CONFORMAL MAP PROJECTIONS IN GEODESY

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September 1973

LECTURE NOTES
37

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September 1973
Latest reprinting January 1998

PREFACE

In order to make our extensive series of lecture notes more readily available, we have scanned the old master copies and produced electronic versions in Portable Document Format. The quality of the images varies depending on the quality of the originals. The images have not been converted to searchable text.

PREFACE

The purpose of these notes is to give the theory and use of some common conformal map projections. The approach used is straightforward. It begins with the basics of differential geometry and conformal mapping. Then, given the conditions for a particular map projection, the particular conformal mapping equations are derived. This is a self-contained analytical approach.

The author is aware of attempts by at least one mathematician [Wray, 1973] to unify all classes of map projections such that the same set of mapping equations can be used to obtain all or many of the well-known projections, simply by assigning appropriate values to certain parameters in some generalized equations. This is somewhat of a simplified explanation of what actually happens, but it at least illustrates the generality of this contemporary approach. As of the date of writing these notes, the details (necessary for instruction at the undergraduate level) were not yet worked out and thus no use was made of it.

There is yet another attempt to obtain a generalized set of equations for conformal map projections alone. As explained in Section 9, this approach consists of one set of mapping equations which is capable of producing the well-known conformal projections Mercator, Transverse Mercator, Lambert Conformal Conic, Stereographic and even other unnamed projections. This is achieved simply by assigning specific values for certain constants in a generalized set of equations. This approach is not readily usable since the expressions have not been developed to a sufficient degree of accuracy, and thus are not satisfactory for the practising surveyor. Only a brief description of this approach is given in these notes.

More on the approach used herein. Complex arithmetic is exploited at every opportunity. Series expansions are avoided at occasions when the closed form exists. The reason being that computer centres nowadays have routines to evaluate natural logs, exponentiation, etc. Derivations are given to show the origin and important steps in the development of the main equations. Lengthy and detailed derivations are omitted from the text and reference made to an appropriate source or an appendix added.

These notes have been written under the assumption that the reader has knowledge of differential and integral calculus, complex arithmetic, ellipsoid geometry, and some knowledge of computer programming.

These notes have evolved from the author's lecture notes over the past few years. Two main sources are acknowledged at the outset as being the starting point for these notes - Thomas'"Conformal Projections in Geodesy and Cartography", U.S. Geodetic Survey Special Publication No. 251, and Dr. Richard H. Rapp's lecture notes on "Advanced Map Projections", Department of Geodetic Science, The Ohio State University, Columbus. Other sources used for important details are referenced within the text.

The author wishes to acknowledge the contribution made by the Surveying Engineering undergraduate class of 1974-75 to improving the notes by finding typographical errors. Messrs. Donald B. Thomson and M. Nassar, graduate students, are acknowledged for their constructive criticism of the notes. Mr. Thomson is particularly acknowledged for the preparation of the appendices.

E. Krakiwsky
December 9, 1974

The author wishes to acknowledge Kresho Frankich for his detailed critique of the December, 1974 version of this work. His involvement has lead to the improvement of section 2.4 and Chapters 7, 8 and 10. Dr. Donald B. Thomson helped to clarify aspects of the stereographic projection. Robin Steeves extended various formulae to meet a higher degree of accuracy. Ms. Wendlynn Wells is thanked for her editorial assistance in preparing this version.

E. Krakiwsky
July 18, 1977

TABLE OF CONTENTS

TABLE OF CONTENTS (cont'd)

GENERAL

Conformal map projections are the class of projections in which angles on the surface to be mapped are preserved, that is, corresponding angles on the map plane and the surface are equal. We will restrict the surface to be mapped to be the ellipsoid as defined in geodesy. The map plane is regarded as a flat - two dimensional surface. The two corresponding sets of coordinates for any point are the geodetic latitude ϕ and longitude λ, and the mapping coordinates x and y.

There are two main problem areas in conformal map projections. The first area is concerned with the transformation (mapping) of the geodetic latitude (ϕ) and longitude (λ) into a pair of mapping coordinates x and y, and vice versa, that is the mapping of x and y into ϕ and λ. The second problem area involves the computations of geodetic positions on the conformal mapping plane from observed quantities that have been appropriately projected onto the map plane. Because of the clear identification of these two tasks, the notes have been structured into two sections. Section I is called Conformal Mapping, while Section II is called Computations on a Mapping Plane.

SECTION I

CONFORMAL MAPPING

1. INTRODUCTION TO CONFORMAL MAPPING

It was stated in the general introduction that the problem at hand is the determination of the conformal transformation equations for mapping ϕ, λ into x, y, and x, y into ϕ, λ. Employing complex arithmetic notation, the problem is symbolized as

$$(x + iy) = f_1(\lambda + i\phi) \ , \qquad 1\text{-}1$$

and

$$(\lambda + i\phi) = f_2(x + iy) \ , \qquad 1\text{-}2$$

where f_1 and f_2 are the analytic functions to be determined, and $i = \sqrt{-1}$. The first equation describes the direct problem and the second the inverse problem.

The solution of these problems requires the use of some elementary complex algebra which is reviewed in Section 2. Also, some differential geometry is needed to characterize differential elements on the ellipsoid and plane surfaces - this material is reviewed in Section 3. The principles of conformal mapping theory needed for our problems are briefly given in Section 4. Sections 5 through 8 respectively treat the

Mercator, Transverse Mercator, Lambert Conformal Conic and Stereographic conformal projections. A brief description of the implementation of some of these projections in practise as official coordinate systems is given. The generalized set of mapping equations mentioned in the preface is briefly described in Section 9.

2. REVIEW OF COMPLEX VARIABLES

Complex algebra is used liberally in the development of the various conformal mapping equations. The purpose of this section is to briefly review, thus summarize, those parts of complex algebra needed herein. The formal and complete treatment can be found in many mathematical texts.

Reviewed in this section are: rules of complex arithmetic; the complex plane; the complex function; differentiation of a complex function and the derivation of the Cauchy-Riemann equations; special complex mapping functions.

2.1 Rules of Complex Arithmetic

For a complex number

$$z = a + ib; \quad i = \sqrt{(-1)} \quad , \qquad \text{2-1}$$

where a is the real part and b the imaginary part, the following rules and definitions apply:

1. The sum of two complex numbers is a complex number, that is,

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$$

$$= (a_1 + a_2) + i(b_1 + b_2) = z_3 \ . \qquad 2\text{-}2$$

2. Addition is associative (grouping is immaterial), that is,

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \ . \qquad 2\text{-}3$$

3. Every complex number has a unique negative, that is

$$z = a + ib$$

and

$$-z = -a - ib \ . \qquad 2\text{-}4$$

4. Addition is commutative (order is immaterial), that is

$$z_1 + z_2 = z_2 + z_1 \qquad 2\text{-}5$$

5. Multiplication is defined as

$$z_1 \cdot z_2 = (a_1 + ib_1)(a_2 + ib_2)$$

$$= a_1 a_2 - b_1 b_2 + i(a_1 b_2 + a_2 b_1) = z_3 \ . \qquad 2\text{-}6$$

6. Multiplication is associative, that is

$$z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3 = z_1 \cdot z_2 \cdot z_3 . \qquad 2\text{-}7$$

7. Every non-zero complex number z has a unique inverse, that is

$$z^{-1} = \frac{1}{z} = \frac{1}{(a+ib)} \cdot \frac{(a-ib)}{(a-ib)} = \frac{a-ib}{a^2+b^2}$$

$$= \frac{a}{a^2+b^2} - i\,\frac{b}{a^2+b^2} \ . \qquad 2\text{-}8$$

8. The complex conjugate of z is

$$\bar{z} = a - ib. \qquad 2\text{-}9$$

2.2 Complex Plane

The rectangular form of a complex number is

$$z = x_1 + iy_1 \; . \qquad \text{2-10}$$

The geometric interpretation is a vector in two dimensional space, with the real part (x) as the abscissa and the imaginary part (y) as the ordinate (Figure 2-1).

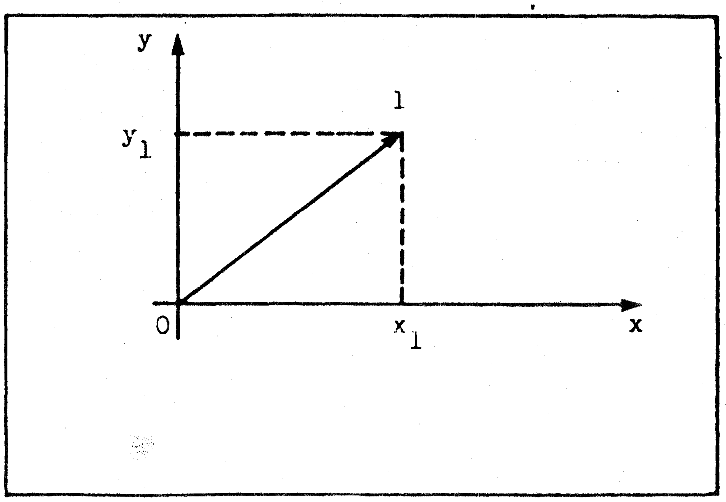


Figure 2-1. Rectangular Form

The polar form of a complex number is (Figure 2-2)

$$z_1 = r_1 \cos\theta_1 + i \, r_1 \sin \theta_1 \qquad \text{2-11}$$

since

$$x_1 = r_1 \cos\theta_1 \; ,$$
$$y_1 = r_1 \sin\theta_1 \; . \qquad \text{2-12}$$

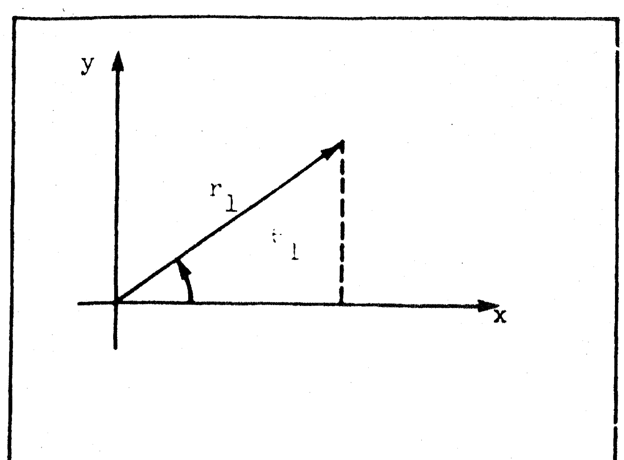


Figure 2-2 Polar Form

The argument of z is denoted as θ_1, and defined by

$$\theta_1 = \tan^{-1} \frac{y_1}{x_1} \quad . \qquad 2\text{-}13$$

The modulus of z_1 is denoted as r_1, and defined by

$$r_1 = |z_1| = |x_1 + iy_1| = (x_1^2 + y_1^2)^{1/2} \quad . \qquad 2\text{-}14$$

The x-y plane is called the complex plane.

2.3 Complex Function

w is said to be a function of z, that is

$$w = f(z) \;, \qquad 2\text{-}15$$

if there exists one or two values of the complex variable w for every value of the complex variable z. In this arrangement, z is the independent variable and w the dependent variable.

An example of a single valued function is

$$w = z^2 \;, \qquad 2\text{-}16$$

and of a many valued (two in this example) function is

$$w = z^{1/2} \quad . \qquad 2\text{-}17$$

In mapping the ellipsoid onto a plane, we are interested only in single valued functions.

In general

$$w = f(z) = u + iv \;,$$

$$= f(x+iy) = u(x, y) + iv(x, y) . \qquad 2\text{-}18$$

The corresponding conjugate is

$$\bar{w} = f(\bar{z}) = u(x, y) - iv(x, y) \; . \qquad 2\text{-}19$$

2.4 Differentiation of a Complex Function

This section first defines the derivative of a complex function, that is $f'(z)$, and then the very useful and important Cauchy-Riemann equations.

Consider a set of complex numbers $\{z_o, z_1, \ldots, z_n\}$, where z_o and z_n are particular values. It is said that $f(z)$ is differentiable at z_o if

$$\lim_{z_n \to z_o} \frac{f(z_n) - f(z_o)}{z_n - z_o} \qquad \text{2-20}$$

or

$$\lim_{\Delta z \to 0} \frac{f(z_o + \Delta z) - f(z_o)}{\Delta z} \quad \text{exists.} \qquad \text{2-21}$$

The basic equation for defining the derivative $f'(z)$ follows from above after recognizing that from

$$z = x + iy$$

$$\Delta z = \Delta x + i\Delta y \qquad \text{2-22}$$

and

$$f(z + \Delta z) = u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) \ . \qquad \text{2-23}$$

The basic equation is

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left\{\frac{u(x + \Delta x, y + \Delta y) - u(x, y)}{\Delta x + i\Delta y} + \right.$$

$$\left. + i \frac{v(x + \Delta x, y + \Delta y) - v(x, y)}{\Delta x + i\Delta y}\right\} \qquad \text{2-24}$$

and is independent of the approach taken. But, for the approach y = constant, that is, $\Delta y = 0$, and $\Delta z = \Delta x$, the result is

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} +$$

$$+ \lim_{\Delta x \to 0} i \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} , \qquad 2\text{-}25$$

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}} \qquad 2\text{-}26$$

For the approach $\Delta z \to 0$ along x = constant, that is $\Delta x = 0$, $\Delta z = i\Delta y$. The result is:

$$f'(z) = \lim_{\Delta y \to 0} \frac{1}{i} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y} +$$

$$+ \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y} , \qquad 2\text{-}27$$

$$\boxed{f'(z) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} .} \qquad 2\text{-}28$$

The Cauchy-Riemann equations are obtained by equating the two definitions of the derivative, equations 2-28 and 2-26, namely

$$\frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \qquad 2\text{-}29$$

$$(-i^2)\frac{1}{i} \frac{\partial u}{\partial y} + (-i^2) \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$-i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} . \qquad 2\text{-}30$$

Equating real and imaginary parts yields the Cauchy-Riemann equations

$$\boxed{\begin{array}{c} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} , \\ \hline \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} . \end{array}} \qquad 2\text{-}31$$

A function of complex number z which is single valued and differentiable at every point of its domain, is said to be regular. The necessary condition for the function f(z) to be differentiable at the point z is that the four partial derivatives u_x, u_y, v_x, v_y must exist and satisfy the Cauchy-Riemann equations.

Since the partial derivatives of u and v are connected by the Cauchy-Riemann equations (2-31) and assuming that these derivatives exist and satisfy the relation $u_{xy} = u_{yx}$, $v_{xy} = v_{yx}$, it follows by partial differentiation that

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2} \quad . \qquad 2\text{-}32$$

Thus both u and v satisfy Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ,$$

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad . \qquad 2\text{-}33$$

By separating any regular function of z, i.e. a function with existing derivatives, into its real and imaginary parts, we obtain immediately two solutions of Laplace's equation.

From the equations (2-31) we can also have the following relationship

$$\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = 0 \quad , \qquad 2\text{-}34$$

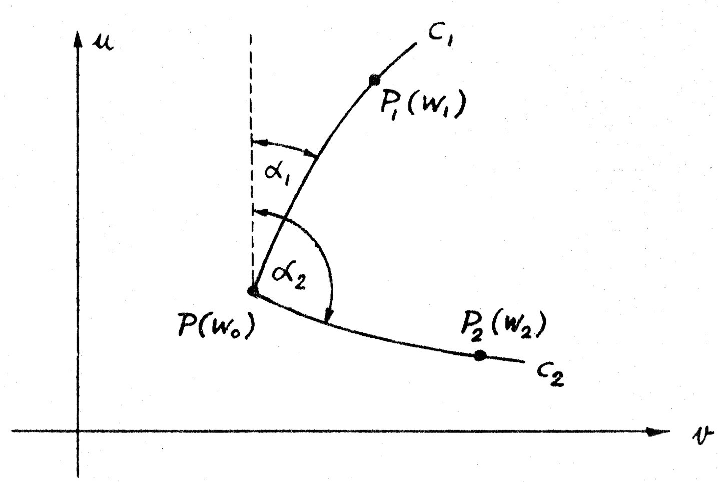
whose geometrical interpretation is that the families of curves in the x, y plane, corresponding to constant values of u and v, intersect at right angles at all their points.

Let us now take an analytic function f which maps the domain D of the independent complex variable w = u + i v into the domain of D' of the dependent complex variable $z = x + i\,y$. If two arbitrary curves in the domain D intersect at the point $P(u_o, v_o)$ at an angle θ, then the mapping is called isogonal if the corresponding curves intersect in the domain D' also at the angle θ. If the sense of the rotation as well as the magnitude of the angle is preserved, the mapping is said to be conformal.

To prove this statement let us take in the w plane an arbitrary point $P(w_o)$ and two continuous curves passing through the point and let tangents to these curves at the point P make angles α_1 and α_2 with the real axis of the coordinate system. Suppose we select on each curve additional points close to P and at the distance r from P. Then

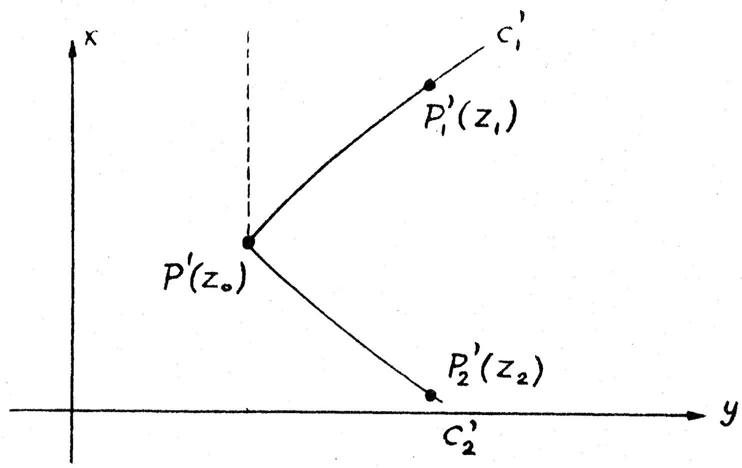
$$w_1 - w_o = r\,e^{i\theta_1} \quad ,$$
$$w_2 - w_o = r\,e^{i\theta_2} \quad , \qquad 2\text{-}35$$

and as r approaches zero, θ_1 and θ_2 become the angles α_1 and α_2.



w-plane

The point $P(w_o)$ is transformed from the domain D by the analytic function into $P'(z_o)$ of the domain D'. The projected curves are denoted by c_1' and c_2'.



z-plane

Then

$$z_1 - z_o = \rho_1 e^{i\phi_1} \quad ,$$
$$z_2 - z_o = \rho_2 e^{i\phi_2} \quad . \qquad 2\text{-}36$$

By the definition of derivatives we have

$$\lim_{w_1 \to w_o} \frac{z_1 - z_o}{w_1 - w_o} = f'(z_o) \quad \text{and} \quad \lim_{w_2 \to w_o} \frac{z_2 - z_o}{w_2 - w_o} = f'(z_o), \quad \text{or}$$

$$\lim \frac{\rho_1 e^{i\phi_1}}{r e^{i\theta_1}} = k \cdot e^{i\alpha} \quad \text{and} \quad \lim \frac{\rho_2 e^{i\phi_2}}{r e^{i\theta_2}} = k \cdot e^{i\alpha} \quad . \qquad 2\text{-}37$$

It is obvious from the above relations that

$$\lim (\phi_1 - \theta_1) = \alpha \quad \text{and} \quad \lim (\phi_2 - \theta_2) = \alpha \quad , \qquad 2\text{-}38$$

or the angles between the corresponding curves in the conformal mapping remain unchanged.

At the same time it is important to notice that

$$\lim \frac{\rho_1}{r} = \lim \frac{\rho_2}{r} = k = |f'(w_o)| \quad , \qquad 2\text{-}39$$

is the scale factor and has the same value in all directions through the same point but it varies from point to point. This is an extremely important property of conformal mapping.

Other expressions related to the derivative of a complex function are needed. The first of these is

$$
\begin{aligned}
d\omega \cdot d\bar{\omega} &= (du + idv)(du - idv) \\
&= du^2 + dv^2 \\
&= f'(x + iy)\ f'(x - iy)(dx + idy)(dx - idy) \\
&= f'(z)\ f'(\bar{z})(dx^2 + dy^2) \quad , \qquad \text{2-40}
\end{aligned}
$$

and realizing that

$$
f'(\bar{z}) = f'(x - iy) = \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial u}{\partial y} \quad ,
$$

and that

$$
f'(z)\ f'(\bar{z}) = (\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2 \qquad \text{2-41}
$$

$$
= (\frac{\partial v}{\partial y})^2 + (\frac{\partial u}{\partial y})^2 \quad , \qquad \text{2-42}
$$

$$
\boxed{
\begin{aligned}
d\omega \cdot d\bar{\omega} &= [(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2](dx^2 + dy^2) \\
&= [(\frac{\partial v}{\partial y})^2 + (\frac{\partial u}{\partial y})^2](dx^2 + dy^2)
\end{aligned}
} . \qquad \text{2-43}
$$

2.5 Special Complex Functions

The following identities are useful in simplifying conformal mapping equations. The first identity is a result of De Moivre's Theorem which gives a relationship between multiples and powers of trigonometric functions. The theorem is briefly stated. For

$$
\begin{aligned}
z &= x + iy \\
&= r\ (\cos\theta + i \sin\theta) \quad ,
\end{aligned}
$$

the product of two complex numbers is

$$
\begin{aligned}
z_1 \cdot z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\ r_2(\cos\theta_2 + i\sin\theta_2) \\
&= r_1 r_2[\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)] \\
&= r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \quad , \qquad \text{2-44}
\end{aligned}
$$

and an extrapolation to n complex numbers for the case $r_1 = r_2 - \ldots = r_n$, and $\theta_1 = \theta_2 = \ldots = \theta_n$, yields

$$z^n = r^n[\cos n\theta + i \sin n\theta] = [r(\cos\theta + i \sin\theta)]^n$$

and finally

$$\boxed{\cos n\theta + i \sin n\theta = [\cos\theta + i \sin\theta]^n} \quad . \qquad 2\text{-}45$$

The second useful formulae is Euler's which relates exponential and trigonometric functions. It is employed in the derivation of the Lambert conformal conic mapping equations. Begin by considering

$$e^\theta = 1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots + \frac{\theta^n}{n!} \quad , \qquad 2\text{-}46$$

and substituting $i\theta$ for θ, the above becomes

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \ldots + \frac{(i\theta)^n}{n!} \quad , \qquad 2\text{-}47$$

and after grouping real and imaginary parts

$$e^{i\theta} = (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots) \quad ,$$

which is of course

$$\boxed{e^{i\theta} = \cos\theta + i \sin\theta} \quad . \qquad 2\text{-}48$$

The above is known as Euler's formula. Substituting $-\theta$ for θ in the above yields

$$\boxed{e^{-i\theta} = \cos\theta - i \sin\theta} \quad . \qquad 2\text{-}49$$

Together the above two formulae yield two more useful formulae, namely

$$\boxed{\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})} \quad , \qquad 2\text{-}50$$

$$\boxed{\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})} \quad . \qquad 2\text{-}51$$

The last set of formulae are the sin and cos of complex numbers. Substituting z for θ in 2-50 and 2-51, yields

$$\boxed{\cos z = \frac{1}{2}(e^{iz} + e^{-iz})} \quad , \qquad 2\text{-}52$$

$$\boxed{\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})} \quad . \qquad 2\text{-}53$$

Further

$$\boxed{\begin{aligned} e^z &= e^{x+iy} = e^x e^{iy} \\ &= e^x (\cos y + i \sin y) \quad . \end{aligned}} \qquad 2\text{-}54$$

3. REVIEW OF DIFFERENTIAL GEOMETRY

The purpose of this section is to review and write in our notation some well-known results of differential geometry. No attempt has been made to be exhaustive, in fact, only those aspects needed herein are given. We treat: parametric equations for a surface; the Gaussian fundamental quantities; angles between parametric curves.

3.1 Parametric Equations

Let us now discuss how to mathematically describe the surface to be mapped.

The general equation describing any surface* is

$$F(X, Y, Z) = 0 \ , \qquad \text{3-1}$$

where

$$X = X(\phi, \lambda) \ ,$$
$$Y = Y(\phi, \lambda) \ , \qquad \text{3-2}$$
$$Z = Z(\phi, \lambda) \ ,$$

are known as the parametric equations. X, Y and Z are understood to be

---

* X, Y, Z used as coordinates on surface to be mapped; x, y used as as coordinates on map plane.

Cartesian coordinates while ϕ and λ are regarded as curvilinear coordinates on the particular surface in question.

For example, the equation of a spherical surface is

$$F(X, Y, Z) = X^2 + Y^2 + Z^2 - R^2 = 0 \quad . \tag{3-3}$$

The corresponding parametric equations are

$$\begin{aligned} X &= X(\phi, \lambda) = R \cos\phi \cos\lambda \quad , \\ Y &= Y(\phi, \lambda) = R \cos\phi \sin\lambda \quad , \\ Z &= Z(\phi) = R \sin\phi \quad , \end{aligned} \tag{3-3a}$$

where R is the radius, and ϕ and λ the spherical latitude and longitude, respectively.

The equation of an ellipsoid surface is

$$F = X^2 + Y^2 + Z^2 - N^2 = 0 \quad . \tag{3-4}$$

The parametric equations are

$$\begin{aligned} X &= N \cos\phi \cos\lambda \quad , \\ Y &= N \cos\phi \sin\lambda \quad , \\ Z &= N(1-e^2) \sin\phi \quad , \end{aligned} \tag{3-4a}$$

where ϕ and λ are the geodetic latitude and longitude respectively, and

$$N = \frac{a^2}{(a^2 \cos^2\phi + b^2 \sin^2\phi)^{1/2}} = \frac{a}{(1-e^2 \sin^2\phi)^{1/2}} \tag{3-4b}$$

is the radius of curvature of the ellipsoid in the prime vertical plane, a and b are the semi-major and semi-minor axes, respectively, and $e^2 = (a^2-b^2)/a^2$,

Parametric equations allow us to describe, in a mathematical way, certain curves on a surface. For example when ϕ is equal to some constant c_1, then the parametric equations become

$$\begin{aligned} X &= X(c_1, \lambda) \quad , \\ Y &= Y(c_1, \lambda) \quad , \\ Z &= Z(c_1, \lambda) \quad , \end{aligned} \tag{3-5}$$

which can be imagined to describe some sort of parallel of latitude. Analogously a meridian arc is described by

$$X = X\ (\phi,\ c_2)\ ,$$
$$Y = Y\ (\phi,\ c_2)\ , \qquad 3\text{-}6$$
$$Z = Z\ (\phi,\ c_2)\ .$$

The equation of an arbitrary curve is denoted by $S(\phi,\ \lambda)$, which is in essence an expression of some functional relationship between ϕ and λ.

3.2 Gaussian Fundamental Quantities

The Gaussian fundamental quantities are a means of describing the geometrical properties of the surface to be mapped. The derivation is as follows.

Consider an arbitrary curve on any surface (Figure 3-1) with

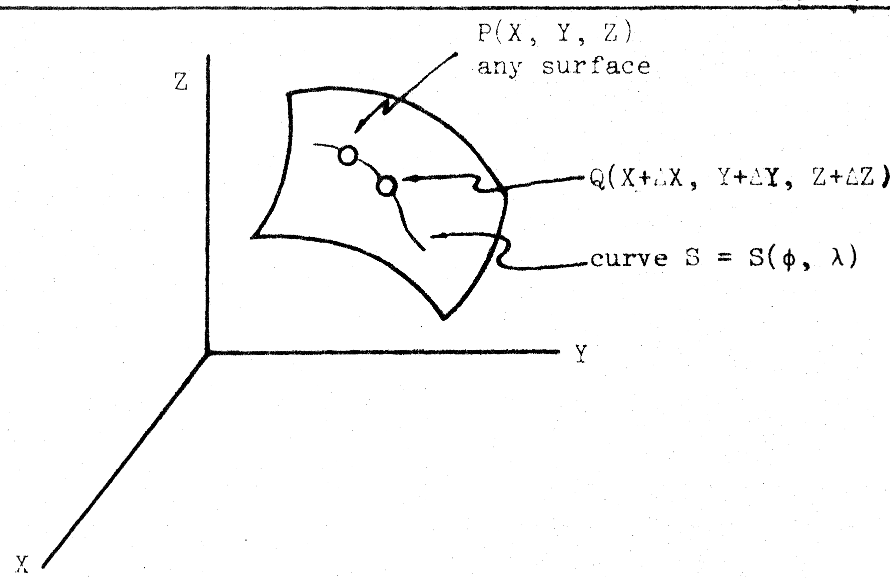


Figure 3-1. Surface Element

a point P(X, Y, Z) on it and a second point Q(X+ΔX, Y+ΔY, Z+ΔZ) located infinitesimally close to P. Let ΔL be the chord length between P and Q and ΔS the corresponding length on the surface. We can then write that

$$\Delta L^2 = \Delta X^2 + \Delta Y^2 + \Delta Z^2\ , \qquad 3\text{-}7$$

and

$$\left(\frac{\Delta L}{\Delta S}\right)^2 = \left(\frac{\Delta X}{\Delta S}\right)^2 + \left(\frac{\Delta Y}{\Delta S}\right)^2 + \left(\frac{\Delta Z}{\Delta S}\right)^2 . \qquad 3\text{-}8$$

As the point Q approaches P, ΔL approaches ΔS and we get

$$1 = \left(\frac{dX}{dS}\right)^2 + \left(\frac{dY}{dS}\right)^2 + \left(\frac{dZ}{dS}\right)^2 , \qquad 3\text{-}9$$

which can be interpreted as the direction cosines of a line on the surface.

From the above we can get an expression for the change in arc length as a function of changes in the Cartesian coordinates, namely

$$dS^2 = dX^2 + dY^2 + dZ^2 . \qquad 3\text{-}10$$

Recall that

$$\begin{aligned} X &= X(\phi, \lambda) , \\ Y &= Y(\phi, \lambda) , \\ Z &= Z(\phi, \lambda) , \end{aligned} \qquad 3\text{-}11$$

and thus the differentials in 3-10 are

$$\begin{aligned} dX &= \frac{\partial X}{\partial \phi} d\phi + \frac{\partial X}{\partial \lambda} d\lambda , \\ dY &= \frac{\partial Y}{\partial \phi} d\phi + \frac{\partial Y}{\partial \lambda} d\lambda , \\ dZ &= \frac{\partial Z}{\partial \phi} d\phi + \frac{\partial Z}{\partial \lambda} d\lambda . \end{aligned} \qquad 3\text{-}12$$

Squaring 3-12 and substituting into 3-10 yields an expression for the changes in arc length on a surface as a function of changes in curvilinear coordinates, the result is

$$\boxed{dS^2 = Ed\phi^2 + 2Fd\phi d\lambda + Gd\lambda^2 ,} \qquad 3\text{-}13$$

where

$$\boxed{E = \left(\frac{\partial X}{\partial \phi}\right)^2 + \left(\frac{\partial Y}{\partial \phi}\right)^2 + \left(\frac{\partial Z}{\partial \phi}\right) ,} \qquad 3\text{-}14$$

$$\boxed{F = \frac{\partial X}{\partial \phi}\frac{\partial X}{\partial \lambda} + \frac{\partial Y}{\partial \phi}\frac{\partial Y}{\partial \lambda} + \frac{\partial Z}{\partial \phi}\frac{\partial Z}{\partial \lambda} ,}$$

$$\boxed{G = \left(\frac{\partial X}{\partial \lambda}\right)^2 + \left(\frac{\partial Y}{\partial \lambda}\right)^2 + \left(\frac{\partial Z}{\partial \lambda}\right)^2 .}$$

E, F, and G are known as the <u>Gaussian fundamental quantities</u> for a surface. Their specific use is in Sections 4.2 and 4.3, which respectively deal with the definition of the scale factor and condition for conformality.

For the ellipsoid (from 3-4a and 4-14)

$$E = M^2$$

$$F = 0 \qquad 3\text{-}15$$

$$G = N^2 \cos^2\phi$$

where

$$M = \frac{a(1-e^2)}{(1-e^2 \sin^2\phi)^{3/2}}$$

is the radius of curvature of the meridian of the ellipsoid. The differential arc length squared is

$$\boxed{dS^2 = (Md\phi)^2 + (N \cos\phi d\lambda)^2 ,} \qquad 3\text{-}16$$

and a useful form is

$$\boxed{dS^2 = N^2 \cos^2\phi \left[\frac{M^2}{N^2} \sec^2\phi \; d\phi^2 + d\lambda^2\right] .} \qquad 3\text{-}17$$

3.3 Angle Between Parametric Curves

In this section we treat the description of the angle between curves in terms of the Gaussian fundamental quantities. In particular the angle between meridians and parallels is described.

To do this we need the evaluation of the Cartesian total differentials (equation 3-12) for $\phi = C_1$ and $\lambda = C_2$(Figure 3-2).

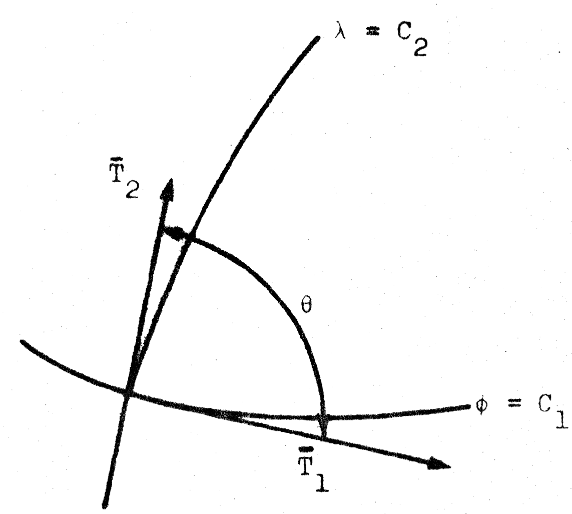


Figure 3-2. Angle Between a Meridian and a Parallel

The result is tabulated in Table 3-1.

Also needed is the evaluation of the total differential of the length of a curve S (equation 3-13). For $\phi = C_1$ and $d\phi = 0$,

$$dS_\phi = \sqrt{G}\, d\lambda \quad , \qquad \text{3-13a}$$

and for $\lambda = C_2$, $d\lambda = 0$,

$$dS_\lambda = \sqrt{E}\, d\phi \quad . \qquad \text{3-13b}$$

	$\phi = C_1$	$\lambda = C_2$
dX	$\frac{\partial X}{\partial \lambda} d\lambda$	$\frac{\partial X}{\partial \phi} d\phi$
dY	$\frac{\partial Y}{\partial \lambda} d\lambda$	$\frac{\partial Y}{\partial \phi} d\phi$
dZ	$\frac{\partial Z}{\partial \lambda} d\lambda$	$\frac{\partial Z}{\partial \phi} d\phi$

Table 3-1. Evaluation of Cartesian Total Differentials

The angle θ between the meridian and parallel is the angle between the two tangent vectors $\bar{T}_1$ and $\bar{T}_2$ (Figure 3-2), and is computed from

$$\cos \theta = J_1 J_2 + K_1 K_2 + L_1 L_2 \quad , \qquad 3\text{-}18$$

where J, K, and L are direction cosines of the two vectors. The direction cosines are (using 3-13a and 3-13b)

$$J_1 = \frac{dX}{dS_\phi} = \frac{\partial X}{\partial \lambda} \cdot \frac{1}{\sqrt{G}} \quad ,$$

$$K_1 = \frac{dY}{dS_\phi} = \frac{\partial Y}{\partial \lambda} \cdot \frac{1}{\sqrt{G}} \quad , \qquad 3\text{-}18a$$

$$L_1 = \frac{dZ}{dS_\phi} = \frac{\partial Z}{\partial \lambda} \cdot \frac{1}{\sqrt{G}} \quad ,$$

$$J_2 = \frac{dX}{dS\lambda} = \frac{\partial X}{\partial \phi} \cdot \frac{1}{\sqrt{E}} ,$$

$$K_2 = \frac{dY}{dS\lambda} = \frac{\partial Y}{\partial \phi} \cdot \frac{1}{\sqrt{E}} , \qquad 3\text{-}18b$$

$$L_2 = \frac{dZ}{dS\lambda} = \frac{\partial Z}{\partial \phi} \cdot \frac{1}{\sqrt{E}} .$$

After substitution of 3-18a and 3-18b into 3-18, we get

$$\cos\theta = \left[\frac{\partial X}{\partial \lambda}\frac{\partial X}{\partial \phi} + \frac{\partial Y}{\partial \lambda}\frac{\partial Y}{\partial \phi} + \frac{\partial Z}{\partial \lambda}\frac{\partial Z}{\partial \phi}\right]\frac{1}{\sqrt{EG}} , \qquad 3\text{-}19$$

and noting that the term in brackets is the Gaussian fundamental quantity F, the final result is achieved, that is

$$\boxed{\cos\theta = \frac{F}{\sqrt{EG}} .} \qquad 3\text{-}20$$

The above equation is valid for any surface (e.g. sphere, ellipsoid, plane). It defines the angle between a meridian and a parallel. It can be used to determine the value of an angle or to help enforce a certain value for θ in the way of a condition.

Applying it to the ellipsoid simply tells us that $\theta = 90^\circ$, since $F = 0$, $E \neq 0$, and $G \neq 0$ (equation 3-15).

4. CONFORMAL PROJECTIONS IN GENERAL

This section describes the general problem of conformal mapping of the ellipsoid surface onto the plane. The expressions developed are applicable to all conformal map projections alike. Discussed immediately below are the isometric plane, scale factor definition, condition for conformality, scale factor evaluation, geometry of projected curves, and meridian convergence.

4.1 Isometric Plane

We are concerned with three surfaces in the subject of conformal map projections - the ellipsoid surface which is to be mapped, the map plane surface which is the surface where the mapping takes place, and the isometric plane surface which is the intermediate plane surface through which all the conformal map projections are derived.

The isometric plane is mathematically described by first considering the expression (equation 3-17) for the total differential of the length of a curve on the ellipsoid surface.

Recall

$$dS^2 = N^2\cos^2\phi \; [\frac{M^2}{N^2} \sec^2\phi \; d\phi^2 + d\lambda^2] \quad . \qquad 4\text{-}1$$

By defining the total differential of a new quantity q as

$$dq = \frac{M}{N} \sec \phi \; d\phi \quad , \qquad 4\text{-}2$$

4-1 can be written as

$$dS^2 = N^2 \cos^2\phi \, [dq^2 + d\lambda^2] \quad . \qquad 4\text{-}3$$

By inspection one can see that the term in brackets in 4-3 resembles a distance element squared on some sort of plane surface. This plane surface is called the <u>isometric plane</u>. The two coordinate values on this plane are the geodetic longitude λ and <u>isometric latitude q</u>.

An expression for the isometric latitude q, in terms of the geodetic latitude ϕ, is obtained by integrating 4-2, that is

$$\int_o^q dq = \int_o^\phi \frac{M}{N} \sec \phi \, d\phi \quad , \qquad 4\text{-}4$$

where

$$M = \frac{a(1-e^2)}{(1-e^2 \sin^2\phi)^{3/2}} \quad , \qquad 4\text{-}5$$

$$N = \frac{a}{(1-e^2 \sin^2\phi)^{1/2}} \quad . \qquad 4\text{-}6$$

Continuing the integration we get

$$\int_o^q dq = \int_o^\phi \frac{(1-e^2)\, d\phi}{(1-e^2 \sin^2\phi) \cos\phi} = \int_o^\phi \frac{(1-e^2)(\cos^2\phi+\sin^2\phi)}{(1-e^2 \sin^2\phi)\cos\phi(\cos^2\phi+\sin^2\phi)} d\phi \qquad 4\text{-}7$$

$$= \int_o^\phi \frac{d\phi}{\cos \phi} - e^2 \int_o^\phi \frac{\cos \phi \, d\phi}{(1-e^2 \sin^2\phi)} \quad . \qquad 4\text{-}8$$

The first integral yields

$$\int_o^\phi \frac{d\phi}{\cos\phi} = \ln \left[\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right] \quad , \qquad 4\text{-}9$$

where $\ln$ denotes the natural logarithm.

To evaluate the second integral let

$$\sin \phi = p, \quad \cos \phi \, d\phi = dp \,, \tag{4-10}$$

then

$$e^2 \int \frac{dp}{1-e^2p^2} = \frac{e}{2} \ln \left[\frac{1 + ep}{1 - ep}\right]$$

$$= \ln \left[\frac{1 + e \sin \phi}{1 - e \sin \phi}\right]^{e/2} . \tag{4-11}$$

Combining the results of the first two integrals gives

$$q = \ln \left[\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\right] - \ln \left[\frac{1 + e \sin \phi}{1 - e \sin \phi}\right]^{e/2} ,$$

$$\boxed{q = \ln \left[\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \left(\frac{1 - e \sin \phi}{1 + e \sin \phi}\right)^{e/2} \right] ,} \tag{4-12}$$

where $e^2 = (a^2-b^2)/a^2$.

Thus we see that there is a value of q for each value of ϕ. Listed in Table 4-1 are pairs of values for the isometric and geodetic latitudes from ϕ equals $0°$ to $89°$. Note that the value of q is less than ϕ up to about $\phi = 11°$, and is greater beyond about $\phi = 12°$; At $\phi = 89°$, q is several times greater.

The geometric interpretation of the situation on the isometric plane is that meridians are parallel with constant spacing, while the spacings of the parallels vary. In fact, we witness for example, large spacings between the northern or southern parallels (Figure 4-1).

There is the task of computing the geodetic latitude ϕ from the isometric latitude q. Since 4-12 cannot be inverted, that is, ϕ expressed explicitly as a function of q, some other method is needed. We give below an iterative method of conversion of the isometric latitude

Geodetic Latitude (ϕ)	Isometric Latitude (q)
1	0.9934132219
10	9.9851128986
11	10.9956288708
12	12.0096232035
20	20.2888725073
30	31.2726570656
40	43.4668126053
50	57.6161578380
60	75.1262119163
70	99.0738773214
75	115.8041916752
80	139.2112650896
85	179.0306399306
86	191.8283989049
87	208.3211451844
88	231.5595670367
89	271.2781638574

Table 1. Isometric and Geodetic Latitudes (units-degrees)

Everest Ellipsoid ($1/f = 300.8017$
$a = 6377276.345$

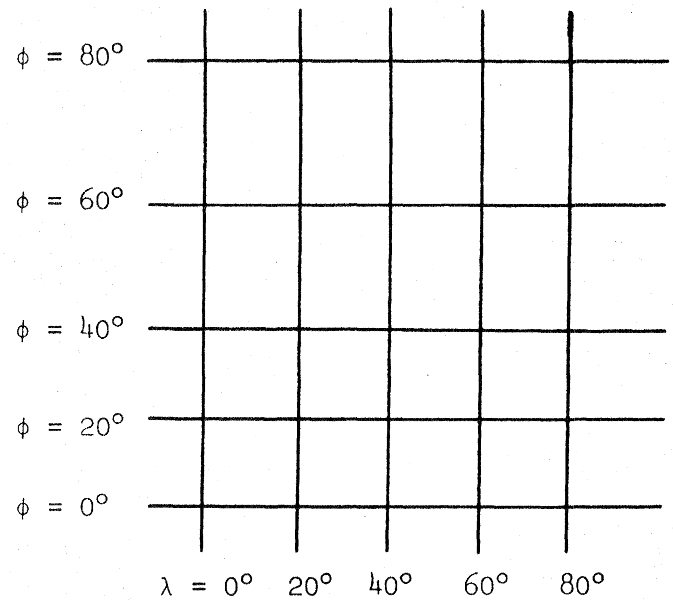


Figure 4-1.
Plot of Meridians and Parallels on the Isometric Plane.

into geodetic latitude. The iteration method employed is that of Newton-Raphson [Conte and Boor 1972; Wells 1971].

A summary of the method follows. Given the non-linear equation $f(x) = 0$, the steps for the solution are:

(1) Select an approximate solution p_o;

(2) Evaluate the iterative improved solutions p_n from

$$p_n = g(p_{n-1});\ n = 1, 2, \ldots, \qquad 4\text{-}13$$

where

$$g(x) = x - \frac{f(x)}{f'(x)}\ ; \qquad 4\text{-}14$$

(3) Stop the iteration when $|p_n - p_{n-1}| < \varepsilon$, where ε is a predetermined accuracy limit (say 10^{-6} for single precision on the IBM 360, or 10^{-12} for double precision);

(4) The solution to $f(x) = 0$ is then $x = p_n$, and is accurate to within $\pm \varepsilon$.

The application of the above to the isometric latitude q (equation 4-12) follows. Recall

$$q = \ln \left[\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\left(\frac{1 - e \sin \phi}{1 + e \sin \phi}\right)^{e/2}\right] \qquad 4\text{-}15$$

and

$$q = \frac{1}{2}\left[\ln(1+\sin\phi) - \ln(1-\sin\phi) + e\ln(1 - e\sin\phi) - \right.$$

$$\left. - e\ln(1 + e\sin\phi)\right] \,. \qquad 4\text{-}16$$

Our function f(x) that equals zero is

$$f(\phi) = 0 \;,$$

$$\boxed{f(\phi) = \frac{1}{2}\left[\ln(1+\sin\phi) - \ln(1-\sin\phi) + e\ln(1-e\sin\phi) - e\ln(1+e\sin\phi)\right] - q = 0 \;,} \qquad 4\text{-}17$$

and

$$\boxed{f'(\phi) = \frac{df(\phi)}{d\phi} = \frac{(1-e^2)}{(1-e^2\sin^2\phi)\cos\phi} \;.} \qquad 4\text{-}18$$

(1) For an initial approximation (ϕ_o), assume a spherical case in 4-15, that is e = 0. Then

$$q = \ln\left[\tan\left(\frac{\pi}{4} + \frac{\phi_o}{2}\right)\right] \;, \qquad 4\text{-}19$$

and inverting the equation;

$$\phi_o = 2\tan^{-1}\left[\exp(q)\right] - \frac{\pi}{2} \;. \qquad 4\text{-}20$$

(2) Evaluate the iterative improved solutions ϕ_n from

$$\phi_n = g(\phi_{n-1}),\; n = 1, 2, \ldots \qquad 4\text{-}21$$

where in general

$$g(\phi) = \phi - \frac{f(\phi)}{f'(\phi)} , \qquad 4\text{-}22$$

namely

$$\boxed{\begin{array}{c} \phi_n = \phi_{n-1} - \dfrac{f(\phi_{n-1})}{f'(\phi_{n-1})} , \\ \hline \phi_n = \phi_{n-1} - (\Delta\phi)_{n-1} \quad . \end{array}} \qquad 4\text{-}23$$

(3) Stop iteration when $|(\Delta\phi)|_{n-1} < \varepsilon$. (For $\varepsilon = 10^{-12}$ radians, convergence is achieved within about three iterations for $0^\circ < \phi < 89^\circ$).

4.2 Scale Factor Definition

The scale factor describes, at each point on the map projection, the amount of distortion in length. This distortion is of course due to maintaining conformality and fulfilling other conditions prescribed for the projection. It should not be confused with the "line scale" (discussed in Section II), which is concerned with the scale distortion over a finite length of line.

We now focus our attention on two surfaces - the ellipsoid and the map plane surfaces. It was proved in Section 3.3 that the meridians and parallels on the ellipsoid surface are perpendicular. Since the projection is conformal, these two curves are also perpendicular on the map plane. In mathematical language, the Gaussian fundamental quantity F equals zero. It follows then from 3-13 that the square of the differential change in the length of a curve on the ellipsoid surface is

$$dS_1^2 = M^2 d\phi^2 + N^2 \cos^2\phi \; d\lambda^2 \;, \tag{4-24}$$

and on the map surface*the corresponding equation is

$$ds_2^2 = dx^2 + dy^2 \;. \tag{4-25}$$

The <u>scale factor</u> k is defined through the ratio

$$\boxed{k^2 = \frac{ds_2^2}{dS_1^2} = \frac{dx^2 + dy^2}{M^2 d\phi^2 + N^2\cos^2\phi \; d\lambda^2} \;.} \tag{4-26}$$

We now define the scale factor for the isometric and ellipsoid surfaces with the aid of the following two parametric equations:

$$x = f_1(q, \lambda) \;,$$

and 4-27

$$y = f_2(q, \lambda) \;.$$

Note the replacement of ϕ by q in the parametric equations which formerly related the map coordinates x and y with the ellipsoid coordinates ϕ and λ.

The linear distance element squared on the map plane in terms of q and λ is (see equation 3-13)

$$ds_2^2 = e \; dq^2 + 2f \; dq \; d\lambda \;\; + q d\lambda^2 \;\;, \tag{4-28}$$

where

$$e = \left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2 \tag{4-29}$$

$$f = \left(\frac{\partial x}{\partial q}\right)\left(\frac{\partial x}{\partial \lambda}\right) + \left(\frac{\partial y}{\partial q}\right)\left(\frac{\partial y}{\partial \lambda}\right) \;, \tag{4-30}$$

$$g = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 \;. \tag{4-31}$$

---

* Note the use of lower case x, y for the coordinates on the map plane.

If angles on the ellipsoid are to be preserved on the map plane, then $f = 0$, and

$$ds_2^2 = e\,dq^2 + g\,d\lambda^2 \; . \qquad 4\text{-}32$$

The scale factor is now defined through the ratio

$$\boxed{k^2 = \frac{ds_2^2}{dS_1^2} = \frac{e\,dq^2 + g\,d\lambda^2}{N^2\cos^2\phi(dq^2 + d\lambda^2)} \; ,} \qquad 4\text{-}33$$

where the denominator is the square of the differential change in the length of a curve on the ellipsoid surface in terms of the isometric latitude (see equation 4-3). Note the usage of the isometric latitude - first on the mapping surface (numerator 4-33) and secondly on the ellipsoid surface (denominator 4-33).

4.3 Condition for Conformality

In this section we derive the equation from which the definition of conformality is obtained. This condition is then represented by the Cauchy-Riemann equations, in terms of the map coordinates x and y, and the isometric latitude q and geodetic longitude λ.

From Figure 4-2,

$$\tan\alpha = \frac{N\cos\phi}{M\,d\phi}\,d\lambda \; , \qquad 4\text{-}34$$

where the newly introduced quantity α is the geodetic azimuth of the diagonal of the differentially small figure. Recall the definition of

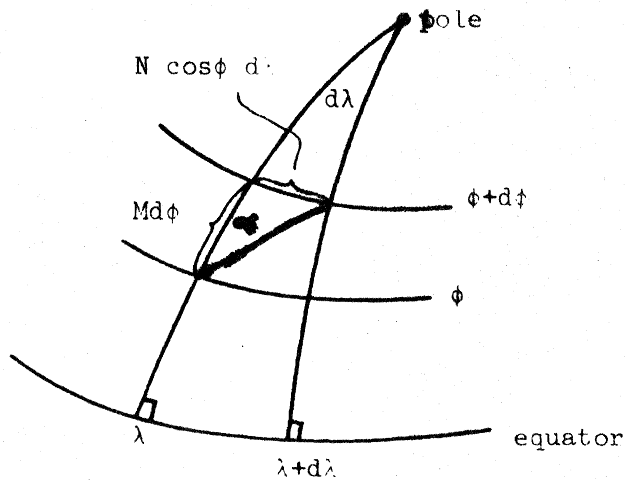


Figure 4-2. Differential Area on the Ellipsoid Surface

dq (equation 4-2)

$$dq = \frac{M\ d\phi}{N\ \cos\phi} \quad . \qquad 4\text{-}35$$

Substituting 4-35 into 4-34 yields

$$\tan\alpha = \frac{d\lambda}{dq} \quad , \qquad 4\text{-}36$$

or

$$d\lambda = \tan\alpha\ dq \ . \qquad 4\text{-}37$$

Now substituting 4-37 into the definition of the scale factor (equation 4-33), we get an expression for the scale factor as a function of the direction, namely

$$k^2 = \frac{e\cos^2\alpha + g\sin^2\alpha}{N^2\cos^2\phi} \ . \qquad 4\text{-}38$$

We know, however that for conformal projections the scale factor cannot be a function of the direction if angles are to be preserved.

Therefore, to force 4-38 to satisfy this condition, we choose e equal to g in 4-38. Thus

$$\boxed{k^2 = \frac{e}{N^2\cos^2\phi} = \frac{g}{N^2\cos^2\phi}\ .} \tag{4-38a}$$

Now we can describe conformal projections in terms of Gaussian fundamental quantities, namely

$$\boxed{f = 0\ ,} \tag{4-39}$$

and

$$\boxed{e = g\ .} \tag{4-40}$$

Recall, that the first was a result of requiring the meridians and parallels to intersect at 90° on the map plane (see Section 3.3).

The Cauchy-Riemann equations can be derived by application of 4-39 and 4-40 along with equations 4-29, 4-30, and 4-31, which respectively define e, f and g in terms of x, y, q and λ. Firstly

$$f = 0\ ,$$

$$\frac{\partial x}{\partial q}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial q}\frac{\partial y}{\partial \lambda} = 0\ , \tag{4-41}$$

and

$$\frac{\partial y}{\partial \lambda} = \frac{-\dfrac{\partial x}{\partial q}\dfrac{\partial x}{\partial \lambda}}{\dfrac{\partial y}{\partial q}}\ . \tag{4-42}$$

Secondly

$$e = g$$

$$\left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2 = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2\ . \tag{4-43}$$

Substituting 4-42 in 4-43, yields

$$\left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2 = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \frac{\left(-\frac{\partial x}{\partial q}\right)^2\left(\frac{\partial x}{\partial \lambda}\right)^2}{\left(\frac{\partial y}{\partial q}\right)^2} , \qquad 4\text{-}44$$

$$\left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2 = \frac{\left(\frac{\partial x}{\partial \lambda}\right)^2}{\left(\frac{\partial y}{\partial q}\right)^2}\left[\left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2\right] . \qquad 4\text{-}45$$

These are only two possible cases for which the above equation is satisfied, namely

$$\left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2 = 0 \qquad 4\text{-}46$$

and

$$\left(\frac{\partial x}{\partial \lambda}\right)^2 = \left(\frac{\partial y}{\partial q}\right)^2 . \qquad 4\text{-}47$$

From 4-47, we get

$$\frac{\partial x}{\partial \lambda} = \pm \frac{\partial y}{\partial q} . \qquad 4\text{-}48$$

Taking the positive root and substituting into 4-41 ($f = 0$), results in

$$\frac{\partial x}{\partial q} = - \frac{\partial y}{\partial \lambda} . \qquad 4\text{-}49$$

The above equation, along with the positive root of 4-48, are known as the Cauchy-Riemann equations, which we label as set number one:

$$\boxed{\begin{array}{c} \dfrac{\partial x}{\partial \lambda} = \dfrac{\partial y}{\partial q} , \\ \hline \dfrac{\partial x}{\partial q} = - \dfrac{\partial y}{\partial \lambda} . \end{array}} \qquad 4\text{-}50$$

By choosing the negative root of 4-48, we arrive at the alternative set of Cauchy-Riemann equations, namely

$$\boxed{\begin{array}{l} \dfrac{\partial x}{\partial \lambda} = -\dfrac{\partial y}{\partial q} \ , \\ \hline \dfrac{\partial x}{\partial q} = \dfrac{\partial y}{\partial \lambda} \ . \end{array}} \qquad 4\text{-}51$$

As stated earlier these equations represent the conformality condition and are used to either help derive the mapping function f_1 or f_2 in

$$x + iy = f_1 \, (\lambda + iq) \ . \qquad 4\text{-}52$$

and

$$\lambda + iq = f_2 \, (x + iy) \ , \qquad 4\text{-}53$$

or corroborate whether these functions, as determined by other means, result in a conformal transformation.

4.4 Scale Factor Evaluation

We now can evaluate the general expression for the scale factor (equation 4-26). Recall

$$k^2 = \frac{ds_2^2}{dS_1^2} = \frac{dx^2 + dy^2}{N^2 \cos^2 \phi (dq^2 + d\lambda^2)} \ . \qquad 4\text{-}54$$

Also, recall from complex algebra (equation 2-40) that the distance squared is given by

$$dw \cdot d\bar{w} = du^2 + dv^2 = f'(z) \ f'(\bar{z}) \ (dx^2 + dy^2) \ , \qquad 4\text{-}55$$

(general-notation). In map projection-notation and for the map plane, the distance element squared is

$$ds_2^2 = dx^2 + dy^2 = f'(\lambda+iq)\ f'(\lambda-iq)(d\lambda^2+dq^2). \qquad 4\text{-}56$$

The scale factor expression, in terms of derivatives of complex functions, is

$$k^2 = \frac{f'(\lambda+iq)\ f'(\lambda-iq)\ (dq^2 + d\lambda^2)}{N^2\ \cos^2\phi\ (dq^2 + d\lambda^2)}\ , \qquad 4\text{-}57$$

$$= \frac{f'(\lambda+iq)\ f'(\lambda-iq)}{N^2\cos^2\phi}\ . \qquad 4\text{-}58$$

The above expression is simplified by recalling that (equation 2-33)

$$f'(z)\ f'(\bar{z}) = (\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2 = (\frac{\partial v}{\partial y})^2 + (\frac{\partial u}{\partial y})^2\ , \qquad 4\text{-}59$$

(general-notation). In map projection-notation

$$z = \lambda + iq \quad ;$$

therefore

$$f'(\lambda+iq)\ f'(\lambda-iq) = (\frac{\partial x}{\partial \lambda})^2 + (\frac{\partial y}{\partial \lambda})^2\ , \qquad 4\text{-}60$$

$$= (\frac{\partial x}{\partial q})^2 + (\frac{\partial y}{\partial q})^2\ . \qquad 4\text{-}61$$

Substituting 4-60 and 4-61 into 4-58, yields the final result for the scale factor:

$$\boxed{k = \frac{\sqrt{(\frac{\partial x}{\partial \lambda})^2 + (\frac{\partial y}{\partial \lambda})^2}}{N\ \cos\ \phi}}\ , \qquad 4\text{-}62$$

$$\boxed{k = \frac{\sqrt{(\frac{\partial x}{\partial q})^2 + (\frac{\partial y}{\partial q})^2}}{N\ \cos\ \phi}}\ . \qquad 4\text{-}63$$

Note the two forms. For example, the first is used to evaluate the scale factor for the Transverse Mercator (since $x = x(\lambda)$), while the second is used for the Lambert Conformal Conic (since $x = x(q)$).

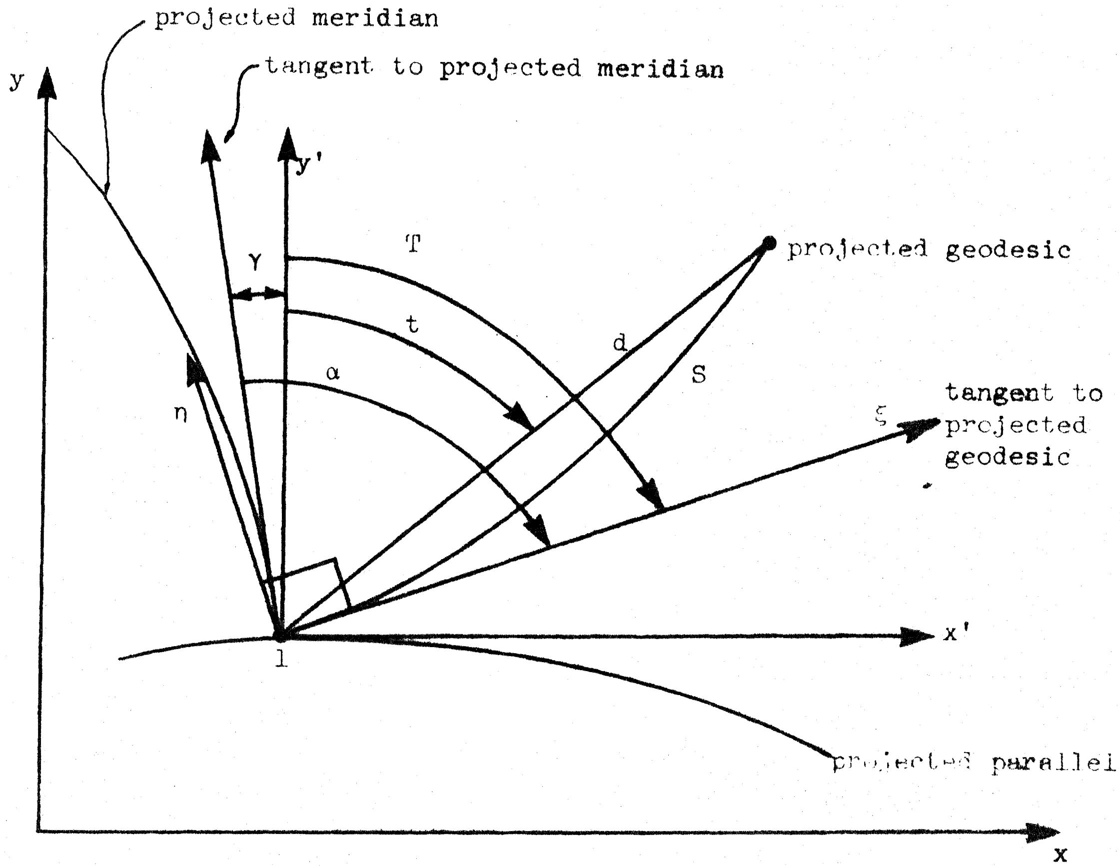
4.5 Geometry of Projected Curves

At this stage in the notes, it is necessary to describe the geometry of projected curves. Specifically, we show the configuration of meridians, parallels, and geodesics on the map plane, and their relationships to the map grid system. Also, numerous terms are defined for later use.

Consider a map plane, as depicted in Figure 4-3, upon which two points have been mapped (transformed) from the ellipsoid surface. The parallel and meridian passing through point 1, along with the geodesic from points 1 to 2, have also been mapped. Shown on the map are tangents to the latter two curves.

Three coordinate systems are shown: the map coordinate system (grid system) denoted by x and y; the translated map system denoted by x' and y'; and the geodesic system denoted by ξ and η. The latter system is orientated with the abscissa axis tangent to the projected geodesic. The ordinate axis of the grid system makes an angle of γ with the tangent to the projected meridian. This angle is called the meridian convergence.

The geodetic azimuth α of the projected geodesic is the clockwise angle from the tangent to the meridian, to the tangent to the projected geodesic. The grid azimuth T of the projected geodesic is the clockwise angle from the ordinate axis to the tangent to the projected geodesic.



x, y - map plane coordinates (grid coordinates)

α - geodetic azimuth of projected geodesic

T - grid azimuth of projected geodesic

t - grid azimuth of chord

γ - meridian convergence

S - length of projected geodesic on the map plane

d - chord length

Figure 4-3. Geometry of Projected Curves.

The grid azimuth t is the clockwise angle from the crdinate axis to the straight line (chord) connecting the two points. The chord length d is the straight line distance between points 1 and 2. The projected length S is the length of the geodesic on the map plane.

4.6 Meridian Convergence

In this section we derive a general expression for the meridian convergence in terms of general functions of the isometric latitude q and geodetic longitude λ.

From Figure 4-3, we define the angle β between the x and ξ abscissae axes (Figure 4-4). We can write

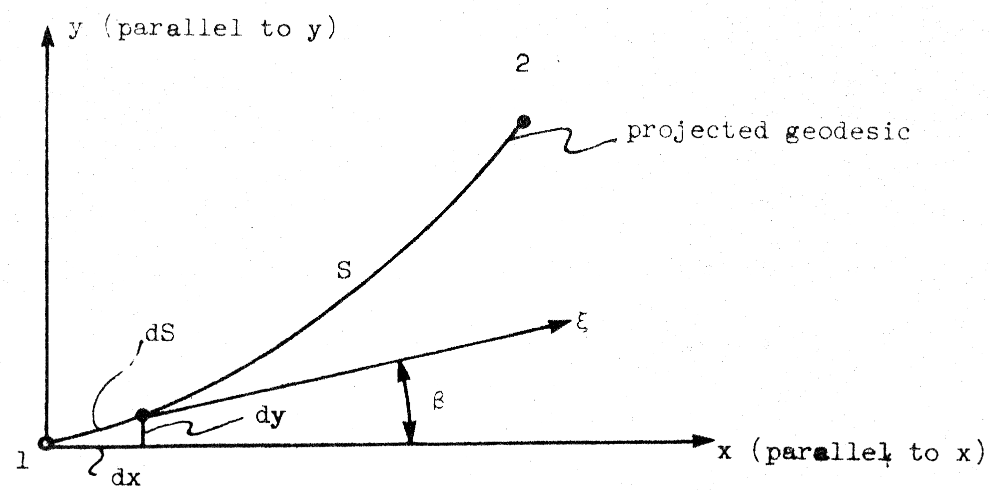


Figure 4-4. Angle of Geodesic

$$\tan \beta = \frac{dy}{dx} \, . \qquad \text{4-64}$$

The equation for the projected geodesic in terms of q and λ is

$$f(q, \lambda) = 0 \, , \qquad \text{4-65}$$

and the parametric equations are

$$x = x(q, \lambda),$$
$$y = y(q, \lambda). \qquad 4\text{-}66$$

The total differentials are

$$dx = \frac{\partial x}{\partial q}\, dq + \frac{\partial x}{\partial \lambda}\, d\lambda \ , \qquad 4\text{-}67$$

and

$$dy = \frac{\partial y}{\partial q}\, dq + \frac{\partial y}{\partial \lambda}\, d\lambda \ . \qquad 4\text{-}68$$

Substituting the above in 4-64, yields

$$\tan \beta = \frac{\frac{\partial y}{\partial q}\, dq + \frac{\partial y}{\partial \lambda}\, d\lambda}{\frac{\partial x}{\partial q}\, dq + \frac{\partial x}{\partial \lambda}\, d\lambda} \ , \qquad 4\text{-}69$$

and changing its form gives

$$\tan \beta = \frac{\frac{\partial y}{\partial q} + \frac{\partial y}{\partial \lambda}\frac{d\lambda}{dq}}{\frac{\partial x}{\partial q} + \frac{\partial x}{\partial \lambda}\frac{d\lambda}{dq}} \ . \qquad 4\text{-}70$$

Recall from 4-36, that

$$\frac{d\lambda}{dq} = \tan \alpha \cdot \qquad 4\text{-}71$$

Substituting 4-71 into 4-70, yields

$$\boxed{\tan \beta = \frac{\frac{\partial y}{\partial q} + \frac{\partial y}{\partial \lambda}\tan \alpha}{\frac{\partial x}{\partial q} + \frac{\partial x}{\partial \lambda}\tan \alpha} \ ,} \qquad 4\text{-}72$$

and

$$\boxed{\tan \alpha = -\frac{\frac{\partial y}{\partial q} - \frac{\partial x}{\partial q}\tan \beta}{\frac{\partial y}{\partial \lambda} - \frac{\partial x}{\partial \lambda}\tan \beta} \ .} \qquad 4\text{-}73$$

The above two equations are rather basic and need further development before an expression for the meridian convergence can be obtained.

Consider the case (Figure 4-5)

$$\gamma = \beta \tag{4-74}$$

and

$$\alpha = 90^\circ . \tag{4-75}$$

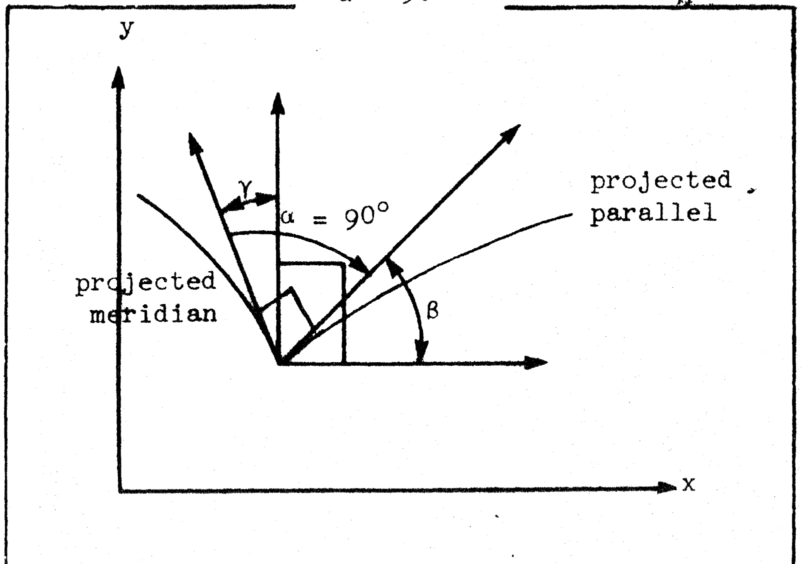


Figure 4-5. Meridian Convergence

Rewriting 4-72 to handle tan 90° = ∞, we get

$$\tan\beta = \frac{\frac{1}{\tan\alpha}\frac{\partial y}{\partial q} + \frac{\partial y}{\partial \lambda}}{\frac{1}{\tan\alpha}\frac{\partial x}{\partial q} + \frac{\partial x}{\partial \lambda}} . \tag{4-76}$$

Substituting 4-74 and 4-75 in 4-76, yields

$$\boxed{\tan\gamma = \frac{\frac{\partial y}{\partial \lambda}}{\frac{\partial x}{\partial \lambda}} .} \tag{4-77}$$

The Cauchy-Riemann equations

$$\frac{\partial y}{\partial q} = \frac{\partial x}{\partial \lambda} ,$$

$$\frac{\partial y}{\partial \lambda} = -\frac{\partial x}{\partial q} ,$$

allow us to write

$$\boxed{\tan\gamma = \frac{-\dfrac{\partial x}{\partial q}}{\dfrac{\partial y}{\partial q}}\ .} \tag{4-78}$$

Both of the above expressions are used to obtain the value of the meridian convergence for specific map projections. For example, the latter is used for the Lambert Conformal Conic, while the former is used in the Transverse Mercator.

5. MERCATOR PROJECTION

In about 1550, Mercator created this projection imperically as a result of attempting to have the loxodrome on the globe appear as a straight line on the map. About 40 years later, Wright gave the mathematical description of the projection.

The requirements for the Mercator projection are:

(1) The scale is true along the equator;

(2) The origin for the ordinate y is at the equator.

In this section we treat the direct problem, the scale factor, the meridian convergence, the appearance of the Mercator projection, the loxodrome, and the inverse problem.

5.1 Direct Problem

The direct problem is the determination of the map coordinates x and y from the geodetic coordinates ϕ and λ. In other words, we are to determine the mapping function f in

$$x + iy = f(\lambda + iq) \ . \qquad 5\text{-}1$$

Notice ϕ is replaced by q, as will be the case for all projections, since the isometric plane is employed as an intermediate mapping plane.

The first requirement implies that

$$x = a\lambda \ , \qquad 5\text{-}2$$

where a is the semi-major axis of the ellipsoid. The equator portion of the ellipsoid surface is mapped at its true size.

The second requirement implies a particular correspondence between three quantities at the equator, namely

$$\phi = q = y = 0 \ . \qquad 5\text{-}3$$

Under these conditions our complex mapping function becomes

$$\boxed{x + iy = a(\lambda + iq) \ .} \qquad 5\text{-}4$$

The above mapping function is conformal since the Cauchy-Riemann equations are satisfied, namely

$$\frac{\partial x}{\partial \lambda} = \frac{\partial y}{\partial q} = a \ ,$$

and

$$\frac{\partial x}{\partial q} = - \frac{\partial y}{\partial \lambda} = 0$$

This complex function can be programmed using complex arithmetic routines, or explicitly separated into real and imaginary parts to give

$$\boxed{\begin{array}{c} x = a\lambda \ , \\ \hline y = aq \ . \end{array}} \qquad 5\text{-}5$$

Note that the units of x and y are those of a, since the units of λ and q are radians.

5.2 Scale Factor

Recall one of the general equations (equation 4-62) for evaluating the scale factor:

$$k = \frac{\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2}}{N \cos \phi} \ . \qquad 5\text{-}6$$

The partial derivatives are evaluated by resorting to the parametric equations of the Mercator projection (equation 5-5). The result is

$$\frac{\partial x}{\partial \lambda} = a; \quad \frac{\partial y}{\partial \lambda} = 0 \ . \qquad 5\text{-}7$$

The scale factor becomes

$$\boxed{k = \frac{a}{N \cos \phi} \ .} \qquad 5\text{-}8$$

The scale distortion is zero on the equator and increases with latitude. At $\phi = 90^\circ$ the distortion is infinite which simply means the pole is represented as a line like a parallel.

5.3 Meridian Convergence

Recall the general expression (equation 4-77) for evaluating the meridian convergence:

$$\tan \gamma = \frac{\frac{\partial y}{\partial \lambda}}{\frac{\partial x}{\partial \lambda}} \ . \qquad 5\text{-}9$$

For the Mercator projection

$$\tan \gamma = \frac{0}{a} = 0 \ ,$$

and

$$\boxed{\gamma = 0 \ .}$$

That is, the grid ordinate axis coincides with the tangent to the projected meridian.

5.4 Appearance of the Mercator Projection

The appearance of the Mercator projection is deduced from the several factors listed below (see Figure 5-1).

Appearance:

$$x = a\lambda \qquad q = \ln \left\{ \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{\frac{e}{2}} \right\}$$

$$y = aq \qquad @ \ \phi = 90 \quad q = \infty$$

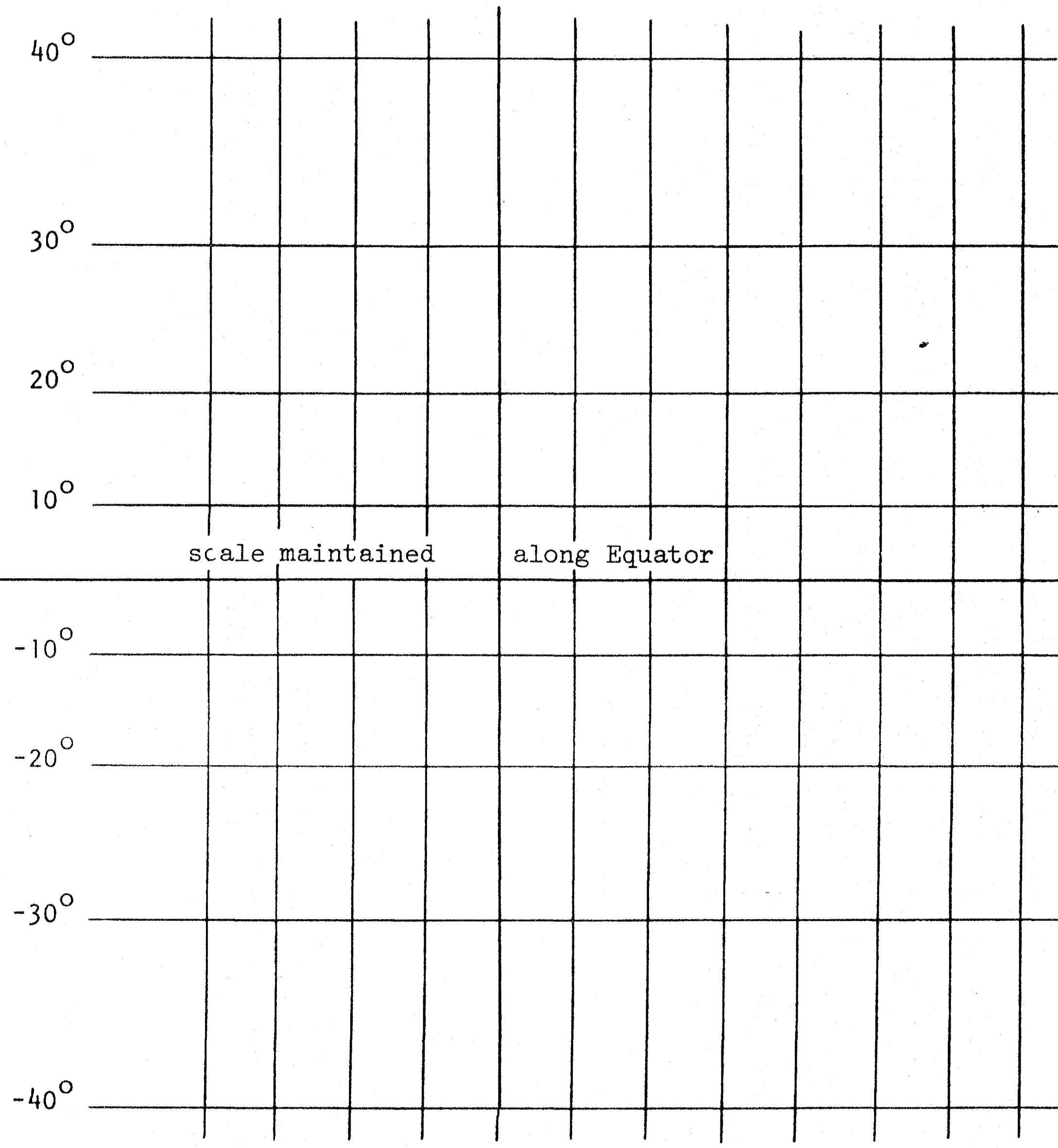


Figure 5-1. Appearance of the Mercator Projection

(1) $\gamma = 0$;

(2) $k = 1$ along equator;

(3) meridians and parallels intersect at $90°$;

(4) meridians are equally spaced;

(5) the spacings of parallels increase as one goes away from the equator.

5.5 The Loxodrome

Let us now show that a loxodrome or rumb-line is a straight line on the Mercator projector. Recall that a loxodrome is a curve on the surface of the earth (more strictly on the ellipsoid surface) that meets the meridians at the same angle that is constant azimuth (Figure 5-2).

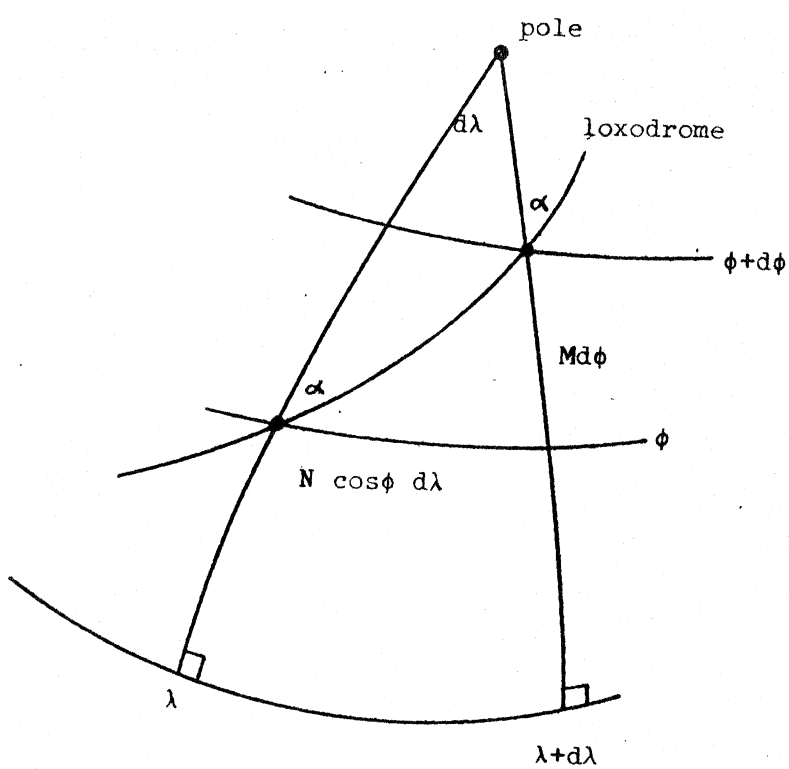


Figure 5-2. Loxodrome

From the figure

$$\tan \alpha = \frac{N \cos \phi}{M} \frac{d\lambda}{d\phi} \qquad 5\text{-}10$$

and

$$d\lambda = \tan \alpha \frac{M}{N} \sec \phi \, d\phi \, . \qquad 5\text{-}11$$

Integrating the above, yields

$$\lambda - \lambda_o = \tan \alpha \, q$$

or

$$\lambda = q \tan \alpha + \lambda_o \quad , \qquad 5\text{-}12$$

which is the basic equation for the loxodrome on the surface of ellipsoid in terms of q. Given the mapping equations

$$q = \frac{y}{a} \, , \qquad \lambda = \frac{x}{a} \, , \qquad 5\text{-}13$$

and substituting these into 5-12, yields the equation of the loxodrome on the map surface, namely

$$\boxed{x = y \tan \alpha + a\lambda_o \, .} \qquad 5\text{-}14$$

This is nothing else but the equation of a straight line, where $\tan \alpha$ is the slope and $a\lambda_o$ is the x-intercept.

5.6 Inverse Problem

The inverse problem is to compute ϕ and λ given x and y. The steps for achieving this are:

(1) Compute the longitude λ from

$$\boxed{\lambda = \frac{x}{a} \; ;} \qquad 5\text{-}15$$

(2) Compute the isometric latitude q from

$$\boxed{q = \frac{y}{a} \quad ;} \qquad 5\text{-}16$$

(3) Compute the value of the geodetic latitude ϕ corresponding to q by the iterative method described in section 4.1 .

6. TRANSVERSE MERCATOR PROJECTION

This very widely used conformal map projection was invented by Johann Heinrich Lambert in 1772. His development was based on elementary considerations (probably a geometric approach). Some 50 years later, Gauss gave an analytical derivation for the projection, showing that it was a special case of the conformal mapping of one surface onto another. Then L. Kruger, in 1912, completed the development of the Transverse Mercator projection by developing the formulae further in order that they would be suitable for numerical calculations (similar to those of 6-25 and 6-26). What a beautiful example of creating - formalizing - implementing, all three processes taking over a century.

The requirements for the Transverse Mercator projection are:

(1) The scale is true along the central meridian;

(2) The origin of the ordinate y is at the equator;

(3) The origin of the abscissa x is at central meridian.

Treated in this section are: the direct problem; the scale factor in terms of geodetic coordinates; the meridian convergence in terms of geodetic coordinates; the appearance of the Transverse Mercator projection; the inverse problem; the scale factor in terms of map coordinates; the meridian convergence in terms of map coordinates; the Universal Transverse Mercator Projection; and the Transverse Mercator in three degree zones.

6.1 Direct Problem

Given the general complex mapping function

$$x + iy = f(\lambda + iq) \ , \qquad (6\text{-}1)$$

the problem is to determine the specific mapping function f which fulfills the requirements stated above.

The third requirement implies that for the central meridian

$$x = \lambda = 0 \quad , \tag{6-2}$$

thus for the central meridian (from equation 6-1)

$$iy = f(iq) \quad ,$$

$$y = f(q) \quad . \tag{6-3}$$

On the other hand, the first requirement implies that the ordinate value y on the projection be equivalent to the length of the meridian S_ϕ on the ellipsoid (Figure 6-1).

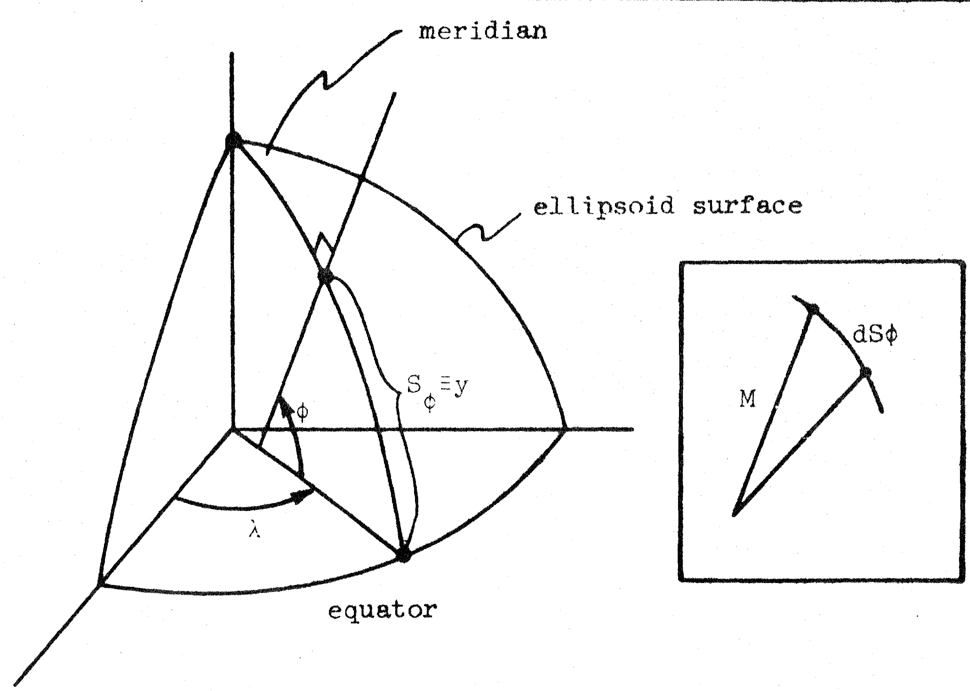


Figure 6-1. Length of Meridian on Ellipsoid

The length of the meridian, from the equator to the point in question, is given by the integral

$$y = S_\phi = \int_0^\phi M d\phi \quad . \tag{6-4}$$

Appendix I contains the solution of this integral.

Further,

$$y = \int_{o}^{\phi} N \cos \phi \, dq \ , \qquad 6\text{-}5$$

$$y = f(q) \ ,$$

since, through the definition of the isometric latitude,

$$Md\phi = N \cos \phi \, dq \ . \qquad 6\text{-}6$$

We thus have been able to determine the function f in 6-1, but only for points on the meridian.

The next step is to generalize the approach for points away from the central meridian, but still close - say within 3° or so. For these points, the abscissa value will be non-zero and the ordinate will not be equal to the length of the meridian from the equator to the point.

To help solve this problem, we approximate the mapping function

$$x + iy = f(\lambda + iq)$$

by a Taylor series, choosing some arbitrary point with coordinates λ_o and q_o as the point of expansion (Figure 6-2). First,

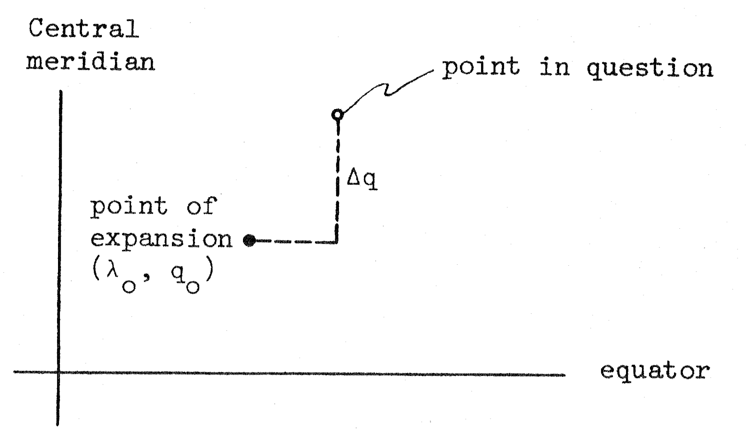


Figure 6-2. Arbitrary Point of Expansion

$$x + iy = f(\lambda + iq) \quad ,$$
$$= f[\lambda_o + \Delta\lambda + i(q_o + \Delta q)] \quad ,$$
$$= f[(\lambda_o + iq_o) + (\Delta\lambda + i\Delta q)] \quad ,$$
$$= f(z_o + \Delta z) = f(z) \quad . \qquad 6\text{-}7$$

Then, approximating the complex function f(z) with a Taylor series

$$f(z) = f(z_o) + f^{I}(z_o)\Delta z + \frac{f^{II}(z_o)(\Delta z)^2}{2!} + \frac{f^{III}(z_o)(\Delta z)^3}{3!} + \ldots \qquad 6.8$$

or

$$x+iy = f(\lambda_o+iq_o) + f^{I}(\lambda_o+iq_o)(\Delta\lambda+i\Delta q) + f^{II}(\lambda_o+iq_o)\frac{(\Delta\lambda+i\Delta q)^2}{2!} \qquad 6\text{-}9$$

The above equations are made practical by choosing a different point of expansion. This time the point is chosen to have coordinates q of the point in question and to be on the central meridian (Figure 6-3). This implies

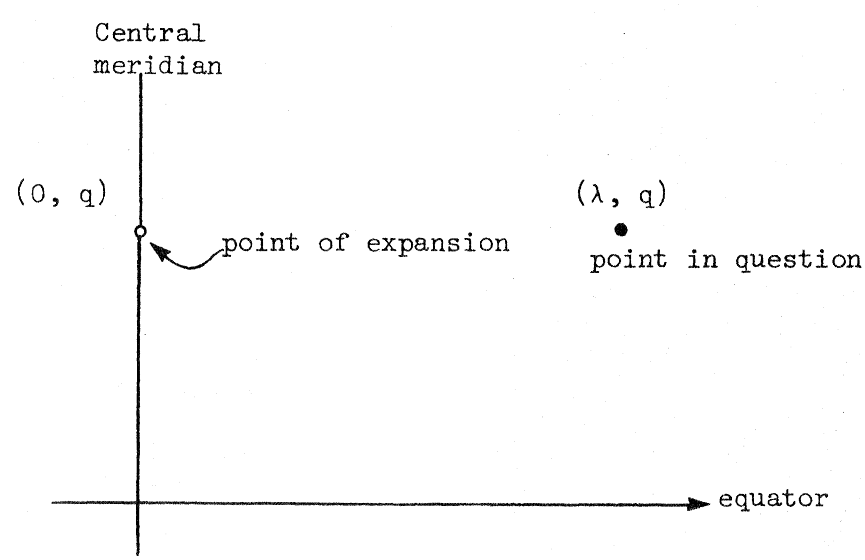


Figure 6-3. Transverse Mercator Point of Expansion

$$\lambda_o = 0 \ ,$$ 6-10

$$\Delta q = 0 \ ,$$

and

$$q = q_o + \Delta q = q_o \ ,$$ 6-11

$$\lambda = \lambda_o + \Delta\lambda = \Delta\lambda \ .$$

Substituting 6-11 into 6-9, yields

$$x+iy = f(iq) + f^{I}(iq)\lambda + \frac{f^{II}(iq)\lambda^2}{2!} + \frac{f^{III}(iq)\lambda^3}{3!} + \ldots$$ 6-12

To evaluate the function and its derivatives, we return to the fact that the length of the meridian is (6-5)

$$S_\phi = f(q) = \int_o^\phi N \cos\phi \, dq \ .$$ 6-13

The function itself is

$$f(iq) = if(q) = i\int_o^\phi N \cos\phi \, dq \ .$$ 6-14

The first derivative is

$$f^{I}(iq) = f^{I}(z) = \frac{df(z)}{dz} = \frac{df(z)}{dq} \cdot \frac{dq}{dz} = \frac{idf(q)}{dq} \cdot \frac{dq}{dz}$$ 6-15

Since

$$z = iq$$ 6-16

$$q = \frac{1}{i} z \ ,$$

and

$$\frac{dq}{dz} = \frac{1}{i} = -i \ , \qquad 6\text{-}17$$

$$f^{I}(iq) = i\,\frac{df(q)}{dq} \cdot (-i) = \frac{df(q)}{dq} \qquad 6\text{-}18$$

$$\boxed{f^{I}(iq) = f'(q)} \qquad 6\text{-}19$$

The second derivative is

$$f^{II}(iq) = \frac{df^{I}(iq)}{dq} = \frac{d}{dz}\left\{\frac{df(q)}{dq}\right\}$$

$$= \frac{d}{dq}\left\{\frac{df(q)}{dq}\right\}\frac{dq}{dz}$$

$$\boxed{f^{II}(iq) = f^{II}(q)(-i) \ .} \qquad 6\text{-}20$$

The higher derivatives, are derived in an analogous manner, that is

$$\boxed{\begin{aligned} f^{III}(iq) &= -f^{III}(q) \ , \\ f^{IV}(iq) &= if^{IV}(q) \ , \\ f^{V}(iq) &= f^{V}(q) \ , \\ &\ \ \vdots \end{aligned}} \qquad 6\text{-}21$$

Substituting the evaluation of the derivatives into 6-12, yields

$$\boxed{\begin{aligned} x+iy = f(\lambda+iq) &= f(iq) + f'(q)\lambda + \\ &+ \frac{(-i)f^{II}(q)\lambda^2}{2!} - \frac{f^{III}(q)\lambda^3}{3!} + \\ &+ \frac{if^{IV}(q)\lambda^4}{4!} + \dots \ . \end{aligned}} \qquad 6\text{-}22$$

All that we have achieved up to this stage is to replace the complex derivatives with real ones (compare 6-22 with 6-12).

After separating into real and imaginary parts

$$\boxed{\begin{aligned} x &= \lambda f'(q) - \frac{f^{III}(q)\lambda^3}{3!} + \frac{f^{V}(q)\lambda^5}{5!} - \frac{f^{VII}(q)\lambda^7}{7!} + \dots , \\ y &= f(q) - \frac{f^{II}(q)\lambda^2}{2!} + \frac{f^{IV}(q)\lambda^4}{4!} - \frac{f^{VI}(q)\lambda^6}{6!} + \dots . \end{aligned}} \qquad \text{6-22a}$$

x and y can be solved for in the above after specifying the values of λ (longitude positive east of point relative to the central meridian, i.e. $\lambda = \lambda_{CM} - \lambda_{point}$) and q (computed from ϕ). The only problem left to be solved is the evaluation of $f(iq)$, $f^{I}(iq)$, etc.

We have to evaluate the real derivatives still further to get an expression that is suitable for computations.

The first derivative is (from 6-5)

$$f^{I}(q) = \frac{dS_\phi}{dq} = \frac{d\{f(q)\}}{dq} = N \cos \phi .$$

The second derivative is

$$f^{II}(q) = [\frac{dN}{d\phi} \cos \phi - N \sin \phi] \frac{d\phi}{dq} ,$$

where, from the definition of the isometric latitude,

$$\frac{d\phi}{dq} = \frac{N}{M} \cos \phi ,$$

$$\frac{dN}{d\phi} = (N-M) \tan \phi \quad , \tag{6-23}$$

so

$$f''(q) = \frac{-N}{2} \sin 2\phi \; . \tag{6-24}$$

The higher derivatives are computed in Thomas [1952]. Substituting these into 6-22a yields

$$\begin{aligned} \frac{X}{N} = {} & \lambda \cos \phi + \frac{\lambda^3 \cos^3\phi}{6} (1 - t^2 + \eta^2) \\ & + \frac{\lambda^5 \cos^5\phi}{120} (5 - 18t^2 + t^4 + 14\eta^2 - 58t^2\eta^2 + 13\eta^4 \\ & \qquad + 4\eta^6 - 64\eta^4 t^2 - 24\eta^6 t^2) \\ & + \frac{\lambda^7 \cos^7\phi}{5040} (61 - 479t^2 + 179t^4 - t^6) \; , \end{aligned} \tag{6-25}$$

$$\begin{aligned} \frac{Y}{N} = {} & \frac{S_\phi}{N} + \frac{\lambda^2}{2} \sin \phi \cos \phi + \frac{\lambda^4}{24} \sin \phi \cos^3\phi \, (5 - t^2 + 9\eta^2 + 4\eta^4) \\ & + \frac{\lambda^6}{720} \sin \phi \cos^5\phi (61 - 58t^2 + t^4 + 270\eta^2 - 330t^2\eta^2 \\ & \qquad + 445\eta^4 + 324\eta^6 - 680\eta^4 t^2 + 88\eta^8 - 600\eta^6 t^2 - 192\eta^8 t^2) \\ & + \frac{\lambda^8}{40320} \sin \phi \cos^7\phi (1385 - 311t^2 + 543t^4 - t^6) \; , \end{aligned} \tag{6-26}$$

where the longitude λ is expressed in radians, and

$$S_\phi = \int_0^\phi M d\phi \; , \text{ (from Appendix I)}$$

$$t = \tan \phi \; , \tag{6-27}$$

$$\eta^2 = (e')^2 \cos^2\phi \quad , \tag{6-28}$$

$$(e')^2 = \frac{a^2 - b^2}{b^2} \quad . \tag{6-29}$$

The above mapping equations yield x and y values accurate to 0.001 metres for $\lambda = \pm 3^\circ$. This completes the direct problem.

6.2 Meridian Convergence in Terms of Geodetic Coordinates

Recall the general equation (equation 4-77) defining the meridian convergence

$$\tan\gamma = \frac{\frac{\partial y}{\partial \lambda}}{\frac{\partial x}{\partial \lambda}} \,. \qquad 6\text{-}30$$

If we differentiate equations 6-25 and 6-26 with respect to λ, and substitute into 6-30, we get [Thomas 1952, p. 97]

$$\tan\gamma = \lambda \sin\phi \, [1 + \frac{\lambda^2}{3}\cos\phi(1 + t^2 + 3\eta^2 + 2\eta^4) + + \frac{\lambda^4\cos^4\phi}{15}\,(2 + 4t^2 + 2t^4 + 15\eta^2 + 35\eta^4 - - 40\,t^2\eta^4 + 33\eta^6 - 60t^2\eta^6 + 18\eta^8 - - 24\,t^2\eta^8) + \frac{17}{315}\,(1 + t^2)^3\lambda^6\cos^6\phi] \,. \qquad 6\text{-}31$$

This expression is developed by a series expansion of the arc tan, that is $\gamma = \tan^{-1}$ {R.H.S.} in the above. The result is [U.T.M., 1958]

$$\boxed{\gamma = \lambda \sin\phi[1 + \frac{\lambda^2\cos^2\phi}{3(\rho'')^2}\,(1 + 3\eta^2 + 2\eta^4) + \frac{\lambda^4\cos^4\phi}{15(\rho'')^4}(2 - t^2)],} \qquad 6\text{-}32$$

where γ and λ are in radians; $\rho'' = \text{cosec } 1''$. The above expression has been truncated and can be applied out to 3° from the central meridian with an expected accuracy of 0".01 in γ. We note that the meridian convergence increases as we go away from the central meridian.

6.3 Scale Factor in Terms of Geodetic Coordinates

Recall the general expression for evaluating the scale factor on a conformal projection (equation 4-62)

$$k = \frac{\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2}}{N \cos \phi} . \qquad 6\text{-}33$$

We first change the form of the above to be a function of tan γ by squaring 6-30, namely

$$\tan^2 \gamma = \frac{\left(\frac{\partial y}{\partial \lambda}\right)^2}{\left(\frac{\partial x}{\partial \lambda}\right)^2}$$

and

$$\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 = \left(\frac{\partial x}{\partial \lambda}\right)^2 (1 + \tan^2 \gamma). \qquad 6\text{-}34$$

Substituting 6-34 into 6-33, yields

$$k = \frac{\frac{\partial x}{\partial \lambda}\sqrt{1 + \tan^2 \gamma}}{N \cos \phi} . \qquad 6\text{-}35$$

Since $\tan^2 \gamma$ is small, we can expand the square root term into a series, and after evaluating the partial derivative, the result is

$$k = 1 + \frac{\lambda^2}{2} \cos^2 \phi (1 + \eta^2) + \frac{\lambda^4 \cos^4 \phi}{24} (5 - 4t^2 + 14\eta^2 + 13\eta^4 - 28t^2\eta^2 + 4\eta^6 - 48t^2\eta^4 - 24t^2\eta^6) + \frac{\lambda^6 \cos^6 \phi}{720} (61 - 148t^2 + 16t^4) . \qquad 6\text{-}36$$

A somewhat truncated expression is

$$\boxed{k = 1 + \frac{\lambda^2}{2} \cos^2\phi \, (1+\eta^2) + \frac{\lambda^4 \cos^4\phi}{24} (5-4t^2) \,,} \qquad 6\text{-}37$$

where the eighth significant digit may be in error by one or two units. Note the scale factor increases as one's longitude relative to the central meridian increases, for a constant latitude. Along the same meridian an increase in latitude causes a decrease in the scale factor.

6.4 Appearance of the Transverse Mercator Projection

The following facts govern the appearance of the Transverse Mercator projection (Figure 6-4):

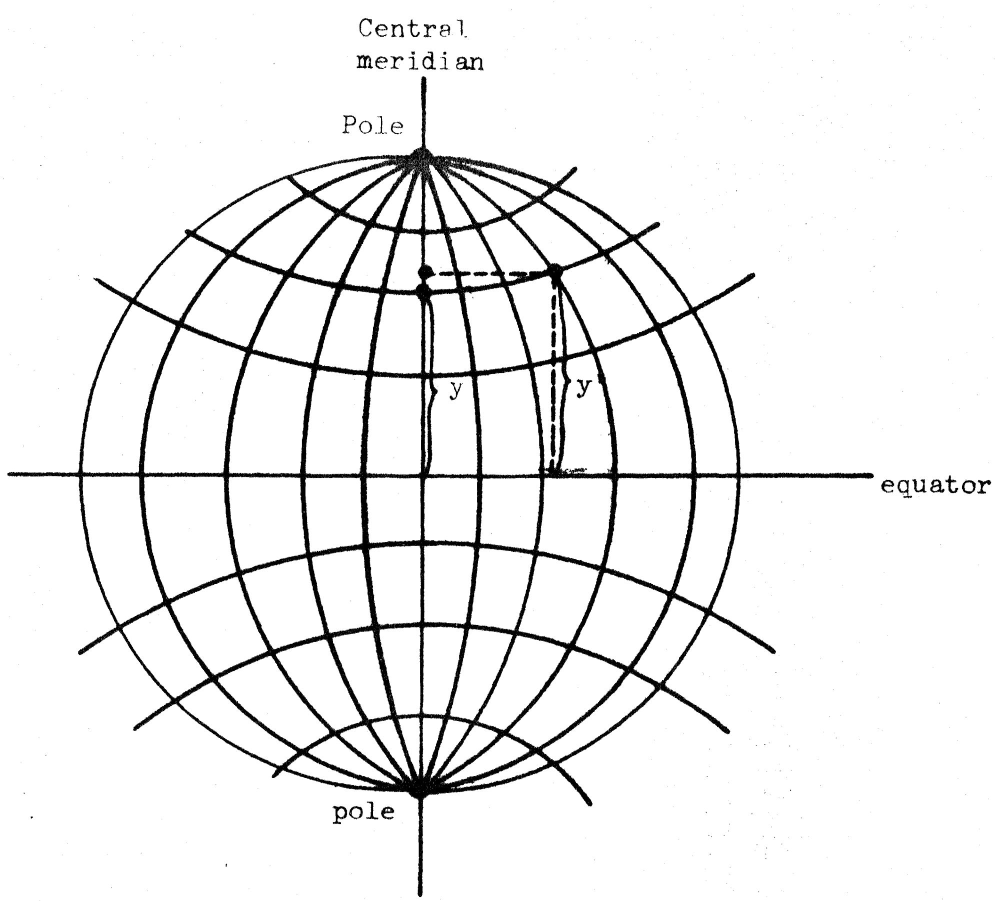


Figure 6-4. Appearance of the Transverse Mercator

(1) $k = 1$ along the central meridian, thus the two poles are plotted at ordinate values equivalent to the length of the meridian between the pole and the equator;

(2) for points off the central meridian the ordinate value y' is greater than the ordinate value y, a point on the same parallel of latitude but on the central meridian;

(3) γ increases with longitude away from central meridian;

(4) meridians and parallels intersect at 90°.

6.5 Inverse Problem

The direct problem was stated in equation form as

$$x + iy = f(\lambda + iq) \quad . \qquad 6\text{-}38$$

We write the inverse problem analogously as

$$\lambda + iq = F(x + iy) \quad , \qquad 6\text{-}39$$

where F is the ampping function to be determined and is different from f o f the direct problem.

We begin by approximating $F(x + iy)$ in 6-39 by a Taylor series. The point of expansion will, at first, be chosen to be some arbitrary point on the map plane with coordinates x_o and y_o (Figure 6-5).

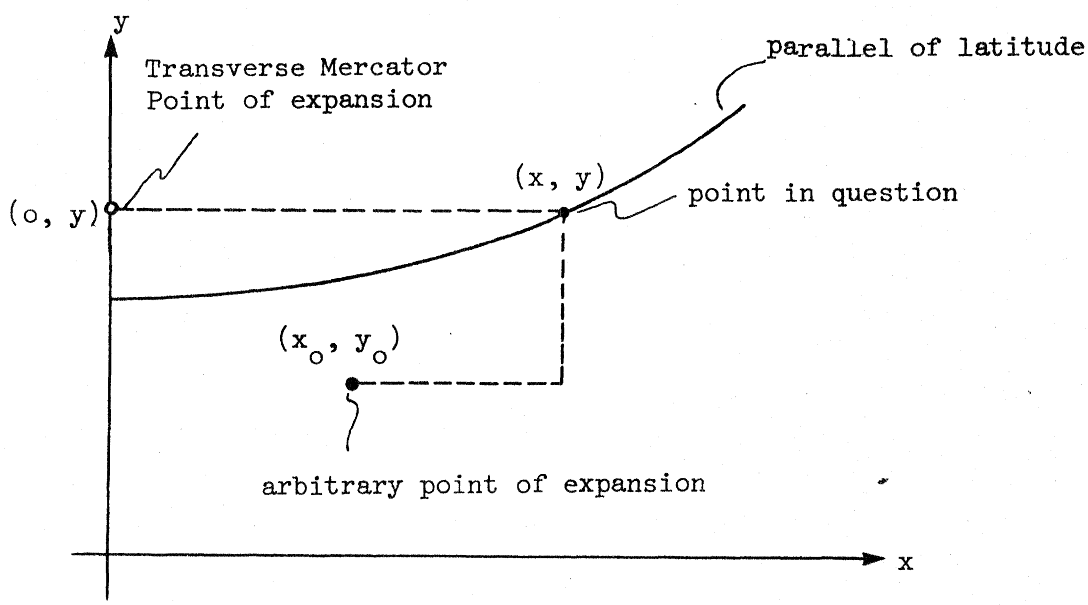


Figure 6-5. Arbitrary and Transverse Mercator Points of Expansion (Inverse Problem)

Thus

$$\lambda + iq = (\lambda_o + iq_o) + (\Delta\lambda + i\Delta q)$$

$$= F(x_o+iy_o) + F^{I}(x_o+iy_o)(\Delta x+i\Delta y) + \frac{F^{II}(x_o+iy_o)}{2!}(\Delta x+i\Delta y)^2 + \ldots \qquad 6\text{-}40$$

$$= F(z_o) + F^{I}(z_o)(\Delta z) + \frac{F^{II}(z_o)(\Delta z)^2}{2!} + \frac{F^{III}(z_o)(\Delta z)^3}{3!} + \ldots \qquad 6\text{-}41$$

Now we choose a more practical point of expansion for our Taylor series - this time one with coordinates (o, y) (Figure 6-5).

$$x_o = 0 \text{ implies } x = x_o + \Delta x = \Delta x \text{ and } (\lambda \equiv \Delta\lambda); \qquad 6\text{-}42$$

$$y_o = y \text{ implies } \Delta y = 0 \text{ and } (\Delta q = 0)\ . \qquad 6\text{-}43$$

Taking into account the above implications in 6-40, we get

$$\lambda + iq = F(iy) + F^{I}(iy)x + \frac{F^{II}(iy)x^2}{2!} + \frac{F^{III}(iy)x^3}{3!} + \ldots \quad . \qquad 6\text{-}44$$

Note that on the central meridian

$$x = 0$$

$$iq = F(iy) \quad . \qquad 6\text{-}45$$

The geometrical situation associated with the above equation is depicted in Figure 6-6. Shown are two parallels of latitude - the parallel through the point in question with latitude ϕ , and the parallel through the ordinate value y on the central meridian. The latitude of the latter parallel is ϕ_1, and is called the <u>footpoint latitude</u> in Thomas [1952]. Corresponding to ϕ_1 is the <u>footpoint isometric latitude</u> <u>q_1</u> . Accordingly, 6-45 is written as

$$iq_1 = F(iy) \quad . \qquad 6\text{-}46$$

As a first step in making the mapping equations (6-44) practical for numerical computations, we evaluate F(iy) and its derivatives, leaving off the subscript 1 for the time being.

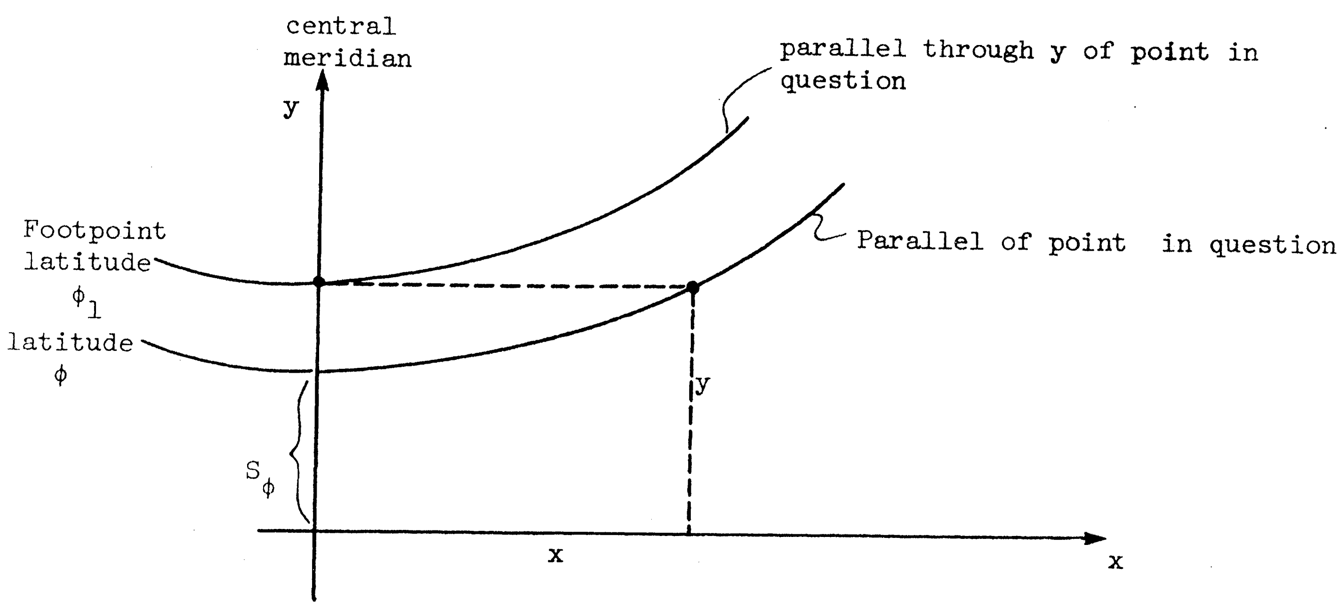


Figure 6-6. Footpoint Latitude.

The first derivative is

$$F^{I}(iy) = \frac{dF(iy)}{d(iy)} = \frac{d(iq)}{d(iy)} = i\,\frac{dq}{dSq} \cdot \frac{dSq}{d(iy)} , \tag{6-47}$$

where in general

$$iy = iS_{\phi} , \tag{6-48}$$

and in particular

$$iy \ = iS\phi_1 = iSq_1 \cdot \tag{6-49}$$

T hen the last derivative in the above becomes

$$\frac{dS_q}{i\ dS_q} = \frac{1}{i} \ . \tag{6-50}$$

If we define

$$\frac{dq}{dS_q} = q^{I} , \tag{6-51}$$

then the result is

$$\boxed{F^{I}(iy) = q^{I} \ .} \tag{6-52}$$

The second derivative is

$$F^{II}(iy) = \frac{d(\frac{dq}{dS_q})}{d(iy)} = \frac{d(\frac{dq}{dS_q})}{dS_q}\ \frac{dS_q}{d(iS_q)} = q^{II}\,\frac{1}{i} \tag{6-53}$$

$$\boxed{F^{II}(iy) = -\ iq^{II} \ .} \tag{6-54}$$

The higher derivatives can be derived in an analogous fashion, they are:

$$\boxed{\begin{array}{l} F^{III}(iy) = -\ q^{III} \\ F^{IV}\ (iy) = iq^{IV} \ , \\ F^{V}\ \ (iy) = q^{V} \quad , \\ \\ \text{(sequence repeats)} \end{array}} \tag{6-55}$$

Replacing the subscripts for the footpoint latitude in the function and its derivatives, 6-44 becomes

$$\lambda+iq = iq_1+xq_1^{I} - \frac{iq_1^{II}x^2}{2!} - \frac{q_1^{III}x^3}{3!} + \frac{i\,q_1^{IV}x^4}{4!} + \ldots , \qquad 6\text{-}56$$

and separating into real and imaginary parts the above becomes

$$\boxed{\lambda = xq_1^{I} - \frac{x^3}{3!}\,q_1^{III} + \frac{x^5}{5!}\,q_1^{V} + \ldots ,} \qquad 6\text{-}57$$

$$\boxed{q = q_1 - \frac{x^2}{2!}\,q_1^{II} + \frac{x^4}{4!}\,q_1^{IV} + \ldots .} \qquad 6\text{-}58$$

Note the last equation in the above gives the difference, $\Delta q = q-q_1$, between the two isometric latitudes. This is used below in deriving the final expression for the geodetic latitude.

Derivatives of complex functions in 6-44 have been replaced by real derivatives as shown above. Now we evaluate the real derivatives.

Recall

$$S_q = \int_o^{\phi} N \cos\phi \; dq \; . \qquad 6\text{-}59$$

The first derivative is

$$q^{I} = \frac{dq}{dS_q} = \frac{1}{N \cos\phi} \; . \qquad 6\text{-}60$$

The second derivative is

$$q^{II} = \frac{d^2q}{dS_q^2} = \frac{t}{N^2\cos\phi} \; . \qquad 6\text{-}61$$

The third derivative is

$$q^{III} = \frac{d^3q}{dS_q^3} = \frac{1}{N^3 \cos\phi}\,(1+2t^2+\eta^2) \; . \qquad 6\text{-}62$$

The higher derivatives have been worked out in Thomas [1952, p. 101].

Substituting the derivatives in 6-57 and 6-58 along with the above expressions results in the following expression for the longitude λ:

$$\lambda = \sec\phi_1\left[\frac{X}{N_1} - \frac{1}{6}\left(\frac{X}{N_1}\right)^3(1 + 2t_1^2 + \eta_1^2)\right.$$

$$+ \frac{1}{120}\left(\frac{X}{N_1}\right)^5(5 + 6\eta_1^2 + 28t_1^2 - 3\eta_1^4 + 8t_1^2\eta_1^2 + 24t_1^4$$

$$- 4\eta_1^6 + 4t_1^2\eta_1^4 + 24t_1^2\eta_1^6)$$

$$\left. - \frac{1}{5040}\left(\frac{X}{N_1}\right)^7(61 + 662t_1^2 + 1320t_1^4 + 720t_1^6)\right] , \qquad 6\text{-}63$$

where the subscript 1 denotes that the functions be evaluated using the footpoint latitude ϕ_1 (Appendix II), and t_1, η_1 are given by 6-27 and 6-29. The accuracy of the above formula is plus or minus 0.00001 arcsecond for λ less than 3° from the central meridian.

The derivation of the corresponding expression for the latitude ϕ is more involved. It is necessary to obtain an expression for $\Delta\phi = \phi - \phi_1$ as a function of the difference $\Delta q = q - q_1$ (6-58). Then the solution for the latitude is

$$\phi = \phi_1 + \Delta\phi = \phi_1 + g(\Delta q) \quad , \qquad 6\text{-}64$$

where ϕ_1 is the footpoint latitude corresponding to a meridian of length y. The determination of ϕ_1 given y is documented in Appendix II.

We now expand $\Delta\phi$ by a Taylor series in Δq,

$$\Delta\phi = \phi - \phi_1 = g(\Delta q) = \Delta q\,\frac{d\phi_1}{dq_1} + \frac{\Delta q_1^2}{2!}\,\frac{d^2\phi_1}{dq_1^2} + \frac{\Delta q^3}{3!}\,\frac{d^3\phi_1}{dq_1^3} + \ldots \qquad 6\text{-}65$$

Noting that, from the definition of the isometric latitude,

$$\frac{d\phi_1}{dq_1} = \frac{N_1 \cos\phi_1}{M_1} = (1 + \eta_1^2) \cos \phi_1 \quad . \qquad 6\text{-}66$$

The second derivative is

$$\frac{d^2\phi_1}{dq_1^2} = \frac{N_1}{M_1} t_1 \cos^2\phi_1 \left(2 - 3\frac{N_1}{M_1}\right) \quad ,$$

$$= -(1 + \eta_1^2)(1 + 3\eta_1^2)\, t_1 \cos^2\phi_1 \quad . \qquad 6\text{-}67$$

The higher derivatives are given in Thomas [1952, p. 102]. Substituting the derivatives of q (6-60, 6-61, 6-62) into 6-58, and the result for Δq into 6-65 along with the derivatives of ϕ_1 (6-66, 6-67) into 6-65, the final equation for determining the latitude is obtained:

$$\begin{aligned}
\phi = \phi_1 &- \frac{t_1 X^2}{2M_1N_1} + \frac{t_1 X^4}{24M_1N_1^3}(5 + 3t_1^2 + \eta_1^2 - 4\eta_1^4 - 9\eta_1^2 t_1^2) \\
&- \frac{t_1 X^6}{720M_1N_1^5}(61 - 90t_1^2 + 46\eta_1^2 + 45t_1^4 - 252t_1^2\eta_1^2 - 3\eta_1^4 \\
&\qquad + 100\eta_1^6 - 66t_1^2\eta_1^4 - 90t_1^4\eta_1^2 + 88\eta_1^8 + 225t_1^4\eta_1^4 \\
&\qquad + 84t_1^2\eta_1^6 - 192t_1^2\eta_1^8) \\
&+ \frac{t_1 X^8}{40320M_1N_1^7}(1385 + 3633t_1^2 + 4095t_1^4 + 1575t_1^6) \quad ,
\end{aligned} \qquad 6\text{-}68$$

where

$$t_1 = \tan \phi_1 \quad ,$$

$$\eta_1^2 = (e')^2 \cos^2\phi_1 \quad ,$$

$$(e')^2 = (a^2 - b^2)/b^2 \quad .$$

The accuracy of the formulae is ± 0".00001 within 3° of the central meridian.

6.6 Meridian Convergence in Terms of Map Coordinates

In the inverse problem the map coordinates x and y are given. The aim of this section is to express the meridian convergence as a function of x only.

From equations 4-77 and 4-78,

$$\tan\gamma = \frac{\frac{\partial y}{\partial \lambda}}{\frac{\partial x}{\partial \lambda}} = \frac{\frac{\partial x}{\partial q}}{\frac{\partial y}{\partial q}} = \frac{dy}{dx} \qquad 6\text{-}69$$

for q = constant. Knowing that in general

$$q = q\ (x,\ y)$$

and that the total differential for the case when q is constant is

$$dq = \frac{\partial q}{\partial x}\,dx + \frac{\partial q}{\partial y}\,dy = 0\ , \qquad 6\text{-}70$$

and

$$\frac{dy}{dx} = -\frac{\partial q}{\partial x}\Big/\frac{\partial q}{\partial y}\ .$$

Thus

$$\tan\gamma = \frac{dy}{dx} = -\frac{\frac{\partial q}{\partial x}}{\frac{\partial q}{\partial y}} = -\frac{\frac{\partial q}{\partial x}}{\frac{\partial \lambda}{\partial x}}\ , \qquad 6\text{-}71$$

since from the Cauchy-Rieman equations (equation 4-50)

$$\frac{\partial q}{\partial y} = \frac{\partial \lambda}{\partial x}\ . \qquad 6\text{-}72$$

The derivatives of 6-71 are obtained by differentiating 6-57 ($\lambda = \lambda(x)$) and 6-58 ($q = q(x)$). The result is

$$\tan\gamma = -\frac{\frac{\partial q}{\partial x}}{\frac{\partial \lambda}{\partial x}} = \frac{x q_1^{II} - \frac{x^3}{6} q_1^{IV} + \frac{x^5}{120} q_1^{VI} - \ldots}{q_1^{I} - \frac{x^2}{2} q_1^{III} + \frac{x^4}{24} q_1^{V} - \ldots} . \qquad 6\text{-}73$$

Substituting the expressions for the partial derivatives, from 6-60 through 6-62, into the above, yields [Thomas, 1950, p. 104]

$$\tan\gamma = \frac{t_1}{N_1} x - \frac{t_1}{3} \left(\frac{x}{N_1}\right)^3 (1 - \eta_1^2 - 2\eta_1^4) +$$

$$+ \frac{t_1}{15} \left(\frac{x}{N_1}\right)^5 (2 + 2\eta_1^2 + 9\eta_1^4 + 6t_1^2\eta_1^2 + 20\eta_1^6 + \qquad 6\text{-}74$$

$$+ 3t_1^2\eta_1^4 - 27t_1^2\eta_1^6 + 11\eta_1^8 - 24t_1^2\eta_1^8) - \frac{17t_1}{315} \left(\frac{x}{N_1}\right)^7 .$$

γ may be computed knowing x and the footpoint latitude ϕ_1 corresponding to y.

6.7 Scale Factor in Terms of Map Coordinates

It is often convenient to compute the scale factor as a function of the map coordinates. You will witness this is Section II when we deal with computations on a conformal map plane.

From equation 6-35, the reciprocal of the scale factor is

$$\frac{1}{k} = N \cos\phi \frac{\partial \lambda}{\partial x} / (1+\tan^2\gamma)^{1/2} . \qquad 6\text{-}75$$

After computing the partial derivative from 6-57 ($\lambda = \lambda(x)$), expressing the square root term in a series, and expanding N cos ϕ in a Taylor series with the point of expansion being the footpoint latitude ϕ_1, the equation for the reciprocal of the scale factor becomes [Thomas 1952, p. 105]

$$\frac{1}{k} = 1 - \frac{1+\eta^2}{2}\left(\frac{x}{N_1}\right)^2 + \frac{5 + 6\eta_1^2 - 3\eta_1^4 - 4\eta_1^6 + 24t_1^2\eta_1^4 + 24t_1^2\eta_1^2}{24}\left(\frac{x}{N_1}\right)^4 - \frac{61}{720}\left(\frac{x}{N_1}\right)^6 . \qquad 6\text{-}76$$

The reciprocal of both sides yields the expression for the scale factor, namely

$$\boxed{k = 1 + \frac{1+\eta_1^2}{2}\left(\frac{x}{N_1}\right)^2 + \frac{1 + 6\eta_1^2 + 9\eta_1^4 + 4\eta_1^6 - 24t_1^2\eta_1^4 - 24t_1^2\eta_1^6}{24}\left(\frac{x}{N_1}\right)^4 + \frac{1}{720}\left(\frac{x}{N_1}\right)^6 . \qquad 6\text{-}77}$$

6.8 Universal Transverse Mercator (UTM)

The Universal Transverse Mercator projection is based completely on the transverse mercator projection. In this section we give the specifications for the UTM, and the equations for the direct and inverse problems.

Its specifications are as follows [UTM 1958]:

(1) transverse mercator in zones 6° wide;

(2) reference ellipsoid - Clark 1866 in North America, with other ellipsoids used elsewhere;

(3) longitude origin - the central meridian;

(4) latitude origin - the equator;

(5) unit - metre;

(6) false northing (y): 0 metres for northern hemisphere; 10,000,000 metres for southern hemisphere;

(7) false easting (x): 500,000 metres;

(8) scale factor at the central meridian: 0.9996;

(9) zone numbering: beginning with 1 for zone between 180°W and 174° W meridians and increasing to 60 for the zone bounded by meridians 174°E and 180° E (Figure 6-7).

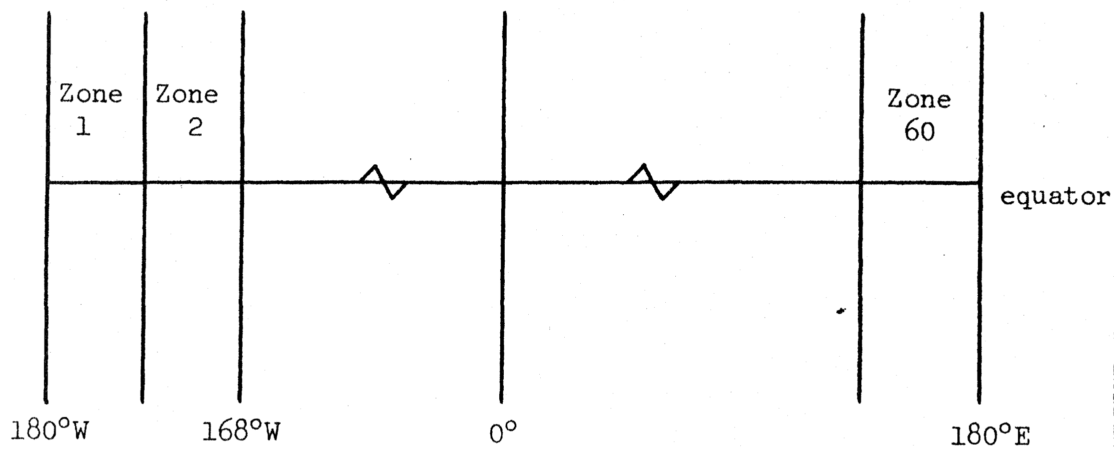


Figure 6-7. UTM Zones

(See Figure 6-8 for the zones for North America).

(10) The latitude limits for the system are 80°N and 80°S due to the many zones that would be involved when working within an area of any appreciable extent.

Scale Factor on the UTM

By choosing a scale factor of 0.9996 on the central meridian we expect from equation 6-37

$$k = 1 + \frac{\lambda^2}{2} \cos^2\phi + \dots , \qquad 6\text{-}78$$

that the scale factor increases as we go away from the central meridian, reaches a value of 1, and then increases beyond a value of 1 (Figure 6-9).

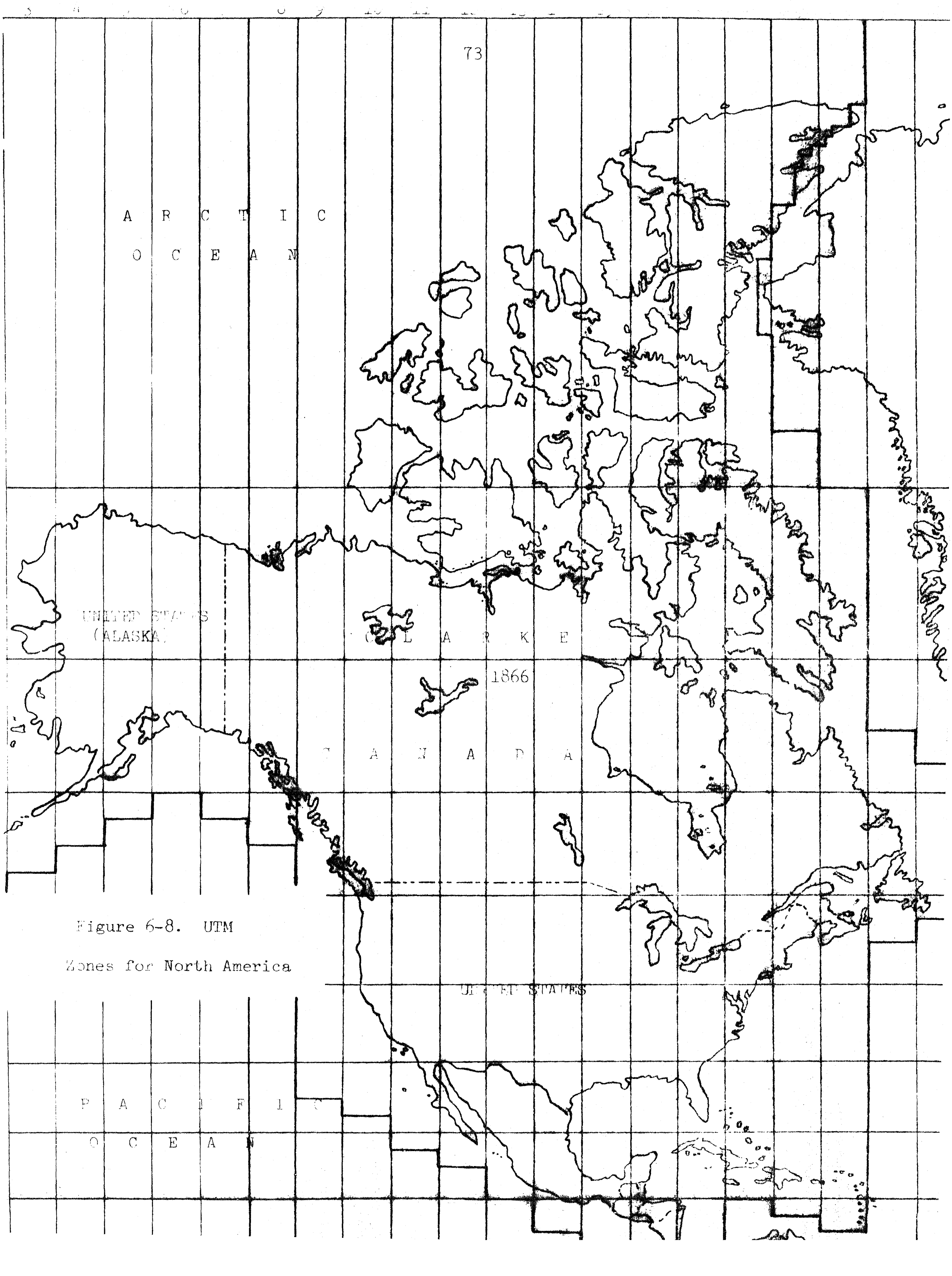


Figure 6-8. UTM Zones for North America

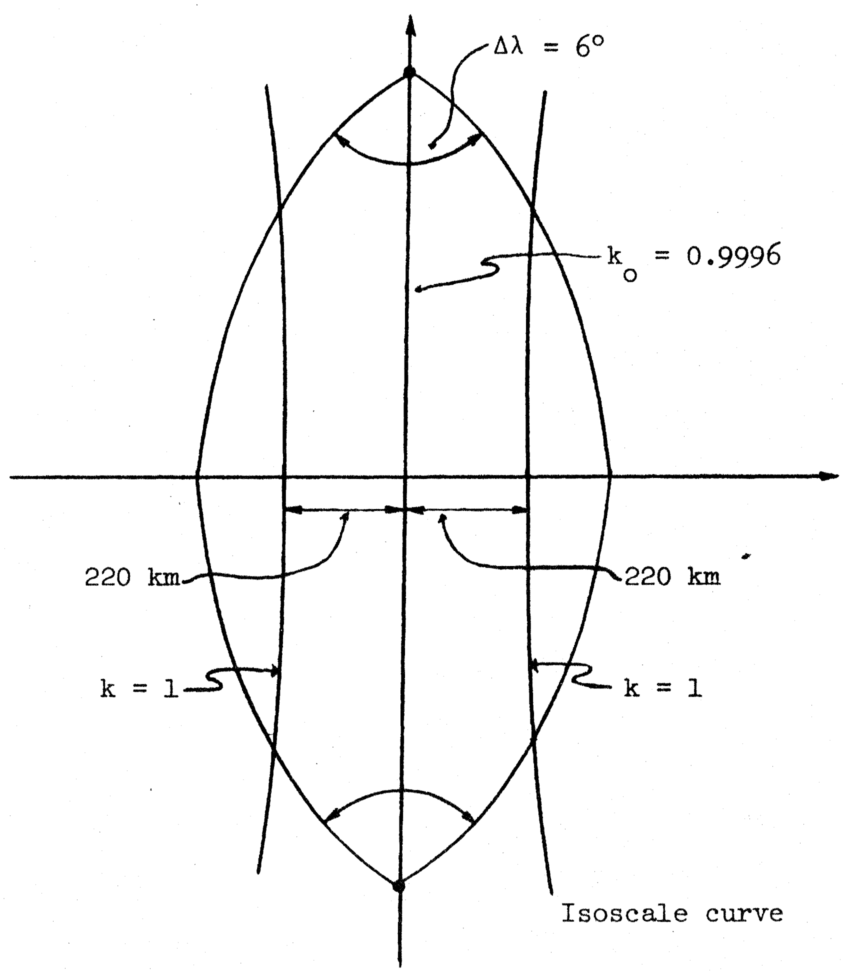


Figure 6-9. Scale Factor on the UTM

At what distance away from the central meridian, say along the equator, is the scale factor equal to 1? To answer this question we first write 6-78 as

$$k = k_o[1 + \lambda^2 \frac{\cos^2\phi}{2}] , \qquad 6\text{-}79$$

where k_o is the scale factor on the central meridian (for the UTM $k_o = 0.9996$). If Δk is the scale difference between that at the central meridian k_o and k, then we get

$$\lambda^{\circ} = \rho^{\circ} \frac{\sqrt{2}}{\cos \phi} \left[\frac{\Delta k}{k_o}\right]^{1/2} \qquad 6\text{-}80$$

For our case, $\Delta k = 1 - 0.9996 = 1/2500 = 4 \times 10^{-4}$. Therefore at

$$\phi \doteq 0^{\circ}, \quad \lambda = 2^{\circ}$$

$$\phi = 40^{\circ}, \lambda = 3^{\circ} .$$

Along the equator, at a distance of about 220,000 metres, the scale factor is equal to unity.

Mapping Equations for the UTM

Let us trace through the equations of the transverse mercator and see how they are affected by the stipulation of $k_o = 0.9996$ on the central meridian. First, the meridian distance is

$$y = k_o S_{\phi} = k_o \int_o^{\phi} M \cdot d\phi = k_o \int_o^{\phi} N \cos\phi \, dq = k_o f(q). \qquad 6\text{-}81$$

We see that the function $f(q)$ is modified by k_o, and thus all derivatives will be also modified, namely

$$k_o \frac{d^n f(q)}{dq^n} . \qquad 6\text{-}82$$

Then the mapping equations for the direct problem for the UTM in terms of the Transverse Mercator equation is

$$\left\{ \begin{matrix} x \\ y \end{matrix} \right\}_{UTM} = k_o \left\{ \begin{matrix} x \\ y \end{matrix} \right\}_{TM} \quad . \qquad 6\text{–}83$$

The meridian convergence for the UTM is the same as the TM since in

$$\tan \gamma = \frac{\partial y}{\partial \lambda} \Big/ \frac{\partial x}{\partial \lambda} \quad , \qquad 6\text{–}84$$

the k_o cancels. Therefore

$$\gamma_{UTM} \equiv \gamma_{TM} \quad . \qquad 6\text{–}85$$

The scale factor for the UTM becomes

$$k_{UTM} = k_o \, k_{TM} \; . \qquad 6\text{–}86$$

For the inverse problem we wish to use the TM expressions but with UTM coordinates. This means $x_{TM} = \frac{x_{UTM}}{k_o}$, so in the expression for λ we write

$$\lambda = \frac{x_{UTM}}{k_o} q_1^{I} - \frac{x_{UTM}^3}{3!k_o^3} q_1^{III} - \frac{x_{UTM}^5}{5!k_o^5} q_1^{V} + \ldots \quad . \qquad 6\text{–}87$$

Thus we see that the derivatives need appropriate division when UTM coordinates are used in the expression. The same holds for the equation for the latitude.

Tables have been prepared for the computation of the direct and inverse problems. The interested reader is referred to [UTM 1957].

6.9 Transverse Mercator in Three Degree Zones

We have seen that the U.T.M. is the transverse mercator projection in 6° zones. In Canada the transverse mercator in 3° zones is also being used. For example, in Ontario [Dept. of Highway 1968] Canadian numbered zones 8 through 17 are used as a basis for the Ontario Coordinate System (Figure 6-10). In fact, the boundaries of these 3° zones are slightly modified (see Figure 6-11).

Since the zone width is chosen to be only 3°, and a scale factor of $k_o = 0.9999$ assigned to the central meridian, we can expect a smaller scale error throughout the zone. In fact, the error is 1/10,000 on the central meridian as compared to 1/2,500 for the UTM. We can compute the distance away from the central meridian where the scale factor is 1. This can be done by simply employing equation 6-80, where, in this case, $\Delta k = 1-0.9999 = 1/10{,}000$. Therefore at

$$\phi = 0^\circ, \quad \lambda = 0^\circ.81,$$

$$\phi = 50^\circ, \quad \lambda = 1^\circ.26.$$

At $\lambda = 1^\circ 30'$, and $\phi = 42^\circ.712$, $k = 1.000085$. (see figure 6-12).

The transverse mercator equations can be employed here as well by simply using the scale factor $k_o = 0.9999$.

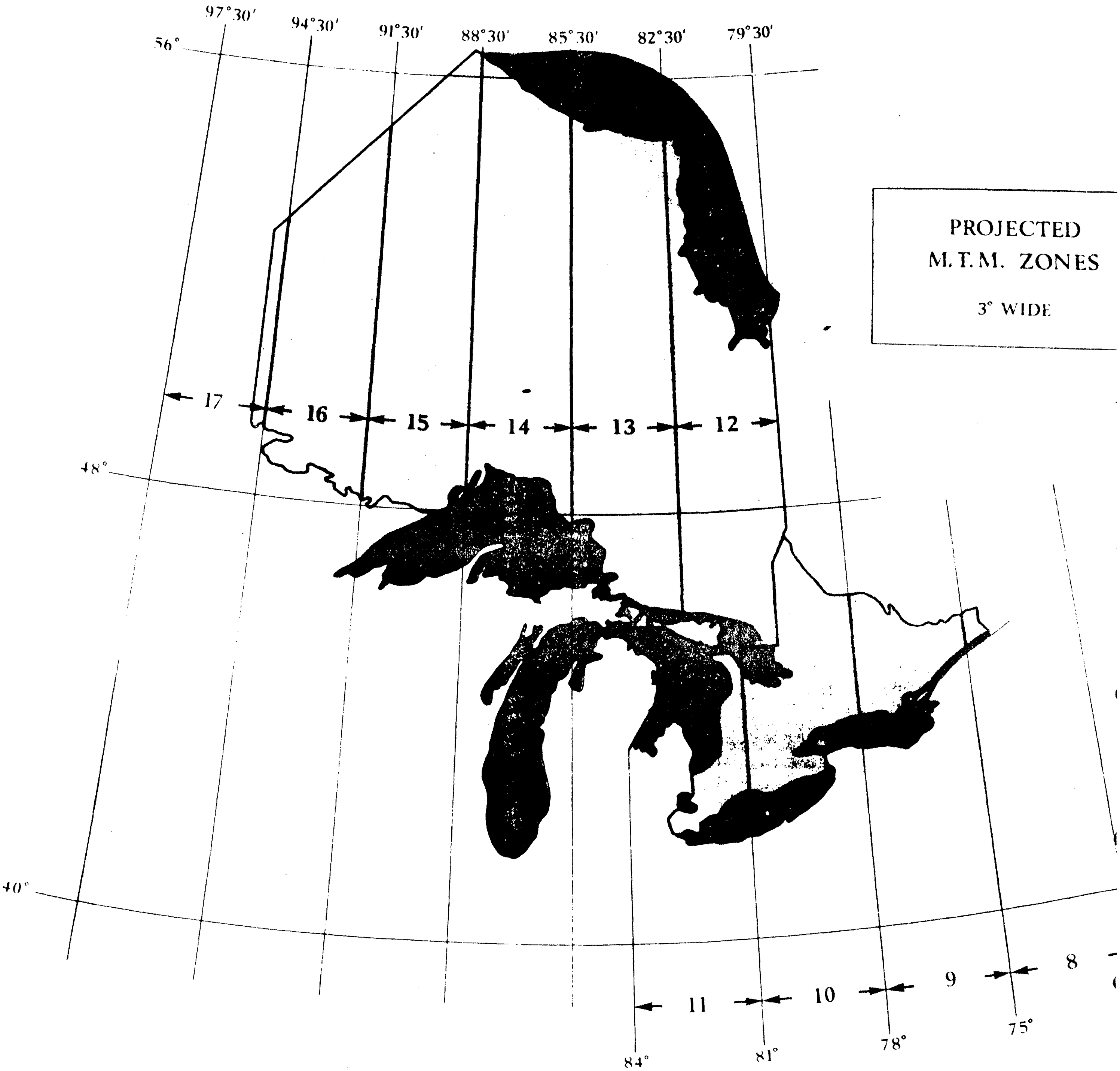
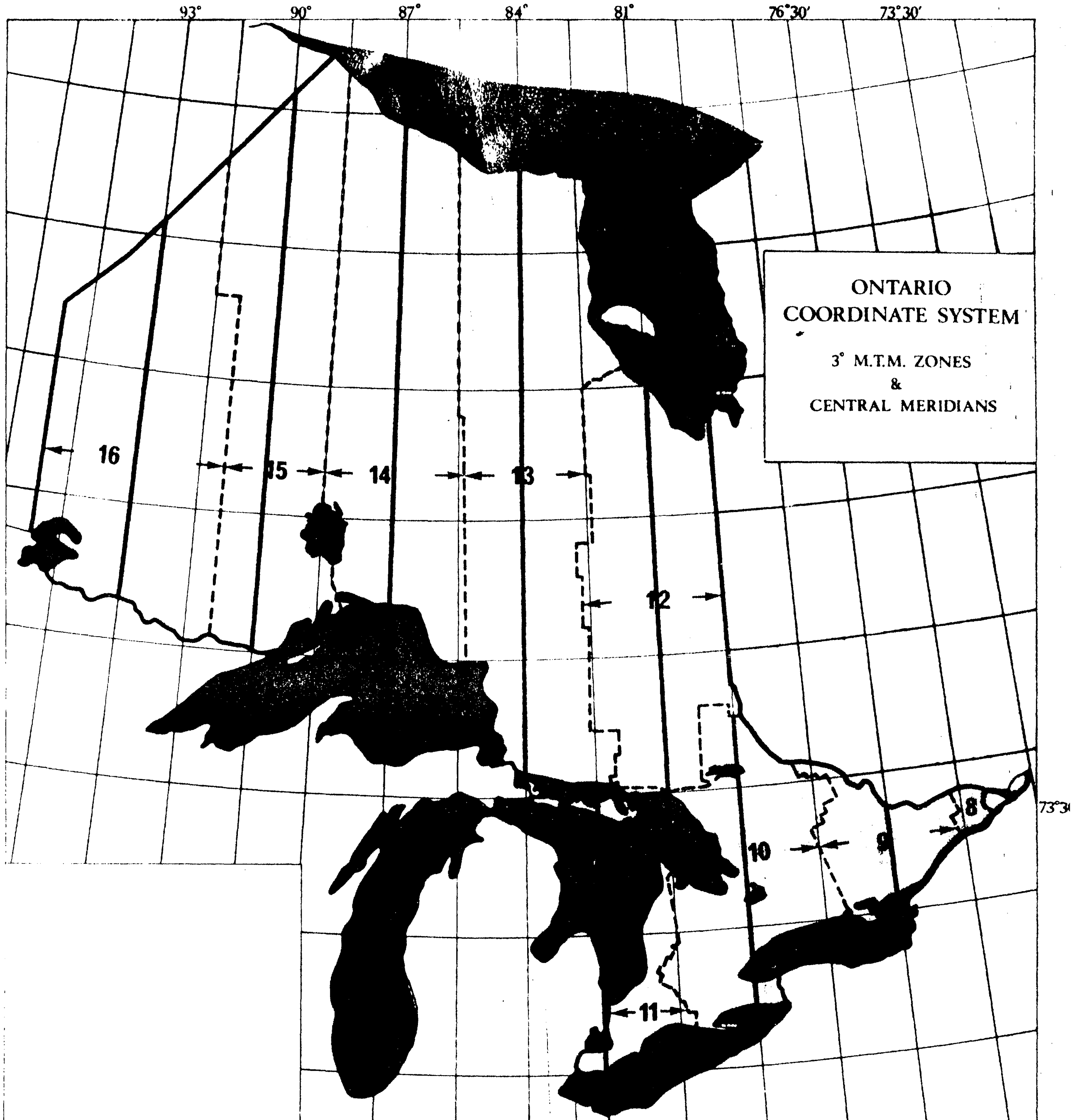

PROJECTED
M.T.M. ZONES
3° WIDE
97°30'
94°30'
91°30'
88°30'
85°30'
82°30'
79°30'
56°
48°
40°
17
16
15
14
13
12
11
10
9
8
84°
81°
78°
75°

Figure 6-10.



ONTARIO
COORDINATE SYSTEM
3° M.T.M. ZONES
&
CENTRAL MERIDIANS
93°
90°
87°
84°
81°
76°30′
73°30′
16
15
14
13
12
11
10
9
8
82°30′
79°30′
76°30′

Figure 6–11.

CROSS SECTION OF A 3° ZONE

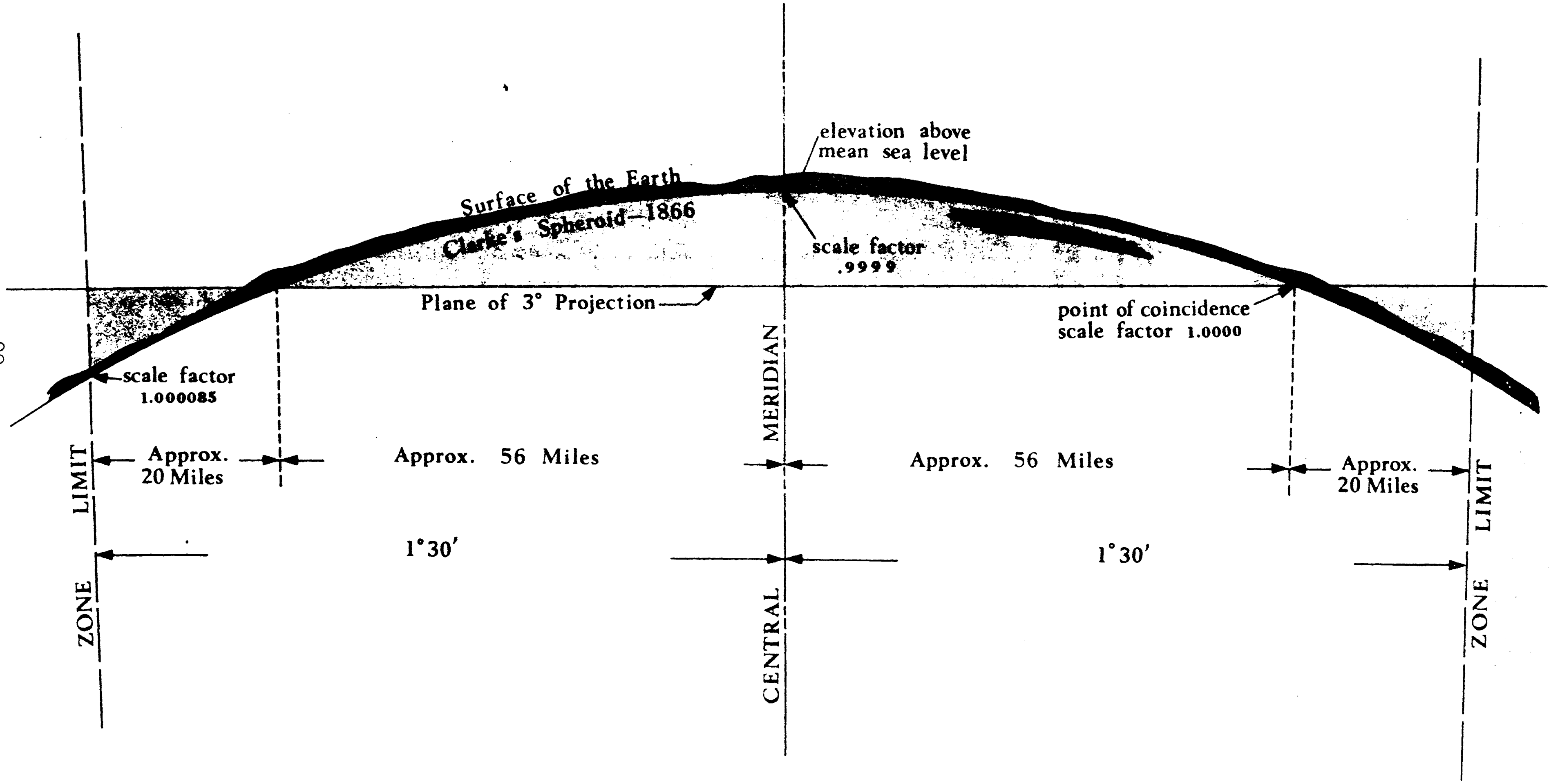


Figure 6-12

7. LAMBERT CONFORMAL CONIC PROJECTION

Lambert developed his conformal conic preojection in 1772 - the same year in which he created the Transverse Mercator projection. The Conformal Conic is used worldwide.

The requirements of this projection are:

(1) parallels are to be parts of concentric circles;

(2) meridians are to be radii of concentric circles.

Treated in this section are: the direct problem; the scale factor; the meridian convergence; the one and two standard parallel cases; and the inverse problem.

7.1 Direct Problem

As before we state the direct problem as

$$x + iy = f(\lambda + iq) \ ,$$

where the complex mapping function f is to be determined. f is determined by first separating into two functions f_1 and f_2. These two functions are determined by the help of the Cauchy-Rieman equations.

The following equation of a circle is a mathematical characterization of the first requirement for circular parallels:

$$x^2 + y^2 = K^2 f_1(q) \qquad 7\text{-}1$$

where K is the constant for the radius and is to be solved for, $f_1(q)$ is a function of the isometric latitude q and is to be determined, and x and y are the map coordinates corresponding to the coordinate system shown in Figure 7.1.

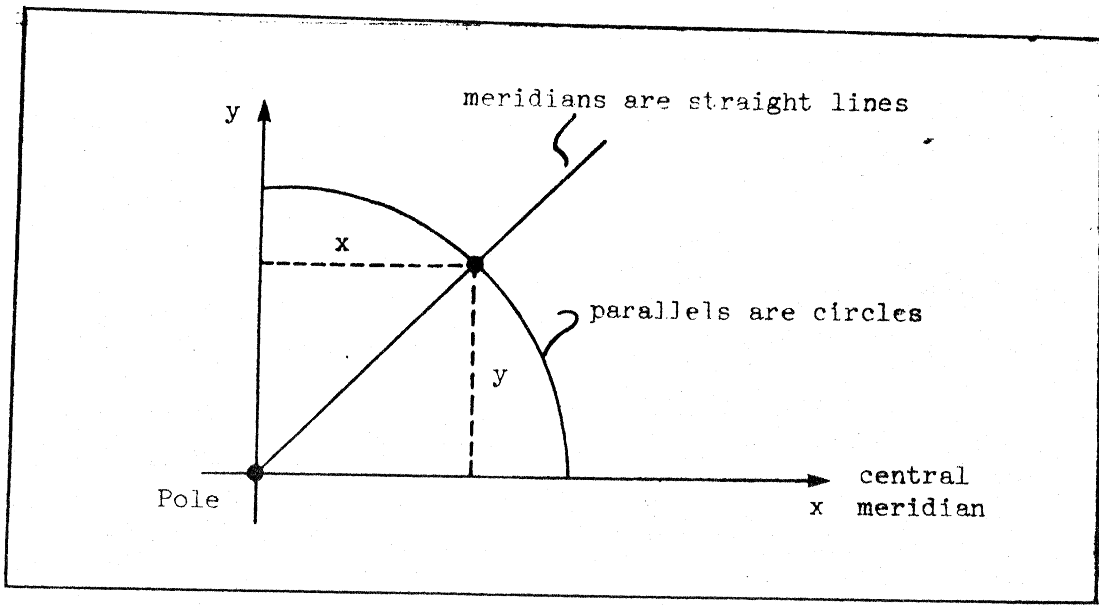


Figure 7-1. Parallels and Meridians on the LCC

The second requirement for straight line meridians is given by

$$y = f_2(\lambda)x + 0. \qquad 7\text{-}2$$

where $f_2(\lambda)$ is the second function to be determined.

Substituting 7-2 into 7-1 yields

$$x^2 + f_2^2(\lambda)\, x^2 = K^2 f_1(q) \; .$$

From the above, the following parametric equations can be written:

$$x = x(\lambda, q) \quad ,$$

$$x = \frac{K[f_1(q)]^{1/2}}{[1 + f_2^2(\lambda)]^{1/2}} \quad , \qquad 7\text{-}3$$

and

$$y = y(\lambda,q),$$

$$= \frac{K\, f_2(\lambda)\, [f_1(q)]^{1/2}}{[1 + f_2^{\,2}(\lambda)]^{1/2}} \quad . \qquad 7\text{-}4$$

We now employ the Cauchy-Rieman equations to help us solve for the $f_1(q)$ and $f_2(\lambda)$.

Recall from equation 4-50

$$\frac{\partial x}{\partial q} = -\frac{\partial y}{\partial \lambda}\ ;\ \frac{\partial x}{\partial \lambda} = \frac{\partial y}{\partial q} \quad . \qquad 7\text{-}5$$

The above partial derivatives of 7-3 and 7-4 are (note variable is left off function from this point forward):

$$\frac{\partial x}{\partial q} = \frac{K\, f_1^{I}}{2(f_1)^{1/2}(1+f_2^{\,2})^{1/2}} \quad ; \qquad 7\text{-}6$$

$$\frac{\partial x}{\partial \lambda} = \frac{-K(f_1)^{1/2}\, f_2 f_2^{\,I}}{(1 + f_2^{\,2})^{3/2}} \quad ;$$

$$\frac{\partial y}{\partial q} = \frac{1}{2}\ \frac{K\, f_2 f_1^{\,I}}{(f_1)^{1/2}(1+f_2^{\,2})^{1/2}} \quad ; \qquad 7\text{-}8$$

$$\frac{\partial y}{\partial \lambda} = \frac{K\,(f_1)^{1/2}\, f_2^{\,I}}{(1+f_2^{\,2})^{3/2}} \quad . \qquad 7\text{-}9$$

From the first set of Cauchy-Rieman equations we get

$$-\frac{f_1^{\,I}}{f_1} = \frac{2f_2^{\,I}}{1 + f_2^{\,2}} = 2\ell, \qquad 7\text{-}10$$

where ℓ is a new constant to be determined. So from 7-10, we can write two equations:

$$\boxed{\begin{array}{c} \dfrac{f_1^I}{f_1} = -2\ell \\ \hline \dfrac{f_2^I}{1+f_2^{\,2}} = \ell . \end{array}}$$ 7-11, 7-12

We shall see later that the constant ℓ is associated with the longitude and is a function of the latitude ϕ.

From the second set of Cauchy-Rieman equations we get

$$-\frac{f_2^{\,I}}{(1+f_2^{\,2})} = \frac{1}{2}\,\frac{f_1^{\,I}}{f_1} = -\ell , \qquad \text{7-13}$$

where ℓ is the same constant as for the first set. The two equations arising from the above are:

$$\boxed{\begin{array}{c} \dfrac{f_1^{\,I}}{f_1} = -2\ell , \\ \hline \dfrac{f_2^{\,I}}{1+(f_2)^2} = \ell . \end{array}}$$ 7-14, 7-15

Up to this point we have determined two equations from which we shall be able to compute expressions for $f_1(q)$ and $f_2(\lambda)$ in terms of the constant K and ℓ.

To begin with, let us determine $f_1(q)$ from 7-14 by integration, namely

$$\frac{df_1}{f_1} = -2\ell dq \qquad \text{7-16}$$

$$\int \frac{df_1}{f_1} = -2\ell \int dq , \qquad \text{7-17}$$

$$\ln f_1 + C_2 = -2\ell q + C_1 , \qquad \text{7-18}$$

$$\ln f_1 = -2\ell q + C_3 , \qquad 7\text{-}19$$

where C_3 is the combined constant of integration. C_3 is evaluated by considering that

$$\ln f_1 = 0, \qquad 7\text{-}20$$

when $$q = 0, \qquad 7\text{-}21$$

thus $$C_3 = 0 . \qquad 7\text{-}22$$

We have then

$$\boxed{\ln f_1 = -2\ell q,} \qquad 7\text{-}23$$

$$\boxed{f_1 = e^{-2\ell q}} \qquad 7\text{-}24$$

The second function $f_2(\lambda)$ is determined from 7-15 by integration, namely

$$\int \frac{1}{1 + (f_2)^2} \, df_2 = \int \ell d\lambda \qquad 7\text{-}25$$

$$\boxed{\tan^{-1} f_2 = \ell\lambda ,} \qquad 7\text{-}26$$

$$\boxed{f_2 = \tan (\ell\lambda).} \qquad 7\text{-}27$$

Introducing the expressions for f_1 and f_2 into the parametric equations given by 7-3 and 7-4, yields

$$x = \frac{K \, (e^{-2\ell q})^{1/2}}{[1 + \tan^2(\ell\lambda)]^{1/2}} \qquad 7\text{-}28$$

$$\boxed{x = K \, e^{-\ell q} \cos\ell\lambda ,} \qquad 7\text{-}29$$

$$\boxed{y = K \, e^{-\ell q} \sin\ell\lambda .} \qquad 7\text{-}30$$

From the above mapping equations, we can compute the radius of any parallel from:

$$r^2 = x^2 + y^2,$$

$$r^2 = K^2 e^{-2\ell q} (\cos^2 \ell\lambda + \sin^2 \ell\lambda),$$

$$\boxed{r = K e^{-\ell q}.} \qquad 7\text{-}31$$

The mapping equations can be now rewritten as

$$x = r \cos\ell\lambda \quad , \qquad 7\text{-}32$$

$$y = r \sin\ell\lambda \quad , \qquad 7\text{-}33$$

where r and $\ell\lambda$ are shown in Figure 7-2.

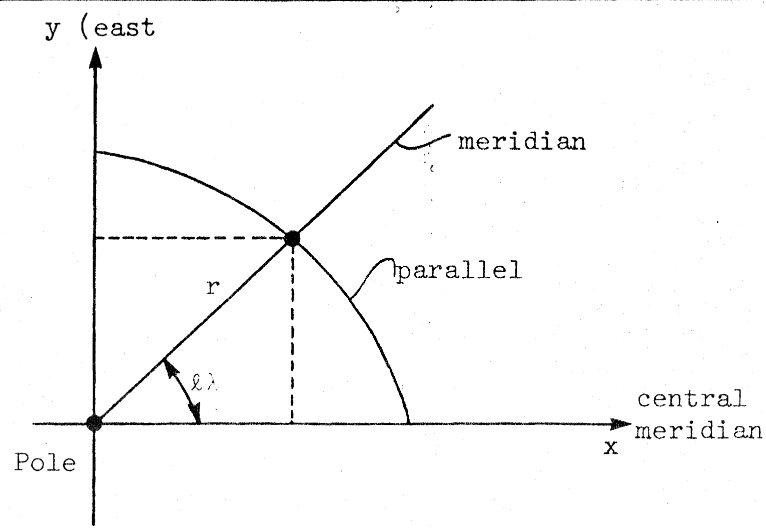


Figure 7-2. Polar Coordinates for LCC

Figure 7-2. Polar Coordinates for LCC

The complex mapping function f is finally determined by combining 7-29 and 7-30. We get

$$x + iy = f(\lambda + iq)$$

$$= Ke^{-\ell q} (\cos\ell\lambda + i \sin \ell\lambda) \qquad 7\text{-}34$$

and by using Euler's equation (2-40),

$$x + iy = Ke^{-\ell q}\, e^{i\ell\lambda}$$

$$x + iy = Ke^{\ell(-q+i\lambda)}$$

$$\boxed{\mathbf{x + iy = Ke^{i\ell(\lambda+iq)}}.} \qquad 7\text{-}35$$

Note the above complex mapping equation is in closed form. This expression can be programmed using complex arithmetic once the constants K and ℓ are defined. K and ℓ are defined below for the one and two standard parallel cases.

7.2 Scale Factor

The general expression for the scale factor is (from equation 4-63).

$$k = [(\frac{\partial x}{\partial q})^2 + (\frac{\partial y}{\partial q})^2]^{1/2} / N \cos \phi \ . \qquad 7\text{-}36$$

For the LCC

$$\frac{\partial x}{\partial q} = - K\ell \, . \, e^{-\ell q} \cos \ell\lambda \qquad 7\text{-}37$$

$$\frac{\partial y}{\partial q} = - K \ell \, e^{-\ell q} \sin \ell \lambda \qquad 7\text{-}38$$

and

$$\boxed{k = \frac{K l e^{-\ell q}}{N \cos \phi} \ .} \qquad 7\text{-}39$$

The scale factor can be computed for any point on the projection with known geodetic coordinates, again, once the constants K and ℓ defined.

7.3 Meridian Convergence

One of the gereral expressions for the meridian convergence derived earlier (equation 4-77) was

$$\tan \gamma = \frac{\partial y}{\partial \lambda} / \frac{\partial x}{\partial \lambda} \ . \qquad 7\text{-}40$$

For the LCC

$$\frac{\partial y}{\partial \lambda} = \ell\, r \sin \ell \lambda, \qquad 7\text{-}41$$

$$\frac{\partial x}{\partial \lambda} = \ell\, r \cos \ell \lambda, \qquad 7\text{-}42$$

thus

$$\tan \gamma = \tan \ell \lambda$$

$$\boxed{\gamma = \ell \lambda} \qquad 7\text{-}43$$

7.4 Two Standard Parallel LCC

In this section we develop the mapping equations for the two standard parallel LCC.

If we stipulate the scale factor to be equal to one along two parallels, then we can write (using 7-39)

$$\frac{K}{N_1}\frac{\ell\, e^{-\ell q_1}}{\cos\phi_1} = \frac{K\ell\, e^{-\ell q_2}}{N_2 \cos\phi_2} = \ell \qquad 7\text{-}44$$

$$\frac{e^{-\ell q_1}}{e^{-lq_2}} = \frac{N_1 \cos\phi_1}{N_2 \cos\phi_2} \qquad 7\text{-}45$$

where the subscript quantities pertain to the two parallels.

Taking natural log of each side of the above equation yields

$$-\ell q_1 + \ell q_2 = \ln N_1 + \ln \cos\phi_1 - \ln N_2 - \ln \cos\phi_2 \,, \qquad 7\text{-}46$$

which defines the constant ℓ:

$$\boxed{\ell = \frac{\ln N_1 - \ln N_2 + \ln \cos\phi_1 - \ln \cos\phi_2}{q_2 - q_1} \,.} \qquad 7\text{-}47$$

From 7-44, the following expression defines the constant K:

$$\boxed{K = \frac{N_1 \cos\phi_1}{\ell e^{-\ell q_1}} = \frac{N_2 \cos\phi_2}{\ell e^{-\ell q_2}} \quad .} \qquad 7\text{-}48$$

The above two expressions for the constants K and ℓ complete the description of the direct problem - recall the complex mapping equation (7-35)

$$\boxed{x + iy = K\, e^{i\ell(\lambda+iq)} \quad .} \qquad 7\text{-}49$$

7.5 One Standard Parallel LCC

The one standard paralled LCC has as its basis the stipulation that the scale factor be equal to one along only one parallel of latitude ϕ_o . We can then write

$$k = 1 = \frac{K\,\ell\, e^{-\ell q_o}}{N_o \cos\phi_o} \quad , \qquad 7\text{-}50$$

$$N_o \cos\phi_o = \ell\, K e^{-\ell q_o} \quad . \qquad 7\text{-}51$$

But from 7-31, the radius of any parallel is given by

$$r = K\, e^{-\ell q} \quad , \qquad 7\text{-}52$$

so we can write an expression for the radius of the parallel with latitude ϕ_o, namely

$$r_o = K\, e^{-\ell q_o} \ . \qquad 7\text{-}53$$

7-51 then becomes

$$N_o \cos\phi_o = \ell r_o . \qquad 7\text{-}54$$

This special radius also has a geometric interpretation as illustrated in Figure 7-3.

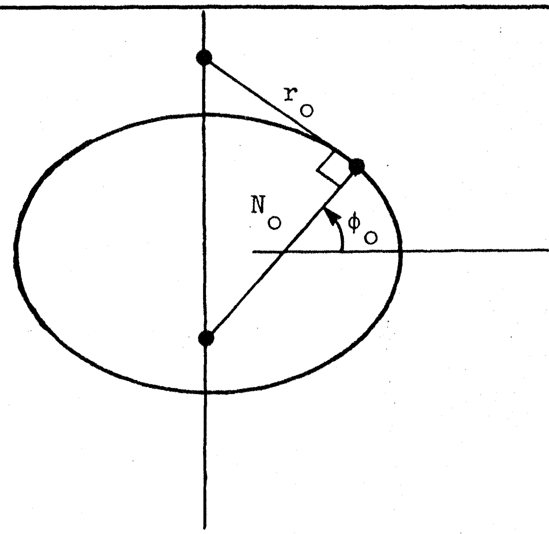


Figure 7-3: Geometric Interpretation of Radius of Standard Parallel

Another expression for standard radius is

$$r_o = N_o \cot \phi_o \ . \tag{7-55}$$

The constant l can be determined by substituting 7-55 into 7-54,

$$N_o \cos\phi_o = \ell N_o \cot\phi_o$$

$$\boxed{\ell = \sin\phi_o \ .} \tag{7-56}$$

The expression for the constant K becomes (from 7-31)

$$\boxed{K = N_o \cot\phi_o \, e^{q_o \sin\phi_o}} \ . \tag{7-57}$$

The above two expressions along with the complex mapping funtion

$$\boxed{x + iy = K e^{i\ell(\lambda + iq)} \ ,} \tag{7-58}$$

constitute the equations for the direct problem for the LCC one standard parallel projection.

7.6 Modification to Coordinate System

The coordinate system exhibited in Figure 7-2 is not convenient in practise. We make two modifications: interchange the x and y, and change the location of the origin to decrease the magnitude of one of the coordinate values.

We had from 7-32 and 7-33

$$x = r \cos \ell \lambda \;, \qquad \text{7-59}$$

$$y = r \sin \ell \lambda \;,$$

Now we interchange the coordinates which results in

$$x = r \sin \ell \lambda \;,$$

$$y = r \cos \ell \lambda \;. \qquad \text{7-60}$$

The equations which reflect a change in the origin are (Figure 7-4)

$$x = r \sin \ell \lambda \;,$$

$$y = r_o - r \cos \ell \lambda , \qquad \text{7-61}$$

where r_o is usually selected such that it is slightly larger than the second term to prevent negative and large values of y.

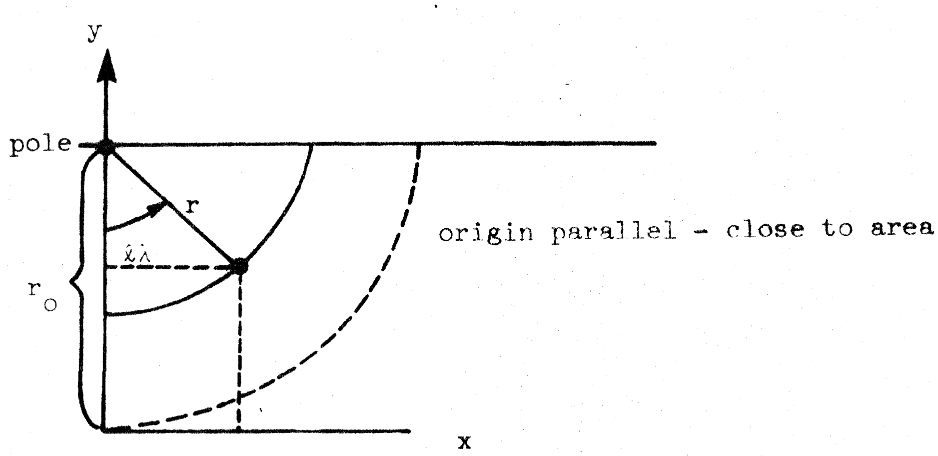


Figure 7-4: Modification to Original LCC Coordinate System

Two ways in which to select r_o is to select some ϕ_o below the area in question and compute

$$r_o = K\, e^{-\ell q_o} \qquad (q_o \text{ from } \phi_o), \qquad 7\text{-}62$$

or

$$r_o = N_o \cot\phi_o \qquad 7\text{-}63$$

7.7 Inverse Problem

In the inverse problem the map coordinates x and y are given along with the constants r_o, ℓ., and K. Required are the geodetic coordinates ϕ and λ.

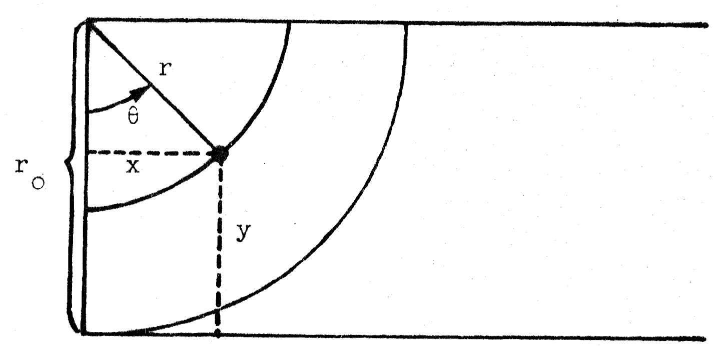


Figure 7-5. Inverse Problem

From Figure 7-5, we can get the expression for the longitude by writing

$$\tan\theta = \frac{x}{r_o - y}\,. \qquad 7\text{-}64$$

and

$$\theta = \ell\lambda \qquad 7\text{-}65$$

$$\boxed{\lambda = \frac{\theta}{\ell}\,.} \qquad 7\text{-}66$$

The latitude is obtained in the following steps. First compute r from

$$r = \frac{r_o - y}{\cos\,\theta}\,. \qquad 7\text{-}67$$

We know that

$$r = K\,e^{-\ell q} \qquad 7\text{-}68$$

so

$$\ln r = \ln K - \ell . q \qquad 7\text{-}69$$

and thus

$$\boxed{q = \frac{\ln K - \ln r}{\ell}} \qquad 7\text{-}70$$

Once the isometric latitude is determined, ϕ may be determined by using the Newton - Raphson approach described earlier.

8. STEREOGRAPHIC PROJECTION

In this section we first derive the expressions for polar stereographic projection and its implimented system - the Universal Stereographic Projection. Secondly, we give references pertaining to the stereographic projection as implemented in New Brunswick.

8.1 Polar Stereographic Projection

The polar stereographic projection is a limiting case of the LCC. Meridians are straight lines and parallels are concentric circles. In this section we deduce expressions for the direct problem, scale factor, meridian convergence, and inverse problem from the expressions already derived for the LCC.

The constant ℓ for the one standard parallel is (7-56)

$$\ell = \sin \phi_o \quad . \tag{8-1}$$

For the pole, $\phi_o = 90$, and $\boxed{\ell = 1}$. The expression for the scale factor then becomes (7-39)

$$k = \frac{K\ e^{-q}}{N \cos \phi} \quad , \tag{8-2}$$

and

$$K = k\ N \cos \phi\ e^{q} \quad . \tag{8-3}$$

We have seen that q increases without limit as ϕ approaches $\frac{\pi}{2}$, so the above expression breaks down at the pole. We transform the expression as follows:

$$K = k\,N \cos\phi\, e^{q} = k\,N\cos\phi\,\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{e/2} . \qquad 8\text{-}4$$

We make the trigonometric substitution

$$\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right) = \frac{\cos\phi}{1-\sin\phi} \qquad 8\text{-}5$$

$$K = k\,\frac{N\cos^2\phi}{1-\sin\phi}\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{e/2} \qquad 8\text{-}6$$

$$= k\,\frac{a\,(1+\sin\phi)}{(1-e^2\sin^2\phi)^{1/2}}\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{e/2} . \qquad 8\text{-}7$$

At the pole $\phi = 90^\circ$ and $k = 1$, so

$$K = \frac{2a}{\sqrt{(1-e^2)}}\left(\frac{1-e}{1+e}\right)^{e/2} . \qquad 8\text{-}8$$

$$\boxed{K = \frac{2a^2}{b}\left(\frac{1-e}{1+e}\right)^{e/2} .} \qquad 8\text{-}9$$

The expression for the scale factor is

$$\boxed{k = \frac{2a^2}{b\,N\cos\phi}\left(\frac{1-e}{1+e}\right)^{e/2} e^{-q} .} \qquad 8\text{-}10$$

The meridian convergence is

$$\boxed{\gamma = \ell\lambda = \lambda} \qquad 8\text{-}11$$

The complex mapping equation for the direct problem is

$$\boxed{x + iy = K\,e^{i\ell(\lambda+iq)} ,} \qquad 8\text{-}12$$

where ℓ and K are defined above. The inverse problem is solved as for the LCC.

The Polar stereographic system is applied in practice as the universal stereographic projection (U.P.S.). It has the following characteristics [UPS 1958]:

1) units - metres

2) ellipsoid - international;

3) pole coordinates: $x = 2{,}000{,}000$; $y = 2{,}000{,}000$;

4) $k_{pole} = 0.994$.

The UPS was designed to be used in conjunction with the UTM to cover the entire globe: UTM to 80° latitude and USP from 80° to 90°.

8.2 The Stereographic Projection System in New Brunswick

The above is not a complete treatment of stereographic projection. For a more detailed coverage, see the following two Department of Surveying Engineering, University of New Brunswick Technical Reports:

(1) Frankich, K. (1977). A Study in Conformal Mapping. Department of Surveying Engineering, Technical Report No. 45, University of New Brunswick, Fredericton.

(2) Thomson, D.B., M.P. Mepham, R.R. Steeves (1977). The Stereographic Double Projection. Department of Surveying Engineering, Technical Report No. 46, University of New Brunswick, Fredericton.

9. A GENERALIZED SET OF CONFORMAL MAPPING EQUATIONS

9.1 Introduction

In the previous sections, we indicated which conformal map projection is best suited for areas of a particular shape such that the scale error would be a minimum. For example:

(a) North-south extent, Transverse Mercator,

(b) East-west extent, Lambert Conformal Conic.

A legitimate question to pose is: What map projection is to be employed when an area is located obliquely to the north-south or east-west directions. An oblique projection? This is one possible answer, but how about areas which are arc-shaped, e.g. circular, hyperbolic? A possible answer to the above questions is given below in connection with the analysis of a generalized set of formulae for the conformal mapping of the ellipsoid onto the plane.

9.2 Generalized Formulae for Conformal Projections

The generalized set of formulae were formulated after a careful analysis of the equations belonging to the various particular conformal map projections [Krakiwsky 1967]. Given below is the generalized set of formulas in our notation where C* is a constant, $\bar{R}$ is the radius of curvature of the ellipsoid at the central parallel, S_ϕ the length of the meridian, and the length of a parallel given by $S_p = \lambda N \cos \phi$. 9-1

$$x = \left(1 + \frac{3 S_\phi^2}{C^* \bar{R}^2}\right) S_p + \left(\frac{C^* - 6}{6C^* \bar{R}^2} - \frac{t^2}{6 \bar{R}^2}\right) S_p^2 + \cdots \qquad 9\text{-}2$$

$$y = S_\phi + \frac{S_\phi^3}{C^* \bar{R}^2} + \left(\frac{t}{2N} - \frac{3 S_\phi}{C^* \bar{R}^2}\right) S_p^2 + \cdots \qquad 9\text{-}3$$

The corresponding general scale factor equation is

$$k = 1 + \frac{6S_\phi^2 + (C^* - 6)\, S_p^2}{2\, C^* \bar{R}^2} \quad . \qquad 9\text{-}4$$

The main result is the above equation for the scale factor. The above set of formulas are good for maps at a scale of 1:50,000. In order to insure an accuracy of 1 to 4 meters in x and y, the area is limited to 450 km in latitude and longitude. More accuracy in x and y can possibly be achieved simply by including more terms in the expansion.

9.3 Application of the General Scale Formula

Specific map projections along with their varying shaped isoscale curves are derived by simply stipulating particular values for the constant C* in 9-4. An interesting analogy exists in physics where the path of motion of a particle in a central force field is determined to be an ellipse, hyperbola, etc. by simply stipulating a certain value of a constant in the solution of the differential equation of motion.

A wise selection of the constant C* makes it possible to obtain projections whose isoscale curves have the shape approaching that of the area to be mapped.

It should be noted that 9-4 is to be used only to investigate possible conformal projections; the actual computation of the scale should be performed with extended formulas giving more accuracy.

Immediately below, seven possible conformal projections are discussed as special cases of the general, among which will appear the four basic (Mercator, T.M., L.C.C., Polar Stereographic) conformal projections described earlier in these notes.

Case $C^* = \infty$ (Transverse Mercator):

The scale factor equation becomes

$$\mathbf{k} = 1 + \frac{6\, S_\phi^2}{2\, C^*\, \bar{R}^2} + \frac{C^*\, S_p^2}{2\, C^*\, \bar{R}^2} - \frac{6\, S_p^2}{2\, C^*\, \bar{R}^2}$$

$$= 1 + \frac{S_p^2}{2\, \bar{R}^2}$$

$$= 1 + \frac{\lambda^2 \cos^2 \phi}{2}\,, \qquad 9\text{-}5$$

which is identical to the truncated form for the Transverse Mercator derived earlier. Similarly, expressions for x and y are obtained by substituting $C^* = \infty$ in 9-2 and 9-3:

$$x = S_p - \frac{t^2}{6\bar{R}^2} S_p + \frac{S_p^3}{6\bar{R}^2} + \ldots , \qquad 9\text{-}6$$

$$y = S_\phi + \frac{t}{2N} S_p^2 + \ldots \; . \qquad 9\text{-}7$$

For the remaining cases, the x-y equations will not be given.

The main point of interest lies in the shape of the isoscale curves. From 9-5, the longitude from the central meridian of all points having a scale of $1 + \Delta k$ is

$$\lambda^o = \rho^o \frac{\sqrt{2}}{\cos\phi} \left(\frac{\Delta k}{k_o}\right)^{1/2} . \qquad 9\text{-}8$$

Basically, the curve has the shape $\frac{1}{\cos\phi}$ curve which changes slowly near the equator and rapidly near the poles (Figure 9-1).

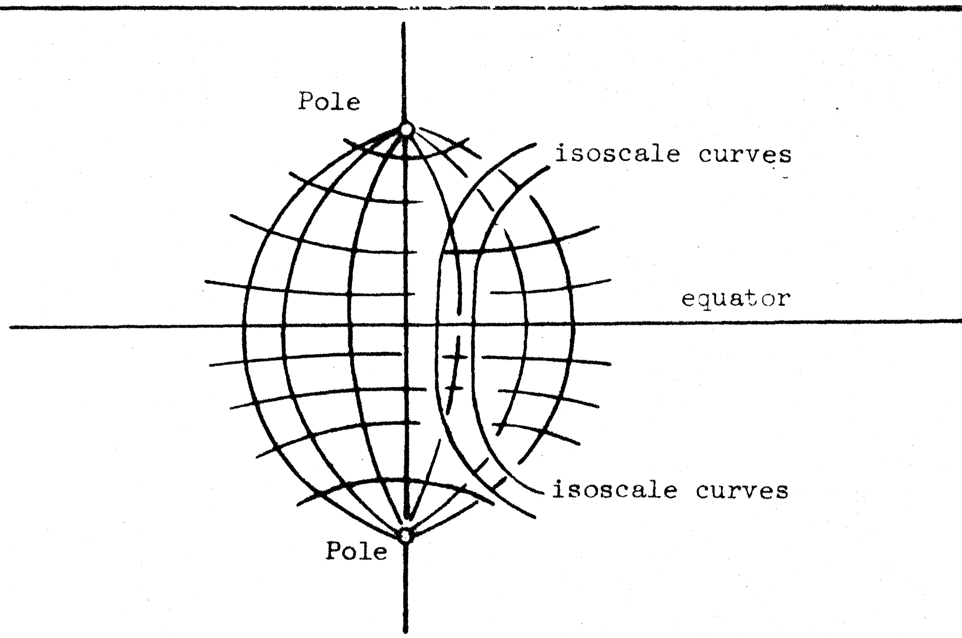


Figure 9-1: Isoscale Curves for Transverse Mercator C* = ∞

<u>Conclusion</u>. The above figure illustrates that an area great in north-south extent but not too close to the poles may enjoy constant scale error throughout its length and further the scale error may be made minimal if east-west zones are devised like in the U.T.M. It has already been mentioned that zones near the poles may be made as wide as 30° simply because of the shape of the isoscale curves in this area (e.g., at $\phi = 85°$ and $k_o = 0.9996$, $k = 1$ at $\lambda = 18°$).

<u>Case $12 < C^* < \infty$, $C^* = 18$</u>:

The equation for the scale error becomes

$$k = 1 + \frac{S_\phi^{\,2} + 2\,S_p^{\,2}}{6\,\overline{R}^{\,2}} , \qquad 9\text{-}9$$

or

$$\frac{S_\phi^{\,2}}{6\,\overline{R}^{\,2}} + \frac{S_p^{\,2}}{3\,\overline{R}^{\,2}} = k - 1 . \qquad 9\text{-}10$$

According to quadratic surfaces in solid analytical geometry,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{K^2}{C^2} \qquad 9\text{-}11$$

is the trace of an ellipsoid in the x-y plane with semi-major and minor axes of a and b, respectively. In our problem the elliptical trace has parameters

$$a = \sqrt{6}\,\overline{R} , \qquad 9\text{-}12$$

$$b = \sqrt{3}\,\overline{R} . \qquad 9\text{-}13$$

Thus the isoscale curves are ellipses elongated approximately two times more along the meridians than along the parallels (Figure 9-2).

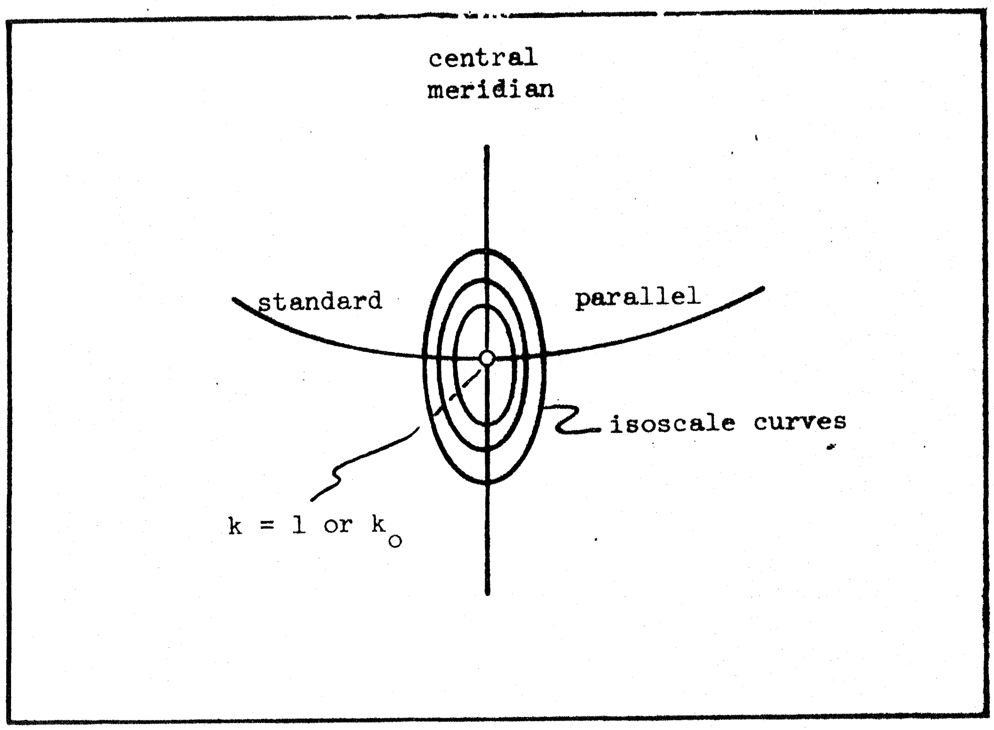


Figure 9-2: Isoscale Curves for $12 < C^* < \infty$

Conclusion. According to 9-9, k = 1 at one point only and from that general point the scale error increases in a specified manner. In order to have a nearly constant and minimal scale error in an elliptically shaped area, a value less than unity could be chosen for the central point. Such a selection could be based on a least square adjustment.

Case $C^* = 12$ (stereographic):

The scale error equation becomes

$$K = 1 + \frac{S_\phi^{\ 2}}{\overline{\frac{R}{3}}^{\,2}} + \frac{S_p^{\ 2}}{\overline{\frac{R}{3}}^{\,2}} , \qquad 9\text{-}14$$

and clearly the traces on the mapping plane are circles centered at a central point within the area to be mapped (Figure 9-3).

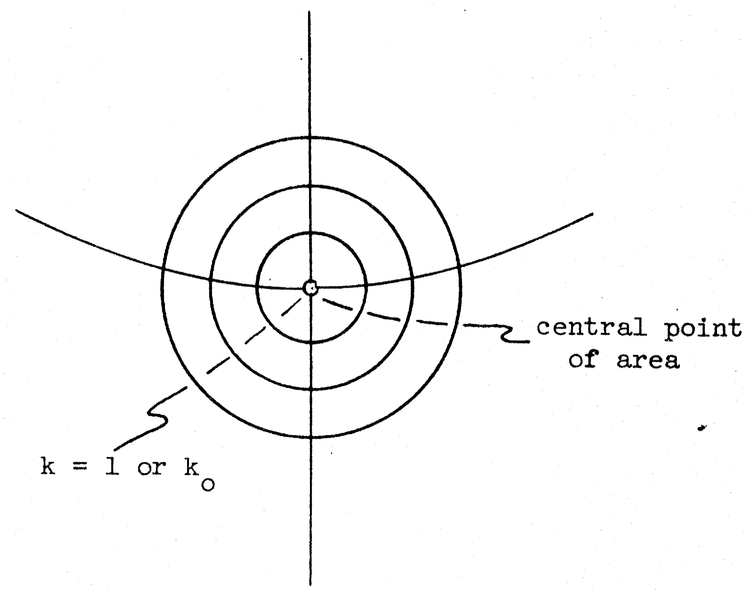


Figure 9-3: Stereographic Projection, C* = 12

The Polar Stereographic Projection is obtained by choosing the central point as the pole. Further, the U.S.P. is obtained by choosing $k_o = 0.994$ at the pole.

Conclusion. The stereographic projection or U.S.P. seems well suited for small circular areas, which is undoubtedly the reason for using it in the polar areas.

Case 6 < C* < 12, C* = 9.

The scale error equation becomes

$$k = 1 + \frac{S_\phi^{\,2}}{3\,\bar{R}^2} + \frac{S_p^{\,2}}{6\,\bar{R}^2} \quad . \qquad 9\text{-}15$$

This case is simply Case C* = 19, rotated 90°.

Conclusion. This case of conformal map projection should be used for small areas elliptically elongated in the longitudinal direction.

Case C* = 6 (Lambert Conformal Conic)

The scale error equation becomes

$$k = 1 + \frac{S_{\phi}^{2}}{2\,\bar{R}^{2}} \qquad 9\text{-}16$$

The isoscale curves are functions solely of the meridian distance from the standard parallel and coincide with the parallels of latitude. This is obviously the case in the Lambert Conformal Conic Projection.

Conclusion. Since the scale can be kept constant along a given parallel, the use is obvious in any area of great longitudinal extent. In order to increase the latitudinal coverage, two standard parallels are introduced with scale factors different from unity.

Case C* = 6, C* = 3

The scale error equation is

$$K = 1 + \frac{S^{2}}{\bar{R}^{2}} - \frac{S_{p}^{2}}{2\,\bar{R}^{2}}. \qquad 9\text{-}17$$

The unequal denominators and the minus sign in the above equation signify that the traces of the isoscale curves on the map plane are hyperbolic curves whose vertical axis coincides with the central meridian (Figure 9-4).

The Transverse Mercator case which is similar to this one can be obtained from Equation 9-17 by omitting the second term and changing the sign of the third, however without mathematical justification.

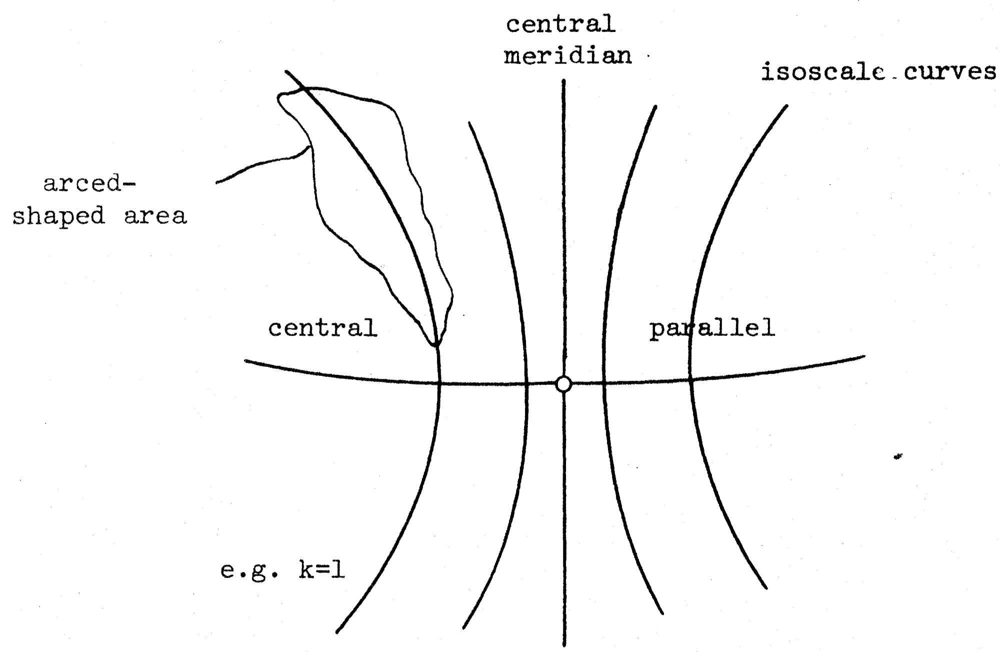


Figure 9-4: Isoscale Curves Case C* = 3

Conclusion. The above figure depicts the odd shape of the isoscale curves. A stipulation of unity for the scale factor at the central point in this case would be a bad choice since there is a rapid change of the scale factor from this point. A better choice would be some value less than unity such that a curve away from the central point and which passes through the centre of an arced-shaped area would have a value of k = 1 (Figure 9-4).

Case $C^* < 0$, $C^* = -3$

$$K = 1 + \frac{S_p^2}{\frac{2}{3}\bar{R}^2} - \frac{S_\phi^2}{\bar{R}^2} \quad . \qquad 9\text{-}18$$

The isoscale curves in this case are hyperbolic with the main axis coinciding with the central parallel.

This case slightly resembles the Mercator. Equation 9-18 could be revamped to fit the Mercator Projection by omitting the 2nd last term (making the scale factor dependent on latitude only), thus

$$k = 1 + \frac{S_{\phi}^{2}}{\bar{R}^{2}} \simeq 1 + \frac{\bar{R}^{2}\phi^{2}}{\bar{R}^{2}} \simeq 1 + \phi^{2} \qquad 9\text{-}19$$

as compared to

$$k = \frac{a}{N\cos\phi} = 1 + \frac{a - N\cos\phi}{N\cos\phi} \simeq 1 + \frac{1 - \cos\phi}{\cos\phi} . \qquad 9\text{-}20$$

These two equations are not completely equivalent.

Conclusion. This projection could be used for areas which are arced-shaped and extend slightly obliquely to the E-W direction, i.e., East-East-North (EEN).

9.4 Summary

By studying the gerneral equation for the scale factor, it was demonstrated how projections with quite distinctly different shaped isoscale curves could be derived. It was also stated which case or shape of isoscale curve may be best suited for an area of particular shape. Table 9-1 serves as a summary of the foregoing.

In general, it may be stated that when mapping small areas, projections with closely knit isoscale curves should be used, thus achieving minimal scale error within the area with a minimum number of zones. Medium and large sized areas of varying shape may be mapped with projections possessing more sweeping isoscale curves like hyperbolas. When the entire earth is to be mapped, the projection with the most sweeping isoscale curves used in conjunction with zones is the most satisfactory, namely the Transverse Mercator.

Table 9-1: Suggested Conformal Projection

	Area Description	Case	Projection Name	Shape of Isoscale Curves
Large to Medium Extent	N - S	$C^* = \infty$	Transverse Mercator	Oblong along central meridian
	E - W	$C^* = 6$	Lambert Conformal Conic	Corresponding to parallels
		$C^* = -3$?	Mercator	Straight lines (infinitely large hyperbolas)
	Oblique or Arced: NNE and other quadrants	$C^* = 3$	________ ?	Hyperbolas with central meridian as main axis
	Oblique or Arced: EEN and other quadrants	$C^* = -3$	________ ?	Hyperbolas with central parallel as main axis
Small Extent	Circular (e.g. polar areas)	$C^* = 12$	Stereographic	Concentric circles
	Elliptical N - S	$C^* = 18$	________?	Ellipses extended along meridian
	Elliptical E - W	$C^* = 9$	________ ?	Ellipses extended along parallel

Note: Based on the assumption that minimal error is desired within area.

SECTION II

COMPUTATIONS ON A CONFORMAL MAP PROJECTION PLANE

10. INTRODUCTION TO COMPUTATIONS ON A CONFORMAL MAP PROJECTION PLANE

In Section I we treated the problem of transformation of geodetic coordinates (ϕ, λ) on the ellipsoid to map coordinates (x, y) on the conformal map plane, and vice versa. The first objective of this section is to show how geodetic position computations, such as the direct and inverse problem, can be made in terms of map coordinates. The advantage of this approach is that the mathematical models can be characterized by plane geometry. The disadvantage is that observations, such as distances, directions, astronomic or gyro-theodolite azimuths, must be reduced to the conformal map projection plane. The development of the formulae for the reductions is the second objective of Section II.

It is worthwhile to place the map projection method of geodetic position computation in context by outlining at least two other methods of computation. First, recall that computations in three-dimensions required that observations remain in space, as observed, with correction made only for the effects of the gravity field and the atmosphere. No surface such as an ellipsoid, sphere, or plane was explicitly employed, thus no reductions of the observations was necessary. Mathematical models were characterized by three-dimensional geometry. Coordinates of points were expressed in terms of three dimensional coordinates (x, y, z, or ϕ,

λ, h). Also, recall that computations on the surface of the ellipsoid required that observations be reduced to the surface of the ellipsoid by first taking into account the effects of the gravity field and the atmosphere, and then the effect of the height of target and geodesic-normal section separation. Mathematical models were characterized in terms of ellipsoid surface geometry. Coordinates for points were given in terms of horizontal geodetic coordinates (ϕ, λ).

Why compute geodetic positions on a conformal projection plane instead of, for example, on an equiareal or an equidistant map projection plane? The answer to this question is straightforward. Conformal projections preserve angles and therefore measured angles can be directly used for computational purposes. However, sometimes these angles need to be corrected in order to transform them into the corresponding plane angles required for plane computations (see equation 11-1). The scale factor at a point in conformal mapping is a function of the position of the point only.

$$k = k(\phi, \lambda) \quad . \qquad 10\text{-}1$$

In other words, all distances in a relatively small area must be multiplied by one and the same scale factor. However, for longer lines some average scale factor should be used.

In equiareal and equidistant projections, the scale factor is also a function of the azimuth at a point, i.e.

$$k = k(\phi, \lambda, \alpha) \quad , \qquad 10\text{-}2$$

which implies that the angles are not preserved.

The third important advantage of conformal projections is the fact that with a priori defined relative accuracy of linear scale factor conformal maps will cover the largest domain compared to all other conceivable map projections. That means with conformal maps we shall have a minimum number of zones or projection systems.

In the chapters to follow, we discuss: the problem of reduction of observations onto the map projection plane (section 11); the various mathematical models for computation on a map projection plane (section 12); the general formulae for reductions (section 13); the reduction formulae for specific projections (section 14).

11. REDUCTION OF OBSERVATIONS

In this section we treat the problem of the reduction of distance, direction, and azimuth observations onto the conformal map plane. First we describe why these reductions are needed.

During the conformal mapping process, lengths of ellipsoid geodesics are altered. The directional characteristics of these geodesics on the map plane is dictated by the conformality condition, that is to say, geodesics are projected onto the map plane in a manner that preserves the angle which a given pair of geodesics define.

11.1 Reduction of Angles

Let us examine two specific angles on the map plane. The first is the angle between two projected geodesics at a point to two other points, and the second is the angle between the two chords at a point to two other points.

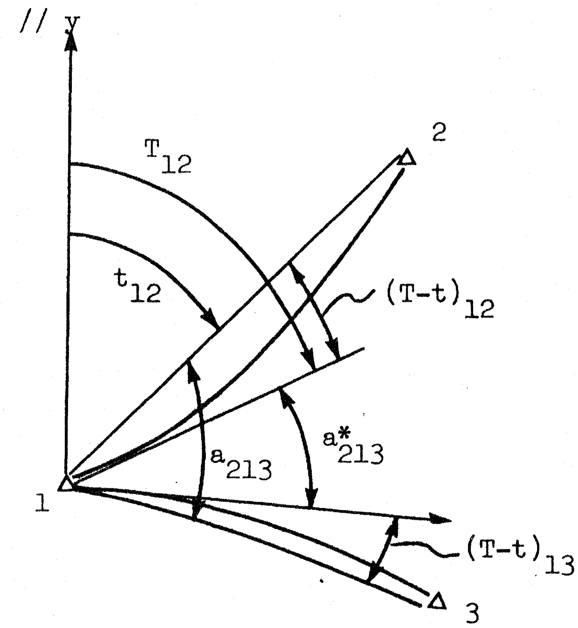


Figure 11-1. Angles.

Because of conformality the ellipsoid angle is equivalent to the angle between the two projected geodesics (a^*_{213}). The grid azimuth of the projected geodesic is denoted by T_{12}, while the grid azimuth of the chord is denoted by t_{12}. The difference between these two azimuths is known as the "T minus t" correction. A similar situation exists between points 1 and 3. The angle between the two geodesics is related to the grid angle by the equation

$$\boxed{a_{213} = a^*_{213} + (T-t)_{12} + (T-t)_{13}\ .} \qquad 11\text{-}1$$

a_{213} is the angle needed in the computation of positions of unknown points. In general $(T-t)_{12}$ does not equal $(T-t)_{13}$, and further the formulae for computing the $(T-t)$ correction is different for each conformal map projection.

11.2 Reduction of Azimuths

There are three different azimuths to contend with. Two have already been defined (T and t). The third is the angle between the tangent to the projected meridian and the tangent to the projected geodesic and is denoted as α_{12} (Figure 11-2). The following relationship is valid

$$T_{12} = \alpha_{12} - \gamma \; , \qquad 11\text{-}2$$

where γ is the meridian convergence. Further

$$t_{12} = T_{12} - (T-t)_{12},$$

$$\boxed{t_{12} = \alpha_{12} - \gamma - (T-t)_{12} \; .} \qquad 11\text{-}3$$

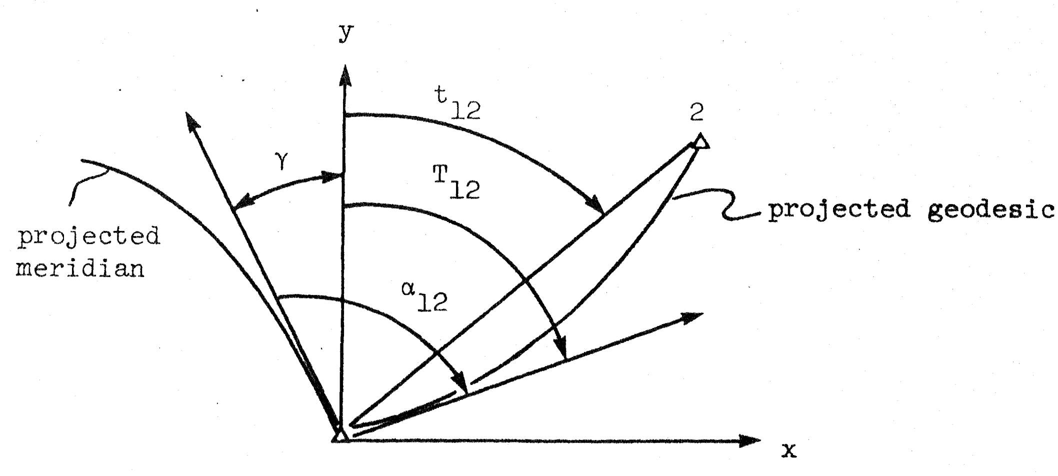


Figure 11-2. Azimuths

The above expression gives the value of the grid azimuth on the plane in terms of the "observed" ellipsoid azimuth. t_{12} is needed in computations on the plane. Expressions for γ have already been derived. What remains to be done is the derivation of expressions for (T-t) for the various projections.

11.3 Reduction of Distances

Three types of distances are involved: the "observed" geodesic distance S*; the length of the projected geodesic distance S; the chord length d (Figure 11-3). The chord distance is

$$d_{12} = S^*_{12} - (S^*-S)_{12} - (S-d)_{12} \,, \qquad 11\text{-}4$$

where $(S^*-S)_{12}$ is the correction due to the length distortion of the geodesic, while $(S-d)_{12}$ is due to the difference in length between the projected geodesic and the chord. These two corrections are usually combined into one as we will see later when it is derived for the various map projections.

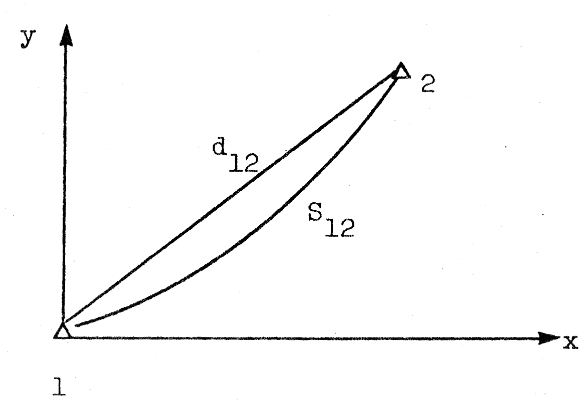


Figure 11-3. Distance

12. MATHEMATICAL MODELS FOR COMPUTATION OF POSITIONS

Let us turn away from the conformal mapping problem pursue to the problem of computing on the mapping plane. The reason for doing this is purely stategical. We use this application-oriented section as motivation, which will undoubtedly help us endure the rather tedious but interesting, development of the reduction formulae.

The mathematical models are not new to a student at this level. The intent is to recapituate the models and show where and when the various reductions are necessary. For an exhaustive treatment of models for the computations on a plane, see for example, Faig [1972].

We treat the problems of: intersection, traverse with astronomic azimuth orientation; traverse with grid azimuth orientation; adjustment of a network on a plane; and calibration of a gyro-theodolite using conformal map projection coordinates.

12.1 Intersection

The intersection problem is depicted in Figure 12-1 along with the projected geodesics. The knowns are the map coordinates of the two points 1 and 2; the unknowns are the map coordinates of the third point 3; the observed quantities are the two angles at 1 and 2.

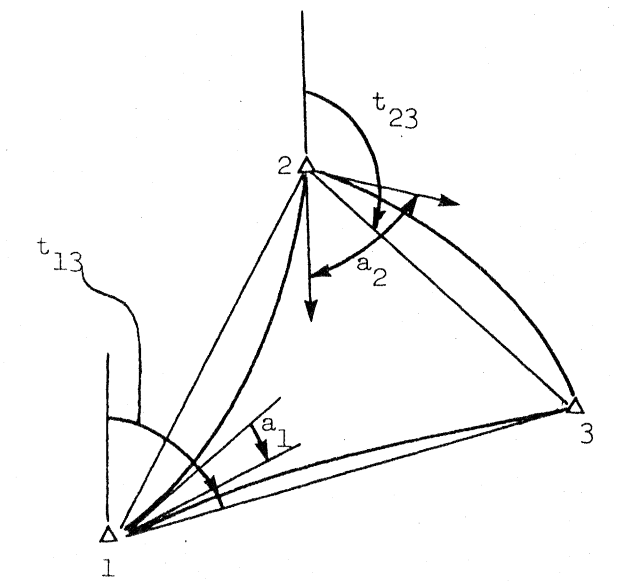


Figure 12-1. Intersection and Projected Geodesics

The solution for the two unknown coordinates x_3, and y_3 is made from the two equations:

$$\tan t_{13} = \frac{x_3 - x_1}{y_3 - y_1} \quad ,$$

$$\tan t_{23} = \frac{x_3 - x_2}{y_3 - y_2} \quad ,$$

where

$$t_{13} = t_{12} + (T-t)_{12} + a_1 + (T-t)_{13} \ ,$$

and

$$t_{23} = t_{21} - (T-t)_{21} - a_2 + (T-t)_{23} \ .$$

Note t_{12} and t_{21} are computed from known coordinates; a_1 and a_2 observed (ellipsoid values); while the (T-t) correction needs to be computed (expressions to be derived in Section 13).

12.2 Traverse with Astronomic Azimuth Orientation

The traverse problem along with the projected meridians and geodesics is depicted in Figure 12-2. The knowns are the map coordinates of a starting point 1. The observed quantities are the astronomic azimuth of the geodesic (1-2), the angles and the distances. The unknowns are the coordinates of points 2 to n.

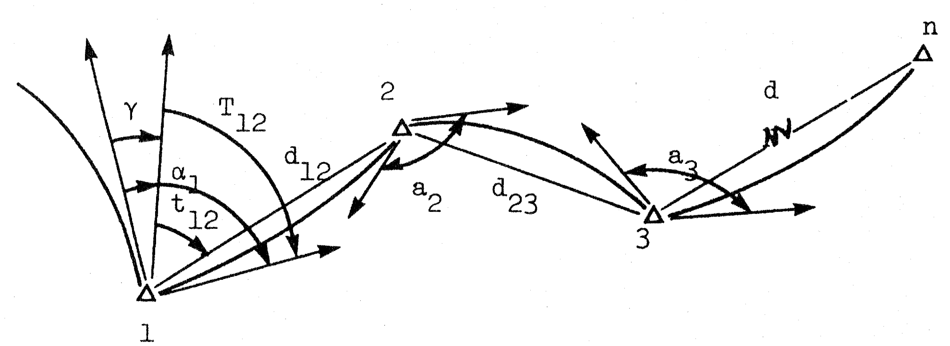


Figure 12-2. Traverse

The coordinates of point 2 (e.g. x_2) can be computed from

$$\sin t_{12} = \frac{x_2 - x_1}{d_{12}} \quad ,$$

where

$$t_{12} = \alpha_{12} - \gamma - (T-t)_{12},$$

and

$$d_{12} = S^*_{12} - (S^* - S)_{12} - (S - d)_{12} \ .$$

Again we see that (T-t) correction is necessary along with the meridian convergence. Distances also need reduction.

12.3 Traverse with Grid Azimuth Orientation

The solution to this problem is similar to the previous case, with one change, that is

$$t_{12} = t_1 + a_1 + (T-t)_1 - (T-t)_{12} \ .$$

Note an observed angle at 1 is necessary. This requires a computation of a (T-t) correction for the backsight. All other corrections are identical to that explained immediately before.

12.4 Adjustment of a Network on a Conformal Map Plane

The unknowns in the adjustment of a network on a conformal map plane are the usual unknown coordinates (possibly some nuisance parameters such as an orientation unknown at each station). The knowns are the map coordinates of the stations to be fixed. The observed quantities are directions, distances and azimuths. These need to be reduced to the map plane as expressed by 11-1, 11-3, 11-4. The adjustment then proceeds as usual.

12.5 Gyro-Theodolite Calibration Using Map Coordinates

The basic idea is to compare the azimuth observed by the gyro with the azimuth computed from map coordinates. They are, of course, two different kinds of azimuths, thus one must be "reduced" to the other. What reductions are necessary? All necessary concepts needed to answer this question have already been dealt with - thus this problem is left as an exercise.

Assume that a gyro-azimuth is equivalent to an astro-azimuth.

13. GENERAL FORMULAE FOR REDUCTION TO THE MAP PROJECTION PLANE

The general formulae for reduction of distances and directions to the conformal map projection plane are discussed in this section. These general formulae are valid for most conformal projections. The purpose of this discussion is to give the reader an appreciation for the fundamentals underlying the computational versions of these formulae given in Section 14 for some specific projections. By understanding the fundamentals, the users of the computational formulae can better understand their application and limitations.

Discussed immediately below are:

(1) the radius of curvature of a projected geodesic;

(2) the parametric equations of the projected geodesic;

(3) the difference in length between the projected geodesic and the chord;

(4) the difference in length between the geodesic and the projected geodesic;

(5) the angle between the projected geodesic and the chord.

Just the main steps of the derivations are given. More details can be found in Thomas [1952].

13.1 Curvature of the Projected Geodesic

By studying the curvature of the projected geodesic we gain insight into the geometry of these curves on the map plane. From calculus we know that the curvature is defined as

$$\sigma = \frac{1}{R} = \frac{x^{I}y^{II} - y^{I}x^{II}}{[(x^{I})^2 + (y^{I})^2]^{3/2}} , \tag{13-1}$$

where R is the radius of curvature, and

$$\begin{aligned} x &= x(s) \\ y &= y(s) \end{aligned} \tag{13-2}$$

are functions of the variable s, which is the length of the projected geodesic. From Figure 4-4,

$$\begin{aligned} &x^{I} = \frac{dx}{ds} = \cos\beta , \qquad y^{I} = \frac{dy}{ds} = \sin\beta , \\ &x^{II} = \frac{d^2x}{ds^2} = -\sin\beta \frac{d\beta}{ds} , \; y^{II} = \frac{d^2y}{ds^2} = \cos\beta \frac{d\beta}{ds} . \end{aligned} \tag{13-3}$$

Substituting 13-3 into 13-1 yields

$$\begin{aligned} \sigma = \frac{1}{R} &= \frac{\cos^2\beta \frac{d\beta}{ds} + \sin^2\beta \frac{d\beta}{ds}}{(\cos^2\beta + \sin^2\beta)^{3/2}} \\ &= \frac{d\beta}{ds} . \end{aligned} \tag{13-4}$$

Thomas derives an expression for the curvature as a function of the scale factor k and coordinate η (figure 4-4). Namely

$$\sigma = \frac{d\beta}{ds} = \frac{1}{k} \left(\frac{\partial k}{\partial x} \sin \beta - \frac{\partial k}{\partial y} \cos \beta \right),$$
$$= \frac{1}{k} \left(\frac{\partial k}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial k}{\partial y} \cdot \frac{\partial y}{\partial \eta} \right) = \frac{1}{k} \frac{\partial k}{\partial \eta} \qquad 13\text{-}5$$

Note in the above all quantities are evaluated at the point where σ is desired.

13.2 Parametric Equations of the Projected Geodesic

The projected geodesic is described by two parametric equations of the form

$$\eta = \eta(s),$$
$$\xi = \xi(s). \qquad 13\text{-}6$$

Approximating the above equations with a Maclaurin series, with point 1 as the point of expansion, we get (Figure 13-1)

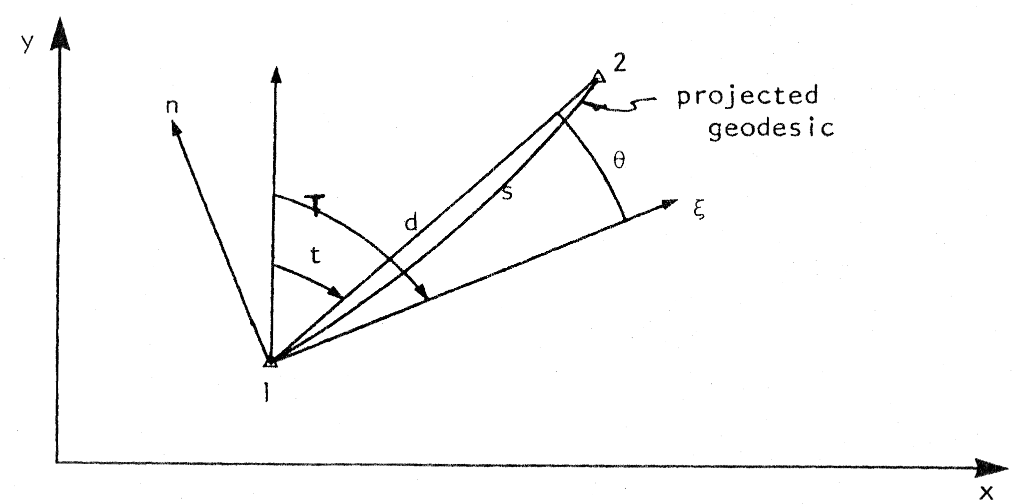


Figure 13-1: Coordinate Systems for Parametric Equations

$$\xi = s\,\frac{d\xi(o)}{ds} + \frac{s^2}{2!}\,\frac{d^2\xi(o)}{ds^2} + \frac{s^3}{3!}\,\frac{d^3\xi(o)}{ds^3} + \ldots,$$

$$\eta = s\,\frac{d\eta(o)}{ds} + \frac{s^2}{2!}\,\frac{d^2\eta(o)}{ds^2} + \frac{s^3}{3!}\,\frac{d^3\eta(o)}{ds^3} + \ldots. \qquad 13\text{-}5$$

From 13-1

$$\sigma = \frac{\dfrac{d\xi}{ds}\,\dfrac{d^2\eta}{ds} - \dfrac{d\eta}{ds}\,\dfrac{d^2\xi}{ds}}{\left[\left(\dfrac{d\xi}{ds}\right)^2 + \left(\dfrac{d\eta}{ds}\right)^2\right]^{3/2}}. \qquad 13\text{-}6$$

By first evaluating the derivatives in 13-5 and then replacing them with curvatures (13-6), we get

$$\boxed{\begin{aligned}\xi &= s - \frac{s^3}{6}\,\sigma_o^{\,2} - \frac{s^4}{8}\,\sigma_o\sigma_o^{\,I} - \frac{s^5}{120}\,(4\sigma_o\sigma_o^{\,II} + 3(\sigma_o^{\,I})^2 - \sigma_o^{\,4}) - \ldots,\\ \eta &= \frac{s^3}{2}\,\sigma_o + \frac{s^3}{6}\,\sigma_o^{\,I} + \frac{s^4}{24}\,(\sigma_o^{\,II} - \sigma_o^{\,3}) + \frac{s^5}{120}\,(\sigma_o^{\,III} - 6\sigma_o^{\,2}\sigma_o^{\,I}) + \ldots,\end{aligned}} \qquad 13\text{-}7$$

where

$$\sigma^I = \frac{d\,\sigma}{ds},\quad \sigma^{II} = \frac{d^2\sigma}{ds^2}$$

are evaluated from

$$\sigma = \sigma_o + \sigma_o^{\,I}s + \sigma_o^{\,II}\,\frac{s^2}{2} + \sigma_o^{\,III}\,\frac{s^3}{6} + \ldots \qquad 13\text{-}8$$

and where the subscript "o" means evaluated at the point of expansion.

13.3 Difference in Length Between Projected Geodesic and Chord

Since we compute with chords, we need to determine the difference in length between the projected geodesic and chord. From Figure 13-1, we can write and expression for the chord length d in terms of the coordinates ξ and η, (13-7), namely

$$d^2 = \eta^2 + \xi^2 \qquad 13\text{-}9$$

$$= s^2 - s^4 \frac{\sigma_o^2}{12} - \frac{s^5 \sigma_o \sigma_o^I}{12} -$$

$$- \frac{s^6}{360} [9\sigma_o \sigma^{II} + 8(\sigma_o^I)^2 - \sigma_o^4) - \ldots \qquad 13\text{-}10$$

$$d = (\xi^2 + \eta^2)^{1/2}$$

$$= s \{ 1 - \frac{s^2}{12} [\sigma_o^2 + \sigma_o \sigma_o^I s + \frac{s^2}{30} (9\sigma_o \sigma_o^{II} + 8(\sigma_o^I)^2 - \sigma_o^4) + ..]\}^{1/2} \qquad 13\text{-}11$$

Since the second term in brackets is small the expression to the 1/2 power may be expanded. The result gives the difference

$$s - d = s^3 \frac{\sigma_o^2}{24} + s^4 \frac{\sigma_o \sigma_o^I}{24} + \frac{s^5}{5760} (72\sigma_o \sigma_o^{II} +$$

$$+ 64(\sigma_o^I)^2 - 3\sigma_o^4) + \ldots.. \qquad 13\text{-}12$$

If we expand about the mid-point of s instead of at one end, then

$$\boxed{s\text{-}d = \frac{s^3 \sigma_2^2}{24} + \frac{s^5}{5760} \; (\ldots..),} \qquad 13\text{-}13$$

where σ_2 is the radius of curvature at the midpoint of the projected geodesic. The value of this difference is usually small.

13.4 Difference in Length Between the Projected Geodesic and the Geodesic

Combined with the difference s-d, the difference in length between the projected geodesic and the geodesic (s), that is S - s allows one to reduce geodesic lengths on the ellipsoid to chord lengths on the projection.

The basic relationship used in this context follows from the definition of the scale factor, namely

$$ds = k \, dS \, . \qquad 13\text{-}14$$

The distance on the map projection (projected geodesic) is then

$$s = \int k \, ds \qquad 13\text{-}15$$

Thomas derives an expression for the above which involves a series development of the scale factor with point of expansion being one end of the line. The result is

$$s = k_o S + \frac{1}{2} k_o k_o^{I} S^2 + \frac{1}{6} (k_o^2 k_o^{I} + k_o k_o^{I}) S^3 + \ldots, \qquad 13\text{-}16$$

where $k^{I} = \frac{dk}{ds}$. The second and higher terms account for the change of scale away from the point of expression. The equation for the difference is

$$\boxed{S - s = (\frac{1}{k_o} - 1) s + (\frac{1}{k})_o^{I} \frac{s^2}{2} + (\frac{1}{k})_o^{II} \frac{s^3}{6} + \ldots ,} \qquad 13\text{-}17$$

where $1/k$ is the reciprocal for the scale factor.

Analogous expressions to 3-16 and 3-17 for a midpoint-point of expansion are

$$S = \frac{1}{k_M} s + (\frac{1}{k_M})^{II} \frac{s^3}{24} + \ldots \quad , \qquad 13\text{-}18$$

and

$$\boxed{S - s = (\frac{1}{k_M} - 1) s + (\frac{1}{k_M})^{I} \frac{s^3}{24} - \frac{1}{8k_M} \sigma_M^2 s^3 + \ldots .} \qquad 13\text{-}19$$

13.5 The Angle Between the Projected Geodesic and the Chord

In conformal projections, the azimuths of the geodesic and projected geodesics are identical, but not so for distances as we have just seen. For this reason we need concern ourselves only with the one angle - that between the chord and projected geodesic.

The angle we seek is defined by (Figure 13-1) $\theta = T - t$.

In terms of the coordinates ξ and η

$$\tan\theta = \frac{\eta}{\xi} \quad \text{13-20}$$

Expanding the $\tan\frac{\eta}{\xi}$ in a series, we get

$$\boxed{\theta = \frac{s}{2}\sigma_o + \frac{s^2}{6}\sigma_o^{I} + \frac{s^3}{24}\sigma_o^{II} + \frac{s^4}{720}(16\sigma_o^{III} - \sigma_o^{2}\sigma_o^{I} + \dots} \quad \text{13-21}$$

14. SPECIFIC FORMULAE FOR REDUCTION TO VARIOUS MAP PROJECTION PLANES

In Section 13 we derived the general formulae for the reduction of ellipsoidal distances and directions to the map projection plane. These formulae are expressed in terms of the curvature of the projected geodesic and the scale factor. In this section we derive reduction formulae for the Transverse Mercator projection by evaluating the curvature and scale factor. We also state the reduction formulae of the Stereographic Projection in New Brunswick.

Reduction formulae are not given for the Mercator or Universal Stereographic Projections as geodetic computations of positions are not usually performed on these projections. The reasons being is that the USP is used in the polar regions while the Mercator is used mainly for navigation.

14.1 Reduction Formulae for the Transverse Mercator

Evaluated immediately below is the curvature of the projected geodesic. Then the reduction formulae for directions and distances are derived.

14.1.1 Curvature

Recall the expression for the curvature (13-5)

$$\sigma = \frac{1}{k}\left(\frac{\partial k}{\partial x}\sin\beta - \frac{\partial k}{\partial y}\cos\beta\right) . \qquad 14\text{-}1$$

For the TM, k has the form

$$k = 1 + C_1x^2 + C_2x^4 + C\,x^6 + \ldots , \qquad 14\text{-}2$$

where, for example

$$C_1 = \frac{1 + \eta_1^{\,2}}{2N_1^{\,2}} . \qquad 14\text{-}3$$

We recognize that k depends mainly on x and only slightly on y because y is only implicitly involved in η_1 and N_1 through ϕ_1 the foot point latitude Equation 14-1 can then be approximated by

$$\sigma = \frac{1}{k}\frac{\partial k}{\partial x}\sin\beta . \qquad 14\text{-}4$$

For further use we need the following expressions which are derived in Thomas [1952]

$$k^{I} = \frac{\partial k}{\partial x}\cos\beta , \qquad 14\text{-}5$$

$$k^{II} = \frac{\partial^2 k}{\partial x^2}\cos^2\beta - k\sigma^2 , \qquad 14\text{-}6$$

$$\left(\frac{1}{k}\right)^{I} = \frac{d(\frac{1}{k})}{dx}\cos\beta , \qquad 14\text{-}7$$

$$\left(\frac{1}{k}\right)^{II} = \frac{d^2}{dx^2}\left(\frac{1}{k}\right)\cos^2\beta - \frac{d}{dx}\left(\frac{1}{k}\right)\sigma\sin\beta . \qquad 14\text{-}8$$

14.1.2 Arc to Chord Evaluation

Recall the general formulae giving the angle between the arc of the projected geodesic and the chord (13-21)

$$\theta = T - t = \frac{s}{2}\ \sigma_o + \frac{s^2}{6}\sigma_o{}^I + \ldots \qquad 14\text{-}8$$

We first evaluate σ_o for the TM. For 1/k in 14-4 we get from (14-2 and 6-77).

$$\frac{1}{k} \doteq 1 - \frac{1}{2}\ \frac{x}{N_1{}^2} \qquad 14\text{-}9$$

and

$$k \doteq 1 + \frac{1}{2}\ \frac{x^2}{N_1{}^2} \qquad 14\text{-}10$$

Also

$$\frac{dk}{dx} = \frac{x}{N_1{}^2} \qquad 14\text{-}11$$

Substituting 14-9 and 14-11 into 14-4 yields

$$\sigma = (1 - \frac{1}{2}\frac{x}{N_1{}^2})\ \frac{x}{N_1{}^2}\ \sin\beta\ \cdot \qquad 14\text{-}12$$

The curvature at one end of the line (say point 1) can then be evaluated as

$$\sigma_o = (1 - \frac{1}{2}\ \frac{x_1}{N_1{}^2})\ \frac{x_1}{N_1{}^2}\ \sin\beta_1\ . \qquad 14\text{-}13$$

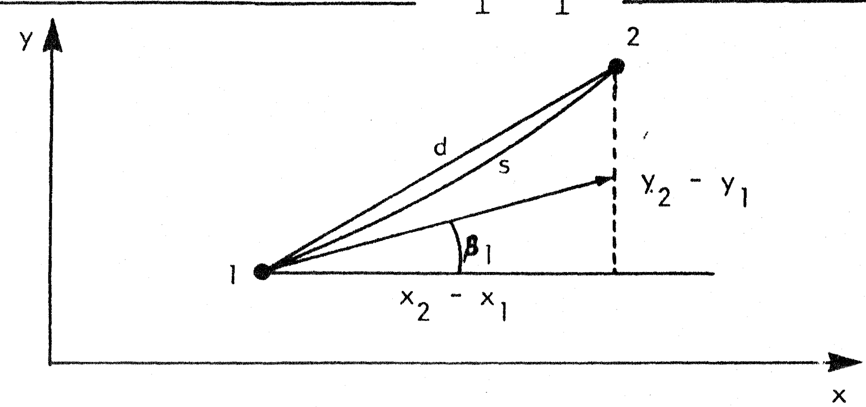


Figure 14-1 Chord and Arc

We now wish to approximate x in 14-12 with (Figure 14-1)

$$x \doteq x_1 + s \cos \beta \quad , \qquad 14\text{-}14$$

where $s \doteq d$ at any point between 1 and 2. Substituting 14-14 in 14-12 yields

$$\sigma = \left[1 - \frac{1}{2}\frac{(x_1 + s\cos\beta)^2}{N_1^{\ 2}}\right]\left(\frac{x_1 + s\cos\beta}{N_1^{\ 2}}\right)\sin\beta$$

$$= \frac{(x_1 + s\cos\beta)\sin\beta}{N_1^{\ 2}} - \frac{1}{2}\,\frac{(x_1 + s\cos\beta)^3\sin\beta\cos\beta}{N_1^{\ 4}} + \ldots \qquad 14\text{-}15$$

The other quantity reading evaluation is the change of the curvature $\sigma_o^{\ I}$ in 14-18. From 14-15

$$\sigma^I = \frac{d\sigma}{ds} = \frac{\cos\beta\sin\beta}{N_1^{\ 2}} - \frac{3}{2}\,\frac{(x_1 + s\cos\beta)^2\sin\beta\cos^2\beta}{N_1^{\ 4}} \qquad 14\text{-}16$$

Thus $\sigma_o^{\ I}$ at point 1 is obtained from above by replacing β by β_1 (Figure 14-1).

The expression for T - t is obtained by substitution of 14-16 (with $\beta = \beta_1$) and 14-13 into 14-8.

The result is

$$\theta = T - t = \frac{s}{2}\frac{x_1\sin\beta_1}{N_1^{\ 2}} - \frac{1}{2}\frac{x_1^{\ 3}}{N_1^{\ 4}}\sin\beta_1 +$$

$$+ \frac{s^2}{6}\left(\frac{\sin\beta_1\cos\beta_1}{N_1^{\ 2}} - \frac{3}{2}\,x_1^{\ 2}\,\frac{\sin\beta_1\cos^2\beta_1}{N_1^{\ 4}} + \ldots \qquad 14\text{-}17\right.$$

$$= \frac{s}{2}\frac{x_1\sin\beta_1}{N_1^{\ 2}} + \frac{s^2}{6}\,\frac{\sin\beta_1\cos\beta_1}{N_1^{\ 2}} + \ldots \; . \qquad 14\text{-}18$$

Eliminating β_1 from the above by (Figure 14-1)

$$\sin \beta_1 \doteq \frac{y_2 - y_1}{s} \; ,$$
$$\cos \beta_1 \doteq \frac{x_2 - x_1}{s} \; , \qquad 14\text{-}19$$

we get

$$\boxed{\theta = T - t = \frac{y_2 - y_1}{6N_1^{\,2}} (x_2 + 2x_1) + \ldots} \qquad 14\text{-}20$$

where N_1 can be approximated by the Gaussian mean radius at point 1.

The above equation is of sufficient accuracy for most work. It can be compared to the following formulae of Bomford [1962]

$$\boxed{T - t = \frac{(y_2 - y_1)\,(x_2 + 2x_1)}{6R_m^{\,2}} \left[1 - \frac{(2x_1 + x_2)^2}{27R_m^{\,2}}\right]} \qquad 14\text{-}21$$

where

$$R_m = \sqrt{M\,N} \; ,$$

M and N evaluated at $(\phi_1 + \phi_2)/\,2$. The accuracy of this formulae is claimed to be 0".02 for a line 100 km in length but within 3° of the central meridian.

The correct application of the correction is made by realizing that projected geodesics are concave towards the central meridian (see Figure 6-4).

14.1.3 Line Scale

The line scale is the average scale over the line. It allows us to get the difference between the lengths of the geodesic (S) and the projected geodesic (s). Recall the general expression (13-17)

$$S - s = (\frac{1}{k_o} - 1)\, s + (\frac{1}{k})_o^{I} \frac{s^2}{2} + (\frac{1}{k})_o^{II} \frac{s^3}{6} + \ldots , \qquad 14\text{-}22$$

where the derivatives are defined by 14-7 and 14-8. We now evaluate

$$\frac{1}{k_o} = 1 - \frac{1}{2} (\frac{x_1}{N_1})^2 , \qquad 14\text{-}23$$

$$\frac{d}{dx} (\frac{1}{k})_o = - \frac{x_1}{N_1^{\,2}} , \qquad 14\text{-}24$$

$$\frac{d^2}{dx^2} (\frac{1}{k})_o = - \frac{1}{N_1^{\,2}} , \qquad 14\text{-}25$$

$$\sigma_o = \frac{x_1}{N_1^{\,2}} \sin \beta_1 , \qquad 14\text{-}26$$

$$\sin \beta_1 = \frac{y_2 - y_1}{d} , \qquad 14\text{-}27$$

$$\cos \beta_1 = \frac{x_2 - x_1}{d} . \qquad 14\text{-}28$$

Substitution of 14-24 to 14-28 in 14-7 and 14-8 yields

$$(\frac{1}{k})_o^{I} = \frac{-x_1}{N_1^{\,2}} \; \frac{x_2 - x_1}{d} , \qquad 14\text{-}29$$

$$(\frac{1}{k})_o^{II} = - \frac{(x_2 - x_1)^2}{N_1^{\,2} d^2} + \ldots . \qquad 14\text{-}30$$

Substitution of 14-29, 14-30, 14-23, in 14-22, along with the approximation that $d \doteq s$, we get

$$S = d \, [\, 1 - \frac{1}{2} \frac{x_1^{\,2}}{N_1^{\,2}} - \frac{x_1}{N_1^{\,2}} \; \frac{(x_2 - x_1)}{2} - \frac{(x_2 - x_1)^2}{6 N_1^{\,2}} + \ldots \,] \qquad 14\text{-}31$$

$$\boxed{\frac{S}{d} = [1 - \frac{1}{6N_1^2}(x_1^2 + x_1 x_2 + x_2^2)]}\ , \qquad 14\text{-}32$$

where N_1 can be approximated by $R = \sqrt{M\ N}$. The formula is of sufficient accuracy for most work.

The following more accurate formula for the line scale ($\bar{k}$) is given in Bomford [1962]:

$$\boxed{\bar{k} = 1 + \frac{x_u^2}{6R_m^2}(1 + \frac{x_u^2}{36R_m^2})}\ , \qquad 14\text{-}33$$

where

$$x_u^2 = x_1^2 + x_1\ x_2 + x_2^2$$
$$R_m = \sqrt{M\ N}\ , \qquad 14\text{-}34$$

M and N evaluated at $(\phi_1 + \phi_2)/2$. The accuracy of the above formula is about $0.1/10^6$ for lines of about 150 km in length and within 3° of the central meridian. Thus $d = \bar{k}S$, without computing s (see 14-31).

14.2 Reduction Formulae for the New Brunswick Stereographic Projection

The reduction formulae for the New Brunswick stereographic projection are given here without derivation. The full derivation is given in Thomson et al. [1977].

14.2.1 Arc to chord (T-t) reduction

The arc to chord correction is given by [Thomson et al., 1977];

$$(T-t)_{ij} = \tan^{-1}\left[\frac{x_j y_i - x_i y_j}{x_i x_j + y_i y_j + (k_o R)^2}\right]\ ,$$

where, x_i, y_i, x_j, y_j are the map coordinates of points P_i and P_j; k_o is the scale factor at the origin and

$$R = \sqrt{M\,N} \quad ,$$

evaluated at ϕ_o, the latitude of the origin.

14.2.2 Geodesic to Chord Length Reduction

The line scale factor m_{ij} used to obtain the plane length d from the geodesic length S between two points P_i and P_j is given by [Thomson et al., 1977],

$$\frac{1}{m_{ij}} = \frac{S}{d} = \frac{1}{6}\left(\frac{1}{m_i} + \frac{4}{m_m} + \frac{1}{m_j}\right)$$

where, m_i, m_m and m_j are the point scale factors at P_i, the mid-point of the line (P_m), and P_j respectively. The point scale factor at P_i, for example, is given by[Thomson et al., 1977],

$$k_o = k_o + \frac{x_i^2 + y_i^2}{4k_o R^2} \quad .$$

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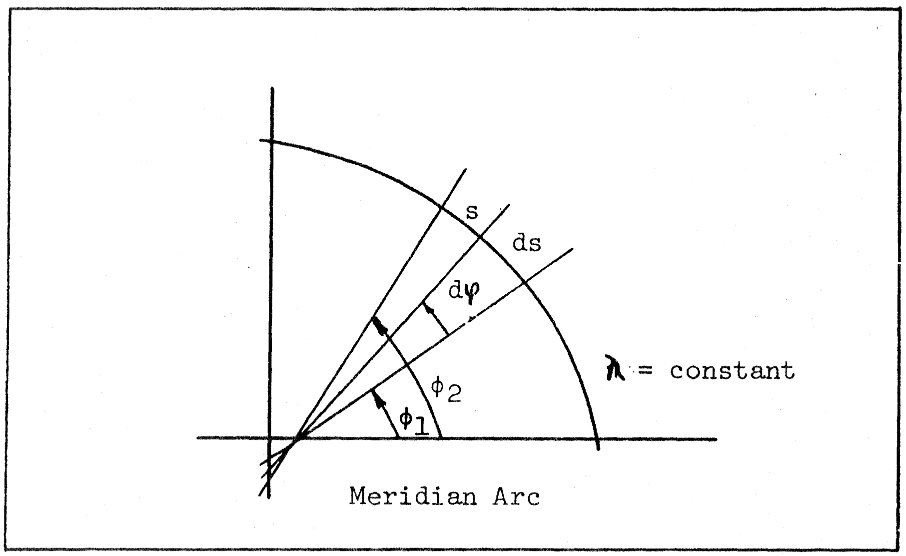
APPENDIX I: EVALUATION OF MERIDIAN ARC LENGTH



The length of a differential part of a meridian arc S is given by

$$dS = Md\phi \ , \tag{1}$$

where

$$M = \frac{a(1-e^2)}{(1-e^2 \sin^2\phi)^{3/2}}$$

is the corresponding radius of curvature. For the total arc, S, we can write

$$S = \int_{\phi_1}^{\phi_2} Md\ \phi \ . \tag{2}$$

In the case of the TM projection, we are interested in the length of the meridian arc from the equator ($\phi_1 = 0$) to our point of interest ($\phi_2 = \phi$). Then we write

$$S = \int_o^\phi Md\phi = \int_o^\phi \frac{a(1-e^2)}{(1-e^2 \sin^2\phi)^{3/2}} d\phi \ . \tag{2a}$$

The expression for M can be expanded in a series and written as

$$M = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}} = a(1-e^2)(1 + \frac{3}{2}e^2\sin^2\phi + \frac{15}{8}e^4\sin^4\phi + \frac{35}{16}e^6\sin^6\phi + \ldots). \tag{3}$$

We treat $a(1-e^2)$ as a constant outside the integration and obtain

$$S = a(1-e^2)\int_o^\phi [1 + \frac{3}{2}e^2\sin^2\phi + \frac{15}{8}e^4\sin^4\phi + \frac{35}{16}e^6\sin^6\phi + \ldots]d\phi\ . \tag{4}$$

Splitting this into parts yields

$$S = a(1-e^2)\ [\int_o^\phi d\phi + \int_o^\phi(\frac{3}{2}e^2\sin^2\phi)d\phi + \int_o^\phi(\frac{15}{8}e^4\sin^4\phi)d\phi + \ldots]. \tag{5}$$

Evaluating each integral separately

$$\int_o^\phi d\phi = \phi\Big|_o^\phi = \phi\ , \tag{5a}$$

$$\int_o^\phi(\frac{3}{2}e^2\sin^2\phi)d\phi = \frac{3}{2}e^2[\frac{\phi}{2} - \frac{\sin 2\phi}{4}]_o^\phi \tag{5b}$$

$$= \frac{3}{4}e^2\phi - \frac{3}{8}e^2\sin 2\phi\ ,$$

$$\int_o^\phi(\frac{15}{8}e^4\sin^4\phi)d\phi = \frac{15}{8}e^4[\frac{3\phi}{8} - \frac{\sin 2\phi}{4} + \frac{\sin 4\phi}{32}]_o^\phi \tag{5c}$$

$$= \frac{45}{64}e^4\phi - \frac{15}{32}e^4\sin 2\phi + \frac{15}{256}e^4\sin 4\phi\ .$$

Thus,

$$S = a(1-e^2)[\phi + \frac{3}{4}e^2\phi - \frac{3}{8}e^2\sin 2\phi + \frac{45}{64}e^4\phi - \frac{15}{32}e^4\sin 2\phi + \frac{15}{256}e^4\sin 4\phi + \ldots] \tag{6}$$

and

$$S = a(\phi + \frac{3}{4}e^2\phi - \frac{3}{8}e^2\sin 2\phi + \frac{45}{64}e^4\phi - \frac{15}{32}e^4\sin 2\phi + \frac{15}{256}e^4\phi\sin 4\phi - \tag{7}$$

$$-e^2\phi - \frac{3}{4}e^4\phi + \frac{3}{8}e^4\sin 2\phi - \frac{45}{64}e^6\phi + \frac{15}{32}e^6\sin 2\phi - \frac{15}{256}e^6\sin 4\phi + \ldots).$$

Collecting terms for ϕ, $\sin 2\phi$, $\sin 4\phi$, etc.

$$S = a((\phi + \frac{3}{4} e^2\phi - e^2\phi + \frac{45}{64} e^4\phi - \frac{3}{4} e^4\phi - \frac{45}{64} e^6\phi + \ldots)$$

$$+ (-\frac{3}{8} e^2\sin 2\phi - \frac{15}{32} e^4\sin 2\phi + \frac{3}{8} e^4\sin 2\phi + \frac{15}{32} e^6\sin 2\phi + \ldots)$$

$$+ (\frac{15}{256} e^4\sin 4\phi - \frac{15}{256} e^6\sin 4\phi + \ldots)). \tag{7a}$$

Including more terms in our series expansion results in the following formula for the meridian arc length,

$$S = a[A_o\phi - A_2\sin 2\phi + A_4\sin 4\phi - A_6\sin 6\phi + A_8\sin 8\phi] \tag{8}$$

where

$$A_o = 1 - \frac{1}{4} e^2 - \frac{3}{64} e^4 - \frac{5}{256} e^6 - \frac{175}{16384} e^8 \quad , \tag{9}$$

$$A_2 = \frac{3}{8} (e^2 + \frac{1}{4} e^4 + \frac{15}{128} e^6 - \frac{455}{4096} e^8) \quad , \tag{10}$$

$$A_4 = \frac{15}{256} (e^4 + \frac{3}{4} e^6 - \frac{77}{128} e^8) \quad , \tag{11}$$

$$A_6 = \frac{35}{3072} (e^6 - \frac{41}{32} e^8) \quad , \tag{12}$$

$$A_8 = - \frac{315}{131072} e^8 \quad . \tag{13}$$

This formula for the meridian arc length is accurate to less than 0.001 m.

APPENDIX II. EVALUATION OF FOOTPOINT LATITUDE

When dealing with the T.M. conformal projection we remember that the inverse formulae for $\Delta\lambda$ and $\Delta\phi$ are given in terms of the "footpoint" latitude, ϕ_1. Thus, we need an expression for determining ϕ_1 from y.

Remembering that the length of a meridian arc is given by

$$S = \int_0^{\phi} M d\phi$$

and that along the central meridian,

$$y = S = \int_0^{\phi} M d\phi \ ,$$

then our footpoint latitude will be given by the meridian arc length "y" along the central meridian.

To solve this problem, we use the Newton-Raphson iterative procedure (see pp. 26-28). In brief, we require the following three steps:

(1) Obtain an initial appropriate value for ϕ (ϕ_0);

(2) Evaluate the iterative improved solution from

$$\phi_n = \phi_{n-1} - \frac{f(\phi)}{f'(\phi)} \ ; \qquad (14)$$

(3) Continue the iterative procedure until

$$|\phi_n - \phi_{n-1}| < \varepsilon \ ,$$

where $\varepsilon = 10^{-12}$ radians (which is approximately 2×10^{-7} arcsec).

Specifically, we have (from 8)

$$S = y = a(A_0\phi - A_2\sin 2\phi + A_4\sin 4\phi - A_6\sin 6\phi + A_8\sin 8\phi - \ldots).$$

Then

$$f(\phi) = a(A_o\phi - A_2\sin 2\phi + A_4\sin 4\phi - A_6\sin 6\phi + A_8\sin 8\phi) - y \ , \qquad (15)$$

and

$$f'(\phi) = a(A_o - 2A_2\cos 2\phi + 4A_4\cos 4\phi - 6A_6\cos 6\phi + 8A_8\cos 8\phi) \ , \qquad (16)$$

where A_o, A_2, A_4, A_6, A_8 are given by (9), (10), (11), (12), (13) respectively. For an initial approximation, we use a spherical approximation, that is $e = 0$, and (8) becomes

$$y = a\phi \qquad (A_o = 1;\ A_i = 0,\ i = 2, 4, 6, 8, \ldots) \ . \qquad (17)$$

Then

$$\phi_o = \frac{y}{a} \ . \qquad (18)$$

Using ϕ_o, we get first approximations for $f(\phi)$ and $f'(\phi)$ (equations (15) and (16) respectively). For $n = 1$

$$\phi_1 = \phi_o - \frac{f(\phi_o)}{f'(\phi_o)} \ , \qquad (19)$$

and

$$\varepsilon_1 = |\phi_1 - \phi_o| \ ; \qquad \varepsilon_1 < 10^{-12} \text{ rad.?} \qquad (20)$$

Continuing, we then get $n = 2$ and

$$\phi_2 = \phi_1 - \frac{f(\phi_1)}{f'(\phi_1)} \ ,$$

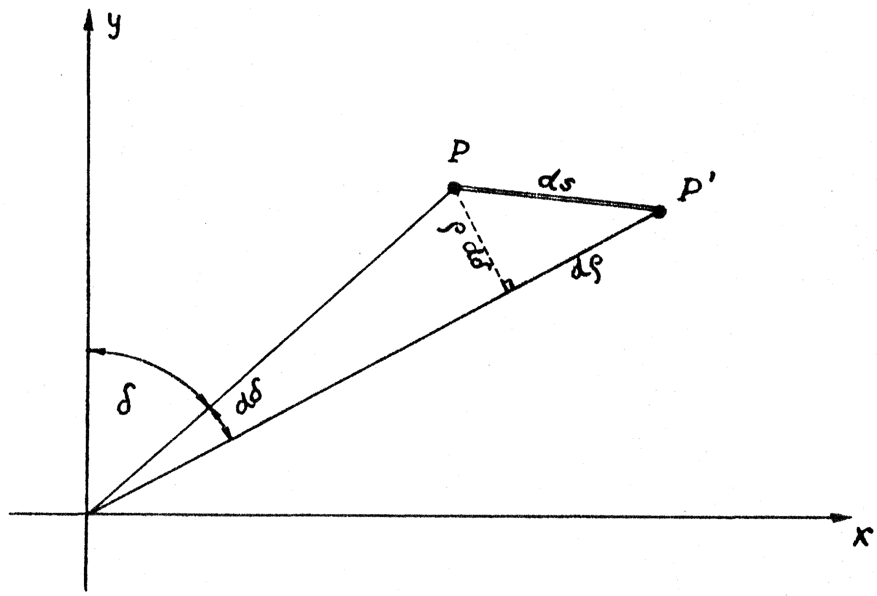
and

$$\varepsilon_2 = |\phi_2 - \phi_1| \ ; \qquad \varepsilon_2 < 10^{-12} \text{ rad.}$$

Thus, we compute the footpoint latitude, and using this we can complete the inverse problem for the Transverse Mercator projection (e.g. solve for $\Delta\lambda$, $\Delta\phi$, thence ϕ and λ).

APPENDIX III

ALTERNATIVE DERIVATION OF LAMBERT CONFORMAL CONICAL PROJECTION



In the polar coordinate system, the linear element ds is computed by the well-known formula

$$ds^2 = d\rho^2 + \rho^2\, d\delta^2 = \rho^2[(\frac{d\rho}{\rho})^2 + d\delta^2] \quad . \tag{1}$$

It is obvious from the above expression that δ and ρ are not isometric coordinates. The isometric coordinates must generally have the following form

$$ds^2 = [f(u, v)]^2\, (du^2 + dv^2) \quad . \tag{2}$$

However, by a simple substitution

$$- d\bar{\rho} = \frac{d\rho}{\rho} \tag{3}$$

we can transform into an isometric system, where new variable $\bar{\rho}$ is directly determined by the integration of the above equation

$$- \bar{\rho} = \ell n\ \rho - \ell n\ K \quad , \tag{4}$$

or

$$\rho = K \cdot e^{-\bar{\rho}} \quad . \tag{5}$$

Now, using the obtained isometric coordinates we can set the fundamental expression of conformal mapping

$$\bar{\rho} + i\,\delta = f(g + i\,\ell) \quad , \tag{6}$$

where

$$\ell = \lambda_o - \lambda \quad . \tag{7}$$

Conical projections are defined as projections in which meridians are transformed into straight lines intersecting at one point and parallels become concentric circles with the centre in the intersection point of meridians. The angles between meridians are proportional to the corresponding angles on the ellipsoid.

Mathematically translated, these conditions mean

$$\bar{\rho} + i\,\delta = K(\rho + i\,\ell), \quad \bar{\rho} = K\,\rho\ , \quad \delta = K\,\ell \quad . \tag{8}$$

Substituting the derived value for $\bar{\rho}$ from (5) we have

$$\rho = K \cdot e^{-Kq} \quad \delta = K \cdot \ell \quad , \tag{9}$$

or for rectangular coordinates

$$\boxed{\begin{aligned} x &= \rho \sin \delta = K\, e^{-Kq} \sin K\,\ell\ , \\ y &= \rho \cos \delta = K\, e^{-Kq} \cos K\,\ell\ . \end{aligned}} \tag{10}$$