2005 Kaohsiung Invitational World Youth Mathematics Intercity Competition

Individual Contest   Time limit: 120 minutes   2005/8/3 Kaohsiung

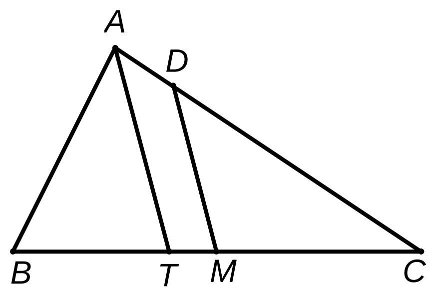
Section I:
In this section, there are 12 questions, fill in the correct answers in the spaces provided at the end of each question. Each correct answer is worth 5 points.

1. The sum of a four-digit number and its four digits is 2005. What is this four-digit number ?

Answer: ______1979______________

Since 2001+2+1<2005<2002+2+2, the four-digit number we seek is 1900+10a+b where a and b are digits. From 1900+11a+2b+1+9=2005, we have 11a+2b=95. Note that a must be odd and less than 9. If $a \leqq 5$, then $b \geqq 20$ is not a digit.
Hence a=7, b=9 and the four-digit number is 1979.

2. In triangle ABC, AB=10 and AC=18. M is the midpoint of of BC, and the line through M parallel to the bisector of $\angle CAB$ cuts AC at D. Find the length of AD.



Answer: ________4_______________

Let the bisector of $\angle CAB$ intersect BC at T. Then BT=10t and CT=18t for some positive number t. Hence CM=14t so that MT=4t. Since MD is parallel to AT, we have AD=4.

3. Let x, y and z be positive numbers such that $\begin{cases} x+y+xy=8, \\ y+z+yz=15, \\ z+x+zx=35. \end{cases}$ Find the value of $x+y+z+xy$.

Answer: _________15_____________

We have (1+x)(1+y)=9, (1+y)(1+z)=16 and (1+z)(1+x)=36. Hence $(1+x)^2$=81/4. Since x is positive, 1+x=9/2 and x=7/2. Similarly, y=1 and z=7. Hence x+y+z+xy=15.

4. The total number of mushroom gathered by 11 boys and n girls is n^2+9n-2, with each gathering exactly the same number. Determine the positive integer n.

Answer: _________9_____________

Dividing n^2+9n-2 by $n+11$, the remainder is 20. Hence $n+11$ must divide 20, and this is only possible if $n=9$.

5. The positive integer x is such that both x and $x + 99$ are squares of integers. Find the total value of all such integers x.

Answer: _______2627_____________

Let $x=n^2$ and $x+99=m^2$. Then $99=m^2-n^2=(m+n)(m-n)$. If $m+n=99$ and $m-n=1$, $n=49$ and $x=2401$. If $m+n=33$ and $m-n=3$, $n=15$ and $x=225$. Finally, if $m+n=11$ and $m-n=9$, $n=1$ and $x=1$. The sum is $2401+225+1=2627$.

6. The lengths of all sides of a right triangle are positive integers, and the length of one of the legs is at most 20. The ratio of the circumradius to the inradius of this triangle is 5:2. Determine the maximum value of the perimenter of this triangle.

Answer: ________72______________

Let the triangle have legs a and b and hypotenuse c. Then its circumradius is $c/2$ while its inradius is $(a+b-c)/2$. Hence $5(a+b)=7c$. By Pythagoras' Theorem, $25(a+b)^2=49(a^2+b^2)$, which simplifies to $0=12a^2-25ab+12b^2=(3a-4b)(4a-3b)$. This means that the two legs are in the ratio 4:3, so that we have a 3-4-5 triangle. Since one of the legs is at most 20, and we wish to maximize the perimeter, let the shorter leg be of length 18, the largest multiple of 3 under 20. Then the other leg has length 24 and the hypotenuse has length 30, yielding the maximum perimeter of 72.

7. Let α be the larger root of $(2004x)^2-2003\cdot 2005x-1=0$ and β be the smaller root of $x^2+2003x-2004=0$. Determine the value of $\alpha-\beta$.

Answer: ________2005_____________

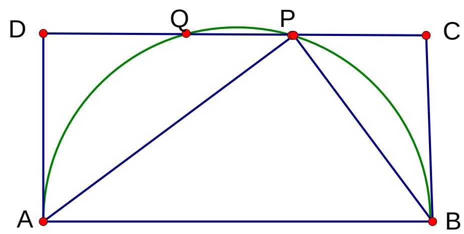
We have $0=(2004x)^2-2003\times 2005x-1=(x-1)(2004^2x+1)$ so that $\alpha=1$. We also have $0=x^2+2003x-2004=(x-1)(x+2004)$ so that $\beta=-2004$. Hence $\alpha-\beta=2005$

8.Let a be a positive real number such that $a^2+\frac{1}{a^2}=5$, Determine the value of $a^3+\frac{1}{a^3}$.

Answer: __________$4\sqrt{7}$______________

We have $(a+1/a)^2=a^2+1/a^2+2=7$. Hence $7\sqrt{7}=(a+1/a)^3=a^3+1/a^3+3\sqrt{7}$ so that $a^3+1/a^3=4\sqrt{7}$.

9.In the figure, $ABCD$ is a rectangle with AB=5 such that the semicircle on AB as diameter cuts CD at two points. If the distance from one of them to A is 4, find the area of $ABCD$.



Answer: __________12______________

Let P be the point of intersection of the semicircle with CD such that AP=4. Now $\angle APB$=90°. By Pythagoras' Theorem, $BP=\sqrt{AB^2-AP^2}$ =3. Hence the area of triangle PAB is $\frac{1}{2}$(4)(3)=6. The area of $ABCD$, which is double the area of PAB, is therefore 12.

10.Let $a=9\left[n\left(\frac{10}{9}\right)^n-1-\left(\frac{10}{9}\right)-\left(\frac{10}{9}\right)^2-\cdots-\left(\frac{10}{9}\right)^{n-1}\right]$ where n is a positive integer. If a is an integer, determine the maximum value of a.

Answer: __________81_______________

We have $1+\left(\frac{10}{9}\right)+\left(\frac{10}{9}\right)^2+\ldots+\left(\frac{10}{9}\right)^{n-1}=\frac{((\frac{10}{9})^n-1)}{(\frac{10}{9}-1)}=9((\frac{10}{9})^n-1)$.

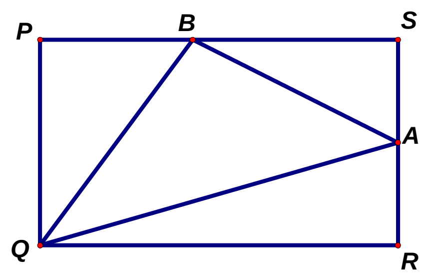
If $a=9(n-9)(\frac{10}{9})^n+81$ is to be an integer, we must have n=9 or n=1 whereby a=81 or a=9. Hence the maximum value of a is 81.

11. In a two-digit number, the tens digit is greater than the units digit. The product of these two digits is divisible by their sum. What is this two-digit number?

Answer: ___________63______________

Let the two-digit number be 10a+b with a>b. We have $ab/(a+b)=k$ for some positive integer k. This expression may be rewritten as $(a-k)(b-k)=ab-ak-bk+k^2=k^2$. For $k \geqq 5$, $a \geqq 10$ and is not a single digit. Suppose k=4. If a-4=4=b-4, then a=b=8 but this is given not to be the case. Hence a-4=8 or 16, but then $a \geqq 12$. If k=3, we have either a=b=6 or $a \geqq 12$. If k=2, we have either a=b=4 or a=6 and b=3. Finally, k=1 leads only to a=b=2. Hence the desired number is 63.

12. In Figure, *PQRS* is a square of area 10. *A* is a point on *RS* and *B* is a point on *PS* such that the area of triangle *QAB* is 4. Determine the largest possible value of *PB*+*AR*.



Let *BP*=*x* , *AR*=*y* and *PS*=z. Then *BS*=*z*-*x*, *RS*=10/*z* and *AS*=10/*z* - *y*. The combined area of triangles *BPQ*, *ABS* and *ARQ* is 6=$\frac{1}{2}\left[x\cdot\frac{10}{z}+(z-x)(\frac{10}{z}-y)+yz\right]$. Hence *xy*=2. The minimum value of *x*+*y* occurs when *x*=*y*=$\sqrt{2}$, and the value of *x*+*y* is $2\sqrt{2}$.

Answer: __________$2\sqrt{2}$______________

Section II:

Answer the following 3 questions, and show your detailed solution in the space provided after each question. Write down the question number in each paper. Each question is worth 20 points.

1. Let a, b and c be real numbers such that $a+bc=b+ca=c+ab=501$. If M is the maximum value of $a+b+c$ and m is the minimum value of $a+b+c$. Determine the value of M+2m.

We have a-b=c(a-b) so that either c=1 or a=b. If c=1, then a+b=501 while ab=500. Hence (a,b)=(1,500) or (500,1), and we have a+b+c=502. If a=b, we have $a(1+c)=c+a^2=501$ so that $a(1+501-a^2)=501$. This factors into $(a-1)(a^2+a-501)=0$. If a=1, then b=1 and c=500 and we have a+b+c=502 again. Otherwise, $a=501-a^2=c$ so that a=b=c. Their common value is $\frac{1}{2}(-1\pm\sqrt{2005})$, so that $a+b+c=\frac{3}{2}(-1\pm\sqrt{2005})$. It follows that M=502 while $m=\frac{3}{2}(-1-\sqrt{2005})$, so that $M+2m=499-3\sqrt{2005}$.

2. The distance from a point inside a quadrilateral to the four vertices are 1, 2, 3 and 4. Determine the maximum value of the area of such a quadrilateral.

Let P be the point inside the quadrilateral $ABCD$. The area of $ABCD$ is the product of $\frac{1}{2}AC$ and the sum of the distances from AC to B and D. This sum is less than or equal to PB+PD, with equality if and only if P lies on BD and BD is perpendicular to AC. On the other hand, AC is less than PA+PC, with equality if and only if P lies on AC. It follows that maximum area occurs when AC and BD intersect at P, and are perpendicular to each other. The area is then given by $\frac{1}{2}(PA+PC)(PB+PD)$. Now PA, PB, PC and PD are 1, 2, 3 and 4 in some order.
To maximize the product of (PA+PC)(PB+PD), we want the values of the products to be as close to each other as possible. Hence the maximum area is $\frac{1}{2}(1+4)(2+3)=\frac{25}{2}$.

3. We have an open-ended table with two rows. Initially, the numbers 1, 2, …, 1024 are written in the first 2005 squares of the first row. In each move, we write down the sum of the first two numbers of the first row as a new number which is then added to the end of this row, and drop the two numbers used in the addition to the corresponding squares in the second row. We continue until there is only one number left in the first row, and drop it to the corresponding square in the second row. Determine the sum of all numbers in the second row. (For example, if 1, 2, 3, 4 and 5 are written in the first row, at the end, we have 1, 2, 3, 4, 5, 3, 7, 8 and 15 in the second row. Hence its sum is 48.)

Lemma:

Initially, the numbers $a_1, a_2, \ldots, a_{2^n}$ are written in the first 2^n squares of the first row. Then the sum of all numbers in the second row is $(n+1)(a_1+a_2+\ldots+a_{2^n})$.

For $n=1$, the numbers in the second row is a_1, a_2, $\underline{a_1+a_2}$, the sum is $2(a_1+a_2)$
For $n=2$, the numbers in the second row is a_1, a_2, a_3, a_4, $\underline{a_1+a_2}$, $\underline{a_3+a_4}$, $\underline{a_1+a_2}+\underline{a_3+a_4}$, the sum is $3(a_1+a_2+a_3+a_4)$.
……
Assume $n=k$, the numbers in the second row is $a_1, a_2, a_3, a_4, \ldots, a_{2^k}$, $\underline{a_1+a_2}$, $\underline{a_3+a_4}$, …, $\underline{a_{2^k-1}+a_{2^k}}$, $\underline{a_1+a_2+a_3+a_4}$,….., $\underline{a_{2^k-3}+a_{2^k-2}+a_{2^k-1}+a_{2^k}}$,..., …,$\underline{a_1+a_2}+\underline{a_3+a_4}+\ldots+\underline{a_{2^k-1}+a_{2^k}}$ the sum of the numbers in the second row is $(k+1)(a_1+a_2+\ldots+a_{2^k})$.
When $n=k+1$, the numbers $a_1, a_2, \ldots, a_{2^{k+1}}$ are written in the first 2^{k+1} squares of the first row. After 2^k moves, we have 2^k numbers : $\underline{a_1+a_2}$, $\underline{a_3+a_4}$, …, $\underline{a_{2^{k+1}-1}+a_{2^{k+1}}}$, in the first row , and we have 2^{k+1} numbers :$a_1, a_2, \ldots, a_{2^{k+1}}$ in the second row.
By induction, at the end,the sum of the numbers in the second row is $(k+1+1)(a_1+a_2+\ldots+a_{2^{k+1}})$.

Now, the numbers 1, 2, … , 2005 are written in the first 2005 squares of the first row. After 981 moves, we have 1024 numbers : 1963, 1964, ... , 2005, 3, 7, 11, … , 3919, 3923 in the first row, and we have 1962 numbers : 1, 2, …. , 1961, 1962 in the second row.
We only look at $1024=2^{10}$ numbers in first row, from Lemma, we obtain the sum of the numbers in the second row is $(10+1)(1963+1964+\ldots+2005+3+7+\ldots+3919+3923)$ $=11(1+2+3+\ldots+2004+2005)=22121165$. We must add first part 1, 2, …, 1961, 1962 in the second row to the total sum. It follows that the sum of all the numbers in the second row is $22121165+(1+2+\ldots+1962)=24046868$..