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1 Solutions of Equations in One Variable

In this chapter we consider

one of the most basic problems

of numerical approximation,

the **root-finding problem**

This process involves

finding a **root**,

or solution,

of an equation

of the form $f(x) = 0$,

1 The Bisection Method

$\boxed{\text{2 Fixed-Point Iteration}}$

3 Newton's Method

مقدمه

فصل۱)
ریشه یابی

فصل۲)
درونیابی

فصل۳) حل
عددی انتگرال

فصل۴) حل عددی
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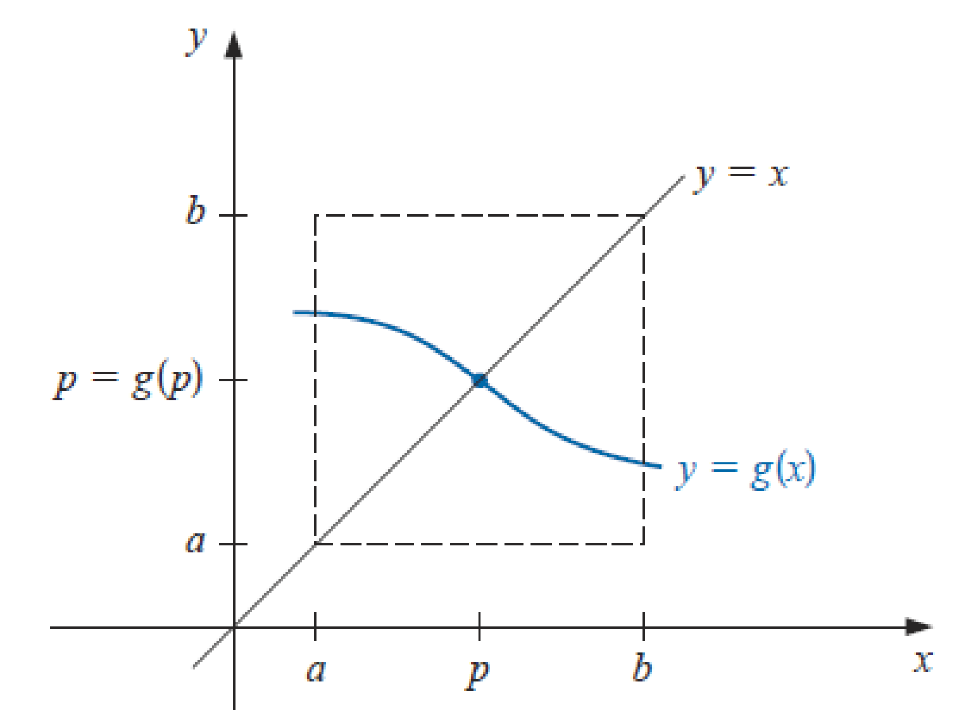
فصل۶)
برازش منحنی

Fixed-Point Iteration

A *fixed point*
for a function
is a number
at which
the value of the function
does not change
when the function is applied

Suppose $\begin{cases} \text{If } g \in C[a,b] \quad g(x) \in [a,b] \\ g'(x) \text{ exists on } (a,b) \quad |g'(x)| \leq 1, \quad \text{for all } x \in (a,b) \end{cases}$



The number p
is a **fixed point**
for a given function g
if $g(p) = p$

Example 1

Show that $g(x) = (x^2 - 1)/3$ has a unique fixed point on the interval $[-1, 1]$.

Solution The maximum and minimum values of $g(x)$ for x in $[-1, 1]$ must occur either when x is an endpoint of the interval or when the derivative is 0.

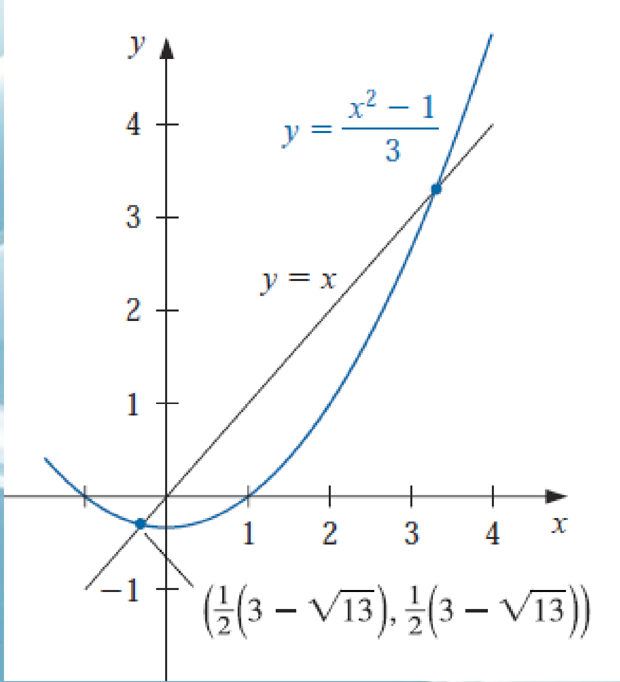
Since $g'(x) = 2x/3$, the function g is continuous and $g'(x)$ exists on $[-1, 1]$

The maximum and minimum values of $g(x)$ occur at $x = -1$, $x = 0$, or $x = 1$.

$$g(-1) = 0,\ g(1) = 0, \text{ and } g(0) = -1/3.$$

$$|g'(x)| = \left|\frac{2x}{3}\right| \le \frac{2}{3}, \quad \text{for all } x \in (-1, 1).$$

$$p = g(p) = \frac{p^2 - 1}{3} \qquad p^2 - 3p - 1 = 0, \qquad p = \frac{1}{2}(3 - \sqrt{13})$$



Fixed-Point Iteration

Example 2 The equation $x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$.

Solution There are many ways to change the equation to the fixed-point form $x = g(x)$ using simple algebraic manipulation.

For example, we can manipulate the equation $x^3 + 4x^2 - 10 = 0$ as follows:

(a) $x = g_1(x) = x - x^3 - 4x^2 + 10$

(b) $x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$

(c) $x = g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$

(d) $x = g_4(x) = \left(\frac{10}{4+x}\right)^{1/2}$

(e) $x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$

Fixed-Point Iteration

With $p_0 = 1.5$

Table 2.2 lists the results of the fixed-point iteration for all five choices of g.

Table 2.2

n	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	−0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	−469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
6			1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

- Question: How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a solution to a given root-finding problem?

Suppose $\begin{cases} \text{If } g \in C[a,b] \quad g(x) \in [a,b] \\ g'(x) \text{ exists on } (a,b) \quad |g'(x)| \le 1, \quad \text{for all } x \in (a,b) \end{cases}$

Then for any number p_0 in $[a,b]$

the sequence defined by

$$\boxed{p_n = g(p_{n-1}), \quad n \ge 1,}$$

converges to the unique fixed point p in $[a,b]$.

(a) $x = g_1(x) = x - x^3 - 4x^2 + 10$

we have $g_1(1) = 6$ and $g_1(2) = -12$ so g_1 does not map $[1, 2]$ into itself.

Moreover, $g_1'(x) = 1 - 3x^2 - 8x$ so $|g_1'(x)| > 1$ for all x in $[1, 2]$

Although Theorem does not guarantee that the method must fail for this choice of g there is no reason to expect convergence.

Example 3 Show that Theorem does not ensure a unique fixed point of $g(x) = 3^{-x}$ on the interval $[0, 1]$, even though a unique fixed point on this interval does exist.

(b) $x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$ we can see that g_2 does not map $[1, 2]$ into $[1, 2]$ $|g_2'(p)| \approx 3.4$

(c) $x = g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$ $g_3'(x) = -\frac{3}{4}x^2(10 - x^3)^{-1/2} < 0$ on $[1, 2]$ $|g_3'(2)| \approx 2.12$

with $p_0 = 1.5$ This shows that g_3 maps the interval $[1, 1.5]$ into itself.

It is also true that $|g_3'(x)| \le |g_3'(1.5)| \approx 0.66$ on this interval.

(d) $x = g_4(x) = \left(\dfrac{10}{4+x}\right)^{1/2}$

$$|g_4'(x)| = \left|\frac{-5}{\sqrt{10}(4+x)^{3/2}}\right| \le \frac{5}{\sqrt{10}(5)^{3/2}} < 0.15, \quad \text{for all} \quad x \in [1,2].$$

$$\boxed{\text{(e)} \quad x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}}$$

converges much more rapidly than our other choices. In the next sections we will see where this choice came from and why it is so effective. □

EXERCISE SET 2.

The following four methods are proposed to compute $21^{1/3}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

a. $p_n = \dfrac{20p_{n-1} + 21/p_{n-1}^2}{21}$

b. $p_n = p_{n-1} - \dfrac{p_{n-1}^3 - 21}{3p_{n-1}^2}$

c. $p_n = p_{n-1} - \dfrac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21}$

d. $p_n = \left(\dfrac{21}{p_{n-1}}\right)^{1/2}$

Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^3 - x - 1 = 0$ on $[1, 2]$. Use $p_0 = 1$.

Use a fixed-point iteration method to determine a solution accurate to within 10^{-4} for $x = \tan x$, for x in $[4, 5]$.

مقدمه

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فصل۶) برازش منحنی

پایان جلسه دوم

۱۲ فروردین ۹۹

با تشکر از توجه شما