# Chapter 9: Center of Mass and Linear Momentum

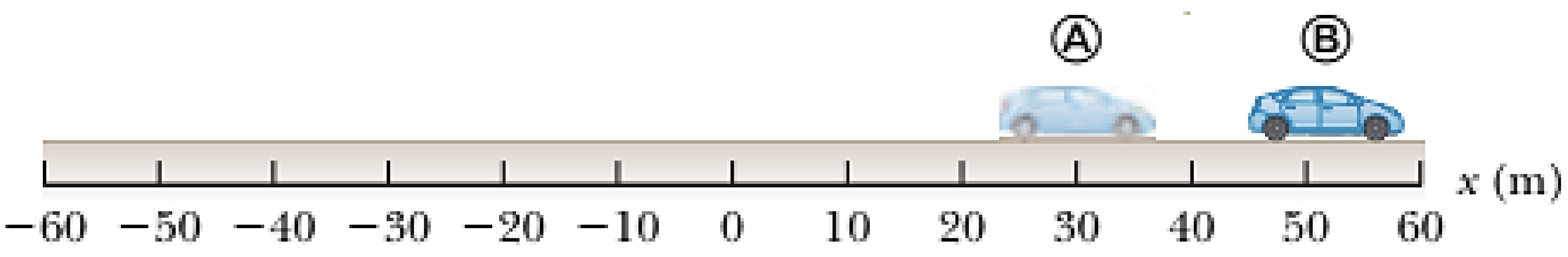
- ✓ **Center of Mass**
- ✓ **Newton's Second Law for a System of Particles**
- ✓ **Linear Momentum**
- ✓ **Impulse**
- ✓ **Collision**

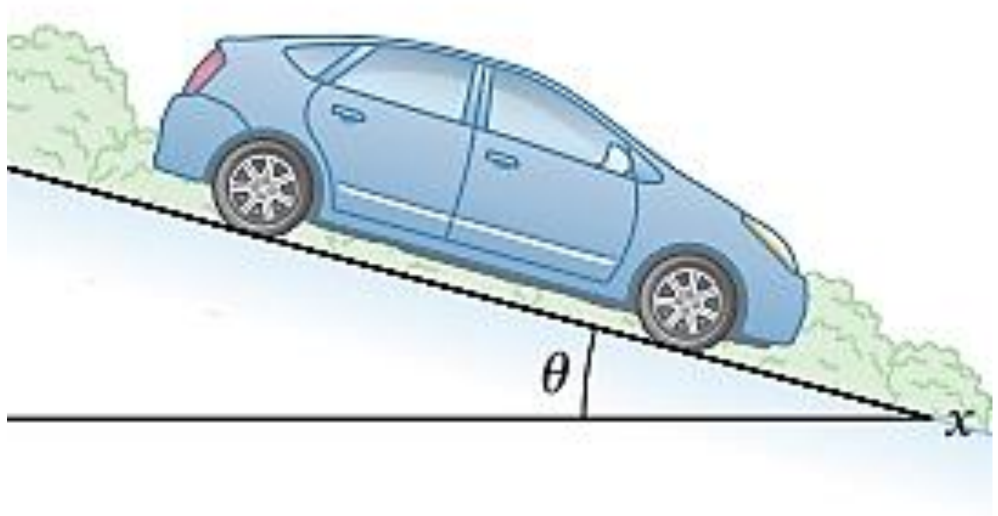
## Session 17:

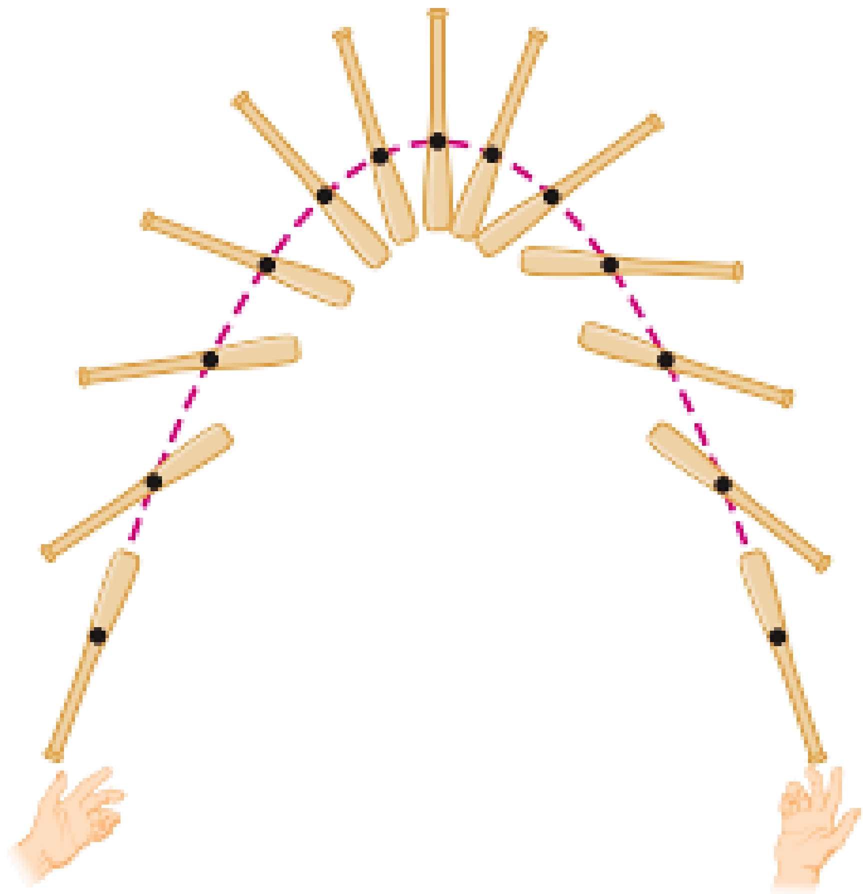
- ✓ **Center of Mass**
- ✓ **Examples**

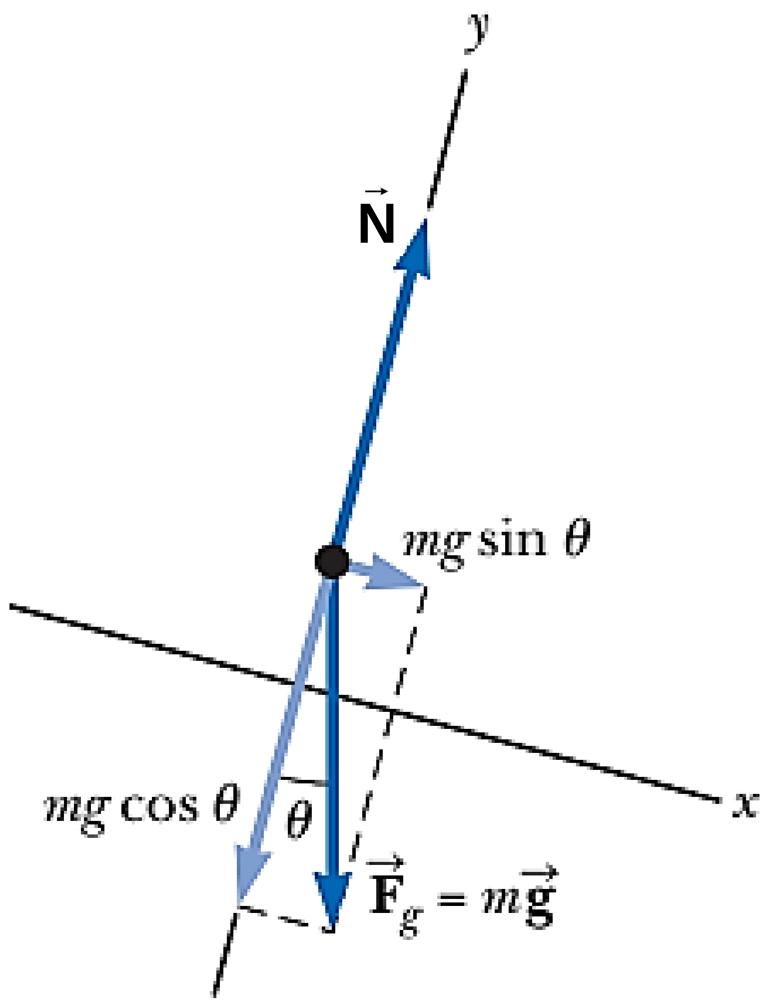
# Introduction

❖ **Chapters 2-8: we use the particle model (A particle is a point-like object; has mass but infinitesimal size)**



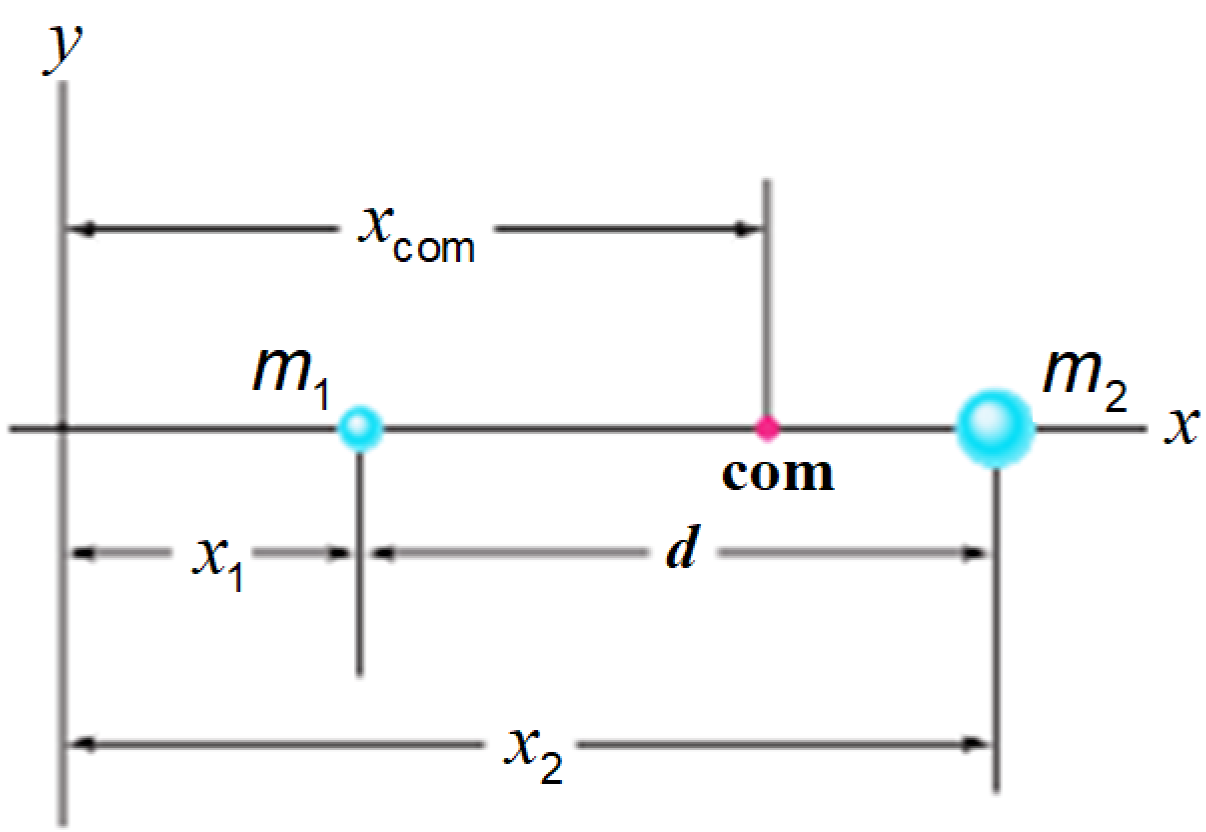


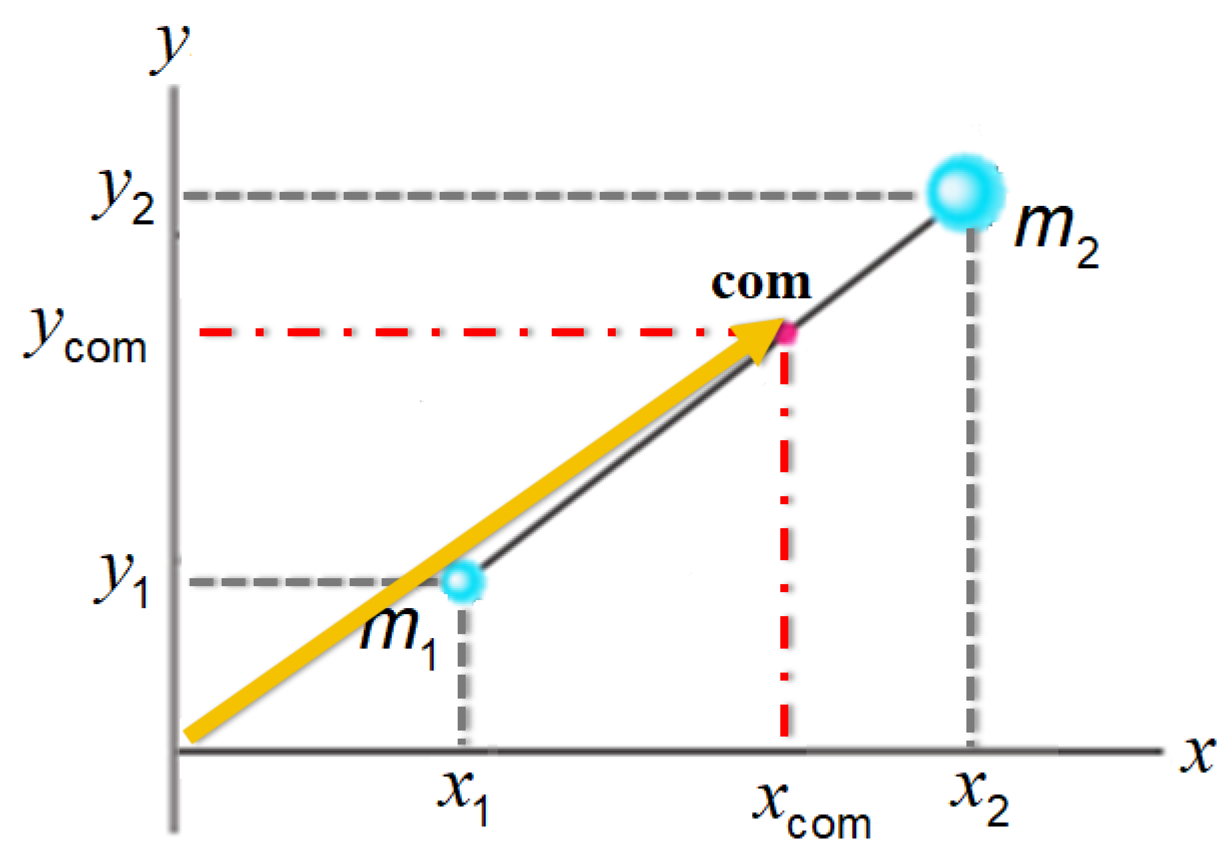
# Center of Mass

❖ **Center of mass:**

1) **All of the system's mass were concentrated there**
2) **All external forces were applied there.**

➢ **Center of mass for Systems of Particles:**




$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

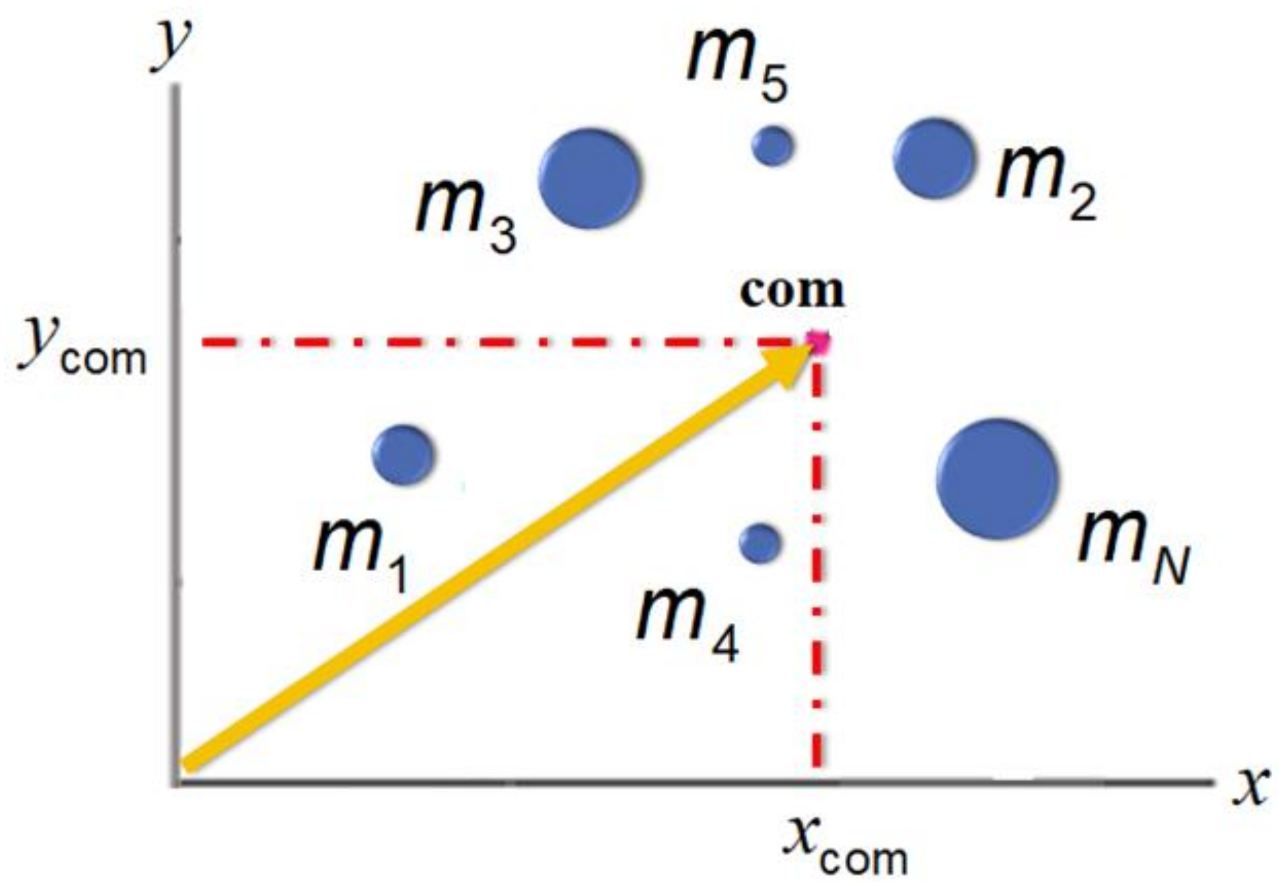
$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$y_{com} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

$$\vec{r}_{com} = x_{com}\,\hat{\mathbf{i}} + y_{com}\,\hat{\mathbf{j}}$$

# Center of Mass

- **Center of mass for Systems of Particles:**




$$x_{com} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_N x_N}{m_1 + m_2 + \cdots + m_N} = \frac{\sum_{i=1}^{N} m_i x_i}{M}$$

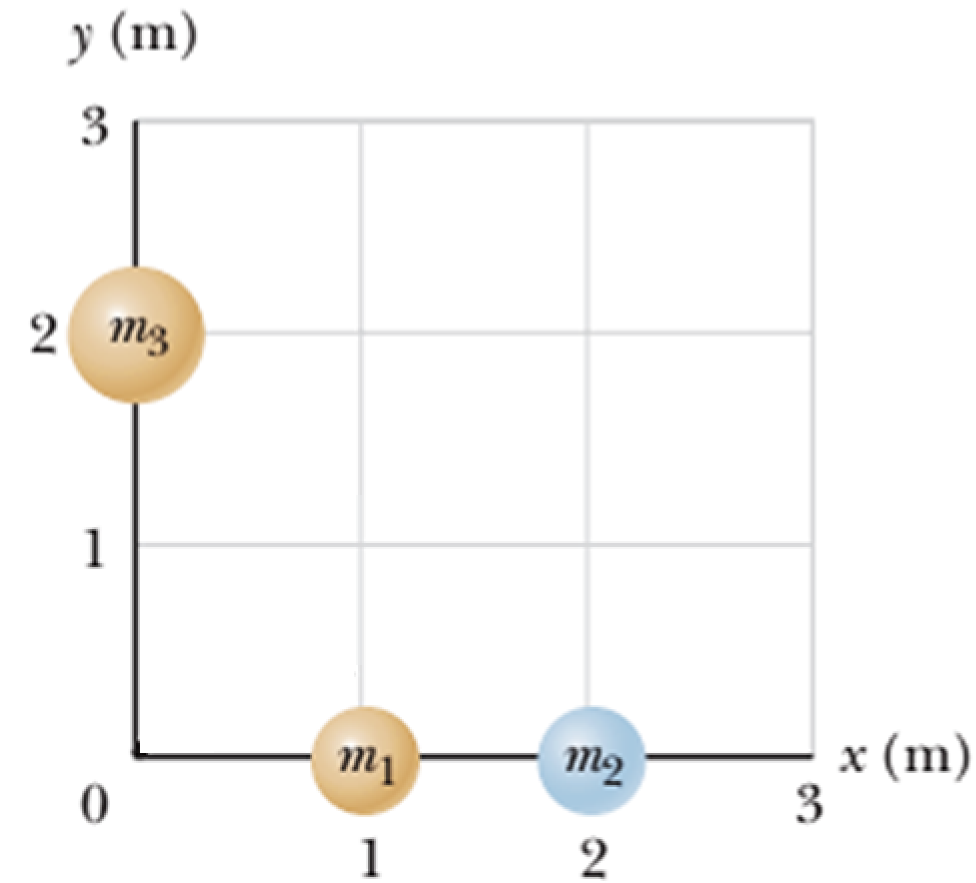
$$y_{com} = \frac{\sum_{i=1}^{N} m_i y_i}{M}$$

$$z_{com} = \frac{\sum_{i=1}^{N} m_i z_i}{M}$$

$$\vec{r}_{com} = x_{com}\,\hat{\mathbf{i}} + y_{com}\,\hat{\mathbf{j}} + z_{com}\,\hat{\mathbf{k}}$$

**Ex 1:** A system consists of three particles located as shown in Figure. Find the **center of mass of the system**. The masses of the particles are $\mathbf{m_1 = m_2 = 1\ kg}$ and $\mathbf{m_3 = 2\ kg}$.

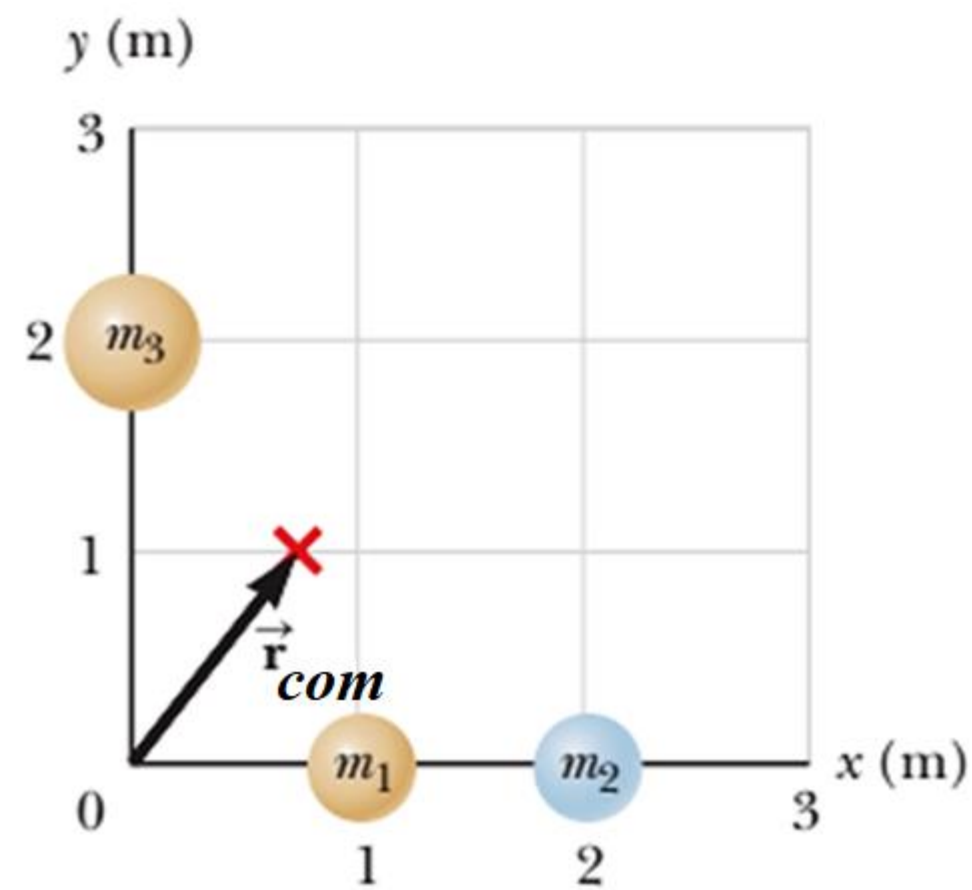
$$x_{com} = \frac{\sum_{i=1}^{3} m_i x_i}{M} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$




$$\boxed{x_{com} = \frac{(1)(1)+(1)(2)+(2)(0)}{1+1+2} = \frac{3}{4} = 0.75\ m}$$

$$y_{com} = \frac{\sum_{i=1}^{3} m_i y_i}{M} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

$$\boxed{y_{com} = \frac{(1)(0)+(1)(0)+(2)(2)}{1+1+2} = \frac{4}{4} = 1\ m}$$




$$\boxed{\vec{r}_{com} = 0.75\,\hat{\mathbf{i}} + \hat{\mathbf{j}}}$$

# Center of Mass

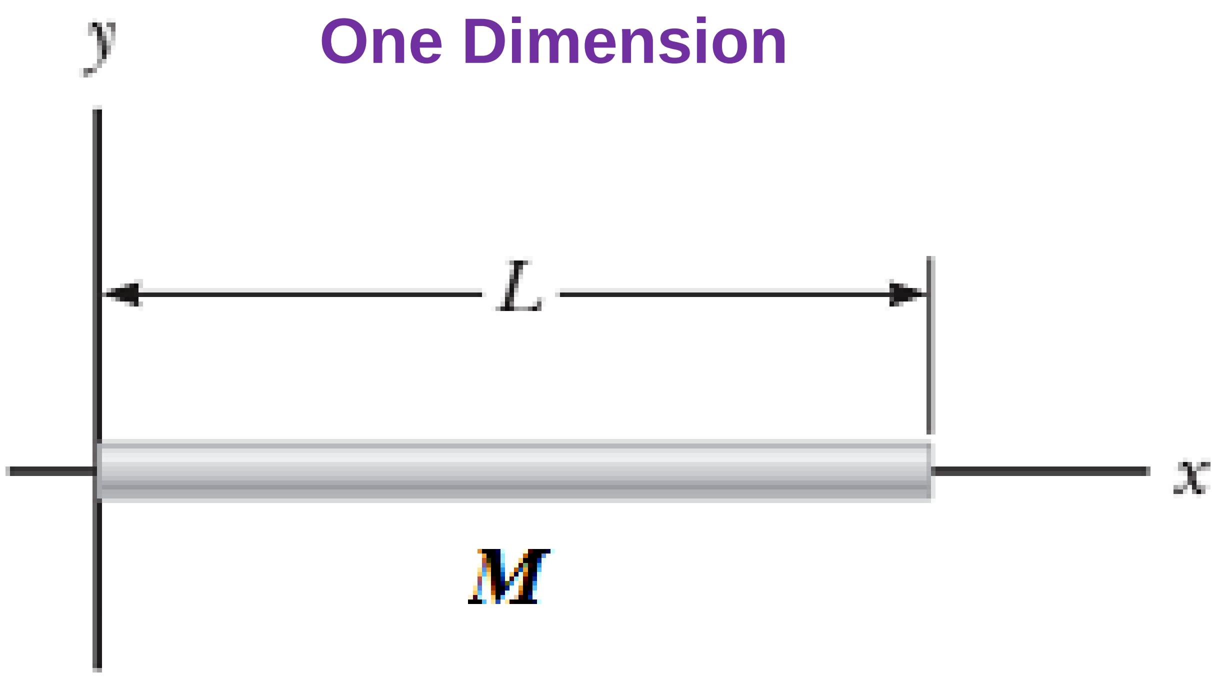
- **Center of mass for an extended object (solid body):**

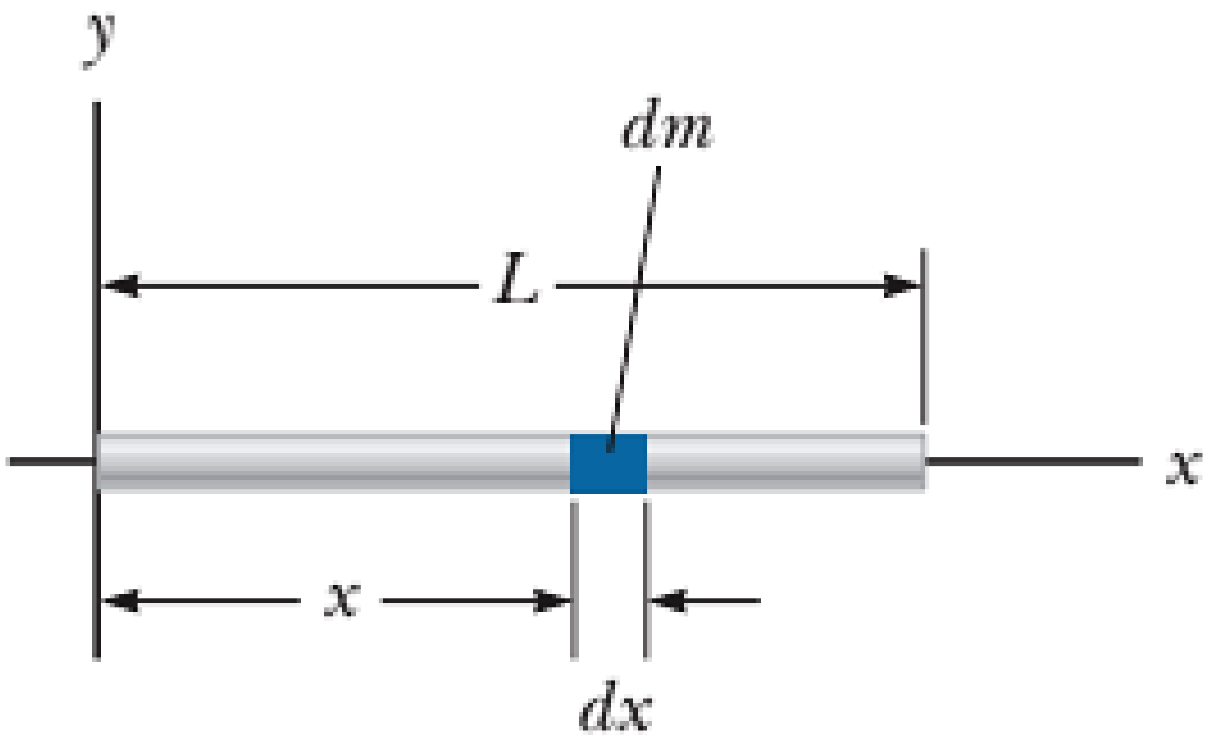
$$x_{\text{com}} = \frac{\sum_{i=1}^{N} (dm_i)\, x_i}{M}$$

$$x_{\text{com}} = \lim_{\substack{dm \to 0 \\ N \to \infty}} \frac{\sum_{i=1}^{N} (dm_i)\, x_i}{M}$$

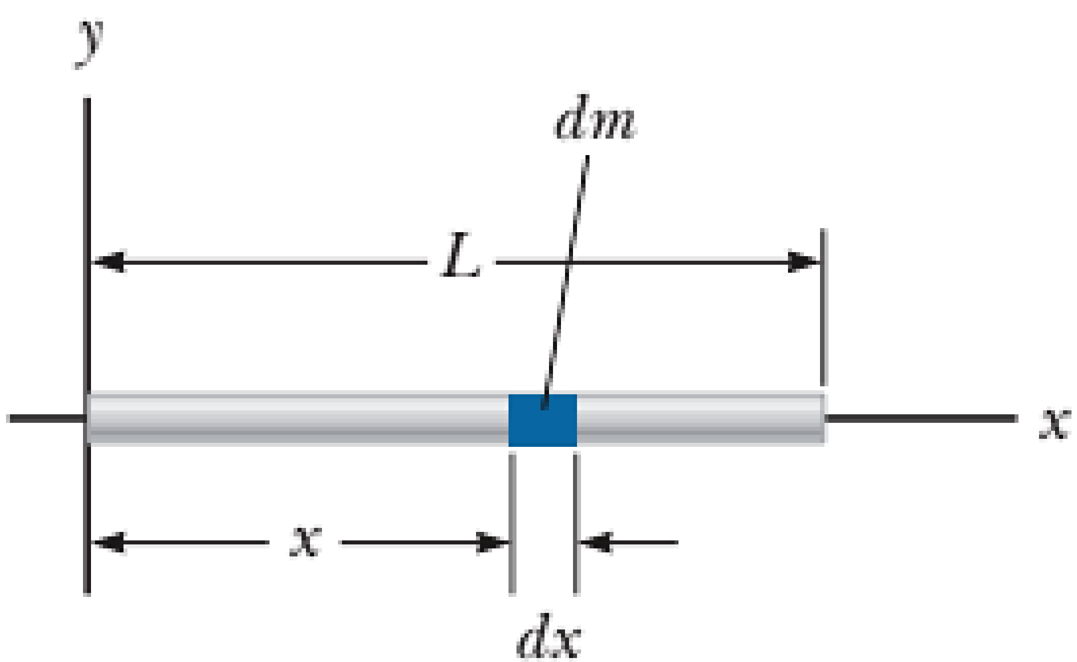
$$\boxed{x_{\text{com}} = \frac{1}{M} \int x\, dm}$$

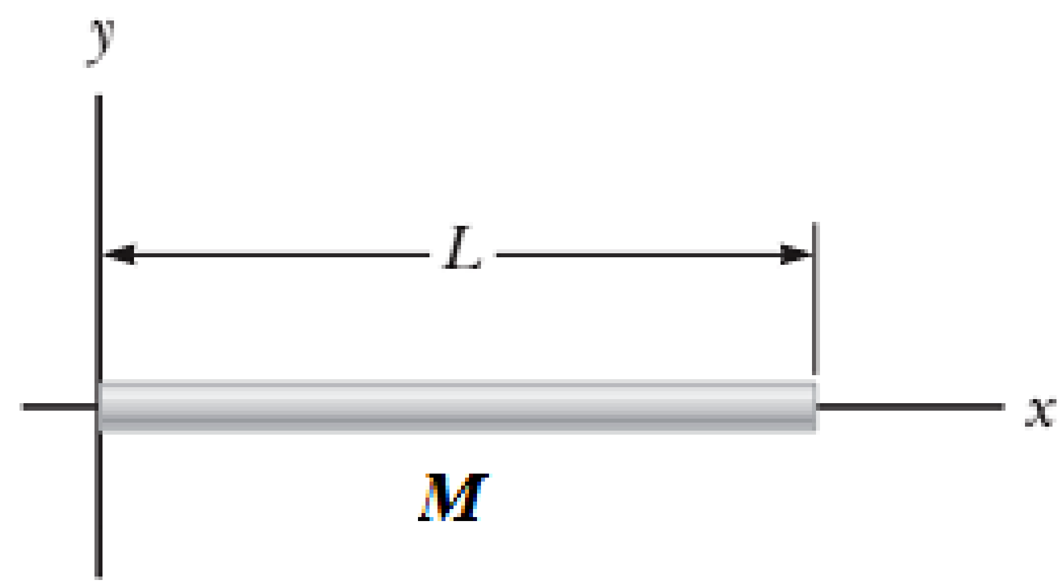
One Dimension



**Ex 2:** Find the **center of mass** of a rod of mass **M** and length **L**. (assuming the rod has a **uniform mass per unit length**)




$$x_{com} = \frac{1}{M}\int x\,dm \qquad \lambda = \frac{M}{L} = \frac{dm}{dx}$$

**Linear mass density**

$$dm = \frac{M}{L}dx = \lambda dx$$

$$x_{com} = \frac{1}{M}\int_0^L x(\lambda dx) = \frac{\lambda}{M}\int_0^L x\,dx \Rightarrow \boxed{x_{com} = \frac{1}{L}\left(\frac{L^2}{2}\right) = \frac{L}{2}}$$

$$\lambda(x) = \alpha x \Rightarrow x_{com} = \frac{1}{M}\int_0^L x[\lambda(x)dx] = \frac{\alpha}{M}\int_0^L x^2\,dx = \frac{\alpha L^3}{3M}$$

$$M = \int dm = \int_0^L \lambda(x)dx = \alpha\int_0^L x\,dx = \frac{\alpha L^2}{2} \Rightarrow \boxed{x_{com} = \frac{2}{3}L}$$

# Center of Mass

➢ **Center of mass for an extended object (solid body):**

$$\boxed{x_{com} = \frac{1}{M}\int x\, dm}$$

$$\boxed{y_{com} = \frac{1}{M}\int y\, dm}$$

$$\sigma = \frac{M}{A} = \frac{dm}{dA}$$

**Surface mass density**

$$dm = \frac{M}{A} dA = \sigma\, dA$$

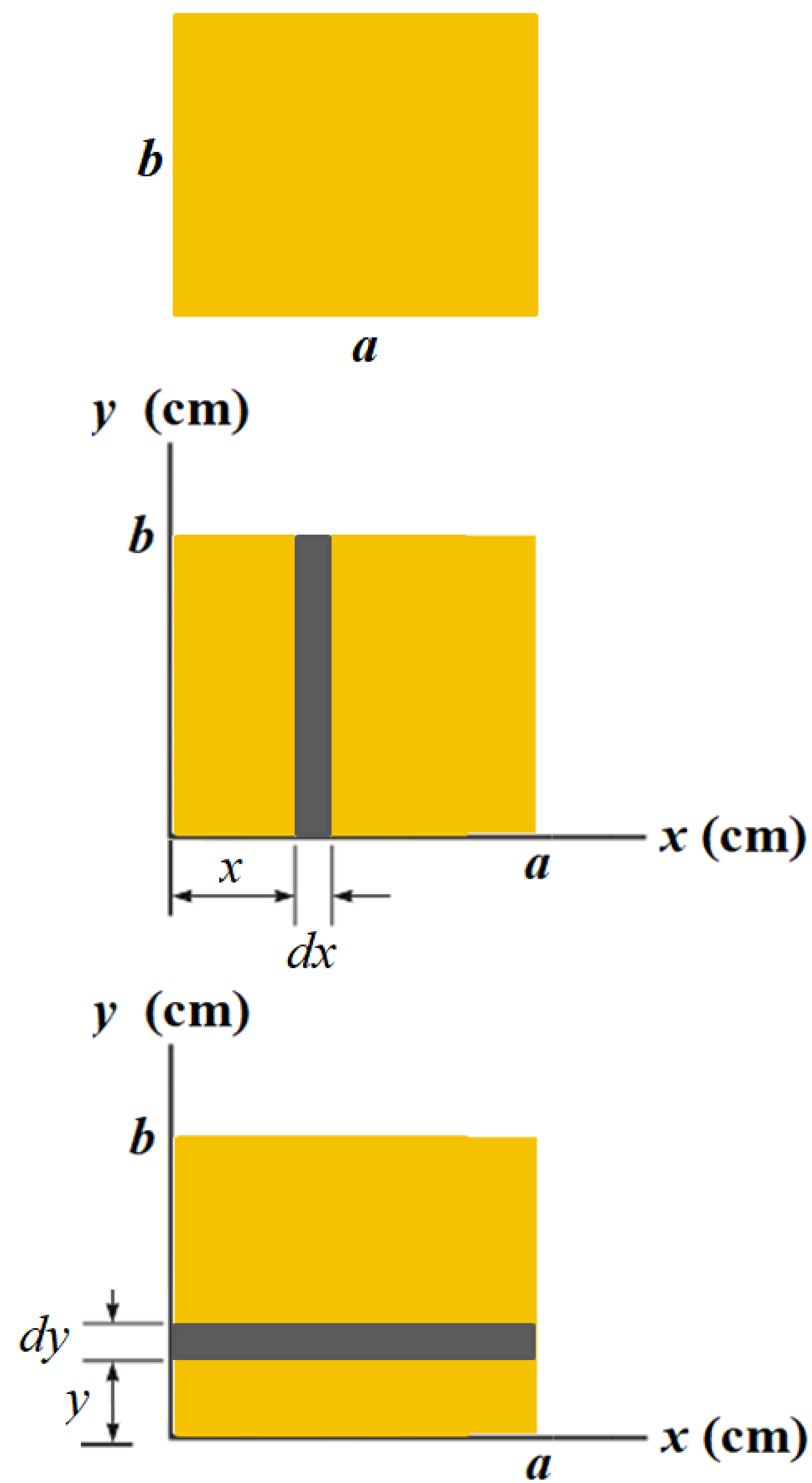
$$x_{com} = \frac{1}{M}\int x(\sigma\, dA) = \frac{\sigma}{M}\int_0^a x(b\, dx) = \frac{\sigma b a^2}{2M}$$

$$\boxed{x_{com} = \frac{\sigma A a}{2\sigma A} = \frac{a}{2}}$$

$$y_{com} = \frac{1}{M}\int y(\sigma\, dA) = \frac{\sigma}{M}\int_0^b y(a\, dy) = \frac{\sigma a b^2}{2M}$$

$$\boxed{y_{com} = \frac{\sigma A b}{2\sigma A} = \frac{b}{2}}$$

**Two Dimensions**

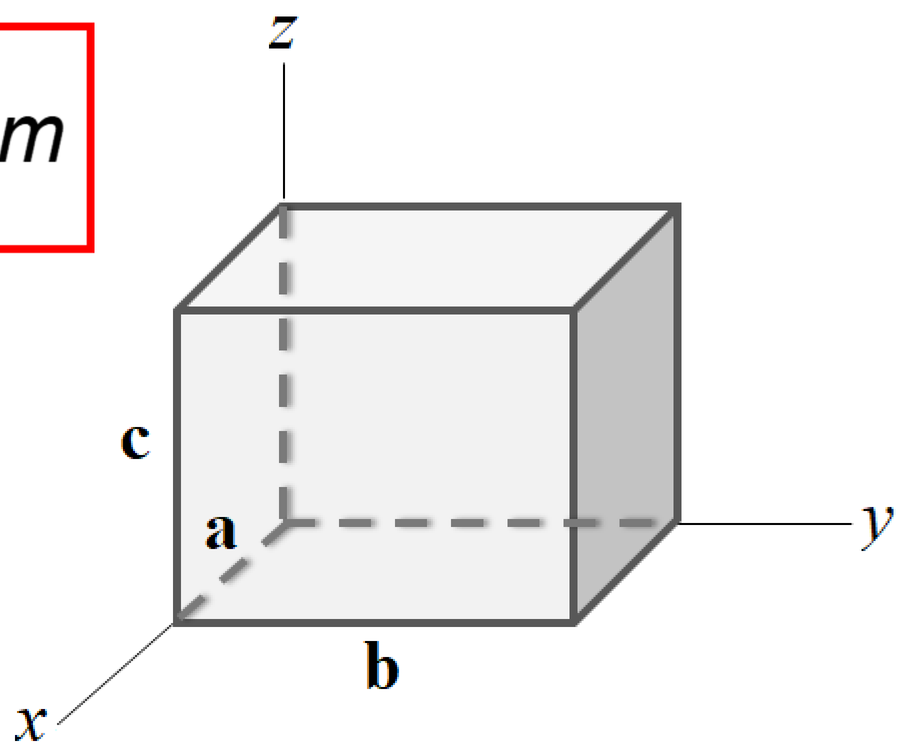


# Center of Mass

➢ **Center of mass for an extended object (solid body):**

**Three Dimensions**

$$\boxed{x_{com} = \frac{1}{M}\int x\, dm} \quad \boxed{y_{com} = \frac{1}{M}\int y\, dm} \quad \boxed{z_{com} = \frac{1}{M}\int z\, dm}$$

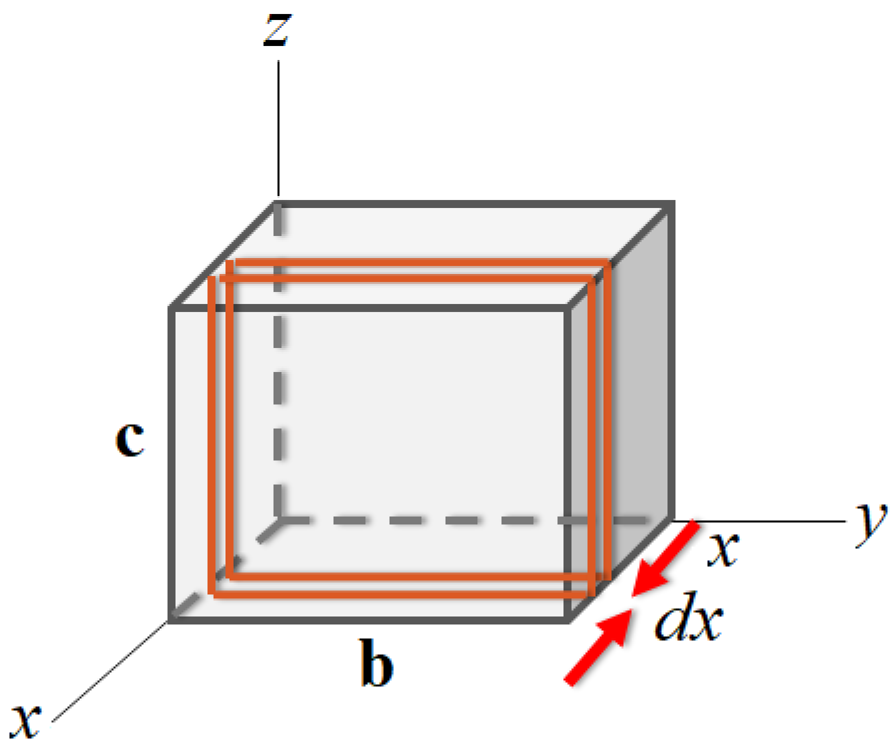



$$\rho = \frac{M}{V} = \frac{dm}{dV}$$

**Volume mass density**

$$dm = \frac{M}{V} dV = \rho\, dV$$

$$x_{com} = \frac{1}{M}\int x(\rho\, dV) = \frac{\rho}{M}\int_0^a x(bc\, dx) = \frac{\rho bc\, a^2}{2M}$$

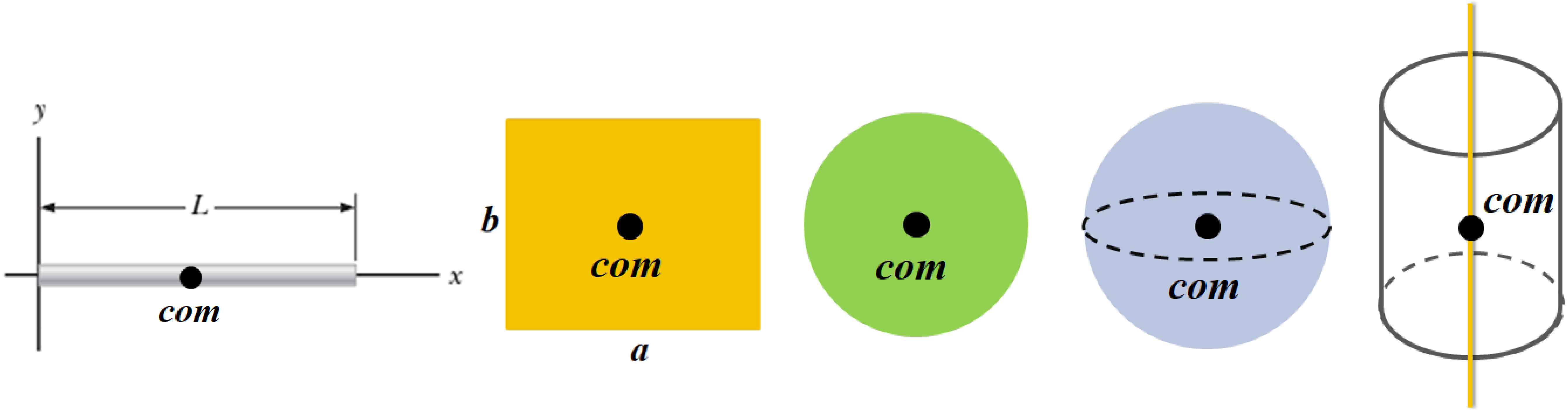



$$\boxed{x_{com} = \frac{\rho V a}{2\rho V} = \frac{a}{2}}$$

$$\boxed{y_{com} = \frac{b}{2}} \quad \boxed{z_{com} = \frac{c}{2}}$$

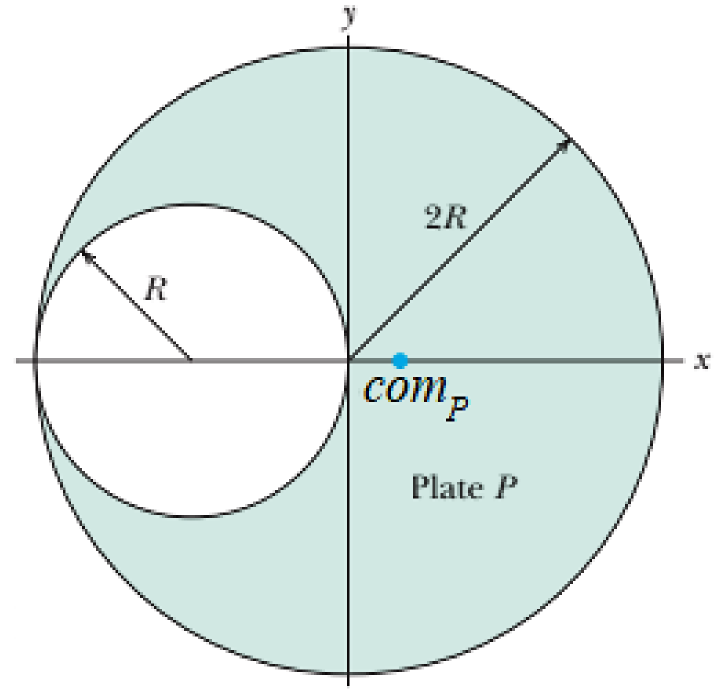
# Center of Mass

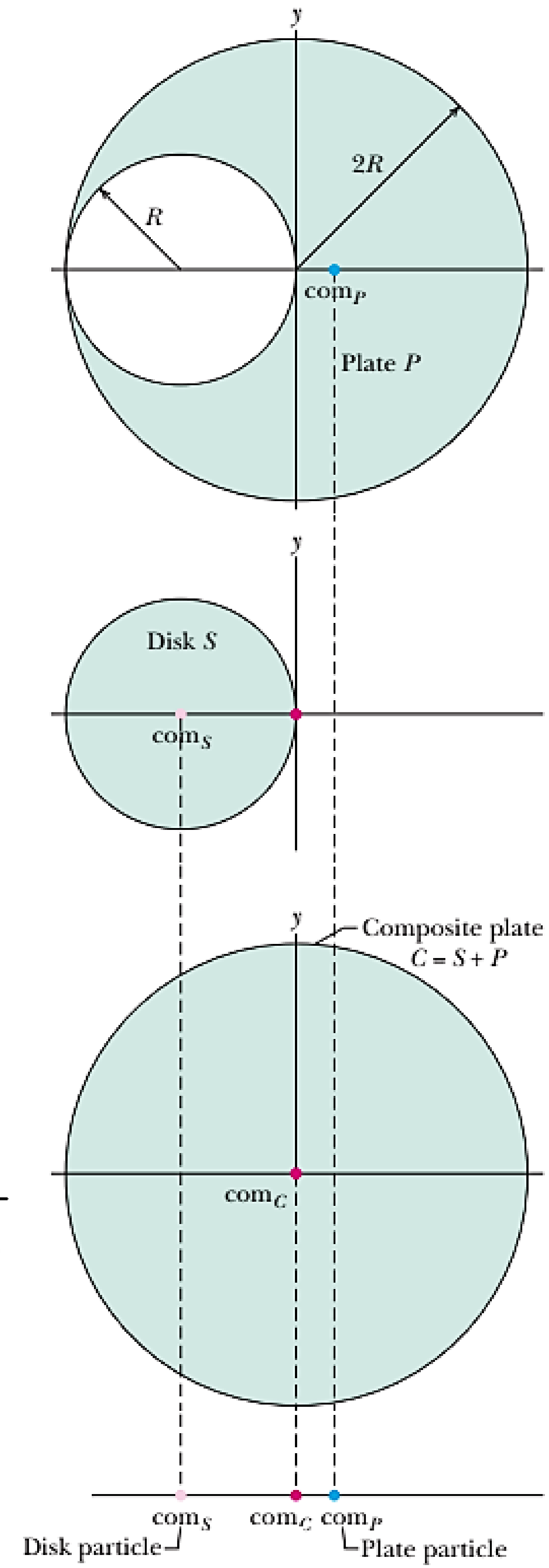
**The center of mass of any symmetric object of uniform mass distribution lies on an axis of symmetry and on any plane of symmetry**




If **g** is constant over the mass distribution, the **center of gravity** coincides with the **center of mass**.

**Ex 3:** Figure below shows a **uniform metal plate P** of radius 2R from which a **disk of radius R** has been **removed** in an assembly line. Using the xy coordinate system shown, locate the **center of mass** ***com*$_P$** of the remaining plate.




$$\boxed{x_{\text{com}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

$$x_{\text{com c}} = \frac{m_p x_p + m_s x_s}{m_p + m_s} = 0 \quad \Rightarrow \quad x_p = -\frac{m_s}{m_p} x_s$$

$$\sigma = \frac{M}{A} \quad \Rightarrow \quad \frac{m_s}{m_p} = \frac{\sigma A_s}{\sigma A_p} = \frac{A_s}{A_p} = \frac{\pi R^2}{\pi(4R^2 - R^2)} = \frac{1}{3}$$

$$\boxed{x_p = -\frac{1}{3} x_s = \frac{1}{3} R}$$

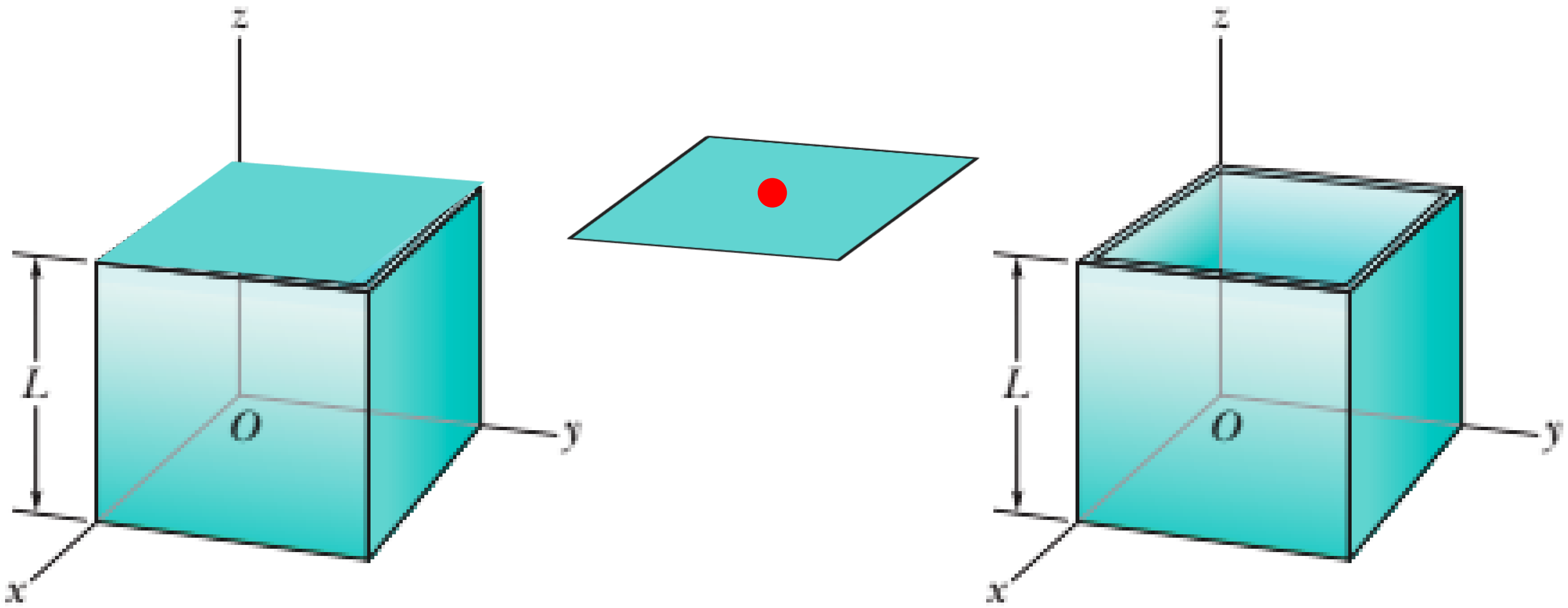
**Ex 4: (Problem 9.6 Halliday)**

Figure 9-39 shows a **cubical box** that has been constructed from **uniform metal plate** of negligible thickness. The box is open at the top and has edge length **L = 40 cm**. Find (a) the x coordinate, (b) the y coordinate, and (c) the z coordinate of the center of mass of the box.

$$x_{com_T} = \frac{L}{2} = 20\ cm$$

$$y_{com_T} = \frac{L}{2} = 20\ cm$$

$$z_{com_T} = \frac{L}{2} = 20\ cm$$




$$x_{com} = y_{com} = \frac{L}{2} = 20\ cm$$

**Due to the symmetry**

$$z_{com_T} = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2} \Rightarrow \frac{L}{2} = \frac{\frac{5M}{6} z_1 + \frac{M}{6}(L)}{M} = \frac{5}{6} z_1 + \frac{L}{6}$$

$$z_{com} = z_1 = \frac{2}{5} L = 16\ cm$$