Nonlinear Control

Lecture1 : Introduction & Phase Plane Analysis

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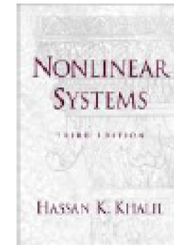
Reference Books

- **Text Book:** Nonlinear Systems, H. K. Khalil, 3^{rd} edition, Prentice-Hall, 2002
- **Other reference Books:**
- Applied Nonlinear Control, J. J. E. Slotine, and W. Li, Prentice-Hall, 1991
- Nonlinear System Analysis, M. Vidyasagar, 2^{nd} edition, Prentice-Hall, 1993
- Nonlinear Control Systems, A. Isidori, 3^{rd} edition Springer-Verlag, 1995




Introduction

Motivations

- A system is called linear if its behavior set satisfies linear superposition laws: .i.e. $\forall\ z_1,\ z_2 \in \mathcal{B}$ and constant $c \in \mathcal{R} \rightsquigarrow z_1 + z_2 \in \mathcal{B},\ cz_1 \in \mathcal{B}$
- A nonlinear system is simply a system which is not linear.
- Powerful tools founded based on superposition principle make analyzing the linear systems simple.
- All practical systems posses nonlinear dynamics.
- Sometimes it is possible to describe the operation of physical systems by linear model around its operating points
- Linearized system can provide us an approximate behavior of the nonlinear system
- But in analyzing the overall system behavior, often linearized model *inadequate* or *inaccurate*.

- Linearization is an approximation in the neighborhood of an operating system $\rightsquigarrow$ it can only predict local behavior of nonlinear system. (No info regarding *nonlocal* or *global* behavior of system)
- Due to richer dynamics of nonlinear systems comparing to the linear ones, there are some *essentially nonlinear phenomena* that can take place only in presence of nonlinearity
- Essentially nonlinear phenomena
  - Finite escape time: The state of linear system goes to infinity as $t \to \infty$; nonlinear system's state can go to infinity in *finite time*.
  - Multiple isolated equilibria: linear system can have only one isolated equilibrium point which attracts the states irrespective on the initial state; nonlinear system can have more than one isolated equilibrium point, the state may converge to each depending on the initial states.
  - Limit cycle: There is no robust oscillation in linear systems. To oscillate there should be a pair of eigenvalues on the imaginary axis which due to presence of perturbations it is almost impossible in practice; For nonlinear systems, there are some oscillations named limit cycle with fixed amplitude and frequency.

Essentially nonlinear phenomena

- Subharmonic,harmonic or almost periodic oscillations: A stable linear system under a periodic input $\rightsquigarrow$ output with the same frequency; A nonlinear system under a periodic input $\rightsquigarrow$ can oscillate with submultiple or multiple frequency of input or almost-periodic oscillation.
- Chaos: A nonlinear system may have a different steady-state behavior which is not equilibrium point, periodic oscillation or almost-periodic oscillation. This chaotic motions exhibit random, despite of deterministic nature of the system.
- Multiple modes of behavior: A nonlinear system may exhibit multiple modes of behavior based on type of excitation:
  - an unforced system may have one limit cycle.
  - Periodic excitation may exhibit harmonic, subharmonic,or chaotic behavior based on amplitude and frequency of input.
  - if amplitude or frequency is smoothly changed, it may exhibit discontinuous jump of the modes as well.

- Linear systems: can be described by a set of ordinary differential equations and usually the *closed-form expressions* for their solutions are derivable. Nonlinear systems: In general this is not possible $\rightsquigarrow$ It is desired to make a prediction of system behavior even in absence of closed-form solution. This type of analysis is called qualitative analysis.
- Despite of linear systems, no tool or methodology in nonlinear system analysis is *universally* applicable $\rightsquigarrow$ their analysis requires a wide verity of tools and higher level of mathematic knowledge
- $\therefore$ stability analysis and stabilizablity of such systems and getting familiar with associated control techniques is the basic requirement of graduate studies in control engineering.
- The aim of this course are
  - developing a basic understanding of nonlinear control system theory and its applications.
  - introducing tools such as Lyapunov's method analyze the system stability
  - Presenting techniques such as feedback linearization to control nonlinear systems.

- At this course we consider dynamical systems modeled by a finite number of coupled first-order ordinary differential equations:

$$\dot{x} = f(t, x, u) \tag{1}$$

where $x = [x_1, \ldots, x_n]^T$: state vector, $u = [u_1, \ldots, u_p]^T$: input vector, and $f(.) = [f_1(.), \ldots, f_n(.)]^T$: a vector of nonlinear functions.
- Euq. (1) is called *state equation*.
- Another equation named *output equation*:

$$y = h(t, x, u) \tag{2}$$

where $y = [y_1, \ldots, y_q]^T$: output vector.
- Equ (2) is employed for particular interest in analysis such as
  - variables which can be measured physically
  - variables which are required to behave in a desirable manner
- Equs (1) and (2) together are called state-space model.

- Most of our analysis are dealing with unforced state equations where u does not present explicitly in Equ (1):

$$\dot{x} = f(t, x)$$

  - In unforced state equations, input to the system is **NOT** necessarily zero.
  - Input can be a function of time: $u = \gamma(t)$, a feedback function of state: $u = \gamma(x)$, or both $u = \gamma(t, x)$ where is substituted in Equ (1).
- **Autonomous** or **Time-invariant Systems**:

$$\dot{x} = f(x) \tag{3}$$

  - function of f does not explicitly depend on t.
  - Autonomous systems are invariant to shift in time origin, i.e. changing t to $\tau = t - a$ does not change f.
  - The system which is not autonomous is called **nonautonomous** or **time-varying**.

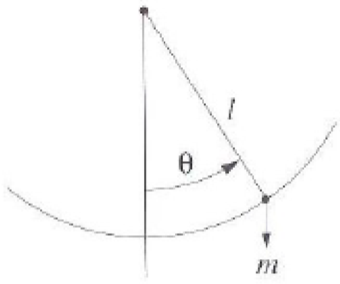
- **Equilibrium Point $x = x^*$**
  - x^* in state space is equilibrium point if whenever the state starts at x^*, it will remain at x^* for all future time.
  - for autonomous systems (3), the equilibrium points are the real roots of equation: $f(x) = 0$.
  - Equilibrium point can be
    - Isolated: There are no other equilibrium points in its vicinity.
    - a continuum of equilibrium points

Pendulum

- Employing Newton's second law of motion, equation of pendulum motion is:

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

l: length of pendulum rod;
m: mass of pendulum bob;
k: coefficient of friction;
θ: angle subtended by rod and vertical a›



Pendulum.

- To obtain state space model, let $x_1 = \theta$, $x_2 = \dot{\theta}$:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l} sin x_1 - \frac{k}{m} x_2 \end{aligned}$$

Pendulum

- To find equilibrium point: $\dot{x}_1 = \dot{x}_2 = 0$

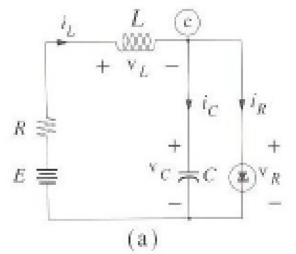
$$\begin{aligned} 0 &= x_2 \\ 0 &= -\frac{g}{l} sin x_1 - \frac{k}{m} x_2 \end{aligned}$$

- The Equilibrium points are at $(n\pi, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$
  - Pendulum has two equilibrium points: $(0, 0)$ and $(\pi, 0)$,
  - Other equilibrium points are repetitions of these two which correspond to number of pendulum full swings before it rests
- Physically we can see that the pendulum rests at $(0, 0)$, but hardly maintain rest at $(\pi, 0)$

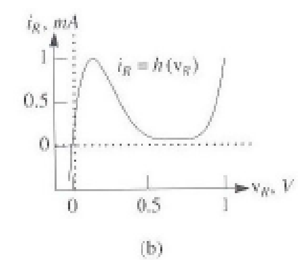
Tunnel Diode Circuit

- The tunnel diode is characterized by $i_R = h(v_R)$
- The energy-storing elements are C and L which assumed are linear and time-invariant $i_C = C\frac{dv_C}{dt}$, $v_L = L\frac{di_L}{dt}$.
- Employing Kirchhoff's current law: $i_C + i_R - i_L = 0$
- Employing Kirchhoff's voltage law: $v_C - E + Ri_L + v_L = 0$
- for state space model, let $x_1 = v_C$, $x_2 = i_L$ $u = E$ as a constant input:

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C}[-h(x_1) + x_2] \\ \dot{x}_2 &= \frac{1}{L}[-x_1 - Rx_2 + u] \end{aligned}$$



(a) Tunnel diode circuit



(b) tunnel diode $v_R - i_R$ characteristic.

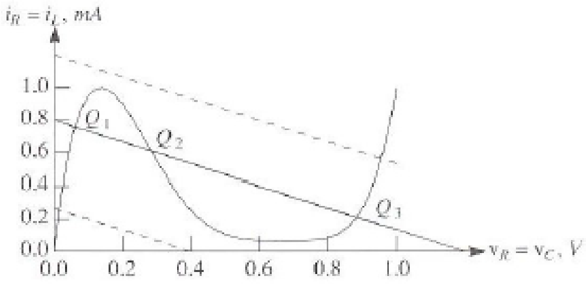
Tunnel Diode Circuit

- To find equilibrium point: $\dot{x}_1 = \dot{x}_2 = 0$

$$0 = \frac{1}{C}[-h(x_1) + x_2]$$
$$0 = \frac{1}{L}[-x_1 - Rx_2 + u]$$

- Equilibrium points depends on E and R

$$x_2 = h(x_1) = \frac{E}{R} - \frac{1}{R}x_1$$



Equilibrium points of the tunnel diode circuit.

  - For certain E and R, it may have 3 points (Q_1, Q_2, Q_3).
  - if $E\uparrow$, and same R $\rightsquigarrow$ only Q_3 exists.
  - if $E\downarrow$, and same R $\rightsquigarrow$ only Q_1 exists.

Phase Plane

- **Phase Plane Analysis**: is a graphical method for studying second-order systems by
  - providing motion trajectories corresponding to various initial conditions.
  - then examine the qualitative features of the trajectories.
  - finally obtaining information regarding the stability and other motion patterns of the system.
- It was introduced by mathematicians such as Henri Poincare in 19th century.

Motivations

- Importance of Knowing **Phase Plane Analysis:**
  - Since it is on second-order, the solution trajectories can be represented by carves in plane $\rightsquigarrow$ provides easy visualization of the system qualitative behavior.
  - Without solving the nonlinear equations analytically, one can study the behavior of the nonlinear system from various initial conditions.
  - It is not restricted to small or smooth nonlinearities and applies equally well to strong and hard nonlinearities.
  - There are lots of practical systems which can be approximated by second-order systems, and apply phase plane analysis.
- Disadvantage of **Phase Plane Method:** It is restricted to at most second-order and graphical study of higher-order is computationally and geometrically complex.

Concept of Phase Plane

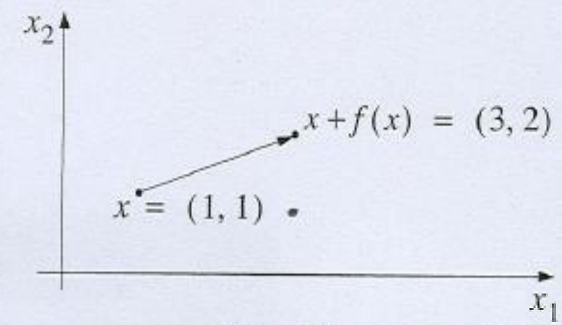
- Phase plane method is applied to autonomous 2*nd* order system described as follows:

$$\dot{x}_1 = f_1(x_1, x_2) \quad (1)$$
$$\dot{x}_2 = f_2(x_1, x_2) \quad (2)$$

- $f_1,\ f_2 : \mathcal{R}^2 \rightarrow \mathcal{R}$.
- System response $(x(t) = (x_1(t),\ x_2(t)))$ to initial condition $x_0 = (x_{10},\ x_{20})$ is a mapping from $\mathcal{R}$ to $\mathcal{R}^2$.
- The $x_1 - x_2$ plane is called **State plane** or **Phase plane**
- The locus in the $x_1 - x_2$ plane of the solution $x(t)$ for all $t \geq 0$ is a curve named **trajectory** or **orbit** that passes through the point x_0
- The family of phase plane trajectories corresponding to various initial conditions is called **Phase protrait** of the system.

How to Construct Phase Plane Trajectories?

- Despite of exiting several routines to generate the phase portraits by computer, it is useful to learn roughly sketch the portraits or quickly verify the computer outputs.
- Some methods named: Isocline, Vector field diagram, delta method, Pell's method, etc
- Vector Field Diagram:
  - Revisiting (1) and (2):
    $\dot{x} = f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}, \dot{x} = (\dot{x}_1, \ \dot{x}_2)$
  - To each vector (x_1, x_2), a corresponding vector $(f_1(x_1, x_2), \ f_2(x_1, x_2))$ known as a **vector field** is associated.
  - Example: If $f(x) = (2x_1^2, x_2)$, for $x = (1, 1)$, next point is $(1, 1) + (2, 1) = (3, 2)$

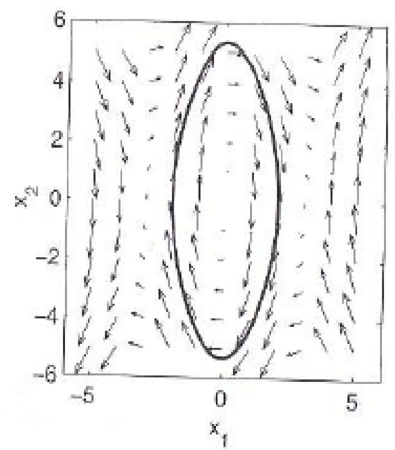


Vector field representation.

Vector Field Diagram

- By repeating this for sufficient point in the state space, a **vector field diagram** is obtained.
- Noting that $\frac{dx_2}{dx_1} = \frac{f_2}{f_1} \rightsquigarrow$ vector field at a point is tangent to trajectory through that point.
  - $\therefore$ starting from x_0 and by using the vector field with sufficient points, the trajectory can be constructed.
- Example: Pendulum without friction

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10 \sin x_1 \end{aligned}$$



Vector field diagram of the pendulum equation without friction.

Isocline Method

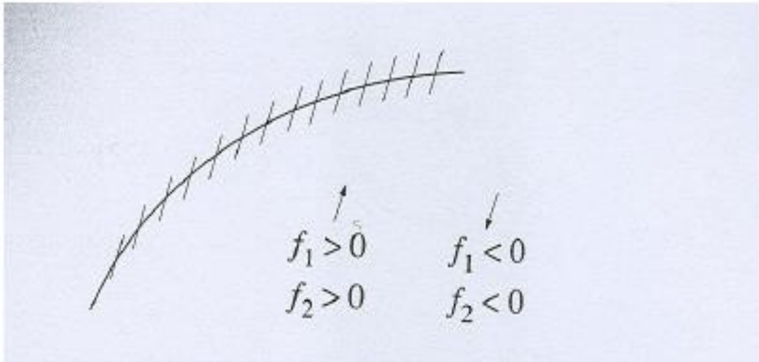
- The term isocline derives from the Greek words for "*same slope.*"
- Consider again Eqs (1) and (2), the slope of the trajectory at point x:

$$S(x) = \frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

- An isocline with slope α is defined as $S(x) = \alpha$
- $\therefore$ all the points on the curve $f_2(x_1, x_2) = \alpha f_1(x_1, x_2)$ have the same tangent slope α.
- Note that the "time" is eliminated here $\Rrightarrow$ The responses $x_1(t)$ and $x_2(t)$ cannot be obtained directly.
- Only qualitative behavior can be concluded, such as stable or oscillatory response.

Isocline Method

- The algorithm of constructing the phase portrait by isocline method:
  1. Plot the curve $S(x) = \alpha$ in state-space (phase plane)
  2. Draw small line with slope α. Note that the direction of the line depends on the sign of f_1 and f_2 at that point.

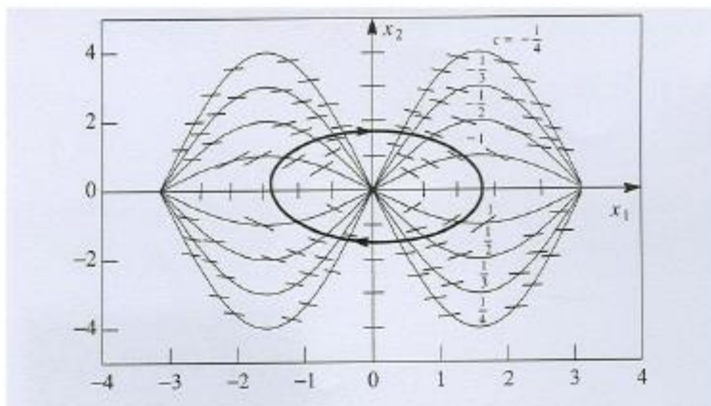


Isocline with positive slope.

  3. Repeat the process for sufficient number of α s.t. the phase plane is full of isoclines.

Example: Pendulum without Friction

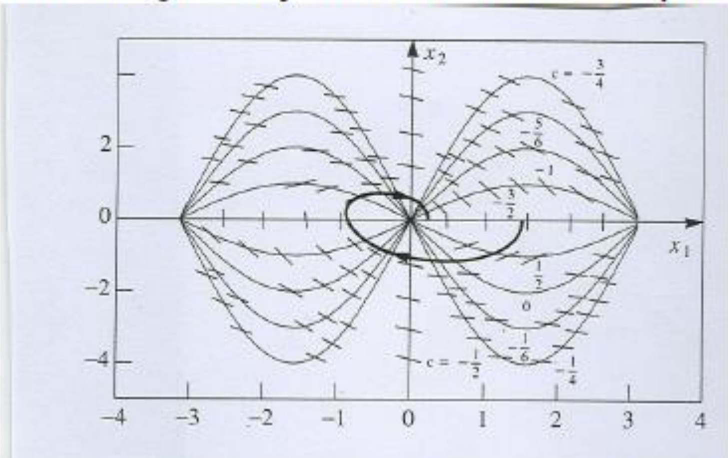
- Consider the dynamics $\dot{x}_1 = x_2, \quad \dot{x}_2 = -sinx_1 \;\therefore S(x) = \frac{-sinx_1}{x_2} = c$
- Isoclines: $\quad x_2 = \frac{-1}{c} sinx_1$
- Trajectories for different init. conditions can be obtained by using the given algorithm
- The response for $x_0 = (\frac{\pi}{2},\ 0)$ is depicted in Fig.
- The closed curve trajectory confirms marginal stability of the system.



Graphical construction of the phase portrait of the pendulum equation (without friction) by the isocline method.

Example: Pendulum with Friction

- Dynamics of pendulum with friction:
  $\dot{x}_1 = x_2, \quad \dot{x}_2 = -0.5x_2 - sinx_1 \;\therefore S(x) = \frac{-0.5 - sinx_1}{x_2} = c$
- Isoclines: $\quad x_2 = \frac{-1}{0.5+c} sinx_1$
- Similar Isoclines but with different slopes
- Trajectory is drawn for $x_0 = (\frac{\pi}{2},\ 0)$
- The trajectory shrinks like an spiral converging to the origin



Graphical construction of the phase portrait of the pend equation (with friction) by the isocline method.

Qualitative Behavior of Linear Systems

Qualitative Behavior of Linear Systems

- First we analyze the phase plane of linear systems since the behavior of nonlinear systems around equilibrium points is similar of linear ones
- For LTI system:
  $\dot{x} = Ax, \quad A \in \mathcal{R}^{2\times 2}, \; x_0$: initial state $\rightsquigarrow x(t) = Me^{J_r t}M^{-1}x_0$
  J_r : Jordan block of A, M : Matrix of eigenvectors $M^{-1}AM = J_r$
- Depending on the eigenvalues of A, J_r has one of the following forms:

$$\lambda_i: \text{ real \& distinct} \rightsquigarrow J_r = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\lambda_i: \text{ real \& multiple} \rightsquigarrow J_r = \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix}, \; k = 0, 1,$$

$$\lambda_i: \text{ complex} \rightsquigarrow J_r = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

- The system behavior is different at each case

Case 1: $\lambda_1 \neq \lambda_2 \neq 0$

- In this case $M = [v_1 \;\; v_2]$ where v_1 and v_2 are real eigenvectors associated with λ_1 and λ_2
- To transform the system into two decoupled first-order diff equations, let $z = M^{-1}x$:

$$\begin{aligned} \dot{z}_1 &= \lambda_1 z_1 \\ \dot{z}_2 &= \lambda_2 z_2 \end{aligned}$$

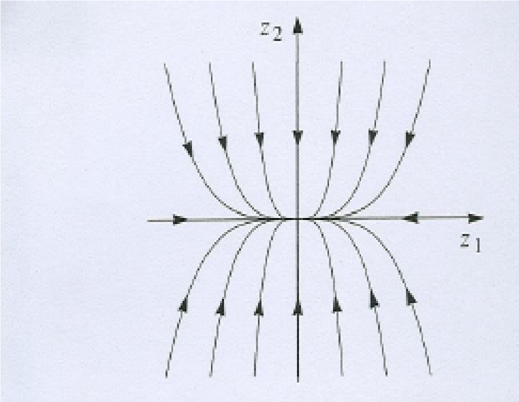
- The solution for initial states $(z_{01},\; z_{02})$:

$$\begin{aligned} z_1(t) &= z_{10}e^{\lambda_1 t}, \quad z_2(t) = z_{20}e^{\lambda_2 t} \\ \text{eliminating } t \rightsquigarrow z_2 &= C z_1^{\lambda_2/\lambda_1}, \quad C = z_{20}/(z_{10})^{\lambda_2/\lambda_1} \end{aligned} \tag{3}$$

- Phase portrait is obtained by changing $C \in \mathcal{R}$ and plotting (3).
- The phase portrait depends on the sign of λ_1 and λ_2.

Case 1.1: $\lambda_2 < \lambda_1 < 0$

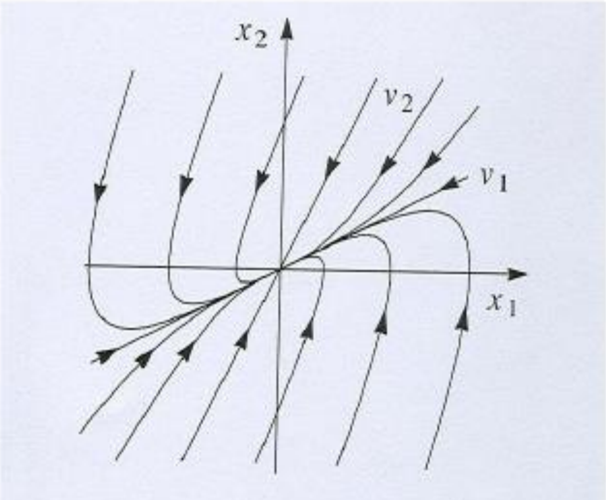
- $t \to \infty \Rightarrow$ the terms $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ tend to zero
  - Trajectories from entire state-space tend to origin $\rightsquigarrow$ the equilibrium point $x = 0$ is **stable node**.
- $e^{\lambda_2 t} \to 0$ faster $\rightsquigarrow$ λ_2 is fast eignevalue and v_2 is fast eigenvector.
- Slope of the curves: $\frac{dz_2}{dz_1} = C\frac{\lambda_2}{\lambda_1} z_1^{(\lambda_2/\lambda_1 - 1)}$
- $\lambda_2 < \lambda_1 < 0 \rightsquigarrow \lambda_2/\lambda_1 > 1$, so slope is
  - **zero** as $z_1 \longrightarrow 0$
  - **infinity** as $z_1 \longrightarrow \infty$.
- $\therefore$ The trajectories are
  - tangent to z_1 axis, as they approach to origin
  - parallel to z_2 axis, as they are far from origin.



Phase portrait of a stable node in modal coo

Case 1.1: $\lambda_2 < \lambda_1 < 0$

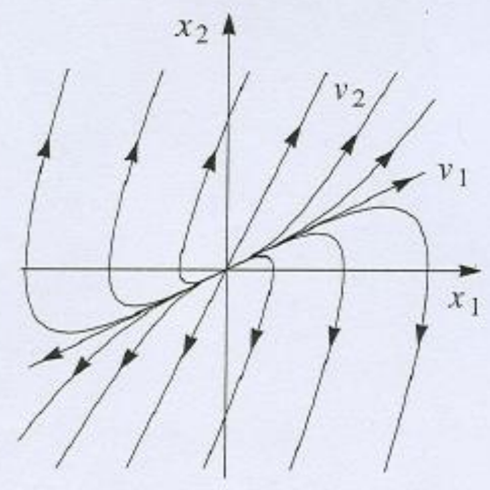
- Since z_2 approaches to zero faster than z_1, trajectories are sliding along z_1 axis
- In X plane also trajectories are:
  - tangent to the slow eigenvector v_1 for near origin
  - parallel to the fast eigenvector v_2 for far from origin



Phase portrait of a stable node.

Case 1.2: $\lambda_2 > \lambda_1 > 0$

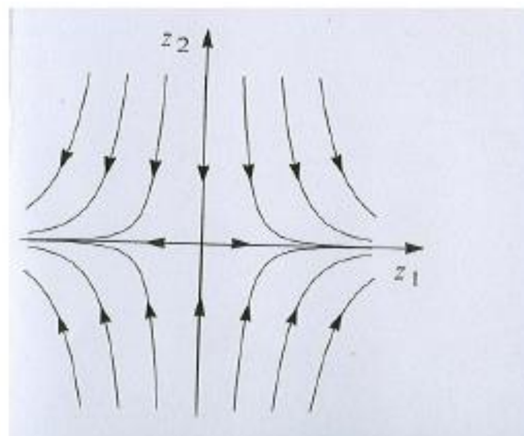
- $t \to \infty \Rightarrow$ the terms $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ grow exponentially, so
  - The shape of the trajectories are the same, with opposite directions
  - The equilibrium point is socalled unstable node



Phase portrait of an unstable node.

Case 1.3: $\lambda_2 < 0 < \lambda_1$

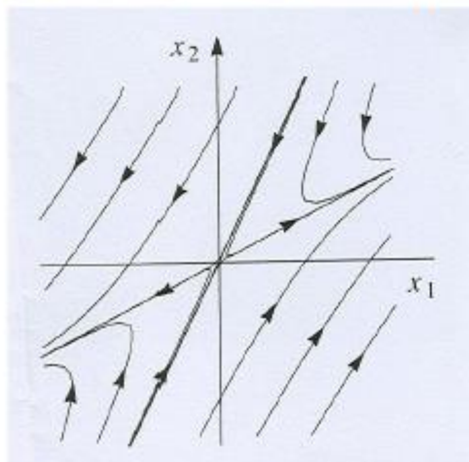
- $t \to \infty \;\Rightarrow\; e^{\lambda_2 t} \longrightarrow 0$, but $e^{\lambda_1 t} \longrightarrow \infty$,so
  - λ_2 : stable eigenvalue, v_2: stable eigenvector
  - λ_1 : unstable eigenvalue, v_1: unstable eigenvector
- Trajectories are negative exponentials since $\frac{\lambda_2}{\lambda_1}$ is negative.
- Trajectories are
  - decreasing in z_2 direction, but increasing in z_1 direction
  - tangent to z_1 as $|z_1| \to \infty$ and tangent to z_2 as $|z_1| \to 0$



Phase portrait of a saddle point in modal coordina

Case 1.3: $\lambda_2 < 0 < \lambda_1$

- The exceptions of this hyperbolic shape:
  - two trajectories along z_2-axis $\rightarrow 0$ as $t \rightarrow 0$, called stable trajectories
  - two trajectories along z_1-axis $\rightarrow \infty$ as $t \rightarrow 0$, called unstable trajectories
- This equilibrium point is called saddle point
- Similarly in X plane, stable trajectories are along v_2, but unstable trajectories are along the v_1
- For $\lambda_1 < 0 < \lambda_2$ the direction of the trajectories are changed.



Phase portrait of a saddle point.

Case 2: Complex Eigenvalues, $\lambda_{1,2} = \alpha \pm j\beta$

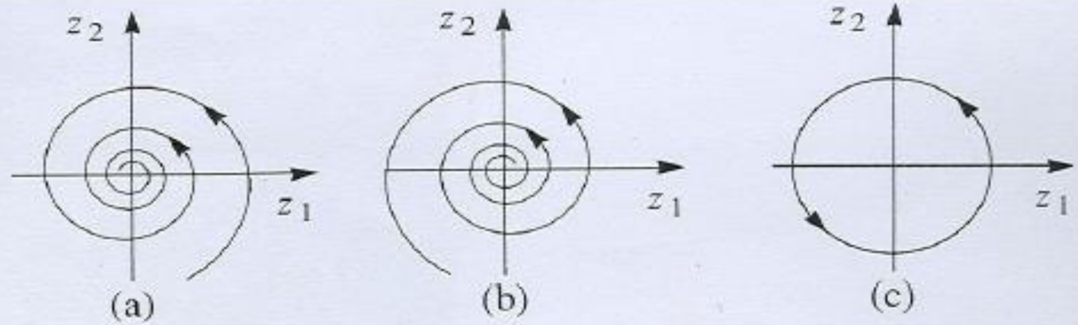
$$\dot{z}_1 = \alpha z_1 - \beta z_2$$
$$\dot{z}_2 = \beta z_1 + \alpha z_2$$

- The solution is oscillatory $\Longrightarrow$ polar coordinates $(r = \sqrt{z_1^2 + z_2^2}, \quad \theta = \tan^{-1}(\frac{z_2}{z_1}))$

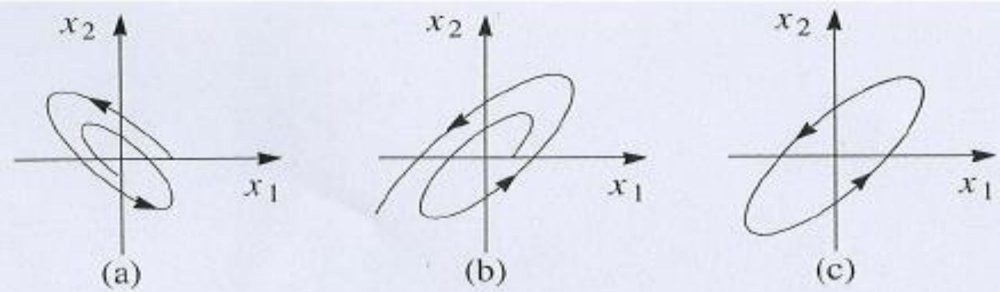
$$\dot{r} = \alpha r \rightsquigarrow r(t) = r_0 e^{\alpha t}$$
$$\dot{\theta} = \beta \rightsquigarrow \theta(t) = \theta_0 + \beta t$$

- This results in Z plane is a logarithmic spiral where α determines the form of the trajectories:
  - $\alpha < 0$: as $t \rightarrow \infty \rightsquigarrow r \rightarrow 0$ and angle θ is rotating. The spiral converges to origin $\Longrightarrow$ Stable Focus.
  - $\alpha > 0$: as $t \rightarrow \infty \rightsquigarrow r \rightarrow \infty$ and angle θ is rotating. The spiral diverges away from origin $\Longrightarrow$ Unstable Focus.
  - $\alpha = 0$: Trajectories are circles with radius $r_0 \Longrightarrow$ Center

Case 2: Complex Eigenvalues, $\lambda_{1,2} = \alpha \pm j\beta$



Typical trajectories in the case of complex eigenvalues. (a) $\alpha < 0$; (b) $\alpha > 0$; (c) $\alpha = 0$.



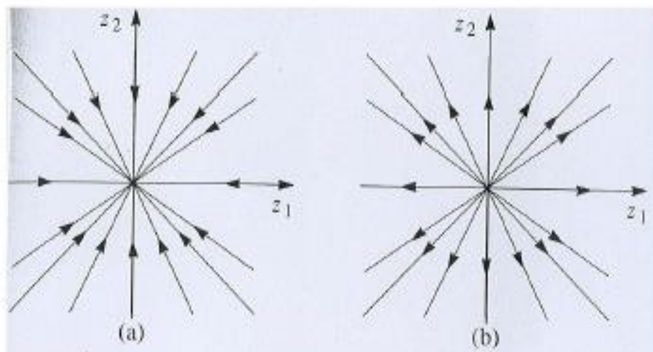
Phase portraits for (a) a stable focus; (b) an unstable focus; (c) a center.

Case 3: Nonzero Multiple Eigenvalues $\lambda_1 = \lambda_2 = \lambda \neq 0$

- Let $z = M^{-1}x$:

$$\begin{aligned} \dot{z}_1 &= \lambda z_1 + k z_2, \quad \dot{z}_2 = \lambda z_2 \\ \text{the solution is :} z_1(t) &= e^{\lambda t}(z_{10} + k z_{20} t), \quad z_2(t) = z_{20} e^{\lambda t} \rightsquigarrow \\ z_1 &= z_2 \left[\frac{z_{10}}{z_{20}} + \frac{k}{\lambda} ln \left(\frac{z_2}{z_{20}} \right) \right] \end{aligned}$$

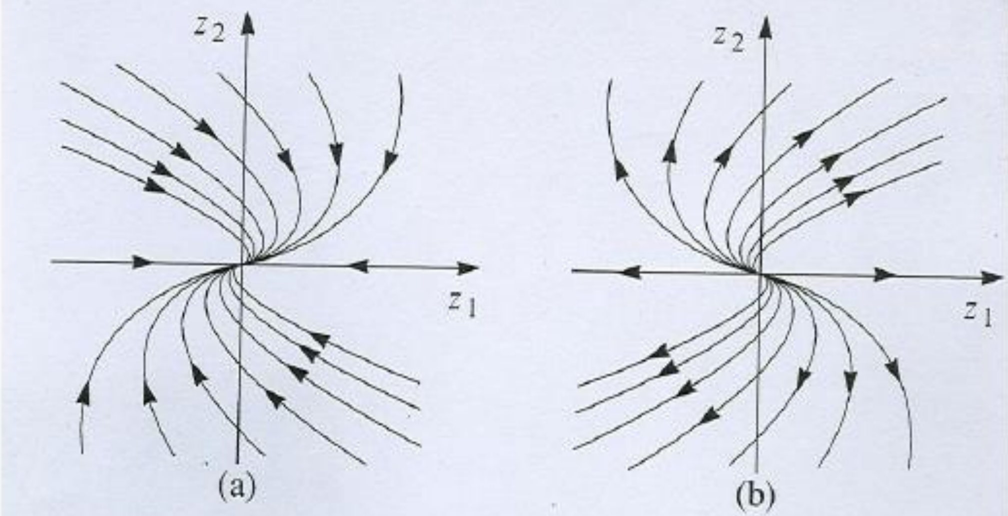
- Phase portrait are depicted for $k = 0$ and $k = 1$.
- When the eignevectors are different $\rightsquigarrow k = 0$:
  - similar to Case 1, for $\lambda < 0$ is stable, $\lambda > 0$ is unstable.
  - Decaying rate is the same for both modes ($\lambda_1 = \lambda_2$) $\rightsquigarrow$ trajectories are lines



Phase portraits for the case of nonzero multiple eigenvalues when $k = 0$: (a) $\lambda < 0$; (b) $\lambda > 0$.

Case 3: Nonzero Multiple Eigenvalues $\lambda_1 = \lambda_2 = \lambda \neq 0$

- There is no fast-slow asymptote.
- $k = 1$ is more complex, but it is still similar to Case 1:



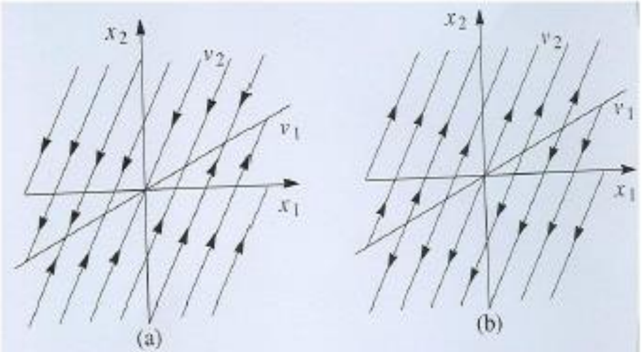
Phase portraits for the case of nonzero multiple eigenvalues when $k = 1$: (a) $\lambda < 0$; (b) $\lambda > 0$.

Case 4.1: One eigenvalue is zero $\lambda_1 = 0, \quad \lambda_2 \neq 0$

- A is singular in this case
- Every vector in null space of A is an equilibrium point
- There is a line (subspace) of equilibrium points
- $M = [v_1 \ v_2]$, $v_1, \ v_2$: corresponding eigenvectors, $v_1 \in \mathcal{N}(A)$.

$$\dot{z}_1 = 0, \ \dot{z}_2 = \lambda_2 z_2$$

$$\text{solution: } z_1(t) = z_{10}, \ z_2(t) = z_{20} e^{\lambda_2 t}$$



Phase portraits for (a) $\lambda_1 = 0$, $\lambda_2 < 0$; (b) $\lambda_1 = 0$, $\lambda_2 > 0$.

- Phase portrait depends on sign of λ_2:
  - $\lambda_2 < 0$: Trajectories converge to equilibrium line
  - $\lambda_2 > 0$: Trajectories diverge from equilibrium line

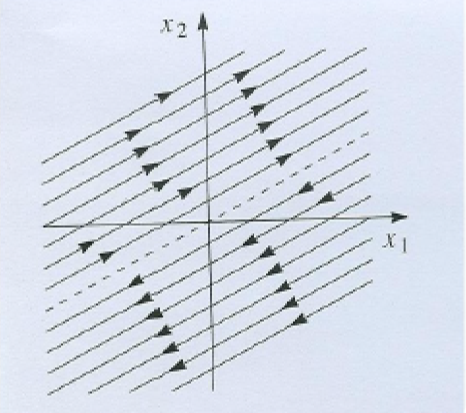
Case 4.2: Both eigenvalues zero $\lambda_1 = \lambda_2 = 0$

- Let $z = M^{-1}x$

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = 0$$

$$\text{solution: } z_1(t) = z_{10} + z_{20}t, \quad z_2(t) = z_{20}$$

- z_1 linearly increases/decreases base on the sign of z_{20}
- z_2 axis is equilibrium subspace in Z-plane
- Dotted line is equilibrium subspace
- The difference between Case 4.1 and 4.2: all trajectories start off the equilibrium set move parallel to it.



Phase portraits when $\lambda_1 = \lambda_2 = 0$.

As Summary:

- Six types of equilibrium points can be identified:
  - stable/unstable node
  - saddle point
  - stable/ unstable focus
  - center
- Type of equilibrium point depends on sign of the eigenvalues
  - If real part of eignevalues are Positive $\rightsquigarrow$ unstability
  - If real part of eignevalues are Negative $\rightsquigarrow$ stability
- All properties for linear systems hold globally
- Properties for nonlinear systems only hold locally

Local Behavior of Nonlinear Systems

Local Behavior of Nonlinear Systems

- Qualitative behavior of nonlinear systems is obtained locally by linearization around the equilibrium points
- Type of the perturbations and reaction of the system to them determines the degree of validity of this analysis
- A simple example: Consider the linear perturbation case $A \longrightarrow A + \Delta A$, where $\Delta A \in \mathcal{R}^{2\times 2}$: small perturbation
- Eigenvalues of a matrix continuously depend on its parameters
  - Positive (Negative) eigenvalues of A remain positive (negative) under small perturbations.
  - For eigenvalues on the $j\omega$ axis no matter how small perturbation is, it changes the sign of eigenvalue.
- Therefore
  - node or saddle point or focus equilibrium point remains the same under small perturbations
  - This analysis is not valid for a center equilibrium point

- Multiple Equilibria
  - Linear systems can have
    - an isolated equilibrium point or
    - a continuum of equilibrium points (When $detA = 0$)
  - Unlike linear systems, nonlinear systems can have multiple isolated equilibria.
- Qualitative behavior of second-order nonlinear system can be investigated by
  - generating phase portrait of system globally by computer programs
  - linearize the system around equilibria and study the system behavior near them without drawing the phase portrait
    - Let (x_{10}, x_{20}) are equilibrium points of

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

    - $f_1,\ f_2$ are continuously differentiable about (x_{10}, x_{20})
    - Since we are interested in trajectories near (x_{10}, x_{20}), define $x_1 = y_1 + x_{10}, \quad x_2 = y_2 + x_{20}$
    - y_1, y_2 are small perturbations form equilibrium point.

Qualitative Behavior Near Equilibrium Points

- Expanding into its Taylor series

$$\dot{x}_1 = \dot{x}_{10} + \dot{y}_1 = \underbrace{f_1(x_{10}, x_{20})}_{0} + \left.\frac{\partial f_1}{\partial x_1}\right|_{(x_{10}, x_{20})} y_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_{(x_{10}, x_{20})} y_2 + H.O.T.$$

$$\dot{x}_2 = \dot{x}_{20} + \dot{y}_2 = \underbrace{f_2(x_{10}, x_{20})}_{0} + \left.\frac{\partial f_2}{\partial x_1}\right|_{(x_{10}, x_{20})} y_1 + \left.\frac{\partial f_2}{\partial x_2}\right|_{(x_{10}, x_{20})} y_2 + H.O.T.$$

- For sufficiently small neighborhood of equilibrium points, H.O.T. are negligible

$$\begin{cases} \dot{y}_1 = a_{11} y_1 + a_{12} y_2 \\ \dot{y}_2 = a_{21} y_1 + a_{22} y_2 \end{cases}, \quad a_{ij} = \left.\frac{\partial f_i}{\partial x}\right|_{x_0}, \quad i = 1, 2$$

- The equilibrium point of the linear system is $(y_1 = y_2 = 0)$

$$\dot{y} = Ay, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \left.\frac{\partial f}{\partial x}\right|_{x_0}$$

Qualitative Behavior Near Equilibrium Points

- Matrix $\frac{\partial f}{\partial x}$ is called Jacobian Matrix.
- The trajectories of the nonlinear system in a small neighborhood of an equilibrium point are close to the trajectories of its linearization about that point:
- if the origin of the linearized state equation is a
  - stable (unstable) node, or a stable (unstable) focus or a saddle point,
- then in a small neighborhood of the equilibrium point, the trajectory of the nonlinear system will behave like a
  - stable (unstable) node, or a stable (unstable) focus or a saddle point.

Example : Tunnel Diode Circuit

$$
\begin{aligned}
\dot{x}_1 &= \frac{1}{C}[-h(x_1) + x_2] \\
\dot{x}_2 &= \frac{1}{L}[-x_1 - Rx_2 + u]
\end{aligned}
$$

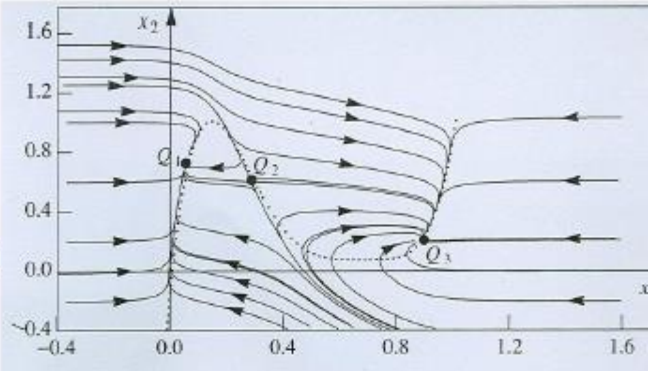
- $u = 1.2v$, $R = 1.5K\Omega$, $C = 2pF$, $L = 5\mu H$, time in nanosecond, current in mA

$$
\begin{aligned}
\dot{x}_1 &= 0.5[-h(x_1) + x_2] \\
\dot{x}_2 &= 0.2[-x_1 - 1.5x_2 + 1.2]
\end{aligned}
$$

- Suppose $h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5$
- equilibrium points ($\dot{x}_1 = \dot{x}_2 = 0$):
  $Q_1 = (0.063, 0.758)$, $Q_2 = (0.285, 0.61)$, $Q_3 = (0.884, 0.21)$

Example : Tunnel Diode Circuit

- The global phase portrait is generated by a computer program is shown in Fig.
- Except for two special trajectories which approach Q_2, all trajectories approach either Q_1 or Q_3.
- Near equilibrium points Q_1 and Q_3 are stable nodes, Q_2 is like saddle point.
- The two special trajectories from a curve that divides the plane into two halves with different behavior (separatrix curves).
- All trajectories originating from left side of the curve approach to Q_1
- All trajectories originating from left side of the curve approach to Q_3



Phase portrait of the tunnel diode circuit

Example : Tunnel Diode Circuit

- Jacobian matrix

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0.5\acute{h}(x_1) & 0.5 \\ -0.2 & -0.3 \end{bmatrix}$$

$$\acute{h}(x_1) = \frac{dh}{dx_1} = 17.76 - 207.58x_1 + 688.86x_1^2 - 905.24x_1^3 + 418.6x_1^4$$

- Evaluate the Jacobian matrix at the equilibriums Q_1, Q_2, Q_3:

$$Q_1 = (0.063, 0.758),\ A_1 = \begin{bmatrix} -3.598 & 0.5 \\ -0.2 & -0.3 \end{bmatrix},\ \lambda_1 = -3.57, \lambda_2 = -0.33 \text{ stable node}$$

$$Q_2 = (0.285, 0.61),\ A_2 = \begin{bmatrix} 1.82 & 0.5 \\ -0.2 & -0.3 \end{bmatrix},\ \lambda_1 = 1.77, \lambda_2 = -0.25 \text{ saddle point}$$

$$Q_3 = (0.884, 0.21),\ A_3 = \begin{bmatrix} -1.427 & 0.5 \\ -0.2 & -0.3 \end{bmatrix},\ \lambda_1 = -1.33, \lambda_2 = -0.4 \text{ stable node}$$

- $\therefore$ similar results given from global phase portrait.

Qualitative Behavior Near Equilibrium Points

- In practice, There are only two stable equilibrium points: Q_1 or Q_3.
- Equilibrium point at Q_2 in never observed,
  - Even if set up the exact initial conditions corresponding t Q_2, the ever-present physical noise causes the trajectory to diverge from Q_2
- Such circuit is called bistable, since it has two steady-state operating points.

Limit Cycle

- A system oscillates when it has a nontrivial periodic solution

$$x(t+T) \quad = \quad x(t),\ \forall t \geq 0,\ \text{for some}\, T > 0$$

- The word "nontrivial" is used to exclude the constant solutions.
- The image of a periodic solution in the phase portrait is a closed trajectory, calling periodic orbit or closed orbit.
- We have already seen oscillation of linear system with eigenvalues $\pm j\beta$.
- The origin of the system is a center, and the trajectories are closed
- the solution in Jordan form:

$$z_1(t) \quad = \quad r_0 \cos(\beta t + \theta_0), \quad z_2 = r_0 \sin(\beta t + \theta_0)$$

$$r_0 \quad = \quad \sqrt{z_{10}^2 + z_{20}^2},\ \theta_0 = \tan^{-1} \frac{z_{20}}{z_{10}}$$

  - r_0: amplitude of oscillation
- Such oscillation where there is a continuum of closed orbits is referred to harmonic oscillator.

Limit Cycle

- A system oscillates when it has a nontrivial periodic solution
$$x(t+T) = x(t), \; \forall t \geq 0, \; \text{for some } T > 0$$
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- the solution in Jordan form:
$$z_1(t) = r_0 \cos(\beta t + \theta_0), \quad z_2 = r_0 \sin(\beta t + \theta_0)$$
$$r_0 = \sqrt{z_{10}^2 + z_{20}^2}, \; \theta_0 = \tan^{-1} \frac{z_{20}}{z_{10}}$$
  - r_0: amplitude of oscillation
- Such oscillation where there is a continuum of closed orbits is referred to harmonic oscillator.

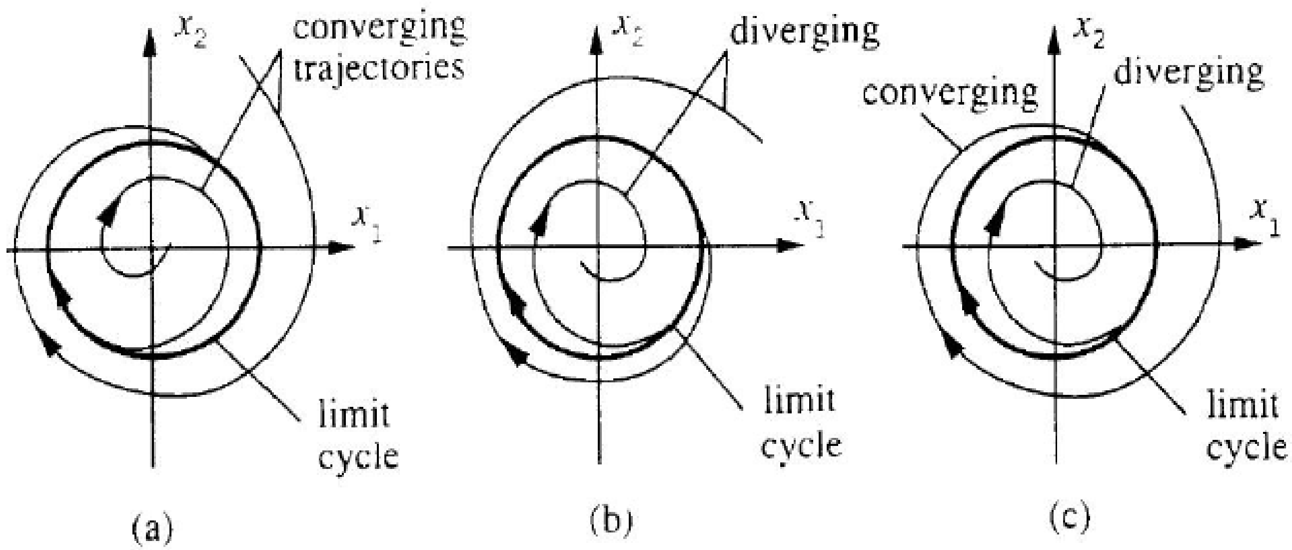
Limit Cycle

- The physical mechanism leading to these oscillations is a periodic exchange of energy stored in the capacitor (electric field) and the inductor (magnetic field).
- We have seen that such oscillation is not robust $\rightsquigarrow$ any small perturbations destroy the oscillation.
- The linear oscillator is not structurally stable
- The amplitude of the oscillation depends on the initial conditions.
- These problems can be eliminated in nonlinear oscillators. A practical nonlinear oscillator can be build such that
  - The nonlinear oscillator is structurally stable
  - The amplitude of oscillation (at steady state) is independent of initial conditions.

Limit Cycle

- On phase plane, a limit cycle is defined as an isolated closed orbit.
- For limit cycle the trajectory should be
  1. closed: indicating the periodic nature of the motion
  2. isolated: indicating limiting nature of the cycle with nearby trajectories converging to/ diverging from it.
- The mass spring damper does not have limit cycle; they are not isolated.
- Depends on trajectories motion pattern in vicinity of limit cycles, there are three type of limit cycle:
  - Stable Limit Cycles: as $t \to \infty$ all trajectories in the vicinity converge to the limit cycle.
  - Unstable Limit Cycles: as $t \to \infty$ all trajectories in the vicinity diverge from the limit cycle.
  - Semi-stable Limit Cycles: as $t \to \infty$ some trajectories in the vicinity converge to/ and some diverge from the limit cycle.

Limit Cycle



Stable, unstable, and semi-stable limit cycles

Example1.a: stable limit cycle

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2 - 1)\\ \dot{x}_2 &= -x_1 - x_2(x_1^2 + x_2^2 - 1)\end{aligned}$$

- Polar coordinates $(x_1 := rcos(\theta),\ x_2 =: rsin(\theta))$

$$\begin{aligned}\dot{r} &= -r(r^2 - 1)\\ \dot{\theta} &= -1\end{aligned}$$

- If trajectories start on the unit circle $(x_1^2(0) + x_2^2(0) = r^2 = 1)$, then $\dot{r} = 0 \implies$ The trajectory will circle the origin of the phase plane with period of $\frac{1}{2\pi}$.
- $r < 1 \implies \dot{r} > 0 \implies$ trajectories converges to the unit circle from inside.
- $r > 1 \implies \dot{r} < 0 \implies$ trajectories converges to the unit circle from outside.
- Unit circle is a stable limit cycle for this system

Example1.b: unstable limit cycle

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1(x_1^2 + x_2^2 - 1)\\ \dot{x}_2 &= -x_1 + x_2(x_1^2 + x_2^2 - 1)\end{aligned}$$

- Polar coordinates $(x_1 := rcos(\theta),\ x_2 =: rsin(\theta))$

$$\begin{aligned}\dot{r} &= r(r^2 - 1)\\ \dot{\theta} &= -1\end{aligned}$$

- If trajectories start on the unit circle $(x_1^2(0) + x_2^2(0) = r^2 = 1)$, then $\dot{r} = 0 \implies$ The trajectory will circle the origin of the phase plane with period of $\frac{1}{2\pi}$.
- $r < 1 \implies \dot{r} < 0 \implies$ trajectories diverges from the unit circle from inside.
- $r > 1 \implies \dot{r} > 0 \implies$ trajectories diverges from the unit circle from outside.
- Unit circle is an unstable limit cycle for this system

Example1.c: semi stable limit cycle

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)^2$$
$$\dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)^2$$

- Polar coordinates $(x_1 := rcos(\theta),\ x_2 =: rsin(\theta))$

$$\dot{r} = -r(r^2 - 1)^2$$
$$\dot{\theta} = -1$$

- If trajectories start on the unit circle $(x_1^2(0) + x_2^2(0) = r^2 = 1)$, then $\dot{r} = 0 \implies$ The trajectory will circle the origin of the phase plane with period of $\frac{1}{2\pi}$.
- $r < 1 \implies \dot{r} < 0 \implies$ trajectories diverges from the unit circle from inside.
- $r > 1 \implies \dot{r} < 0 \implies$ trajectories converges to the unit circle from outside.
- Unit circle is a semi-stable limit cycle for this system

Bendixson's Criterion: Nonexistence Theorem of Limit Cycle

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

$$\nabla f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

Theorem (Bendixson) *For the nonlinear system* **given** *no limit cycle can exist in a region* Ω *of the phase plane in which* $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ *does not vanish and does not change sign.*

Limit Cycle

- Example for nonexistence of limit cycle

$$\dot{x}_1 = g(x_2) + 4x_1x_2^2$$
$$\dot{x}_2 = h(x_1) + 4x_1^2x_2$$

- $\therefore \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 4(x_1^2 + x_2^2) > 0 \ \forall x \in \mathcal{R}^2$
- No limit cycle exist in $\mathcal{R}^2$ for this system.
- Note that: there is no equivalent theorem for higher order systems.

Poincare-Bendixson Criterion: Existence Theorem of Limit Cycle

- Existence of Limit Cycles
  - Relation between L.C. and Eq.points

Theorem (Poincare) *If a limit cycle exists in the second-order autonomous system (2.1), then* $N = S + 1$.

N: The No. of nodes, centers and foci enclosed by a L.C.

S: The No. of saddle points enclosed by a L.C.

The Limit cycle must enclose at least one eq. point

  - Eq. point, limit cycle, and trajectory

Theorem (Poincare-Bendixson) *If a trajectory of the second-order autonomous system remains in a finite region* Ω*, then one of the following is true:*

(a) the trajectory goes to an equilibrium point

(b) the trajectory tends to an asymptotically stable limit cycle

(c) the trajectory is itself a limit cycle