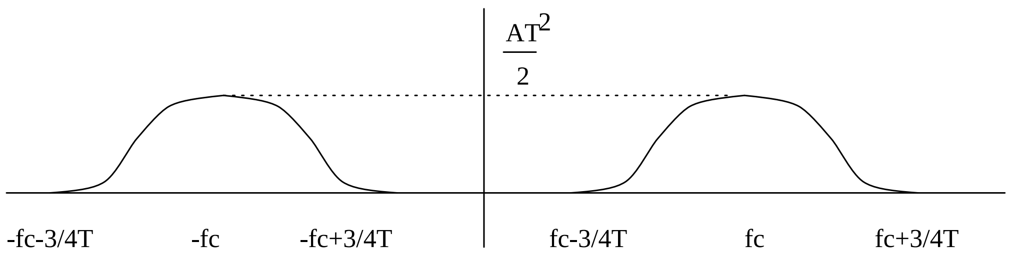
where $T = \frac{1}{2400}$ and

$$X_{rc}(f) = \begin{cases} T & 0 \le |f| \le \frac{1}{4T} \\ \frac{T}{2}(1 + \cos(2\pi T(|f| - \frac{1}{4T})) & \frac{1}{4T} \le |f| \le \frac{3}{4T} \\ 0 & \text{otherwise} \end{cases}$$

If the carrier signal has the form $c(t) = A\cos(2\pi f_c t)$, then the spectrum of the DSB-SC modulated signal, $S_U(f)$, is

$$S_U(f) = \frac{A}{2}[S_V(f - f_c) + S_V(f + f_c)]$$

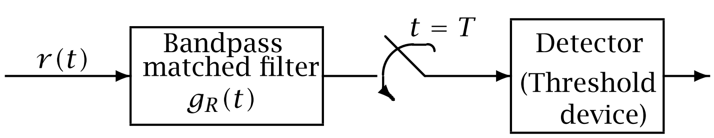
A sketch of $S_U(f)$ is shown in the next figure.



2) Assuming bandpass coherent demodulation using a matched filter, the received signal $r(t)$ is first passed through a linear filter with impulse response

$$g_R(t) = Ax_{rc}(T - t)\cos(2\pi f_c(T - t))$$

The output of the matched filter is sampled at $t = T$ and the samples are passed to the detector. The detector is a simple threshold device that decides if a binary 1 or 0 was transmitted depending on the sign of the input samples. The following figure shows a block diagram of the optimum bandpass coherent demodulator.



Problem 10.18

1) The bandwidth of the bandpass channel is

$$W = 3000 - 600 = 2400 \text{ Hz}$$

Since each symbol of the QPSK constellation conveys 2 bits of information, the symbol rate of transmission is

$$R = \frac{2400}{2} = 1200 \text{ symbols/sec}$$

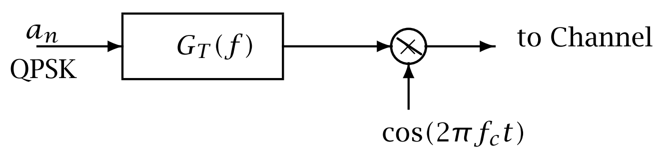
Thus, for spectral shaping we can use a signal pulse with a raised cosine spectrum and roll-off factor $\alpha = 1$, that is

$$X_{rc}(f) = \frac{T}{2}[1 + \cos(\pi T|f|)] = \frac{1}{2400}\cos^2\left(\frac{\pi|f|}{2400}\right)$$

If the desired spectral characteristic is split evenly between the transmitting filter $G_T(f)$ and the receiving filter $G_R(f)$, then

$$G_T(f) = G_R(f) = \sqrt{\frac{1}{1200}}\cos\left(\frac{\pi|f|}{2400}\right), \qquad |f| < \frac{1}{T} = 1200$$

A block diagram of the transmitter is shown in the next figure.



2) If the bit rate is 4800 bps, then the symbol rate is

$$R = \frac{4800}{2} = 2400 \text{ symbols/sec}$$

In order to satisfy the Nyquist criterion, the the signal pulse used for spectral shaping, should have the spectrum

$$X(f) = T\Pi\left(\frac{f}{W}\right)$$

Thus, the frequency response of the transmitting filter is $G_T(f) = \sqrt{T}\Pi\left(\frac{f}{W}\right)$.

Problem 10.19

The bandwidth of the bandpass channel is

$$W = 3300 - 300 = 3000 \text{ Hz}$$

In order to transmit 9600 bps with a symbol rate $R = 2400$ symbols per second, the number of information bits per symbol should be

$$k = \frac{9600}{2400} = 4$$

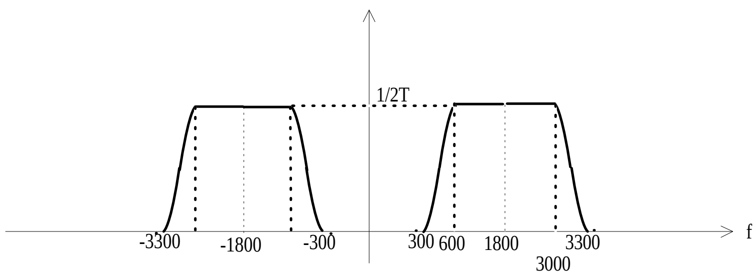
Hence, a $2^4 = 16$ QAM signal constellation is needed. The carrier frequency f_c is set to 1800 Hz, which is the mid-frequency of the frequency band that the bandpass channel occupies. If a pulse with raised cosine spectrum and roll-off factor α is used for spectral shaping, then for the bandpass signal with bandwidth W

$$R = 1200(1+\alpha) = 1500$$

and

$$\alpha = 0.25$$

A sketch of the spectrum of the transmitted signal pulse is shown in the next figure.



Problem 10.20

1) The number of bits per symbol is

$$k = \frac{4800}{R} = \frac{4800}{2400} = 2$$

Thus, a 4-QAM constellation is used for transmission. The probability of error for an M-ary QAM system with $M = 2^k$, is

$$P_M = 1 - \left(1 - 2\left(1 - \frac{1}{\sqrt{M}}\right) Q\left[\sqrt{\frac{3k\mathcal{E}_b}{(M-1)N_0}}\right]\right)^2$$

With $P_M = 10^{-5}$ and $k = 2$ we obtain

$$Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = 5 \times 10^{-6} \Longrightarrow \frac{\mathcal{E}_b}{N_0} = 9.7682$$

2 If the bit rate of transmission is 9600 bps, then

$$k = \frac{9600}{2400} = 4$$

In this case a 16-QAM constellation is used and the probability of error is

$$P_M = 1 - \left(1 - 2\left(1 - \frac{1}{4}\right) Q\left[\sqrt{\frac{3 \times 4 \times \mathcal{E}_b}{15 \times N_0}}\right]\right)^2$$

Thus,

$$Q\left[\sqrt{\frac{3 \times \mathcal{E}_b}{15 \times N_0}}\right] = \frac{1}{3} \times 10^{-5} \Longrightarrow \frac{\mathcal{E}_b}{N_0} = 25.3688$$

3 If the bit rate of transmission is 19200 bps, then

$$k = \frac{19200}{2400} = 8$$

In this case a 256-QAM constellation is used and the probability of error is

$$P_M = 1 - \left(1 - 2\left(1 - \frac{1}{16}\right) Q\left[\sqrt{\frac{3 \times 8 \times \mathcal{E}_b}{255 \times N_0}}\right]\right)^2$$

With $P_M = 10^{-5}$ we obtain

$$\frac{\mathcal{E}_b}{N_0} = 659.8922$$

4) The following table gives the SNR per bit and the corresponding number of bits per symbol for the constellations used in parts a)-c).

k	2	4	8
SNR (db)	9.89	14.04	28.19

As it is observed there is an increase in transmitted power of approximately 3 dB per additional bit per symbol.

Problem 10.21

The channel bandwidth is $W = 4000$ Hz.

(1) Binary PSK with a pulse shape that has $\alpha = \frac{1}{2}$. Hence

$$\frac{1}{2T}(1 + \alpha) = 2000$$

and $\frac{1}{T} = 2667$, the bit rate is 2667 bps.

(2) Four-phase PSK with a pulse shape that has $\alpha = \frac{1}{2}$. From (a) the symbol rate is $\frac{1}{T} = 2667$ and the bit rate is 5334 bps.

(3) $M = 8$ QAM with a pulse shape that has $\alpha = \frac{1}{2}$. From (a), the symbol rate is $\frac{1}{T} = 2667$ and hence the bit rate $\frac{3}{T} = 8001$ bps.

(4) Binary FSK with noncoherent detection. Assuming that the frequency separation between the two frequencies is $\Delta f = \frac{1}{T}$, where $\frac{1}{T}$ is the bit rate, the two frequencies are $f_c + \frac{1}{2T}$ and $f_c - \frac{1}{2T}$. Since $W = 4000$ Hz, we may select $\frac{1}{2T} = 1000$, or, equivalently, $\frac{1}{T} = 2000$. Hence, the bit rate is 2000 bps, and the two FSK signals are orthogonal.

(5) Four FSK with noncoherent detection. In this case we need four frequencies with separation of $\frac{1}{T}$

between adjacent frequencies. We select $f_1 = f_c - \frac{1.5}{T}$, $f_2 = f_c - \frac{1}{2T}$, $f_3 = f_c + \frac{1}{2T}$, and $f_4 = f_c + \frac{1.5}{T}$, where $\frac{1}{2T} = 500$ Hz. Hence, the symbol rate is $\frac{1}{T} = 1000$ symbols per second and since each symbol carries two bits of information, the bit rate is 2000 bps.
(6) $M = 8$ FSK with noncoherent detection. In this case we require eight frequencies with frequency separation of $\frac{1}{T} = 500$ Hz for orthogonality. Since each symbol carries 3 bits of information, the bit rate is 1500 bps.

Problem 10.22

1) The output of the matched filter demodulator is

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} g_T(\tau - kT_b) g_R(t-\tau) d\tau + \nu(t) \\ &= \sum_{k=-\infty}^{\infty} a_k x(t - kT_b) + \nu(t) \end{aligned}$$

where,

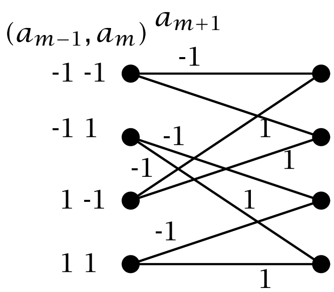
$$x(t) = g_T(t) \star g_R(t) = \frac{\sin\frac{\pi t}{T}}{\frac{\pi t}{T}} \frac{\cos\frac{\pi t}{T}}{1 - 4\frac{t^2}{T^2}}$$

Hence,

$$\begin{aligned} y(mT_b) &= \sum_{k=-\infty}^{\infty} a_k x(mT_b - kT_b) + v(mT_b) \\ &= a_m + \frac{1}{\pi} a_{m-1} + \frac{1}{\pi} a_{m+1} + \nu(mT_b) \end{aligned}$$

The term $\frac{1}{\pi}a_{m-1} + \frac{1}{\pi}a_{m+1}$ represents the ISI introduced by doubling the symbol rate of transmission.

2) In the next figure we show one trellis stage for the ML sequence detector. Since there is postcursor ISI, we delay the received signal, used by the ML decoder to form the metrics, by one sample. Thus, the states of the trellis correspond to the sequence (a_{m-1}, a_m), and the transition labels correspond to the symbol a_{m+1}. Two branches originate from each state. The upper branch is associated with the transmission of -1, whereas the lower branch is associated with the transmission of 1.



Problem 10.23

1) The output of the matched filter at the time instant mT is

$$y_m = \sum_k a_m x_{k-m} + v_m = a_m + \frac{1}{4}a_{m-1} + v_m$$

The autocorrelation function of the noise samples v_m is

$$E[v_k v_j] = \frac{N_0}{2} x_{k-j}$$

Thus, the variance of the noise is

$$\sigma_v^2 = \frac{N_0}{2} x_0 = \frac{N_0}{2}$$

If a symbol by symbol detector is employed and we assume that the symbols $a_m = a_{m-1} = \sqrt{\mathcal{E}_b}$ have been transmitted, then the probability of error $P(e|a_m = a_{m-1} = \sqrt{\mathcal{E}_b})$ is

$$\begin{aligned} P(e|a_m = a_{m-1} = \sqrt{\mathcal{E}_b}) &= P(y_m < 0|a_m = a_{m-1} = \sqrt{\mathcal{E}_b}) \\ &= P(v_m < -\frac{5}{4}\sqrt{\mathcal{E}_b}) = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{-\frac{5}{4}\sqrt{\mathcal{E}_b}} e^{-\frac{v_m^2}{N_0}} dv_m \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{5}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}} e^{-\frac{v^2}{2}} dv = Q\left[\frac{5}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] \end{aligned}$$

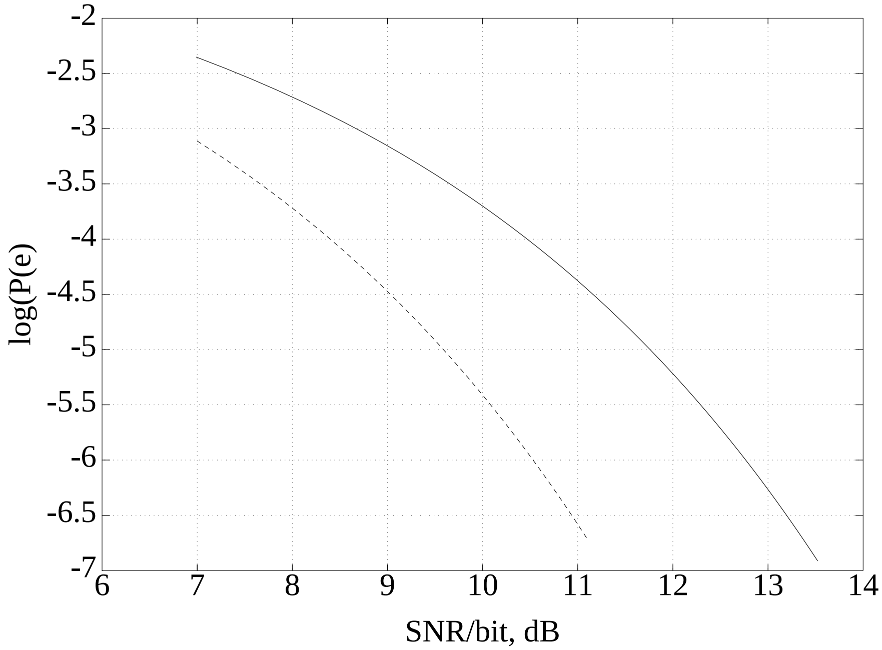
If however $a_{m-1} = -\sqrt{\mathcal{E}_b}$, then

$$P(e|a_m = \sqrt{\mathcal{E}_b}, a_{m-1} = -\sqrt{\mathcal{E}_b}) = P(\frac{3}{4}\sqrt{\mathcal{E}_b} + v_m < 0) = Q\left[\frac{3}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]$$

Since the two symbols $\sqrt{\mathcal{E}_b}$, $-\sqrt{\mathcal{E}_b}$ are used with equal probability, we conclude that

$$\begin{aligned} P(e) &= P(e|a_m = \sqrt{\mathcal{E}_b}) = P(e|a_m = -\sqrt{\mathcal{E}_b}) \\ &= \frac{1}{2}Q\left[\frac{5}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] + \frac{1}{2}Q\left[\frac{3}{4}\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] \end{aligned}$$

2) In the next figure we plot the error probability obtained in part (a) ($\log_{10}(P(e))$) vs. the SNR per bit and the error probability for the case of no ISI. As it observed from the figure, the relative difference in SNR of the error probability of 10^{-6} is 2 dB.



Problem 10.24

The frequency response of the RC filter is

$$C(f) = \frac{\frac{1}{j2\pi RCf}}{R + \frac{1}{j2\pi RCf}} = \frac{1}{1 + j2\pi RCf}$$

The amplitude and the phase spectrum of the filter are

$$|C(f)| = \left(\frac{1}{1 + 4\pi^2 (RC)^2 f^2}\right)^{\frac{1}{2}}, \qquad \Theta_c(f) = \arctan(-2\pi RCf)$$

The envelope delay is

$$T_c(f) = -\frac{1}{2\pi}\frac{d\Theta_c(f)}{df} = -\frac{1}{2\pi}\frac{-2\pi RC}{1 + 4\pi^2 (RC)^2 f^2} = \frac{RC}{1 + 4\pi^2 (RC)^2 f^2}$$

where we have used the formula

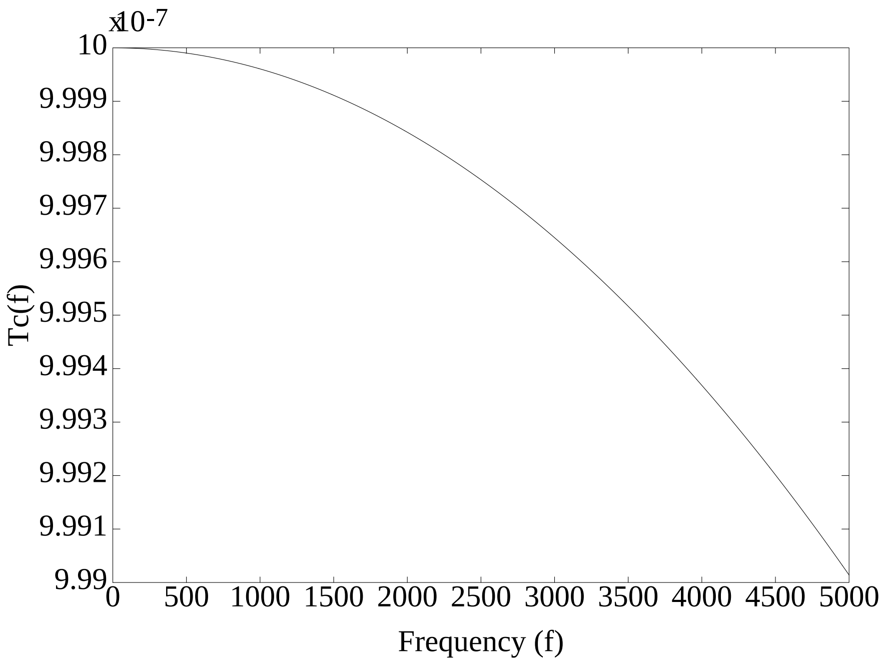
$$\frac{d}{dx}\arctan u = \frac{1}{1+u^2}\frac{du}{dx}$$

Problem 10.25

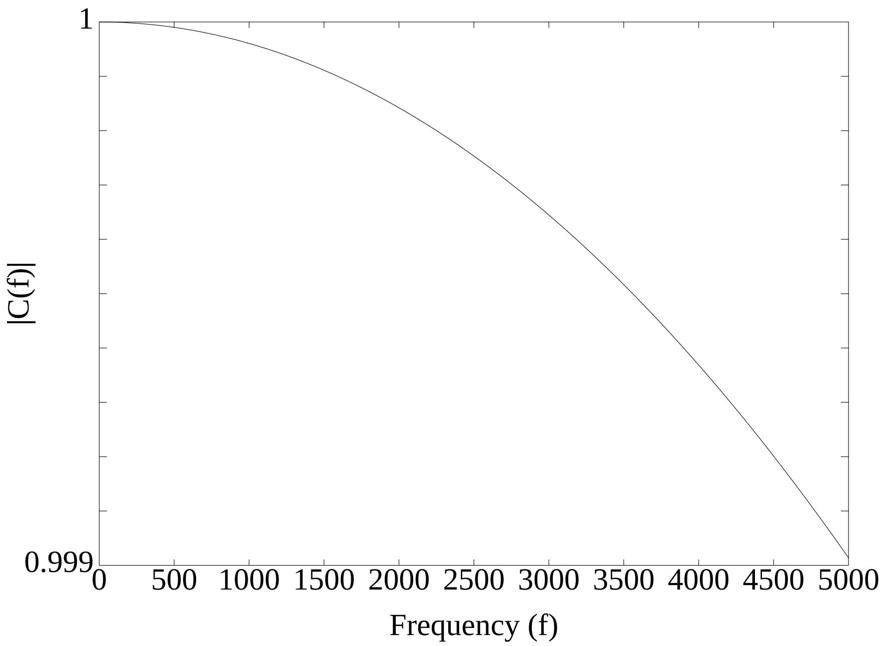
1) The envelope delay of the RC filter is (see Problem 9.19)

$$T_c(f) = \frac{RC}{1 + 4\pi^2 (RC)^2 f^2}$$

A plot of $T(f)$ with $RC = 10^{-6}$ is shown in the next figure



2) The following figure is a plot of the amplitude characteristics of the RC filter, $|C(f)|$. The values of the vertical axis indicate that $|C(f)|$ can be considered constant for frequencies up to 2000 Hz. Since the same is true for the envelope delay, we conclude that a lowpass signal of bandwidth $\Delta f = 1$ KHz will not be distorted if it passes the RC filter.



Problem 10.26

Let $G_T(f)$ and $G_R(f)$ be the frequency response of the transmitting and receiving filter. Then, the condition for zero ISI implies

$$G_T(f)C(f)G_R(f) = X_{rc}(f) = \begin{cases} T & 0 \le |f| \le \frac{1}{4T} \\ \frac{T}{2}[1+\cos(2\pi T(|f|-\frac{1}{T})] & \frac{1}{4T} \le |f| \le \frac{3}{4T} \\ 0 & |f| > \frac{3}{4T} \end{cases}$$

Since the additive noise is white, the optimum transmitting and receiving filter characteristics are given by (see Example 8.6.1)

$$|G_T(f)| = \frac{|X_{rc}(f)|^{\frac{1}{2}}}{|C(f)|^{\frac{1}{2}}}, \qquad |G_R(f)| = \frac{|X_{rc}(f)|^{\frac{1}{2}}}{|C(f)|^{\frac{1}{2}}}$$

Thus,

$$|G_T(f)| = |G_R(f)| = \begin{cases} \left[\frac{T}{1+0.3\cos 2\pi fT}\right]^{\frac{1}{2}} & 0 \le |f| \le \frac{1}{4T} \\ \left[\frac{T(1+\cos(2\pi T(|f|-\frac{1}{T})}{2(1+0.3\cos 2\pi fT)}\right]^{\frac{1}{2}} & \frac{1}{4T} \le |f| \le \frac{3}{4T} \\ 0 & \text{otherwise} \end{cases}$$

Problem 10.27

A 4-PAM modulation can accommodate $k = 2$ bits per transmitted symbol. Thus, the symbol interval duration is

$$T = \frac{k}{9600} = \frac{1}{4800} \text{ sec}$$

Since, the channel's bandwidth is $W = 2400 = \frac{1}{2T}$, in order to achieve the maximum rate of transmission, $R_{\max} = \frac{1}{2T}$, the spectrum of the signal pulse should be

$$X(f) = T\Pi\left(\frac{f}{2W}\right)$$

Then, the magnitude frequency response of the optimum transmitting and receiving filter is (see Example 9.4.1)

$$|G_T(f)| = |G_R(f)| = \left[1 + \left(\frac{f}{2400}\right)^2\right]^{\frac{1}{4}} \Pi\left(\frac{f}{2W}\right) = \begin{cases} \left[1 + \left(\frac{f}{2400}\right)^2\right]^{\frac{1}{4}}, & |f| < 2400 \\ 0 & \text{otherwise} \end{cases}$$

Problem 10.28

1) The equivalent discrete-time impulse response of the channel is

$$h(t) = \sum_{n=-1}^{1} h_n \delta(t - nT) = 0.3\delta(t + T) + 0.9\delta(t) + 0.3\delta(t - T)$$

If by $\{c_n\}$ we denote the coefficients of the FIR equalizer, then the equalized signal is

$$q_m = \sum_{n=-1}^{1} c_n h_{m-n}$$

which in matrix notation is written as

$$\begin{pmatrix} 0.9 & 0.3 & 0. \\ 0.3 & 0.9 & 0.3 \\ 0. & 0.3 & 0.9 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

The coefficients of the zero-force equalizer can be found by solving the previous matrix equation. Thus,

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} -0.4762 \\ 1.4286 \\ -0.4762 \end{pmatrix}$$

2) The values of q_m for $m = \pm 2, \pm 3$ are given by

$$\begin{aligned} q_2 &= \sum_{n=-1}^{1} c_n h_{2-n} = c_1 h_1 = -0.1429 \\ q_{-2} &= \sum_{n=-1}^{1} c_n h_{-2-n} = c_{-1} h_{-1} = -0.1429 \\ q_3 &= \sum_{n=-1}^{1} c_n h_{3-n} = 0 \\ q_{-3} &= \sum_{n=-1}^{1} c_n h_{-3-n} = 0 \end{aligned}$$

Problem 10.29

1) The output of the zero-force equalizer is

$$q_m = \sum_{n=-1}^{1} c_n x_{m_n}$$

With $q_0 = 1$ and $q_m = 0$ for $m \neq 0$, we obtain the system

$$\begin{pmatrix} 1.0 & 0.1 & -0.5 \\ -0.2 & 1.0 & 0.1 \\ 0.05 & -0.2 & 1.0 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Solving the previous system in terms of the equalizer's coefficients, we obtain

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0.000 \\ 0.980 \\ 0.196 \end{pmatrix}$$

2) The output of the equalizer is

$$q_m = \begin{cases} 0 & m \leq -4 \\ c_{-1}x_{-2} = 0 & m = -3 \\ c_{-1}x_{-1} + c_0 x_{-2} = -0.49 & m = -2 \\ 0 & m = -1 \\ 1 & m = 0 \\ 0 & m = 1 \\ c_0 x_2 + x_1 c_1 = 0.0098 & m = 2 \\ c_1 x_2 = 0.0098 & m = 3 \\ 0 & m \geq 4 \end{cases}$$

Hence, the residual ISI sequence is

$$\text{residual ISI} = \{\ldots, 0, -0.49, 0, 0, 0, 0.0098, 0.0098, 0, \ldots\}$$

and its span is 6 symbols.

Problem 10.30

The MSE performance index at the time instant k is

$$J(\mathbf{c}_k) = E\left[\left|\sum_{n=-N}^{N} c_{k,n} y_{k-n} - a_k\right| 62\right]$$

If we define the gradient vector $\mathbf{g}_k$ as

$$\mathbf{g}_k = \frac{\partial J(\mathbf{c}_k)}{2\partial \mathbf{c}_k}$$

then its l^{th} element is

$$\begin{aligned} g_{k,l} = \frac{\partial J(\mathbf{c}_k)}{2\partial c_{k,l}} &= \frac{1}{2} E\left[2\left(\sum_{n=-N}^{N} c_{k,n} y_{k-n} - a_k\right) y_{k-l}\right] \\ &= E\left[-e_k y_{k-l}\right] = -E\left[e_k y_{k-l}\right] \end{aligned}$$

Thus, the vector $\mathbf{g}_k$ is

$$\mathbf{g}_k = \begin{pmatrix} -E[e_k y_{k+N}] \\ \vdots \\ -E[e_k y_{k-N}] \end{pmatrix} = -E[e_k \mathbf{y}_k]$$

where $\mathbf{y}_k$ is the vector $\mathbf{y}_k = [y_{k+N} \cdots y_{k-N}]^T$. Since $\hat{\mathbf{g}}_k = -e_k \mathbf{y}_k$, its expected value is

$$E[\hat{\mathbf{g}}_k] = E[-e_k \mathbf{y}_k] = -E[e_k \mathbf{y}_k] = \mathbf{g}_k$$

Problem 10.31

If $\{c_n\}$ denote the coefficients of the zero-force equalizer and $\{q_m\}$ is the sequence of the equalizer's output samples, then

$$q_m = \sum_{n=-1}^{1} c_n x_{m-n}$$

where $\{x_k\}$ is the noise free response of the matched filter demodulator sampled at $t = kT$. With $q_{-1} = 0$, $q_0 = q_1 = \mathcal{E}_b$, we obtain the system

$$\begin{pmatrix} \mathcal{E}_b & 0.9\mathcal{E}_b & 0.1\mathcal{E}_b \\ 0.9\mathcal{E}_b & \mathcal{E}_b & 0.9\mathcal{E}_b \\ 0.1\mathcal{E}_b & 0.9\mathcal{E}_b & \mathcal{E}_b \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{E}_b \\ \mathcal{E}_b \end{pmatrix}$$

The solution to the system is

$$\begin{pmatrix} c_{-1} & c_0 & c_1 \end{pmatrix} = \begin{pmatrix} 0.2137 & -0.3846 & 1.3248 \end{pmatrix}$$

Problem 10.32

The optimum tap coefficients of the zero-force equalizer can be found by solving the system

$$\begin{pmatrix} 1.0 & 0.3 & 0.0 \\ 0.2 & 1.0 & 0.3 \\ 0.0 & 0.2 & 1.0 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} -0.3409 \\ 1.1364 \\ -0.2273 \end{pmatrix}$$

b) The output of the equalizer is

$$q_m = \begin{cases} 0 & m \leq -3 \\ c_{-1}x_{-1} = -0.1023 & m = -2 \\ 0 & m = -1 \\ 1 & m = 0 \\ 0 & m = 1 \\ c_1 x_1 = -0.0455 & m = 2 \\ 0 & m \geq 3 \end{cases}$$

Hence, the residual ISI sequence is

$$\text{residual ISI} = \{\ldots, 0, -0.1023, 0, 0, 0, -0.0455, 0, \ldots\}$$

Problem 10.33

1) If we assume that the signal pulse has duration T, then the output of the matched filter at the time instant $t = T$ is

$$\begin{aligned} y(T) &= \int_0^T r(\tau)s(\tau)d\tau \\ &= \int_0^T (s(\tau) + \alpha s(\tau - T) + n(\tau))s(\tau)d\tau \\ &= \int_0^T s^2(\tau)d\tau + \int_0^T n(\tau)s(\tau)d\tau \\ &= \mathcal{E}_s + n \end{aligned}$$

where $\mathcal{E}_s$ is the energy of the signal pulse and n is a zero-mean Gaussian random variable with variance $\sigma_n^2 = \frac{N_0\mathcal{E}_s}{2}$. Similarly, the output of the matched filter at $t = 2T$ is

$$\begin{aligned} y(2T) &= \alpha\int_0^T s^2(\tau)d\tau + \int_0^T n(\tau)s(\tau)d\tau \\ &= \alpha\mathcal{E}_s + n \end{aligned}$$

2) If the transmitted sequence is

$$x(t) = \sum_{n=-\infty}^{\infty} a_n s(t - nT)$$

with a_n taking the values $1, -1$ with equal probability, then the output of the demodulator at the time instant $t = kT$ is

$$y_k = a_k\mathcal{E}_s + \alpha a_{k-1}\mathcal{E}_s + n_k$$

The term $\alpha a_{k-1}\mathcal{E}_s$ expresses the ISI due to the signal reflection. If a symbol by symbol detector is employed and the ISI is ignored, then the probability of error is

$$\begin{aligned} P(e) &= \frac{1}{2}P(\text{error}|a_n = 1, a_{n-1} = 1) + \frac{1}{2}P(\text{error}|a_n = 1, a_{n-1} = -1) \\ &= \frac{1}{2}P((1+\alpha)\mathcal{E}_s + n_k < 0) + \frac{1}{2}P((1-\alpha)\mathcal{E}_s + n_k < 0) \\ &= \frac{1}{2}Q\left[\sqrt{\frac{2(1+\alpha)^2\mathcal{E}_s}{N_0}}\right] + \frac{1}{2}Q\left[\sqrt{\frac{2(1-\alpha)^2\mathcal{E}_s}{N_0}}\right] \end{aligned}$$

3) To find the error rate performance of the DFE, we assume that the estimation of the parameter α is correct and that the probability of error at each time instant is the same. Since the transmitted symbols are equiprobable, we obtain

$$\begin{aligned} P(e) &= P(\text{error at } k|a_k = 1) \\ &= P(\text{error at } k-1)P(\text{error at } k|a_k = 1, \text{error at } k-1) \\ &\quad +P(\text{no error at } k-1)P(\text{error at } k|a_k = 1, \text{no error at } k-1) \\ &= P(e)P(\text{error at } k|a_k = 1, \text{error at } k-1) \\ &\quad +(1-P(e))P(\text{error at } k|a_k = 1, \text{no error at } k-1) \\ &= P(e)p + (1-P(e))q \end{aligned}$$

where

$$\begin{aligned} p &= P(\text{error at } k|a_k = 1, \text{error at } k-1) \\ &= \frac{1}{2}P(\text{error at } k|a_k = 1, a_{k-1} = 1, \text{error at } k-1) \\ &\quad + \frac{1}{2}P(\text{error at } k|a_k = 1, a_{k-1} = -1, \text{error at } k-1) \\ &= \frac{1}{2}P((1+2\alpha)\mathcal{E}_s + n_k < 0) + \frac{1}{2}P((1-2\alpha)\mathcal{E}_s + n_k < 0) \\ &= \frac{1}{2}Q\left[\sqrt{\frac{2(1+2\alpha)^2\mathcal{E}_s}{N_0}}\right] + \frac{1}{2}Q\left[\sqrt{\frac{2(1-2\alpha)^2\mathcal{E}_s}{N_0}}\right] \end{aligned}$$

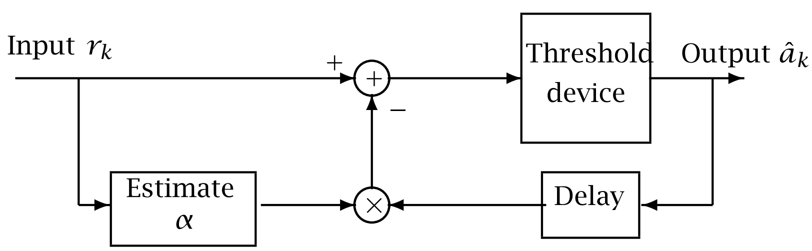
and

$$\begin{aligned} q &= P(\text{error at } k|a_k = 1, \text{no error at } k-1) \\ &= P(\mathcal{E}_s + n_k < 0) = Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right] \end{aligned}$$

Solving for $P(e)$, we obtain

$$P(e) = \frac{q}{1-p+q} = \frac{Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]}{1 - \frac{1}{2}Q\left[\sqrt{\frac{2(1+2\alpha)^2\mathcal{E}_s}{N_0}}\right] - \frac{1}{2}Q\left[\sqrt{\frac{2(1-2\alpha)^2\mathcal{E}_s}{N_0}}\right] + Q\left[\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]}$$

A sketch of the detector structure is shown in the next figure.



Problem 10.34

A discrete time transversal filter equivalent to the cascade of the transmitting filter $g_T(t)$, the channel $c(t)$, the matched filter at the receiver $g_R(t)$ and the sampler, has tap gain coefficients $\{y_m\}$, where

$$y_m = \begin{cases} 0.9 & m = 0 \\ 0.3 & m = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

The noise ν_k, at the output of the sampler, is a zero-mean Gaussian sequence with autocorrelation function

$$E[\nu_k \nu_l] = \sigma^2 y_{k-l}, \qquad |k-l| \leq 1$$

If the $\mathcal{Z}$-transform of the sequence $\{y_m\}$, $Y(z)$, assumes the factorization

$$Y(z) = F(z)F^*(z^{-1})$$

then the filter $1/F^*(z^{-1})$ can follow the sampler to white the noise sequence ν_k. In this case the output of the whitening filter, and input to the MSE equalizer, is the sequence

$$u_n = \sum_k a_k f_{n-k} + n_k$$

where n_k is zero mean Gaussian with variance σ^2. The optimum coefficients of the MSE equalizer, c_k, satisfy (see (9.4.32))

$$\sum_{n=-1}^{1} c_n R_u(n-k) = R_{ua}(k), \qquad k = 0, \pm 1$$

where

$$\begin{aligned} R_u(n-k) &= E[u_{l-k}u_{l-n}] = \sum_{m=0}^{1} f_m f_{m+n-k} + \sigma^2 \delta_{n,k} \\ &= \begin{cases} y_{n-k} + \sigma^2 \delta_{n,k}, & |n-k| \le 1 \\ 0 & \text{otherwise} \end{cases} \\ R_{ua}(k) &= E[a_n u_{n-k}] = \begin{cases} f_{-k}, & -1 \le k \le 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

With

$$Y(z) = 0.3z + 0.9 + 0.3z^{-1} = (f_0 + f_1 z^{-1})(f_0 + f_1 z)$$

we obtain the parameters f_0 and f_1 as

$$f_0 = \begin{cases} \pm\sqrt{0.7854} \\ \pm\sqrt{0.1146} \end{cases}, \qquad f_1 = \begin{cases} \pm\sqrt{0.1146} \\ \pm\sqrt{0.7854} \end{cases}$$

The parameters f_0 and f_1 should have the same sign since $f_0 f_1 = 0.3$. However, the sign itself does not play any role if the data are differentially encoded. To have a stable inverse system $1/F^*(z^{-1})$, we select f_0 and f_1 in such a way that the zero of the system $F^*(z^{-1}) = f_0 + f_1 z$ is inside the unit circle. Thus, we choose $f_0 = \sqrt{0.1146}$ and $f_1 = \sqrt{0.7854}$ and therefore, the desired system for the equalizer's coefficients is

$$\begin{pmatrix} 0.9 + 0.1 & 0.3 & 0.0 \\ 0.3 & 0.9 + 0.1 & 0.3 \\ 0.0 & 0.3 & 0.9 + 0.1 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \sqrt{0.7854} \\ \sqrt{0.1146} \\ 0 \end{pmatrix}$$

Solving this system, we obtain

$$c_{-1} = 0.8596, \quad c_0 = 0.0886, \quad c_1 = -0.0266$$

Problem 10.35

The power spectral density of the noise at the output of the matched filter is

$$S_\nu(f) = S_n(f)|G_R(f)|^2 = \frac{N_0}{2}|X(f)| = \frac{N_0}{2}\frac{1}{W}\cos(\frac{\pi f}{2W})$$

Hence, the autocorrelation function of the output noise is

$$\begin{aligned}
R_\nu(\tau) &= \mathcal{F}^{-1}[S_\nu(f)] = \frac{N_0}{2}\int_{-\infty}^{\infty}\frac{1}{W}\cos(\frac{\pi f}{2W})e^{j2\pi f\tau}df \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty}\frac{1}{W}\cos(\frac{\pi f}{2W})e^{-j\frac{\pi f}{2W}}e^{j2\pi f(\tau+\frac{1}{4W})}df \\
&= \frac{N_0}{2}\int_{-\infty}^{\infty}X(f)e^{j2\pi f(\tau+\frac{1}{4W})}df \\
&= \frac{N_0}{2}x(\tau+\frac{1}{4W})
\end{aligned}$$

and therefore,

$$\begin{aligned}
R_\nu(0) &= \frac{N_0}{2}x(\frac{1}{4W}) = \frac{N_0}{2}\left(\text{sinc}(\frac{1}{2}) + \text{sinc}(-\frac{1}{2})\right) = \frac{2N_0}{\pi} \\
R_\nu(T) &= R_\nu(\frac{1}{2W}) = \frac{N_0}{2}\left(\text{sinc}(\frac{3}{2}) + \text{sinc}(\frac{1}{2})\right) = \frac{2N_0}{3\pi}
\end{aligned}$$

Since the noise is of zero mean, the covariance matrix of the noise is given by

$$\mathbf{C} = \begin{pmatrix} R_\nu(0) & R_\nu(T) \\ R_\nu(T) & R_\nu(0) \end{pmatrix} = \frac{2N_0}{\pi}\begin{pmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & 1 \end{pmatrix}$$

Problem 10.36

a) Each segment of the wire-line can be considered as a bandpass filter with bandwidth $W = 1200$ Hz. Thus, the highest bit rate that can be transmitted without ISI by means of binary PAM is

$$R = 2W = 2400 \text{ bps}$$

b) The probability of error for binary PAM transmission is

$$P_2 = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right]$$

Hence, using mathematical tables for the function $Q[\cdot]$, we find that $P_2 = 10^{-7}$ is obtained for

$$\sqrt{\frac{2\mathcal{E}_b}{N_0}} = 5.2 \Longrightarrow \frac{\mathcal{E}_b}{N_0} = 13.52 = 11.30 \text{ dB}$$

c) The received power P_R is related to the desired SNR per bit through the relation

$$\frac{P_R}{N_0} = R\frac{\mathcal{E}_b}{N_0}$$

Hence, with $N_0 = 4.1 \times 10^{-21}$ we obtain

$$P_R = 4.1 \times 10^{-21} \times 1200 \times 13.52 = 6.6518 \times 10^{-17} = -161.77 \text{ dBW}$$

Since the power loss of each segment is

$$L_s = 50 \text{ Km } \times 1 \text{ dB/Km } = 50 \text{ dB}$$

the transmitted power at each repeater should be

$$P_T = P_R + L_s = -161.77 + 50 = -111.77 \text{ dBW}$$

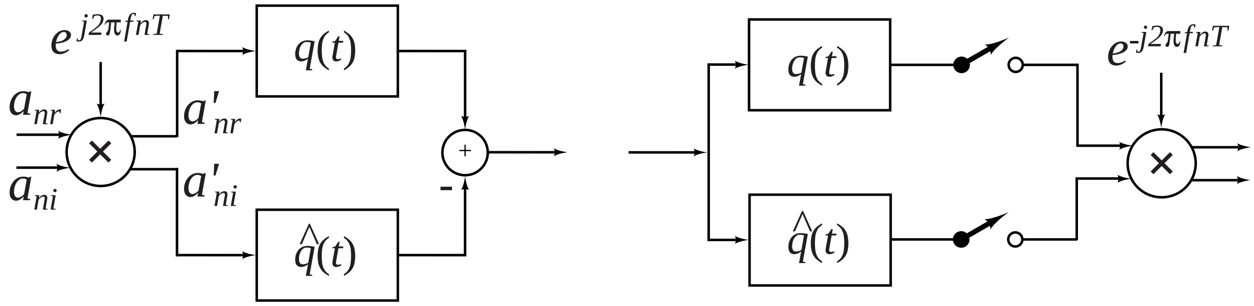
Problem 10.37

1) The alternative expression for $s(t)$ can be rewritten as

$$\begin{aligned}
s(t) &= \text{Re}\left[\sum_n a'_n Q(t-nT)\right] \\
&= \text{Re}\left[\sum_n a_n e^{j2\pi f_c nT} g(t-nT)\left[\cos 2\pi f_c(t-nT) + j\sin 2\pi f_c(t-nT)\right]\right] \\
&= \text{Re}\left[\sum_n a_n g(t-nT)\left[\cos 2\pi f_c nT + j\sin 2\pi f_c nT\right]\left[\cos 2\pi f_c(t-nT) + j\sin 2\pi f_c(t-nT)\right]\right] \\
&= \text{Re}\left[\sum_n a_n g(t-nT)\left[\cos 2\pi f_c nT \cos 2\pi f_c(t-nT) - \sin 2\pi f_c nT \sin 2\pi f_c(t-nT)\right.\right. \\
&\qquad \left.\left. + j\sin 2\pi f_c nT \cos 2\pi f_c(t-nT) + j\cos 2\pi f_c nT \sin 2\pi f_c(t-nT)\right]\right] \\
&= \text{Re}\left[\sum_n a_n g(t-nT)\left[\cos 2\pi f_c t + j\sin 2\pi f_c t\right]\right] \\
&= \text{Re}\left[\sum_n a_n g(t-nT) e^{j2\pi f_c t}\right] \\
&= s(t)
\end{aligned}$$

so indeed the alternative expression for $s(t)$ is a valid one.

2)



Modulator (with phase rotator)

Demodulator (with phase derotator)

Problem 10.38

1) The impulse response of the pulse having a square-root raised cosine characteristic, is an even function, i.e., $x_{SQ}(t) = x_{SQ}(-t)$, i.e., the pulse $g(t)$ is an even function. We know that the product of an even function times an even function is an even function, while the product of an even function times an odd function is an odd function. Hence $q(t)$ is even while $\hat{q}(t)$ is odd and their product $q(t)\hat{q}(t)$ has odd symmetry. Therefore,

$$\int_{-\infty}^{\infty} q(t)\hat{q}(t)\,dt = \int_{-(1+\beta)/2T}^{(1+\beta)/2T} q(t)\hat{q}(t)\,dt = 0$$

2) We notice that when $f_c = k/T$, where k is an integer, then the rotator/derotator of a carrierless QAM system (described in Problem 10.22) gives a trivial rotation of an integer number of full circles ($2\pi kn$), and the carrierless QAM/PSK is equivalent to CAP.

Computer Problems

Computer Problem 10.1

The impulse response and the frequency response of a length $N = 41$ FIR filter that meets these specifications is illustrated in Figures 10.1, 10.2, and 10.3 . Since N is odd, the delay through the filter is $(N+1)/2$ taps, which corresponds to a time delay of $(N+1)/20$ ms at the sampling rate of $F_s = 10$ KHz. In this example, the FIR filter was designed in MATLAB using the Chebyshev approximation method (Remez algorithm).

The MATLAB scripts for this problem are given next.

```
% MATLAB script for Computer Problem 10.1.
```

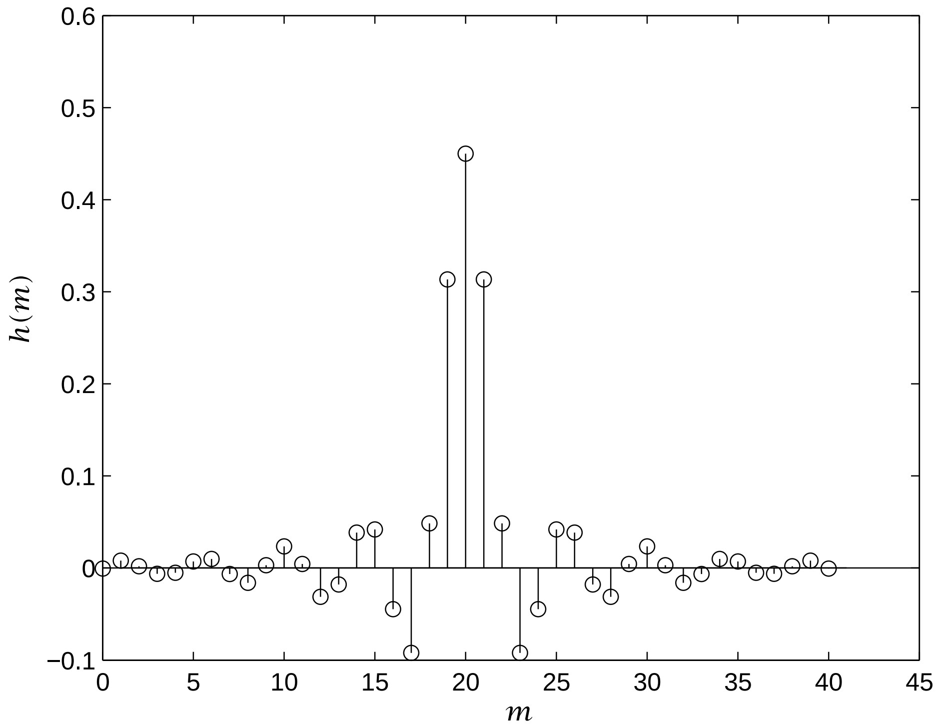


Figure 10.1: Impulse response of linear phase FIR filter in Computer Problem 10.1

```
echo on
f_cutoff=2000;            % the desired cutoff frequency
f_stopband=2500;          % the actual stopband frequency
fs=10000;                 % the sampling frequency
f1=2*f_cutoff/fs;         % the normalized passband frequency
f2=2*f_stopband/fs;       % the normalized stopband frequency
N=41;                     % This number is found by experiment.
F=[0 f1 f2 1];
M=[1 1 0 0];              % describes the lowpass filter
B=remez(N−1,F,M);         % returns the FIR tap coefficients
% Plotting command follows.
figure(1);
[H,W]=freqz(B);
H_in_dB=20*log10(abs(H));
plot(W/(2*pi),H_in_dB);
figure(2);
plot(W/(2*pi),(180/pi)*unwrap(angle(H)));
% Plot of the impulse response follows.
figure(3);
plot(zeros(size([0:N−1])));
hold;
stem([0:N−1],B);
```

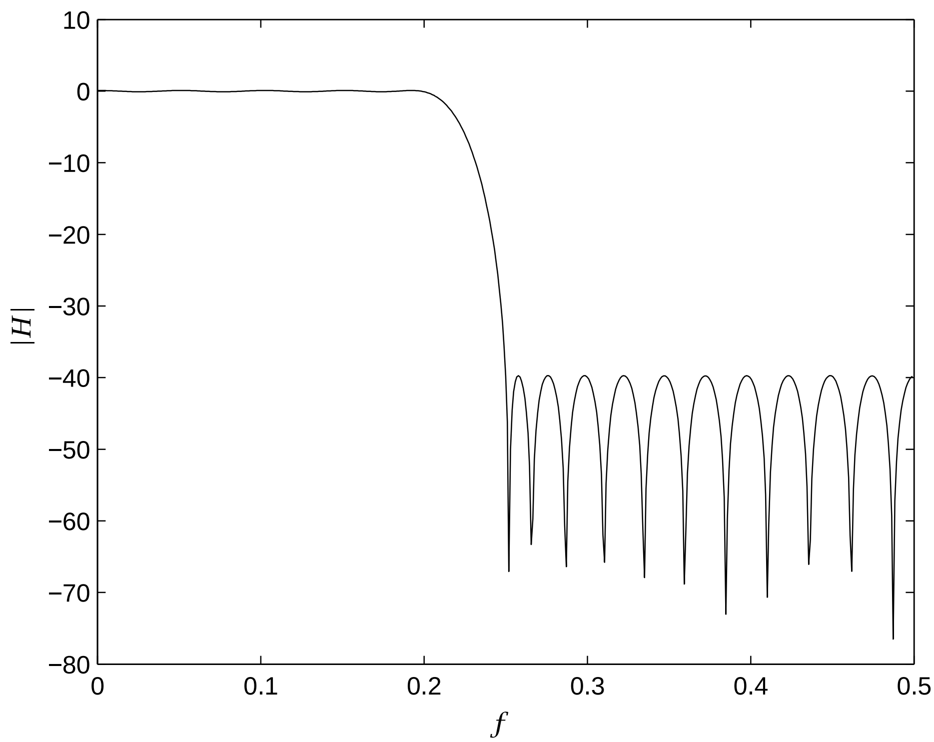


Figure 10.2: Frequency response of linear phase FIR filter in Computer Problem 10.1

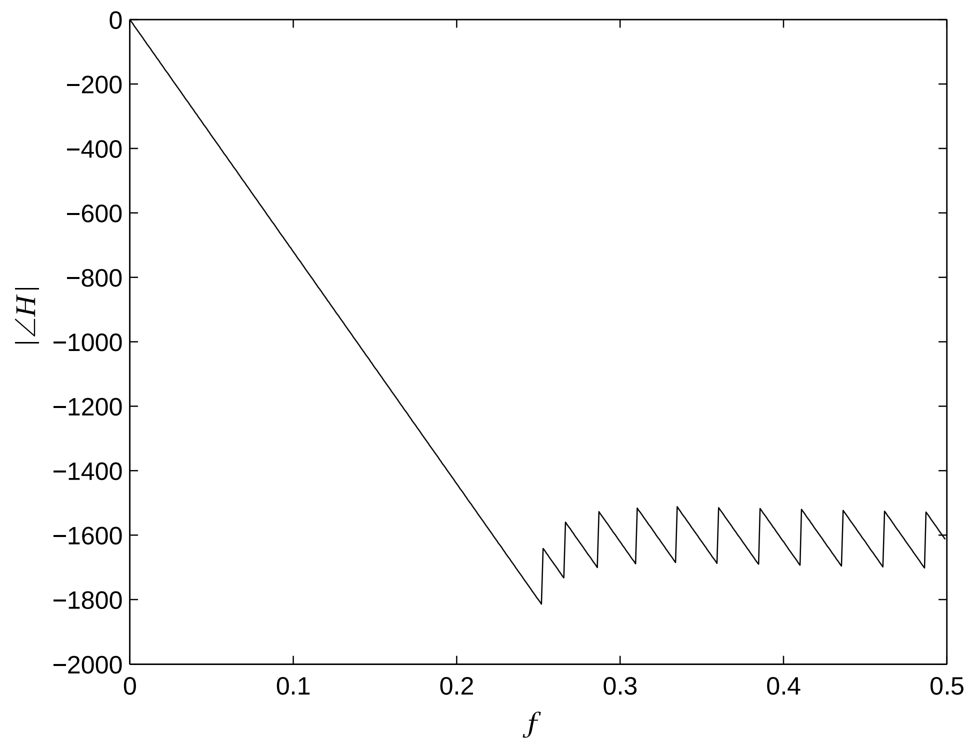


Figure 10.3: Phase response of linear phase FIR filter in Computer Problem 10.1

Computer Problem 10.2

For Channel 1, the received signal sequence $\{y_n\}$ in the absence of noise is shown in Figure 10.4(a), and with additive white Gaussian noise having a variance of $\sigma^2 = 0.1$, the received signal sequence is shown in Figure 10.4(b). We note that in the absence of noise, the ISI alone does not cause errors at the detector that compares the received signal sequence $\{y_n\}$ with the threshold set to zero. Hence, the eye diagram is open in the absence of noise. However, when the additive noise is sufficiently large, errors will occur.

In the case of Channel 2, the noise-free and noisy ($\sigma^2 = 0.1$) sequence $\{y_n\}$ is illustrated in Figure 10.5. Now, we observe that the ISI can cause errors at the detector that compares the received sequence $\{y_n\}$ with the threshold set at zero, even in the absence of noise. Thus, for this channel characteristic, the eye is completely closed.

The MATLAB scripts for this problem are given next.

```
% MATLAB script for Computer Problem 10.2

% channel 1
x_ch1 = [ 0.1 -0.25 1 -0.25 0.1];
% channel 2
x_ch2 = [-0.2 0.5 1  0.5 -0.2];

noise_var=0.1;
sigma=sqrt(noise_var);

for ak=-1:2:1
    for i=0:15
        a = dec2bin(i,4);
        %Output of channel 1
        yk1(i+1) = ak + x_ch1(4)*(2*str2num(a(1))-1) + x_ch1(2)*(2*str2num(a(2))-1) + ...
            x_ch1(5)*(2*str2num(a(3))-1) + x_ch1(1) * (2*str2num(a(4))-1);
        yk1_noise(i+1) = yk1(i+1) + gngauss(sigma);
        %Output of channel 2
        yk2(i+1) = ak + x_ch2(4)*(2*str2num(a(1))-1) + x_ch2(2)*(2*str2num(a(2))-1) + ...
            x_ch2(5)*(2*str2num(a(3))-1) + x_ch2(1) * (2*str2num(a(4))-1);
        yk2_noise(i+1) = yk2(i+1) + gngauss(sigma);
    end
    if (ak==-1)
        figure(1);
        plot(yk1, 0, 'x')
        axis([-2 2 0 1]);
        hold on;
        figure(2);
        plot(yk1_noise, 0, 'x')
        axis([-2 2 0 1]);
        hold on;
        figure(3);
        plot(yk2, 0, 'x')
        axis([-2 2 0 1]);
        hold on;
        figure(4);
        plot(yk2_noise, 0, 'x')
        axis([-2 2 0 1]);
```

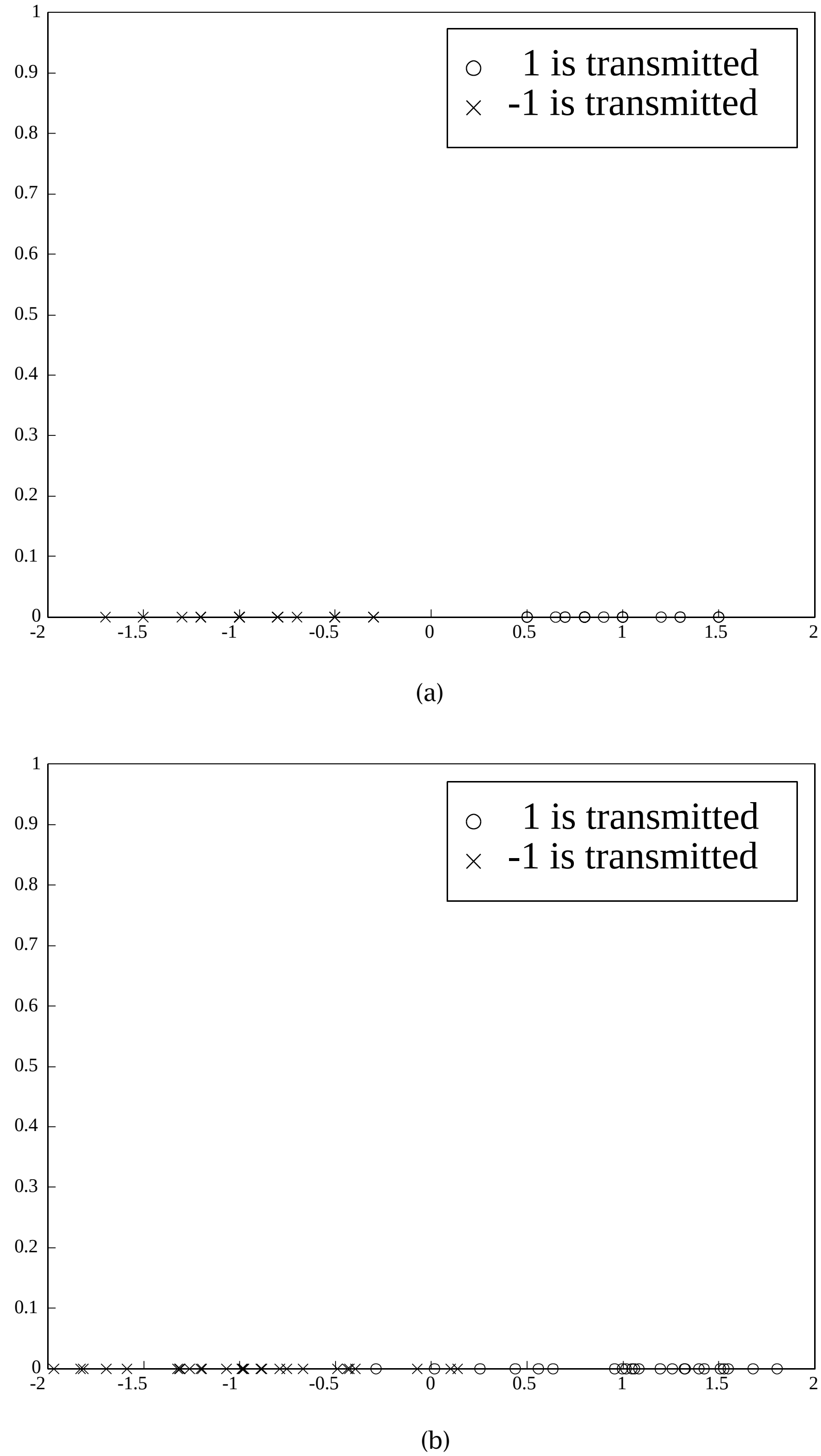
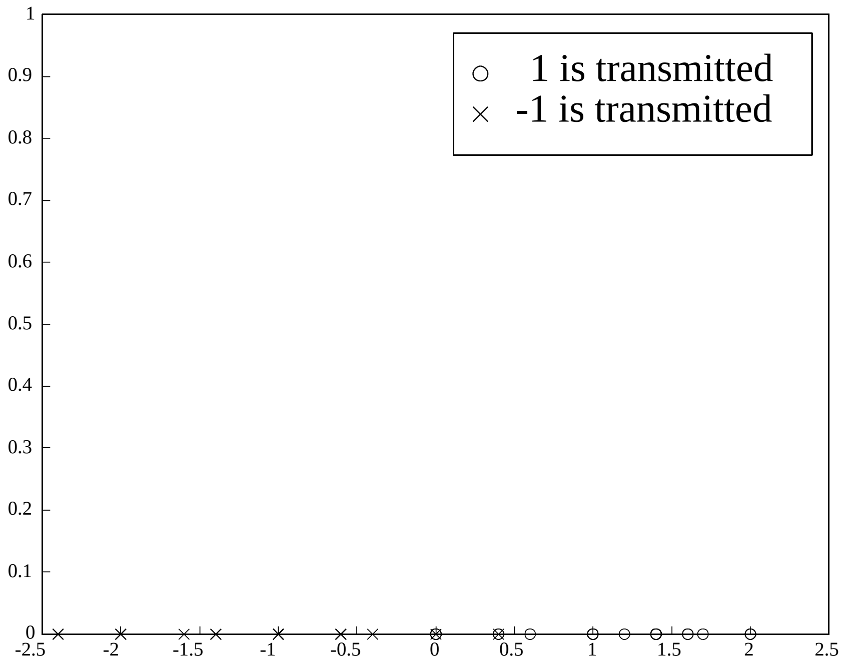
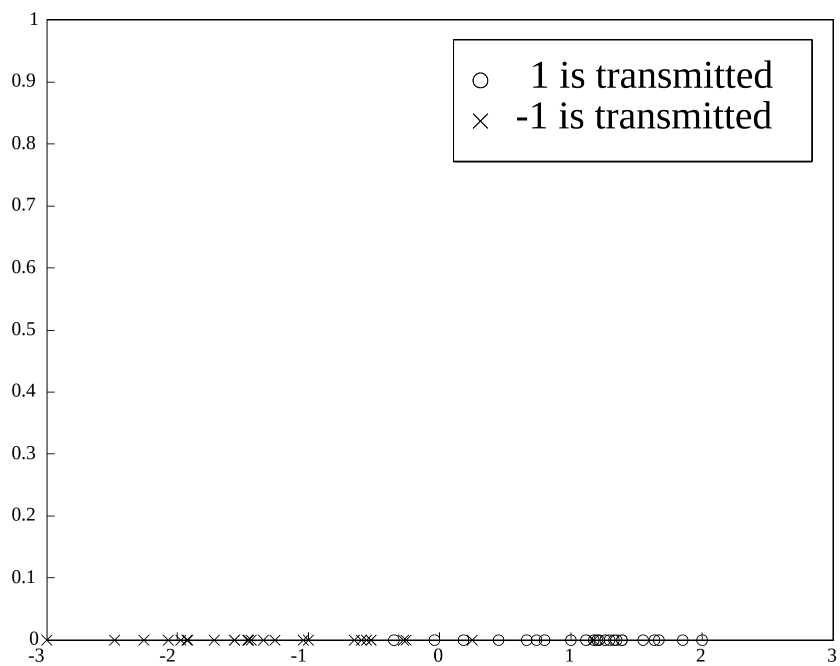


Figure 10.4: Output of channel model 1 without and with AWGN. (a) No noise. (b) Additive Gaussian noise with $\sigma^2 = 0.1$



(a)



(b)

Figure 10.5: Output of channel model 2 without and with AWGN. (a) No noise. (b) Additive Gaussian noise with variance $\sigma^2 = 0.1$

```
        hold on;
    else
        figure(1);
        plot(yk1, 0, 'o')
        axis([-2 2 0 1]);
        figure(2);
        plot(yk1_noise, 0, 'o')
        axis([-2 2 0 1]);
        figure(3);
        plot(yk2, 0, 'o')
        axis([-2 2 0 1]);
        figure(4);
        plot(yk2_noise, 0, 'o')
        axis([-2 2 0 1]);
    end
end
```

Computer Problem 10.3

Figure 10.6 illustrates the impulse response of the transmitter filter $g_T(n-\frac{N-1}{2})$, $n = 0, 1, ..., N-1$ for $\alpha = \frac{1}{4}$ and $N = 31$. The corresponding frequency response characteristics are shown in Figure 10.7. Note that the frequency response is no longer zero for $|f| \geq (1+\alpha)/T$, because the digital filter has finite duration. However, the sidelobes in the spectrum are relatively small. Further reduction in the sidelobes may be achieved by increasing N.

Also in Figure 10.8, we compare the $|G_T(f)|^2$ and $X_{rc}(f)$. Finally we repeat the first three parts using $N = 41$. Figure 10.9 illustrates the impulse response of the transmitter filter $g_T(n-\frac{N-1}{2})$ for $N = 41$. The corresponding frequency response characteristics are shown in Figure 10.10. Note that the sidelobes in the spectrum became smaller by increasing N from $N = 31$ to $N = 41$. In Figure 10.11, we compare the $|G_T(f)|^2$ and $X_{rc}(f)$ and in Figure 10.12, we compare the frequency response characteristics of the filter for different values of N.

The MATLAB script is given next.

```
% MATLAB script for Computer Problem 10.3.
echo on
N=31;
T=1/1800;
alpha=1/4;
n=-(N-1)/2:(N-1)/2;                    % the indices for g_T
% The expression for g_T is obtained next.
for i=1:length(n),
  g_T(i)=0;
  for m=-(N-1)/2:(N-1)/2,
    g_T(i)=g_T(i)+sqrt(xrc(4*m/(N*T),alpha,T))*exp(j*2*pi*m*n(i)/N);
    echo off ;
  end;
end;
echo on ;
g_T=real(g_T) ; % The imaginary part is due to the finite machine precision
% Derive g_T(n-(N-1)/2).
```

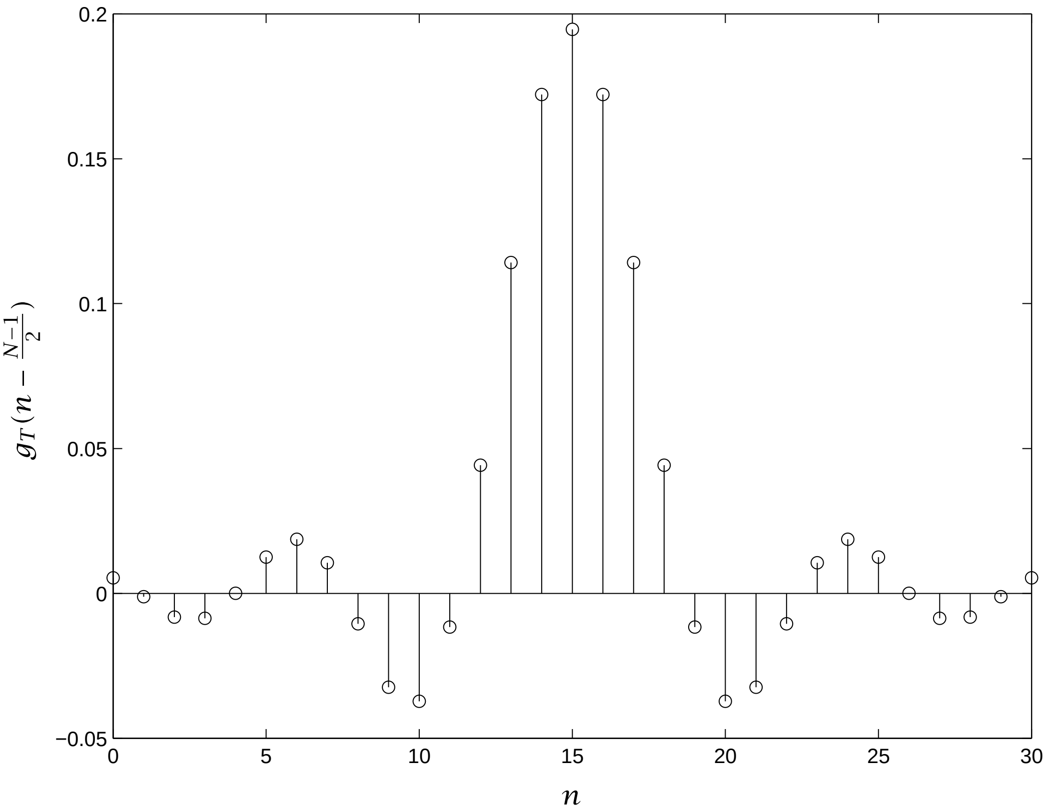


Figure 10.6: Impulse response of truncated discrete-time FIR filter at transmitter

```
n2=0:N−1;
% Get the frequency response characteristics.
[G_T,W]=freqz(g_T,1);
% normalized magnitude response
magG_T_in_dB=20*log10(abs(G_T)/max(abs(G_T)));
G_T2_in_dB=20*log10(abs(G_T).*abs(G_T)/max(abs(G_T).*abs(G_T)));

for l=1:length(W),
    X_rc(l) = xrc(W(l)/(2*T), alpha, T);
end;
X_rc_in_dB = 20*log10(X_rc/T);

% impulse response of the cascade of the transmitter and the receiver filters
g_R=g_T;
imp_resp_of_cascade=conv(g_R,g_T);
% Plotting commands follow.
```

Computer Problem 10.4

Figure 10.13 illustrates $g_T(n - \frac{N-1}{2})$, $n = 0, 1, ..., N-1$ for $W = 1800$ and $N = 31$. The corresponding

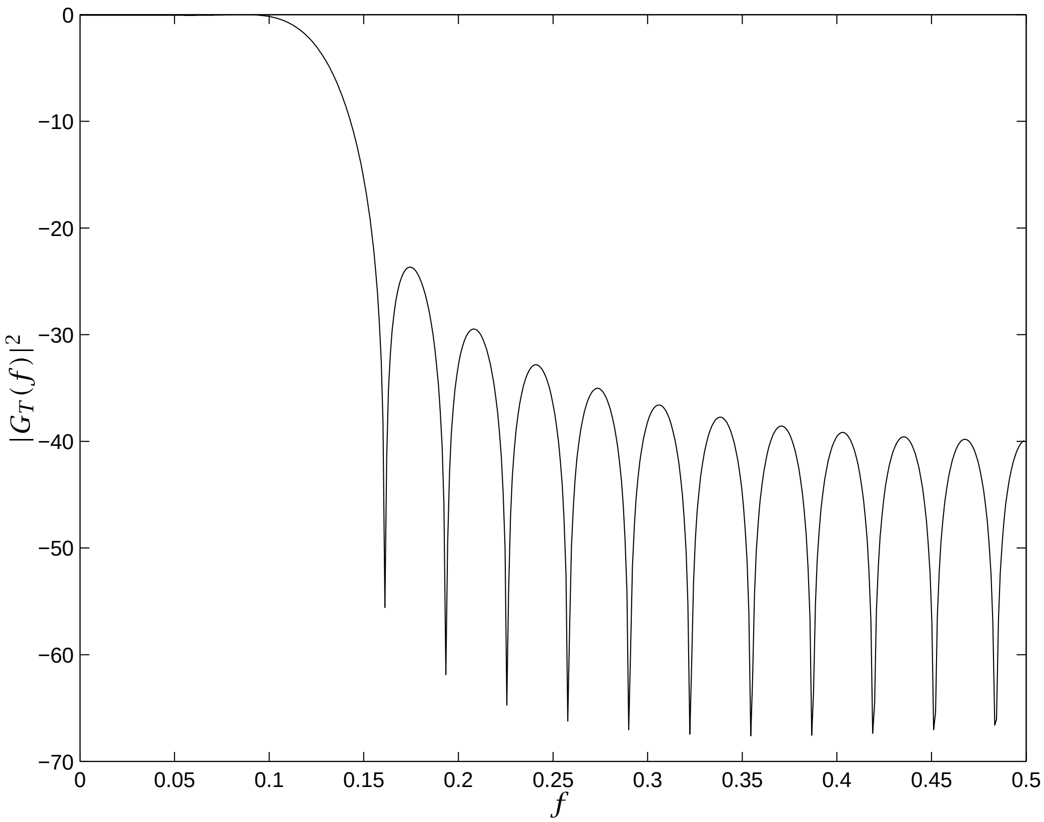


Figure 10.7: Frequency response of truncated discrete-time FIR filter at transmitter

frequency response characteristic is shown in Figure 10.14. Finally in Figure 10.15, we show the impulse response of the cascade of the transmitter and receiver FIR filters. This impulse response may be compared with the ideal impulse response obtained by sampling $x(t)$ at a rate $F_s = 4/T = 8W$.

Note that the frequency response is no longer zero for $|f| \geq W$ because the digital filter has finite duration. However, the sidelobes in the spectrum are relatively small.

The MATLAB script is given next.

```
% MATLAB script for Computer Problem 10.4.
echo on
N=31;
W=1800;
T=1/(2*W);
n=-(N-1)/2:(N-1)/2;                 % the indices for g_T
% The expression for g_T is obtained next.
for i=1:length(n),
  g_T(i)=0;
  for m=-(N-1)/2:(N-1)/2,
    if ( abs((4*m)/(N*T)) <= W ),
      g_T(i)=g_T(i)+sqrt((1/W)*cos((2*pi*m)/(N*T*W)))*exp(j*2*pi*m*n(i)/N);
    end;
    echo off ;
```

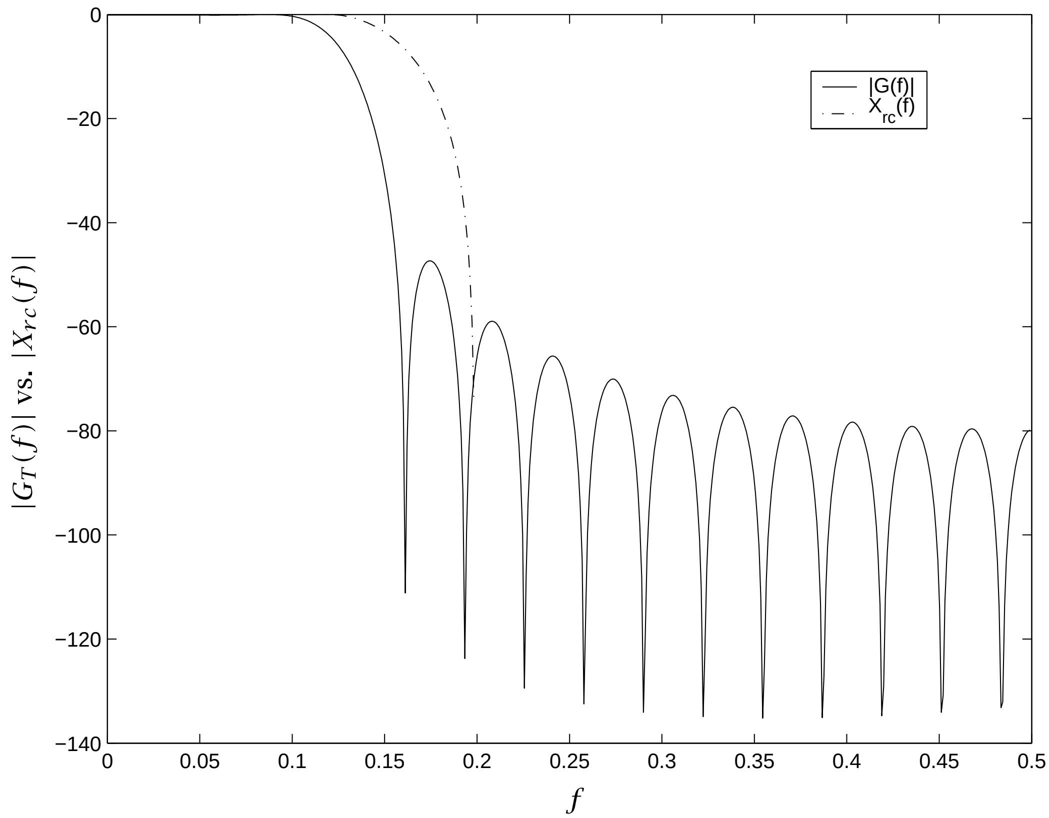


Figure 10.8: $|G_T(f)|^2$ and $X_{rc}(f)$

```
  end;
end;
echo on ;
g_T=real(g_T) ; % The imaginary part is due to the finite machine precision
% Obtain g_T(n-(N-1)/2).
n2=0:N-1;
% Obtain the frequency response characteristics.
[G_T,Wf]=freqz(g_T,1);
% normalized magnitude response
magG_T_in_dB=20*log10(abs(G_T)/max(abs(G_T)));
G_T2 = abs(G_T).*abs(G_T);
G_T2_in_dB = 20*log10(G_T2/max(G_T2));
for m=1:length(Wf),
    f=4*W*m/length(Wf);
    if (f<W)
        X(m) = (1/W)*cos(pi*f/(2*W));
    else
        X(m)=0;
    end
end;
X=X/max(abs(X));
X= 20.* log10(abs(X));
```

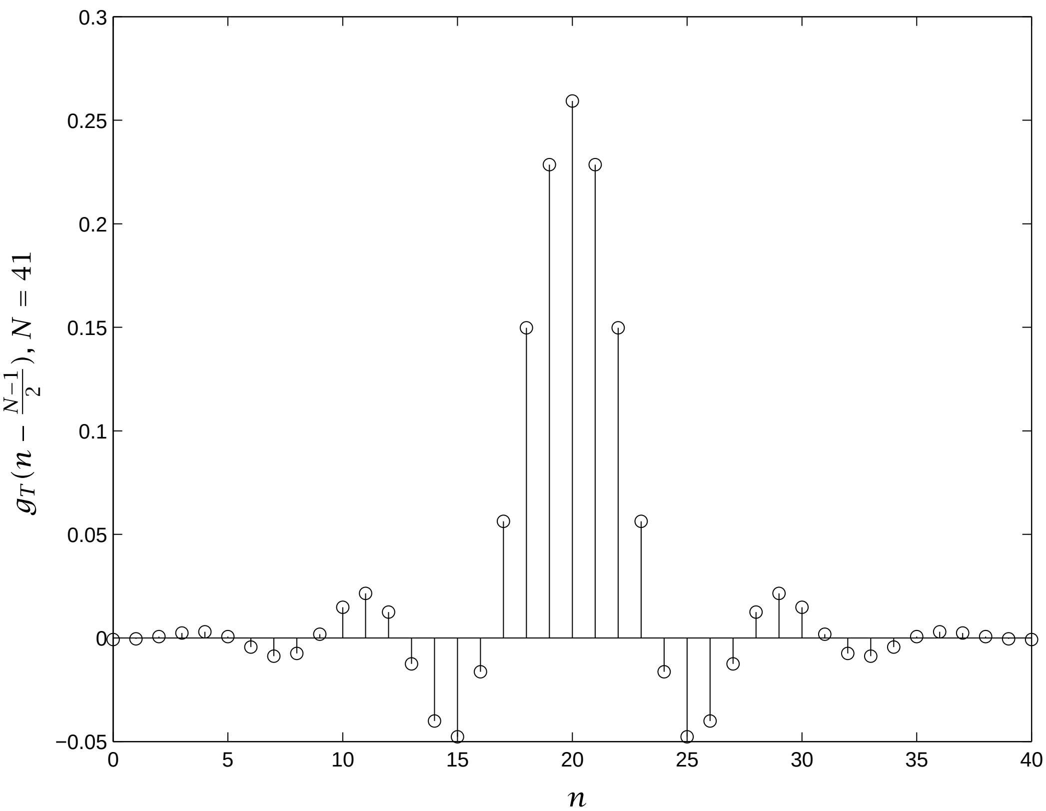


Figure 10.9: Impulse response of truncated discrete-time FIR filter at transmitter $N = 41$

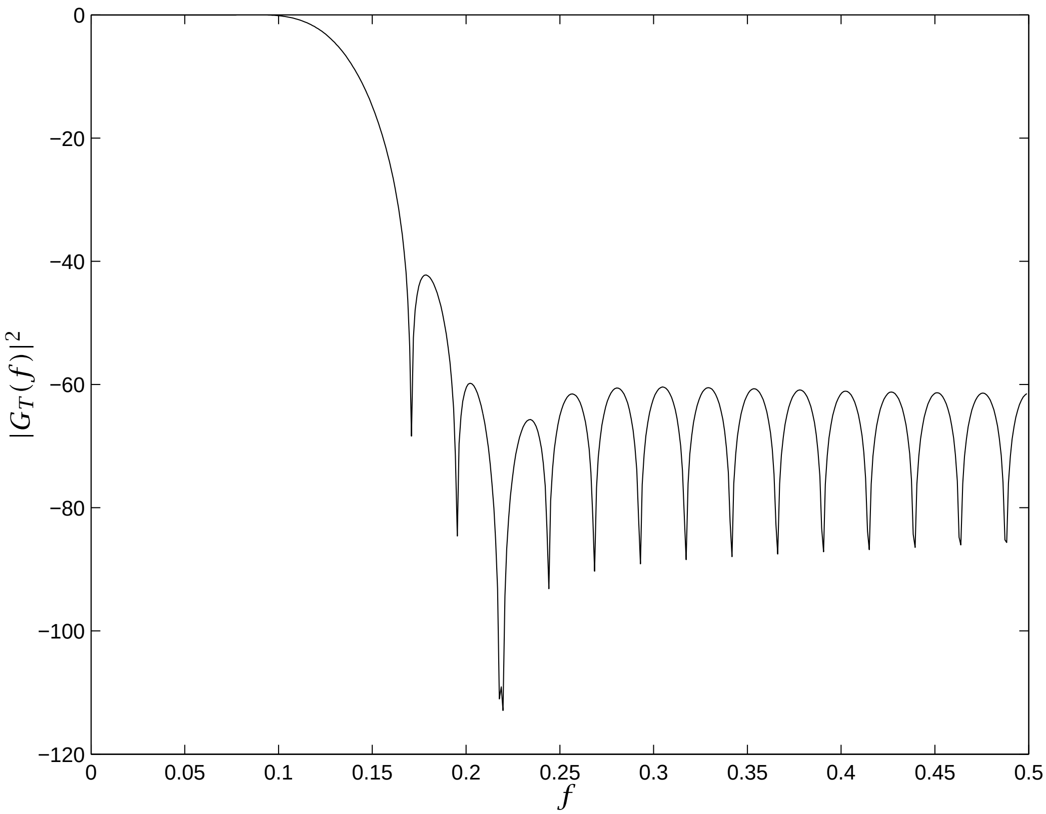


Figure 10.10: Frequency response of truncated discrete-time FIR filter at transmitter $N = 41$

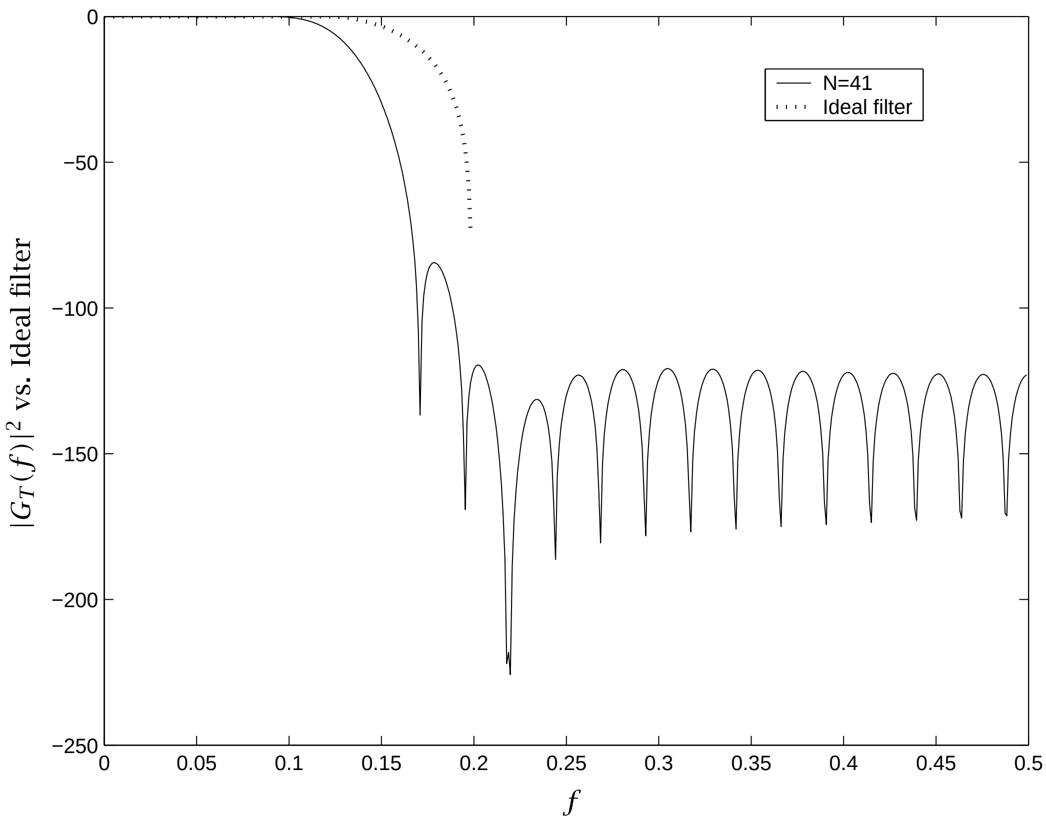


Figure 10.11: $|G_T(f)|^2$ and $X_{rc}(f)$

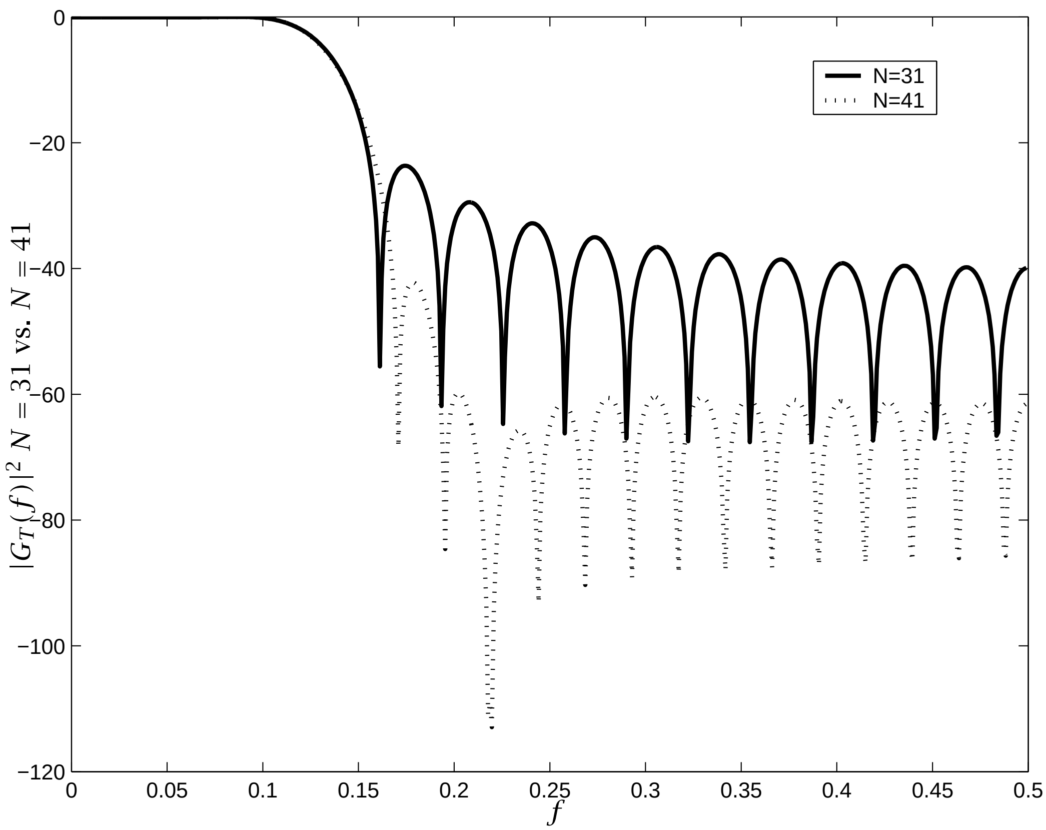


Figure 10.12: $|G_T(f)|^2$ for $N = 31$ and $N = 41$

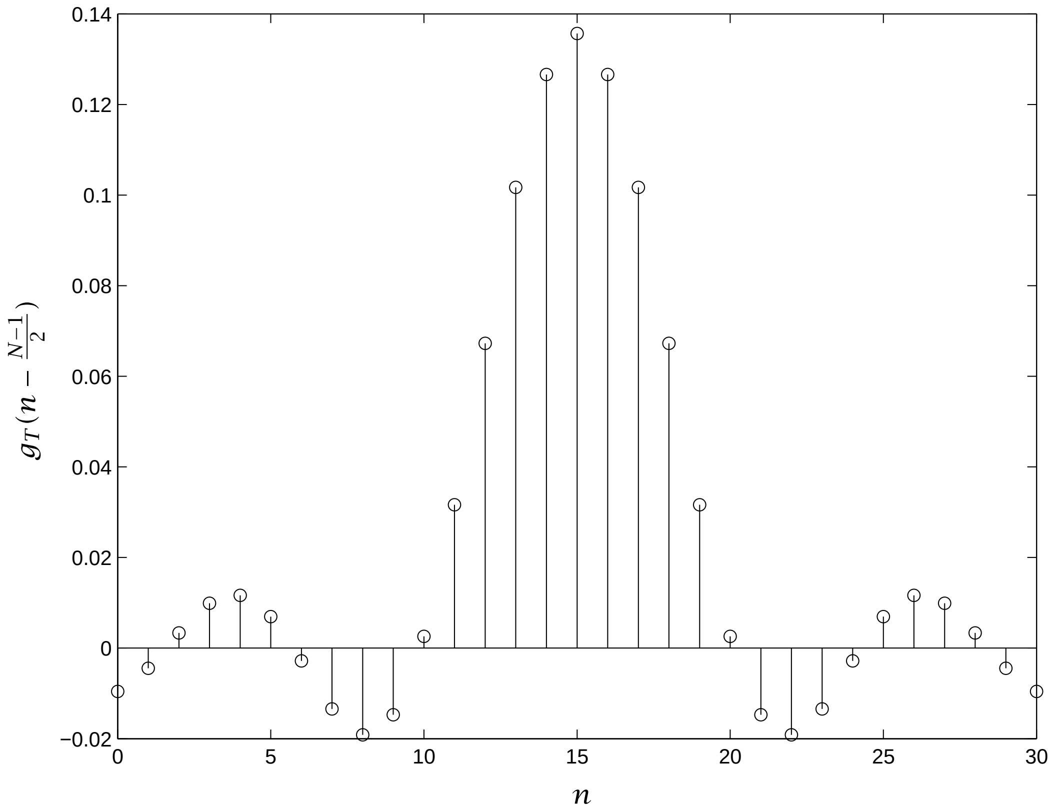


Figure 10.13: Impulse response of truncated discrete-time duobinary FIR filter at the transmitter

```
% impulse response of the cascade of the transmitter and the receiver filters
g_R=g_T;
imp_resp_of_cascade=conv(g_R,g_T);
% Plotting commands follow.
```

Computer Problem 10.5

The MATLAB script for this question is given below.

```
% MATLAB script for Computer Problem 10.5
echo on
d = [1 0 0 1 0 1 1 1 0 1 1 0];
p(1)=0;
for i=1:length(d)
  p(i+1)=rem(p(i)+d(i),2);
  echo off ;
end
echo on ;
a=2.*p−1;
b(1)=0;
```

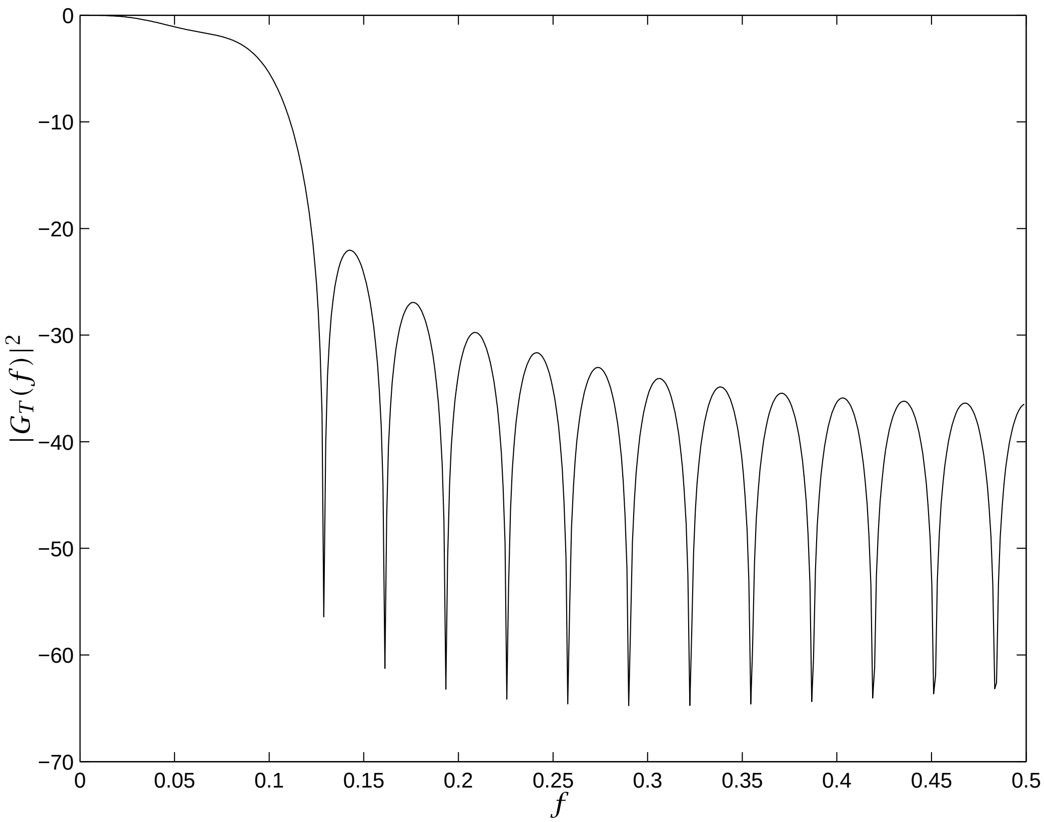


Figure 10.14: Frequency response of truncated discrete-time duobinary FIR filter at the transmitter

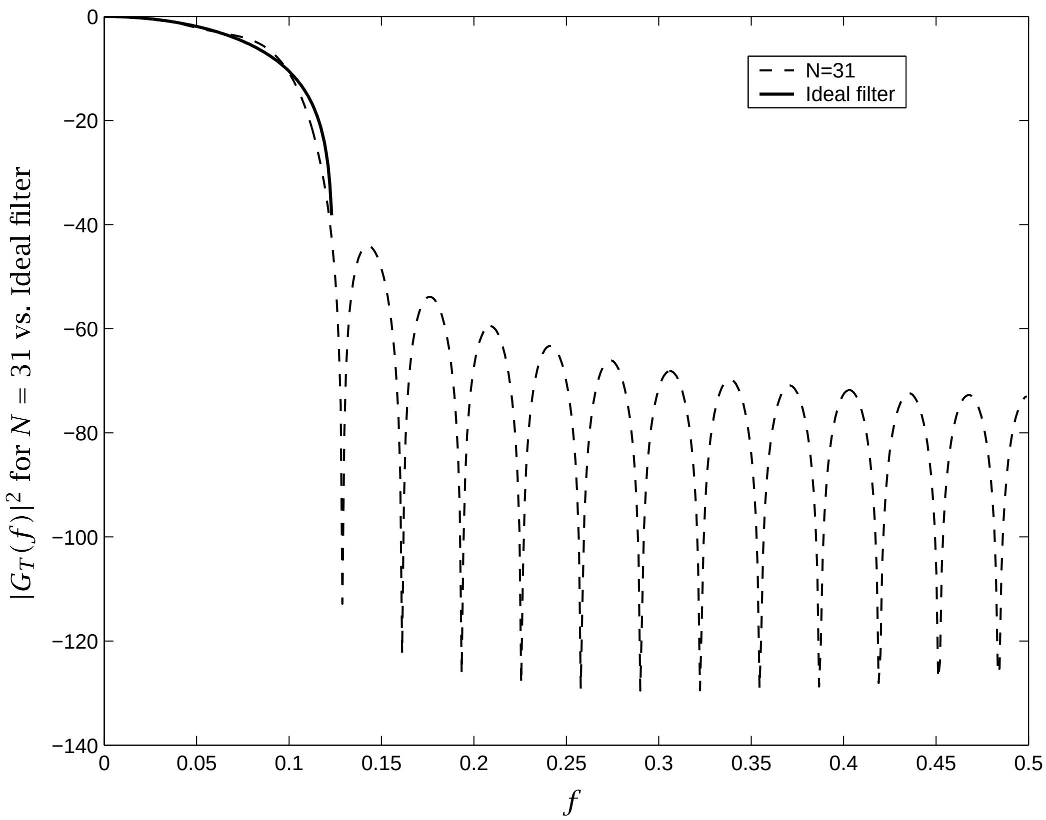


Figure 10.15: $|G_T(f)|$ and ideal $X(f)$

```
dd(1)=0;
for i=1:length(d)
  b(i+1)=a(i+1)+a(i);
  d_out(i+1)=rem(b(i+1)/2+1,2);
  echo off ;
end
echo on ;
d_out=d_out(2:length(d)+1);
```

Computer Problem 10.6

Figure 10.16 illustrates the error probability of the receiver for different values of $\sigma^2 = 0.1$, $\sigma^2 = 0.5$ and $\sigma^2 = 1$. Inter Symbol Interference (ISI) causes degradation in the performance of the system.

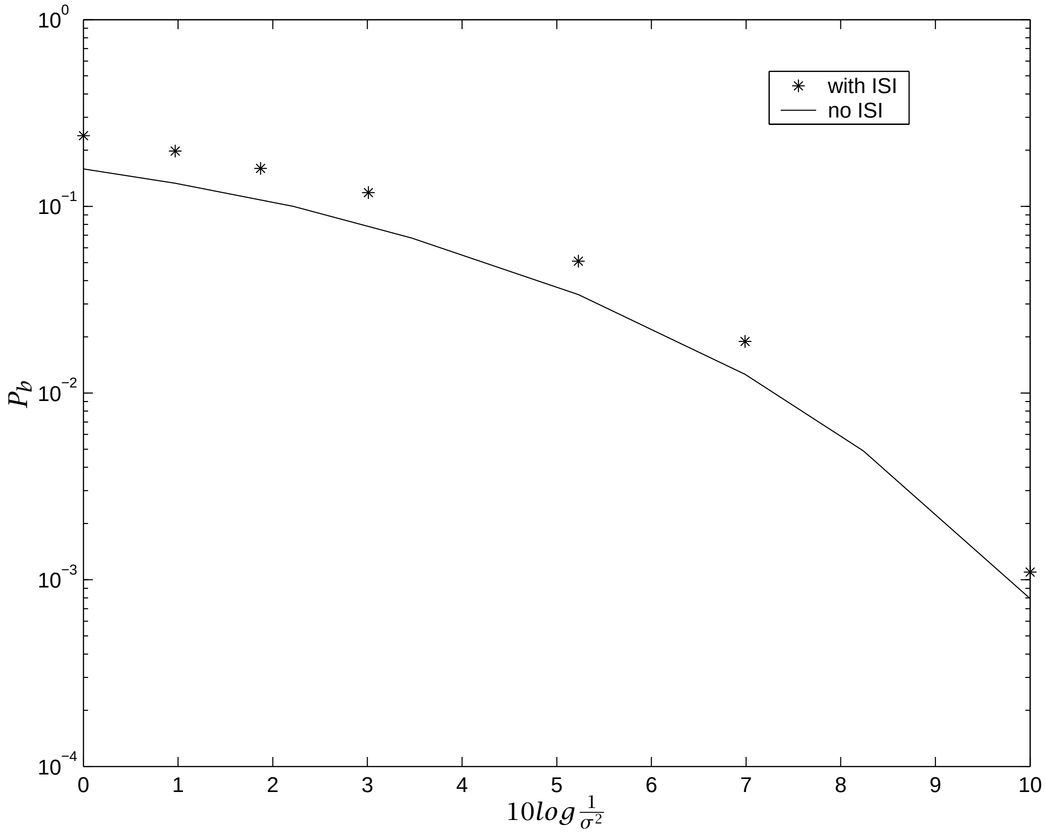


Figure 10.16: Bit Error Probability of duobinary system using binary PAM with and without ISI

The MATLAB script for this question is given next.

```
% MATLAB script for Computer Problem 10.6
echo on
L=100000;
```

```
N=100;
noise_var=0.1;
sigma=sqrt(noise_var);          % standard deviation of the noise
errors = 0;

for m=1:L,
  d=zeros(1,N);
  d( find( rand(1,N) < 0.5 ) )=1;
  p(1)=0;
  for i=1:length(d)
    p(i+1)=rem(p(i)+d(i),2);
    echo off ;
  end
  echo on ;
  a=2.*p-1;

  for i=1:N,
    noise(i)=gngauss(0, sigma);   %  channel noise
    echo off ;
  end;

  b(1)=0;
  dd(1)=0;

  for i=1:length(d)
    b(i+1)= a(i+1) + a(i) + noise(i);
    if (b(i+1) < -1)
      b_rec(i+1)=-2;
    elseif(b(i+1) < 1)
      b_rec(i+1)=0;
    else
      b_rec(i+1)=2;
    end;
    d_out(i+1) = rem(b_rec(i+1)/2+1,2);
    echo off ;
  end

  echo on ;
  d_out=d_out(2:length(d)+1);
  errors = errors + sum(abs(d_out-d));
end

[ errors , errors/(N*L) ]
```

Computer Problem 10.7

The equation

$$q(mT) = \sum_{n=-2}^{2} c_n x(mT - \frac{nT}{2})$$

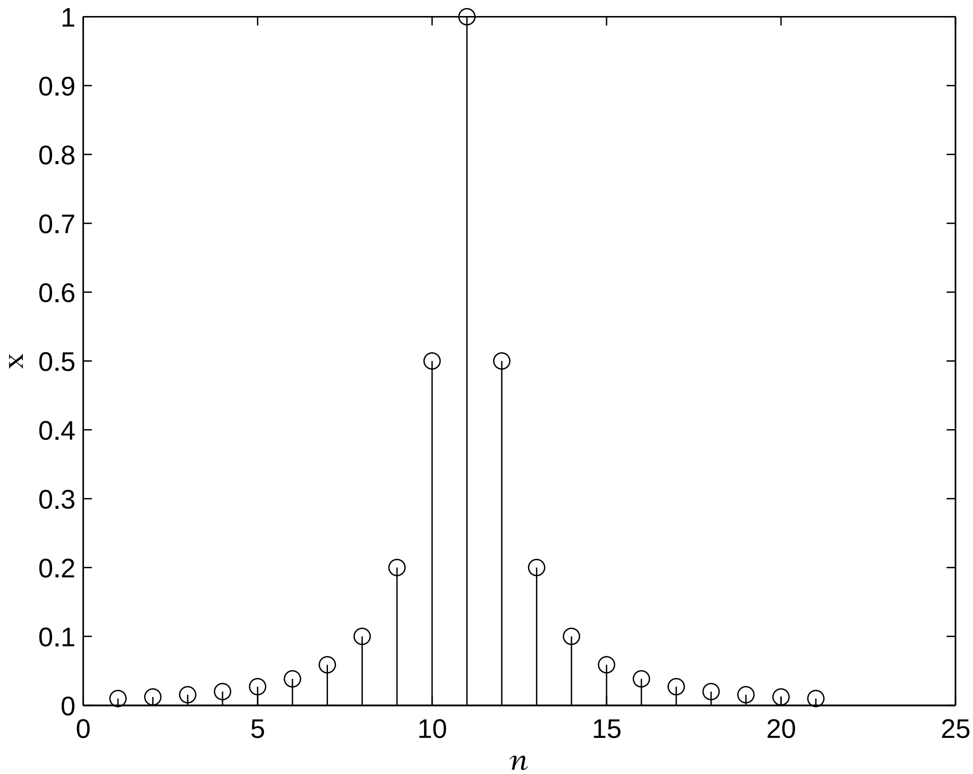


Figure 10.17: Original pulse

may be expressed in matrix form as $\boldsymbol{Xc} = \boldsymbol{q}$, where $\boldsymbol{X}$ is a $(2K+1) \times (2K+1)$ matrix with elements $x(mT - n\tau)$, $\boldsymbol{c}$ is the $(2K+1)$ coefficient vector, and $\boldsymbol{q}$ is the $(2K+1)$ column vector with one nonzero element. Thus, we obtain a set of $2K+1$ linear equations for the coefficients of the zero-forcing equalizer using following equation

$$\boldsymbol{c}_{\text{opt}} = \boldsymbol{X}^{-1}\boldsymbol{q}$$

Figure 10.17 illustrates the original pulse $x(t)$. Figures 10.18, 10.19, and 10.20 present the equalized pulse for $K = 2$, $K = 4$, and $K = 6$, respectively. We should emphasize that the FIR zero-forcing equalizer does not completely eliminate ISI for K = 2. However, as K is increased, the residual ISI is reduced.

The MATLAB scripts for this problem are given next.

```
% MATLAB script for Computer Problem 10.7
T = 1;
Fs=2/T;
Ts=1/Fs;
t=-5*T:T/2:5*T;
x=1./(1+((2/T)*t).^2);                  % sampled pulse
stem(x);
for K = 2:2:6
    X = zeros(2*K+1, 2*K+1);
    m=-K;
```

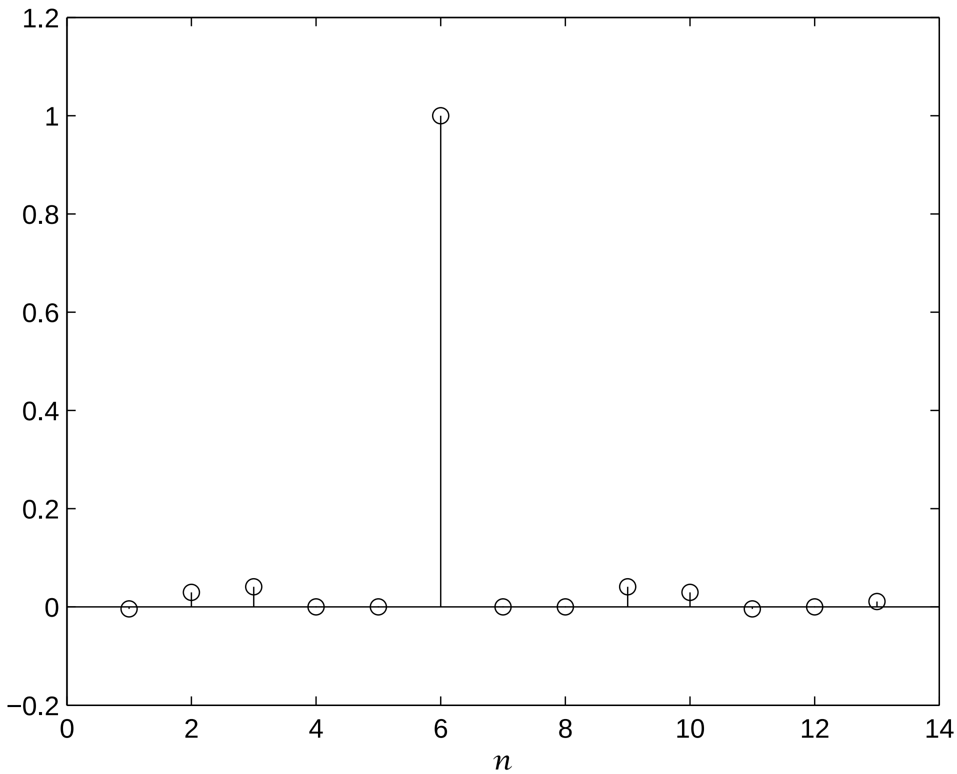


Figure 10.18: Equalized pulse for $K = 2$

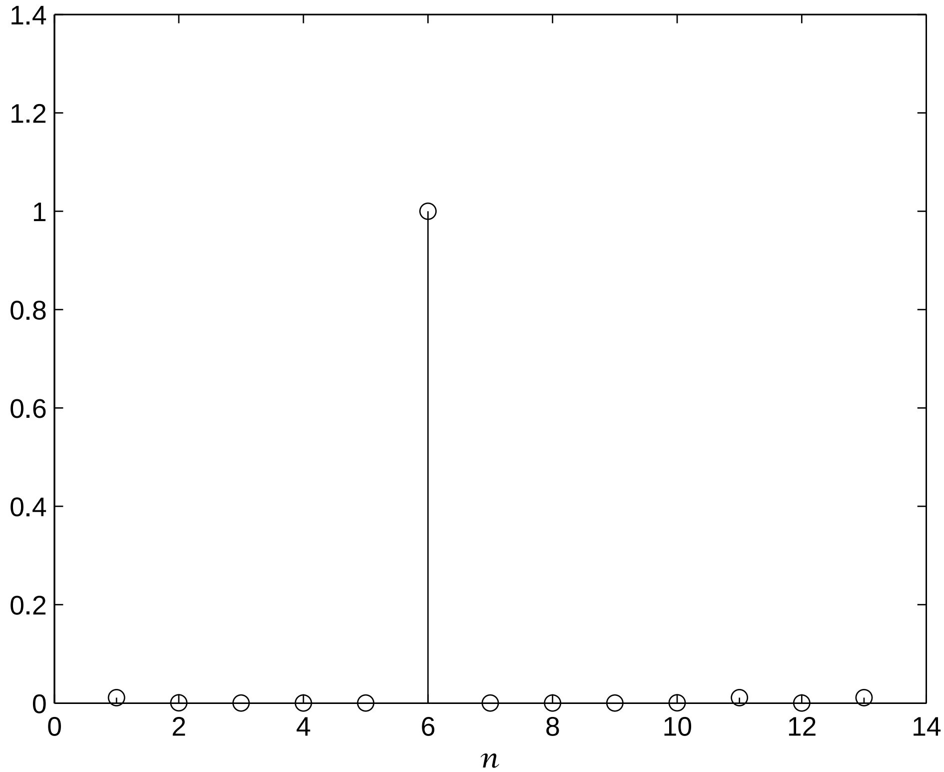


Figure 10.19: Equalized pulse for $K = 4$

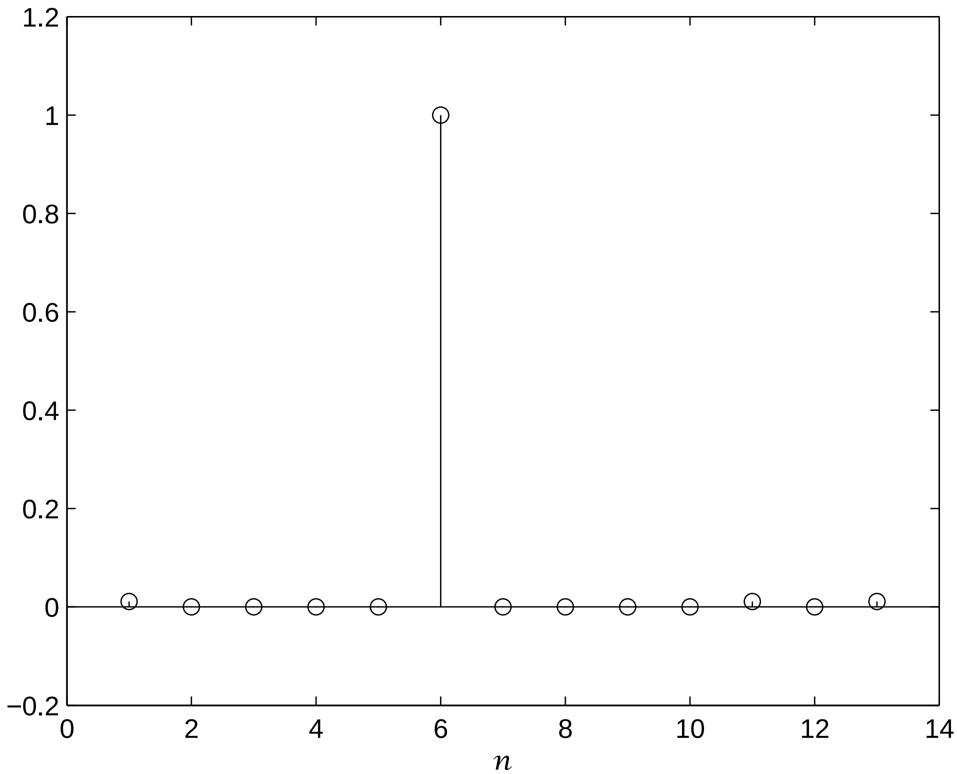


Figure 10.20: Equalized pulse for $K = 6$

```
    for i=1:2*K+1
        n=−K;
        for j = 1:2*K+1
            X(i,j) = 1/(1+(2*(m*T−n*(T/2))/T)^2);
            n= n+1;
        end
        m = m+1;
    end
    q = zeros(1, 2*K+1);
    q(K+1)=1;
    c_opt = inv(X) * q'
    equalized_x=filter(c_opt,1,[x zeros(1, K)]);        % since there will be a delay of two samples at the output
    % to take care of the delay
    equalized_x=equalized_x(K+1:length(equalized_x));
    % Now, let us downsample the equalizer output.
    for i=1:2:length(equalized_x),
        downsampled_equalizer_output((i+1)/2)=equalized_x(i);
    end;
    figure;
    stem(downsampled_equalizer_output);
end
```

Computer Problem 10.8

The error in MSE equalizer is defined as difference between a_m and the equalized symbol $z(mT)$. The mean square error (MSE) between the actual output symbol $z(mT)$ and the desired values a_m is

$$\begin{aligned} \textbf{MSE} &= E|z(mT) - a_m|^2 && (10.32)\\ &= E\left[\left|\sum_{n=-K}^{K} c_n y(mT - n\tau)\right|^2\right] && (10.33)\\ &= \sum_{n=-K}^{K}\sum_{-K}^{K} c_n c_k R_y(n-k) - 2\sum_{k=-K}^{K} c_k R_{ay}(k) + E(|a_m|^2) && (10.34)\end{aligned}$$

where the correlations are defined as

$$\begin{aligned} R_y(n-k) &= E[y^*(mT - n\tau)y(mT - k\tau)] && (10.35)\\ R_{ay}(k) &= E[y^*(mT - k\tau)a_m^*] && (10.36)\end{aligned}$$

and the expectation is taken with respect to the random information sequence $\{a_m\}$ and the additive noise.

The MSE solution is obtained by differentiating with respect to the equalizer coefficients $\{c_n\}$. Thus we obtain the necessary conditions for minimum MSE as

$$\sum_{n=-K}^{K} c_n R_y(n-k) = R_{ay}(k), \quad , k = 0, \pm 1, \pm 2, \ldots, \pm K \tag{10.37}$$

or

$$R_y c = R_{ay}$$

$$c_{opt} = R_y^{-1} R_{ay}$$

First we construct the matrix R_y and vector R_{ay}, then we determine the filter coefficient using $c_{opt} = R_y^{-1}R_{ay}$. For K=2, the matrix R_y with elements $R_y(n-k)$ is simply

$$R_y = X^t X + \frac{N_0}{2} I$$

where X is given in previous problem and I is identity matrix. The vector with elements $R_{ay}(k)$ is given as

$$\boldsymbol{R_{ay}} = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ \frac{1}{5} \end{bmatrix}$$

The equalizer coefficients obtained by inverting the matrix R_y and the results are as follows

$$c_{opt} = X^{-1}q = \begin{bmatrix} 0.0956 \\ -0.7347 \\ 1.6761 \\ -0.7347 \\ 0.0956 \end{bmatrix}$$

Figure 10.21 presents the equalized pulse. Note the small amount of residual ISI in the equalized pulse. Using a MSE linear equalizer with higher number of taps, a better equalization can be achieved. Figures 10.22 and 10.23 illustrates the equalized pulse for linear filters with $K = 4$ and $K = 6$. Note that there is no or little residual ISI in the equalized pulse.

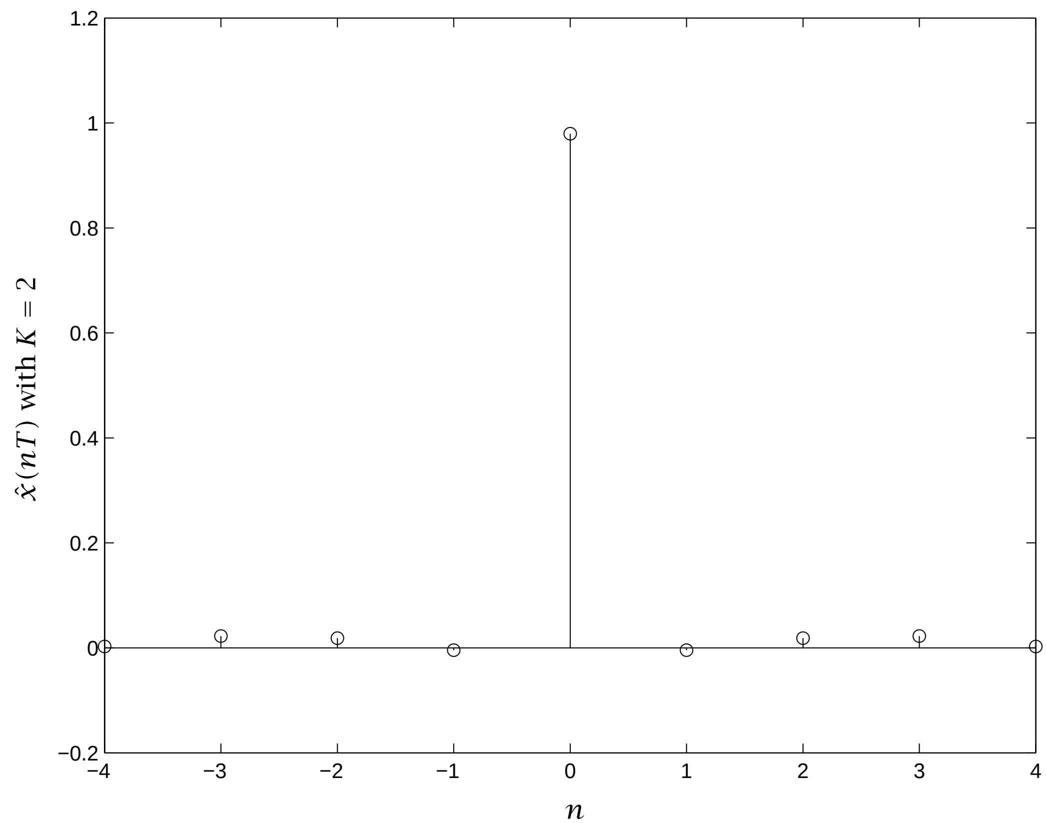


Figure 10.21: Graph of equalized signal by MSE linear equalizer $K = 2$

The MATLAB script for this question is given next.

```
% MATLAB script for Computer Problem 10.8.
echo on
T=1;
K=6;
N0=0.01;                          % assuming that N0=0.01
%   XX and Ry
```

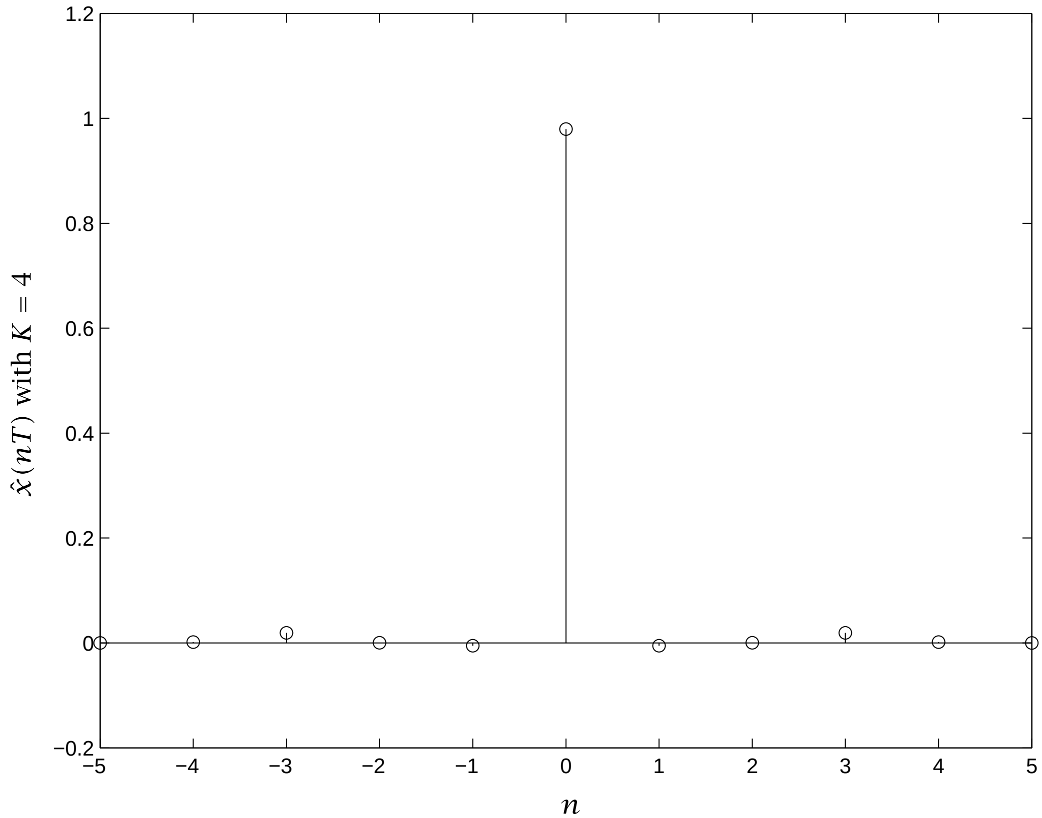


Figure 10.22: Graph of equalized signal by MSE linear equalizer $K = 4$

```
for n=−K:K,
  for k=−K:K,
    temp=0;
    for i=−K:K, temp=temp+(1/(1+(n−i)^2))*(1/(1+(k−i)^2)); end;
    XX(k+K+1,n+K+1)=temp;
    echo off ;
  end;
end;
echo on;
Ry=XX+(N0/2)*eye(2*K+1);
% Riy
t=−K:K;
Riy = (1 ./ (1+(t./T).^2)).';

c_opt=inv(Ry)*Riy;                 % optimal tap coefficients
% find the equalized pulse...
t=−3:1/2:3;
x=1./(1+(2*t/T).^2);               % sampled pulse
equalized_pulse=conv(x,c_opt);
% Decimate the pulse to get the samples at the symbol rate.
decimated_equalized_pulse=equalized_pulse(1:2:length(equalized_pulse));
% Plotting command follows.
```

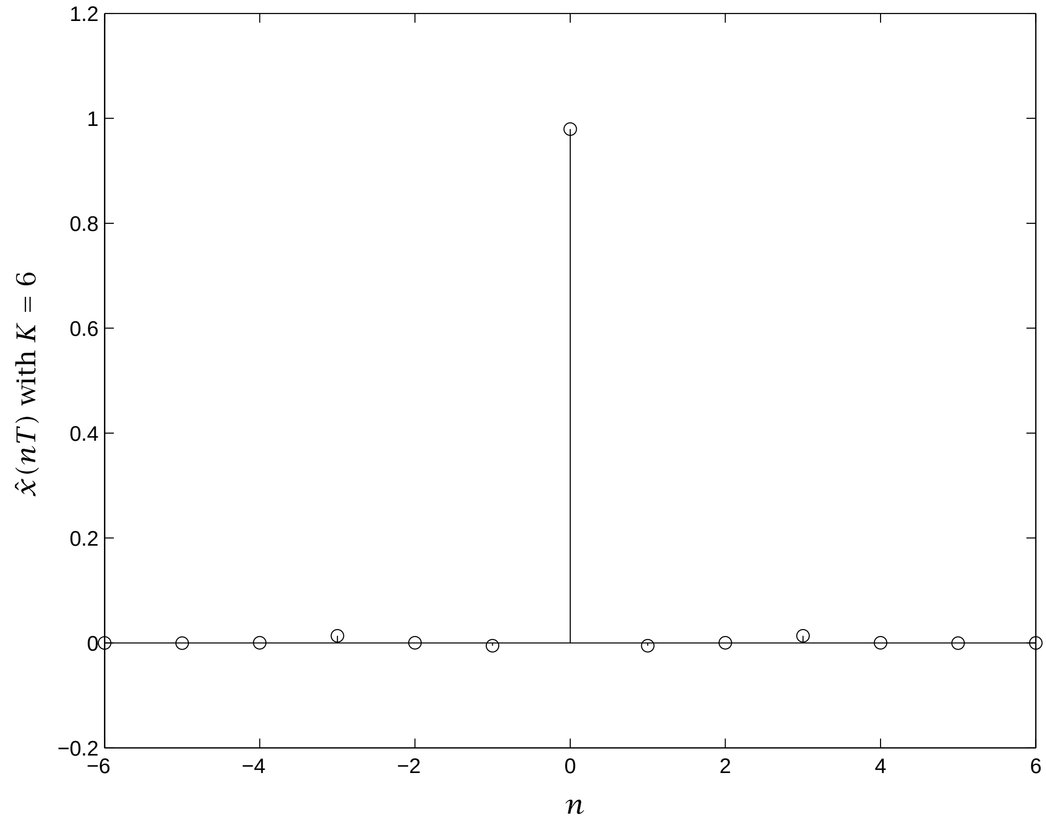


Figure 10.23: Graph of equalized signal by MSE linear equalizer $K = 6$

Computer Problem 10.9

The bit error probability of the adaptive equalizer for various values of σ^2 versus AWGN channel without ISI is shown is Figure 10.24.
The MATLAB script for this question is given next.

```
% MATLAB script for Computer Problem 10.9
T = 1;
Fs=2/T;
Ts=1/Fs;
t=-5*T:T/2:5*T;
x=1./(1+((2/T)*t).^2);                    % sampled pulse
stem(x);
for K = 2:2:6
    X = zeros(2*K+1, 2*K+1);
    m=-K;
    for i=1:2*K+1
        n=-K;
```

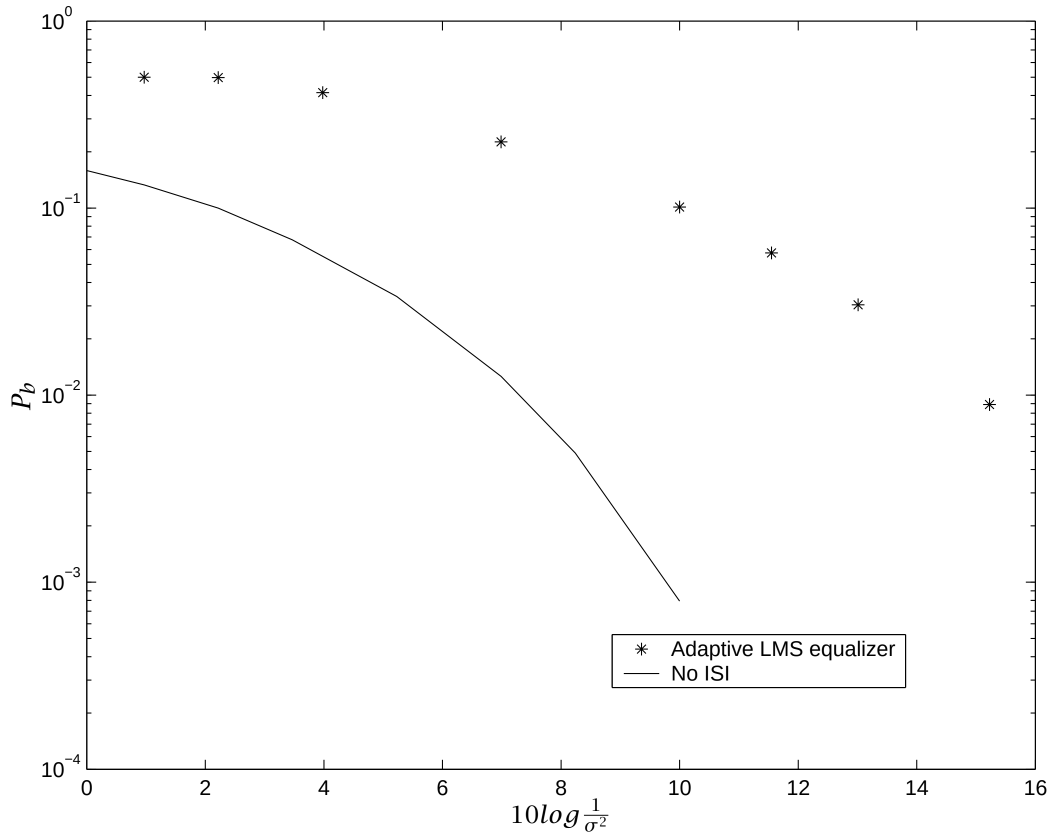


Figure 10.24: Symbol error probability of LMS adaptive equalizer for $\sigma^2 = 0.01$, $\sigma^2 = 0.1$ and $\sigma^2 = 1$

```
        for j = 1:2*K+1
            X(i,j) = 1/(1+(2*(m*T-n*(T/2))/T)^2);
            n= n+1;
        end
        m = m+1;
    end
    q = zeros(1, 2*K+1);
    q(K+1)=1;
    c_opt = inv(X) * q'
    equalized_x=filter(c_opt,1,[x zeros(1, K)]);      % since there will be a delay of two samples at the output
    % to take care of the delay
    equalized_x=equalized_x(K+1:length(equalized_x));
    % Now, let us downsample the equalizer output.
    for i=1:2:length(equalized_x),
        downsampled_equalizer_output((i+1)/2)=equalized_x(i);
    end;
    figure;
    stem(downsampled_equalizer_output);
end
```

Chapter 11

Problem 11.1

The analog signal is

$$x(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k e^{j2\pi kt/T}, \quad 0 \le t < T$$

The subcarrier frequencies are: $F_k = k/T,\ k = 0, 1, \ldots, \tilde{N}$, and, hence, the maximum frequency in the analog signal is: $\tilde{N}/T$. If we sample at the Nyquist rate: $2\tilde{N}/T = N/T$, we obtain the discrete-time sequence:

$$x(n) = x(t = nT/N) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k e^{j2\pi k(nT/N)/T} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}, \quad n = 0, 10, \ldots, N-1$$

which is simply the IDFT of the information sequence $\{X_k\}$.

To show that $x(t)$ is a real-valued signal, we make use of the condition: $X_{N-k} = X_k^*$, for $k = 1.2. \ldots, \tilde{N}-1$. By combining the pairs of complex conjugate terms, we obtain for $k = 1, 2, \ldots, \tilde{N}-1$

$$X_k e^{j2\pi kt/T} + X_k^* e^{-j2\pi kt/T} = 2|X_k| \cos\left(\frac{2\pi kt}{T} + \theta_k\right)$$

where $X_k = |X_k| e^{j\theta_k}$. We also note that X_0 and $X_{\tilde{N}}$ are real. Hence, $x(t)$ is a real-valued signal.

Problem 11.2

The filter with system function $H_n(z)$ has the impulse response $h(k) = e^{j2\pi nk/N},\ k = 0, 1, \ldots$. If we pass the sequence $\{X_k,\ k = 0, 1, \ldots, N-1\}$ through such a filter, we obtain the sequence $y_n(m)$, given as

$$\begin{aligned} y_n(m) &= \sum_{k=0}^{m} X_k h(m-k), \quad m = 0, 1, \ldots \\ &= \sum_{k=0}^{m} X_k e^{j2\pi n(m-k)/N} \end{aligned}$$

At $m = N$, where $y_n(N) = \sum_{k=0}^{N} X_k e^{-j2\pi nk/N} = \sum_{k=0}^{N-1} X_k e^{-j2\pi nk/N}$, since $X_N = 0$. Therefore, the IDFT of $\{X_k\}$ can be computed by passing $\{X_k\}$ through the N filters $H_n(z)$ and sampling their outputs at $m = N$.

Problem 11.3

If T is the time duration of the symbols on each subcarrier and T_c is the ime duration of the channel impulse response, then the cyclic prefix (or time-guard interval) must span T_c seconds. Equivalently, if N is the number of signal samples in the time interval T and m is the number of samples in the cyclic prefix, then:

(a) The channel bandwidth is expanded by the precentage $100m/N\%$ or $100T_c/T\%$.

(b) The transmitted signal energy is expanded by the same percentage.

Problem 11.4

The DFT of $x[n]$ is

$$X(k) = \sum_{n=0}^{N-1} e^{-j2\pi nk/N}, \qquad 0 \le k \le N-1$$

Define

$$y[n] = \begin{cases} x[n], & \text{for } 0 \le n \le N-1 \\ 0, & \text{for } N \le n \le N+L-1 \end{cases}$$

The DFT of $y[n]$ is

$$X'(k) = \sum_{n=0}^{N+L-1} y[n]e^{-j2\pi nk/(N+L)}, \qquad 0 \le k \le N+L-1$$

Then, $X(0) = \sum_{n=0}^{N-1} x[n]$ and $X'(0) = \sum_{n=0}^{N+L-1} y[n] = \sum_{n=0}^{N-1} x[n] = X(0)$

To determine the relationship between $X(k)$ and $X'(k)$, we begin by computing the Fourier transform of $x[n]$,

$$X(f) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi fn}$$

The DFT $\{X(k)\}$ is simply the sampled version of $X(f)$ at the frequencies $f_k = k/N$, $0 \le k \le N-1$. The DFT $\{X'(k)\}$ is the sampled version of $X(f)$ at the frequencies $f_k = k/(N+L)$, $0 \le k \le N+L-1$. In general, $X(k) \ne X'(k)$, except for $k = 0$. Note that if L is selected as $L = N$, then $X(k) = X'(2k)$, $0 \le k \le N-1$.

Computer Problems

Computer Problem 11.1

From Equation (11.2.1) we have $U_6 = \sqrt{2/T}X_6$. Since the channel is noise free, $R_6 = C_6U_6$. Y_6 is obtained by demodulation through correlating with signals given in Equation (11.2.4) and then sampling at time T. Since $\psi_1(t)$ and $\psi_2(t)$ have the channel phase in them, the correlation and sampling process is equivalent to multiplying R_6 by $\sqrt{T/2}e^{-j\angle C_6}$. Therefore,

$$Y_6 = R_6\sqrt{T/2}e^{-j\angle C_6} = \sqrt{T/2}e^{-j\angle C_6}C_6U_6 = e^{-j\angle C_6}C_6X_6 = |C_6|X_6$$

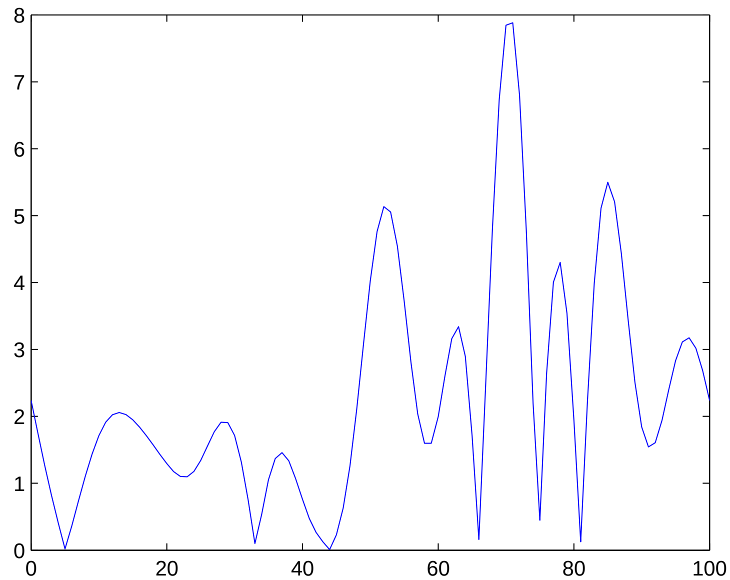
from which we have

$$X_6 = \frac{Y_6}{|C_6|}$$

for this problem $X_6 = 3 + j$ and $C_6 = \frac{1}{2}j$, and $T = 50$. Hence, $U_6 = \frac{1}{5}(3 + j)$ and $R_6 = \frac{1}{10}(-1 + 3j)$. From this we have $Y_6 = -5jR_6 = \frac{1}{2}(3 + j)$ and since $|C_6| = \frac{1}{2}$, we have $Y_6/|C_6| = 3 + j$ which is clearly equal to X_6.

Computer Problem 11.2

Here $K = 10$ and $N = 20$. The plot is shown below and the MATLAB script for this problem is given next.



```
echo on
K=10;N=2*K;T=100;
a=rand(1,36);
a=sign(a−0.5);
b=reshape(a,9,4);
% Generate the 16QAM points
XXX=2*b(:,1)+b(:,2)+j*(2*b(:,3)+b(:,4));
XX=XXX';
X=[0 XX 0 conj(XX(9:−1:1))];
```

```
xt=zeros(1,101);
for t=0:100
    for k=0:N-1
        xt(1,t+1)=xt(1,t+1)+1/sqrt(N)*X(k+1)*exp(j*2*pi*k*t/T);
        echo off
    end
end
echo on
xn=zeros(1,N);
for n=0:N-1
    for k=0:N-1
        xn(n+1)=xn(n+1)+1/sqrt(N)*X(k+1)*exp(j*2*pi*n*k/N);
        echo off
    end
end
echo on
pause % press any key to see a plot of x(t)
plot([0:100],abs(xt))
% Check the difference between xn and samples of x(t)
for n=0:N-1
    d(n+1)=xt(T/N*n+1)-xn(1+n);
    echo off
end
echo on
e=norm(d);
Y=zeros(1,10);
for k=1:9
    for n=0:N-1
        Y(1,k+1)=Y(1,k+1)+1/sqrt(N)*xn(n+1)*exp(-j*2*pi*k*n/N);
        echo off
    end
end
echo on
dd=Y(1:10)-X(1:10);
ee=norm(dd);
```

Computer Problem 11.3

The solution is similar to the solution of Computer Problem 11.2.

Computer Problem 11.4

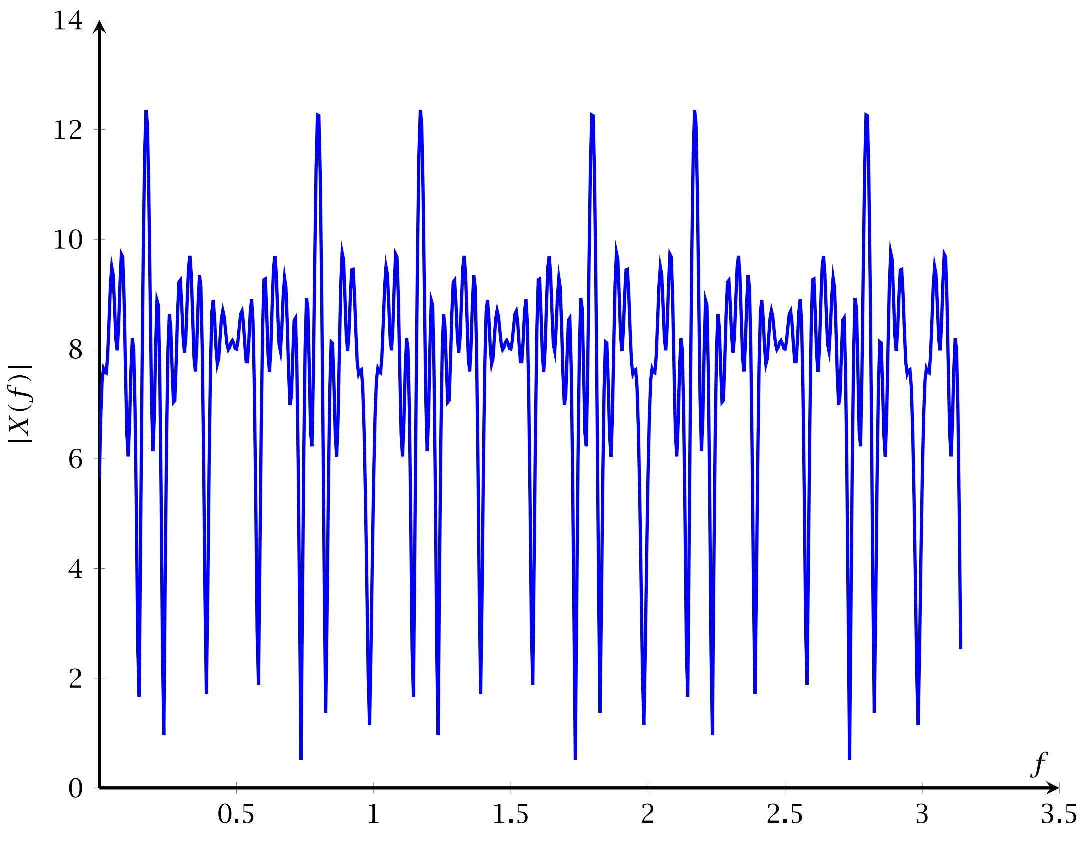
Here $N = 2K = 32$ and X_k's are selected from a QPSK constellation, i.e., $\pm 1 \pm j$. Let us assume we have selected the following values for X_k, $0 \le k \le 15$, $\{1+j, -1+j, -1-j, 1-j, 1+j, 1+j, -1-j, 1-j, 1-j, 1+j, -1+j, -1-j, 1-j, 1-j, 1+j, -1+j\}$. Note that here we have not assumed that $X_0 = 0$. Using Equation (11.3.5) we compute x_n values using the following Matlab code

```
X=[1+j,-1+j,-1-j,1-j,1+j,1+j,-1-j,1-j,1-j,1+j,-1+j,-1-j,1-j,1-j,1+j,-1+j];
Xp=[real(X(1)),X(2:16),conj(X(16:-1:2)),imag(X(1))];
x(1:32)=0;
for n=1:32
  for k=1:32
    x(n)=x(n)+1/sqrt(32)*Xp(k)*exp(2*pi*(k-1)*(n-1)*j/32);
  end
end
```

and then compute $X(f)$ and plot its magnitude using the following Matlab code

```
f=[0:0.005:pi];
Xf=zeros(size(f));
for m=1:32
  Xf=Xf+x(m)*exp(-j*2*pi*(m-1)*f);
end
plot(f,abs(Xf))
```

The plot is shown below



Computer Problem 11.5

In this problem $K = 10$ and $N = 20$. The signal values from the N-point DFT are $x_0, x_1, \ldots, x_{19}$. To this sequence, we append the values x_{16}, x_{17}, x_{18}, and x_{19} at the front end, prior to x_0. Thus, the transmitted signal sequence at the input of the D/A converter is

$$x_{16}, x_{17}, x_{18}, x_{19}, x_0, x_1, \ldots, x_{19}$$

The modification to the MATLAB script is given below.

```
echo on
K=10;N=2*K;T=100;m=4;
a=rand(1,36);
a=sign(a-0.5);
b=reshape(a,9,4);
% Generate the 16QAM points
XXX=2*b(:,1)+b(:,2)+1i*(2*b(:,3)+b(:,4));
XX=XXX';
X=[0 XX 0 conj(XX(9:-1:1))];
xt=zeros(1,101);
for t=0:100
    for k=0:N-1
        xt(1,t+1)=xt(1,t+1)+1/sqrt(N)*X(k+1)*exp(1i*2*pi*k*t/T);
        echo off
    end
end
echo on
xn=zeros(1,N+m);
for n=0:N-1
    for k=0:N-1
        xn(n+m+1)=xn(n+1)+1/sqrt(N)*X(k+1)*exp(1i*2*pi*n*k/N);
        echo off
    end
end
xn(1:m)=xn(N-m+1:N);
echo on
pause % press any key to see a plot of x(t)
plot([0:100],abs(xt))
% Check the difference between xn and samples of x(t)
for n=0:N-1
    d(n+1)=xt(T/N*n+1)-xn(1+n+m);
    echo off
end
echo on
e=norm(d);
Y=zeros(1,10);
for k=1:9
    for n=0:N-1
        Y(1,k+1)=Y(1,k+1)+1/sqrt(N)*xn(n+m+1)*exp(-1i*2*pi*k*n/N);
        echo off
    end
end
echo on
```

```
dd=Y(1:10)-X(1:10);
ee=norm(dd);
```

Computer Problem 11.6

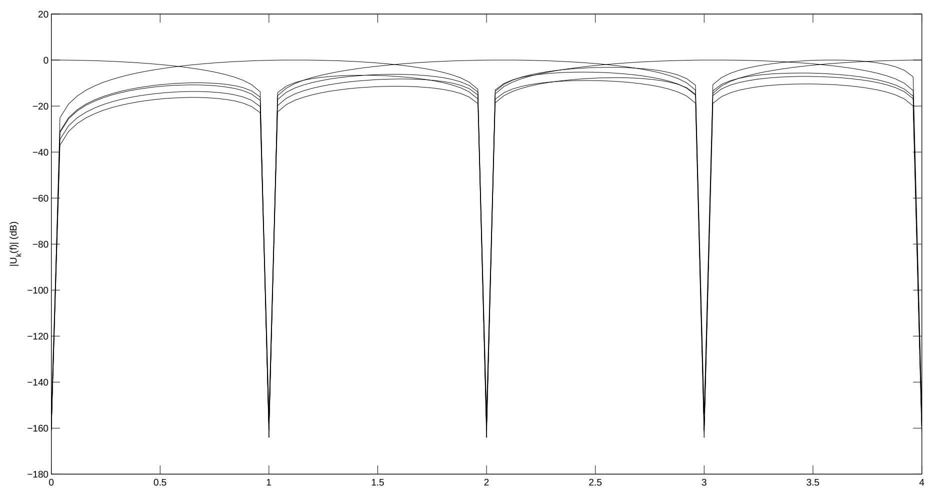
The Fourier transform of $x_k(t) = \sqrt{2/T}\cos 2\pi f_k t$ for $0 \le t \le T$ may be expressed as the convolution of $G(f)$ with $V(f)$ where

$$V(f) = \frac{1}{2}[\delta(f - f_k) + \delta(f + f_k)] \leftrightarrow \cos 2\pi f_k t$$
$$G(f) = \sqrt{2T}e^{-j(\pi f T - \pi/2)}\text{sinc}(fT)$$

Hence,

$$|U_k(f)| = \sqrt{\frac{T}{2}}\,|\,\text{sinc}(f - f_k)T + \text{sinc}(f + f_k)T\,|$$

The plot of $|U_k(f)|$ is shown below. Note the large spectral overlap of the main lobes of each $|U_k(f)|$. Also note that the first sidelobe in the spectrum is only 13 dB down from the main lobe. Therefore, there is a significant amount of spectral overlap among the signals transmitted on different subcarriers. Nevertheless, these signals are orthogonal when transmitted synchronously in time.



The MATLAB script for the problem is given below.

```
T = 1;
k = 0 : 5;
f_k = k/T;
f = -4/T : 0.01*4/T : 4/T;
U_k_abs = zeros(length(k),length(f));
for i = 1 : length(k)
    U_k_abs(i,:) = abs(sqrt(T/2)*(sinc((f-f_k(i))*T) + sinc((f+f_k(i))*T)));
end
```

```
plot(f,U_k_abs(1,:),'.-',f,U_k_abs(2,:),'--',f,U_k_abs(3,:),'c-',f,U_k_abs(4,:),'.',f,U_k_abs(5,:),f,U_k_abs(6,:))
xlabel('f')
ylabel('|U_k(f)|')
```

Computer Problem 11.7

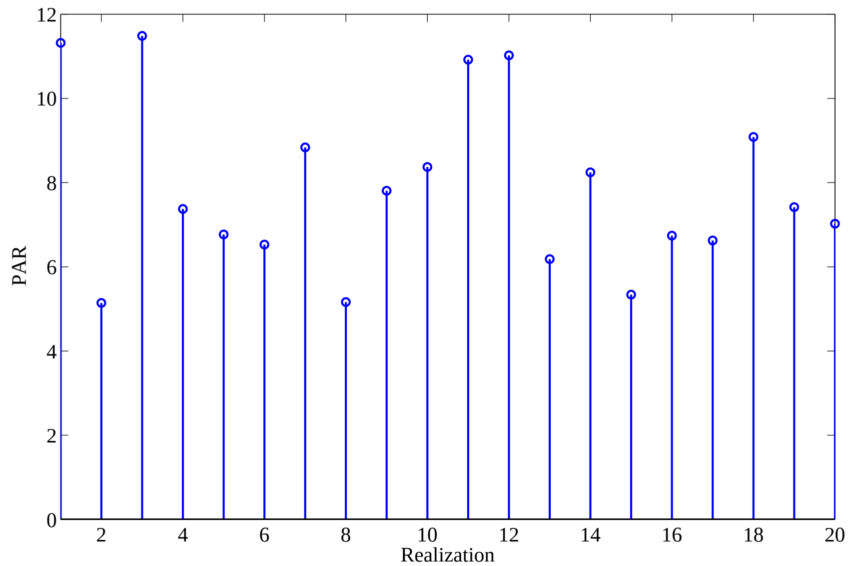
The average power of the sample $\{x_n\}$ is

$$P_{\mathrm{av}} = \frac{1}{200}\sum_{n=0}^{199} x_n^2$$

and the peak power is

$$P_{\mathrm{peak}} = \max_n \{x_n^2\}$$

Hence, the PAR $= P_{\mathrm{peak}}/P_{\mathrm{av}}$. The plot of the PAR is shown here.



The MATLAB script for the problem is given below.

```
T = 1;
Fs = 200;
t = 0 : 1/(Fs*T) : T-1/(Fs*T);
K = 32;
k = 1 : K-1;
rlz = 20;                  % No. of realizations
PAR = zeros(1,rlz);        % Initialization for speed
for j = 1 : rlz
    theta = pi*floor(rand(1,length(k))/0.25)/2;
    x = zeros(1,Fs);       % Initialization for speed
    echo off;
    for i = 1 : Fs
```

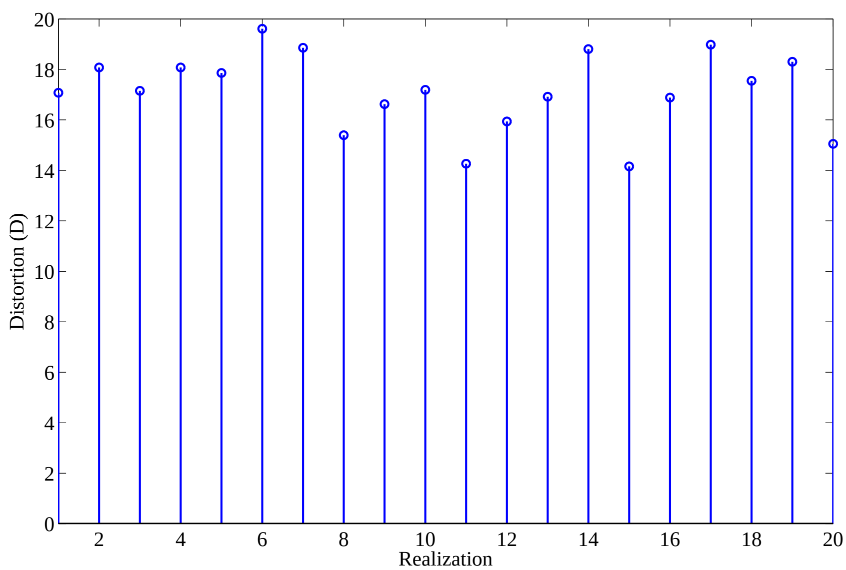
```
            for l = 1 : K−1
                x(i) = x(i) + cos(2*pi*l*t(i)/T+theta(l));
            end
        end
        echo on;
        % Calculation of the PAR:
        P_peak = max(x.^2);
        P_av = sum(x.^2)/Fs;
        PAR(j) = P_peak/P_av;
end
% Plotting the results:
stem(PAR)
axis([1 20 min(PAR) max(PAR)])
xlabel('Realization')
ylabel('PAR')
```

Computer Problem 11.8

Solution is similar to Computer Problem 11.7.

Computer Problem 11.9

The plot of the signal distortion D is shown below



The MATLAB script for the problem is given below.

```
T = 1;
Fs = 200;
```

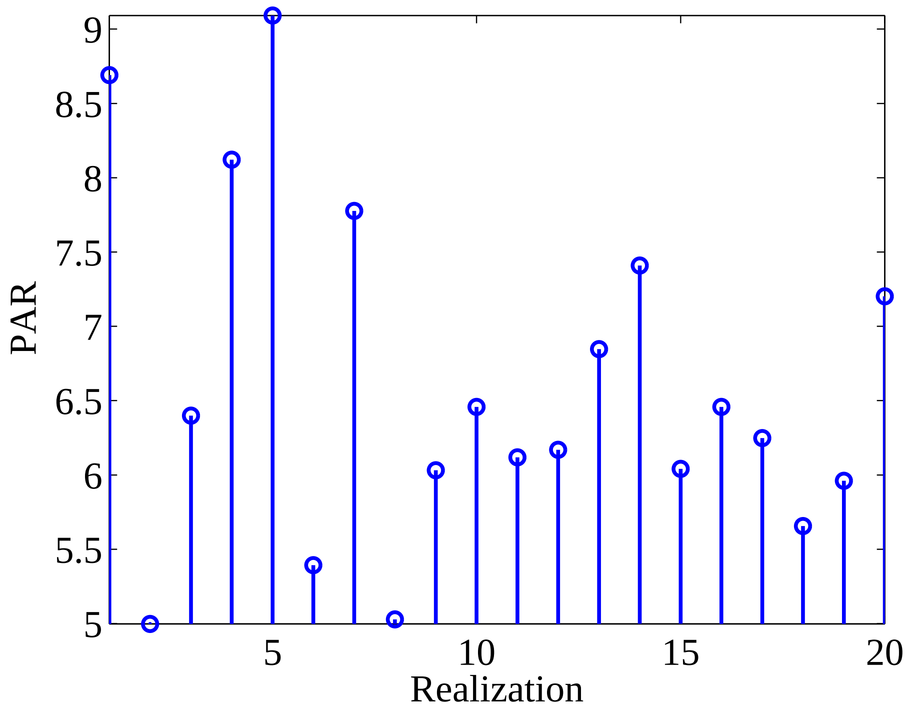
```
t = 0 : 1/(Fs*T) : T-1/(Fs*T);
K = 32;
k = 1 : K-1;
rlz = 20;                    % No. of realizations
% Initialization for speed:
PAR = zeros(1,rlz);
PAR_dB = zeros(1,rlz);
D = zeros(1,rlz);
echo off;
for j = 1 : rlz
    theta = pi*floor(rand(1,length(k))/0.25)/2;
    x = zeros(1,Fs);          % Initialization for speed
    for i = 1 : Fs
        for l = 1 : K-1
            x(i) = x(i) + cos(2*pi*l*t(i)/T+theta(l));
        end
    end
    x_h = x;
    % Calculation of the PAR:
    [P_peak idx] = max(x.^2);
    P_av = sum(x.^2)/Fs;
    PAR(j) = P_peak/P_av;
    PAR_dB(j) = 10*log10(PAR(j));
    % Clipping the peak:
    if P_peak/P_av > 1.9953
        while P_peak/P_av > 1.9953
            x_h(idx) = sqrt(10^0.3*P_av);
            [P_peak idx] = max(x_h.^2);
            P_av = sum(x_h.^2)/Fs;
            PAR_dB(j) = 10*log10(P_peak/P_av);
        end
    end
    D(j) = sum((x-x_h).^2)/Fs;   % Distortion
end
echo on;
% Plotting the results:
stem(D)
axis ([1 20 min(D) max(D)])
xlabel('Realization')
ylabel('Distortion (D)')
```

Computer Problem 11.10

This is similar to Computer Problem 11.9.

Computer Problem 11.11

The plot of the PAR is shown below.



The MATLAB script for the problem is given below.

```
T = 1;
Fs = 200;
t = 0 : 1/(Fs*T) : T−1/(Fs*T);
K = 32;
k = 1 : K−1;
rlz = 20;                       % No. of realizations
PAR = zeros(1,rlz);             % Initialization for speed
echo off;
for j = 1 : rlz
    theta = pi*floor(rand(1,length(k))/0.25)/2;
    phi = 2*pi*rand(4,length(k));
    PAR_phi = zeros(1,size(phi,1)); % Initialization for speed
    for m = 1 : size(phi,1)
        x = zeros(1,Fs);            % Initialization for speed
        for i = 1 : Fs
            for l = 1 : K−1
                x(i) = x(i) + cos(2*pi*l*t(i)/T+theta(l)+phi(m,l));
            end
        end
        % Calculation of the PAR:
        P_peak = max(x.^2);
        P_av = sum(x.^2)/Fs;
        PAR_phi(m) = P_peak/P_av;
    end
    [PAR(j) idx_theta]= min(PAR_phi);
end
echo on;
```

```
% Plotting the results:
stem(PAR)
axis ([1 20 min(PAR) max(PAR)])
xlabel('Realization')
ylabel('PAR')
```

Computer Problem 11.12

This is similar to Computer Problem 11.11.

Chapter 12

Problem 12.1

$$\begin{aligned} H(X) &= -\sum_{i=1}^{6} p_i \log_2 p_i = -(0.1\log_2 0.1 + 0.2\log_2 0.2 \\ &\quad +0.3\log_2 0.3 + 0.05\log_2 0.05 + 0.15\log_2 0.15 + 0.2\log_2 0.2) \\ &= 2.4087 \quad \text{bits/symbol} \end{aligned}$$

If the source symbols are equiprobable, then $p_i = \frac{1}{6}$ and

$$H_u(X) = -\sum_{i=1}^{6} p_i \log_2 p_i = -\log_2 \frac{1}{6} = \log_2 6 = 2.5850 \quad \text{bits/symbol}$$

As it is observed the entropy of the source is less than that of a uniformly distributed source.

Problem 12.2

If the source is uniformly distributed with size N, then $p_i = \frac{1}{N}$ for $i = 1, \ldots, N$. Hence,

$$\begin{aligned} H(X) &= -\sum_{i=1}^{N} p_i \log_2 p_i = -\sum_{i=1}^{N} \frac{1}{N} \log_2 \frac{1}{N} \\ &= -\frac{1}{N} N \log_2 \frac{1}{N} = \log_2 N \end{aligned}$$

Problem 12.3

$$H(X) = -\sum_i p_i \log p_i = \sum_i p_i \log \frac{1}{p_i}$$

By definition the probabilities p_i satisfy $0 < p_i \leq 1$ so that $\frac{1}{p_i} \geq 1$ and $\log \frac{1}{p_i} \geq 0$. It turns out that each term under summation is positive and thus $H(X) \geq 0$. If X is deterministic, then $p_k = 1$ for some k and $p_i = 0$ for all $i \neq k$. Hence,

$$H(X) = -\sum_i p_i \log p_i = -p_k \log 1 = -p_k 0 = 0$$

Note that $\lim_{x\to 0} x \log x = 0$ so if we allow source symbols with probability zero, they contribute nothing in the entropy.

Problem 12.4
1)

$$\begin{aligned} H(X) &= -\sum_{k=1}^{\infty} p(1-p)^{k-1}\log_2(p(1-p)^{k-1}) \\ &= -p\log_2(p)\sum_{k=1}^{\infty}(1-p)^{k-1} - p\log_2(1-p)\sum_{k=1}^{\infty}(k-1)(1-p)^{k-1} \\ &= -p\log_2(p)\frac{1}{1-(1-p)} - p\log_2(1-p)\frac{1-p}{(1-(1-p))^2} \\ &= -\log_2(p) - \frac{1-p}{p}\log_2(1-p) \end{aligned}$$

2) Clearly $p(X=k|X>K)=0$ for $k \le K$. If $k > K$, then

$$p(X=k|X>K) = \frac{p(X=k, X>K)}{p(X>K)} = \frac{p(1-p)^{k-1}}{p(X>K)}$$

But,

$$\begin{aligned} p(X>K) &= \sum_{k=K+1}^{\infty} p(1-p)^{k-1} = p\left(\sum_{k=1}^{\infty}(1-p)^{k-1} - \sum_{k=1}^{K}(1-p)^{k-1}\right) \\ &= p\left(\frac{1}{1-(1-p)} - \frac{1-(1-p)^K}{1-(1-p)}\right) = (1-p)^K \end{aligned}$$

so that

$$p(X=k|X>K) = \frac{p(1-p)^{k-1}}{(1-p)^K}$$

If we let $k = K + l$ with $l = 1, 2, \ldots$, then

$$p(X=k|X>K) = \frac{p(1-p)^K(1-p)^{l-1}}{(1-p)^K} = p(1-p)^{l-1}$$

that is $p(X=k|X>K)$ is the geometrically distributed. Hence, using the results of the first part we obtain

$$\begin{aligned} H(X|X>K) &= -\sum_{l=1}^{\infty} p(1-p)^{l-1}\log_2(p(1-p)^{l-1}) \\ &= -\log_2(p) - \frac{1-p}{p}\log_2(1-p) \end{aligned}$$

Problem 12.5

$$\begin{aligned} H(X,Y) &= H(X,g(X)) = H(X) + H(g(X)|X) \\ &= H(g(X)) + H(X|g(X)) \end{aligned}$$

But, $H(g(X)|X) = 0$, since $g(\cdot)$ is deterministic. Therefore,

$$H(X) = H(g(X)) + H(X|g(X))$$

Since each term in the previous equation is non-negative we obtain

$$H(X) \geq H(g(X))$$

Equality holds when $H(X|g(X)) = 0$. This means that the values $g(X)$ uniquely determine X, or that $g(\cdot)$ is a one to one mapping.

Problem 12.6
The entropy of the source is

$$H(X) = -\sum_{i=1}^{6} p_i \log_2 p_i = 2.4087 \quad \text{bits/symbol}$$

The sampling rate is

$$f_s = 2000 + 2 \cdot 6000 = 14000 \quad \text{Hz}$$

This means that 14000 samples are taken per each second. Hence, the entropy of the source in bits per second is given by

$$H(X) = 2.4087 \times 14000 \ \ \text{(bits/symbol)} \times \text{(symbols/sec)} = 33721.8 \ \ \text{bits/second}$$

Problem 12.7
Consider the function $f(x) = x - 1 - \ln x$. For $x > 1$,

$$\frac{df(x)}{dx} = 1 - \frac{1}{x} > 0$$

Thus, the function is monotonically increasing. Since, $f(1) = 0$, the latter implies that if $x > 1$ then, $f(x) > f(1) = 0$ or $\ln x < x - 1$. If $0 < x < 1$, then

$$\frac{df(x)}{dx} = 1 - \frac{1}{x} < 0$$

which means that the function is monotonically decreasing. Hence, for $x < 1$, $f(x) > f(1) = 0$ or $\ln x < x - 1$. Therefore, for every $x > 0$,

$$\ln x \leq x - 1$$

with equality if $x = 0$. Applying the inequality with $x = \frac{1/N}{p_i}$, we obtain

$$\ln\frac{1}{N} - \ln p_i \le \frac{1/N}{p_i} - 1$$

Multiplying the previous by p_i and adding, we obtain

$$\sum_{i=1}^{N} p_i \ln\frac{1}{N} - \sum_{i=1}^{N} p_i \ln p_i \le \sum_{i=1}^{N}\left(\frac{1}{N} - p_i\right) = 0$$

Hence,

$$H(X) \le -\sum_{i=1}^{N} p_i \ln\frac{1}{N} = \ln N \sum_{i=1}^{N} p_i = \ln N$$

But, $\ln N$ is the entropy (in nats/symbol) of the source when it is uniformly distributed (see Problem 12.2). Hence, for equiprobable symbols the entropy of the source achieves its maximum.

Problem 12.8

Suppose that q_i is a distribution over $1, 2, 3, \ldots$ and that

$$\sum_{i=1}^{\infty} i q_i = m$$

Let $v_i = \frac{1}{q_i m}\left(1 - \frac{1}{m}\right)^{i-1}$ and apply the inequality $\ln x \le x - 1$ (see Problem 12.7) to v_i. Then,

$$\ln\left[\frac{1}{m}\left(1 - \frac{1}{m}\right)^{i-1}\right] - \ln q_i \le \frac{1}{q_i m}\left(1 - \frac{1}{m}\right)^{i-1} - 1$$

Multiplying the previous by q_i and adding, we obtain

$$\sum_{i=1}^{\infty} q_i \ln\left[\frac{1}{m}\left(1 - \frac{1}{m}\right)^{i-1}\right] - \sum_{i=1}^{\infty} q_i \ln q_i \le \sum_{i=1}^{\infty}\frac{1}{m}(1 - \frac{1}{m})^{i-1} - \sum_{i=1}^{\infty} q_i = 0$$

But,

$$\begin{aligned}
\sum_{i=1}^{\infty} q_i \ln\left[\frac{1}{m}\left(1 - \frac{1}{m}\right)^{i-1}\right] &= \sum_{i=1}^{\infty} q_i\left[\ln(\frac{1}{m}) + (i-1)\ln(1 - \frac{1}{m})\right] \\
&= \ln(\frac{1}{m}) + \ln(1 - \frac{1}{m})\sum_{i=1}^{\infty}(i-1)q_i \\
&= \ln(\frac{1}{m}) + \ln(1 - \frac{1}{m})\left[\sum_{i=1}^{\infty} i q_i - \sum_{i=1}^{\infty} q_i\right] \\
&= \ln(\frac{1}{m}) + \ln(1 - \frac{1}{m})(m-1) = -H(\mathbf{p})
\end{aligned}$$

where $H(\mathbf{p})$ is the entropy of the geometric distribution (see Problem 12.4). Hence,

$$-H(\mathbf{p}) - \sum_{i=1}^{\infty} q_i \ln q_i \le 0 \Longrightarrow H(\mathbf{q}) \le H(\mathbf{p})$$

Problem 12.9
1)

$$\begin{aligned} H(X) &= -(.05\log_2 .05 + .1\log_2 .1 + .1\log_2 .1 + .15\log_2 .15 \\ &\quad +.05\log_2 .05 + .25\log_2 .25 + .3\log_2 .3) = 2.5282 \end{aligned}$$

2) After quantization, the new alphabet is $B = \{-4, 0, 4\}$ and the corresponding symbol probabilities are given by

$$\begin{aligned} p(-4) &= p(-5) + p(-3) = .05 + .1 = .15 \\ p(0) &= p(-1) + p(0) + p(1) = .1 + .15 + .05 = .3 \\ p(4) &= p(3) + p(5) = .25 + .3 = .55 \end{aligned}$$

Hence, $H(Q(X)) = 1.4060$. As it is observed quantization decreases the entropy of the source.

Problem 12.10
Using the first definition of the entropy rate, we have

$$\begin{aligned} H &= \lim_{n\to\infty} H(X_n|X_1, \ldots X_{n-1}) \\ &= \lim_{n\to\infty} (H(X_1, X_2, \ldots, X_n) - H(X_1, X_2, \ldots, X_{n-1})) \end{aligned}$$

However, X_1, X_2, $\ldots X_n$ are independent, so that

$$H = \lim_{n\to\infty} \left(\sum_{i=1}^{n} H(X_i) - \sum_{i=1}^{n-1} H(X_i) \right) = \lim_{n\to\infty} H(X_n) = H(X)$$

where the last equality follows from the fact that $X_1, \ldots, X_n$ are identically distributed.

Using the second definition of the entropy rate, we obtain

$$\begin{aligned} H &= \lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, \ldots, X_n) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} H(X_i) \\ &= \lim_{n\to\infty} \frac{1}{n} nH(X) = H(X) \end{aligned}$$

The second line of the previous relation follows from the independence of X_1, X_2, $\ldots X_n$, whereas the third line from the fact that for a DMS the random variables $X_1, \ldots X_n$ are identically distributed independent of n.

Problem 12.11

$$\begin{aligned}
H &= \lim_{n\to\infty} H(X_n|X_1,\ldots,X_{n-1}) \\
&= \lim_{n\to\infty}\left[-\sum_{x_1,\ldots,x_n} p(x_1,\ldots,x_n)\log_2 p(x_n|x_1,\ldots,x_{n-1})\right] \\
&= \lim_{n\to\infty}\left[-\sum_{x_1,\ldots,x_n} p(x_1,\ldots,x_n)\log_2 p(x_n|x_{n-1})\right] \\
&= \lim_{n\to\infty}\left[-\sum_{x_n,x_{n-1}} p(x_n,x_{n-1})\log_2 p(x_n|x_{n-1})\right] \\
&= \lim_{n\to\infty} H(X_n|X_{n-1})
\end{aligned}$$

However, for a stationary process $p(x_n, x_{n-1})$ and $p(x_n|x_{n-1})$ are independent of n, so that

$$H = \lim_{n\to\infty} H(X_n|X_{n-1}) = H(X_n|X_{n-1})$$

Problem 12.12

$$\begin{aligned}
H(X|Y) &= -\sum_{x,y} p(x,y)\log p(x|y) = -\sum_{x,y} p(x|y)p(y)\log p(x|y) \\
&= \sum_{y} p(y)\left(-\sum_{x} p(x|y)\log p(x|y)\right) = \sum_{y} p(y)H(X|Y=y)
\end{aligned}$$

Problem 12.13

1) The marginal distribution $p(x)$ is given by $p(x) = \sum_y p(x,y)$. Hence,

$$\begin{aligned}
H(X) &= -\sum_{x} p(x)\log p(x) = -\sum_{x}\sum_{y} p(x,y)\log p(x) \\
&= -\sum_{x,y} p(x,y)\log p(x)
\end{aligned}$$

Similarly it is proved that $H(Y) = -\sum_{x,y} p(x,y)\log p(y)$.

2) Using the inequality $\ln w \le w - 1$ (see Problem 12.7) with $w = \frac{p(x)p(y)}{p(x,y)}$, we obtain

$$\ln\frac{p(x)p(y)}{p(x,y)} \le \frac{p(x)p(y)}{p(x,y)} - 1$$

Multiplying the previous by $p(x,y)$ and adding over x, y, we obtain

$$\sum_{x,y} p(x,y)\ln p(x)p(y) - \sum_{x,y} p(x,y)\ln p(x,y) \le \sum_{x,y} p(x)p(y) - \sum_{x,y} p(x,y) = 0$$

Hence,

$$\begin{aligned} H(X,Y) &\leq -\sum_{x,y} p(x,y)\ln p(x)p(y) = -\sum_{x,y} p(x,y)(\ln p(x) + \ln p(y)) \\ &= -\sum_{x,y} p(x,y)\ln p(x) - \sum_{x,y} p(x,y)\ln p(y) = H(X) + H(Y) \end{aligned}$$

Equality holds when $\frac{p(x)p(y)}{p(x,y)} = 1$, i.e when X, Y are independent.

Problem 12.14

$$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$

Also, from Problem 12.15, $H(X,Y) \leq H(X) + H(Y)$. Combining the two relations, we obtain

$$H(Y) + H(X|Y) \leq H(X) + H(Y) \Longrightarrow H(X|Y) \leq H(X)$$

Suppose now that the previous relation holds with equality. Then,

$$-\sum_x p(x)\log p(x|y) = -\sum_x p(x)\log p(x) \Rightarrow \sum_x p(x)\log(\frac{p(x)}{p(x|y)}) = 0$$

However, $p(x)$ is always greater or equal to $p(x|y)$, so that $\log(p(x)/p(x|y))$ is non-negative. Since $p(x) > 0$, the above equality holds if and only if $\log(p(x)/p(x|y)) = 0$ or equivalently if and only if $p(x)/p(x|y) = 1$. This implies that $p(x|y) = p(x)$ meaning that X and Y are independent.

Problem 12.15

Let $p_i(x_i)$ be the marginal distribution of the random variable X_i. Then,

$$\begin{aligned} \sum_{i=1}^n H(X_i) &= \sum_{i=1}^n \left[-\sum_{x_i} p_i(x_i)\log p_i(x_i) \right] \\ &= -\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} p(x_1,x_2,\cdots,x_n)\log\left(\prod_{i=1}^n p_i(x_i)\right) \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{i=1}^n H(X_i) - H(X_1,X_2,\cdots X_n) \\ &= \sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} p(x_1,x_2,\cdots,x_n)\log\left(\frac{p(x_1,x_2,\cdots,x_n)}{\prod_{i=1}^n p_i(x_i)}\right) \\ &\geq \sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} p(x_1,x_2,\cdots,x_n)\left(1 - \frac{\prod_{i=1}^n p_i(x_i)}{p(x_1,x_2,\cdots,x_n)}\right) \\ &= \sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} p(x_1,x_2,\cdots,x_n) - \sum_{x_1}\sum_{x_2}\cdots\sum_{x_n} p_1(x_1)p_2(x_2)\cdots p_n(x_n) \\ &= 1 - 1 = 0 \end{aligned}$$

where we have used the inequality $\ln x \geq 1 - \frac{1}{x}$. This inequality is obtained by substituting $y = 1/x$ into $\ln y \leq y - 1$ (see Problem 12.7). Hence,

$$H(X_1, X_2, \cdots X_n) \leq \sum_{i=1}^{n} H(X_i)$$

with equality if $\prod_{i=1}^{n} p_i(x_i) = p(x_1, \cdots, x_n)$, i.e. a memoryless source.

Problem 12.16

1) The probability of an all zero sequence is

$$p(X_1 = 0, X_2 = 0, \cdots, X_n = 0) = p(X_1 = 0)p(X_2 = 0)\cdots p(X_n = 0) = \left(\frac{1}{2}\right)^n$$

2) Similarly with the previous case

$$p(X_1 = 1, X_2 = 1, \cdots, X_n = 1) = p(X_1 = 1)p(X_2 = 1)\cdots p(X_n = 1) = \left(\frac{1}{2}\right)^n$$

3)

$$\begin{aligned} & p(X_1 = 1, \cdots, X_k = 1, X_{k+1} = 0, \cdots X_n = 0) \\ = \ & p(X_1 = 1)\cdots p(X_k = 1)p(X_{k+1} = 0)\cdots p(X_n = 0) \\ = \ & \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} = \left(\frac{1}{2}\right)^n \end{aligned}$$

4) The number of zeros or ones follows the binomial distribution. Hence

$$p(k \text{ ones }) = \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} = \binom{n}{k} \left(\frac{1}{2}\right)^n$$

5) In case that $p(X_i = 1) = p$, the answers of the previous questions change as follows

$$\begin{aligned} p(X_1 = 0, X_2 = 0, \cdots, X_n = 0) &= (1-p)^n \\ p(X_1 = 1, X_2 = 1, \cdots, X_n = 1) &= p^n \\ p(\text{first } k \text{ ones, next } n-k \text{ zeros}) &= p^k(1-p)^{n-k} \\ p(k \text{ ones }) &= \binom{n}{k} p^k (1-p)^{n-k} \end{aligned}$$

Problem 12.17

From the discussion in the beginning of Section 12.2 it follows that the total number of sequences of length n of a binary DMS source producing the symbols 0 and 1 with probability p and $1-p$ respectively is $2^{nH(p)}$. Thus if $p = 0.3$, we will observe sequences having $np = 3000$ zeros and $n(1-p) = 7000$ ones. Therefore,

$$\#\text{ sequences with 3000 zeros } \approx 2^{8813}$$

Another approach to the problem is via the Stirling's approximation. In general the number of binary sequences of length n with k zeros and $n-k$ ones is the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

To get an estimate when n and k are large numbers we can use Stirling's approximation

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Hence,

$$\#\text{ sequences with 3000 zeros } = \frac{10000!}{3000!7000!} \approx \frac{1}{21\sqrt{2\pi 30\cdot 70}}10^{10000}$$

Problem 12.18

1) The total number of typical sequences is approximately $2^{nH(X)}$ where $n = 1000$ and

$$H(X) = -\sum_i p_i \log_2 p_i = 1.4855$$

Hence,

$$\#\text{ typical sequences } \approx 2^{1485.5}$$

2) The number of all sequences of length n is N^n, where N is the size of the source alphabet. Hence,

$$\frac{\#\text{ typical sequences}}{\#\text{ non-typical sequences}} \approx \frac{2^{nH(X)}}{N^n - 2^{nH(X)}} \approx 1.14510^{-30}$$

3) The typical sequences are almost equiprobable. Thus,

$$p(X = \mathbf{x},\ \mathbf{x}\text{ typical}) \approx 2^{-nH(X)} = 2^{-1485.5}$$

4) Since the number of the total sequences is $2^{nH(X)}$ the number of bits required to represent these sequences is $nH(X) \approx 1486$.

5) The most probable sequence is the one with all a_3's that is $\{a_3, a_3, \ldots, a_3\}$. The probability of this sequence is

$$p(\{a_3, a_3, \ldots, a_3\}) = \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{1000}$$

6) The most probable sequence of the previous question is not a typical sequence. In general in a typical sequence, symbol a_1 is repeated $1000p(a_1) = 200$ times, symbol a_2 is repeated approximately $1000p(a_2) = 300$ times and symbol a_3 is repeated almost $1000p(a_3) = 500$ times.

Problem 12.19

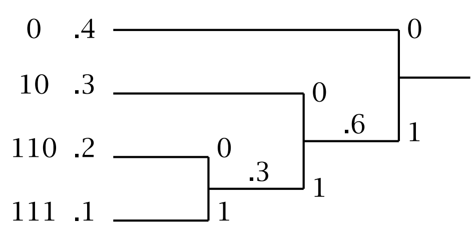
1) The entropy of the source is

$$H(X) = -\sum_{i=1}^{4} p(a_i) \log_2 p(a_i) = 1.8464 \quad \text{bits/output}$$

2) The average codeword length is lower bounded by the entropy of the source for error free reconstruction. Hence, the minimum possible average codeword length is $H(X) = 1.8464$.

3) The following figure depicts the Huffman coding scheme of the source. The average codeword length is

$$\bar{R}(X) = 3 \times (.2 + .1) + 2 \times .3 + .4 = 1.9$$

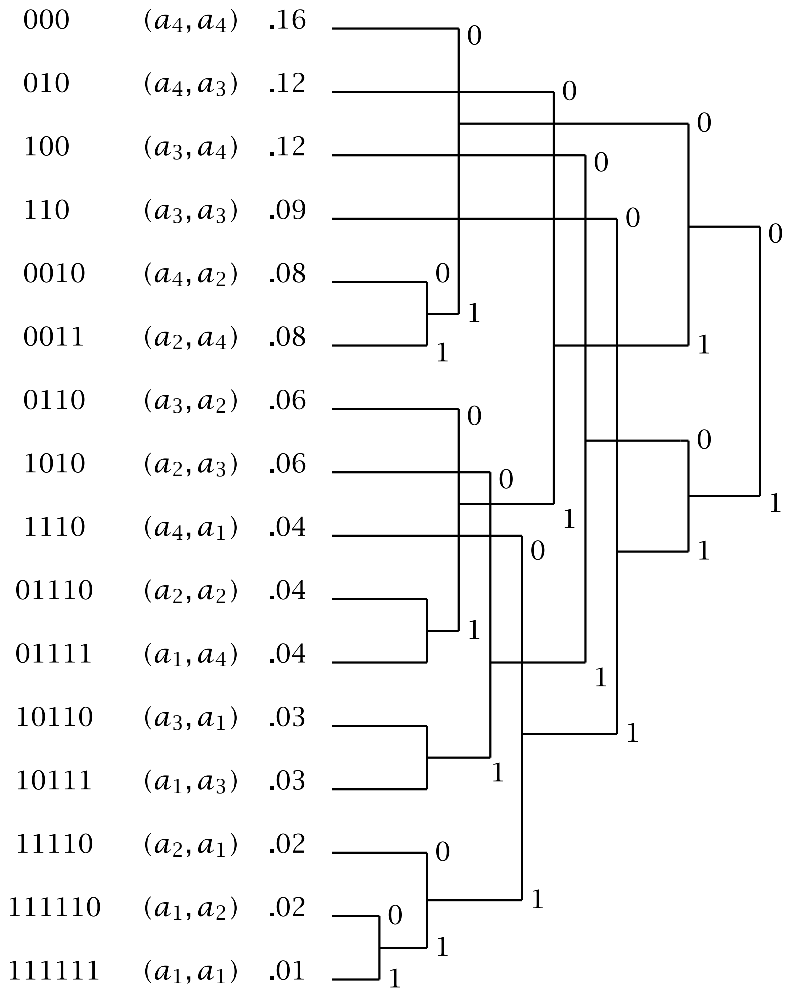


4) For the second extension of the source the alphabet of the source becomes $\mathcal{A}^2 = \{(a_1, a_1), (a_1, a_2), \ldots (a_4, a_4)\}$ and the probability of each pair is the product of the probabilities of each component, i.e. $p((a_1, a_2)) = .2$. A Huffman code for this source is depicted in the next figure. The average codeword length in bits per pair of source output is

$$\bar{R}_2(X) = 3 \times .49 + 4 \times .32 + 5 \times .16 + 6 \times .03 = 3.7300$$

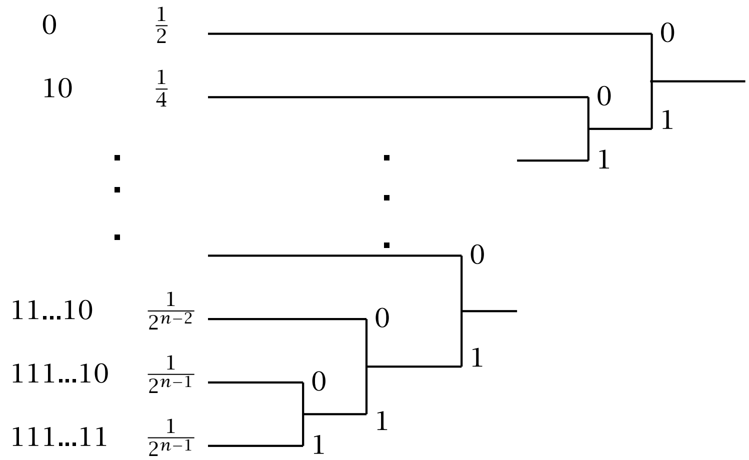
The average codeword length in bits per each source output is $\bar{R}_1(X) = \bar{R}_2(X)/2 = 1.865$.

5) Huffman coding of the original source requires 1.9 bits per source output letter whereas Huffman coding of the second extension of the source requires 1.865 bits per source output letter and thus it is more efficient.



Problem 12.20

The following figure shows the design of the Huffman code. Note that at each step of the algorithm the branches with the lowest probabilities (that merge together) are those at the bottom of the tree.



The entropy of the source is

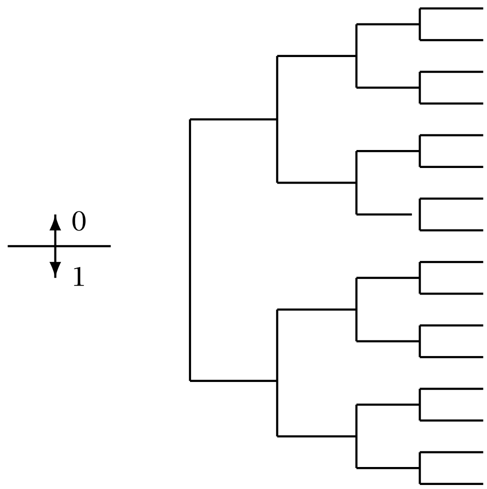
$$
\begin{aligned}
H(X) &= \sum_{i=1}^{n-1} \frac{1}{2^i} \log_2 2^i + \frac{1}{2^{n-1}} \log_2 2^{n-1} \\
&= \sum_{i=1}^{n-1} \frac{1}{2^i} i \log_2 2 + \frac{1}{2^{n-1}} (n-1) \log_2 2 \\
&= \sum_{i=1}^{n-1} \frac{i}{2^i} + \frac{n-1}{2^{n-1}}
\end{aligned}
$$

In the way that the code is constructed, the first codeword (0) has length one, the second codeword (10) has length two and so on until the last two codewords (111...10, 111...11) which have length $n-1$. Thus, the average codeword length is

$$
\begin{aligned}
\bar{R} &= \sum_{x \in X} p(x) l(x) = \sum_{i=1}^{n-1} \frac{i}{2^i} + \frac{n-1}{2^{n-1}} \\
&= 2\left(1 - (1/2)^{n-1}\right) = H(X)
\end{aligned}
$$

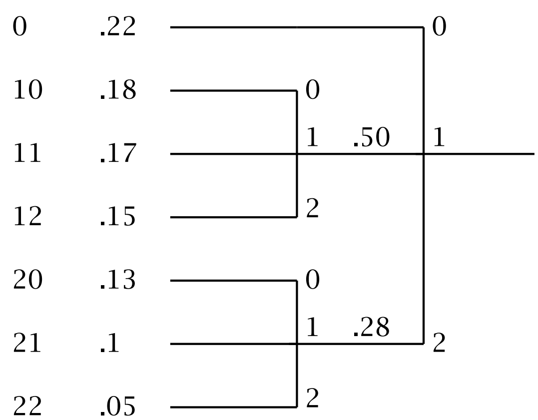
Problem 12.21

The following figure shows the position of the codewords (black filled circles) in a binary tree. Although the prefix condition is not violated the code is not optimum in the sense that it uses more bits that is necessary. For example the upper two codewords in the tree (0001, 0011) can be substituted by the codewords (000, 001) (un-filled circles) reducing in this way the average codeword length. Similarly codewords 1111 and 1110 can be substituted by codewords 111 and 110.



Problem 12.22

The following figure depicts the design of a ternary Huffman code.



The average codeword length is

$$\begin{aligned}\bar{R}(X) &= \sum_x p(x)l(x) = .22 + 2(.18 + .17 + .15 + .13 + .10 + .05) \\ &= 1.78 \quad \text{(ternary symbols/output)}\end{aligned}$$

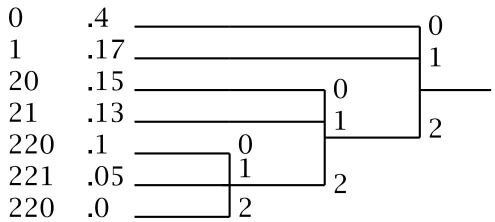
For a fair comparison of the average codeword length with the entropy of the source, we compute the latter with logarithms in base 3. Hence,

$$H(X) = -\sum_x p(x)\log_3 p(x) = 1.7047$$

As it is expected $H(X) \leq \bar{R}(X)$.

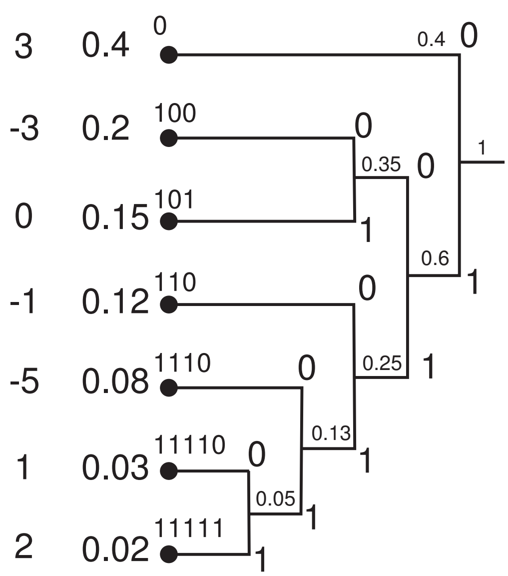
Problem 12.23

If D is the size of the code alphabet, then the Huffman coding scheme takes D source outputs and it merges them to 1 symbol. Hence, we have a decrease of output symbols by $D - 1$. In K steps of the algorithm the decrease of the source outputs is $K(D - 1)$. If the number of the source outputs is $K(D - 1) + D$, for some K, then we are in a good position since we will be left with D symbols for which we assign the symbols $0, 1, \ldots, D - 1$. To meet the above condition with a ternary code the number of the source outputs should be $2K + 3$. In our case that the number of source outputs is six we can add a dummy symbol with zero probability so that $7 = 2 \cdot 2 + 3$. The following figure shows the design of the ternary Huffman code.



Problem 12.24

1. Designing Huffman code



results in

$$\bar{R} = 0.4 \times 1 + 3 \times (0.2 + 0.15 + 0.12) + 4 \times 0.08 + 5 \times (0.03 + 0.02) = 2.38$$

2. $H(X) = -\sum_{i=1}^{7} p_i \log_2 p_i = 2.327$ bits and $\eta = \frac{H(X)}{\bar{R}} = 0.977$

3. We have

$$\begin{cases} P(\hat{X} = -2) = 0.2 + 0.08 = 0.28 \\ P(\hat{X} = 0) = 0.12 + 0.15 + 0.03 = 0.3 \\ P(\hat{X} = 2) = 0.02 + 0.4 = 0.42 \end{cases}$$

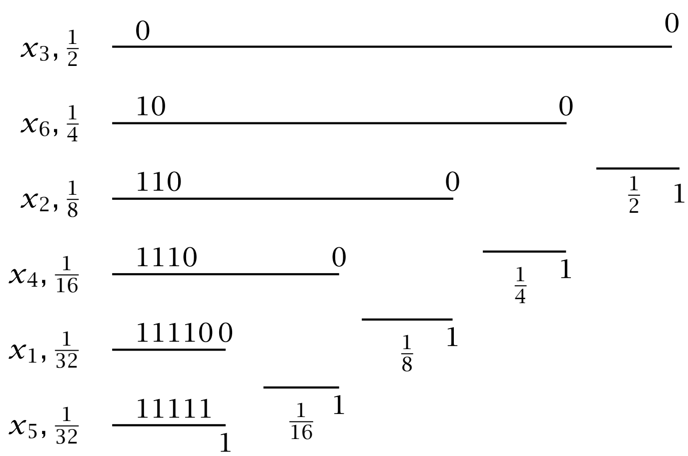
and $H(\hat{X}) = -0.28 \log_2 0.28 - 0.3 \log_2 0.3 - 0.42 \log_2 0.42 = 1.56$ bits.

4. There are a total of 3^{10000} sequences of which roughly $2^{10000 H(\hat{X})} = 2^{15600}$ are typical.

5. In general for the second extension we have $H(X) \le \bar{R} < H(X) + \frac{1}{2}$ and therefore $2.327 \le \bar{R} < 2.827$. But in this case since the second extension will not perform worse than the first extension, the upper bound is the $\bar{R}$ we derived in part 1. Therefore, the tightest bounds are $2.327 \le \bar{R} \le 2.38$.

Problem 12.25

1.



The average codeword length is $\bar{R} = \frac{1}{2} + 2 \times \frac{1}{4} + 3\frac{1}{8} + 4 \times \frac{1}{16} + 5 \times (2 \times \frac{1}{32}) = 1\frac{15}{16}$.

2. We first find the entropy of the source

$$H(X) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{4}\log_2\frac{1}{4} - \frac{1}{8}\log_2\frac{1}{8} - \frac{1}{16}\log_2\frac{1}{16} - \frac{2}{32}\log_2\frac{1}{32} = 1\frac{15}{16}$$

Since the average codeword length is already equal to the entropy no improvement is possible

3. No, in general entropy is the lower limit for the average codeword length and no improvement for lossless coding is possible.

Problem 12.26

1.

$$H(X) = -0.1\log_2 0.1 - 0.2\log_2 0.2 - 0.05\log_2 0.05 - 0.3\log_2 0.3 - 0.35\log_2 0.35 \approx 2.064 \text{ bits/symbol}$$

Since the entropy exceeds 2, lossless encoding of this source at 2 bits per symbol is impossible.

2. The size of the source alphabet is 5, therefore $5^{1000} = 2^{1000\log_2 5} \approx 2^{2322}$ sequences of length 1000 are possible

3. The number of typical sequences is $2^{1000H(X)} \approx 2^{2064}$.

4. The process of merging should be done such that the entropy of the resulting sources (with an alphabet of size 4) is less than 1.5 bits per symbol. To minimize the entropy by combining two letters we have to combine the two letters that have the maximum contribution to the entropy. These are a_4 and a_4 with probabilities of 0.3 and 0.35, respectively. Combining these two results in a single letter b with probability of 0.65. The entropy of the resulting source would be

$$H(Y) = -0.1\log_2 0.1 - 0.2\log_2 0.2 - 0.05\log_2 0.05 - 0.65\log_2 0.65 \approx 1.417 \text{ bits/symbol}$$

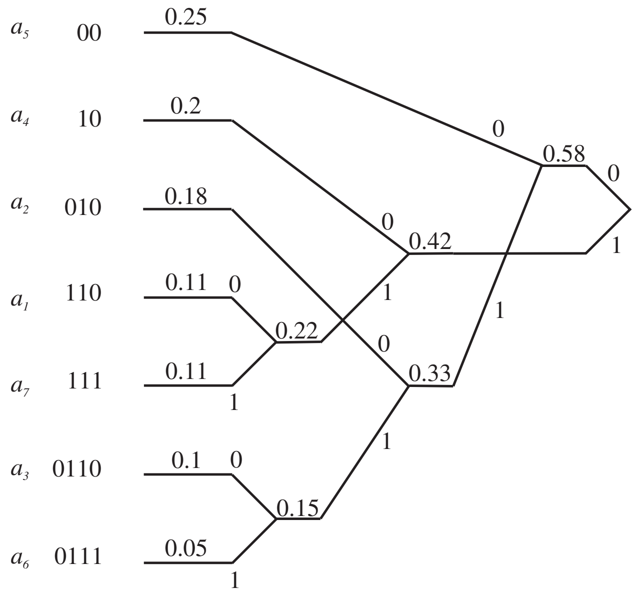
Since the entropy is less than 1.5, it is possible to transmit this source at a rate of 1.5 bits per symbol.

Problem 12.27

1. The minimum rate is equal to the entropy of the source, given by $H(X) = -\sum_i p_i \log_2 p_i$, and this is given by

$$\begin{aligned} H(X) = &-2\times 0.11\log_2 0.11 - 0.18\log_2 0.18 - 0.1\log_2 0.1 - 0.2\log_2 0.2 \\ &- 0.215\log_2 0.25 - 0.05\log_2 0.05 = 2.66 \ \text{ bits/symbol} \end{aligned}$$

2. Following the algorithm for designing Huffman codes we have



The average codeword length is

$$\begin{aligned}\bar{R} &= \sum_i p_i l_i \\ &= 0.25 \times 2 + 0.2 \times 2 + 0.18 \times 3 + 0.11 \times 3 + 0.11 \times 3 + 0.1 \times 4 + 0.05 \times 4 \\ &= 0.5 + 0.4 + 0.54 + 0.33 + 0.33 + 0.4 + 0.2 = 2.7 \text{ binary symbols/source output}\end{aligned}$$

the inequality $H(X) \leq \bar{R} < H(X) + 1$ is satisfied, as required.

3. Here $P(b_1) = P(a_1) + P(a_2) = 0.29$, $P(b_2) = P(a_3) + P(a_4) = 0.2$, $P(b_3) = P(a_5) + P(a_6) = 0.3$, $P(b_4) = P(a_7) = 0.11$, and $H(B) = \sum_{i=1}^{4} -P(b_i) \log_2 P(b_i) = 1.91$ bits/symbol.

Problem 12.28

Parsing the sequence by the rules of the Lempel-Ziv coding scheme we obtain the phrases

0, 00, 1, 001, 000, 0001, 10, 00010, 0000, 0010, 00000, 101, 00001,

000000, 11, 01, 0000000, 110, 0, ...

The number of the phrases is 19. For each phrase we need 5 bits plus an extra bit to represent the new source output.

Dictionary Location	Dictionary Contents	Codeword
1 00001	0	00000 0
2 00010	00	00001 0
3 00011	1	00000 1
4 00100	001	00010 1
5 00101	000	00010 0
6 00110	0001	00101 1
7 00111	10	00011 0
8 01000	00010	00110 0
9 01001	0000	00101 0
10 01010	0010	00100 0
11 01011	00000	01001 0
12 01100	101	00111 1
13 01101	00001	01001 1
14 01110	000000	01011 0
15 01111	11	00011 1
16 10000	01	00001 1
17 10001	0000000	01110 0
18 10010	110	01111 0
19	0	00000

Problem 12.29

$$\begin{aligned}
I(X;Y) &= H(X) - H(X|Y) \\
&= -\sum_x p(x)\log p(x) + \sum_{x,y} p(x,y)\log p(x|y) \\
&= -\sum_{x,y} p(x,y)\log p(x) + \sum_{x,y} p(x,y)\log p(x|y) \\
&= \sum_{x,y} p(x,y)\log\frac{p(x|y)}{p(x)} = \sum_{x,y} p(x,y)\log\frac{p(x,y)}{p(x)p(y)}
\end{aligned}$$

Using the inequality $\ln y \le y - 1$ (see Problem 12.7) with $y = \frac{1}{x}$, we obtain $\ln x \ge 1 - \frac{1}{x}$. Applying this inequality with $x = \frac{p(x,y)}{p(x)p(y)}$ we obtain

$$\begin{aligned} I(X;Y) &= \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} \\ &\geq \sum_{x,y} p(x,y)\left(1 - \frac{p(x)p(y)}{p(x,y)}\right) = \sum_{x,y} p(x,y) - \sum_{x,y} p(x)p(y) = 0 \end{aligned}$$

$\ln x \geq 1 - \frac{1}{x}$ holds with equality if $x = 1$. This means that $I(X;Y) = 0$ if $p(x,y) = p(x)p(y)$ or in other words if X and Y are independent.

Problem 12.30

1) $I(X;Y) = H(X) - H(X|Y)$. Since in general, $H(X|Y) \geq 0$, we have $I(X;Y) \leq H(X)$. Also (see Problem 12.33), $I(X;Y) = H(Y) - H(Y|X)$ from which we obtain $I(X;Y) \leq H(Y)$. Combining the two inequalities, we obtain

$$I(X;Y) \leq \min\{H(X), H(Y)\}$$

2) It can be shown (see Problem 12.7), that if X and Z are two random variables over the same set $\mathcal{X}$ and Z is uniformly distributed, then $H(X) \leq H(Z)$. Furthermore $H(Z) = \log|\mathcal{X}|$, where $|\mathcal{X}|$ is the size of the set $\mathcal{X}$ (see Problem 6.2). Hence, $H(X) \leq \log|\mathcal{X}|$ and similarly we can prove that $H(Y) \leq \log|\mathcal{Y}|$. Using the result of the first part of the problem, we obtain

$$I(X;Y) \leq \min\{H(X), H(Y)\} \leq \min\{\log|\mathcal{X}|, \log|\mathcal{Y}|\}$$

Problem 12.31

By definition $I(X;Y) = H(X) - H(X|Y)$ and $H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$. Combining the two equations we obtain

$$\begin{aligned} I(X;Y) &= H(X) - H(X|Y) = H(X) - (H(X,Y) - H(Y)) \\ &= H(X) + H(Y) - H(X,Y) = H(Y) - (H(X,Y) - H(X)) \\ &= H(Y) - H(Y|X) = I(Y;X) \end{aligned}$$

Problem 12.32

1) The joint probability density is given by

$$\begin{aligned} p(Y=1, X=0) &= p(Y=1|X=0)p(X=0) = \epsilon p \\ p(Y=0, X=1) &= p(Y=0|X=1)p(X=1) = \epsilon(1-p) \\ p(Y=1, X=1) &= (1-\epsilon)(1-p) \\ p(Y=0, X=0) &= (1-\epsilon)p \end{aligned}$$

The marginal distribution of Y is

$$\begin{aligned}
p(Y=1) &= \epsilon p + (1-\epsilon)(1-p) = 1 + 2\epsilon p - \epsilon - p \\
p(Y=0) &= \epsilon(1-p) + (1-\epsilon)p = \epsilon + p - 2\epsilon p
\end{aligned}$$

Hence,

$$\begin{aligned}
H(X) &= -p\log_2 p - (1-p)\log_2(1-p) \\
H(Y) &= -(1+2\epsilon p-\epsilon-p)\log_2(1+2\epsilon p-\epsilon-p) \\
&\quad -(\epsilon+p-2\epsilon p)\log_2(\epsilon+p-2\epsilon p) \\
H(Y|X) &= -\sum_{x,y} p(x,y)\log_2(p(y|x)) = -\epsilon p\log_2\epsilon - \epsilon(1-p)\log_2\epsilon \\
&\quad -(1-\epsilon)(1-p)\log_2(1-\epsilon) - (1-\epsilon)p\log_2(1-\epsilon) \\
&= -\epsilon\log_2\epsilon - (1-\epsilon)\log_2(1-\epsilon) \\
H(X,Y) &= H(X) + H(Y|X) \\
&= -p\log_2 p - (1-p)\log_2(1-p) - \epsilon\log_2\epsilon - (1-\epsilon)\log_2(1-\epsilon) \\
H(X|Y) &= H(X,Y) - H(Y) \\
&= -p\log_2 p - (1-p)\log_2(1-p) - \epsilon\log_2\epsilon - (1-\epsilon)\log_2(1-\epsilon) \\
&\quad (1+2\epsilon p-\epsilon-p)\log_2(1+2\epsilon p-\epsilon-p) \\
&\quad +(\epsilon+p-2\epsilon p)\log_2(\epsilon+p-2\epsilon p) \\
I(X;Y) &= H(X) - H(X|Y) = H(Y) - H(Y|X) \\
&= \epsilon\log_2\epsilon + (1-\epsilon)\log_2(1-\epsilon) \\
&\quad -(1+2\epsilon p-\epsilon-p)\log_2(1+2\epsilon p-\epsilon-p) \\
&\quad -(\epsilon+p-2\epsilon p)\log_2(\epsilon+p-2\epsilon p)
\end{aligned}$$

2) The mutual information is $I(X;Y) = H(Y) - H(Y|X)$. As it was shown in the first question $H(Y|X) = -\epsilon\log_2\epsilon - (1-\epsilon)\log_2(1-\epsilon)$ and thus it does not depend on p. Hence, $I(X;Y)$ is maximized when $H(Y)$ is maximized. However, $H(Y)$ is the binary entropy function with probability $q = 1 + 2\epsilon p - \epsilon - p$, that is

$$H(Y) = H_b(q) = H_b(1 + 2\epsilon p - \epsilon - p)$$

$H_b(q)$ achieves its maximum value, which is one, for $q = \frac{1}{2}$. Thus,

$$1 + 2\epsilon p - \epsilon - p = \frac{1}{2} \Longrightarrow p = \frac{1}{2}$$

3) Since $I(X;Y) \geq 0$, the minimum value of $I(X;Y)$ is zero and it is obtained for independent X and Y. In this case

$$p(Y=1, X=0) = p(Y=1)p(X=0) \Longrightarrow \epsilon p = (1 + 2\epsilon p - \epsilon - p)p$$

or $\epsilon = \frac{1}{2}$. This value of epsilon also satisfies

$$\begin{aligned} p(Y=0,X=0) &= p(Y=0)p(X=0) \\ p(Y=1,X=1) &= p(Y=1)p(X=1) \\ p(Y=0,X=1) &= p(Y=0)p(X=1) \end{aligned}$$

resulting in independent X and Y.

Problem 12.33

$$\begin{aligned} I(X;YZW) &= I(YZW;X) = H(YZW) - H(YZW|X) \\ &= H(Y) + H(Z|Y) + H(W|YZ) \\ &\quad -[H(Y|X) + H(Z|XY) + H(W|XYZ)] \\ &= [H(Y) - H(Y|X)] + [H(Z|Y) - H(Z|YX)] \\ &\quad +[H(W|YZ) - H(W|XYZ)] \\ &= I(X;Y) + I(Z|Y;X) + I(W|ZY;X) \\ &= I(X;Y) + I(X;Z|Y) + I(X;W|ZY) \end{aligned}$$

This result can be interpreted as follows: The information that the triplet of random variables (Y,Z,W) gives about the random variable X is equal to the information that Y gives about X plus the information that Z gives about X, when Y is already known, plus the information that W provides about X when Z, Y are already known.

Problem 12.34

1) Using Bayes rule, we obtain $p(x,y,z) = p(z)p(x|z)p(y|x,z)$. Comparing this form with the one given in the first part of the problem we conclude that $p(y|x,z) = p(y|x)$. This implies that Y and Z are independent given X so that, $I(Y;Z|X) = 0$. Hence,

$$\begin{aligned} I(Y;ZX) &= I(Y;Z) + I(Y;X|Z) \\ &= I(Y;X) + I(Y;Z|X) = I(Y;X) \end{aligned}$$

Since $I(Y;Z) \geq 0$, we have

$$I(Y;X|Z) \leq I(Y;X)$$

2) Comparing $p(x,y,z) = p(x)p(y|x)p(z|x,y)$ with the given form of $p(x,y,z)$ we observe that $p(y|x) = p(y)$ or, in other words, random variables X and Y are independent. Hence,

$$\begin{aligned} I(Y;ZX) &= I(Y;Z) + I(Y;X|Z) \\ &= I(Y;X) + I(Y;Z|X) = I(Y;Z|X) \end{aligned}$$

Since in general $I(Y;X|Z) \geq 0$, we have

$$I(Y;Z) \leq I(Y;Z|X)$$

3) For the first case consider three random variables X, Y and Z, taking the values 0, 1 with equal probability and such that $X = Y = Z$. Then, $I(Y;X|Z) = H(Y|Z) - H(Y|ZX) = 0 - 0 = 0$, whereas $I(Y;X) = H(Y) - H(Y|X) = 1 - 0 = 1$. Hence, $I(Y;X|Z) < I(X;Y)$. For the second case consider two independent random variables X, Y, taking the values 0, 1 with equal probability and a random variable Z which is the sum of X and Y ($Z = X + Y$.) Then, $I(Y;Z) = H(Y) - H(Y|Z) = 1 - 1 = 0$, whereas $I(Y;Z|X) = H(Y|X) - H(Y|ZX) = 1 - 0 = 1$. Thus, $I(Y;Z) < I(Y;Z|X)$.

Problem 12.35

The capacity of the channel is defined as

$$C = \max_{p(x)} I(X;Y) = \max_{p(x)} [H(Y) - H(Y|X)]$$

The conditional entropy $H(Y|X)$ is

$$H(Y|X) = p(X = a)H(Y|X = a) + p(X = b)H(Y|X = b) + p(X = c)H(Y|X = c)$$

However,

$$\begin{aligned} H(Y|X = a) &= -\sum_k p(Y = k|X = a) \log P(Y = k|X = a) \\ &= -(0.2 \log 0.2 + 0.3 \log 0.3 + 0.5 \log 0.5) \\ &= H(Y|X = b) = H(Y|X = c) = 1.4855 \end{aligned}$$

and therefore,

$$H(Y|X) = \sum_k p(X = k)H(Y|X = k) = 1.4855$$

Thus,

$$I(X;Y) = H(Y) - 1.4855$$

To maximize $I(X;Y)$, it remains to maximize $H(Y)$. However, $H(Y)$ is maximized when Y is a uniformly distributed random variable, if such a distribution can be achieved by an appropriate input distribution. Using the symmetry of the channel, we observe that a uniform input distribution produces a uniform output. Thus, the maximum of $I(X;Y)$ is achieved when $p(X = a) = p(X = b) = p(X = c) = \frac{1}{3}$ and the channel capacity is

$$C = \log_2 3 - H(Y|X) = 0.0995 \text{ bits/transmission}$$

Problem 12.36

The capacity of the channel is defined as

$$C = \max_{p(x)} I(X;Y) = \max_{p(x)} [H(Y) - H(Y|X)]$$

If the probability distribution $p(x)$ that achieves capacity is

$$p(X) = \begin{cases} p & X = 0 \\ 1-p & X = 1 \end{cases}$$

then,

$$\begin{aligned} H(Y|X) &= pH(Y|X=0) + (1-p)H(Y|X=1) \\ &= ph(\epsilon) + (1-p)h(\epsilon) = h(\epsilon) \end{aligned}$$

where $h(\epsilon)$ is the binary entropy function. As it is seen $H(Y|X)$ is independent on p and therefore $I(X;Y)$ is maximized when $H(Y)$ is maximized. To find the distribution $p(x)$ that maximizes the entropy $H(Y)$ we reduce first the number of possible outputs as follows. Let V be a function of the output defined as

$$V = \begin{cases} 1 & Y = E \\ 0 & \text{otherwise} \end{cases}$$

Clearly $H(V|Y) = 0$ since V is a deterministic function of Y. Therefore,

$$\begin{aligned} H(Y,V) &= H(Y) + H(V|Y) = H(Y) \\ &= H(V) + H(Y|V) \end{aligned}$$

To find $H(V)$ note that $P(V=1) = P(Y=E) = p\epsilon + (1-p)\epsilon = \epsilon$. Thus, $H(V) = h(\epsilon)$, the binary entropy function at ϵ. To find $H(Y|V)$ we write

$$H(Y|V) = p(V=0)H(Y|V=0) + p(V=1)H(Y|V=1)$$

But $H(Y|V=1) = 0$ since there is no ambiguity on the output when $V=1$, and

$$H(Y|V=0) = -\sum_{k=0,1} p(Y=k|V=0)\log_2 p(Y=k|V=0)$$

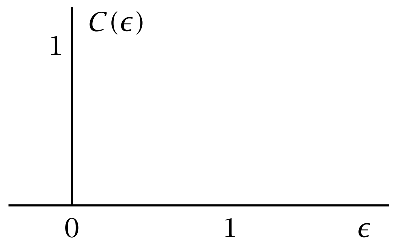
Using Bayes rule, we write the conditional probability $P(Y=0|V=0)$ as

$$P(Y=0|V=0) = \frac{P(Y=0,V=0)}{p(V=0)} = \frac{p(1-\epsilon)}{(1-\epsilon)} = p$$

Thus, $H(Y|V=0)$ is $h(p)$ and $H(Y|V) = (1-\epsilon)h(p)$. The capacity is now written as

$$\begin{aligned} C &= \max_{p(x)} [H(V) + H(Y|V) - h(\epsilon)] \\ &= \max_{p(x)} H(Y|V) = \max_{p(x)} (1-\epsilon)h(p) = (1-\epsilon) \end{aligned}$$

and it is achieved for $p = \frac{1}{2}$. The next figure shows the capacity of the channel as a function of ϵ.



Problem 12.37

The overall channel is a binary symmetric channel with crossover probability p. To find p note that an error occurs if an odd number of channels produce an error. Thus,

$$p = \sum_{k=\text{odd}} \binom{n}{k} \epsilon^k (1-\epsilon)^{n-k}$$

Using the results of Problem 11.24, we find that

$$p = \frac{1}{2}\left[1 - (1-2\epsilon)^2\right]$$

and therefore,

$$C = 1 - h(p)$$

If $n \to \infty$, then $(1-2\epsilon)^n \to 0$ and $p \to \frac{1}{2}$. In this case

$$C = \lim_{n\to\infty} C(n) = 1 - h(\frac{1}{2}) = 0$$

Problem 12.38

Denoting $\bar{\epsilon} = 1-\epsilon$, we have $n! \approx \sqrt{2\pi n} n^n e^{-n}$, $(n\epsilon)! \approx \sqrt{2\pi n\epsilon}(n\epsilon)^{n\epsilon} e^{-n\epsilon}$, and $(n\bar{\epsilon})! \approx \sqrt{2\pi n\bar{\epsilon}}(n\bar{\epsilon})^{n\bar{\epsilon}} e^{-n\bar{\epsilon}}$

$$\begin{aligned}
\binom{n}{n\epsilon} &= \frac{n!}{(n\epsilon)!(n\bar{\epsilon})!} \\
&\approx \frac{\sqrt{2\pi n} n^n e^{-n}}{\sqrt{2\pi n\epsilon}(n\epsilon)^{n\epsilon} e^{-n\epsilon}\sqrt{2\pi n\bar{\epsilon}}(n\bar{\epsilon})^{n\bar{\epsilon}} e^{-n\bar{\epsilon}}} \\
&= \frac{1}{\sqrt{2\pi n\epsilon\bar{\epsilon}}\epsilon^{n\epsilon}\bar{\epsilon}^{n\bar{\epsilon}}}
\end{aligned}$$

From above

$$\begin{aligned}
\frac{1}{n}\log_2\binom{n}{n\epsilon} &\approx -\frac{1}{2n}\log_2(2\pi n\epsilon\bar{\epsilon}) - \epsilon\log_2\epsilon - \bar{\epsilon}\log_2\bar{\epsilon} \\
&\to -\epsilon\log_2\epsilon - \bar{\epsilon}\log_2\bar{\epsilon} \quad \text{as } n\to\infty \\
&= H_b(\epsilon)
\end{aligned}$$

This shows that as $n \to \infty$, $\binom{n}{n\epsilon} \approx 2^{nH_b(\epsilon)}$.

Problem 12.39

Due to the symmetry in channel, the capacity is achieved for uniform input distribution, i.e., for $p(X = A) = p(X = -A) = \frac{1}{2}$. For this input distribution, the output distribution is given by

$$p(y) = \frac{1}{2\sqrt{2\pi\sigma^2}} e^{-(y+A)^2/2\sigma^2} + \frac{1}{2\sqrt{2\pi\sigma^2}} e^{-(y-A)^2/2\sigma^2}$$

and the mutual information between the input and the output is

$$\begin{aligned} I(X;Y) &= \frac{1}{2}\int_{-\infty}^{\infty} p(y \mid X = A) \log_2 \frac{p(y \mid X = A)}{p(y)}\, dy \\ &\quad + \frac{1}{2}\int_{-\infty}^{\infty} p(y \mid X = -A) \log_2 \frac{p(y \mid X = -A)}{p(y)}\, dy \\ &= \frac{1}{2} I_1 + \frac{1}{2} I_2 \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} p(y \mid X = A) \log_2 \frac{p(y \mid X = A)}{p(y)}\, dy \\ I_2 &= \int_{-\infty}^{\infty} p(y \mid X = -A) \log_2 \frac{p(y \mid X = -A)}{p(y)}\, dy \end{aligned}$$

Now consider the first term in the above expression. Substituting for $p(y \mid X = A)$ and $p(y)$, we obtain,

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-A)^2}{2\sigma^2}} \log_2 \frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-A)^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-A)^2}{2\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y+A)^2}{2\sigma^2}}}\, dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y/\sigma - A/\sigma)^2}{2}} \log_2 \frac{2}{1 + e^{-2yA/\sigma^2}}\, dy \end{aligned}$$

using the change of variable $u = y/\sigma$ and denoting A/σ by a we obtain

$$I_1 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(u-a)^2}{2}} \log_2 \frac{2}{1 + e^{-2ua}}\, du$$

A similar approach can be applied to I_2, the second term in the expression for $I(X;Y)$, resulting in

$$I(X;Y) = \frac{1}{2} f\left(\frac{A}{\sigma}\right) + \frac{1}{2} f\left(-\frac{A}{\sigma}\right)$$

where

$$f(a) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(u-a)^2/2} \log_2 \frac{2}{1 + e^{-2au}}\, du$$

Problem 12.40
The capacity of the channel is defined as

$$C = \max_{p(x)} I(X;Y) = \max_{p(x)} [H(Y) - H(Y|X)]$$

However,

$$H(Y|X) = \sum_x p(x) H(Y|X=x) = \sum_x p(x) H(R) = H(R)$$

where $H(R)$ is the entropy of a source with symbols having probabilities the elements of a row of the probability transition matrix. The last equality in the previous equation follows from the fact that $H(R)$ is the same for each row since the channel is symmetric. Thus

$$C = \max_{p(x)} H(Y) - H(R)$$

$H(Y)$ is maximized when Y is a uniform random variable. With a symmetric channel we can always find an input distribution that makes Y uniformly distributed, and thus maximize $H(Y)$. To see this, let

$$p(Y=y) = \sum_x p(x) P(Y=y|X=x)$$

If $p(x) = \frac{1}{|\mathcal{X}|}$, where $|\mathcal{X}|$ is the cardinality of $\mathcal{X}$, then

$$p(Y=y) = \frac{1}{|\mathcal{X}|} \sum_x P(Y=y|X=x)$$

But $\sum_x P(Y=y|X=x)$ is the same for each y since the columns of a symmetric channel are permutations of each other. Thus,

$$C = \log|\mathcal{Y}| - H(R)$$

where $|\mathcal{Y}|$ is the cardinality of the output alphabet.

Problem 12.41
a) The capacity of the channel is

$$C_1 = \max_{p(x)} [H(Y) - H(Y|X)]$$

But, $H(Y|X) = 0$ and therefore, $C_1 = \max_{p(x)} H(Y) = 1$ which is achieved for $p(0) = p(1) = \frac{1}{2}$.

b) Let q be the probability of the input symbol 0, and thus $(1-q)$ the probability of the input symbol 1. Then,

$$\begin{aligned} H(Y|X) &= \sum_x p(x) H(Y|X=x) \\ &= qH(Y|X=0) + (1-q)H(Y|X=1) \\ &= (1-q)H(Y|X=1) = (1-q)h(0.5) = (1-q) \end{aligned}$$

The probability mass function of the output symbols is

$$\begin{aligned} P(Y=c) &= qp(Y=c|X=0)+(1-q)p(Y=c|X=1) \\ &= q+(1-q)0.5 = 0.5+0.5q \\ p(Y=d) &= (1-q)0.5 = 0.5-0.5q \end{aligned}$$

Hence,

$$C_2 = \max_q [h(0.5+0.5q)-(1-q)]$$

To find the probability q that achieves the maximum, we set the derivative of C_2 with respect to q equal to 0. Thus,

$$\begin{aligned} \frac{\partial C_2}{\partial q} = 0 &= 1 - \left[0.5\log_2(0.5+0.5q) + (0.5+0.5q)\frac{0.5}{0.5+0.5q}\frac{1}{\ln 2}\right] \\ &\quad - \left[-0.5\log_2(0.5-0.5q) + (0.5-0.5q)\frac{-0.5}{0.5-0.5q}\frac{1}{\ln 2}\right] \\ &= 1 + 0.5\log_2(0.5-0.5q) - 0.5\log_2(0.5+0.5q) \end{aligned}$$

Therefore,

$$\log_2\frac{0.5-0.5q}{0.5+0.5q} = -2 \Longrightarrow q = \frac{3}{5}$$

and the channel capacity is

$$C_2 = h(\frac{1}{5}) - \frac{2}{5} = 0.3219$$

3) The transition probability matrix of the third channel can be written as

$$\mathbf{Q} = \frac{1}{2}\mathbf{Q}_1 + \frac{1}{2}\mathbf{Q}_2$$

where $\mathbf{Q}_1$, $\mathbf{Q}_2$ are the transition probability matrices of channel 1 and channel 2 respectively. We have assumed that the output space of both channels has been augmented by adding two new symbols so that the size of the matrices $\mathbf{Q}$, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ is the same. The transition probabilities to these newly added output symbols is equal to zero. Now we show that in general, the capacity of a channel is a convex function of the probability transition of the channel, in other words for any two probability transition matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ and any $0 \le \lambda \le 1$, if we define $\mathbf{Q} = \lambda\mathbf{Q}_1 + (1-\lambda)\mathbf{Q}_2$, then $C_{\mathbf{Q}} \le \lambda C_{\mathbf{Q}_1} + (1-\lambda)C_{\mathbf{Q}_2}$, where $C_{\mathbf{Q}_i} = \max_{\mathbf{p}} I(\mathbf{p};\mathbf{Q}_i)$ is the channel capacity when the transition probability is $\mathbf{Q}_i$. To show this we have (note that $\bar{\lambda} = 1 - \lambda$)

$$
\begin{aligned}
I(\mathbf{p}; \lambda\mathbf{Q}_1 + \bar{\lambda}\mathbf{Q}_2) &- \lambda I(\mathbf{p};\mathbf{Q}_1) + \bar{\lambda} I(\mathbf{p};\mathbf{Q}_2) \\
= \; & \sum_x \sum_y p(x)(\lambda p_1(y|x) + \bar{\lambda} p_2(y|x)) \log \frac{\lambda p_1(y|x) + \bar{\lambda} p_2(y|x)}{\sum_x p(x)(\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))} \\
& - \sum_x \sum_y p(x) \lambda p_1(y|x) \log \frac{p_1(y|x)}{\sum_x p(x) p_1(y|x)} \\
& - \sum_x \sum_y p(x) \bar{\lambda} p_2(y|x) \log \frac{p_2(y|x)}{\sum_x p(x) p_2(y|x)} \\
= \; & \sum_x \sum_y p(x) \lambda p_1(y|x) \log \left[\frac{\lambda p_1(y|x) + \bar{\lambda} p_2(y|x)}{\sum_x p(x)(\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))} \frac{\sum_x p(x) p_1(y|x)}{p_1(y|x)} \right] \\
& + \sum_x \sum_y p(x) \bar{\lambda} p_2(y|x) \log \left[\frac{\lambda p_1(y|x) + \bar{\lambda} p_2(y|x)}{\sum_x p(x)(\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))} \frac{\sum_x p(x) p_2(y|x)}{p_2(y|x)} \right] \\
\leq \; & \sum_x \sum_y p(x) \lambda p_1(y|x) \left[\frac{\lambda p_1(y|x) + \bar{\lambda} p_2(y|x)}{\sum_x p(x)(\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))} \frac{\sum_x p(x) p_1(y|x)}{p_1(y|x)} - 1 \right] \\
& + \sum_x \sum_y p(x) \bar{\lambda} p_2(y|x) \left[\frac{\lambda p_1(y|x) + \bar{\lambda} p_2(y|x)}{\sum_x p(x)(\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))} \frac{\sum_x p(x) p_2(y|x)}{p_2(y|x)} - 1 \right] \\
= \; & \sum_y \frac{\sum_x p(x) p_1(y|x)}{\sum_x p(x)(\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))} \sum_x \lambda p(x) p_1(y|x) \frac{\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))}{p_1(y|x)} \\
& - \lambda \sum_x \sum_y p(x) p_1(y|x) \\
& + \sum_y \frac{\sum_x p(x) p_2(y|x)}{\sum_x p(x)(\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))} \sum_x \bar{\lambda} p(x) p_2(y|x) \frac{\lambda p_1(y|x) + \bar{\lambda} p_2(y|x))}{p_2(y|x)} \\
& - \bar{\lambda} \sum_x \sum_y p(x) p_2(y|x) \\
= \; & 0
\end{aligned}
$$

From which we conclude that $C_{\mathbf{Q}} \leq \lambda C_{\mathbf{Q}_1} + (1-\lambda) C_{\mathbf{Q}_2}$. Putting $\lambda = \frac{1}{2}$, we have $C < \frac{1}{2}(C_1 + C_2)$ (Since $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are different, the inequality is strict.)

Problem 12.42

The capacity of a channel is

$$C = \max_{p(x)} I(X;Y) = \max_{p(x)} [H(Y) - H(Y|X)] = \max_{p(x)} [H(X) - H(X|Y)]$$

Since in general $H(X|Y) \geq 0$ and $H(Y|X) \geq 0$, we obtain

$$C \leq \min\{\max[H(Y)], \max[H(X)]\}$$

However, the maximum of $H(X)$ is attained when X is uniformly distributed, in which case $\max[H(X)] = \log|\mathcal{X}|$. Similarly $\max[H(Y)] = \log|\mathcal{Y}|$ and by substituting in the previous inequality, we obtain

$$\begin{aligned} C &\leq \min\{\max[H(Y)], \max[H(X)]\} = \min\{\log|\mathcal{Y}|, \log|\mathcal{X}|\} \\ &= \min\{\log M, \log N\} \end{aligned}$$

Problem 12.43

1) Let q be the probability of the input symbol 0, and therefore $(1-q)$ the probability of the input symbol 1. Then,

$$\begin{aligned} H(Y|X) &= \sum_x p(x) H(Y|X=x) \\ &= qH(Y|X=0) + (1-q)H(Y|X=1) \\ &= (1-q)H(Y|X=1) = (1-q)h(\epsilon) \end{aligned}$$

The probability mass function of the output symbols is

$$\begin{aligned} p(Y=0) &= qp(Y=0|X=0) + (1-q)p(Y=0|X=1) \\ &= q + (1-q)(1-\epsilon) = 1 - \epsilon + q\epsilon \\ p(Y=1) &= (1-q)\epsilon = \epsilon - q\epsilon \end{aligned}$$

Hence,

$$C = \max_q [h(\epsilon - q\epsilon) - (1-q)h(\epsilon)]$$

To find the probability q that achieves the maximum, we set the derivative of C with respect to q equal to 0. Thus,

$$\frac{\partial C}{\partial q} = 0 = h(\epsilon) + \epsilon \log_2(\epsilon - q\epsilon) - \epsilon \log_2(1 - \epsilon + q\epsilon)$$

Therefore,

$$\log_2 \frac{\epsilon - q\epsilon}{1 - \epsilon + q\epsilon} = -\frac{h(\epsilon)}{\epsilon} \Longrightarrow q = \frac{\epsilon + 2^{-\frac{h(\epsilon)}{\epsilon}}(\epsilon - 1)}{\epsilon(1 + 2^{-\frac{h(\epsilon)}{\epsilon}})}$$

and the channel capacity is

$$C = h\left(\frac{2^{-\frac{h(\epsilon)}{\epsilon}}}{1 + 2^{-\frac{h(\epsilon)}{\epsilon}}}\right) - \frac{h(\epsilon) 2^{-\frac{h(\epsilon)}{\epsilon}}}{\epsilon(1 + 2^{-\frac{h(\epsilon)}{\epsilon}})}$$

2) If $\epsilon \to 0$, then using L'Hospital's rule we find that

$$\lim_{\epsilon \to 0} \frac{h(\epsilon)}{\epsilon} = \infty, \qquad \lim_{\epsilon \to 0} \frac{h(\epsilon)}{\epsilon} 2^{-\frac{h(\epsilon)}{\epsilon}} = 0$$

and therefore

$$\lim_{\epsilon \to 0} C(\epsilon) = h(0) = 0$$

If $\epsilon = 0.5$, then $h(\epsilon) = 1$ and $C = h(\frac{1}{5}) - \frac{2}{5} = 0.3219$. In this case the probability of the input symbol 0 is

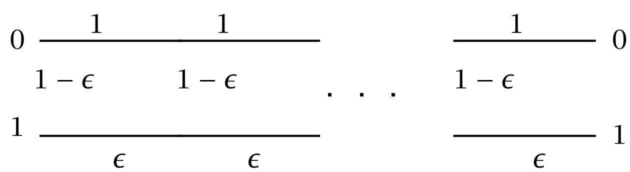
$$q = \frac{\epsilon + 2^{-\frac{h(\epsilon)}{\epsilon}}(\epsilon - 1)}{\epsilon(1 + 2^{-\frac{h(\epsilon)}{\epsilon}})} = \frac{0.5 + 0.25 \times (0.5 - 1)}{0.5 \times (1 + 0.25)} = \frac{3}{5}$$

If $\epsilon = 1$, then $C = h(0.5) = 1$. The input distribution that achieves capacity is $p(0) = p(1) = 0.5$.

3) The following figure shows the topology of the cascade channels. If we start at the input labelled 0, then the output will be 0. If however we transmit a 1, then the output will be zero with probability

$$\begin{aligned} p(Y = 0|X = 1) &= (1-\epsilon) + \epsilon(1-\epsilon) + \epsilon^2(1-\epsilon) + \cdots \\ &= (1-\epsilon)(1 + \epsilon + \epsilon^2 + \cdots) \\ &= 1 - \epsilon\frac{1-\epsilon^n}{1-\epsilon} = 1 - \epsilon^n \end{aligned}$$

Thus, the resulting system is equivalent to a Z channel with $\epsilon_1 = \epsilon^n$.



4) As $n \to \infty$, $\epsilon^n \to 0$ and the capacity of the channel goes to 0.

Problem 12.44
The capacity of Channel A satisfies (see Problem 12.44)

$$C_A \leq \min\{\log_2 M, \log_2 N\}$$

where M, N is the size of the output and input alphabet respectively. Since $M = 2 < 3 = N$, we conclude that $C_A \leq \log_2 2 = 1$. With input distribution $p(A) = p(B) = 0.5$ and $p(C) = 0$, we have a noiseless channel, therefore $C_A = 1$. Similarly, we find that $C_B = 1$, which is achieved when

$$p(a') = p(b') = 0.5,$$

achieved when interpreting B' and C' as a single output. Therefore, the capacity of the cascade channel is $C_{AB} = 1$.

Problem 12.45
The SNR is

$$\text{SNR} = \frac{2P}{N_0 2W} = \frac{P}{2W} = \frac{10}{10^{-9} \times 10^6} = 10^4$$

Thus the capacity of the channel is

$$C = W\log_2(1+\frac{P}{N_0W}) = 10^6\log_2(1+10000) \approx 13.2879\times 10^6 \text{ bits/sec}$$

Problem 12.46
The capacity of the additive white Gaussian channel is

$$C = \frac{1}{2}\log\left(1+\frac{P}{N_0W}\right)$$

For the nonwhite Gaussian noise channel, although the noise power is equal to the noise power in the white Gaussian noise channel, the capacity is higher, The reason is that since noise samples are correlated, knowledge of the previous noise samples provides partial information on the future noise samples and therefore reduces their effective variance.

Problem 12.47
The capacity of the channel of the channel is given by

$$C = \max_{p(x)} I(X;Y) = \max_{p(x)}[H(Y) - H(Y|X)]$$

Let the probability of the inputs C, B and A be p, q and $1-p-q$ respectively. From the symmetry of the nodes B, C we expect that the optimum distribution $p(x)$ will satisfy $p(B) = p(C) = p$. The entropy $H(Y|X)$ is given by

$$\begin{aligned} H(Y|X) &= \sum p(x)H(Y|X=x) = (1-2p)H(Y|X=A) + 2pH(Y|X=B) \\ &= 0 + 2ph(0.5) = 2p \end{aligned}$$

The probability mass function of the output is

$$\begin{aligned} p(Y=1) &= \sum p(x)p(Y=1|X=x) = (1-2p)+p = 1-p \\ p(Y=2) &= \sum p(x)p(Y=2|X=x) = 0.5p + 0.5p = p \end{aligned}$$

Therefore,

$$C = \max_p[H(Y) - H(Y|X)] = \max_p(h(p) - 2p)$$

To find the optimum value of p that maximizes $I(X;Y)$, we set the derivative of C with respect to p equal to zero. Thus,

$$\begin{aligned} \frac{\partial C}{\partial p} = 0 &= -\log_2(p) - p\frac{1}{p\ln(2)} + \log_2(1-p) - (1-p)\frac{-1}{(1-p)\ln(2)} - 2 \\ &= \log_2(1-p) - \log_2(p) - 2 \end{aligned}$$

and therefore

$$\log_2 \frac{1-p}{p} = 2 \Longrightarrow \frac{1-p}{p} = 4 \Longrightarrow p = \frac{1}{5}$$

The capacity of the channel is

$$C = h(\frac{1}{5}) - \frac{2}{5} = 0.7219 - 0.4 = 0.3219 \text{ bits/transmission}$$

Problem 12.48

The capacity of the "product" channel is given by

$$C = \max_{p(x_1,x_2)} I(X_1X_2; Y_1Y_2)$$

However,

$$\begin{aligned} I(X_1X_2;Y_1Y_2) &= H(Y_1Y_2) - H(Y_1Y_2|X_1X_2) \\ &= H(Y_1Y_2) - H(Y_1|X_1) - H(Y_2|X_2) \\ &\leq H(Y_1) + H(Y_2) - H(Y_1|X_1) - H(Y_2|X_2) \\ &= I(X_1;Y_1) + I(X_2;Y_2) \end{aligned}$$

and therefore,

$$\begin{aligned} C = \max_{p(x_1,x_2)} I(X_1X_2;Y_1Y_2) &\leq \max_{p(x_1,x_2)} [I(X_1;Y_1) + I(X_2;Y_2)] \\ &\leq \max_{p(x_1)} I(X_1;Y_1) + \max_{p(x_2)} I(X_2;Y_2) \\ &= C_1 + C_2 \end{aligned}$$

The upper bound is achievable by choosing the input joint probability density $p(x_1, x_2)$, in such a way that

$$p(x_1, x_2) = \tilde{p}(x_1)\tilde{p}(x_2)$$

where $\tilde{p}(x_1)$, $\tilde{p}(x_2)$ are the input distributions that achieve the capacity of the first and second channel respectively.

Problem 12.49

1) Let $\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$, $\mathcal{Y} = \mathcal{Y}_1 + \mathcal{Y}_2$ and

$$p(y|x) = \begin{cases} p(y_1|x_1) & \text{if } x \in \mathcal{X}_1 \\ p(y_2|x_2) & \text{if } x \in \mathcal{X}_2 \end{cases}$$

the conditional probability density function of $\mathcal{Y}$ and $\mathcal{X}$. We define a new random variable M taking the values $1, 2$ depending on the index i of $\mathcal{X}$. Note that M is a function of X or Y. This is because

$\mathcal{X}_1 \cap \mathcal{X}_2 = \emptyset$ and therefore, knowing X we know the channel used for transmission. The capacity of the sum channel is

$$\begin{aligned} C &= \max_{p(x)} I(X;Y) = \max_{p(x)}[H(Y) - H(Y|X)] = \max_{p(x)}[H(Y) - H(Y|X,M)] \\ &= \max_{p(x)}[H(Y) - p(M=1)H(Y|X,M=1) - p(M=2)H(Y|X,M=2)] \\ &= \max_{p(x)}[H(Y) - \lambda H(Y_1|X_1) - (1-\lambda)H(Y_2|X_2)] \end{aligned}$$

where $\lambda = p(M=1)$. Also,

$$\begin{aligned} H(Y) &= H(Y,M) = H(M) + H(Y|M) \\ &= H(\lambda) + \lambda H(Y_1) + (1-\lambda)H(Y_2) \end{aligned}$$

Substituting $H(Y)$ in the previous expression for the channel capacity, we obtain

$$\begin{aligned} C &= \max_{p(x)} I(X;Y) \\ &= \max_{p(x)}[H(\lambda) + \lambda H(Y_1) + (1-\lambda)H(Y_2) - \lambda H(Y_1|X_1) - (1-\lambda)H(Y_2|X_2)] \\ &= \max_{p(x)}[H(\lambda) + \lambda I(X_1;Y_1) + (1-\lambda)I(X_2;Y_2)] \end{aligned}$$

Since $p(x)$ is function of λ, $p(x_1)$ and $p(x_2)$, the maximization over $p(x)$ can be substituted by a joint maximization over λ, $p(x_1)$ and $p(x_2)$. Furthermore, since λ and $1-\lambda$ are nonnegative, we let $p(x_1)$ to maximize $I(X_1;Y_1)$ and $p(x_2)$ to maximize $I(X_2;Y_2)$. Thus,

$$C = \max_{\lambda}[H(\lambda) + \lambda C_1 + (1-\lambda)C_2]$$

To find the value of λ that maximizes C, we set the derivative of C with respect to λ equal to zero. Hence,

$$\frac{dC}{d\lambda} = 0 = -\log_2(\lambda) + \log_2(1-\lambda) + C_1 - C_2 \Longrightarrow \lambda = \frac{2^{C_1}}{2^{C_1} + 2^{C_2}}$$

Substituting this value of λ in the expression for C, we obtain

$$\begin{aligned} C &= H\left(\frac{2^{C_1}}{2^{C_1}+2^{C_2}}\right) + \frac{2^{C_1}}{2^{C_1}+2^{C_2}}C_1 + \left(1 - \frac{2^{C_1}}{2^{C_1}+2^{C_2}}\right)C_2 \\ &= -\frac{2^{C_1}}{2^{C_1}+2^{C_2}}\log_2\left(\frac{2^{C_1}}{2^{C_1}+2^{C_2}}\right) - \left(1 - \frac{2^{C_1}}{2^{C_1}+2^{C_2}}\right)\log_2\left(\frac{2^{C_1}}{2^{C_1}+2^{C_2}}\right) \\ &\quad + \frac{2^{C_1}}{2^{C_1}+2^{C_2}}C_1 + \left(1 - \frac{2^{C_1}}{2^{C_1}+2^{C_2}}\right)C_2 \\ &= \frac{2^{C_1}}{2^{C_1}+2^{C_2}}\log_2(2^{C_1}+2^{C_2}) + \frac{2^{C_2}}{2^{C_1}+2^{C_2}}\log_2(2^{C_1}+2^{C_2}) \\ &= \log_2(2^{C_1}+2^{C_2}) \end{aligned}$$

Hence

$$C = \log_2(2^{C_1}+2^{C_2}) \Longrightarrow 2^C = 2^{C_1} + 2^{C_2}$$

2)

$$2^C = 2^0 + 2^0 = 2 \Longrightarrow C = 1$$

Thus, the capacity of the sum channel is nonzero although the component channels have zero capacity. In this case the information is transmitted through the process of selecting a channel.

3) The channel can be considered as the sum of two channels. The first channel has capacity $C_1 = \log_2 1 = 0$ and the second channel is BSC with capacity $C_2 = 1 - h(0.5) = 0$. Thus

$$C = \log_2(2^{C_1} + 2^{C_2}) = \log_2(2) = 1$$

Problem 12.50

1) The entropy of the source is

$$H(X) = h(0.3) = 0.8813$$

and the capacity of the channel

$$C = 1 - h(0.1) = 1 - 0.469 = 0.531$$

If the source is directly connected to the channel, then the probability of error at the destination is

$$\begin{aligned} P(\text{error}) &= p(X=0)p(Y=1|X=0) + p(X=1)p(Y=0|X=1) \\ &= 0.3 \times 0.1 + 0.7 \times 0.1 = 0.1 \end{aligned}$$

2) For reliable transmission we must have $H(X) = C = 1 - h(\epsilon)$. Hence, with $H(X) = 0.8813$ we obtain

$$0.8813 = 1 - h(\epsilon) \Longrightarrow \epsilon < 0.016 \text{ or } \epsilon > 0.984$$

Computer Problems

Computer Problem 12.1

1) Figure 12.1 shows the Huffman code tree

2) The average codeword length for this code is

$$\begin{aligned} \bar{L} &= 2 \times 0.2 + 3 \times (0.15 + 0.13 + 0.12 + 0.1) + 4 \times (0.09 + 0.08 + 0.07 + 0.06) \\ &= 3.1 \quad \text{bits per source output} \end{aligned}$$

3) The entropy of the source is given as

$$H(X) = -\sum_{i=1}^{9} p_i \log p_i = 3.0371 \quad \text{bits per source output}$$

We observe that $\bar{L} > H(X)$, as expected.

The MATLAB function entropy.m given next calculates the entropy of a probability vector $\boldsymbol{p}$.

```
function h=entropy(p)
%          H=ENTROPY(P) returns the entropy function of
%          the probability vector p.
if length(find(p<0))~=0,
  error('Not a prob. vector, negative component(s)')
end
if abs(sum(p)-1)>10e-10,
  error('Not a prob. vector, components do not add up to 1')
end
h=sum(-p.*log2(p));
```

Computer Problem 12.2

1) The entropy of the source is derived via the entropy.m function and is found to be 2.3549 bits per source symbol.

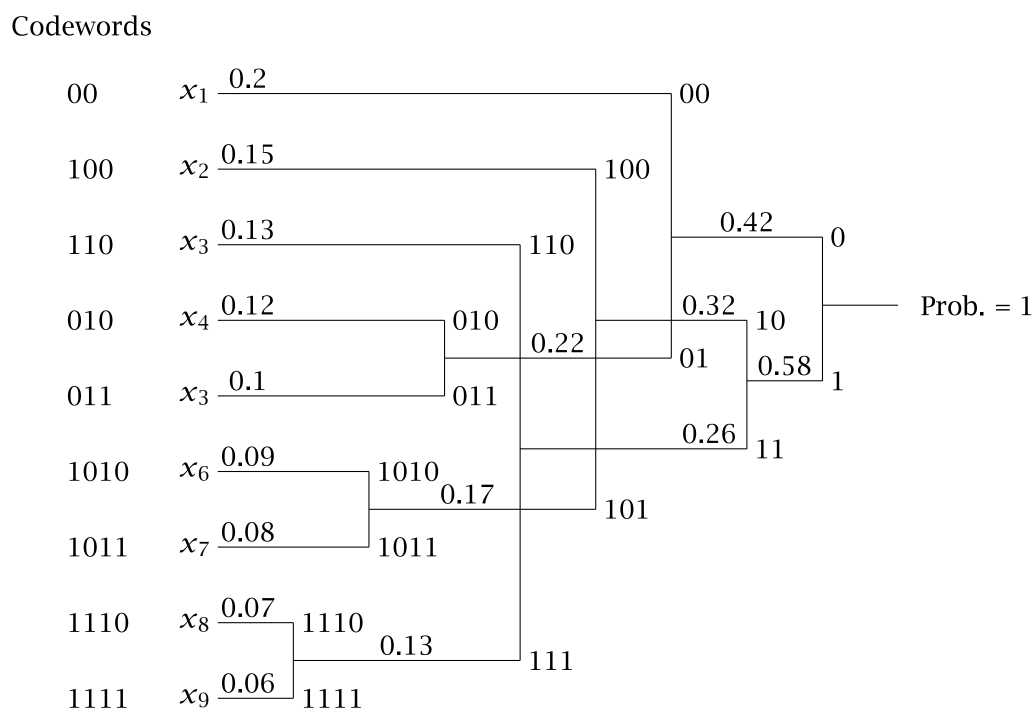


Figure 12.1: Huffman code tree

2) We can solve this problem using the MATLAB function huffman.m, which designs a Huffman code for a discrete-memoryless source with probability vector $\boldsymbol{p}$ and returns both the codewords and the average codeword length. huffman.m function is given next.

```
function [h,l]=huffman(p);
%HUFFMAN      Huffman code generator
%             [h,l]=huffman(p), Huffman code generator
%             returns h the Huffman code matrix, and l the
%             average codeword length for a source with
%             probability vector p.

if length(find(p<0))~=0,
  error('Not a prob. vector, negative component(s)')
end
if abs(sum(p)-1)>10e-10,
  error('Not a prob. vector, components do not add up to 1')
end
n=length(p);
q=p;
m=zeros(n-1,n);
for i=1:n-1
  [q,l]=sort(q);
  m(i,:)=[l(1:n-i+1),zeros(1,i-1)];
  q=[q(1)+q(2),q(3:n),1];
end
for i=1:n-1
  c(i,:)=blanks(n*n);
end
c(n-1,n)='0';
c(n-1,2*n)='1';
for i=2:n-1
  c(n-i,1:n-1)=c(n-i+1,n*(find(m(n-i+1,:)==1))...
  -(n-2):n*(find(m(n-i+1,:)==1)));
  c(n-i,n)='0';
  c(n-i,n+1:2*n-1)=c(n-i,1:n-1);
  c(n-i,2*n)='1';
  for j=1:i-1
    c(n-i,(j+1)*n+1:(j+2)*n)=c(n-i+1,...
    n*(find(m(n-i+1,:)==j+1)-1)+1:n*find(m(n-i+1,:)==j+1));
  end
end
for i=1:n
  h(i,1:n)=c(1,n*(find(m(1,:)==i)-1)+1:find(m(1,:)==i)*n);
  l1(i)=length(find(abs(h(i,:))~=32));
end
l=sum(p.*l1);
```

Using this function, the codewords are found to be 010, 11, 0110, 0111, 00, and 10.

3) The average codeword length for this code is found to be 2.38 binary symbols per source output. Therefore, the efficiency of this code is

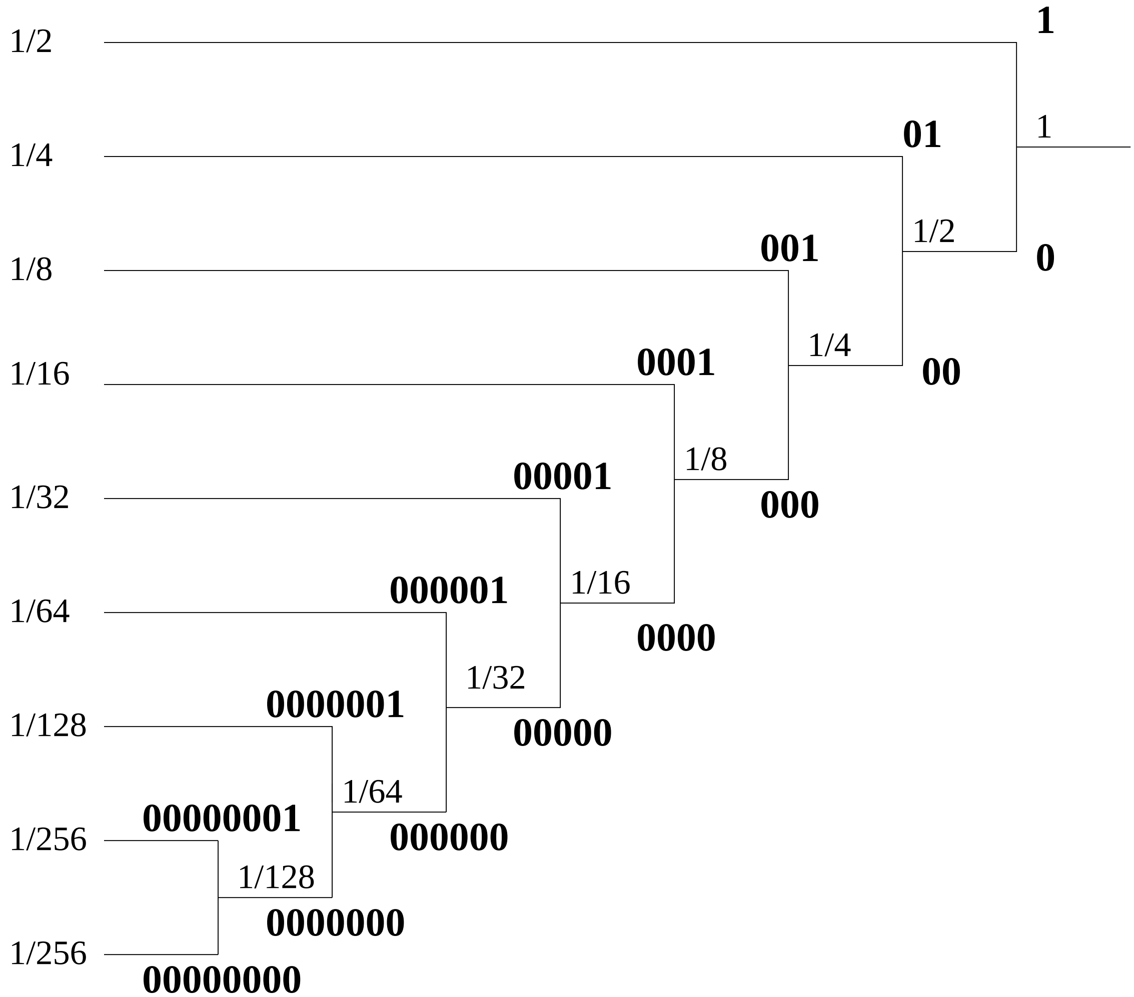
$$\eta_1 = \frac{2.3549}{2.38} = 0.9895$$

4) A new source whose outputs are letter pairs of the original source has 36 output letters of the form $\{(x_i, x_j)\}_{i,j=1}^{6}$. Since the source is memoryless, the probability of each pair is the product of the individual letter probabilities. Thus, in order to obtain the probability vector for the extended source, we must generate a vector with 36 components, each component being the product of two probabilities in the original probability vector $\boldsymbol{p}$. This can be done by employing the MATLAB function kron.m in the form of kron(p, p). The Huffman codewords are given by
1110000, 01110, 10110111, 1011001, 111001, 00101, 01111, 000, 011010, 00111, 1001, 1100, 11101110, 011011, 111011110, 111011111, 1110001, 001000, 1011010, 01100, 10110110, 1011000, 101110, 111110, 111010, 1010, 1110110, 101111, 11110, 0100, 00110, 1101, 001001, 111111, 0101, 1000
The average codeword length for the extended source is 4.7420. The entropy of the extended source is found to be 4.7097, so the efficiency of this Huffman code is

$$\eta_2 = \frac{4.7097}{4.7420} = 0.9932$$

which shows an improvement compared to the efficiency of the Huffman code designed in part 2.

Computer Problem 12.3
1) We use the huffman.m function to determine a Huffman code. The resulting codewords are 1, 01, 001, 0001, 00001, 000001, 0000001, 00000000, and 00000001. The figure below presents the code tree.



2) The average codeword length is found to be 1.9922 binary symbols per source output by using huffman.m function. If we find the entropy of the source using the entropy.m function, we see that the entropy of the source is also 1.9922 bits per source output; hence the efficiency of this code is 1.
3) For the efficiency of the Huffman code to be equal to one, the source must have a diadic distribution, i.e., all probabilities be powers of $\frac{1}{2}$.

Computer Problem 12.4
1) The entropy is found to be 3.7179 using the Matlab function entropy.m introduced in Computer Problem 12.1.
2) We design a Huffman code using the Matlab function huffman.m. that was presented in Computer Problem 12.2. This function generates the resulting code words 1011, 100000, 00000, 10100, 010, 110011, 100001, 0001, 0111, 11000000111,, 11000001, 10101, 110010, 0110, 1001, 100010, 1100000010, 0010, 0011, 1101, 00001, 1100001, 110001, 110000000, 100011, 11000000110, 111.
3) The average code word is found to be 4.1195. In this case efficiency of the code is

$$\eta = \frac{3.7179}{4.1195} = 0.9025$$

The MATLAB script for this problem follows.

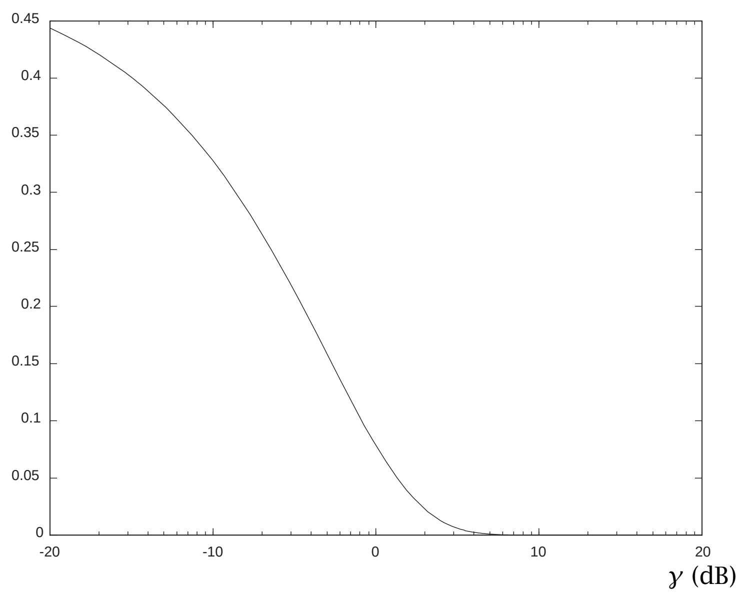
```
% MATLAB script for Computer Problem 12.4.
A = 0.0642; B = 0.0127; C = 0.0218;
D = 0.0317; E = 0.1031; F = 0.0208;
G = 0.0152; H = 0.0467; I = 0.0575;
J = 0.0008; K = 0.0049; L = 0.0321;
M = 0.0198; N = 0.0574; O = 0.0632;
P = 0.0152; Q = 0.0008; R = 0.0484;
S = 0.0514; T = 0.0796; U = 0.0228;
V = 0.0083; W = 0.0175; X = 0.0013;
Y = 0.0164; Z = 0.0005; Space = 0.1859;
p = [A B C D E F G H I J K L M N O P Q R S T U V W X Y Z Space];
% Compute the entropy
H = entropy(p)
% Design a Huffman code
[h l] = huffman(p);
```

Computer Problem 12.5

1. The error probability of the BPSK with optimal detection is given by

$$p = Q\left(\sqrt{2\gamma}\right) \tag{12.38}$$

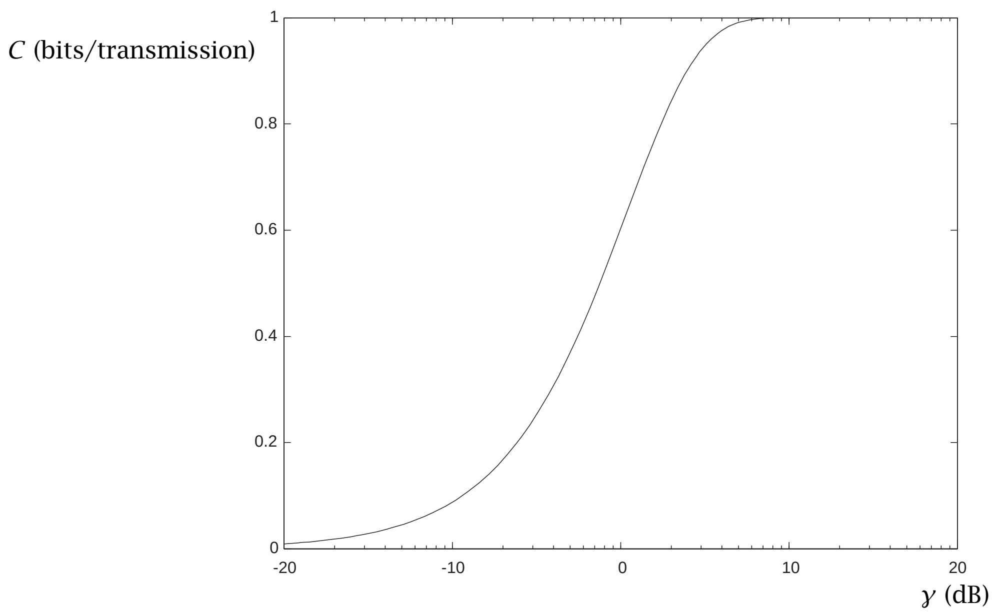
The corresponding plot is shown below.



2. Here we use the relation

$$\begin{aligned} C &= 1 - H_b(p) \\ &= 1 - H_b\left(Q\left(\sqrt{2\gamma}\right)\right) \end{aligned} \tag{12.39}$$

to obtain a plot of C versus γ. This plot is shown below.



The MATLAB script for this problem is given next.

```
echo on
gamma_db=[−20:0.1:20];
gamma=10.^(gamma_db./10);
p_error=q(sqrt(2.*gamma));
capacity=1.−entropy2(p_error);
pause % Press a key to see a plot of error probability vs. SNR/bit.
clf
semilogx(gamma,p_error)
xlabel('SNR/bit')
title('Error probability versus SNR/bit')
ylabel('Error Prob.')
pause % Press a key to see a plot of channel capacity vs. SNR/bit.
clf
semilogx(gamma,capacity)
xlabel('SNR/bit')
title('Channel capacity versus SNR/bit')
ylabel('Channel capacity')
```

Computer Problem 12.6

Due to the symmetry in the problem, the capacity is achieved for uniform input distribution—that is, for $p(X = A) = p(X = -A) = \frac{1}{2}$. For this input distribution, the output distribution is given by

$$p(y) = \frac{1}{2\sqrt{2\pi\sigma^2}} e^{-(y+A)^2/2\sigma^2} + \frac{1}{2\sqrt{2\pi\sigma^2}} e^{-(y-A)^2/2\sigma^2} \tag{12.40}$$

and the mutual information between the input and the output is given by

$$\begin{aligned} I(X;Y) = &\frac{1}{2}\int_{-\infty}^{\infty} p(y \mid X = A) \log_2 \frac{p(y \mid X = A)}{p(y)}\, dy \\ &+ \frac{1}{2}\int_{-\infty}^{\infty} p(y \mid X = -A) \log_2 \frac{p(y \mid X = -A)}{p(y)}\, dy \end{aligned} \tag{12.41}$$

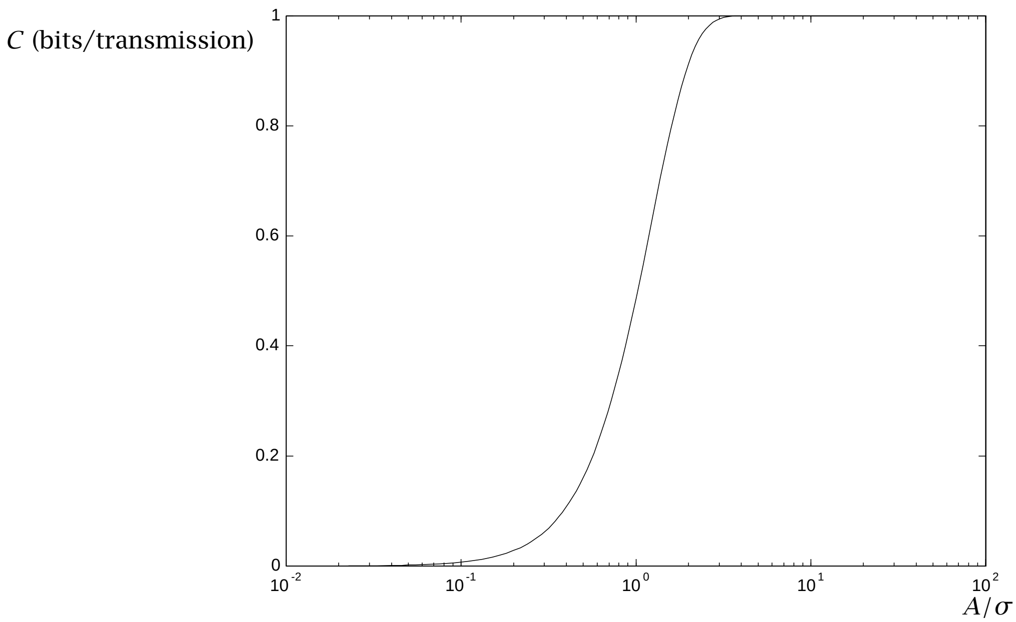
Simple integration and change of variables result in

$$I(X;Y) = f\left(\frac{A}{\sigma}\right) \tag{12.42}$$

where

$$f(a) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(u-a)^2/2} \log_2 \frac{2}{1 + e^{-2au}}\, du \tag{12.43}$$

Using these relations we can calculate $I(X;Y)$ for various values of A/σ and plot the result. A plot of the resulting curve is shown below.



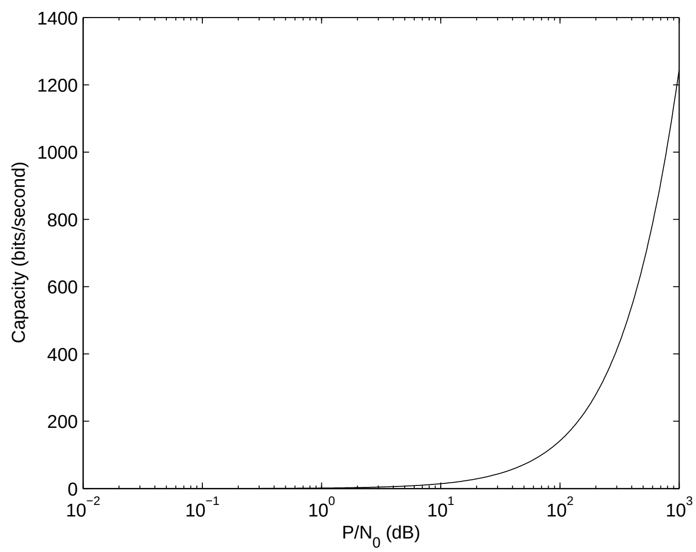
The MATLAB script for this problem follows.

```
echo on
a_db=[−20:0.2:20];
a=10.^(a_db/10);
for i=1:201
```

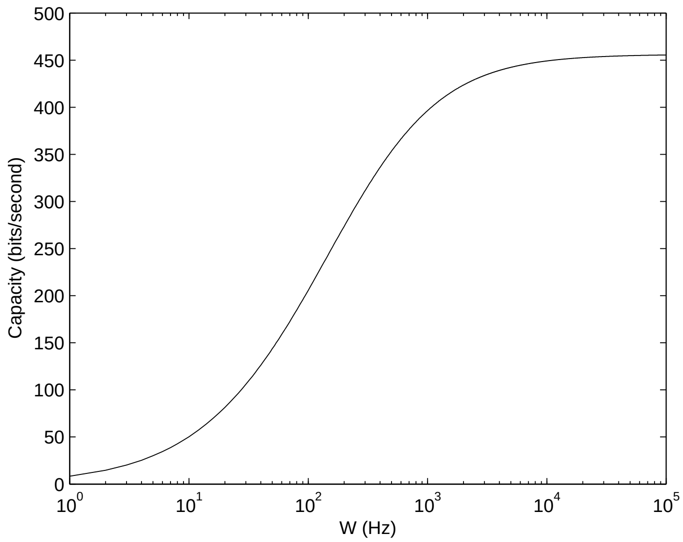
```
    f(i)=quad('il3_8fun',a(i)-5,a(i)+5,1e-3,[],a(i));
    g(i)=quad('il3_8fun',-a(i)-5,-a(i)+5,1e-3,[],-a(i));
    c(i)=0.5*f(i)+0.5*g(i);
    echo off ;
end
echo on ;
pause % Press a key to see capacity vs. SNR plot.
semilogx(a,c)
title('Capacity versus SNR in binary input AWGN channel')
xlabel('SNR')
ylabel('Capacity (bits/transmission)')
```

Computer Problem 12.7

1) The desired plot is given below.



2) The capacity as a function of bandwidth is plotted here.



As is seen in the plots, when either P/N_0 or W tend to zero, the capacity of the channel also tends to zero. However, when P/N_0 or W tends to infinity, the capacity behaves differently. When P/N_0 tends to infinity, the capacity also tends to infinity, as shown in the first figure. However, when W tends to infinity, the capacity does go to a certain limit, which is determined by P/N_0. To determine this limiting value, we have

$$\lim_{W\to\infty} W \log_2\left(1 + \frac{P}{N_0 W}\right) = \frac{P}{N_0 \ln 2} \tag{12.44}$$

$$= 1.4427 \frac{P}{N_0} \tag{12.45}$$

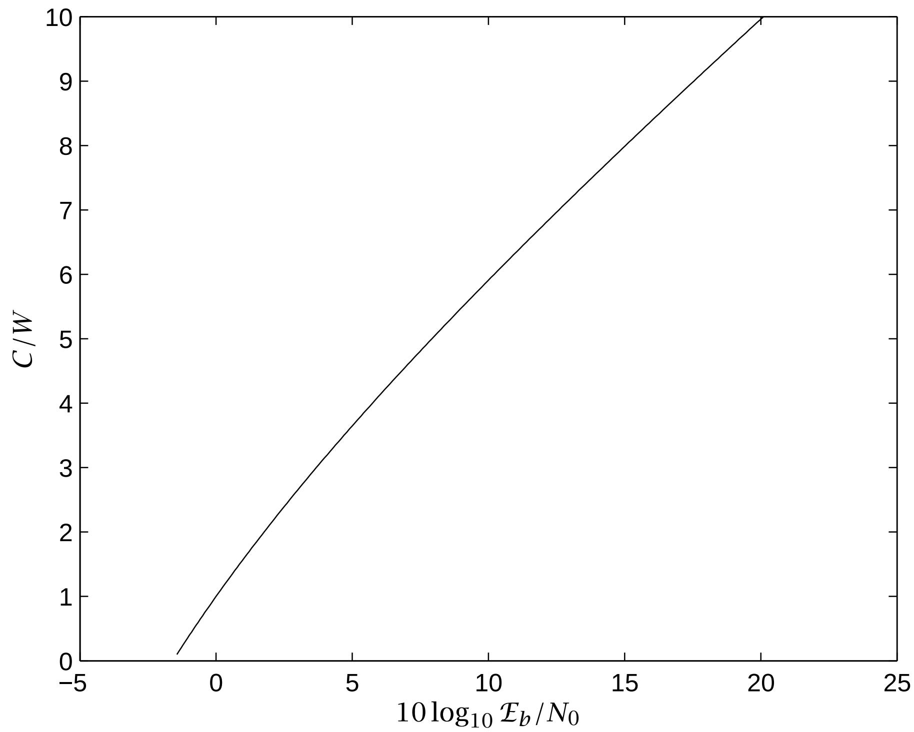
The MATLAB script for this problem is given next.

```
% MATLAB script for Computer Problem 12.7.
echo on
pn0_db=[−20:0.1:30];
pn0=10.^(pn0_db./10);
capacity=3000.*log2(1+pn0/3000);
pause % Press a key to see a plot of channel capacity vs. P/N0.
clf
semilogx(pn0,capacity)
title('Capacity vs. P/N0 in an AWGN channel')
xlabel('P/N0')
ylabel('Capacity (bits/second)')
clear
w=[1:10,12:2:100,105:5:500,510:10:5000,5025:25:20000,20050:50:100000];
pn0_db=25;
pn0=10^(pn0_db/10);
capacity=w.*log2(1+pn0./w);
pause % Press a key to see a plot of channel capacity vs. bandwidth.
clf
semilogx(w,capacity)
title('Capacity vs. bandwidth in an AWGN channel')
xlabel('Bandwidth (Hz)')
```

```
ylabel('Capacity (bits/second)')
```

Computer Problem 12.8

This figure presents the normalized capacity C/W as a function of $10\log_{10} \mathcal{E}_b/N_0$.



The MATLAB script for this problem is given next.

```
% MATLAB script for Computer Problem 12.8
C_W = 0.1:0.05:10; % C_W= C/W;
Eb_No = ((2.^C_W) - 1) ./C_W; % Eb_No = Eb/No
Eb_No_indB = 10*log10(Eb_No);
plot(Eb_No_indB ,C_W );
```

Chapter 13

Problem 13.1

The codewords of the linear code of Example 13.2.1 are

$$\begin{aligned}\mathbf{c}_1 &= [\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ ]\\ \mathbf{c}_2 &= [\ 1\ \ 0\ \ 1\ \ 0\ \ 0\ ]\\ \mathbf{c}_3 &= [\ 0\ \ 1\ \ 1\ \ 1\ \ 1\ ]\\ \mathbf{c}_4 &= [\ 1\ \ 1\ \ 0\ \ 1\ \ 1\ ]\end{aligned}$$

Since the code is linear the minimum distance of the code is equal to the minimum weight of the codewords. Thus,

$$d_{\min} = w_{\min} = 2$$

There is only one codeword with weight equal to 2 and this is $\mathbf{c}_2$.

Problem 13.2

The parity check matrix of the code in Example 13.2.3 is

$$\mathbf{H} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The codewords of the code are

$$\begin{aligned}\mathbf{c}_1 &= [\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ ]\\ \mathbf{c}_2 &= [\ 1\ \ 0\ \ 1\ \ 0\ \ 0\ ]\\ \mathbf{c}_3 &= [\ 0\ \ 1\ \ 1\ \ 1\ \ 1\ ]\\ \mathbf{c}_4 &= [\ 1\ \ 1\ \ 0\ \ 1\ \ 1\ ]\end{aligned}$$

Any of the previous codewords when postmultiplied by $\mathbf{H}^t$ produces an all-zero vector of length 3. For example

$$\begin{aligned}\mathbf{c}_2\mathbf{H}^t &= [\ 1 \oplus 1\ \ \ 0\ \ \ 0\ ] = [\ 0\ \ 0\ \ 0\ ]\\ \mathbf{c}_4\mathbf{H}^t &= [\ 1 \oplus 1\ \ \ 1 \oplus 1\ \ \ 1 \oplus 1\ ] = [\ 0\ \ 0\ \ 0\ ]\end{aligned}$$

Problem 13.3

The following table lists all the codewords of the (7,4) Hamming code along with their weight. Since the Hamming codes are linear $d_{\min} = w_{\min}$. As it is observed from the table the minimum weight is 3 and therefore $d_{\min} = 3$.

No.	Codewords	Weight
1	0000000	0
2	1000110	3
3	0100011	3
4	0010101	3
5	0001111	4
6	1100101	4
7	1010011	4
8	1001001	3
9	0110110	4
10	0101100	3
11	0011010	3
12	1110000	3
13	1101010	4
14	1011100	4
15	0111001	4
16	1111111	7

Problem 13.4

The parity check matrix **H** of the (15,11) Hamming code consists of all binary sequences of length 4, except the all zero sequence. The systematic form of the matrix **H** is

$$\mathbf{H} = [\ \mathbf{P}^t \ \mid \ \mathbf{I}_4\ ] = \left(\begin{array}{ccccccccccc|cccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

The corresponding generator matrix is

$$\mathbf{G} = [\ \mathbf{I}_{11} \ \ | \ \ \mathbf{P}\ ] = \left(\begin{array}{ccccccccccc|cccc}
1 & & & & & & & & & & & 1 & 1 & 0 & 0 \\
 & 1 & & & & & & & & & & 1 & 0 & 1 & 0 \\
 & & 1 & & & & & & \mathbf{0} & & & 1 & 0 & 0 & 1 \\
 & & & 1 & & & & & & & & 0 & 1 & 1 & 0 \\
 & & & & 1 & & & & & & & 0 & 1 & 0 & 1 \\
 & & & & & 1 & & & & & & 0 & 0 & 1 & 1 \\
 & & & & & & 1 & & & & & 1 & 1 & 1 & 0 \\
 & & & & & & & 1 & & & & 1 & 1 & 0 & 1 \\
 & & & \mathbf{0} & & & & & 1 & & & 1 & 0 & 1 & 1 \\
 & & & & & & & & & 1 & & 0 & 1 & 1 & 1 \\
 & & & & & & & & & & 1 & 1 & 1 & 1 & 1
\end{array}\right)$$

Problem 13.5

Let C be an (n, k) linear block code with parity check matrix $\mathbf{H}$. We can express the parity check matrix in the form

$$\mathbf{H} = [\ \mathbf{h}_1 \quad \mathbf{h}_2 \quad \cdots \quad \mathbf{h}_n\ ]$$

where $\mathbf{h}_i$ is an $n - k$ dimensional column vector. Let $\mathbf{c} = [c_1 \cdots c_n]$ be a codeword of the code C with l nonzero elements which we denote as $c_{i_1}, c_{i_2}, \ldots, c_{i_l}$. Clearly $c_{i_1} = c_{i_2} = \ldots = c_{i_l} = 1$ and since $\mathbf{c}$ is a codeword

$$\begin{aligned}
\mathbf{c}\mathbf{H}^t = 0 \quad &= \quad c_1\mathbf{h}_1 + c_2\mathbf{h}_2 + \cdots + c_n\mathbf{h}_n \\
&= \quad c_{i_1}\mathbf{h}_{i_1} + c_{i_2}\mathbf{h}_{i_2} + \cdots + c_{i_l}\mathbf{h}_{i_l} \\
&= \quad \mathbf{h}_{i_1} + \mathbf{h}_{i_2} + \cdots + \mathbf{h}_{i_l} = 0
\end{aligned}$$

This proves that l column vectors of the matrix $\mathbf{H}$ are linear dependent. Since for a linear code the minimum value of l is $w_{\min}$ and $w_{\min} = d_{\min}$, we conclude that there exist $d_{\min}$ linear dependent column vectors of the matrix $\mathbf{H}$.

Now we assume that the minimum number of column vectors of the matrix $\mathbf{H}$ that are linear dependent is $d_{\min}$ and we will prove that the minimum weight of the code is $d_{\min}$. Let $\mathbf{h}_{i_1}, \mathbf{h}_{i_2}, \ldots, \mathbf{h}_{d_{\min}}$ be a set of linear dependent column vectors. If we form a vector $\mathbf{c}$ with non-zero components at positions $i_1, i_2, \ldots, i_{d_{\min}}$, then

$$\mathbf{c}\mathbf{H}^t = c_{i_1}\mathbf{h}_{i_1} + \cdots + c_{i_{d_{\min}}} = 0$$

which implies that $\mathbf{c}$ is a codeword with weight $d_{\min}$. Therefore, the minimum distance of a code is equal to the minimum number of columns of its parity check matrix that are linear dependent.

For a Hamming code the columns of the matrix $\mathbf{H}$ are non-zero and distinct. Thus, no two columns $\mathbf{h}_i$, $\mathbf{h}_j$ add to zero and since $\mathbf{H}$ consists of all the $n-k$ tuples as its columns, the sum $\mathbf{h}_i + \mathbf{h}_j = \mathbf{h}_m$ should also be a column of $\mathbf{H}$. Then,

$$\mathbf{h}_i + \mathbf{h}_j + \mathbf{h}_m = 0$$

and therefore the minimum distance of the Hamming code is 3.

Problem 13.6

The generator matrix of the $(n, 1)$ repetition code is a $1 \times n$ matrix, consisted of the non-zero codeword. Thus,

$$\mathbf{G} = \begin{bmatrix} 1 & | & 1 & \cdots & 1 \end{bmatrix}$$

This generator matrix is already in systematic form, so that the parity check matrix is given by

$$\mathbf{H} = \left(\begin{array}{c|cccc} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{array}\right)$$

Problem 13.7

1) The parity check matrix $\mathbf{H}_e$ of the extended code is an $(n+1-k) \times (n+1)$ matrix. The codewords of the extended code have the form

$$\mathbf{c}_{e,i} = [\ \mathbf{c}_i \quad | \quad x\ ]$$

where x is 0 if the weight of $\mathbf{c}_i$ is even and 1 if the weight of $\mathbf{c}_i$ is odd. Since $\mathbf{c}_{e,i}\mathbf{H}_e^t = [\mathbf{c}_i|x]\mathbf{H}_e^t = 0$ and $\mathbf{c}_i\mathbf{H}^t = 0$, the first $n-k$ columns of $\mathbf{H}_e^t$ can be selected as the columns of $\mathbf{H}^t$ with a zero added in the last row. In this way the choice of x is immaterial. The last column of $\mathbf{H}_e^t$ is selected in such a way that the even-parity condition is satisfied for every codeword $\mathbf{c}_{e,i}$. Note that if $\mathbf{c}_{e,i}$ has even weight, then

$$c_{e,i_1} + c_{e,i_2} + \cdots + c_{e,i_{n+1}} = 0 \Longrightarrow \mathbf{c}_{e,i}[\ 1 \quad 1 \quad \cdots \quad 1\ ]^t = 0$$

for every i. Therefore the last column of $\mathbf{H}_e^t$ is the all-one vector and the parity check matrix of the extended code has the form

$$\mathbf{H}_e = \left(\mathbf{H}_e^t\right)^t = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ \\ 0 & 0 & 0 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

2) The original code has minimum distance equal to 3. But for those codewords with weight equal to the minimum distance, a 1 is appended at the end of the codewords to produce even parity. Thus, the minimum weight of the extended code is 4 and since the extended code is linear, the minimum distance is $d_{e,\min} = w_{e,\min} = 4$.

3) The coding gain of the extended code is

$$G_{\text{coding}} = d_{e,min} R_c = 4 \times \frac{3}{7} = 1.7143$$

Problem 13.8

If no coding is employed, we have

$$p_b = Q\left[\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right] = Q\left[\sqrt{\frac{P}{RN_0}}\right]$$

where

$$\frac{P}{RN_0} = \frac{10^{-6}}{10^4 \times 2 \times 10^{-11}} = 5$$

Thus,

$$p_b = Q[\sqrt{5}] = 1.2682 \times 10^{-2}$$

and therefore, the error probability for 11 bits is

$$P_{\text{error in 11 bits}} = 1 - (1 - p_b)^{11} \approx 0.1310$$

If coding is employed, then since the minimum distance of the $(15, 11)$ Hamming code is 3,

$$p_e \leq (M-1)Q\left[\sqrt{\frac{d_{\min}\mathcal{E}_s}{N_0}}\right] = 10Q\left[\sqrt{\frac{3\mathcal{E}_s}{N_0}}\right]$$

where

$$\frac{\mathcal{E}_s}{N_0} = R_c\frac{\mathcal{E}_b}{N_0} = R_c\frac{P}{RN_0} = \frac{11}{15} \times 5 = 3.6667$$

Thus

$$p_e \leq 10Q\left[\sqrt{3 \times 3.6667}\right] \approx 4.560 \times 10^{-3}$$

As it is observed the probability of error decreases by a factor of 28. If hard decision is employed, then

$$p_e \leq (M-1) \sum_{i=\frac{d_{\min}+1}{2}}^{d_{\min}} \binom{d_{\min}}{i} p_b^i (1-p_b)^{d_{\min}-i}$$

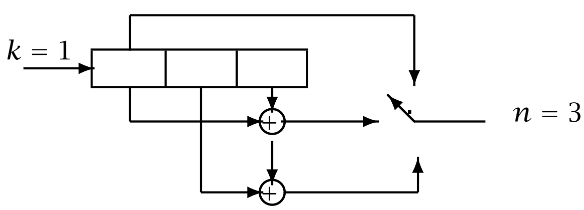
where $M = 10$, $d_{\min} = 3$ and $p_b = Q\left[\sqrt{R_c \frac{P}{RN_0}}\right] = 2.777 \times 10^{-2}$. Hence,

$$p_e = 10 \times (3 \times p_b^2(1-p_b) + p_b^3) = 0.0227$$

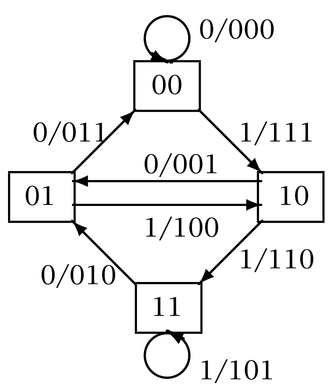
In this case coding has decreased the error probability by a factor of 6.

Problem 13.9

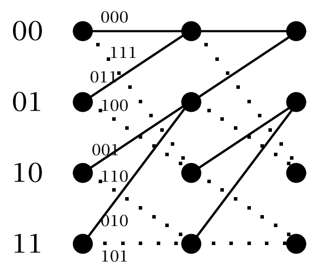
1) The encoder for the $(3,1)$ convolutional code is depicted in the next figure.



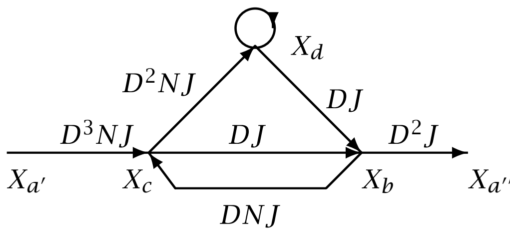
2) The state transition diagram for this code is depicted in the next figure.



3) In the next figure we draw two frames of the trellis associated with the code. Solid lines indicate an input equal to 0, whereas dotted lines correspond to an input equal to 1.



4) The diagram used to find the transfer function is shown in the next figure.



Using the flow graph results, we obtain the system

$$\begin{aligned} X_c &= D^3NJX_{a'} + DNJX_b \\ X_b &= DJX_c + DJX_d \\ X_d &= D^2NJX_c + D^2NJX_d \\ X_{a''} &= D^2JX_b \end{aligned}$$

Eliminating X_b, X_c and X_d results in

$$T(D,N,J) = \frac{X_{a''}}{X_{a'}} = \frac{D^6NJ^3}{1 - D^2NJ - D^2NJ^2}$$

To find the free distance of the code we set $N = J = 1$ in the transfer function, so that

$$T_1(D) = T(D,N,J)|_{N=J=1} = \frac{D^6}{1-2D^2} = D^6 + 2D^8 + 4D^{10} + \cdots$$

Hence, $d_{\text{free}} = 6$

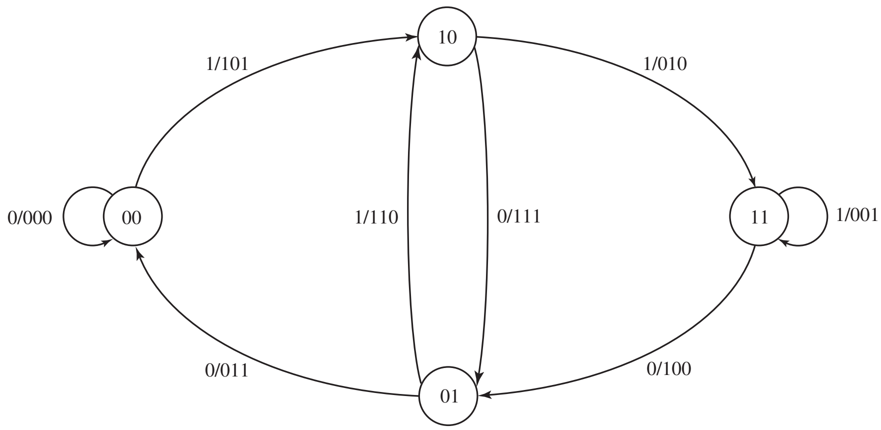
5) Since there is no self loop corresponding to an input equal to 1 such that the output is the all zero sequence, the code is not catastrophic.

Problem 13.10

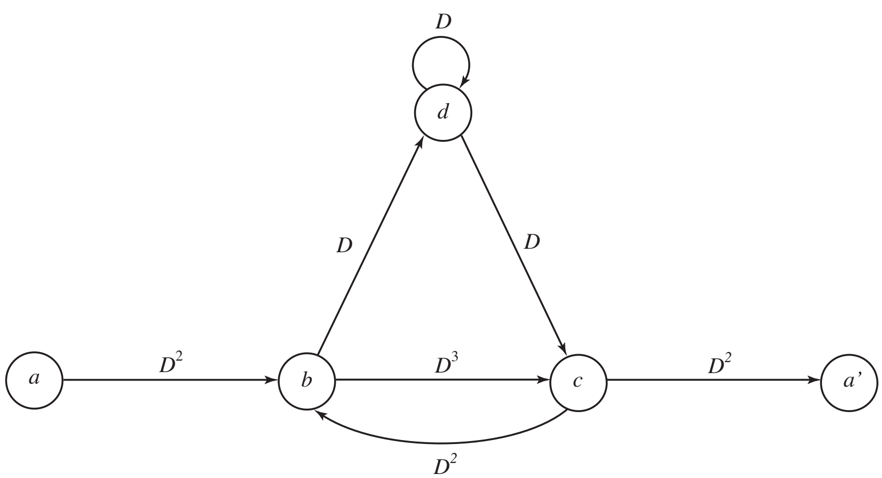
The number of branches leaving each state correspond to the number possible different inputs to the encoder. Since the encoder at each state takes k binary symbols at its input, the number of branches leaving each state of the trellis is 2^k. The number of branches entering each state is the number of possible kL contents of the encoder shift register that have their first $k(L-1)$ bits corresponding to that particular state (note that the destination state for a branch is determined by the contents of the first $k(L-1)$ bits of the shift register). This means that the number of branches is equal to the number of possible different contents of the last k bits of the encoder, i.e., 2^k.

Problem 13.11

1.



2.



we have the following equations

$$\begin{aligned} X_b &= D^2 X_a + D^2 X_c \\ X_c &= D^3 X_b + D X_d \\ X_d &= D X_d + D X_b \\ X_{a'} &= D^2 X_c \end{aligned}$$

from which we obtain

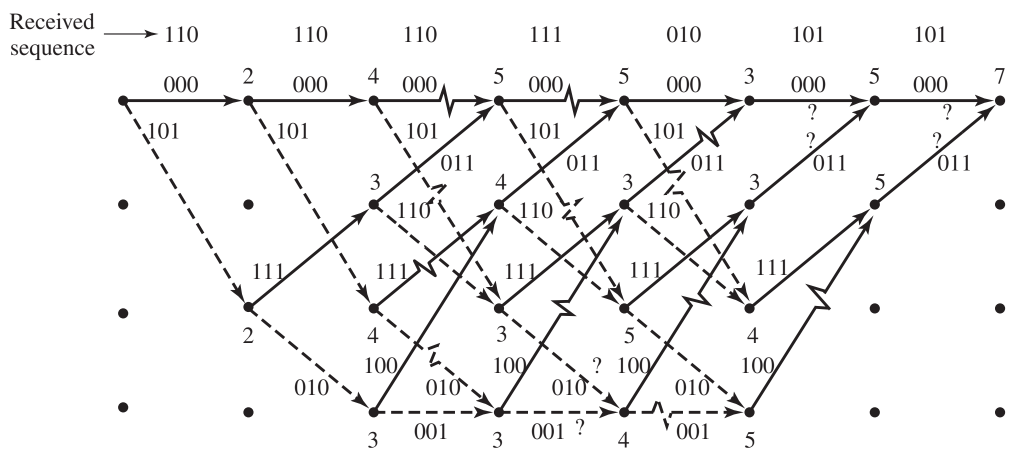
$$T(D) = \frac{X_{a'}}{X_a} = \frac{D^6 + D^7 - D^8}{1 - D - D^4 - D^5 + D^6}$$

3. Expanding $T(D)$ we have

$$T(D) = D^6 + 2D^7 + D^8 + \dots$$

Therefore $d_{\text{free}} = 6$.

4.



From this figure we see that there are many options, one is the sequence 0101000, in which the two last bits are the additional two zeros to reset the memory. Therefore (one of the) most likely transmitted sequences is 01010. Other options are 10101 and 11000. All these sequences result in codewords that are at a Hamming distance of 7 from the received sequence.

5. In general we have to use Equation (13.3.23)

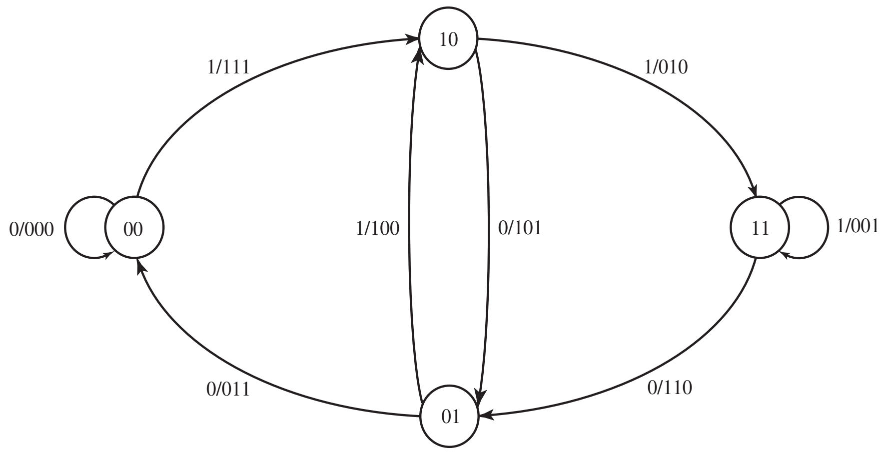
$$\overline{P_b} \leq \frac{1}{k} \left. \frac{\partial T_2(D,N)}{\partial N} \right|_{N=1, D=\sqrt{4p(1-p)}}$$

But here $p = 10^{-5}$ is very small, therefore the dominating term in the expansion on the right side of the inequality will be the first term in expansion of $T_2(D,N)$ corresponding to d_{free}. This term in $T_2(D,N)$ is D^6N, since the path with weight 6 at the output corresponds to the input sequence 100, which is of weight 1. Therefore

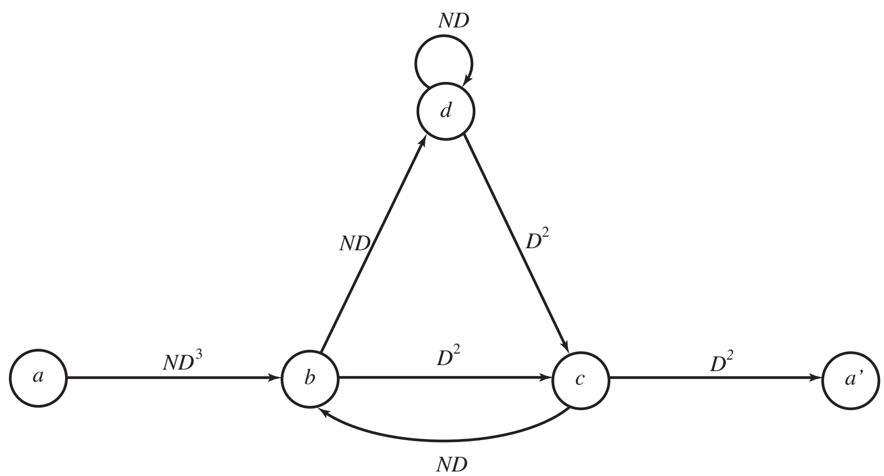
$$\overline{P_b} \leq \frac{1}{k} \left. \frac{\partial D^6 N}{\partial N} \right|_{N=1, D=\sqrt{4p(1-p)}} = \frac{1}{3} \left(4 \times 10^{-5}(1 - 10^{-5})\right)^3 \approx 2 \times 10^{-14}$$

Problem 13.12

1.



2.



From the figure we have the following equations

$$\begin{aligned} X_b &= ND^3 X_a + NDX_c \\ X_c &= D^2 X_b + D^2 X_d \\ X_d &= NDX_d + NDX_b \\ X_{a'} &= D^2 X_c \end{aligned}$$

From which we obtain

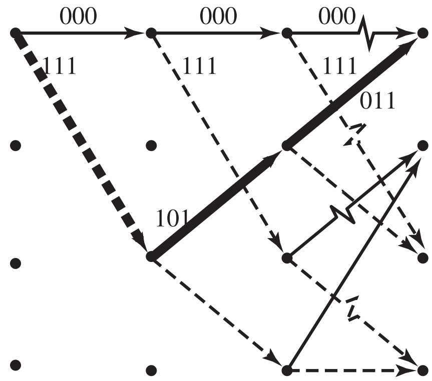
$$T_2(D,N) = \frac{ND^7}{1 - ND - ND^3}$$

Substituting $N = 1$ we have $T(D) = \frac{D^7}{1-D-D^3}$.

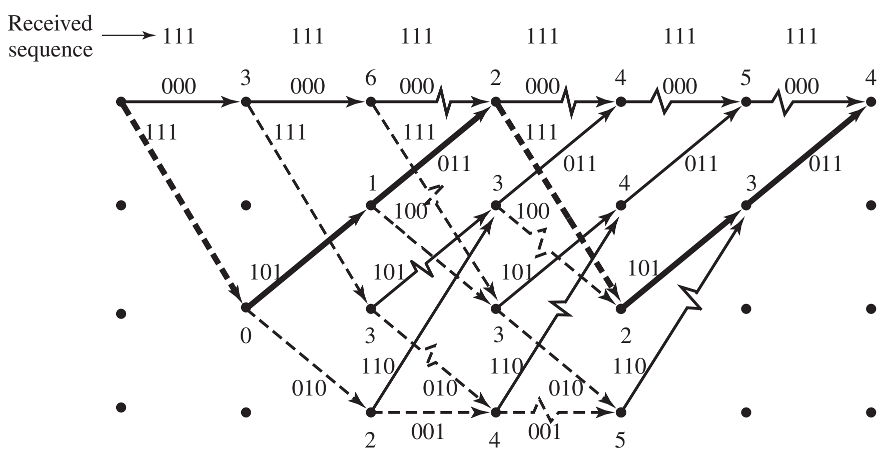
3. Expanding $T_2(N,D)$ we have

$$T_2(N,D) = ND^7 + N^2D^8 + N^3D^9 + \ldots$$

and therefore $d_{\text{free}} = 7$. This path corresponds to the sequence 100. The path is highlighted below



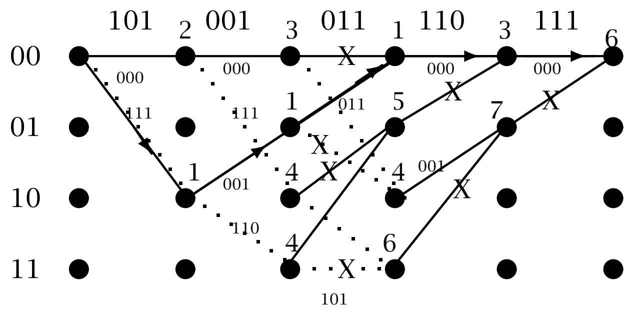
4. From the trellis diagram shown below



We see that the best matching path through the trellis is the highlighted path corresponding to 100100. Therefore the information sequence is 1001.

Problem 13.13

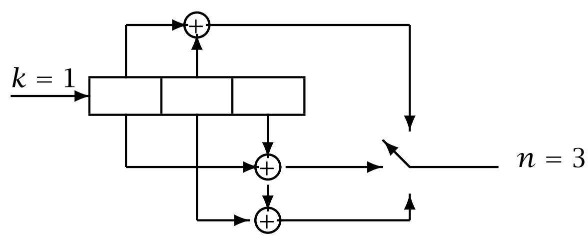
The code of Problem 13.11 is a $(3,1)$ convolutional code with $L = 3$. The length of the received sequence $\mathbf{y}$ is 15. This means that 5 symbols have been transmitted, and since we assume that the information sequence has been padded by two 0's, the actual length of the information sequence is 3. The following figure depicts 5 frames of the trellis used by the Viterbi decoder. The numbers on the nodes denote the metric (Hamming distance) of the survivor paths. In the case of a tie of two merging paths at a node, we have purged the lower path.



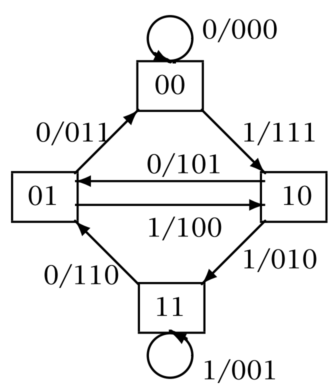
The decoded sequence is $\{111, 001, 011, 000, 000\}$ and corresponds to the information sequence $\{1, 0, 0\}$ followed by two zeros.

Problem 13.14

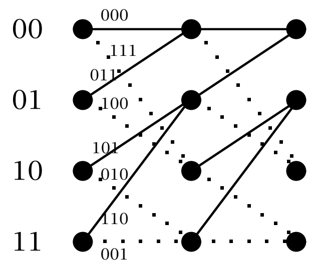
1) The encoder for the $(3, 1)$ convolutional code is depicted in the next figure.



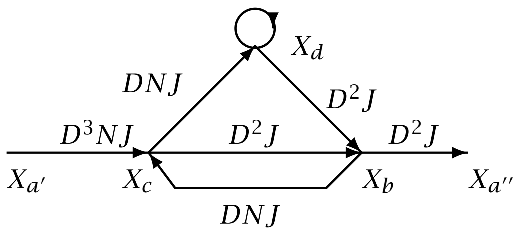
2) The state transition diagram for this code is shown below



3) In the next figure we draw two frames of the trellis associated with the code. Solid lines indicate an input equal to 0, whereas dotted lines correspond to an input equal to 1.



4) The diagram used to find the transfer function is shown in the next figure.



Using the flow graph results, we obtain the system

$$\begin{aligned} X_c &= D^3NJX_{a'} + DNJX_b \\ X_b &= D^2JX_c + D^2JX_d \\ X_d &= DNJX_c + DNJX_d \\ X_{a''} &= D^2JX_b \end{aligned}$$

Eliminating X_b, X_c and X_d results in

$$T(D,N,J) = \frac{X_{a''}}{X_{a'}} = \frac{D^7NJ^3}{1 - DNJ - D^3NJ^2}$$

To find the free distance of the code we set $N = J = 1$ in the transfer function, so that

$$T_1(D) = T(D,N,J)|_{N=J=1} = \frac{D^7}{1 - D - D^3} = D^7 + D^8 + D^9 + \cdots$$

Hence, $d_{\text{free}} = 7$

5) Since there is no self loop corresponding to an input equal to 1 such that the output is the all zero sequence, the code is not catastrophic.

Problem 13.15

Using the diagram of Figure 13.13, we see that there are only two ways to go from state $X_{a'}$ to state $X_{a''}$ with a total number of ones (sum of the exponents of D) equal to 6. The corresponding transitions are:

$$\text{Path 1:} \quad X_{a'} \overset{D^2}{\rightarrow} X_c \overset{D}{\rightarrow} X_d \overset{D}{\rightarrow} X_b \overset{D^2}{\rightarrow} X_{a''}$$

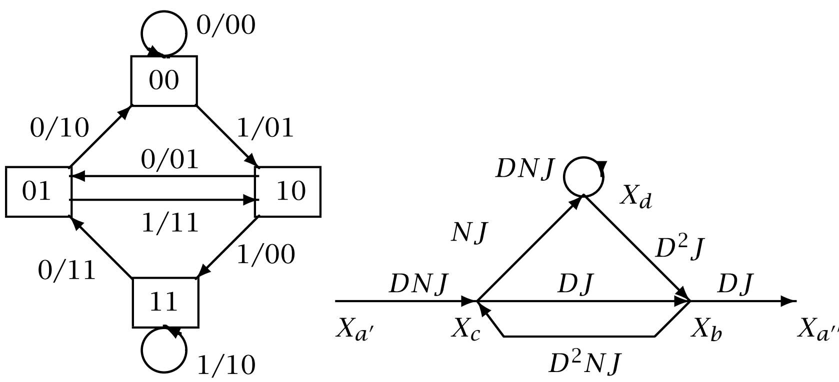
$$\text{Path 2:} \quad X_{a'} \overset{D^2}{\rightarrow} X_c \overset{D}{\rightarrow} X_b \rightarrow X_c \overset{D}{\rightarrow} X_b \overset{D^2}{\rightarrow} X_{a''}$$

These two paths correspond to the codewords

$$\begin{aligned}\mathbf{c}_1 &= 0,0,\ 1,0,\ 1,0,\ 1,1,\ 0,0,\ 0,0,\ldots\\ \mathbf{c}_2 &= 0,0,\ 0,1,\ 0,0,\ 0,1,\ 1,1,\ 0,0,\ldots\end{aligned}$$

Problem 13.16

1) The state transition diagram and the flow diagram used to find the transfer function for this code are depicted in the next figure.



Thus,

$$\begin{aligned}X_c &= DNJX_{a'} + D^2NJX_b\\ X_b &= DJX_c + D^2JX_d\\ X_d &= NJX_c + DNJX_d\\ X_{a''} &= DJX_b\end{aligned}$$

and by eliminating X_b, X_c and X_d, we obtain

$$T(D,N,J) = \frac{X_{a''}}{X_{a'}} = \frac{D^3NJ^3}{1 - DNJ - D^3NJ^2}$$

To find the transfer function of the code in the form $T(D,N)$, we set $J = 1$ in $T(D,N,J)$. Hence,

$$T(D,N) = \frac{D^3N}{1 - DN - D^3N}$$

2) To find the free distance of the code we set $N = 1$ in the transfer function $T(D,N)$, so that

$$T_1(D) = T(D,N)|_{N=1} = \frac{D^3}{1 - D - D^3} = D^3 + D^4 + D^5 + 2D^6 + \cdots$$

Hence, $d_{\text{free}} = 3$

3) An upper bound on the bit error probability, when hard decision decoding is used, is given by

$$\bar{P}_b \leq \frac{1}{k}\frac{\partial T(D,N)}{\partial N}\bigg|_{N=1,D=\sqrt{4p(1-p)}}$$

Since

$$\left.\frac{\partial T(D,N)}{\partial N}\right|_{N=1} = \left.\frac{\partial}{\partial N}\frac{D^3 N}{1-(D+D^3)N}\right|_{N=1} = \frac{D^3}{(1-(D+D^3))^2}$$

with $k=1$, $p=10^{-6}$ we obtain

$$\bar{P}_b \leq \left.\frac{D^3}{(1-(D+D^3))^2}\right|_{D=\sqrt{4p(1-p)}} = 8.0321\times 10^{-9}$$

Problem 13.17

1) Let the decoding rule be that the first codeword is decoded when $\mathbf{y}_i$ is received if

$$p(\mathbf{y}_i|\mathbf{x}_1) > p(\mathbf{y}_i|\mathbf{x}_2)$$

The set of $\mathbf{y}_i$ that decode into $\mathbf{x}_1$ is

$$Y_1 = \{\mathbf{y}_i : p(\mathbf{y}_i|\mathbf{x}_1) > p(\mathbf{y}_i|\mathbf{x}_2)\}$$

The characteristic function of this set $\chi_1(\mathbf{y}_i)$ is by definition equal to 0 if $\mathbf{y}_i \notin Y_1$ and equal to 1 if $\mathbf{y}_i \in Y_1$. The characteristic function can be bounded as (see Problem 9.40)

$$1-\chi_1(\mathbf{y}_i) \leq \left(\frac{p(\mathbf{y}_i|\mathbf{x}_2)}{p(\mathbf{y}_i|\mathbf{x}_1)}\right)^{\frac{1}{2}}$$

Given that the first codeword is sent, then the probability of error is

$$\begin{aligned}
P(\text{error}|\mathbf{x}_1) &= \sum_{\mathbf{y}_i\in Y-Y_1} p(\mathbf{y}_i|\mathbf{x}_1) = \sum_{\mathbf{y}_i\in Y} p(\mathbf{y}_i|\mathbf{x}_1)[1-\chi_1(\mathbf{y}_i)] \\
&\leq \sum_{\mathbf{y}_i\in Y} p(\mathbf{y}_i|\mathbf{x}_1)\left(\frac{p(\mathbf{y}_i|\mathbf{x}_2)}{p(\mathbf{y}_i|\mathbf{x}_1)}\right)^{\frac{1}{2}} = \sum_{\mathbf{y}_i\in Y}\sqrt{p(\mathbf{y}_i|\mathbf{x}_1)p(\mathbf{y}_i|\mathbf{x}_2)} \\
&= \sum_{i=1}^{2^n}\sqrt{p(\mathbf{y}_i|\mathbf{x}_1)p(\mathbf{y}_i|\mathbf{x}_2)}
\end{aligned}$$

where Y denotes the set of all possible sequences $\mathbf{y}_i$. Since, each element of the vector $\mathbf{y}_i$ can take two values, the cardinality of the set Y is 2^n.

2) Using the results of the previous part we have

$$\begin{aligned}
P(\text{error}) &\leq \sum_{i=1}^{2^n}\sqrt{p(\mathbf{y}_i|\mathbf{x}_1)p(\mathbf{y}_i|\mathbf{x}_2)} = \sum_{i=1}^{2^n} p(\mathbf{y}_i)\sqrt{\frac{p(\mathbf{y}_i|\mathbf{x}_1)}{p(\mathbf{y}_i)}}\sqrt{\frac{p(\mathbf{y}_i|\mathbf{x}_2)}{p(\mathbf{y}_i)}} \\
&= \sum_{i=1}^{2^n} p(\mathbf{y}_i)\sqrt{\frac{p(\mathbf{x}_1|\mathbf{y}_i)}{p(\mathbf{x}_1)}}\sqrt{\frac{p(\mathbf{x}_2|\mathbf{y}_i)}{p(\mathbf{x}_2)}} = \sum_{i=1}^{2^n} 2p(\mathbf{y}_i)\sqrt{p(\mathbf{x}_1|\mathbf{y}_i)p(\mathbf{x}_2|\mathbf{y}_i)}
\end{aligned}$$

However, given the vector $\mathbf{y}_i$, the probability of error depends only on those values that $\mathbf{x}_1$ and $\mathbf{x}_2$ are different. In other words, if $x_{1,k} = x_{2,k}$, then no matter what value is the k^{th} element of $\mathbf{y}_i$, it will not produce an error. Thus, if by d we denote the Hamming distance between $\mathbf{x}_1$ and $\mathbf{x}_2$, then

$$p(\mathbf{x}_1|\mathbf{y}_i)p(\mathbf{x}_2|\mathbf{y}_i) = p^d(1-p)^d$$

and since $p(\mathbf{y}_i) = \frac{1}{2^n}$, we obtain

$$P(\text{error}) = P(d) = 2p^{\frac{d}{2}}(1-p)^{\frac{d}{2}} = [4p(1-p)]^{\frac{d}{2}}$$

3) Assuming codeword $\mathbf{c}$ is sent, let $d_1, d_2, \ldots, d_{M-1}$ denote its Hamming distance from other codewords, then by the union bound

$$P(\text{Error}|\mathbf{c}) \leq \sum_{i=1}^{M-1} (4p(1-p))^{\frac{d_i}{2}} \leq (M-1)\left(4p(1-p)\right)^{\frac{d_{\min}}{2}}$$

where in the last step we have used the fact that since $0 \leq p \leq 1$, then $4p(1-p) \leq 1$, and for $x \leq 1$ the function x^p is a decreasing function of p. Therefore since $\frac{d_{\min}}{2} \leq \frac{d_i}{2}$ we have $(4p(1-p))^{\frac{d_i}{2}} \leq (4p(1-p))^{\frac{d_{\min}}{2}}$.

Problem 13.18

1)

$$\begin{aligned} Q(x) &= \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{v^2}{2}}dv \\ &\overset{v=\sqrt{2}t}{=} \frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{2}}}^\infty e^{-t^2}dt \\ &= \frac{1}{2}\frac{2}{\pi}\int_{\frac{x}{\sqrt{2}}}^\infty e^{-t^2}dt \\ &= \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right) \end{aligned}$$

2) The average bit error probability can be bounded as (see (13.3.16))

$$\begin{aligned} \bar{P}_b &\leq \frac{1}{k}\sum_{d=d_{\text{free}}}^\infty a_d f(d) Q\left[\sqrt{2R_c d\frac{\mathcal{E}_b}{N_0}}\right] = \frac{1}{k}\sum_{d=d_{\text{free}}}^\infty a_d f(d) Q\left[\sqrt{2R_c d\gamma_b}\right] \\ &= \frac{1}{2k}\sum_{d=d_{\text{free}}}^\infty a_d f(d)\text{erfc}(\sqrt{R_c d\gamma_b}) \\ &= \frac{1}{2k}\sum_{d=1}^\infty a_{d+d_{\text{free}}} f(d+d_{\text{free}})\text{erfc}(\sqrt{R_c(d+d_{\text{free}})\gamma_b}) \\ &\leq \frac{1}{2k}\text{erfc}(\sqrt{R_c d_{\text{free}}\gamma_b})\sum_{d=1}^\infty a_{d+d_{\text{free}}} f(d+d_{\text{free}}) e^{-R_c d\gamma_b} \end{aligned}$$

But,

$$T(D,N) = \sum_{d=d_{\text{free}}}^\infty a_d D^d N^{f(d)} = \sum_{d=1}^\infty a_{d+d_{\text{free}}} D^{d+d_{\text{free}}} N^{f(d+d_{\text{free}})}$$

and therefore,

$$\begin{aligned}\left.\frac{\partial T(D,N)}{\partial N}\right|_{N=1} &= \sum_{d=1}^{\infty} a_{d+d_{\text{free}}} D^{d+d_{\text{free}}} f(d+d_{\text{free}}) \\ &= D^{d_{\text{free}}} \sum_{d=1}^{\infty} a_{d+d_{\text{free}}} D^{d} f(d+d_{\text{free}})\end{aligned}$$

Setting $D = e^{-R_c \gamma_b}$ in the previous and substituting in the expression for the average bit error probability, we obtain

$$\bar{P}_b \le \frac{1}{2k} \operatorname{erfc}(\sqrt{R_c d_{\text{free}} \gamma_b}) e^{R_c d_{\text{free}} \gamma_b} \left.\frac{\vartheta T(D,N)}{\vartheta N}\right|_{N=1, D=e^{-R_c \gamma_b}}$$

Problem 13.19

1. For the Hamming code $d_{\min} = d_1 = 3$ and for the second code simple inspection shows $d_{\min} = d_2 = 4$. Therefore for the product code we have $d_{\min} = d_1 d_2 = 3 \times 4 = 12$.

2. $e_c = \left\lfloor \frac{d_{\min}-1}{2} \right\rfloor = 5$.

3. For Hamming code we use the result of Example 13.2.4 and for the $(6,2)$ code use the given generator matrix to obtain

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

4. The Hamming code can correct all single errors and the other code, with minimum distance of 4 can also correct all single errors. For three errors, if they are on different rows and columns, they can obviously been corrected. If all three are in a single row, then in each column we do not have more than one error, and the errors can all be corrected, similarly for the case when all errors are in a single column. If two errors are in a row and two errors are in a column, like the figure below

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & X & 1 & X & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}$$

then we can first correct the single error in the fifth row, and then the single errors in the columns. A four error pattern shown below cannot be corrected by applying hard decision to the rows and columns separately (but can be corrected by applying the optimal hard decision decoding to the product code)

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & X & 1 & X & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X & 0 & X & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

Problem 13.20

By definition $\max^*\{x, y\} = \ln(e^x + e^y)$, if $x = y$ the conclusion is obvious, otherwise, if $x > y$ we have

$$\begin{aligned} \max^*\{x, y\} &= \ln[e^x(1 + e^{y-x})] \\ &= \ln(e^x) + \ln(1 + e^{|y-x|}) \\ &= \max\{x, y\} + \ln(1 + e^{|y-x|}) \end{aligned}$$

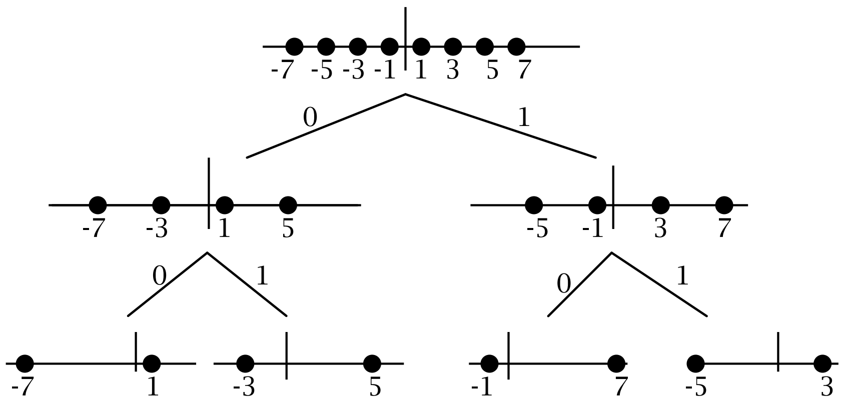
and the result is proved, for $y > x$ we can similarly show the result.

For the second relation, we define $w = \ln(e^x + e^y) = \max^*\{x, y\}$, then

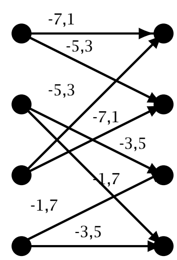
$$\begin{aligned} \max^*\{x, y, z\} &= \ln(e^x + e^y + e^z) \\ &= \ln(e^w + e^z) \\ &= \max^*\{w, z\} \\ &= \max^*\{\max^*\{x, y\}, z\} \end{aligned}$$

Problem 13.21

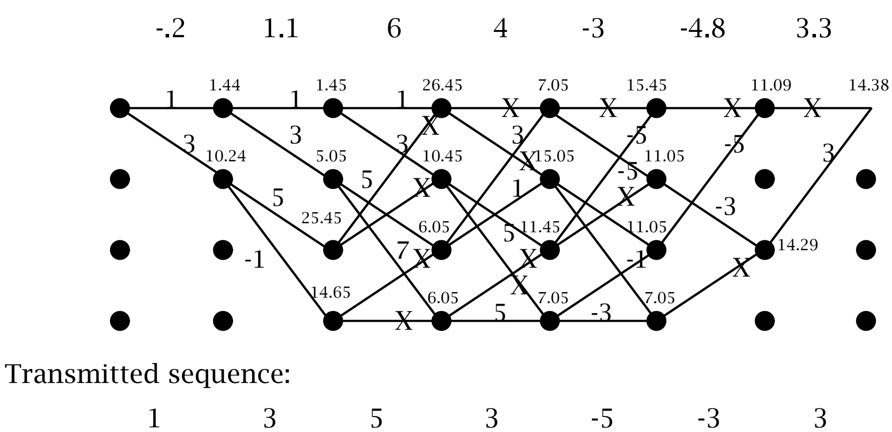
The partition of the 8-PAM constellation in four subsets is depicted in the figure below.



2) The next figure shows one frame of the trellis used to decode the received sequence. Each branch consists of two transitions which correspond to elements in the same coset in the final partition level.



The operation of the Viterbi algorithm for the decoding of the sequence $\{-.2, 1.1, 6, 4, -3, -4.8, 3.3\}$ is shown schematically in the next figure. It has been assumed that we start at the all zero state and that a sequence of zeros terminates the input bit stream in order to clear the encoder. The numbers at the nodes indicate the minimum Euclidean distance, and the branches have been marked with the decoded transmitted symbol. The paths that have been purged are marked with an X.



Computer Problems

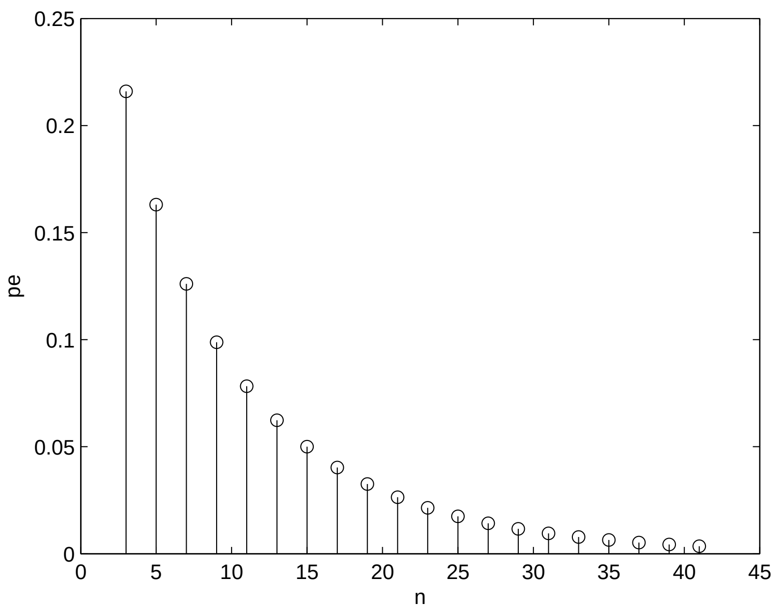
Computer Problem 13.1

We derive p_e for values of n from 1 to 61. The error probability is given by

$$p_e = \sum_{k=(n+1)/2}^{n} \binom{n}{k} 0.3^k \times 0.7^{n-k}$$

and the resulting plot is shown below. The MATLAB script file for this problem is given next.

```
echo on
ep=0.3;
for i=1:2:61
  p(i)=0;
  for j=(i+1)/2:i
    p(i)=p(i)+prod(1:i)/(prod(1:j)*prod(1:(i−j)))*ep^j*(1−ep)^(i−j);
    echo off ;
  end
end
echo on ;
pause % Press a key to see the plot.
stem((3:2:41),p(3:2:41))
xlabel('n')
ylabel('pe')
title('Error probability as a function of n in simple repetition code')
```



Computer Problem 13.2

In order to obtain all codewords, we have to use all information sequences of length 4 and find the corresponding encoded sequences. Since there is a total of 16 binary sequences of length 4, there will be 16 codewords. Let $\boldsymbol{U}$ denote a $2^k \times k$ matrix whose rows are all possible binary sequences of length k, starting from the all-0 sequence and ending with the all-1 sequence. The rows are chosen in such a way that the decimal representation of each row is smaller than the decimal representation of all rows below it. For the case of $k = 4$, the matrix $\boldsymbol{U}$ is given by

$$\boldsymbol{U} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{13.46}$$

We have

$$\boldsymbol{C} = \boldsymbol{U}\boldsymbol{G} \tag{13.47}$$

where $\boldsymbol{C}$ is the matrix of codewords, which in this case is a 16×10 matrix whose rows are the codewords. The matrix of codewords is given by

$$\boldsymbol{C} = \begin{bmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&1&0\\ 0&0&1&1\\ 0&1&0&0\\ 0&1&0&1\\ 0&1&1&0\\ 0&1&1&1\\ 1&0&0&0\\ 1&0&0&1\\ 1&0&1&0\\ 1&0&1&1\\ 1&1&0&0\\ 1&1&0&1\\ 1&1&1&0\\ 1&1&1&1 \end{bmatrix} \begin{bmatrix} 1&0&0&1&1&1&0&1&1&1\\ 1&1&1&0&0&0&1&1&1&0\\ 0&1&1&0&1&1&0&1&0&1\\ 1&1&0&1&1&1&1&0&0&1 \end{bmatrix}$$

$$= \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0\\ 1&1&0&1&1&1&1&0&0&1\\ 0&1&1&0&1&1&0&1&0&1\\ 1&0&1&1&0&0&1&1&0&0\\ 1&1&1&0&0&0&1&1&1&0\\ 0&0&1&1&1&1&0&1&1&1\\ 1&0&0&0&1&1&1&0&1&1\\ 0&1&0&1&0&0&0&0&1&0\\ 1&0&0&1&1&1&0&1&1&1\\ 0&1&0&0&0&0&1&1&1&0\\ 1&1&1&1&0&0&0&0&1&0\\ 0&0&1&0&1&1&1&0&1&1\\ 0&1&1&1&1&1&1&0&0&1\\ 1&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&1&1&0&0\\ 1&1&0&0&1&1&0&1&0&1 \end{bmatrix}$$

A close inspection of the codewords shows that the minimum distance of the code is

$d_{\min} = 2$.

The MATLAB script file for this problem is given next.

```
% Generate U, denoting all information sequences.
k=4;
for i=1:2^k
  for j=k:−1:1
    if rem(i−1,2^(−j+k+1))>=2^(−j+k)
      u(i,j)=1;
    else
      u(i,j)=0;
    end
    echo off ;
  end
end
echo on ;
% Define G, the generator matrix.
g=[1 0 0 1 1 1 0 1 1 1;
   1 1 1 0 0 0 1 1 1 0;
   0 1 1 0 1 1 0 1 0 1;
   1 1 0 1 1 1 1 0 0 1];
% Generate codewords.
c=rem(u*g,2);
% Find the minimum distance.
w_min=min(sum((c(2:2^k,:))'));
```

Computer Problem 13.3

Here

$$
\boldsymbol{H} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{13.48}
$$

and, therefore,

$$\boldsymbol{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{13.49}$$

There is a total of $2^{11} = 2048$ codewords, each of length 15. The rate of the code is $\frac{11}{15} = 0.733$. In order to verify the minimum distance of the code, we use a MATLAB script similar to the one used in Computer Problem 13.2. The MATLAB script is given next, and it results in $d_{\min} = 3$.

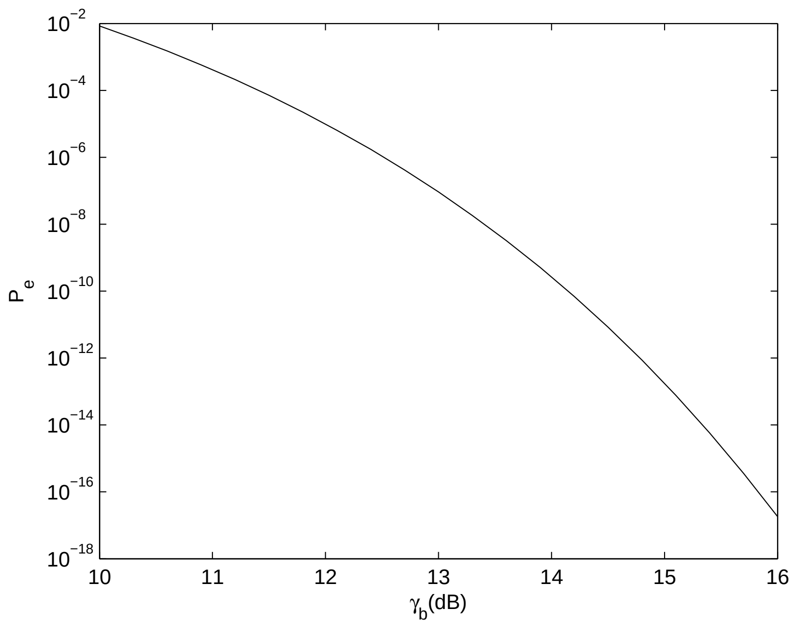
```
echo on
k=11;
for i=1:2^k
  for j=k:−1:1
    if rem(i−1,2^(−j+k+1))>=2^(−j+k)
      u(i,j)=1;
    else
      u(i,j)=0;
    end
    echo off ;
  end
end
echo on ;

g=[1 0 0 0 0 0 0 0 0 0 0 1 1 0 0;
   0 1 0 0 0 0 0 0 0 0 0 0 1 1 0;
   0 0 1 0 0 0 0 0 0 0 0 0 0 1 1;
   0 0 0 1 0 0 0 0 0 0 0 1 0 1 0;
   0 0 0 0 1 0 0 0 0 0 0 1 0 0 1;
   0 0 0 0 0 1 0 0 0 0 0 0 1 0 1;
   0 0 0 0 0 0 1 0 0 0 0 1 1 1 0;
   0 0 0 0 0 0 0 1 0 0 0 0 1 1 1;
   0 0 0 0 0 0 0 0 1 0 0 1 0 1 1;
   0 0 0 0 0 0 0 0 0 1 0 1 1 0 1;
   0 0 0 0 0 0 0 0 0 0 1 1 1 1 1];

c=rem(u*g,2);
w_min=min(sum((c(2:2^k,:))'));
```

Computer Problem 13.4

Since the minimum distance of Hamming codes is 3, we have

$$\begin{aligned} p_e &\leq \left(2^{11}-1\right)\left[4p(1-p)\right]^{d_{\min}/2} \\ &= 2047\left[4Q\left(\sqrt{\frac{1.466\mathcal{E}_b}{N_0}}\right)\left(1-Q\left(\sqrt{\frac{1.466\mathcal{E}_b}{N_0}}\right)\right)\right]^{1.5} \end{aligned} \qquad (13.50)$$

The resulting plot is shown below.



The MATLAB function for computing the bound on message-error probability of a linear block code when hard-decision decoding and antipodal signaling are employed is given next.

```
function [p_err,gamma_db]=p_e_hd_a(gamma_db_l,gamma_db_h,k,n,d_min)
% p_e_hd_a.m   Matlab function for computing error probability in
%              hard-decision decoding of a linear block code
%              when antipodal signaling is used.
%              [p_err,gamma_db]=p_e_hd_a(gamma_db_l,gamma_db_h,k,n,d_min)
%              gamma_db_l=lower E_b/N_0
%              gamma_db_h=higher E_b/N_0
%              k=number of information bits in the code
%              n=code block length
%              d_min=minimum distance of the code

gamma_db=[gamma_db_l:(gamma_db_h-gamma_db_l)/20:gamma_db_h];
gamma_b=10.^(gamma_db/10);
```

```
R_c=k/n;
p_b=q(sqrt(2.*R_c.*gamma_b));
p_err=(2^k-1).*(4*p_b.*(1-p_b)).^(d_min/2);
```

In the MATLAB script given next, the preceding MATLAB function is employed to plot error probability versus γ_b.

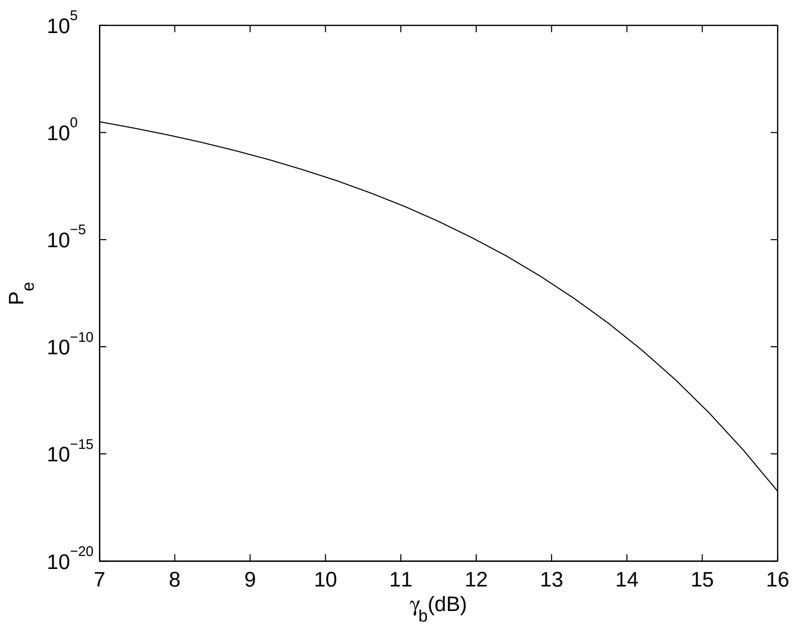
```
[p_err_ha,gamma_b]=p_e_hd_a(10,16,11,15,3);
semilogy(gamma_b,p_err_ha)
```

Computer Problem 13.5

In the problem under study, $d_{\min} = 3$, $R_c = \frac{11}{15}$, and $M = 2^{11} - 1 = 2047$. Therefore, we have

$$P_e \leq 2047Q\left(\sqrt{\frac{22}{5}\frac{\mathcal{E}_b}{N_0}}\right) \tag{13.51}$$

The corresponding plots are shown below.



Two MATLAB functions for computing the error probability for antipodal signaling when soft-decision decoding is employed, are given next

```
function [p_err,gamma_db]=p_e_sd_a(gamma_db_l,gamma_db_h,k,n,d_min)
% p_e_sd_a.m   Matlab function for computing error probability in
%              soft-decision decoding of a linear block code
%              when antipodal signaling is used.
%              [p_err,gamma_db]=p_e_sd_a(gamma_db_l,gamma_db_h,k,n,d_min)
%              gamma_db_l=lower E_b/N_0
%              gamma_db_h=higher E_b/N_0
```

```
%               k=number of information bits in the code
%               n=code block length
%               d_min=minimum distance of the code

gamma_db=[gamma_db_l:(gamma_db_h-gamma_db_l)/20:gamma_db_h];
gamma_b=10.^(gamma_db/10);
R_c=k/n;
p_err=(2^k-1).*q(sqrt(2.*d_min.*R_c.*gamma_b));
```

In the MATLAB script given next, the preceding MATLAB function is employed to plot error probability versus γ_b.

```
[p_err_ha,gamma_b]=p_e_sd_a(7,13,11,15,3);
semilogy(gamma_b,p_err_ha)
```

Computer Problem 13.6

Here, the length of the information sequence is 17, which is not a multiple of $k_0 = 2$; therefore, extra zero-padding will be done. In this case it is sufficient to add one 0, which gives a length of 18. Thus, we have the following information sequence:

$$1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0$$

Now, since we have

$$\boldsymbol{G} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

we obtain $n_0 = 3$ and $L = 4$. The length of the output sequence is, therefore,

$$\left(\frac{18}{2} + 4 - 1\right) \times 3 = 36$$

The zero-padding required to make sure that the encoder starts from the all-0 state and returns to the all-0 state adds $(L-1)k_0$ 0's to the beginning and end of the input sequence. Therefore, the sequence under study becomes

$$0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0$$

Using the function cnv_encd.m, we find the output sequence to be

$$0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1$$

The MATLAB script and the function cnv_encd.m to solve this problem is given next.

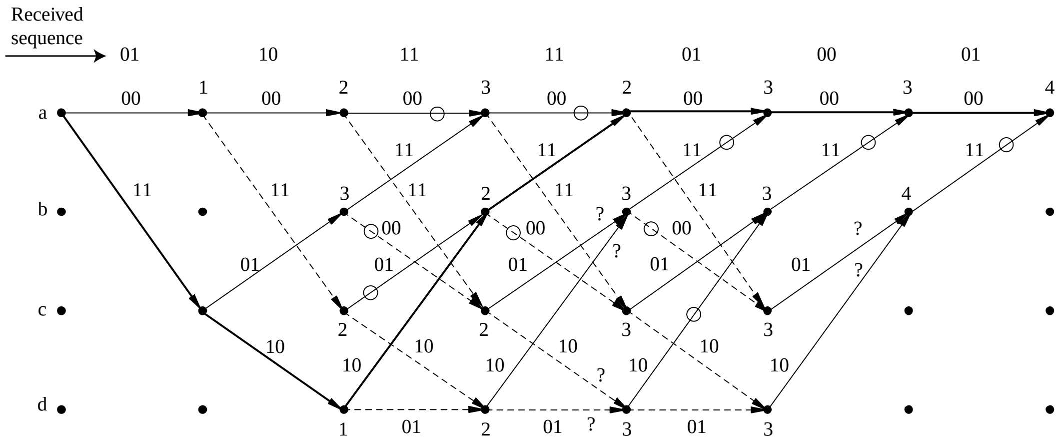
```
k0 = 2;
g = [0 0 1 0 1 0 0 1; 0 0 0 0 0 0 0 1; 1 0 0 0 0 0 0 1];
```

```
input = [1 0 0 1 1 1 0 0 1 1 0 0 0 0 1 1 1];
output = cnv_encd(g, k0, input)
```

```
function output=cnv_encd(g,k0,input)
%           cnv_encd(g,k0,input)
%           determines the output sequence of a binary convolutional encoder
%           g is the generator matrix of the convolutional code
%           with n0 rows and l*k0 columns. Its rows are g1,g2,...,gn.
%           k0 is the number of bits entering the encoder at each clock cycle.
%           input the binary input seq.

% Check to see if extra zero-padding is necessary.
if rem(length(input),k0) > 0
  input=[input,zeros(size(1:k0−rem(length(input),k0)))];
end
n=length(input)/k0;
% Check the size of matrix g.
if rem(size(g,2),k0) > 0
  error('Error, g is not of the right size.')
end
% Determine l and n0.
l=size(g,2)/k0;
n0=size(g,1);
% add extra zeros
u=[zeros(size(1:(l−1)*k0)),input,zeros(size(1:(l−1)*k0))];
% Generate uu, a matrix whose columns are the contents of
% conv. encoder at various clock cycles.
u1=u(l*k0:−1:1);
for i=1:n+l−2
  u1=[u1,u((i+l)*k0:−1:i*k0+1)];
end
uu=reshape(u1,l*k0,n+l−1);
% Determine the output.
output=reshape(rem(g*uu,2),1,n0*(l+n−1));
```

Computer Problem 13.7

The code is a $(2,1)$ code with $L = 3$. The length of the received sequence $\boldsymbol{y}$ is 14. This means that $m = 7$, and we have to draw a trellis of depth 7. Also note that since the input information sequence is padded with $k_0(L-1) = 2$ 0's, for the final two stages of the trellis we will draw only the branches corresponding to all-0 inputs. This also means that the actual length of the input sequence is 5, which, after padding with two 0's, has increased to 7. The trellis diagram for this case is shown below.



The parsed received sequence y is also shown in this figure. Note that in drawing the trellis in the last two stages, we have considered only the 0 inputs to the encoder (notice that in the final two stages, there exist no dashed lines corresponding to 1 inputs). Now the metric of the initial all-0 state is set to 0 and the metrics of the next stage are computed. In this step there is only one branch entering each state; therefore, there is no comparison, and the metrics (which are the Hamming distances between that part of the received sequence and the branches of the trellis) are added to the metric of the previous state. In the next stage there exists no comparison either. In the fourth stage, for the first time we have two branches entering each state. This means that a comparison has to be made here, and survivors are to be chosen. From the two branches that enter each state, one that corresponds to the least total accumulated metric remains as a survivor, and the other branches are deleted (marked by a small circle on the trellis). If at any stage two paths result in the same metric, each one of them can be a survivor. Such cases have been marked by a question mark in the trellis diagram. The procedure is continued to the final all-0 state of the trellis; then, starting from that state we move along the surviving paths to the initial all-0 state. This path, which is denoted by a heavy path through the trellis, is the optimal path. The input-bit sequence corresponding to this path is 1100000, where the last two 0's are not information bits but were added to return the encoder to the all-0 state. Therefore, the information sequence is 11000. The corresponding codeword for the selected path is 11101011000000, which is at Hamming distance 4 from the received sequence. All other paths through the trellis correspond to codewords that are at greater Hamming distance from the received sequence.

For soft-decision decoding a similar procedure is followed, with squared Euclidean distances substituted for Hamming distances.

The MATLAB function viterbi.m given next employs the Viterbi algorithm to decode a channel output. This algorithm can be used for both soft-decision and hard-decision decoding of convolutional codes. The separate file metric.m defines the metric used in the decoding process. For hard-decision decoding this metric is the Hamming distance, and for soft-decision decoding it is the Euclidean distance. For cases where the channel output is quantized, the metric is usually the negative of the log-likelihood, $-\log p(\text{channel output} \mid \text{channel input})$ A number of short m-files called by viterbi.m are also given next.

```
function [decoder_output,survivor_state,cumulated_metric]=viterbi(G,k,channel_output)
%VITERBI      The Viterbi decoder for convolutional codes
```

```
%          [decoder_output,survivor_state,cumulated_metric]=viterbi(G,k,channel_output)
%          G is a n x Lk matrix each row of which
%          determines the connections from the shift register to the
%          n-th output of the code, k/n is the rate of the code.
%          survivor_state is a matrix showing the optimal path through
%          the trellis. The metric is given in a separate function metric(x,y)
%          and can be specified to accommodate hard and soft decision.
%          This algorithm minimizes the metric rather than maximizing
%          the likelihood.

n=size(G,1);
%  check the sizes
if rem(size(G,2),k) ~=0
  error('Size of G and k do not agree')
end
if rem(size(channel_output,2),n) ~=0
  error('channel output not of the right size')
end
L=size(G,2)/k;
number_of_states=2^((L-1)*k);
%  Generate state transition matrix, output matrix, and input matrix.
for j=0:number_of_states-1
  for l=0:2^k-1
    [next_state,memory_contents]=nxt_stat(j,l,L,k);
    input(j+1,next_state+1)=l;
    branch_output=rem(memory_contents*G',2);
    nextstate(j+1,l+1)=next_state;
    output(j+1,l+1)=bin2deci(branch_output);
  end
end
state_metric=zeros(number_of_states,2);
depth_of_trellis=length(channel_output)/n;
channel_output_matrix=reshape(channel_output,n,depth_of_trellis);
survivor_state=zeros(number_of_states,depth_of_trellis+1);
%  Start decoding of non-tail channel outputs.
for i=1:depth_of_trellis-L+1
  flag=zeros(1,number_of_states);
  if i <= L
    step=2^((L-i)*k);
  else
    step=1;
  end
  for j=0:step:number_of_states-1
    for l=0:2^k-1
      branch_metric=0;
      binary_output=deci2bin(output(j+1,l+1),n);
      for ll=1:n
        branch_metric=branch_metric+metric(channel_output_matrix(ll,i),binary_output(ll));
      end
      if((state_metric(nextstate(j+1,l+1)+1,2) > state_metric(j+1,1)...
        +branch_metric) | flag(nextstate(j+1,l+1)+1)==0)
        state_metric(nextstate(j+1,l+1)+1,2) = state_metric(j+1,1)+branch_metric;
        survivor_state(nextstate(j+1,l+1)+1,i+1)=j;
        flag(nextstate(j+1,l+1)+1)=1;
```

```
        end
      end
    end
    state_metric=state_metric(:,2:-1:1);
  end
  % Start decoding of the tail channel outputs.
  for i=depth_of_trellis-L+2:depth_of_trellis
    flag=zeros(1,number_of_states);
    last_stop=number_of_states/(2^((i-depth_of_trellis+L-2)*k));
    for j=0:last_stop-1
        branch_metric=0;
        binary_output=deci2bin(output(j+1,1),n);
        for ll=1:n
          branch_metric=branch_metric+metric(channel_output_matrix(ll,i),binary_output(ll));
        end
        if((state_metric(nextstate(j+1,1)+1,2) > state_metric(j+1,1)...
          +branch_metric) | flag(nextstate(j+1,1)+1)==0)
          state_metric(nextstate(j+1,1)+1,2) = state_metric(j+1,1)+branch_metric;
          survivor_state(nextstate(j+1,1)+1,i+1)=j;
          flag(nextstate(j+1,1)+1)=1;
        end
    end
    state_metric=state_metric(:,2:-1:1);
  end
  % Generate the decoder output from the optimal path.
  state_sequence=zeros(1,depth_of_trellis+1);
  state_sequence(1,depth_of_trellis)=survivor_state(1,depth_of_trellis+1);
  for i=1:depth_of_trellis
    state_sequence(1,depth_of_trellis-i+1)=survivor_state((state_sequence(1,depth_of_trellis+2-i)...
    +1),depth_of_trellis-i+2);
  end
  decodeder_output_matrix=zeros(k,depth_of_trellis-L+1);
  for i=1:depth_of_trellis-L+1
    dec_output_deci=input(state_sequence(1,i)+1,state_sequence(1,i+1)+1);
    dec_output_bin=deci2bin(dec_output_deci,k);
    decoder_output_matrix(:,i)=dec_output_bin(k:-1:1)';
  end
  decoder_output=reshape(decoder_output_matrix,1,k*(depth_of_trellis-L+1));
  cumulated_metric=state_metric(1,1);
```

```
function distance=metric(x,y)
if x==y
  distance=0;
else
  distance=1;
end
```

```
function [next_state,memory_contents]=nxt_stat(current_state,input,L,k)
binary_state=deci2bin(current_state,k*(L-1));
binary_input=deci2bin(input,k);
next_state_binary=[binary_input,binary_state(1:(L-2)*k)];
```

```
next_state=bin2deci(next_state_binary);
memory_contents=[binary_input,binary_state];
```

```
function y=bin2deci(x)
l=length(x);
y=(l-1:-1:0);
y=2.^y;
y=x*y';
```

```
function y=deci2bin(x,l)
y = zeros(1,l);
i = 1;
while x>=0 & i<=l
        y(i)=rem(x,2);
        x=(x-y(i))/2;
        i=i+1;
end
y=y(l:-1:1);
```

Computer Problem 13.8

The parity-check bits are given by

$$\boldsymbol{c}^{(2)} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

and the MATLAB file for the encoder is given next.

```
function [c_sys,c_pc]=RSCC_57_Encoder(u);
% RSCC_57_Encoder  Encoder for 5/7 RSCC
%                  [c_sys,c_pc]=RSCC_57_Encoder(u)
%                  returns c_sys the systematic bits and
%                  c_pc, the parity check bits of the code
%                  when input is u and the encoder is
%                  initiated at 0-state.
u = [0 1 1 1 0 0 1 0 0 1 1 0 0 1 0 0 1 1 1 1];
L = length(u);
l = 1;
% Initializing the values of the shift register:
r1 = 0;
r2 = 0;
r3 = 0;
while l <= L
    u_t = u(l);
    % Generating the systematic bits:
    c1(l) = u_t;
    % Updating the values of the shift register:
    r1_t = mod(mod(r3 + r2,2) + u_t,2);
```

```
    r3 = r2;
    r2 = r1;
    r1 = r1_t;
    % Generating the parity check bits:
    c2(l) = mod(r1 + r3,2);
    l = l + 1;
end
c_cys=c1;
c_pc=c2;
```

Computer Problem 13.9

ïż£The MATLAB script for the problem is given next.

```
function alpha=forward_recursion(gamma);
% FORWARD_RECURSION   computing alpha for 5/7 RSCC
%                     alpha=forward_recursion(gamma);
%                     returns alpha in the form of a matrix.
%                     gamma is a 16XN matrix of gamma_i(sigma_(i-1),sigm_i)

N = size(gamma,2);          % Assuming gamma is given
Ns = 4;                     % Number of states
% Initialization:
alpha = zeros(Ns,N);
alpha_0 = 1;
i = 1;                      % Time index
simga_i = [1 3];            % Set of states at i=1
alpha(simga_i(1),i) = gamma(1,i);
alpha(simga_i(2),i) = gamma(3,i);
i = 2;
simga_i = [1 2 3 4];        % Set of states at i=2
alpha(simga_i(1),i) = gamma(1,i) *alpha(1,i−1);
alpha(simga_i(2),i) = gamma(10,i)*alpha(3,i−1);
alpha(simga_i(3),i) = gamma(3,i) *alpha(1,i−1);
alpha(simga_i(4),i) = gamma(12,i)*alpha(3,i−1);
for i = 3:N−2
    alpha(simga_i(1),i) = gamma(1,i) *alpha(1,i−1) + gamma(5,i) *alpha(2,i−1);
    alpha(simga_i(2),i) = gamma(10,i)*alpha(3,i−1) + gamma(14,i)*alpha(4,i−1);
    alpha(simga_i(3),i) = gamma(3,i) *alpha(1,i−1) + gamma(7,i) *alpha(2,i−1);
    alpha(simga_i(4),i) = gamma(12,i)*alpha(3,i−1) + gamma(16,i)*alpha(4,i−1);
end
i = N − 1;                  % Set of states at i=N-1
simga_i = [1 2];
alpha(simga_i(1),i) = gamma(1,i) *alpha(1,i−1) + gamma(5,i) *alpha(2,i−1);
alpha(simga_i(2),i) = gamma(10,i)*alpha(3,i−1) + gamma(14,i)*alpha(4,i−1);
i = N;
simga_i = 1;                % Set of states at i=N
alpha(simga_i(1),i) = gamma(1,i) *alpha(1,i−1) + gamma(5,i) *alpha(2,i−1);
alpha=[[1 0 0 0]',alpha];
```

Computer Problem 13.10

ï¿£The MATLAB script for the problem is given below.

```
function beta=backward_recursion(gamma);
% BACKWARD_RECURSION  computing beta for 5/7 RSCC
%                     beta=backward_recursion(gamma);
%                     beta in the form of a matrix
%                     gamma is a 16XN matrix of gamma_i(sigma_(i-1),sigm_i)
N = size(gamma,2);          % Assuming gamma is given
Ns = 4;                     % Number of states
% Initialization:
beta = zeros(Ns,N);
beta(1,N) = 1;
i = N;                      % Time index
simga_i_1 = [1 2];          % Set of states at i=N
beta(simga_i_1(1),i−1) = gamma(1,i);
beta(simga_i_1(2),i−1) = gamma(5,i);
i = N − 1;
simga_i_1 = [1 2 3 4];      % Set of states at i=N-1
beta(simga_i_1(1),i−1) = gamma(1,N)*gamma(1,i);
beta(simga_i_1(2),i−1) = gamma(1,N)*gamma(5,i);
beta(simga_i_1(3),i−1) = gamma(5,N)*gamma(10,i);
beta(simga_i_1(4),i−1) = gamma(5,N)*gamma(14,i);
for i = N−2:−1:3
   beta(simga_i_1(1),i−1) = beta(1,i)*gamma(1,i)  + beta(3,i)*gamma(3,i);
   beta(simga_i_1(2),i−1) = beta(1,i)*gamma(5,i)  + beta(3,i)*gamma(7,i);
   beta(simga_i_1(3),i−1) = beta(2,i)*gamma(10,i) + beta(4,i)*gamma(12,i);
   beta(simga_i_1(4),i−1) = beta(4,i)*gamma(16,i) + beta(2,i)*gamma(14,i);
end
i = 2;                      % Set of states at i=2
simga_i_1 = [1 3];
beta(simga_i_1(1),i−1) = beta(1,i)*gamma(1,i)  + beta(3,i)*gamma(3,i);
beta(simga_i_1(2),i−1) = beta(2,i)*gamma(10,i) + beta(4,i)*gamma(12,i);
i = 1;
simga_i_1 = 1;              % Set of states at i=1
beta_0(simga_i_1(1)) = beta(1,i)*gamma(1,i) + beta(3,i)*gamma(3,i);
```

Computer Problem 13.11

The MATLAB scripts for the problem are given next.

```
function [c check] = sp_decoder(H,y,max_it,E,EbN0_dB)
%SP_DECODER is the Sum-Product decoder for a linear block code code with BPSK modulation
%   [c check] = sp_decoder(H,y,max_it,N0)
%   y          channel output
%   H          parity-check matrix of the code
%   max_it     maximum number of iterations
```

```
%   E           symbol energy
%   EbN0_dB     SNR/bit (in dB)
%   c           decoder output
%   check       is 0 if c is a codeword and is 1 otherwise

n = size(H,2);                  % Length of the code
f = size(H,1);                  % Number of parity checks
R = (n-f)/n;                    % Rate
Eb = E/R;                       % Energy/bit
N0 = Eb*10^(-EbN0_dB/10);       % one-sided noise PSD
L_i = 4*sqrt(E)*y/N0;
[j i] = find(H);
nz = length(find(H));
L_j2i = zeros(f,n);
L_i2j = repmat(L_i,f,1) .* H;
L_i2j_vec = L_i + sum(L_j2i,1);
% Decision making:
L_i_total = L_i2j_vec;
for l = 1:n
    if L_i_total(l) <= 0
        c_h(l) = 1;
    else
        c_h(l) = 0;
    end
end
s = mod(c_h*H',2);
if nnz(s) == 0
    c = c_h;
else
    it = 1;
    while ((it <= max_it) && (nnz(s)~=0))
        % Variable node updates:
        for idx = 1:nz
            L_i2j(j(idx),i(idx)) = L_i2j_vec(i(idx)) - L_j2i(j(idx),i(idx));
        end
        % Check node updates:
        for q = 1:f
            F = find(H(q,:));
            L_j2i_vec(q) = prod(tanh(0.5*L_i2j(q,F(:))),2);
        end
        for idx = 1:nz
            L_j2i(j(idx),i(idx)) = 2*atanh(L_j2i_vec(j(idx)) /...
                tanh(0.5*L_i2j(j(idx),i(idx))));
        end
        L_i2j_vec = L_i + sum(L_j2i,1);
        % Decision making:
        L_i_total = L_i2j_vec;
        for l = 1:n
            if L_i_total(l) <= 0
                c_h(l) = 1;
            else
                c_h(l) = 0;
            end
        end
```

```
        s = mod(c_h*H',2);
        it = it + 1;
    end
end
c = c_h;
check = nnz(s);
if (check > 0)
    check = 1;
end
```

Chapter 14

Problem 14.1

1) The wavelength λ is

$$\lambda = \frac{3 \times 10^8}{10^9} \text{ m } = \frac{3}{10} \text{ m}$$

Hence, the Doppler frequency shift is

$$f_D = \pm\frac{u}{\lambda} = \pm\frac{100 \text{ Km/hr}}{\frac{3}{10} \text{ m}} = \pm\frac{100 \times 10^3 \times 10}{3 \times 3600} \text{Hz} = \pm 92.5926 \text{ Hz}$$

The plus sign holds when the vehicle travels towards the transmitter whereas the minus sign holds when the vehicle moves away from the transmitter.

2) The maximum difference in the Doppler frequency shift, when the vehicle travels at speed 100 km/hr and $f = 1$ GHz, is

$$\Delta f_{D\max} = 2 f_D = 185.1852 \text{ Hz}$$

This should be the bandwidth of the Doppler frequency tracking loop.

3) The maximum Doppler frequency shift is obtain when $f = 1$ GHz $+ 1$ MHz and the vehicle moves towards the transmitter. In this case

$$\lambda_{\min} = \frac{3 \times 10^8}{10^9 + 10^6} \text{ m } = 0.2997 \text{ m}$$

and therefore

$$f_{D\max} = \frac{100 \times 10^3}{0.2997 \times 3600} = 92.6853 \text{ Hz}$$

Thus, the Doppler frequency spread is $B_d = 2 f_{D\max} = 185.3706$ Hz.

Problem 14.2

1) Since $T_m = 1$ second, the coherence bandwidth

$$B_{cb} = \frac{1}{2T_m} = 0.5 \quad \text{Hz}$$

and with $B_d = 0.01$ Hz, the coherence time is

$$T_{ct} = \frac{1}{2B_d} = 100/2 = 50 \quad \text{seconds}$$

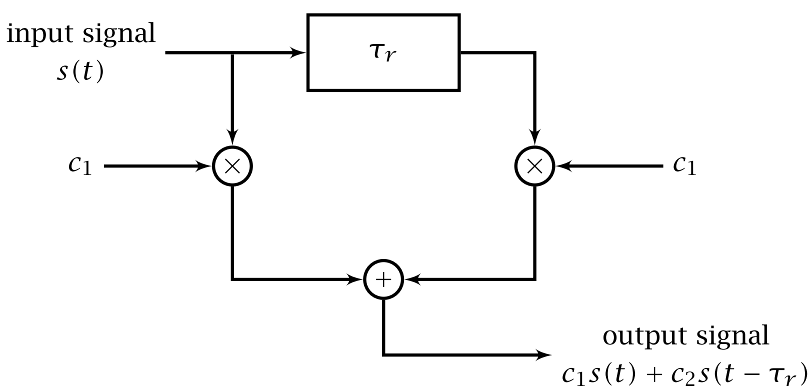
2) Since the channel bandwidth $W \gg b_{cb}$, the channel is frequency selective.
3) Since the signal duration $T \ll T_{ct}$, the channel is slowly fading.

4) The ratio $W/B_{cb} = 10$. Hence, in principle up to tenth order diversity is available by subdividing the channel bandwidth into 10 subchannels, each of width 0.5 Hz. If we employ binary PSK with symbol duration $T = 10$ seconds, then the channel bandwidth can be subdivided into 25 subchannels, each of bandwidth $\frac{2}{T} = 0.2$ Hz. We may choose to have 5^{th} order frequency diversity and for each transmission, thus, have 5 parallel transmissions. Thus, we would have a data rate of 5 bits per signal interval, i.e., a bit rate of 1/2 bps. By reducing the order of diversity, we may increase the data rate, for example, with no diversity, the data rate becomes 2.5 bps.
5) To answer the question we may use the approximate relation for the error probability given by (11.1.32), or we may use the results in the graph shown in Figure 11.5. For example, for binary PSK with $D = 4$, the SNR per bit required to achieve an error probability of 10^{-6} is 18 dB. This the total SNR per bit for the four channels (with maximal ration combining). Hence, the SNR per bit per channel is reduced to 12 dB (a factor of four smaller).

Problem 14.3
The signal bandwidth is $W = 100$ kHz. Therefore, the time resolution is

$$\tau_r = \frac{1}{W} = 10\ \mu\text{sec}.$$

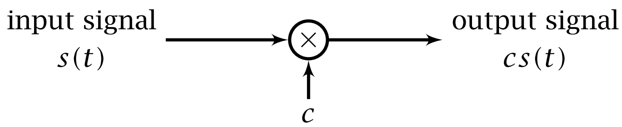
and, hence, the multipath component is resolvable. The appropriate channel model is



Problem 14.4
The signal bandwidth is $W = 10$ kHz. Therefore, the time resolution is

$$\tau_r = \frac{1}{W} = 100\ \mu\text{sec}.$$

in this case, the multipath component is not resolvable. The appropriate channel model is



Problem 14.5

The Rayleigh distribution is

$$p(\alpha) = \begin{cases} \frac{\alpha}{\sigma_\alpha^2} e^{-\alpha^2/2\sigma_\alpha^2}, & \alpha > 0 \\ 0, & \text{otherwise} \end{cases}$$

Hence, the probability of error for the binary FSK and DPSK with noncoherent detection averaged over all possible values of α is

$$\begin{aligned} P_2 &= \int_0^\infty \frac{1}{2} e^{-c\frac{\alpha^2 \mathcal{E}_b}{N_0}} \frac{\alpha}{\sigma_\alpha^2} e^{-\alpha^2/2\sigma_\alpha^2} d\alpha \\ &= \frac{1}{2\sigma_\alpha^2} \int_0^\infty \alpha e^{-\alpha^2\left[\frac{c\mathcal{E}_b}{N_0} + \frac{1}{2\sigma_\alpha^2}\right]} d\alpha \end{aligned}$$

But,

$$\int_0^\infty x^{2n+1} e^{-ax^2} dx = \frac{n!}{2a^{n+1}}, \quad (a > 0)$$

so that with $n = 0$ we obtain

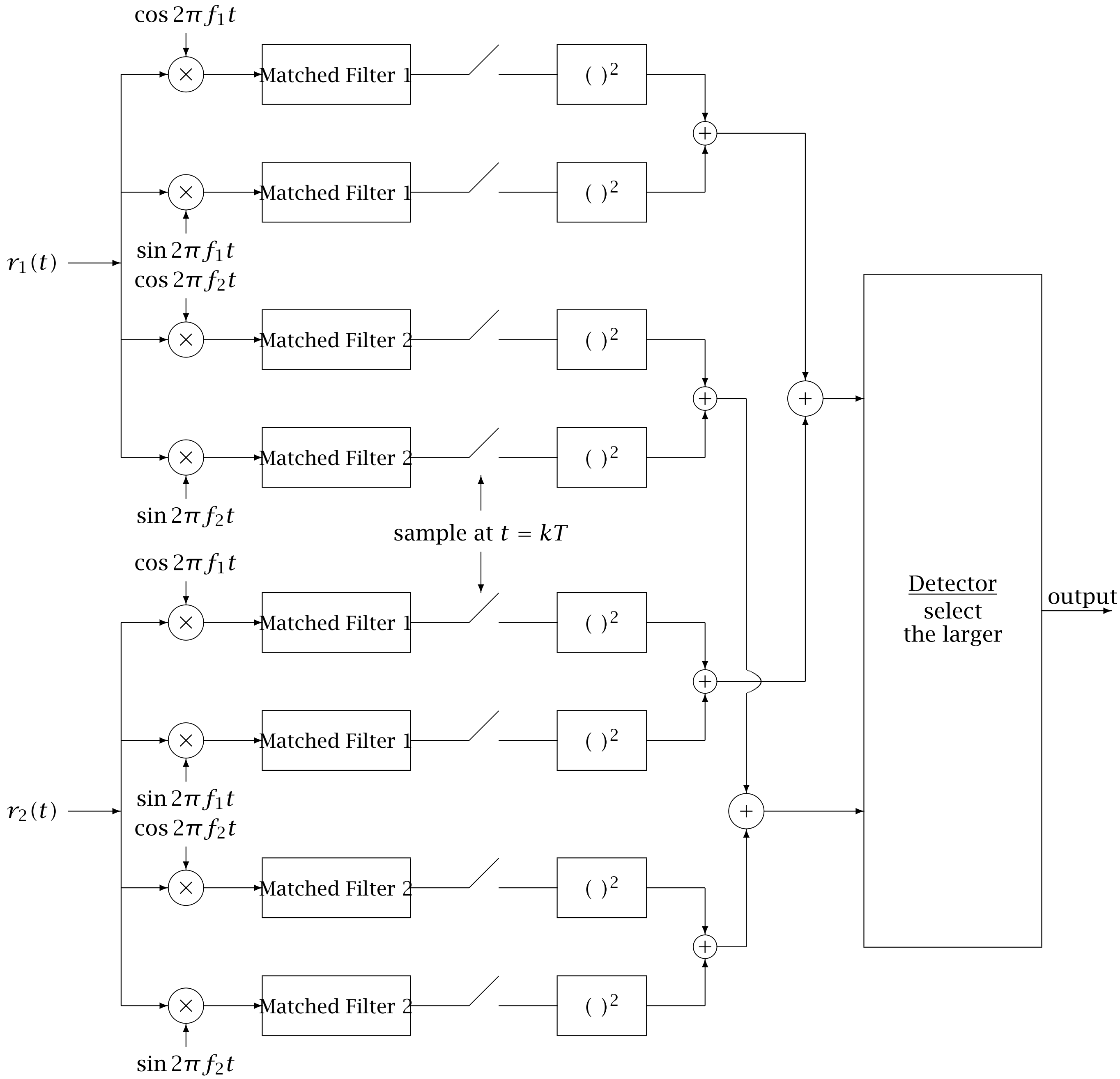
$$\begin{aligned} P_2 &= \frac{1}{2\sigma_\alpha^2} \int_0^\infty \alpha e^{-\alpha^2\left[\frac{c\mathcal{E}_b}{N_0} + \frac{1}{2\sigma_\alpha^2}\right]} d\alpha = \frac{1}{2\sigma_\alpha^2} \frac{1}{2\left[\frac{c\mathcal{E}_b}{N_0} + \frac{1}{2\sigma_\alpha^2}\right]} \\ &= \frac{1}{2\left[c\frac{\mathcal{E}_b 2\sigma_\alpha^2}{N_0} + 1\right]} = \frac{1}{2\left[c\bar{\rho}_b + 1\right]} \end{aligned}$$

where $\bar{\rho}_b = \frac{\mathcal{E}_b 2\sigma_\alpha^2}{N_0}$. With $c = 1$ (DPSK) and $c = \frac{1}{2}$ (FSK) we have

$$P_2 = \begin{cases} \frac{1}{2(1+\bar{\rho}_b)}, & \text{DPSK} \\ \frac{1}{2+\bar{\rho}_b}, & \text{FSK} \end{cases}$$

Problem 14.6

1)



2) The probability of error for binary FSK with square-law combining for $D = 2$ is given in Figure 14.14. The probability of error for $D = 1$ is also given in Figure 14.14. Note that an increase in SNR by a factor of 10 reduces the error probability by a factor of 10 when $D = 1$ and by a factor of 100 when $D = 2$.

Problem 14.7

1) r is a Gaussian random variable. If $\sqrt{\mathcal{E}_b}$ is the transmitted signal point, then

$$E(r) = E(r_1) + E(r_2) = (1+k)\sqrt{\mathcal{E}_b} \equiv m_r$$

and the variance is

$$\sigma_r^2 = \sigma_1^2 + k^2\sigma_2^2$$

The probability density function of r is

$$f(r) = \frac{1}{\sqrt{2\pi}\sigma_r} e^{-\frac{(r-m_r)^2}{2\sigma_r^2}}$$

and the probability of error is

$$\begin{aligned} P_2 &= \int_{-\infty}^{0} f(r)\,dr \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{m_r}{\sigma_r}} e^{-\frac{x^2}{2}}\,dx \\ &= Q\left(\sqrt{\frac{m_r^2}{\sigma_r^2}}\right) \end{aligned}$$

where

$$\frac{m_r^2}{\sigma_r^2} = \frac{(1+k)^2\mathcal{E}_b}{\sigma_1^2 + k^2\sigma_2^2}$$

The value of k that maximizes this ratio is obtained by differentiating this expression and solving for the value of k that forces the derivative to zero. Thus, we obtain

$$k = \frac{\sigma_1^2}{\sigma_2^2}$$

Note that if $\sigma_1 > \sigma_2$, then $k > 1$ and r_2 is given greater weight than r_1. On the other hand, if $\sigma_2 > \sigma_1$, then $k < 1$ and r_1 is given greater weight than r_2. When $\sigma_1 = \sigma_2$, $k = 1$. In this case

$$\frac{m_r^2}{\sigma_r^2} = \frac{2\mathcal{E}_b}{\sigma_1^2}$$

2) When $\sigma_2^2 = 3\sigma_1^2$, $k = \frac{1}{3}$, and

$$\frac{m_r^2}{\sigma_r^2} = \frac{(1+\frac{1}{3})^2\mathcal{E}_b}{\sigma_1^2 + \frac{1}{9}(3\sigma_1^2)} = \frac{4}{3}\left(\frac{\mathcal{E}_b}{\sigma_1^2}\right)$$

On the other hand, if k is set to unity we have

$$\frac{m_r^2}{\sigma_r^2} = \frac{4\mathcal{E}_b}{\sigma_1^2 + 3\sigma_1^2} = \frac{\mathcal{E}_b}{\sigma_1^2}$$

Therefore, the optimum weighting provides a gain of

$$10\log\frac{4}{3} = 1.25 \text{ dB}$$

Problem 14.8

1) The probability of error for a fixed value of a is

$$P_e(a) = Q\left(\sqrt{\frac{2a^2\mathcal{E}}{N_0}}\right)$$

since the given a takes two possible values, namely $a = 0$ and $a = 2$ with probabilities 0.1 and 0.9, respectively, the average probability of error is

$$P_e = \frac{0.1}{2} + Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right) = 0.05 + Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right)$$

As $\frac{\mathcal{E}}{N_0} \to \infty$, $P_e \to 0.05$

3) The probability of error for fixed values of a_1 and a_2 is

$$P_e(a_1, a_2) = Q\left(\sqrt{\frac{2(a_1^2 + a_2^2)\mathcal{E}}{N_0}}\right)$$

In this case we have four possible values for the pair (a_1, a_2), namely, $(0,0)$, $(0,2)$, $(2,0)$, and $(2,2)$, with corresponding probabilities).01, 0.09, 0.09 and 0.81. Hence, the average probability of error is

$$P_e = \frac{0.01}{2} + 0.18Q\left(\sqrt{\frac{8\mathcal{E}}{N_0}}\right) + 0.81Q\left(\sqrt{\frac{16\mathcal{E}}{N_0}}\right)$$

4) As $\frac{\mathcal{E}}{N_0} \to \infty$, $P_e \to 0.005$, which is a factor of 10 smaller than in (2).

Problem 14.9

a)

$$E[\hat{c}] = \frac{1}{N\sqrt{\mathcal{E}_b}} \sum_{k=1}^{N} \left[c\sqrt{\mathcal{E}_b} + E[n_{1k}] + E[n_{2k}]\right] = c$$

since n_{1k} and n_{2k} are zero-mean random variables.

b) In order to find the variance of the estimate we first need $E[\hat{c}^2]$. Note that

$$\begin{aligned}
E\left[\hat{c}^2\right] &= \frac{1}{N^2\mathcal{E}_b}E\left(\sum_{k=1}^{N}\left[c\sqrt{\mathcal{E}_b}+n_{1k}+n_{2k}\right]\right)^2 \\
&= \frac{1}{N^2\mathcal{E}_b}E\left(Nc\sqrt{\mathcal{E}_b}+\sum_{k=1}^{N}n_{1k}+\sum_{j=1}^{N}n_{2j}\right)^2 \\
&= \frac{1}{N^2\mathcal{E}_b}\left[N^2c^2\mathcal{E}_b+N\sigma^2+N\sigma^2\right] \\
&= \frac{1}{N^2\mathcal{E}_b}\left[N^2c^2\mathcal{E}_b+NN_0\right] \\
&= c^2+\frac{N_0}{N\mathcal{E}_b}
\end{aligned}$$

where we have used the facts that $E[n_{1j}n_{1k}] = 0$ for $j \neq k$ because n_{1k} and n_{1k} are independent and zero-mean, $E[n_{1j}n_{2k}] = 0$ for all i, j, and $E[n_{1k}^2] = E[n_{2k}^2] = \sigma^2 = N_0/2$. From above $\text{Var}(\hat{c}) = E\left[\hat{c}^2\right] - (E[\hat{c}])^2 = c^2 + \frac{N_0}{N\mathcal{E}_b} - c^2 = \frac{N_0}{N\mathcal{E}_b}$ which goes to 0 as N increases.

c) $\sigma_c^2 = \frac{N_0}{N\mathcal{E}_b}$, therefore $1/\sigma_c^2 = \frac{N\mathcal{E}_b}{N_0}$, which increases linearly with both N and $\mathcal{E}_{[}$

Problem 14.10
Channel bandwidth is $B = 3200$ Hz.

1. To achieve a rate of 4800 bps, we may use $M = 4$ PSK with a symbol rate of $T_s = 1/2400$ sec., which is 0.42 milliseconds. Since $T_m = 5$ msec., and $T_s < T_m$, an equalizer is needed to combat ISI.

2. To achieve a rate of 20 nits per second, we may use BPSK with a symbol rate of $T_s = 1/20 = 50$ msec. Since $T_m = 5$ msec., and $T_s \gg T_m$, no equalizer is needed. However, a time guard band of T_m may ne used to avoid ISI.

Problem 14.11
For a train traveling at 200 km/ht, the vehicle speed is $v = 56$ m/sec. At a carrier of $f_c = 1$ GHz, the maximum Doppler frequency is

$$\begin{aligned}
f_m &= v f_c/c \\
&= 56\times 10^9/3\times 10^8 \\
&= 186 \text{ Hz}
\end{aligned}$$

The Doppler power spectrum is

$$\mathcal{S}(f) = \begin{cases} \frac{1}{186\pi\sqrt{1-\left(\frac{f}{186}\right)^2}}, & |f| \leq 186 \\ 0, & |f| > 186 \end{cases}$$

Problem 14.12

We may select the symbol duration $T = 100$ μsec. to satisfy the bandwidth loss constraint. Then, $\Delta f = \frac{1}{T} = \frac{1}{10^{-4}} = 10$ kHz and the number of subcarriers is $N = \frac{800}{10} = 80$. The coherence time is $T_{ct} = \frac{1}{B_d} = 100$ msec. Therefore, $T_{ct} \gg T$. The coherence bandwidth is $B_c = \frac{1}{T_m} = 100$ kHz. To combat signal fading in any subchannel, we may transmit the same symbol on multiple subcarriers having a frequency separation of at least 100 kHz.

The symbol throughput rate achieved on this channel is

$$R_s = \frac{N}{TD} = \frac{80}{10^{-4}D} = \frac{800}{D} \text{ k symbols/sec}$$

where D is the order of diversity.

Problem 14.13

We have $T \gg T_m = \frac{1}{B_{cn}}$, then $TW \gg T_m W = \frac{W}{B_{cb}} \gg 1$. Therefore, $TW \gg 1$.

Problem 14.14

The matrix $\boldsymbol{G}$ is given by

$$\boldsymbol{G} = \begin{bmatrix} s_1 & s_2 & s_3 & 0 \\ -s_2^* & s_1^* & 0 & s_3 \\ s_3^* & 0 & -s_1^* & s_2 \\ 0 & s_3^* & -s_2^* & -s1 \end{bmatrix}$$

To show that the code has full diversity (in this case 4) we need to show that the matrix

$$\boldsymbol{D} = \boldsymbol{G} - \boldsymbol{G}'$$

has full rank; where $\boldsymbol{G}'$ is similar to $\boldsymbol{G}$ but is obtained by using the triplet (s_1', s_2', s_3') instead of $(s_, s_2, s_3)$, where $(s_1', s_2', s_3') \neq (s_, s_2, s_3)$. Simple substitution gives

$$\boldsymbol{D} = \boldsymbol{G} - \boldsymbol{G}' = \begin{bmatrix} s_1 - s_1' & s_2 - s_2' & s_3 - s_3' & 0 \\ -s_2^* + s_2'^* & s_1^* - s_1'^* & 0 & s_3 - s_3'^* \\ s_3^* - s_3'^* & 0 & -s_1^* + s_1'^* & s_2 - s_2' \\ 0 & s_3^* - s_3'^* & -s_2^* + s_2'^* & -s_1 + s_1' \end{bmatrix}$$

To show that this matrix is full rank, we can show that its determinant is nonzero unless $(s_1, s_2, s_3) = (s_1', s_2', s_3')$. It is easily verified that

$$\boldsymbol{D}^H \boldsymbol{D} = \left(|s_1 - s_1'|^2 + |s_2 - s_2'|^2 + |s_3 - s_3'|^2 \right) \boldsymbol{I}_4$$

from which we conclude that if $(s_1', s_2', s_3') \neq (s_, s_2, s_3)$, then $\boldsymbol{G}$ has full rank, and hence the code has full diversity.

The orthogonality of code is easy to verify by computing $\boldsymbol{G}^H\boldsymbol{G}$ and showing that

$$\boldsymbol{G}^H\boldsymbol{G} = \left(|s_1|^2 + |s_2|^2 + |s_3|^2\right)\boldsymbol{I}_4$$

Problem 14.15

1) The antenna gain for a parabolic antenna of diameter D is

$$G_R = \eta\left(\frac{\pi D}{\lambda}\right)^2$$

If we assume that the efficiency factor is 0.5, then with

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{10^9} = 0.3 \text{ m} \qquad D = 3\times 0.3048 \text{ m}$$

we obtain

$$G_R = G_T = 45.8458 = 16.61 \text{ dB}$$

2) The effective radiated power is

$$\text{EIRP} = P_T G_T = G_T = 16.61 \text{ dB}$$

3) The received power is

$$P_R = \frac{P_T G_T G_R}{\left(\frac{4\pi d}{\lambda}\right)^2} = 2.995\times 10^{-9} = -85.23 \text{ dB} = -55.23 \text{ dBm}$$

Note that

$$\text{dBm} = 10\log_{10}\left(\frac{\text{actual power in Watts}}{10^{-3}}\right) = 30 + 10\log_{10}(\text{power in Watts })$$

Problem 14.16

1) The antenna gain for a parabolic antenna of diameter D is

$$G_R = \eta\left(\frac{\pi D}{\lambda}\right)^2$$

If we assume that the efficiency factor is 0.5, then with

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{10^9} = 0.3 \text{ m} \qquad \text{and} \qquad D = 1 \text{ m}$$

we obtain

$$G_R = G_T = 54.83 = 17.39 \text{ dB}$$

2) The effective radiated power is

$$\text{EIRP} = P_T G_T = 0.1 \times 54.83 = 7.39 \text{ dB}$$

3) The received power is

$$P_R = \frac{P_T G_T G_R}{\left(\frac{4\pi d}{\lambda}\right)^2} = 1.904 \times 10^{-10} = -97.20 \text{ dB} \ = -67.20 \text{ dBm}$$

Problem 14.17

The wavelength of the transmitted signal is

$$\lambda = \frac{3 \times 10^8}{10 \times 10^9} = 0.03 \text{ m}$$

The gain of the parabolic antenna is

$$G_R = \eta \left(\frac{\pi D}{\lambda}\right)^2 = 0.6 \left(\frac{\pi 10}{0.03}\right)^2 = 6.58 \times 10^5 = 58.18 \text{ dB}$$

The received power at the output of the receiver antenna is

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} = \frac{3 \times 10^{1.5} \times 6.58 \times 10^5}{(4 \times 3.14159 \times \frac{4\times 10^7}{0.03})^2} = 2.22 \times 10^{-13} = -126.53 \text{ dB}$$

Problem 14.18

1) Since $T = 300^0 K$, it follows that

$$N_0 = kT = 1.38 \times 10^{-23} \times 300 = 4.14 \times 10^{-21} \text{ W/Hz}$$

If we assume that the receiving antenna has an efficiency $\eta = 0.5$, then its gain is given by

$$G_R = \eta \left(\frac{\pi D}{\lambda}\right)^2 = 0.5 \left(\frac{3.14159 \times 50}{\frac{3\times 10^8}{2\times 10^9}}\right)^2 = 5.483 \times 10^5 = 57.39 \text{ dB}$$

Hence, the received power level is

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} = \frac{10 \times 10 \times 5.483 \times 10^5}{(4 \times 3.14159 \times \frac{10^8}{0.15})^2} = 7.8125 \times 10^{-13} = -121.07 \text{ dB}$$

2) If $\frac{\mathcal{E}_b}{N_0} = 10 \text{ dB} \ = 10$, then

$$R = \frac{P_R}{N_0} \left(\frac{\mathcal{E}_b}{N_0}\right)^{-1} = \frac{7.8125 \times 10^{-13}}{4.14 \times 10^{-21}} \times 10^{-1} = 1.8871 \times 10^7 = 18.871 \text{ Mbits/sec}$$

Problem 14.19

The overall gain of the system is

$$G_{\text{tot}} = G_{a_1} + G_{\text{os}} + G_{\text{BPF}} + G_{a_2} = 10 - 5 - 1 + 25 = 29 \text{ dB}$$

Hence, the power of the signal at the input of the demodulator is

$$P_{s,\text{dem}} = (-113 - 30) + 29 = -114 \text{ dB}$$

The noise-figure for the cascade of the first amplifier and the multiplier is

$$F_1 = F_{a_1} + \frac{F_{\text{os}} - 1}{G_{a_1}} = 10^{0.5} + \frac{10^{0.5} - 1}{10} = 3.3785$$

We assume that F_1 is the spot noise-figure and therefore, it measures the ratio of the available PSD out of the two devices to the available PSD out of an ideal device with the same available gain. That is,

$$F_1 = \frac{S_{n,o}(f)}{S_{n,i}(f) G_{a_1} G_{\text{os}}}$$

where $S_{n,o}(f)$ is the power spectral density of the noise at the input of the bandpass filter and $S_{n,i}(f)$ is the power spectral density at the input of the overall system. Hence,

$$S_{n,o}(f) = 10^{\frac{-175-30}{10}} \times 10 \times 10^{-0.5} \times 3.3785 = 3.3785 \times 10^{-20}$$

The noise-figure of the cascade of the bandpass filter and the second amplifier is

$$F_2 = F_{\text{BPF}} + \frac{F_{a_2} - 1}{G_{\text{BPF}}} = 10^{0.2} + \frac{10^{0.5} - 1}{10^{-0.1}} = 4.307$$

Hence, the power of the noise at the output of the system is

$$P_{n,\text{dem}} = 2 S_{n,o}(f) B G_{\text{BPF}} G_{a_2} F_2 = 7.31 \times 10^{-12} = -111.36 \text{ dB}$$

The signal to noise ratio at the output of the system (input to the demodulator) is

$$\text{SNR} = \frac{P_{s,\text{dem}}}{P_{n,\text{dem}}} = -114 + 111.36 = -2.64 \text{ dB}$$

Problem 14.20

The wavelength of the transmission is

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{4 \times 10^9} = 0.75 \text{ m}$$

If 1 MHz is the passband bandwidth, then the rate of binary transmission is $R_b = W = 10^6$ bps. Hence, with $N_0 = 4.1 \times 10^{-21}$ W/Hz we obtain

$$\frac{P_R}{N_0} = R_b \frac{\mathcal{E}_b}{N_0} \Longrightarrow 10^6 \times 4.1 \times 10^{-21} \times 10^{1.5} = 1.2965 \times 10^{-13}$$

The transmitted power is related to the received power through the relation

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} \Longrightarrow P_T = \frac{P_R}{G_T G_R}\left(4\pi \frac{d}{\lambda}\right)^2$$

Substituting in this expression the values $G_T = 10^{0.6}$, $G_R = 10^5$, $d = 36 \times 10^6$ and $\lambda = 0.75$ we obtain

$$P_T = 0.1185 = -9.26 \text{ dBW}$$

Problem 14.21

Since $T = 290^0 + 15^0 = 305^0 K$, it follows that

$$N_0 = kT = 1.38 \times 10^{-23} \times 305 = 4.21 \times 10^{-21} \text{ W/Hz}$$

The transmitting wavelength λ is

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{2.3 \times 10^9} = 0.130 \text{ m}$$

Hence, the gain of the receiving antenna is

$$G_R = \eta \left(\frac{\pi D}{\lambda}\right)^2 = 0.55 \left(\frac{3.14159 \times 64}{0.130}\right)^2 = 1.3156 \times 10^6 = 61.19 \text{ dB}$$

and therefore, the received power level is

$$P_R = \frac{P_T G_T G_R}{(4\pi \frac{d}{\lambda})^2} = \frac{17 \times 10^{2.7} \times 1.3156 \times 10^6}{(4 \times 3.14159 \times \frac{1.6 \times 10^{11}}{0.130})^2} = 4.686 \times 10^{-12} = -113.29 \text{ dB}$$

If $\mathcal{E}_b/N_0 = 6 \text{ dB} = 10^{0.6}$, then

$$R = \frac{P_R}{N_0}\left(\frac{\mathcal{E}_b}{N_0}\right)^{-1} = \frac{4.686 \times 10^{-12}}{4.21 \times 10^{-21}} \times 10^{-0.6} = 4.4312 \times 10^9 = 4.4312 \text{ Gbits/sec}$$

Computer Problems

Computer Problem 14.1

Figures 14.1 and 14.2 present $|c_1(n)|$ and $|c_2(n)|$ for $W = 10$ kHz, respectively. The channel output $|y(n)|$ for input sequence $x(n) = 1$ is presented in Figure 14.3 for $\sigma_w^2 = 0, 0.5, 1$ for $W = 10$ kHz. $|c_1(n)|$ and $|c_2(n)|$ for $W = 5$ kHz are presented in Figures 14.4 and 14.5, respectively. The channel output for $W = 5$kHz is presented in Figure 14.6.

The MATLAB script for this problem is given next.

```
% MATLAB script for Computer Problem 14.1
W = 5 * 10^3; % Signal bandwidth
Td = 10^(-3);
timeResolution = 1/W;
delaySamples = Td / timeResolution;
% generate tap weights
% MATLAB script for Computer Problem 14.1
c1(1) = randn + j*randn;
c2(1) = randn + j*randn;
for n = 2:1000
    c1(n) = 0.9*c1(n-1)+ randn + j*randn;
    c2(n) = 0.9*c2(n-1)+ randn + j*randn;
end
x = ones(1,1000);
i = 0;
for variance = 0:0.5:1.5
    i = i + 1;
    c2delayed = zeros(1,n);
    c2delayed(delaySamples+1:1000) = c2(1:1000-delaySamples);
   y(i, :) = (x.*c1 + x.*c2delayed) + (sqrt(variance) * (randn(1,1000) + j*randn(1,1000)));
end
% Plotting commands follow.
```

Computer Problem 14.2

The following figure illustrates the result of the Monte Carlo simulation and the comparison with the theoretical error probability.

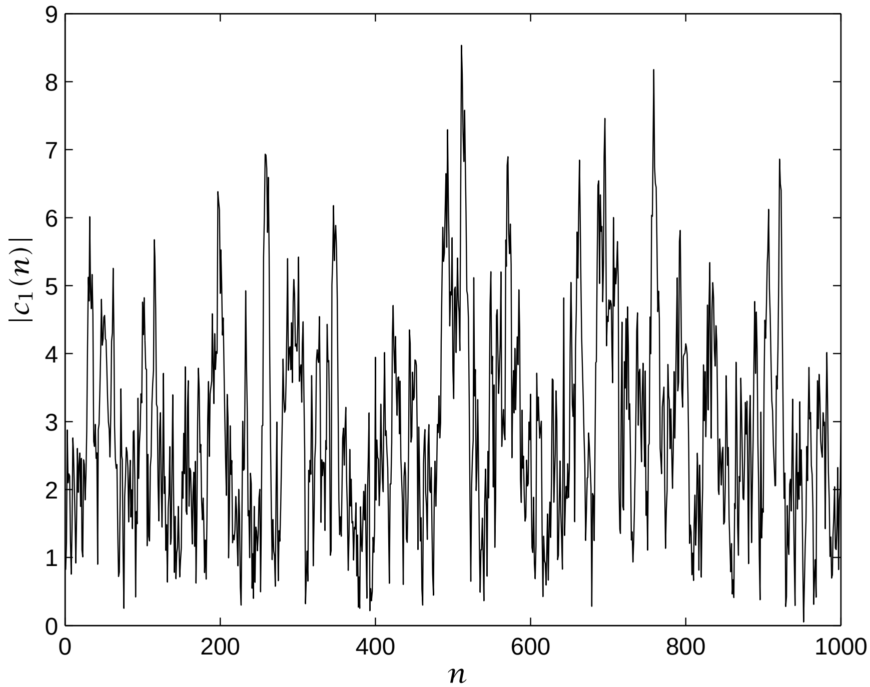


Figure 14.1: The tap weight sequences $|c_1(n|)$ for $W = 10$kHz

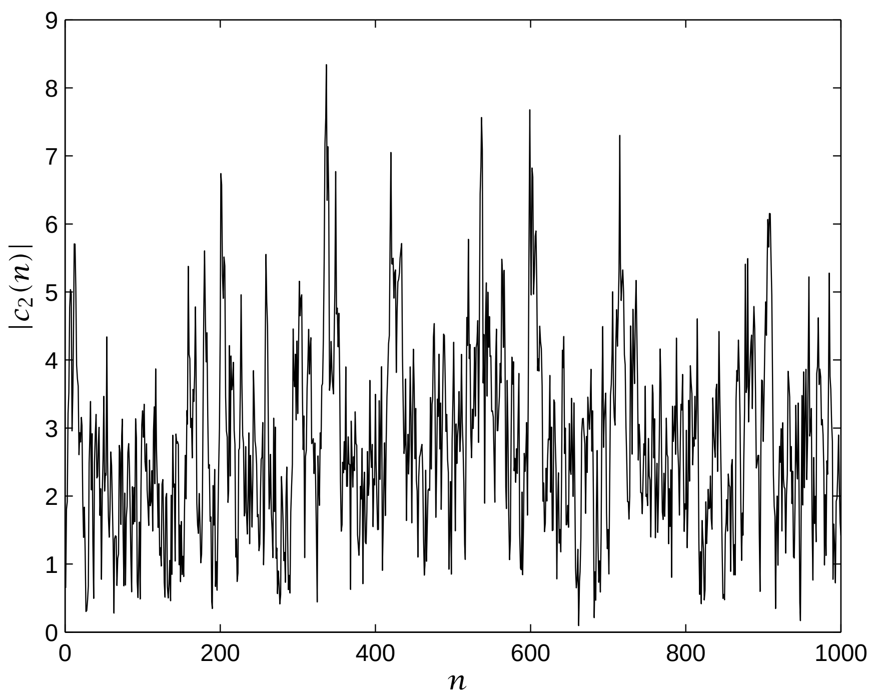


Figure 14.2: The tap weight sequences $|c_2(n)|$ for $W = 10$kHz

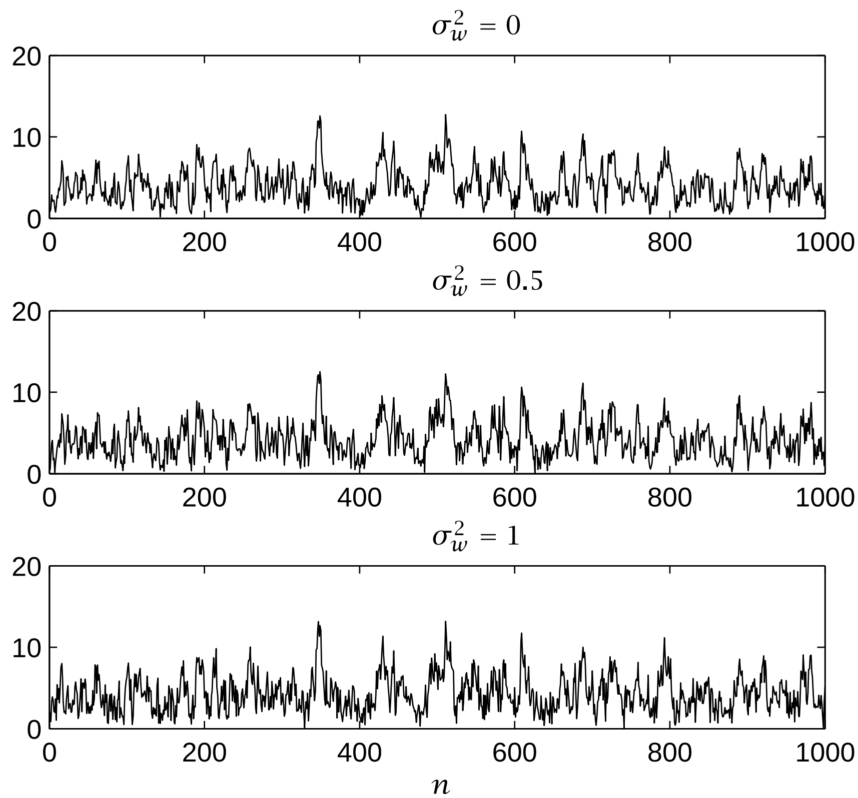


Figure 14.3: Channel outputs for $\sigma_w^2 = 0, 0.5, 1$ for $W = 10$kHz

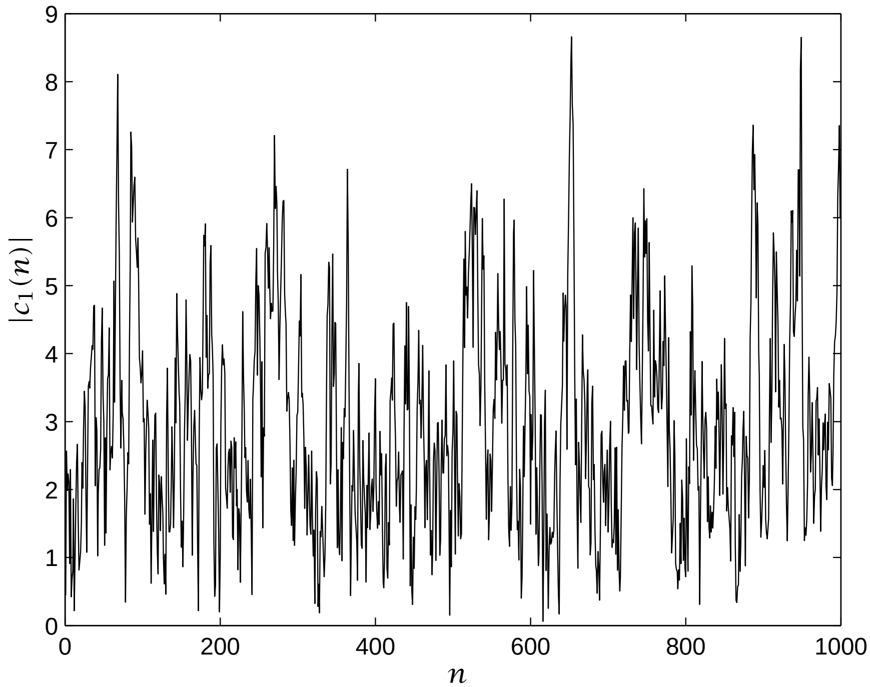


Figure 14.4: The tap weight sequences $|c_1(n)|$ for $W = 5$kHz

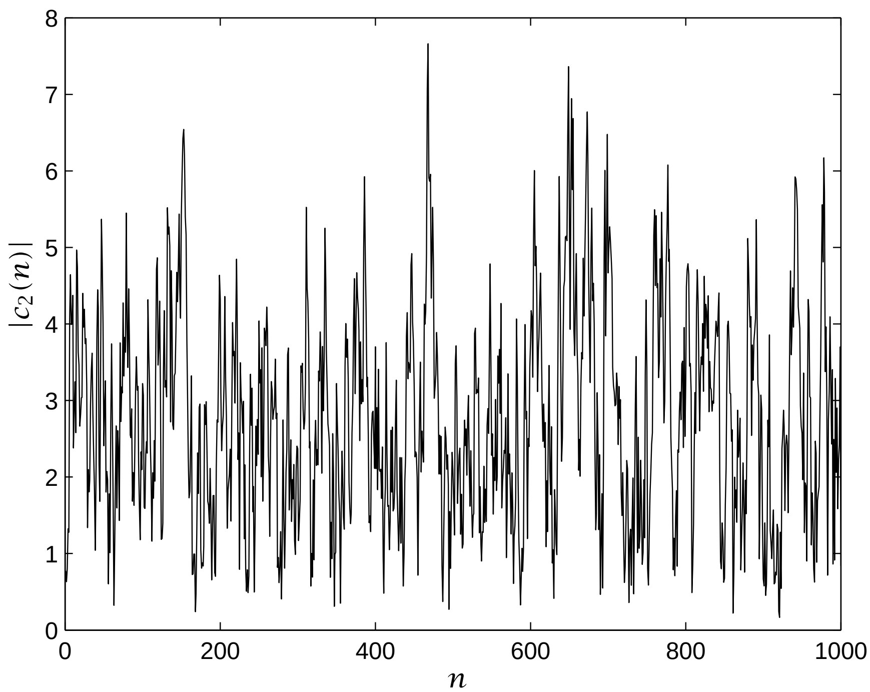


Figure 14.5: The tap weight sequences $|c_2(n)|$ for $W = 5$kHz

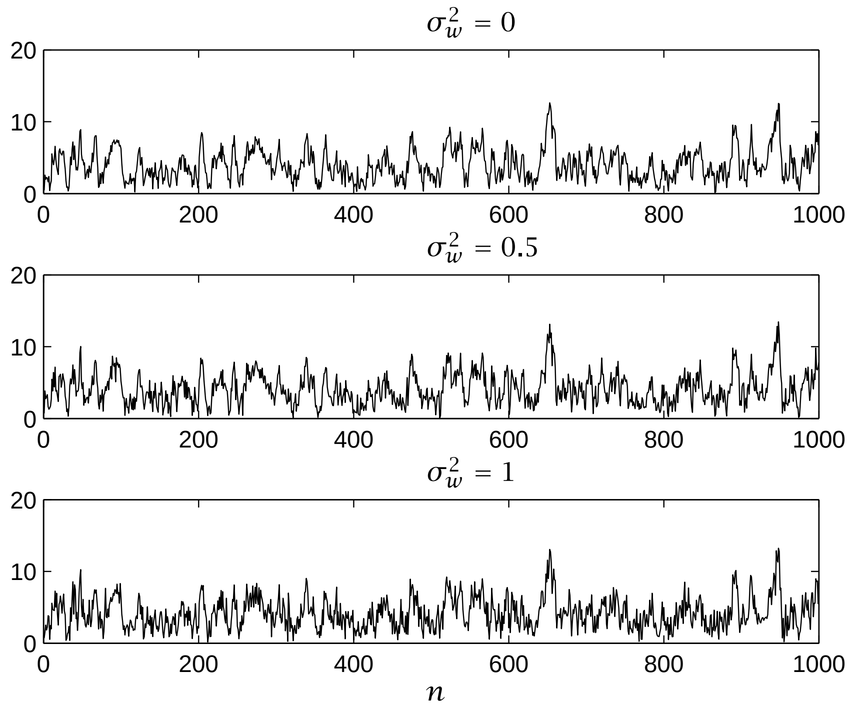
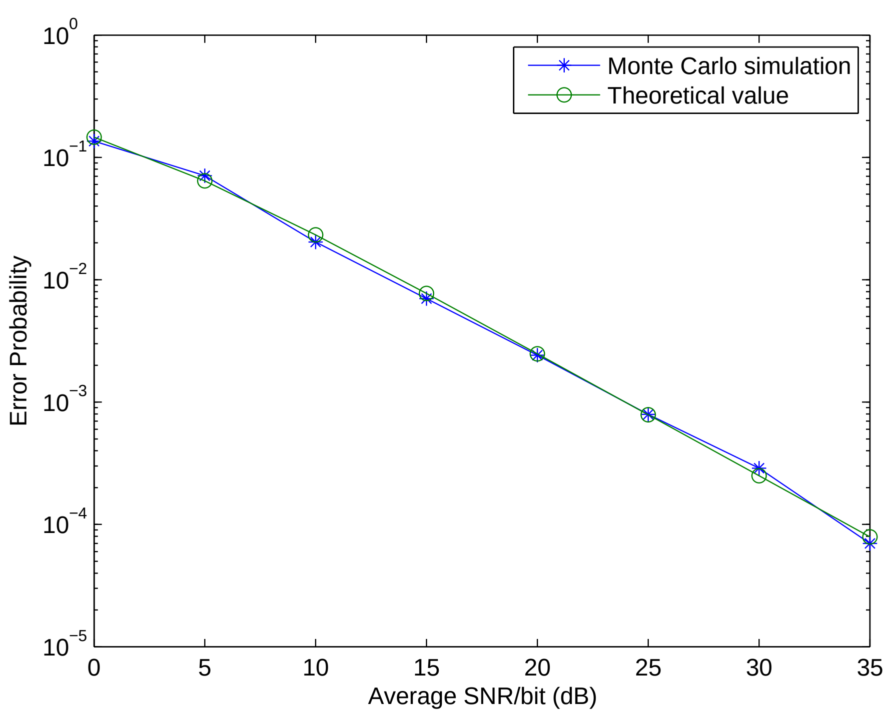


Figure 14.6: Channel outputs for $\sigma_w^2 = 0, 0.5, 1$ for $W = 5\text{kHz}$

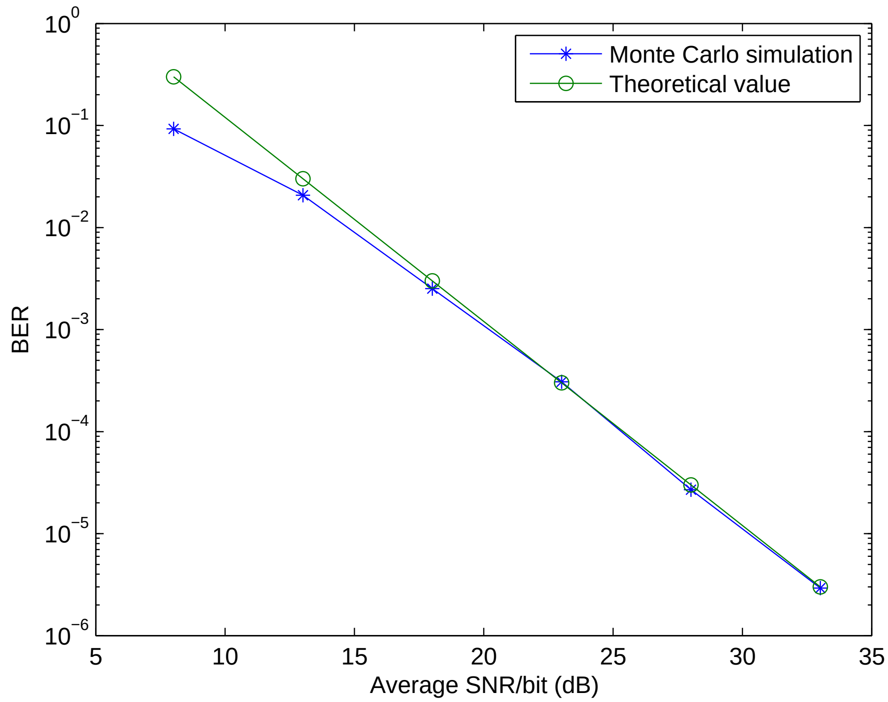


The MATLAB script for the problem is given below.

```
Eb = 1;                              % Energy per bit
EbNo_dB = 0:5:35;
No_over_2 = Eb*10.^(-EbNo_dB/10);    % Noise power
sigma = 1;                           % Rayleigh parameter
BER = zeros(1,length(EbNo_dB));
% Calculation of error probability using Monte Carlo simulation:
for i = 1:length(EbNo_dB)
    no_errors = 0;
    no_bits = 0;
    % Assumption: m = 0 (All zero codeword is transmitted):
    while no_errors <= 100
        u = rand;
        alpha = sigma*sqrt(-2*log(u));
        noise = sqrt(No_over_2(i))*randn;
        y = alpha*sqrt(Eb) + noise;
        if y <= 0
            y_d = 1;
        else
            y_d = 0;
        end
        no_bits = no_bits + 1;
        no_errors = no_errors + y_d;
    end
    BER(i) = no_errors/no_bits;
end
% Calculation of error probability using the theoretical formula:
rho_b = Eb./No_over_2;
P2 = 1/2*(1-sqrt(rho_b./(1+rho_b)));
% Plot the results:
semilogy(EbNo_dB,BER,'-*',EbNo_dB,P2,'-o')
xlabel('Average SNR/bit (dB)')
ylabel('Error Probability')
legend('Monte Carlo simulation','Theoretical value')
```

Computer Problem 14.3

The figure shown below illustrates the result of the Monte Carlo simulation and a comparison with the theoretical error probability. We note that the agreement is very good for large SNR.



The MATLAB script for the problem is given below.

```
D = 2;
sigma = 1;
Eb = 1/sqrt(2);
EbNo_rx_per_ch_dB = 5:5:30;
EbNo_rx_per_ch = 10.^(EbNo_rx_per_ch_dB/10);
No = Eb*2*sigma^2*10.^(-EbNo_rx_per_ch_dB/10);
BER = zeros(1,length(No));
SNR_rx_per_b_per_ch = zeros(1,length(No));
% Calculation of error probability using Monte Carlo simulation:
for i = 1:length(No)
    no_bits = 0;
    no_errors = 0;
    P_rx_t = 0;                % Total rxd power
    P_n_t = 0;                 % Total noise power
    r = zeros(2,2);
    R = zeros(1,2);
    % Assumption: m = 1 (All one codeword is transmitted):
    while no_errors <= 100
        no_bits = no_bits + 1;
        u = rand(1,2); alpha = sigma*sqrt(-2*log(u)); phi = 2*pi*rand(1,2);
        noise = sqrt(No(i)/2)*(randn(2,2) + 1i*randn(2,2));
        r(1,1) = alpha(1)*sqrt(Eb)*exp(1i*phi(1))+noise(1,1);
        r(1,2) = noise(1,2);
        r(2,1) = alpha(2)*sqrt(Eb)*exp(1i*phi(2))+noise(2,1);
        r(2,2) = noise(2,2);
        R(1) = abs(r(1,1))^2 + abs(r(2,1))^2;
```

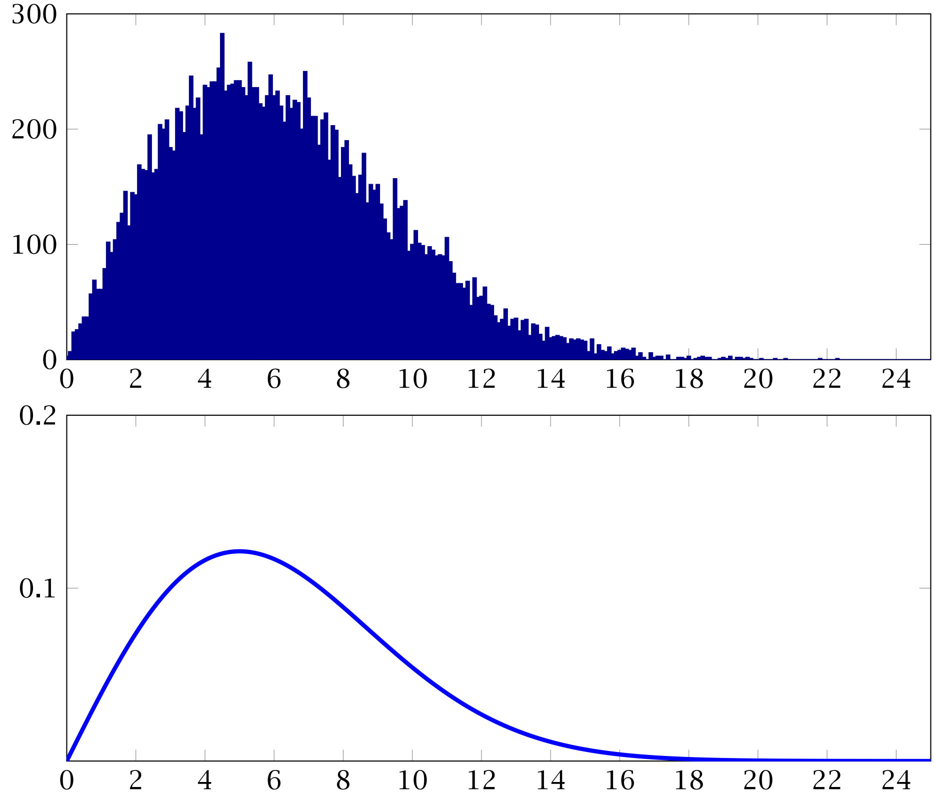
```
            R(2) = abs(r(1,2))^2 + abs(r(2,2))^2;
            if R(1) <= R(2)
                m_h = 0;
            else
                m_h = 1;
            end
            P_n_t = P_n_t + No(i);
            P_rx_t = P_rx_t + 0.5*(abs(r(1))^2 + abs(r(2))^2);
            no_errors = no_errors + (1-m_h);
        end
        SNR_rx_per_b_per_ch(i) = (P_rx_t-P_n_t)/P_n_t;
        BER(i) = no_errors/no_bits;
end
% Calculation of error probability using the theoretical formula:
rho = EbNo_rx_per_ch;
rho_dB = 10*log10(rho);
rho_b =D*rho;
rho_b_dB = 10*log10(rho_b);
K_D = factorial((2*D-1))/factorial(D)/factorial((D-1));
P_2 = K_D./rho.^D;
% Plot the results:
semilogy(rho_b_dB,BER,'-*',rho_b_dB,P_2,'-o')
xlabel('Average SNR/bit (dB)'); ylabel('BER')
legend('Monte Carlo simulation','Theoretical value')
```

Computer Problem 14.4

We use the relation $R = \sqrt{2\sigma^2 \ln\left(\frac{1}{1-A}\right)}$ to generate the the samples from a Rayleigh distribution, where the parameter A is generated from a uniform distribution in the interval $(0,1)$ and σ^2 may be arbitrary selected as 1,5,10. Then the actual Rayleigh PDF is given by

$$f(x) = \frac{x}{\sigma^2} e^{-x^2/\sigma^2}, \quad x \geq 0$$

The figure below illustrates the histogram and the comparison with the actual Rayleigh PDF for $\sigma = 5$.

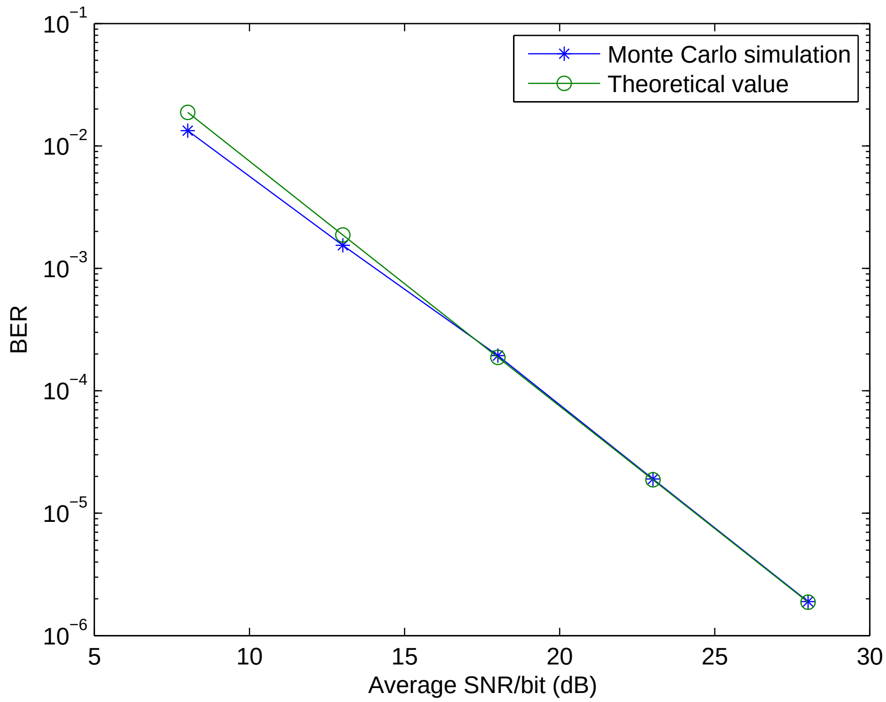


The MATLAB script for the problem is given below.

```
N=20000;
x=0:0.1:25;
u=rand(1,N);
sigma=5;
r=sigma*sqrt(-2*log(u));
r_ac=x/sigma^2.*exp(-(x/sigma).^2/2);
subplot(2,1,1)
hist(r,x)
axis([0 25 0 300])
subplot(2,1,2)
plot(x,r_ac)
```

Computer Problem 14.5

The following figure illustrates the result of the Monte Carlo simulation and comparison with the theoretical error probability for a dual diversity system ($D = 2$) at large SNR. We note that the agreement is very good for large SNR.



]

The MATLAB script for the problem is given below.

```
D = 2;
sigma = 1/sqrt(2);
Eb = 1;
EbNo_rx_per_ch_dB = 5:5:25;
EbNo_rx_per_ch = 10.^(EbNo_rx_per_ch_dB/10);
No = Eb*2*sigma^2*10.^(-EbNo_rx_per_ch_dB/10);
BER = zeros(1,length(No));
SNR_rx_per_b_per_ch = zeros(1,length(No));
% Calculation of error probability using Monte Carlo simulation:
for i = 1:length(No)
    no_bits = 0;
    no_errors = 0;
    % Assumption: m = 0 (All zero codeword is transmitted):
    while no_errors <= 100
        no_bits = no_bits + 1;
        u = rand(1,2);
        alpha = sigma*sqrt(-2*log(u));
        phi = 2*pi*rand(1,2);
        c = alpha.*exp(1i*phi);
        noise = sqrt(No(i)/2)*(randn(1,2) + 1i*randn(1,2));
        r = c*sqrt(Eb) + noise;
        R = real(conj(c(1))*r(1)+conj(c(2))*r(2));
        if R <= 0
            m_h = 1;
        else
            m_h = 0;
```

```
        end
        no_errors = no_errors + m_h;
    end
    BER(i) = no_errors/no_bits;
end
% Calculation of error probability using the theoretical formula:
rho = EbNo_rx_per_ch;
rho_b = D*rho;
rho_b_dB = 10*log10(rho_b);
K_D = factorial((2*D-1))/factorial(D)/factorial((D-1));
P_2 = K_D./(4*rho).^D;
% Plot the results:
semilogy(rho_b_dB,BER,'-*',rho_b_dB,P_2,'-o')
xlabel('Average SNR/bit (dB)'); ylabel('BER')
legend('Monte Carlo simulation','Theoretical value')
```

Computer Problem 14.6

ïż£The inputs to the detectors at the receive antennas are given by Equation (14.4.7), where the channel coefficients complex-valued, independent, zero-mean Gaussian random variables with identical variance σ^2.

The MATLAB script for the problem for the case of $N_T = N_R = 2, \sigma = 5$ is given below.

```
Nt = 2;                         % No. of transmit antennas
Nr = 2;                         % No. of receive antennas
sigma = 5;                      % Variance of fading coefficients
H = (randn(Nr,Nt) + 1i*randn(Nr,Nt))*sigma/sqrt(2);      % Channel coefficients
```

Computer Problem 14.7

ïż£The elements of $\boldsymbol{H}$ are h_{11}, h_{12}, h_{21}, and h_{22}. For the Rayleigh fading channel, these parameters are complex-valued, statistically independent, zero-mean Gaussian random variables with identical variances σ^2. Hence, the two inputs to the detectors at the two antennas are

$$y_1 = h_{11}s_1 + h_{12}s_2 + \eta_1$$
$$y_2 = h_{21}s_1 + h_{22}s_2 + \eta_2$$

where s_1 and s_2 are the transmitted symbols from the two transmit antennas and (η_1, η_2) are the statistically independent additive Gaussian noise terms with zero mean and equal variances σ_n^2.

The MATLAB script for the problem when $\sigma^2 = 5$ and $\sigma_n^2 = 1$ is given below.

```
Nt = 2;                         % No. of transmit antennas
Nr = 2;                         % No. of receive antennas
```

```
sigma = 5;                          % Variance of fading coefficients
No = 1;                             % Noise variance
s = 2*randi([0 1],Nt,1) − 1;      % Binary transmitted symbols
H = (randn(Nr,Nt) + 1i*randn(Nr,Nt))*sigma/sqrt(2);        % Channel coefficients
noise = sqrt(No/2)*(randn(Nr,1) + 1i*randn(Nr,1));  % AWGN noise
y = H*s + noise;                    % Inputs to the detectors
disp(['The inputs to the detectors are:    ', num2str(y')])
```

Computer Problem 14.8

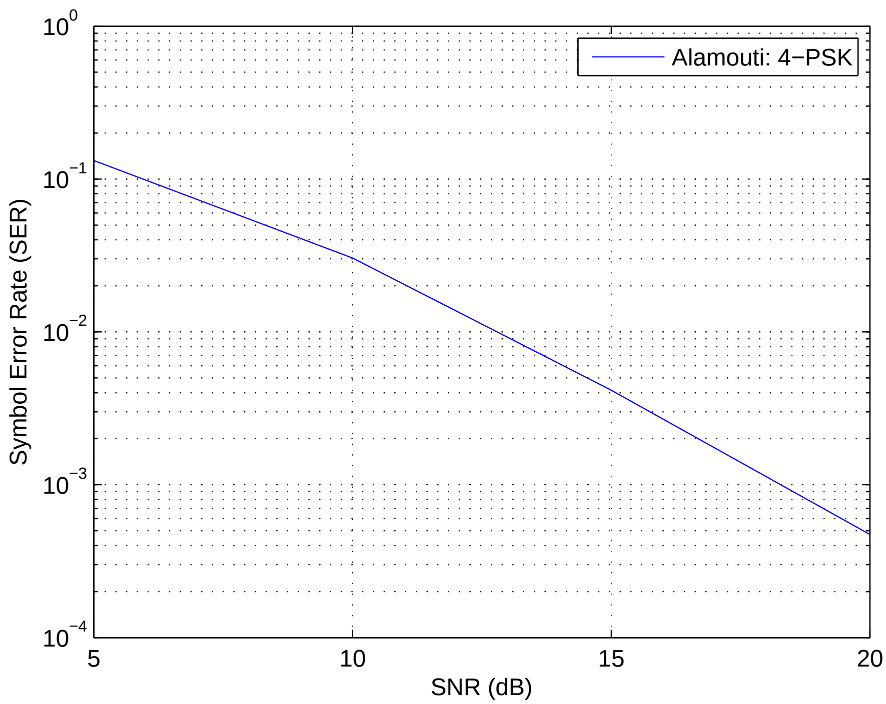
ïż£The MATLAB script for the computations in each of the three detectors is given below.

```
Nt = 2;                                            % No. of transmit antennas
Nr = 2;                                            % No. of receive antennas
S = [1 1 −1 −1; 1 −1 1 −1];                        % Reference codebook
H = (randn(Nr,Nt) + 1i*randn(Nr,Nt))/sqrt(2);    % Channel coefficients
s = 2*randi([0 1],Nt,1) − 1;                     % Binary transmitted symbols
No = 0.1;                                          % Noise Noiance
noise = sqrt(No/2)*(randn(Nr,1) + 1i*randn(Nr,1)); % AWGN noise
y = H*s + noise;                                   % Inputs to the detectors
disp(['The transmitted symbols are:                    ',num2str(s')])

% Maximum Likelihood Detector:
mu = zeros(1,4);
for i = 1:4
    mu(i) = sum(abs(y − H*S(:,i)).^2);            % Euclidean distance metric
end
[Min idx] = min(mu);
s_h = S(:,idx);
disp(['The detected symbols using the ML method are:   ',num2str(s_h')])

% MMSE Detector:
w1 = (H*H' + No*eye(2))^(-1) * H(:,1);          % Optimum weight vector 1
w2 = (H*H' + No*eye(2))^(-1) * H(:,2);          % Optimum weight vector 2
W = [w1 w2];
s_h = W'*y;
for i = 1:Nt
    if s_h(i) >= 0
        s_h(i) = 1;
    else
        s_h(i) = −1;
    end
end
disp(['The detected symbols using the MMSE method are:  ',num2str(s_h')])

% Inverse Channel Detector:
s_h = H\y;
for i = 1:Nt
    if s_h(i) >= 0
        s_h(i) = 1;
```

```
    else
        s_h(i) = -1;
    end
end
disp(['The detected symbols using the ICD method are:   ',num2str(s_h')])
```

Computer Problem 14.9

ïż£The graph for the estimated error rates as a function of SNR is shown below.



The MATLAB script for the problem is given below.

```
Nt = 2;                          % No. of transmit antennas
Nr = 1;                          % No. of receive antennas
codebook = [1+1i 1-1i -1+1i -1-1i]; % Reference codebook
Es = 2;                          % Energy per symbol
SNR_dB = 5:5:20;                   % SNR in dB
No = Es*10.^(-1*SNR_dB/10);           % Noise variance
% Preallocation for speed:
Dist1 = zeros(1,4);              % Distance vector for s1
Dist2 = zeros(1,4);              % Distance vector for s1
BER = zeros(1,length(SNR_dB));
% Maximum Likelihood Detector:
echo off;
for i = 1:length(SNR_dB)
```

```
    no_errors = 0;
    no_symbols = 0;
    while no_errors <= 100
        s = 2*randi([0 1],1,2)-1 + 1i*(2*randi([0 1],1,2)-1);
        no_symbols = no_symbols + 2;
        % Channel coefficients
        h = 1/sqrt(2) * (randn(1,2) + 1i*randn(1,2));
        % Noise generation:
        noise = sqrt(No(i))*(randn(2,1) + 1i*randn(2,1));
        % Correlator outputs:
        y(1) = h(1)*s(1) + h(2)*s(2) + noise(1);
        y(2) = -h(1)*conj(s(2)) + h(2)*conj(s(1)) + noise(2);
        % Estimates of the symbols s1 and s2:
        s_h(1) = y(1)*conj(h(1)) + conj(y(2))*h(2);
        s_h(2) = y(1)*conj(h(2)) - conj(y(2))*h(1);
        % Maximum-Likelihood detection:
        for j = 1 : 4
            Dist1(j) = abs(s_h(1)-codebook(j));
            Dist2(j) = abs(s_h(2)-codebook(j));
        end
        [Min1 idx1] = min(Dist1);
        [Min2 idx2] = min(Dist2);
        s_t(1) = codebook(idx1);
        s_t(2) = codebook(idx2);
        % Calculation of error numbers:
        if s_t(1) ~= s(1)
            no_errors = no_errors + 1;
        end
        if s_t(2) ~= s(2)
            no_errors = no_errors + 1;
        end
    end
    BER(i) = no_errors/no_symbols;
end
echo on;
semilogy(SNR_dB,BER)
xlabel('SNR (dB)')
ylabel('Symbol Error Rate (SER)')
legend('Alamouti: 4-PSK')
```

Computer Problem 14.10

ïż£Figures 14.7 and 14.8 illustrate the binary error rate (BER) for binary PSK modulation with $(N_T, N_R) = (2, 2)$ and $(N_T, N_R) = (2, 3)$, respectively. In both cases, the variances of the channel gains are identical and their sum is normalized to unity; that is,

$$\sum_{n,m} E[|h_{mn}|^2] = 1 \tag{14.52}$$

The BER for binary PSK modulation is plotted as a function of the average SNR per bit. With the normalization of the variances in the channel gains $\{h_{mn}\}$ as given by Equation (14.52), the average received energy is simply the transmitted signal energy per symbol.

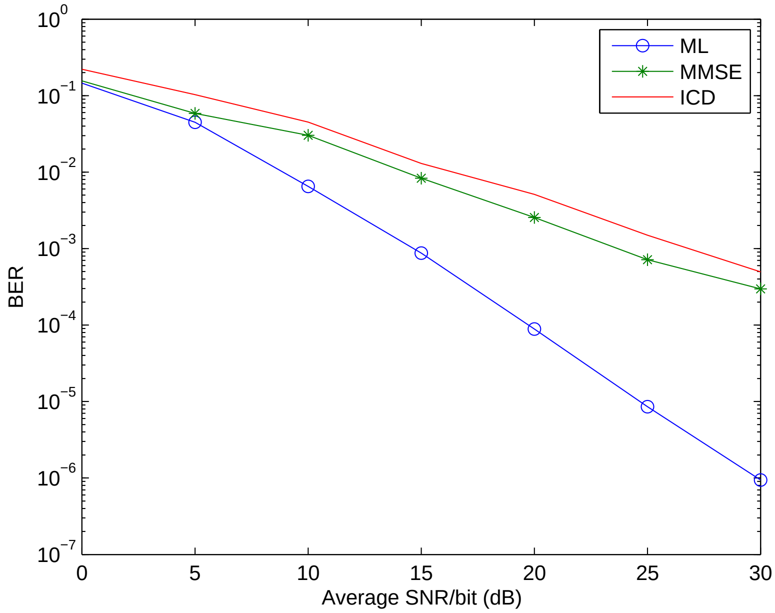


Figure 14.7: Performance of MLD, MMSE, and ICD (detectors) with $N_R = 2$ receiving antennas

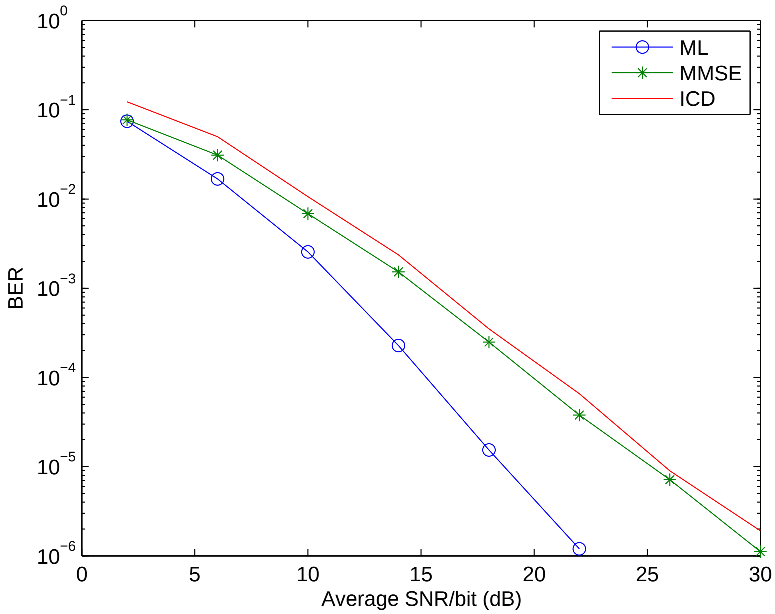


Figure 14.8: Performance of MLD and MMSE detectors with $N_R = 3$ receiving antennas

The performance results in Figures 14.7 and 14.8 illustrate that the MLD exploits the full diversity of order N_R available in the received signal and, thus, its performance is comparable to that of a maximal ratio combiner (MRC) of the N_R received signals, without the presence of interchannel interference; that is, $(N_T, N_R) = (1, N_R)$. The two linear detectors, the MMSE detector and the ICD, achieve an error rate that decreases inversely as the SNR raised to the $(N_R - 1)$ power for $N_T = 2$ transmitting antennas. Thus, when $N_R = 2$, the two linear detectors achieve no diversity, and when $N_R = 3$, the linear detectors achieve dual diversity. We also note that the MMSE detector outperforms the ICD, although both achieve the same order of diversity. In general, with spatial multiplexing (N_T antennas transmitting independent data streams), the MLD detector achieves a diversity of order N_R and the linear detectors achieve a diversity of order $N_R - N_T + 1$, for any $N_R \geq N_T$. In effect, with N_T antennas transmitting independent data streams and N_R receiving antennas, a linear detector has N_R degrees of freedom. In detecting any one data stream, in the presence of $N_T - 1$ interfering signals from the other transmitting antennas, the linear detectors utilize $N_T - 1$ degrees of freedom to cancel the $N_T - 1$ interfering signals. Therefore, the effective order of diversity for the linear detectors is $N_R - (N_T - 1) = N_R - N_T + 1$.

The MATLAB script for the problem is given below.

```
Nt = 2;                                  % No. of transmit antennas
Nr = 2;                                  % No. of receive antennas
S = [1 1 −1 −1; 1 −1 1 −1];              % Reference codebook
Eb = 1;                                  % Energy per bit
EbNo_dB = 0:5:30;                        % Average SNR per bit
No = Eb*10.^(−1*EbNo_dB/10);             % Noise variance
BER_ML = zeros(1,length(EbNo_dB));       % Bit-Error-Rate Initialization
BER_MMSE = zeros(1,length(EbNo_dB));     % Bit-Error-Rate Initialization
BER_ICD = zeros(1,length(EbNo_dB));      % Bit-Error-Rate Initialization

% Maximum Likelihood Detector:
echo off;
for i = 1:length(EbNo_dB)
    no_errors = 0;
    no_bits = 0;
    while no_errors <= 100
        mu = zeros(1,4);
        s = 2*randi([0 1],Nt,1) − 1;
        no_bits = no_bits + length(s);
        H = (randn(Nr,Nt) + 1i*randn(Nr,Nt))/sqrt(2*Nr);
        noise = sqrt(No(i)/2)*(randn(Nr,1) + 1i*randn(Nr,1));
        y = H*s + noise;
        for j = 1:4
            mu(j) = sum(abs(y − H*S(:,j)).^2);  % Euclidean distance metric
        end
        [Min idx] = min(mu);
        s_h = S(:,idx);
        no_errors = no_errors + nnz(s_h−s);
    end
    BER_ML(i) = no_errors/no_bits;
end
echo on;
```

```
% Minimum Mean-Sqaure-Error (MMSE) Detector:
echo off;
for i = 1:length(EbNo_dB)
    no_errors = 0;
    no_bits = 0;
    while no_errors <= 100
        s = 2*randi([0 1],Nt,1) − 1;
        no_bits = no_bits + length(s);
        H = (randn(Nr,Nt) + 1i*randn(Nr,Nt))/sqrt(2*Nr);
        noise = sqrt(No(i)/2)*(randn(Nr,1) + 1i*randn(Nr,1));
        y = H*s + noise;
        w1 = (H*H' + No(i)*eye(Nr))^(-1) * H(:,1); % Optimum weight vector 1
        w2 = (H*H' + No(i)*eye(Nr))^(-1) * H(:,2); % Optimum weight vector 2
        W = [w1 w2];
        s_h = W'*y;
        for j = 1:Nt
            if s_h(j) >= 0
                s_h(j) = 1;
            else
                s_h(j) = −1;
            end
        end
        no_errors = no_errors + nnz(s_h−s);
    end
    BER_MMSE(i) = no_errors/no_bits;
end
echo on;

% Inverse Channel Detector:
echo off;
for i = 1:length(EbNo_dB)
    no_errors = 0;
    no_bits = 0;
    while no_errors <= 100
        s = 2*randi([0 1],Nt,1) − 1;
        no_bits = no_bits + length(s);
        H = (randn(Nr,Nt) + 1i*randn(Nr,Nt))/sqrt(2*Nr);
        noise = sqrt(No(i)/2)*(randn(Nr,1) + 1i*randn(Nr,1));
        y = H*s + noise;
        s_h = H\y;
        for j = 1:Nt
            if s_h(j) >= 0
                s_h(j) = 1;
            else
                s_h(j) = −1;
            end
        end
        no_errors = no_errors + nnz(s_h−s);
    end
    BER_ICD(i) = no_errors/no_bits;
end
echo on;
% Plot the results:
semilogy(EbNo_dB,BER_ML,'-o',EbNo_dB,BER_MMSE,'-*',EbNo_dB,BER_ICD)
```

```
xlabel('Average SNR/bit (dB)','fontsize',10)
ylabel('BER','fontsize',10)
legend('ML','MMSE','ICD')
```

Computer Problem 14.11

ï¿£The MATLAB script for the problem is given below.

```
no_bits = 10;                    % Determine the length of input vector
input = randi([0 3],1,no_bits); % Define the input as a random vector
if mod(no_bits,2) ~= 0
    input = [input 0];
end
L = size(input,2);
st_0 = 0;                        % Initial state
st_c = st_0;                     % Initialization of the current state
ant_1 = [];                      % Output of antenna 1
ant_2 = [];                      % Output of antenna 2
% Update the current state as well as outputs of antennas 1 and 2:
for i = 1:L
    st_p = st_c;
    if input(i) == 0
        st_c = 0;
    elseif input(i) == 1
        st_c = 1;
    elseif input(i) == 2
        st_c = 2;
    else
        st_c = 3;
    end
    ant_1 = [ant_1 st_p];
    ant_2 = [ant_2 st_c];
end
if st_c ~= 0
        st_p = st_c;
    st_c = 0;
    ant_1 = [ant_1 st_p];
    ant_2 = [ant_2 st_c];
end
% Display the input vector and outputs of antennas 1 and 2:
disp(['The input sequence is:                  ', num2str(input)])
disp(['The transmitted sequence by antenna 1 is:   ', num2str(ant_1)])
disp(['The transmitted sequence by antenna 2 is:   ', num2str(ant_2)])
```

Chapter 15

Problem 15.1

The probability of error for DS spread spectrum with binary PSK may be expressed as

$$P_2 = Q\left(\sqrt{\frac{2W/R_b}{P_J/P_S}}\right)$$

where W/R is the processing gain and P_J/P_S is the jamming margin. If the jammer is a broadband, WGN jammer, then

$$\begin{aligned} P_J &= WJ_0 \\ P_S &= \mathcal{E}_b/T_b = \mathcal{E}_b R_b \end{aligned}$$

Therefore,

$$P_2 = Q\left(\sqrt{\frac{2\mathcal{E}_b}{J_0}}\right)$$

which is identical to the performance obtained with a non-spread signal.

Problem 15.2

We assume that the interference is characterized as a zero-mean AWGN process with power spectral density J_0. To achieve an error probability of 10^{-5}, the required $\mathcal{E}_b/J_0 = 10$. Then, by using the relation in (11.3.33) and (11.3.37), we have

$$\begin{aligned} \frac{W/R}{P_N/P_S} &= \frac{W/R}{N_u - 1} = \frac{\mathcal{E}_b}{J_0} \\ W/R &= \left(\frac{\mathcal{E}_b}{J_0}\right)(N_u - 1) \\ W &= R\left(\frac{\mathcal{E}_b}{J_0}\right)(N_u - 1) \end{aligned}$$

where $R = 10^4 bps$, $N_u = 30$ and $\mathcal{E}_b/J_0 = 10$. Therefore,

$$W = 2.9 \times 10^6 \ Hz$$

The minimum chip rate is $1/T_c = W = 2.9 \times 10^6$ chips/sec.

Problem 15.3

To achieve an error probability of 10^{-6}, we require

$$\left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} = 10.5 dB$$

Then, the number of users of the CDMA system is

$$\begin{aligned} N_u &= \frac{W/R_b}{\mathcal{E}_b/J_0} + 1 \\ &= \frac{1000}{11.3} + 1 = 89 \text{ users} \end{aligned}$$

If the processing gain is reduced to $W/R_b = 500$, then

$$N_u = \frac{500}{11.3} + 1 = 45 \text{users}$$

Problem 15.4

We are given a system where $(P_J/P_S)_{dB} = 20\ dB, R = 1000\ bps$ and $(\mathcal{E}_b/J_0)_{dB} = 10\ dB$. Hence, using the relation in (11.3.33) we obtain

$$\begin{aligned} \left(\frac{W}{R}\right)_{dB} &= \left(\frac{P_J}{P_S}\right)_{dB} + \left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} = 30\ dB \\ \frac{W}{R} &= 1000 \\ W &= 1000R = 10^6 Hz \end{aligned}$$

Problem 15.5

The radio signal propagates at the speed of light, $c = 3 \times 10^8 m/\sec$. The difference in propagation delay for a distance of 300 meters is

$$T_d = \frac{300}{3 \times 10^8} = 1\mu\sec$$

The minimum bandwidth of a DS spread spectrum signal required to resolve the propagation paths is $W = 1\ MHz$. Hence, the minimum chip rate is 10^6 chips per second.

Problem 15.6

1. We have $N_u = 15$ users transmitting at a rate of $10,000\ bps$ each, in a bandwidth of $W = 1\ MHz$. The $\mathcal{E}_b/J_0$ is

$$\begin{aligned}\frac{\mathcal{E}}{J_0} &= \frac{W/R}{N_u - 1} = \frac{10^6/10^4}{14} = \frac{100}{14} \\ &= 7.14\,(8.54\ dB)\end{aligned}$$

1. The processing gain is 100.

2. With $N_u = 30$ and $\mathcal{E}_b/J_0 = 7.14$, the processing gain should be increased to

$$W/R = (7.14)\,(29) = 207$$

Hence, the bandwidth must be increased to $W = 2.07 MHz$.

Problem 15.7

1. The length of the shift-register sequence is

$$\begin{aligned}L &= 2^m - 1 = 2^{15} - 1 \\ &= 32767 \text{ bits}\end{aligned}$$

For binary FSK modulation, the minimum frequency separation is $2/T$, where $1/T$ is the symbol (bit) rate. The hop rate is 100 $hops/\sec$. Since the shift register has $N = 32767$ states and each state utilizes a bandwidth of $2/T = 200\ Hz$, then the total bandwidth for the FH signal is $6.5534\ MHz$.

2. The processing gain is W/R. We have,

$$\frac{W}{R} = \frac{6.5534 \times 10^6}{100} = 6.5534 \times 10^4 \text{ bps}$$

3. If the noise is AWG with power spectral density N_0, the probability of error expression is

$$P_2 = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right) = Q\left(\sqrt{\frac{W/R}{P_N/P_S}}\right)$$

Problem 15.8

The processing gain is given as

$$\frac{W}{R_b} = 500 \ \ (27\ dB)$$

The $(\mathcal{E}_b/J_0)$ required to obtain an error probability of 10^{-5} for binary PSK is 9.5 dB. Hence, the jamming margin is

$$\begin{aligned}\left(\frac{P_J}{P_S}\right)_{dB} &= \left(\frac{W}{R_b}\right)_{dB} - \left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} \\ &= 27 - 9.5 \\ &= 17.5\ dB\end{aligned}$$

Problem 15.9

Without loss of generality, let us assume that $L_1 < L_2$. Then, the period of the sequence obtained by forming the modulo-2 sum of the two periodic sequences is

$$L_3 = kL_2$$

where k is the smallest integer multiple of L_2 such that kL_2/L_1 is an integer. For example, suppose that $L_1 = 15$ and $L_2 = 63$. Then, we find the smallest multiple of 63 which is divisible by $L_1 = 15$, without a remainder. Clearly, if we take $k = 5$ periods of L_2, which yields a sequence of $L_3 = 315$, and divide L_3 by L_1, the result is 21. Hence, if we take $21L_1$ and $5L_2$, and modulo-2 add the resulting sequences, we obtain a single period of length $L_3 = 21L, = 5L_2$ of the new sequence.

Problem 15.10

1. The period of the maximum length shift register sequence is

$$L = 2^{10} - 1 = 1023$$

Since $T_b = LT_c$, then the processing gain is

$$L = \frac{T_b}{T_c} = 1023\ (30dB)$$

2. The jamming margin is

$$\begin{aligned}\left(\frac{P_J}{P_S}\right)_{dB} &= \left(\frac{W}{R_b}\right)_{dB} - \left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} \\ &= 30 - 10 \\ &= 20dB\end{aligned}$$

where $J_{av} = J_0 W \approx J_0/T_c = J_0 \times 10^6$

Problem 15.11

At the bit rate of 270.8 Kbps, the bit interval is

$$T_b = \frac{10^{-6}}{.2708} = 3.69\mu\text{sec}$$

a) For the suburban channel model, the delay spread is 7 μsec. Therefore, the number of bits affected by intersymbol interference is at least 2. The number may be greater than 2 if the signal pulse extends over more than one bit interval, as in the case of partial response signals, such as CPM.

b) For the hilly terrain channel model, the delay spread is approximately 20 μ sec. Therefore, the number of bits affected by ISI is at least 6. The number may be greater than 6 if the signal pulse extends over more than one bit interval.

Problem 15.12

In the case of the urban channel model, the number of RAKE receiver taps will be at least 2. If the signal pulse extends over more than one bit interval, the number of RAKE taps must be further increased to account for the ISI over the time span of the signal pulse. For the hilly terrain channel model, the minimum number of RAKE taps is at least 6 but only three will be active, one for the first arriving signal and 2 for the delayed arrivals.

If the signal pulse extends over more than one bit interval, the number of RAKE taps must be further increased to account for the ISI over the same span of the signal pulse. For this channel, in which the multipath delay characteristic is zero in the range of 2 μsec to 15 μsec, as many as 3 RAKE taps between the first signal arrival and the delayed signal arrivals will contain no signal components.

Problem 15.13

For an automobile traveling at a speed of 100 Km/hr,

$$f_m = \frac{vf_0}{c} = \frac{10^5}{3600} \times \frac{9 \times 10^8}{3^8} = 83.3\text{Hz}$$

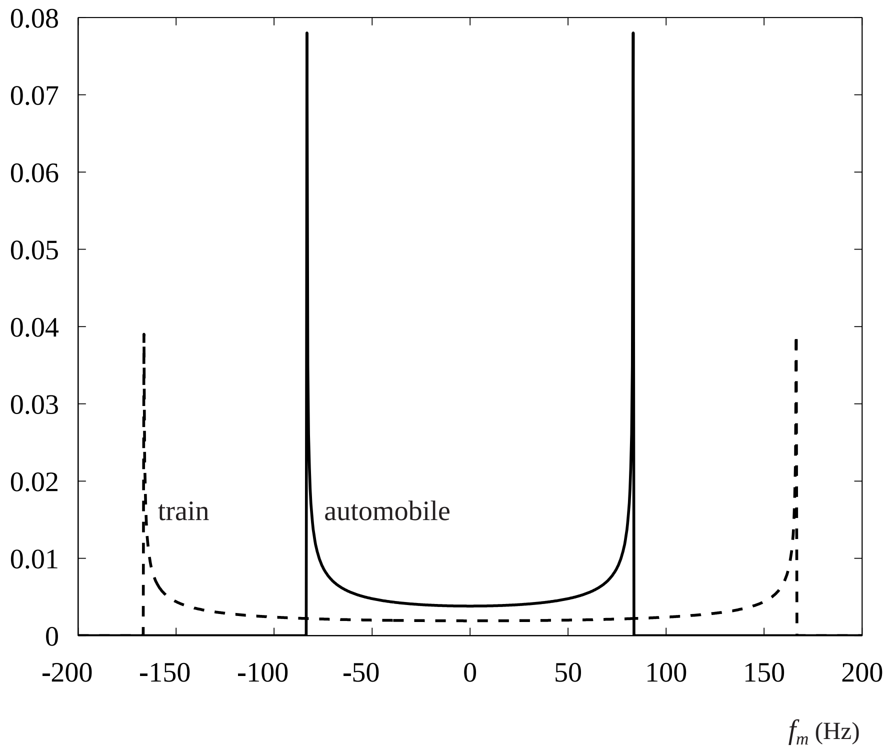
For a train traveling at a speed of 200 Km/hr,

$$f_m = 166.6\text{Hz}$$

The corresponding spread factors are

$$T_m B_d = T_m f_m = \begin{cases} 5.83 \times 10^{-4}, & \text{automobile} \\ 1.166 \times 10^{-3}, & \text{train} \end{cases}$$

The plots of the power spectral density for the automobile and the train are shown below



Problem 15.14
The expression for the received signal power is

$$P_{R\text{dB}} = P_{T\text{dB}} - L_{s\text{dB}} + G_{T\text{dB}}$$

where $L_{s\text{dB}}$ is the free-space path loss and $G_{T\text{dB}}$ is the antenna gain. The path loss is

$$L_{s\text{dB}} = 20 \log\left(\frac{4\pi d}{2}\right)$$

where the wavelength $\lambda = 100$ meters. Hence,

$$L_{s\text{dB}} = 20 \times \log(8\pi \times 10^4) = 108 \text{ dB}$$

Therefore,

$$\begin{aligned} P_{T\text{dB}} &= P_{R\text{dB}} + 108 - 20 \\ &= P_{R\text{dB}} + 88 \end{aligned}$$

The received power level can be obtained from the condition $P_R/P_N = 10^{-2}$. First of all, $P_N = WN_0$, where $N_0 = kT = 4 \times 10^{-21}$ W/Hz and $W = 10^5$ Hz. Hence,

$$P_N = 4.1 \times 10^{-16} \text{ W}$$

and

$$P_R = 4.1 \times 10^{-18} \text{ W}$$

or, equivalently, $P_{R\text{dB}} = -174$ dBW. Therefore,

$$P_{T\text{dB}} = P_{R\text{dB}} + 99 = -86 \text{ dBW}$$

or, equivalently, $P_T = 2.5 \times 10^{-9}$ W. The bit rate is $R = W/L_c = 10^5/10^3 = 100$ bps.

Problem 15.15

(a) The coding gain is

$$R_c d_{\min}^H = \frac{1}{2} \times 10 = 5 \ (7dB)$$

(b) The processing gain is W/R, where $W = 10^7 Hz$ and $R = 2000 bps$. Hence,

$$\frac{W}{R} = \frac{10^7}{2 \times 10^3} = 5 \times 10^3 \ (37dB)$$

(c) The jamming margin given by (10.3.43) is

$$\begin{aligned} \left(\frac{P_J}{P_s}\right)_{dB} &= \left(\frac{W}{R}\right)_{dB} + (CG)_{dB} - \left(\frac{\mathcal{E}_b}{J_0}\right)_{dB} \\ &= 37 + 7 - 10 = 34dB \end{aligned}$$

Problem 15.16

(a) If the hopping rate is 2 hops/bit and the bit rate is 100 bits/sec, then, the hop rate is 200 hops/sec. The minimum frequency separation for orthogonality $2/T = 400Hz$. Since there are $N = 32767$ states of the shift register and for each state we select one of two frequencies separated by 400 Hz, the hopping bandwidth is 13.1068 MHz.

(b) The processing gain is W/R, where $W = 13.1068\ MHz$ and $R = 100 bps$. Hence

$$\frac{W}{R} = 0.131068 MHz$$

(c) The probability of error in the presence of AWGN is given by (10.3.61) with $N = 2$ chips per hop.

Problem 15.17

a) The total SNR for three hops is $20 \sim 13$ dB.Therefore the SNR per hop is $20/3$. The probability of a chip error with noncoherent detection is

$$p = \frac{1}{2}e^{-\frac{\mathcal{E}_c}{2N_0}}$$

where $\mathcal{E}_c/N_0 = 20/3$. The probability of a bit error is

$$\begin{aligned} P_b &= 1-(1-p)^2 \\ &= 1-(1-2p+p^2) \\ &= 2p-p^2 \\ &= e^{-\frac{\mathcal{E}_c}{2N_0}} - \frac{1}{2}e^{-\frac{\mathcal{E}_c}{N_0}} \\ &= 0.0013 \end{aligned}$$

b) In the case of one hop per bit, the SNR per bit is 20, Hence,

$$\begin{aligned} P_b &= \frac{1}{2}e^{-\frac{\mathcal{E}_c}{2N_0}} \\ &= \frac{1}{2}e^{-10} \\ &= 2.27 \times 10^{-5} \end{aligned}$$

Therefore there is a loss in performance of a factor 57 AWGN due to splitting the total signal energy into three chips and, then, using hard decision decoding.

Problem 15.18

(a) We are given a hopping bandwidth of 2 GHz and a bit rate of 10 kbs. Hence,

$$\frac{W}{R} = \frac{2 \times 10^9}{10^4} = 2 \times 10^5\ (53dB)$$

(b) The bandwidth of the worst partial-band jammer is $\alpha^* W$, where

$$\alpha^* = 2/\left(\mathcal{E}_b/J_0\right) = 0.2$$

Hence

$$\alpha^* W = 0.4GHz$$

(c) The probability of error with worst-case partial-band jamming is

$$P_2 = \frac{e^{-1}}{(\mathcal{E}_b/J_0)} = \frac{e^{-1}}{10}$$

$$= 3.68 \times 10^{-2}$$

Computer Problems

Computer Problem 15.1

The results of Monte Carlo simulation are shown in Figure 15.1 for three different values of amplitude of the sinusoidal interference with $L_c = 20$.

The Matlab script for the simulation program is given next.

```
% MATLAB script for Computer Problem 15.1.
echo on
Lc=20;                          % number of chips per bit
A1=3;                           % amplitude of the first sinusoidal interference
A2=10;                          % amplitude of the second sinusoidal interference
A3=12;                          % amplitude of the third sinusoidal interference
A4=0;                           % fourth case: no interference
w0=1;                           % frequency of the sinusoidal interference in radians
SNRindB=0:2:30;
for i=1:length(SNRindB),
  % measured error rates
  smld_err_prb1(i)=ss_Pe(SNRindB(i),Lc,A1,w0);
  smld_err_prb2(i)=ss_Pe(SNRindB(i),Lc,A2,w0);
  smld_err_prb3(i)=ss_Pe(SNRindB(i),Lc,A3,w0);
  echo off ;
end;
echo on ;
SNRindB4=0:1:8;
for i=1:length(SNRindB4),
  % measured error rate when there is no interference
  smld_err_prb4(i)=ss_Pe(SNRindB4(i),Lc,A4,w0);
  echo off ;
end;
echo on ;
% Plotting commands follow.
```

```
function [p]=ss_Pe(snr_in_dB, Lc, A, w0)
% [p]=ss_Pe(snr_in_dB, Lc, A, w0)
%               SS_PE finds the measured error rate. The function
%               that returns the measured probability of error for the given value of
%               the snr_in_dB, Lc, A and w0.
snr=10^(snr_in_dB/10);
sgma=1;                              % Noise standard deviation is fixed.
Eb=2*sgma^2*snr;                     % signal level required to achieve the given
                                     % signal-to-noise ratio
E_chip=Eb/Lc;                        % energy per chip
N=100000;                            % number of bits transmitted
% The generation of the data, noise, interference, decoding process and error
% counting is performed all together in order to decrease the run time of the
% program. This is accomplished by avoiding very large sized vectors.
num_of_err=0;
for i=1:N,
  % Generate the next data bit.
  temp=rand;
  if (temp<0.5),
    data=-1;
  else
    data=1;
  end;
  % Repeat it Lc times, i.e. divide it into chips.
  for j=1:Lc,
    repeated_data(j)=data;
  end;
  % pn sequence for the duration of the bit is generated next
  for j=1:Lc,
    temp=rand;
    if (temp<0.5),
      pn_seq(j)=-1;
    else
      pn_seq(j)=1;
    end;
  end;
  % the transmitted signal is
  trans_sig=sqrt(E_chip)*repeated_data.*pn_seq;
  % AWGN with variance sgma^2
  noise=sgma*randn(1,Lc);
  % interference
  n=(i-1)*Lc+1:i*Lc;
  interference=A*sin(w0*n);
  % received signal
  rec_sig=trans_sig+noise+interference;
  % Determine the decision variable from the received signal.
  temp=rec_sig.*pn_seq;
  decision_variable=sum(temp);
  % making decision
  if (decision_variable<0),
    decision=-1;
  else
    decision=1;
  end;
```

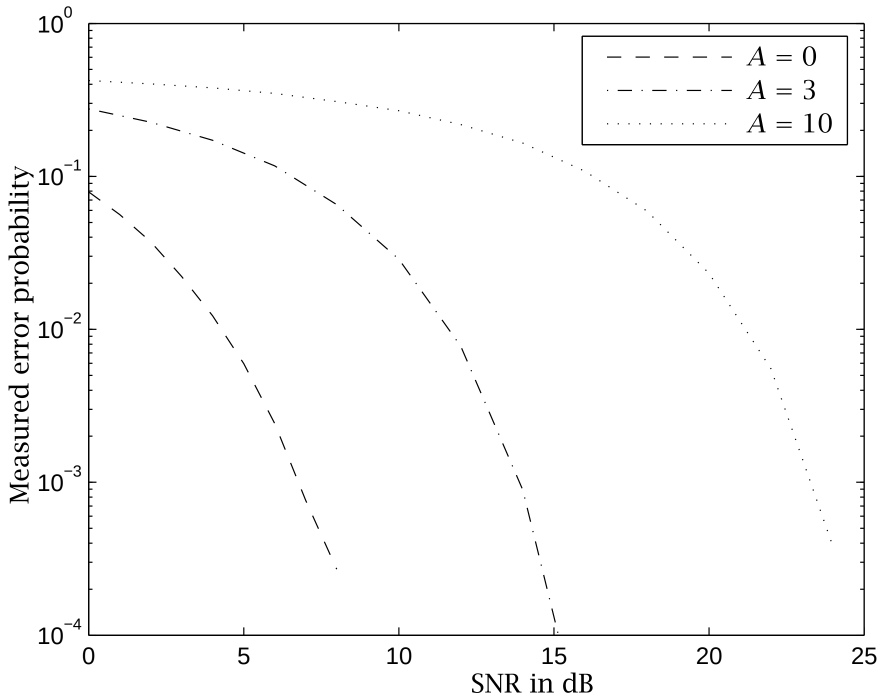


Figure 15.1: The results of Monte Carlo simulation

```
  % If it is an error, increment the error counter.
  if (decision~=data),
    num_of_err=num_of_err+1;
  end;
end;
% then the measured error probability is
p=num_of_err/N;
```

Computer Problem 15.2

The results of Monte Carlo simulation are shown in Figure 15.2.

The Matlab script for the simulation program is given next.

```
% MATLAB script for Computer Problem 15.2
snrMin = 1;
snrMax = 14;
numOfUsers = 4;
N = 10000; % Number of simulation bits
% Assign the gold sequences
gs{1} = [1 0 0 1 1 0 0 0 0 0 0 1 0 0 1 1 1 1 0 0 1 1 0 0 1 1 1 1 0 1 1];
gs{2} = [1 0 0 0 1 0 1 0 1 0 1 0 0 0 1 1 0 0 1 0 1 0 1 0 0 0 0 0 1 1 0];
gs{3} = [1 0 1 0 1 1 1 1 1 1 0 0 0 0 1 0 1 1 1 0 0 1 1 1 1 1 1 1 1 0 0];
gs{4} = [1 1 1 0 0 1 0 1 0 0 0 0 0 0 0 1 0 1 1 1 1 1 0 0 0 0 0 1 0 0 0];
```

```
for i=1:4
    gs{i} = 2*gs{i}-1;
end
chip = length(gs{1});
err = zeros(numOfUsers, (snrMax - snrMin)+1);
j = 0;
for SNR = snrMin:snrMax
    j = j + 1;
    std_dev = sqrt(chip/((10^(SNR/10))*2));
    for inData=1:N
        channelOutput = zeros(1, chip);
        for i =1:numOfUsers
            infoData(i) = 2.*(round(rand(1))) - 1;
            channelOutput = channelOutput + gs{i}*infoData(i);
        end
        noise = std_dev * randn(1,chip);
        channelOutput = channelOutput + noise;
        for i = 1:numOfUsers
            hardDec = sign(sum(channelOutput .* gs{i}));
            if hardDec ~= infoData(i);
                err(i, j) = err(i, j) + 1;
            end
        end
    end
end
% Plotting commands follow.
ber = err./N;
snr = snrMin:snrMax;
semilogy(snr, ber(1,:), '--');
hold on;
semilogy(snr, ber(2,:), '-.');
semilogy(snr, ber(3,:), ':');
semilogy(snr, ber(4,:), '-');
legend('User 1', 'User 2', 'User 3', 'User 4');
```

Computer Problem 15.3

The results of Monte Carlo simulation are shown in Figure 15.3.
The Matlab script for the simulation program is given next.

```
% MATLAB script for Computer Problem 15.3
snrMin = 1;
snrMax = 14;
numOfUsers = 4;
N = 10000;
% Assign the gold sequences
gs{1} = [1 0 0 1 1 0 0 0 0 0 0 1 0 0 1 1 1 1 0 0 1 1 0 0 1 1 1 1 0 1 1];
gs{2} = [1 0 0 0 1 0 1 0 1 0 1 0 0 0 1 1 0 0 1 0 1 0 1 0 0 0 0 0 1 1 0];
gs{3} = [1 0 1 0 1 1 1 1 1 1 0 0 0 0 1 0 1 1 1 0 0 1 1 1 1 1 1 1 1 0 0];
gs{4} = [1 1 1 0 0 1 0 1 0 0 0 0 0 0 0 1 0 1 1 1 1 1 0 0 0 0 0 1 0 0 0];
for i=1:4
```

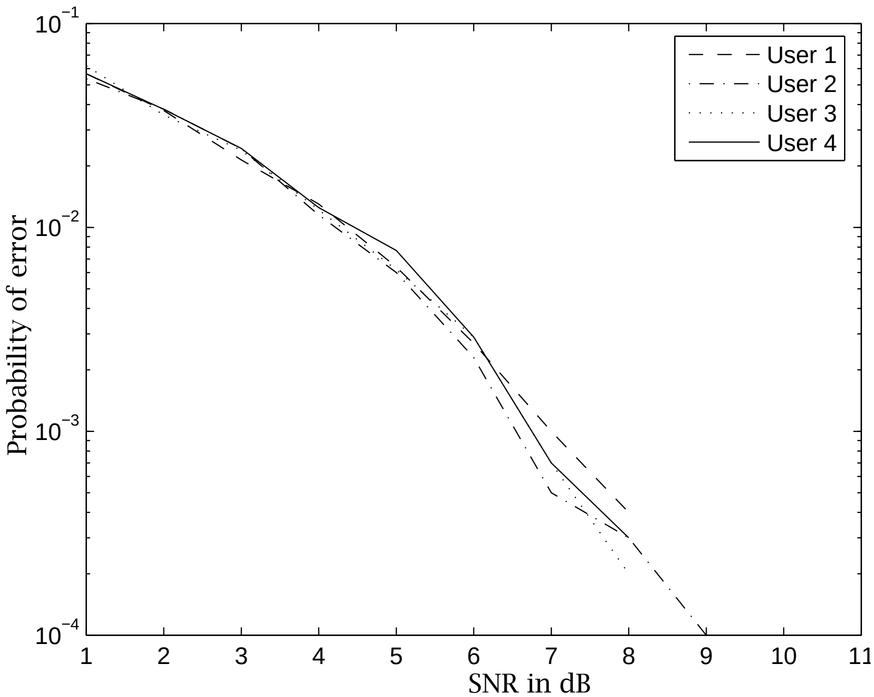


Figure 15.2: The results of Monte Carlo simulation of four time synchronous CDMA users

```
    gs{i} = 2*gs{i}−1;
end
chip = length(gs{1});
err = zeros(numOfUsers, (snrMax − snrMin)+1);
j = 0;
for SNR = snrMin:snrMax
    j = j + 1;
    std_dev = sqrt(chip/((10^(SNR/10))*2));
    previosInfoData = 2.*(round(rand(1, 4))) − 1;
    infoData= 2.*(round(rand(1, 4))) − 1;
    for inData=1:N
        channelOutput = zeros(1, chip+3);
        temp1 = zeros(1, chip+3);
        temp2 = zeros(1, chip+3);
        temp3 = zeros(1, chip+3);
        temp4 = zeros(1, chip+3);

        nextInfoData = 2.*(round(rand(1, 4))) − 1;

        temp1(1:chip) = gs{1}*infoData(1);
        temp1(chip+1:chip+3) = nextInfoData(1) * gs{1}(1:3);

        temp2(1) = previosInfoData(2) * gs{2}(chip);
        temp2(2:chip+1) = infoData(2) * gs{2};
        temp2(chip+2:chip+3) = nextInfoData(2) * gs{2}(1:2);
```

```
        temp3(1:2) = previosInfoData(3) * gs{3}(chip-1: chip);
        temp3(3:chip+2) = infoData(3) * gs{3};
        temp3(chip+3) = nextInfoData(3) * gs{3}(1);

        temp4(1:3) = previosInfoData(4) * gs{4}(chip-2: chip);
        temp4(4:chip+3) = infoData(4) * gs{4};

        noise = std_dev * randn(1,chip+3);
        channelOutput = temp1 + temp2 + temp3 + temp4+ noise;


        for user = 1:4
            hardDec = sign(sum(channelOutput(user:(chip+(user-1))).*gs{user}));
            if hardDec ~= infoData(user);
                err(user, j) = err(user, j) + 1;
            end
        end
        previousInfoData = infoData;
        infoData = nextInfoData;

    end

end
% Plotting commands follow.
ber = err./N;
snr = snrMin:snrMax;
semilogy(snr, ber(1,:), '--');
hold on;
semilogy(snr, ber(2,:), '-.');
semilogy(snr, ber(3,:), ':');
semilogy(snr, ber(4,:), '-');
legend('User 1', 'User 2', 'User 3', 'User 4');
```

Computer Problem 15.4

The period of the sequence is

$$L = 2^m - 1 = 4095$$

The periodic autocorrelation function of the equivalent bipolar sequence is presented in Figure 15.4
The Matlab script for this problem is given next.

```
% MATLAB script for Computer Problem 15.4
connections=zeros(1,12);
m = 12;
L = 2^m -1;
connections(1) = 1;
connections(7) = 1;
```

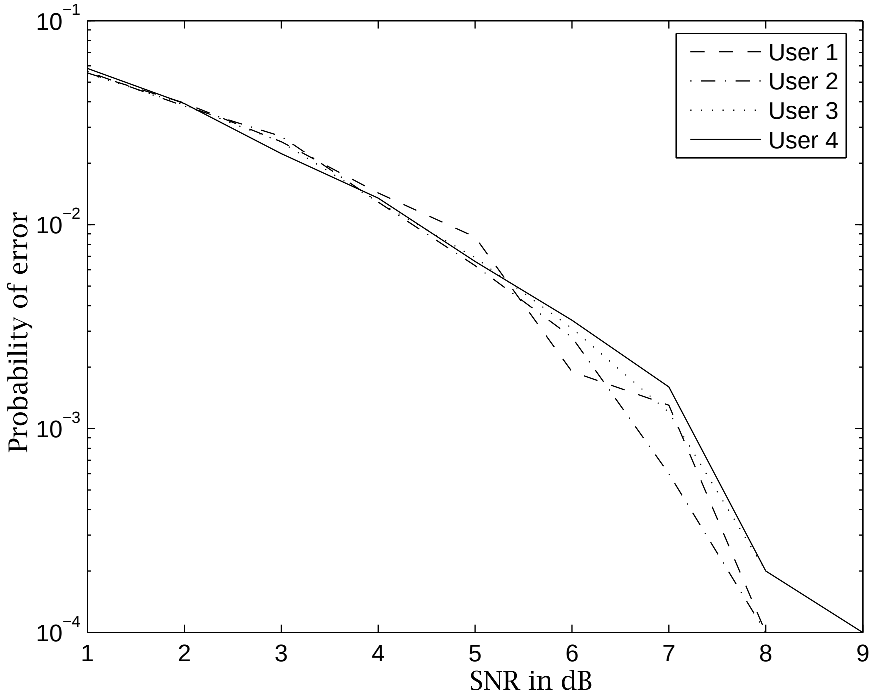


Figure 15.3: The results of Monte Carlo simulation of four time asynchronous CDMA users

```
connections(9) = 1;
connections(12) = 1;
sequence=ss_mlsrs(connections);
c = 2.* sequence − 1;
Rc = zeros(1, L);
for m=1:L
    for n=1:L

        Rc(m) = Rc(m)+ c(n)*c(n+m−1);
    end
end
% Plotting commands follow.

plot(Rc);
axis([−500 4500 −500 4500]);
```

```
function [seq]=ss_mlsrs(connections);
% [seq]=ss_mlsrs(connections)
%           SS_MLSRS generates the maximal length shift-register sequence when the
%           shift-register connections are given as input to the function. A “zero”
%           means not connected, whereas a “one” represents a connection.
m=length(connections);
L=2^m−1                        % length of the shift register sequence requested
registers=[1 zeros(1,m−1)];           % initial register contents
seq(1)=registers(1);                  % first element of the sequence
```

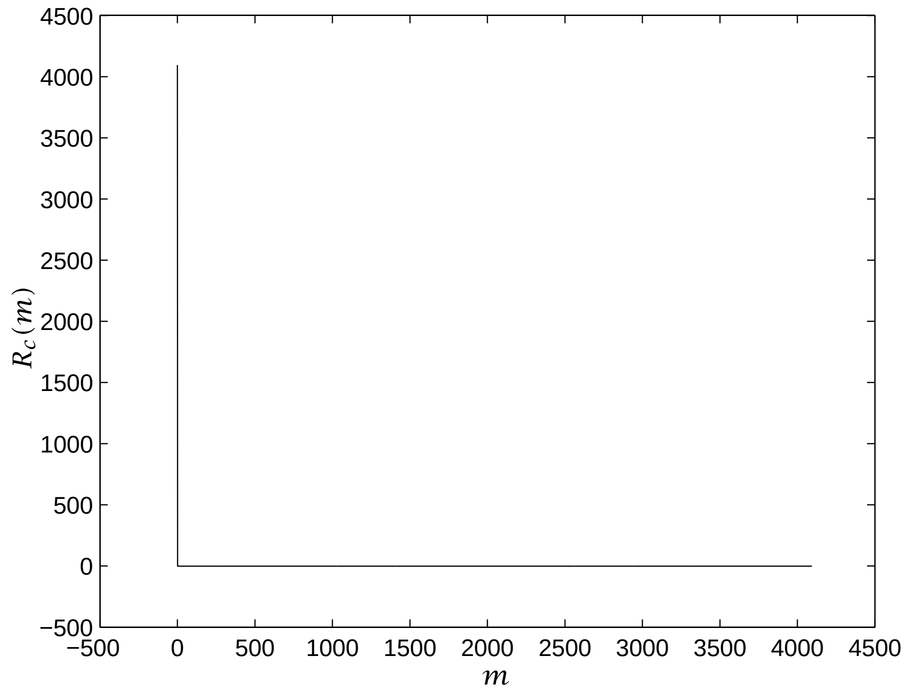


Figure 15.4: The autocorrelation of the bipolar sequence

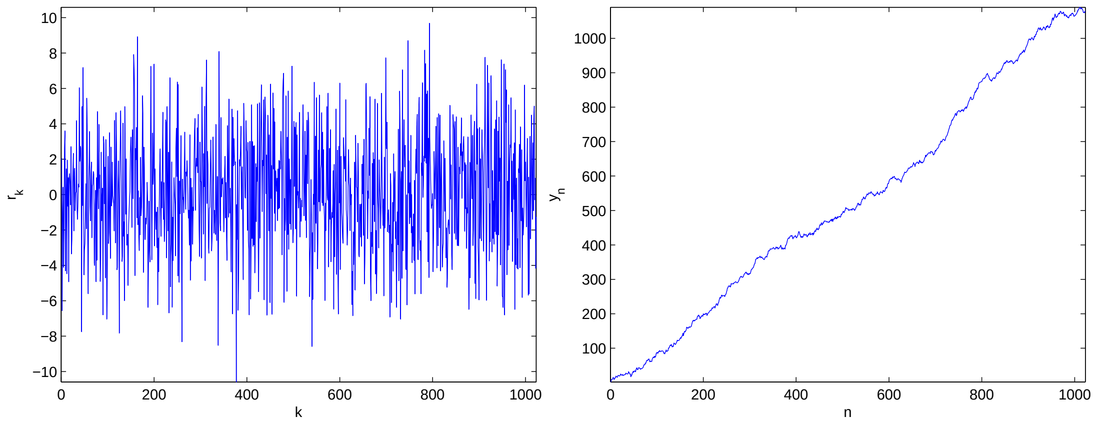
```
for i=2:2*L,
  new_reg_cont(m)=mod(sum(registers.*connections), 2);
  for j=m-1:-1:1,
    new_reg_cont(j)=registers(j+1);
  end;
  registers=new_reg_cont;          % current register contents
  seq(i)=registers(1);             % the next element of the sequence
end;
```

Computer Problem 15.5

The figure shown below illustrates the result of this crosscorrelation of $\{r_k\}$ with $\{c_k\}$. Although the signal component is not observable in the high-level noise, the signal is clearly detectable at the output of the correlator.

The MATLAB script for the problem is given below.

```
N = 1000;              % Number of samples
M = 50;                % Length of the autocorrelation function
p = [0.9 0.99];        % Pole positions
w = 1/sqrt(2)*(randn(1,N) + 1i*randn(1,N)); % AWGN sequence
% Preallocation for speed:
c = zeros(length(p),N);
```




```
Rx = zeros(length(p),M+1);
Sx = zeros(length(p),M+1);
for i = 1:length(p)
    for n = 3:N
        c(i,n) = 2*p(i)*c(n−1) − power(p(i),2)*c(n−2) + power((1−p(i)),2)*w(n);
    end
    % Calculation of autocorrelations and power spectra:
    Rx(i,:) = Rx_est(c(i,:),M);
    Sx(i,:)=fftshift(abs(fft(Rx(i,:))));
end
% Plot the results:
subplot(3,2,1)
plot(real(c(1,:)))
axis([0 N −max(abs(real(c(1,:)))) max(abs(real(c(1,:))))])
title('\it{p} = 0.9')
xlabel('\it{n}')
ylabel('\it{c_{nr}}')
subplot(3,2,2)
plot(real(c(2,:)))
axis([0 N −max(abs(real(c(2,:)))) max(abs(real(c(2,:))))])
title('\it{p} = 0.99')
xlabel('\it{n}')
ylabel('\it{c_{nr}}')
subplot(3,2,3)
plot(imag(c(1,:)))
axis([0 N −max(abs(imag(c(1,:)))) max(abs(imag(c(1,:))))])
title('\it{p} = 0.9')
xlabel('\it{n}')
ylabel('\it{c_{ni}}')
subplot(3,2,4)
plot(imag(c(2,:)))
axis([0 N −max(abs(imag(c(2,:)))) max(abs(imag(c(2,:))))])
title('\it{p} = 0.99')
xlabel('\it{n}')
ylabel('\it{c_{ni}}')
subplot(3,2,5)
plot(abs(c(1,:)))
axis([0 N 0 max(abs(c(1,:)))])
title('\it{p} = 0.9')
```

```
xlabel('\it{n}')
ylabel('\it{|c_n |}')
subplot(3,2,6)
plot(abs(c(2,:)))
axis([0 N 0 max(abs(c(2,:)))])
title('\it{p} = 0.99')
xlabel('\it{n}')
ylabel('\it{|c_n |}')

figure
subplot(2,2,1)
plot(abs(Rx(1,:)))
axis([0 M 0 max(abs(Rx(1,:)))])
title('\it{p} = 0.9')
xlabel('\it{n}'); ylabel('\it{|R_{c}(n)|}')
subplot(2,2,2)
plot(abs(Rx(2,:)))
title('\it{p} = 0.99')
xlabel('\it{n}'); ylabel('\it{|R_{c}(n)|}')
axis([0 M 0 max(abs(Rx(2,:)))])
subplot(2,2,3)
plot(Sx(1,:))
title('\it{p} = 0.9')
xlabel('\it{f}'); ylabel('\it{S_{c}(f)}')
axis([0 M 0 max(abs(Sx(1,:)))])
subplot(2,2,4)
plot(Sx(2,:))
title('\it{p} = 0.99')
xlabel('\it{f}'); ylabel('\it{S_{c}(f)}')
axis([0 M 0 max(abs(Sx(2,:)))])
```

Computer Problem 15.6

Figure 15.5 presents the frequency selection pattern for the first ten bit interval. The Matlab script for this problem is given next.

```
% MATLAB script for Computer Problem 15.6
W = 127;                          % frequency band of width
Df = 2;                           % frequency seperation for fsk
m=7;
L=2^m−1                                     % length of the shift register sequence requested
connections=zeros(1,7);
connections(1) = 1;
connections(7) = 1;
registers=[1 zeros(1,m−1)];              % initial register contents
seq(1)=registers(1);                     % first element of the sequence
frequency(1) = bin2dec(num2str(registers)); % first frequency
% select the first two frequencies
f0(1) = frequency(1) − Df/2;
f1(1) = frequency(1) + Df/2;
for i=2:2*L,
```

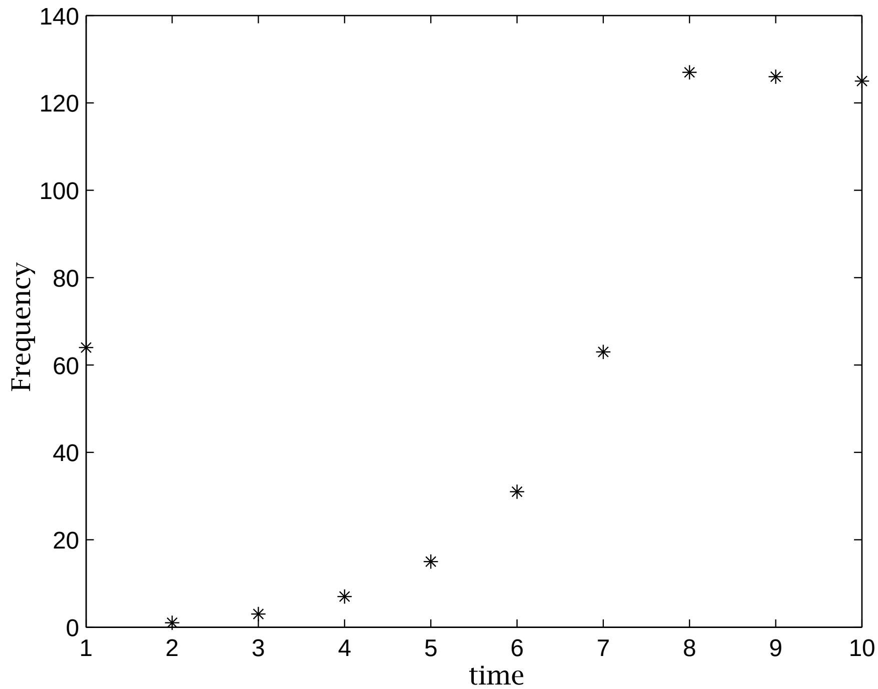


Figure 15.5: Frequency selection pattern

```
    new_reg_cont(m)=mod(sum(registers.*connections), 2);
    for j=m−1:−1:1,
      new_reg_cont(j)=registers(j+1);
    end;
    registers=new_reg_cont;                % current register contents
    seq(i)=registers(1);                   % the next element of the sequence
    frequency(i) = bin2dec(num2str(registers)); % select the frequency
    % select two frequencies
     f0(i) = frequency(i) − Df/2;
     f1(i) = frequency(i) + Df/2;
end;
plot(frequency(1:10), '*');
```

Computer Problem 15.7

Figure 15.6 illustrates the error rate that results from the Monte Carlo simulation. Also shown in the figure is the theoretical value of the probability of error.

The MATLAB scripts for the simulation program are given next.

```
echo on
rho_b1=0:5:35;                 % rho in dB for the simulated error rate
rho_b2=0:0.1:35;               % rho in dB for theoretical error rate computation
for i=1:length(rho_b1),
```

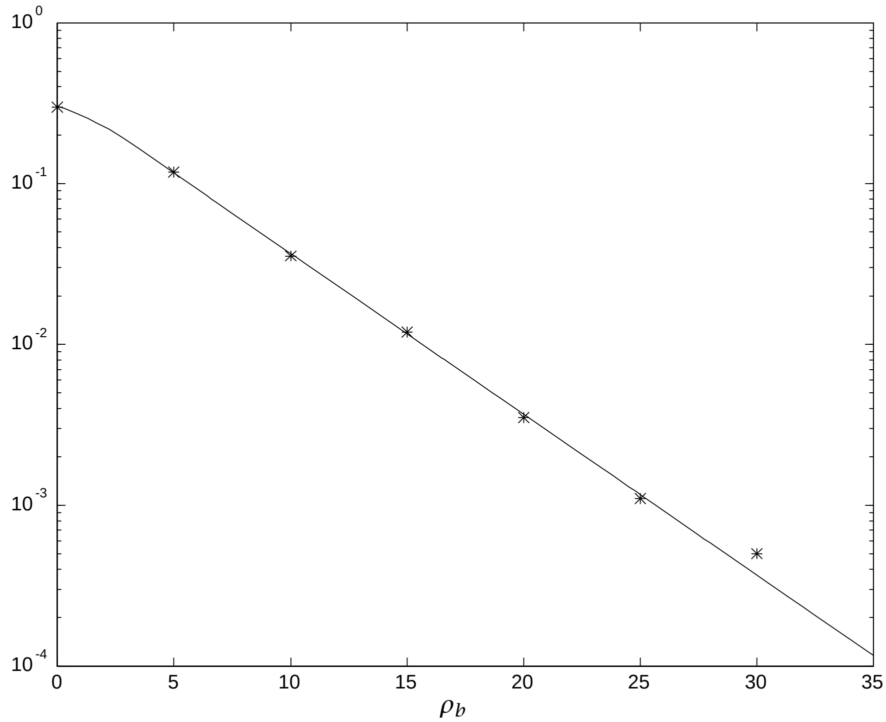


Figure 15.6: Error-rate performance of FH binary FSK system with partial-band interference—Monte Carlo simulation

```
    smld_err_prb(i)=ss_pe96(rho_b1(i)); % simulated error rate
    echo off ;
end;
echo on ;
for i=1:length(rho_b2),
    temp=10^(rho_b2(i)/10);
    if (temp>2)
        theo_err_rate(i)=1/(exp(1)*temp);        % theoretical error rate if rho>2
    else
        theo_err_rate(i)=(1/2)*exp(-temp/2);% theoretical error rate if rho<2
    end;
    echo off ;
end;
echo on ;
% Plotting commands follow.
```

```
function [p]=ss_Pe96(rho_in_dB)
% [p]=ss_Pe96(rho_in_dB)
%           SS_PE96  finds the measured error rate. The value of
%           signal per interference ratio in dB is given as an
%           input to the function.
rho=10^(rho_in_dB/10);
Eb=rho;                             % energy per bit
if (rho>2),
```

```
  alpha=2/rho;                    % optimal alpha if rho>2
else
  alpha=1;                        % optimal alpha if rho<2
end;
sgma=sqrt(1/(2*alpha));           % noise standard deviation
N=10000;                          % number of bits transmitted
% generation of the data sequence
for i=1:N,
  temp=rand;
  if (temp<0.5)
    data(i)=1;
  else
    data(i)=0;
  end;
end;
% Find the received signals.
for i=1:N,
  % the transmitted signal
  if (data(i)==0),
    r1c(i)=sqrt(Eb);
    r1s(i)=0;
    r2c(i)=0;
    r2s(i)=0;
  else
    r1c(i)=0;
    r1s(i)=0;
    r2c(i)=sqrt(Eb);
    r2s(i)=0;
  end;
  % The received signal is found by adding noise with probability alpha.
  if (rand<alpha),
    r1c(i)=r1c(i)+gngauss(sgma);
    r1s(i)=r1s(i)+gngauss(sgma);
    r2c(i)=r2c(i)+gngauss(sgma);
    r2s(i)=r2s(i)+gngauss(sgma);
  end;
end;
% Make the decisions and count the number of errors made.
num_of_err=0;
for i=1:N,
  r1=r1c(i)^2+r1s(i)^2;           % first decision variable
  r2=r2c(i)^2+r2s(i)^2;           % second decision variable
  % Decision is made next.
  if (r1>r2),
    decis=0;
  else
    decis=1;
  end;
  % Increment the counter if this is an error.
  if (decis~=data(i)),
    num_of_err=num_of_err+1;
  end;
end;
% measured bit error rate is then
```

```
p=num_of_err/N;
```