Chapter 15

Matrices

DEFINITION OF A MATRIX

A *matrix of order* $m \times n$, or *m by n matrix*, is a rectangular array of numbers having m rows and n columns. It can be written in the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \tag{1}$$

Each number a_{jk} in this matrix is called an *element*. The subscripts j and k indicate respectively the row and column of the matrix in which the element appears.

We shall often denote a matrix by a letter, such as A in (1), or by the symbol (a_{jk}) which shows a representative element.

A matrix having only one row is called a *row matrix* [or *row vector*] while a matrix having only one column is called a *column matrix* [or *column vector*]. If the number of rows m and columns n are equal the matrix is called a *square matrix* of order $n \times n$ or briefly n. A matrix is said to be a *real matrix* or *complex matrix* according as its elements are real or complex numbers.

SOME SPECIAL DEFINITIONS AND OPERATIONS INVOLVING MATRICES

1. **Equality of Matrices.** Two matrices $A = (a_{jk})$ and $B = (b_{jk})$ of the same order [i.e. equal numbers of rows and columns] are *equal* if and only if $a_{jk} = b_{jk}$.

2. **Addition of Matrices.** If $A = (a_{jk})$ and $B = (b_{jk})$ have the same order we define the *sum* of A and B as $A + B = (a_{jk} + b_{jk})$.

   Example 1. If $A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -5 & 1 \\ 2 & 1 & 3 \end{pmatrix}$ then

$$A + B = \begin{pmatrix} 2+3 & 1-5 & 4+1 \\ -3+2 & 0+1 & 2+3 \end{pmatrix} = \begin{pmatrix} 5 & -4 & 5 \\ -1 & 1 & 5 \end{pmatrix}$$

   Note that the commutative and associative laws for addition are satisfied by matrices, i.e. for any matrices A, B, C of the same order

$$A + B = B + A, \qquad A + (B + C) = (A + B) + C \tag{2}$$

3. **Subtraction of Matrices.** If $A = (a_{jk})$, $B = (b_{jk})$ have the same order, we define the *difference* of A and B as $A - B = (a_{jk} - b_{jk})$.

   Example 2. If A and B are the matrices of Example 1, then

$$A - B = \begin{pmatrix} 2-3 & 1+5 & 4-1 \\ -3-2 & 0-1 & 2-3 \end{pmatrix} = \begin{pmatrix} -1 & 6 & 3 \\ -5 & -1 & -1 \end{pmatrix}$$

4. **Multiplication of a Matrix by a Number.** If $A = (a_{jk})$ and λ is any number [or *scalar*], we define the *product* of A by λ as $\lambda A = A\lambda = (\lambda a_{jk})$.

Example 3. If A is the matrix of Example 1 and $\lambda = 4$, then

$$\lambda A = 4\begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 4 & 16 \\ -12 & 0 & 8 \end{pmatrix}$$

5. **Multiplication of Matrices.** If $A = (a_{jk})$ is an $m \times n$ matrix while $B = (b_{jk})$ is an $n \times p$ matrix, then we define the *product* $A \cdot B$ or AB of A and B as the matrix $C = (c_{jk})$ where

$$c_{jk} = \sum_{l=1}^{n} a_{jl} b_{lk} \tag{3}$$

and where C is of order $m \times p$.

Note that matrix multiplication is defined if and only if the number of columns of A is the same as the number of rows of B. Such matrices are sometimes called *conformable*.

Example 4. Let $A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 5 \\ 2 & -1 \\ 4 & 2 \end{pmatrix}$. Then

$$AD = \begin{pmatrix} (2)(3) + (1)(2) + (4)(4) & (2)(5) + (1)(-1) + (4)(2) \\ (-3)(3) + (0)(2) + (2)(4) & (-3)(5) + (0)(-1) + (2)(2) \end{pmatrix} = \begin{pmatrix} 24 & 17 \\ -1 & -11 \end{pmatrix}$$

Note that in general $AB \neq BA$, i.e. the commutative law for multiplication of matrices is not satisfied in general. However, the associative and distributive laws are satisfied, i.e.

$$A(BC) = (AB)C, \quad A(B+C) = AB + AC, \quad (B+C)A = BA + CA \tag{4}$$

A matrix A can be multiplied by itself if and only if it is a square matrix. The product $A \cdot A$ can in such case be written A^2. Similarly we define powers of a square matrix, i.e. $A^3 = A \cdot A^2$, $A^4 = A \cdot A^3$, etc.

6. **Transpose of a Matrix.** If we interchange rows and columns of a matrix A, the resulting matrix is called the *transpose* of A and is denoted by A^T. In symbols, if $A = (a_{jk})$ then $A^T = (a_{kj})$.

Example 5. The transpose of $A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & 0 & 2 \end{pmatrix}$ is

$$A^T = \begin{pmatrix} 2 & -3 \\ 1 & 0 \\ 4 & 2 \end{pmatrix}$$

We can prove that

$$(A+B)^T = A^T + B^T, \quad (AB)^T = B^T A^T, \quad (A^T)^T = A \tag{5}$$

7. **Symmetric and Skew-Symmetric Matrices.** A square matrix A is called *symmetric* if $A^T = A$ and *skew-symmetric* if $A^T = -A$.

Example 6. The matrix $E = \begin{pmatrix} 2 & -4 \\ -4 & 3 \end{pmatrix}$ is symmetric while $F = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ is skew-symmetric.

Any real square matrix [i.e. one having only real elements] can always be expressed as the sum of a real symmetric matrix and a real skew-symmetric matrix.

8. **Complex Conjugate of a Matrix.** If all elements a_{jk} of a matrix A are replaced by their complex conjugates $\bar{a}_{jk}$, the matrix obtained is called the *complex conjugate* of A and is denoted by $\bar{A}$.

9. **Hermitian and Skew-Hermitian Matrices.** A square matrix A which is the same as the complex conjugate of its transpose, i.e. if $A = \bar{A}^T$, is called *Hermitian*. If $A = -\bar{A}^T$, then A is called *skew-Hermitian*. If A is real these reduce to symmetric and skew-symmetric matrices respectively.

10. **Principal Diagonal and Trace of a Matrix.** If $A = (a_{jk})$ is a square matrix, then the diagonal which contains all elements a_{jk} for which $j = k$ is called the *principal* or *main diagonal* and the sum of all such elements is called the *trace* of A.

Example 7. The principal or main diagonal of the matrix

$$\begin{pmatrix} 5 & 2 & 0 \\ 3 & 1 & -2 \\ -1 & 4 & 2 \end{pmatrix}$$

is indicated by the shading, and the trace of the matrix is $5 + 1 + 2 = 8$.

A matrix for which $a_{jk} = 0$ when $j \neq k$ is called a *diagonal matrix*.

11. **Unit Matrix.** A square matrix in which all elements of the principal diagonal are equal to 1 while all other elements are zero is called the *unit matrix* and is denoted by I. An important property of I is that

$$AI = IA = A, \quad I^n = I, \quad n = 1, 2, 3, \ldots \tag{6}$$

The unit matrix plays a role in matrix algebra similar to that played by the number one in ordinary algebra.

12. **Zero or Null Matrix.** A matrix whose elements are all equal to zero is called the *null* or *zero matrix* and is often denoted by O or simply 0. For any matrix A having the same order as 0 we have

$$A + 0 = 0 + A = A \tag{7}$$

Also if A and 0 are square matrices, then

$$A0 = 0A = 0 \tag{8}$$

The zero matrix plays a role in matrix algebra similar to that played by the number zero of ordinary algebra.

DETERMINANTS

If the matrix A in (*1*) is a square matrix, then we associate with A a number denoted by

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix} \tag{9}$$

called the *determinant* of A of *order* n, written $\det(A)$. In order to define the value of a determinant, we introduce the following concepts.

1. **Minor.** Given any element a_{jk} of Δ we associate a new determinant of order $(n-1)$ obtained by removing all elements of the jth row and kth column called the minor of a_{jk}.

Example 8. The minor corresponding to the element 5 in the 2nd row and 3rd column of the fourth order determinant

$$\begin{vmatrix} 2 & -1 & 1 & 3 \\ -3 & 2 & 5 & 0 \\ 1 & 0 & -2 & 2 \\ 4 & -2 & 3 & 1 \end{vmatrix} \quad \text{is} \quad \begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 4 & -2 & 1 \end{vmatrix}$$

which is obtained by removing the elements shown shaded.

2. **Cofactor.** If we multiply the minor of a_{jk} by $(-1)^{j+k}$, the result is called the *cofactor* of a_{jk} and is denoted by A_{jk}.

Example 9. The cofactor corresponding to the element 5 in the determinant of Example 8 is $(-1)^{2+3}$ times its minor, or

$$-\begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 4 & -2 & 1 \end{vmatrix}$$

The value of a determinant is then defined as the sum of the products of the elements in any row [or column] by their corresponding cofactors and is called the *Laplace expansion.* In symbols,

$$\det A \quad = \quad \sum_{k=1}^{n} a_{jk} A_{jk} \tag{10}$$

We can show that this value is independent of the row [or column] used [see Problem 15.7].

THEOREMS ON DETERMINANTS

Theorem 15-1. The value of a determinant remains the same if rows and columns are interchanged. In symbols, $\det(A) = \det(A^T)$.

Theorem 15-2. If all elements of any row [or column] are zero except for one element, then the value of the determinant is equal to the product of that element by its cofactor. In particular, if all elements of a row [or column] are zero the determinant is zero.

Theorem 15-3. An interchange of any two rows [or columns] changes the sign of the determinant.

Theorem 15-4. If all elements in any row [or column] are multiplied by a number, the determinant is also multiplied by this number.

Theorem 15-5. If any two rows [or columns] are the same or proportional, the determinant is zero.

Theorem 15-6. If we express the elements of each row [or column] as the sum of two terms, then the determinant can be expressed as the sum of two determinants having the same order.

Theorem 15-7. If we multiply the elements of any row [or column] by a given number and add to corresponding elements of any other row [or column], then the value of the determinant remains the same.

Theorem 15-8. If A and B are square matrices of the same order, then

$$\det(AB) \quad = \quad \det(A)\det(B) \tag{11}$$

Theorem 15-9. The sum of the products of the elements of any row [or column] by the cofactors of another row [or column] is zero. In symbols,

$$\sum_{k=1}^{n} a_{qk} A_{pk} = 0 \quad \text{or} \quad \sum_{k=1}^{n} a_{kq} A_{kp} = 0 \quad \text{if } p \neq q \tag{12}$$

If $p = q$, the sum is $\det(A)$ by (*10*).

Theorem 15-10. Let $v_1, v_2, \ldots, v_n$ represent row vectors [or column vectors] of a square matrix A of order n. Then $\det(A) = 0$ if and only if there exist constants [scalars] $\lambda_1, \lambda_2, \ldots, \lambda_n$ not all zero such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = O \tag{13}$$

where O is the null or zero row matrix. If condition (*13*) is satisfied we say that the vectors $v_1, v_2, \ldots, v_n$ are *linearly dependent*. Otherwise they are *linearly independent*. A matrix A such that $\det(A) = 0$ is called a *singular matrix*. If $\det(A) \neq 0$, then A is a *non-singular matrix*.

In practice we evaluate a determinant of order n by using Theorem 15-7 successively to replace all but one of the elements in a row or column by zeros and then using Theorem 15-2 to obtain a new determinant of order $n-1$. We continue in this manner, arriving ultimately at determinants of orders 2 or 3 which are easily evaluated.

INVERSE OF A MATRIX

If for a given square matrix A there exists a matrix B such that $AB = I$, then B is called an *inverse* of A and is denoted by A^{-1}. The following theorem is fundamental.

Theorem 15-11. If A is a non-singular square matrix of order n [i.e. $\det(A) \neq 0$], then there exists a unique inverse A^{-1} such that $AA^{-1} = A^{-1}A = I$ and we can express A^{-1} in the following form

$$A^{-1} = \frac{(A_{jk})^T}{\det(A)} \tag{14}$$

where (A_{jk}) is the matrix of cofactors A_{jk} and $(A_{jk})^T = (A_{kj})$ is its transpose.

The following express some properties of the inverse:

$$(AB)^{-1} = B^{-1}A^{-1}, \quad (A^{-1})^{-1} = A \tag{15}$$

ORTHOGONAL AND UNITARY MATRICES

A real matrix A is called an *orthogonal matrix* if its transpose is the same as its inverse, i.e. if $A^T = A^{-1}$ or $A^T A = I$.

A complex matrix A is called a *unitary matrix* if its complex conjugate transpose is the same as its inverse, i.e. if $\bar{A}^T = A^{-1}$ or $\bar{A}^T A = I$. It should be noted that a real unitary matrix is an orthogonal matrix.

ORTHOGONAL VECTORS

In Chapter 5 we found that the scalar or dot product of two vectors $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is $a_1b_1 + a_2b_2 + a_3b_3$ and that the vectors are perpendicular or orthogonal if $a_1b_1 + a_2b_2 + a_3b_3 = 0$. From the point of view of matrices we can consider these vectors as column vectors

$$A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

from which it follows that $\quad A^T B = a_1 b_1 + a_2 b_2 + a_3 b_3$

This leads us to define the *scalar product of real column vectors* A and B as $A^T B$ and to define A and B to be *orthogonal* if $A^T B = 0$.

It is convenient to generalize this to cases where the vectors can have complex components and we adopt the following definition:

Definition 1. Two column vectors A and B are called *orthogonal* if $\bar{A}^T B = 0$, and $\bar{A}^T B$ is called the *scalar product* of A and B.

It should be noted also that if A is a unitary matrix then $\bar{A}^T A = 1$, which means that the scalar product of A with itself is 1 or equivalently A is a *unit vector,* i.e. having length 1. Thus a unitary column vector is a unit vector. Because of these remarks we have the following

Definition 2. A set of vectors $X_1, X_2, \ldots$ for which

$$\bar{X}_j^T X_k = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

is called a *unitary set or system of vectors* or, in the case where the vectors are real, an *orthonormal set* or an *orthogonal set of unit vectors.*

SYSTEMS OF LINEAR EQUATIONS

A set of equations having the form

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = r_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = r_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = r_n \end{array}\right\} \tag{16}$$

is called a *system of m linear equations in the n unknowns* $x_1, x_2, \ldots, x_n$. If $r_1, r_2, \ldots, r_n$ are all zero the system is called *homogeneous.* If they are not all zero it is called *non-homogeneous.* Any set of numbers $x_1, x_2, \ldots, x_n$ which satisfies (*16*) is called a *solution* of the system.

In matrix form (*16*) can be written

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \tag{17}$$

or more briefly

$$AX = R \tag{18}$$

where A, X, R represent the corresponding matrices in (*17*).

SYSTEMS OF n EQUATIONS IN n UNKNOWNS. CRAMER'S RULE

If $m = n$ and if A is a non-singular matrix so that A^{-1} exists, we can solve (*17*) or (*18*) by writing

$$X = A^{-1}R \tag{19}$$

and the system has a unique solution.

Alternatively we can express the unknowns $x_1, x_2, \ldots, x_n$ as

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad \ldots, \quad x_n = \frac{\Delta_n}{\Delta} \tag{20}$$

where $\Delta = \det(A)$, called the *determinant of the system*, is given by (9) and Δ_k, $k = 1, 2, \ldots, n$ is the determinant obtained from Δ by removing the kth column and replacing it by the column vector R. The rule expressed in (20) is called *Cramer's rule.*

The following four cases can arise.

Case 1, $\Delta \neq 0$, $R \neq 0$. In this case there will be a unique solution where not all x_k will be zero.

Case 2, $\Delta \neq 0$, $R = 0$. In this case the only solution will be $x_1 = 0, x_2 = 0, \ldots, x_n = 0$, i.e. $X = 0$. This is often called the *trivial solution.*

Case 3, $\Delta = 0$, $R = 0$. In this case there will be infinitely many solutions other than the trivial solution. This means that at least one of the equations can be obtained from the others, i.e. the equations are linearly dependent.

Case 4, $\Delta = 0$, $R \neq 0$. In this case infinitely many solutions will exist if and only if all of the determinants Δ_k in (20) are zero. Otherwise there will be no solution.

The cases where $m \neq n$ are considered in Problems 15.93-15.96.

EIGENVALUES AND EIGENVECTORS

Let $A = (a_{jk})$ be an $n \times n$ matrix and X a column vector. The equation

$$AX = \lambda X \tag{21}$$

where λ is a number can be written as

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{22}$$

or

$$\left.\begin{aligned} (a_{11} - \lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots& \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn} - \lambda)x_n &= 0 \end{aligned}\right\} \tag{23}$$

The equation (23) will have non-trivial solutions if and only if

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda \end{vmatrix} = 0 \tag{24}$$

which is a polynomial equation of degree n in λ. The roots of this polynomial equation are called *eigenvalues* or *characteristic values* of the matrix A. Corresponding to each eigenvalue there will be a solution $X \neq 0$, i.e. a non-trivial solution, which is called an *eigenvector* or *characteristic vector* belonging to the eigenvalue. The equation (24) can also be written

$$\det(A - \lambda I) = 0 \tag{25}$$

and the equation in λ is often called the *characteristic equation.*

THEOREMS ON EIGENVALUES AND EIGENVECTORS

Theorem 15-12. The eigenvalues of a Hermitian matrix [or symmetric real matrix] are real. The eigenvalues of a skew-Hermitian matrix [or skew-symmetric real matrix] are zero or pure imaginary. The eigenvalues of a unitary [or real orthogonal matrix] all have absolute value equal to 1.

Theorem 15-13. The eigenvectors belonging to different eigenvalues of a Hermitian matrix [or symmetric real matrix] are orthogonal.

Theorem 15-14 **[Cayley-Hamilton].** A matrix satisfies its own characteristic equation [see Problem 15.40].

Theorem 15-15 **[Reduction of matrix to diagonal form].** If a non-singular matrix A has distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$ with corresponding eigenvectors written as columns in the matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \ldots \\ b_{21} & b_{22} & b_{23} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

then

$$B^{-1}AB = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots \\ 0 & \lambda_2 & 0 & \ldots \\ 0 & 0 & \lambda_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

i.e. $B^{-1}AB$, called the *transform* of A by B, is a diagonal matrix containing the eigenvalues of A in the main diagonal and zeros elsewhere. We say that A has been *transformed* or *reduced to diagonal form.* See Problem 15.41.

Theorem 15-16 **[Reduction of quadratic form to canonical form].**

Let A be a symmetric real matrix, for example,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad a_{12} = a_{21},\ a_{13} = a_{31},\ a_{23} = a_{32}$$

Then if $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, we obtain the *quadratic form*

$$X^TAX = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}^2x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$$

The cross product terms of this quadratic form can be removed by letting $X = BU$ where U is the column vector with elements u_1, u_2, u_3 and B is an orthogonal matrix which diagonalizes A. The new quadratic form in u_1, u_2, u_3 with no cross product terms is called the *canonical form.* See Problem 15.43. A generalization can be made to Hermitian quadratic forms [see Problem 15.114].

OPERATOR INTERPRETATION OF MATRICES

If A is an $n \times n$ matrix, we can think of it as an *operator* or *transformation* acting on a column vector X to produce AX which is another column vector. With this interpretation equation (*21*) asks for those vectors X which are transformed by A into constant multiples of themselves [or equivalently into vectors which have the same direction but possibly different magnitude].

If case A is an orthogonal matrix, the transformation is a *rotation* and explains why the absolute value of all the eigenvalues in such case are equal to one [Theorem 15-12], since an ordinary rotation of a vector would not change its magnitude.

The ideas of transformation are very convenient in giving interpretations to many properties of matrices.

Solved Problems

OPERATIONS WITH MATRICES

15.1. If $A = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 4 \\ -2 & -1 \end{pmatrix}$ find (*a*) $A+B$, (*b*) $A-B$, (*c*) $2A-3C$, (*d*) $3A+2B-4C$, (*e*) AB, (*f*) BA, (*g*) $(AB)C$, (*h*) $A(BC)$, (*i*) A^T+B^T, (*j*) B^TA^T.

(*a*) $$A+B = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$$

(*b*) $$A-B = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & 7 \end{pmatrix}$$

(*c*) $$2A-3C = 2\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix} - 3\begin{pmatrix} 1 & 4 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 8 & 6 \end{pmatrix} + \begin{pmatrix} -3 & -12 \\ 6 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -14 \\ 14 & 9 \end{pmatrix}$$

(*d*) $$\begin{aligned} 3A+2B-4C &= 3\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix} + 2\begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} - 4\begin{pmatrix} 1 & 4 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 6 & -3 \\ 12 & 9 \end{pmatrix} + \begin{pmatrix} -2 & 2 \\ 4 & -8 \end{pmatrix} + \begin{pmatrix} -4 & -16 \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} 0 & -17 \\ 24 & 5 \end{pmatrix} \end{aligned}$$

(*e*) $$AB = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} (2)(-1)+(-1)(2) & (2)(1)+(-1)(-4) \\ (4)(-1)+(3)(2) & (4)(1)+(3)(-4) \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ 2 & -8 \end{pmatrix}$$

(*f*) $$BA = \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} (-1)(2)+(1)(4) & (-1)(-1)+(1)(3) \\ (2)(2)+(-4)(4) & (2)(-1)+(-4)(3) \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -12 & -14 \end{pmatrix}$$

Note that $AB \neq BA$ using (*e*), illustrating the fact that the commutative law for products does not hold in general.

(*g*) $$(AB)C = \begin{pmatrix} -4 & 6 \\ 2 & -8 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} -16 & -22 \\ 18 & 16 \end{pmatrix}$$

(*h*) $$A(BC) = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}\left[\begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ -2 & -1 \end{pmatrix}\right] = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} -3 & -5 \\ 10 & 12 \end{pmatrix} = \begin{pmatrix} -16 & -22 \\ 18 & 16 \end{pmatrix}$$

Note that $(AB)C = A(BC)$ using (*g*), illustrating the fact that the associative law for products holds.

(*i*) $$A^T+B^T = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}^T + \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}^T = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & -1 \end{pmatrix}$$

Note that $A^T+B^T = (A+B)^T$ using (*a*).

(*j*) $$B^TA^T = \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}^T \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}^T = \begin{pmatrix} -1 & 2 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 6 & -8 \end{pmatrix}$$

Note that $B^TA^T = (AB)^T$ using (*e*).

15.2. If $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ -2 & 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix}$ show that

$$(A+B)^2 = A^2 + AB + BA + B^2$$

We have
$$A + B = \begin{pmatrix} 3 & 0 & 1 \\ -1 & -1 & 6 \\ 0 & 0 & 3 \end{pmatrix}$$

Then
$$(A+B)^2 = (A+B)(A+B) = \begin{pmatrix} 3 & 0 & 1 \\ -1 & -1 & 6 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 3 & 0 & 1 \\ -1 & -1 & 6 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} (3)(3)+(0)(-1)+(1)(0) & (3)(0)+(0)(-1)+(1)(0) & (3)(1)+(0)(6)+(1)(3) \\ (-1)(3)+(-1)(-1)+(6)(0) & (-1)(0)+(-1)(-1)+(6)(0) & (-1)(1)+(-1)(6)+(6)(3) \\ (0)(3)+(0)(-1)+(3)(0) & (0)(0)+(0)(-1)+(3)(0) & (0)(1)+(0)(6)+(3)(3) \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 0 & 6 \\ -2 & 1 & 11 \\ 0 & 0 & 9 \end{pmatrix}$$

Now
$$A^2 = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ -2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & -1 & -1 \\ -6 & 8 & -1 \\ -7 & -2 & 9 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 6 \\ 11 & -6 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ -2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -3 & 5 & 0 \\ -9 & -1 & 11 \\ 1 & 5 & -3 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -4 & 1 \\ 2 & 0 & 2 \\ 6 & -4 & 2 \end{pmatrix}$$

Thus
$$A^2 + AB + BA + B^2 = \begin{pmatrix} 9 & 0 & 6 \\ -2 & 1 & 11 \\ 0 & 0 & 9 \end{pmatrix} = (A+B)^2$$

15.3. Prove that any real square matrix can always be expressed as the sum of a real symmetric matrix and a real skew-symmetric matrix.

If A is any real square matrix, then

$$A = \tfrac{1}{2}(A+A^T) + \tfrac{1}{2}(A-A^T)$$

But since $(A+A^T)^T = A^T + A = A + A^T$, it follows that $\frac{1}{2}(A+A^T)$ is symmetric. Also, since $(A-A^T)^T = A^T - A = -(A-A^T)$, it follows that $\frac{1}{2}(A-A^T)$ is skew-symmetric. The required result is thus proved.

15.4. Show that the matrix $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is Hermitian.

We have $A^T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ and $\overline{A^T} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = A$. Thus A is Hermitian.

15.5. Prove that a unit matrix I of order n commutes with any square matrix A of order n and the resulting product is A.

We illustrate the proof for $n = 3$. In such case

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Then
$$IA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A$$

$$AI = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = A$$

i.e. $IA = AI = A$.

Extensions are easily made for $n > 3$.

DETERMINANTS

15.6. Use the definition of a determinant [Laplace expansion] as given on page 345 to evaluate a determinant of (*a*) order 2, (*b*) order 3.

(*a*) Let the determinant be $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$. Use the elements of the first row. The corresponding cofactors are

$$A_{11} = (-1)^{1+1}a_{22} = a_{22}, \qquad A_{12} = (-1)^{2+1}a_{21} = -a_{21}$$

Then by the Laplace expansion the determinant has the value

$$a_{11}A_{11} + a_{12}A_{12} = a_{11}a_{22} - a_{12}a_{21}$$

The same value is obtained by using the elements of the second row [or first and second columns].

(*b*) Let the determinant be $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ The cofactors of the elements in the first row are

$$A_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{23}a_{31} - a_{21}a_{33}$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$$

Then the value of the determinant is

$$\begin{aligned} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) \\ &\quad + a_{12}(a_{23}a_{31} - a_{21}a_{33}) \\ &\quad\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

The same value is obtained by using elements of the second or third rows [or first, second and third columns].

15.7. Prove that the value of a determinant remains the same regardless of which row [or column] is taken for the Laplace expansion.

Consider the determinant $\Delta = (a_{jk})$ of order n. The result is true for $n = 2$ by Problem 15.6. We use proof by induction, i.e. assuming it to be true for order $n-1$ we shall prove it true for order n. The plan will be to expand Δ using two different rows p and q and show that the expansions are the same.

Let us first expand Δ by elements in the pth row. Then a typical term in the expansion is

$$a_{pk}A_{pk} = a_{pk}(-1)^{p+k}M_{pk} \tag{1}$$

where M_{pk} is the minor corresponding to the cofactor A_{pk} of a_{pk}. Since this minor is of order $n-1$, any row can be used in its expansion.

We shall use the qth row where we assume that $q > p$ since a similar argument holds if $q < p$. This row consists of elements a_{qr} where $r \neq k$ and corresponds to the $(q-1)$st row of M_{pk}.

Now if $r < k$, a_{qr} is located in the rth column of M_{pk} so that in the expansion the term corresponding to a_{qr} is

$$a_{qr}(-1)^{(q-1)+r}M_{pkqr} \tag{2}$$

where M_{pkqr} is the minor corresponding to the element a_{qr} in M_{pk}. From (1) and (2) it follows that a typical term in the expansion of Δ is

$$a_{pk}(-1)^{p+k}a_{qr}(-1)^{q-1+r}M_{pkqr} = a_{pk}a_{qr}(-1)^{p+k+q+r-1}M_{pkqr} \tag{3}$$

If $r > k$ then a_{qr} is located in the $(r-1)$st column and so there is an additional minus sign in (3).

If we now expand Δ by elements in the qth row, a typical term is

$$a_{qr}A_{qr} = a_{qr}(-1)^{q+r}M_{qr} \tag{4}$$

We can expand M_{qr} by elements in the pth row where $p > q$. As before if $k > r$, a typical term in the expansion of M_{qr} is

$$a_{pk}(-1)^{p+(k-1)}M_{pkqr} \tag{5}$$

From (4) and (5) we see that a typical term in the expansion of Δ is

$$a_{qr}(-1)^{q+r}\,a_{pk}(-1)^{p+k-1}M_{pkqr} = a_{pk}a_{qr}(-1)^{p+k+q+r-1}M_{pkqr} \tag{6}$$

which is the same as (3). If $k < r$ an additional minus sign appears in (6), agreeing with the case corresponding to $r > k$ using the first expansion. Thus the required result is proved.

In a similar manner we can prove that expansion by columns is the same and gives the same result as the expansion by rows [Theorem 15-1, page 345].

15.8. Evaluate by the Laplace expansion the determinant $\begin{vmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{vmatrix}$ (a) using elements in the first row and (b) using elements in the second row.

(a) Using elements in the first row, the expansion is

$$(3)\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} + (2)\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}$$
$$= (3)(7) - (-2)(14) + (2)(-7) = 35$$

(b) Using elements in the second row, the expansion is

$$-(1)\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} + (2)\begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} - (-3)\begin{vmatrix} 3 & -2 \\ 4 & 1 \end{vmatrix}$$
$$= -(1)(-6) + (2)(-2) - (-3)(11) = 35$$

15.9. Prove Theorem 15-4, page 345.

Let the determinant be

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{1}$$

and suppose that the elements in the kth row are multiplied by λ to give the determinant

$$\Delta_1 = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda a_{k1} & \lambda a_{k2} & \cdots & \lambda a_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{2}$$

Expanding (*1*) and (*2*) according to elements in the kth row, we find respectively

$$\Delta = a_{k1}A_{k1} + a_{k2}A_{k2} + \cdots + a_{kn}A_{kn} \tag{3}$$

$$\Delta_1 = (\lambda a_{k1})A_{k1} + (\lambda a_{k2})A_{k2} + \cdots + (\lambda a_{kn})A_{kn} \tag{4}$$

from which $\Delta_1 = \lambda\Delta$ as required.

15.10. Prove Theorem 15-5, page 345.

(*a*) If two rows have the same elements, then the value of the determinant will not change if the rows are interchanged. However, according to Theorem 15-3, page 345, the sign must change. Thus we have $\Delta = -\Delta$ or $\Delta = 0$.

(*b*) If the two rows have proportional elements, then they can be made the same by factoring out the proportionality constants and thus the determinant must be zero by (*a*).

15.11. Prove Theorem 15-6, page 345.

Write the determinant as

$$\Delta = \begin{vmatrix} a_{11}+b_1 & a_{12}+b_2 & \cdots & a_{1n}+b_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

in which the first row has each element expressed as the sum of two terms. Then by the Laplace expansion we have

$$\Delta = (a_{11}+b_1)A_{11} + (a_{12}+b_2)A_{12} + \cdots + (a_{1n}+b_n)A_{1n} \tag{1}$$

where $A_{11}, A_{12}, \ldots, A_{1n}$ are the cofactors of the corresponding elements in the first row. But (*1*) can be written as

$$\begin{aligned} \Delta &= (a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}) + (b_1A_{11} + \cdots + b_nA_{1n}) \\ &= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} b_1 & b_2 & \cdots & b_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \end{aligned}$$

as required. A similar procedure proves the result if any other row [or column] is chosen.

15.12. Prove Theorem 15-7, page 345.

Suppose we multiply the elements of the second row of $\Delta = (a_{jk})$ by λ and add to the elements of the first row [a similar proof can be used for any other rows or columns]. Then the determinant can be written as

$$\begin{vmatrix} a_{11}+\lambda a_{21} & a_{12}+\lambda a_{22} & \cdots & a_{1n}+\lambda a_{2n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

But by Problem 15.11 this can be written as

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Then the required result follows since the second determinant is zero because the elements of its first and second rows are proportional [Theorem 15-5].

15.13. Evaluate $\begin{vmatrix} 2 & 1 & -1 & 4 \\ -2 & 3 & 2 & -5 \\ 1 & -2 & -3 & 2 \\ -4 & -3 & 2 & -2 \end{vmatrix}$.

Multiplying the elements of the first row by $-3, 2, 3$ and adding to the elements of the second, third and fourth rows respectively, we find

$$\begin{vmatrix} 2 & 1 & -1 & 4 \\ -8 & 0 & 5 & -17 \\ 5 & 0 & -5 & 10 \\ 2 & 0 & -1 & 10 \end{vmatrix}$$

which by Theorem 15-7 has a value equal to that of the given determinant. Note that this new determinant has three zeros in the 2nd column, which was precisely our intention in choosing the numbers $-3, 2, 3$ in the first place.

Multiplying each element in the second column by its cofactor, we see that the value of the determinant is

$$-\begin{vmatrix} -8 & 5 & -17 \\ 5 & -5 & 10 \\ 2 & -1 & 10 \end{vmatrix} = -5\begin{vmatrix} -8 & 5 & -17 \\ 1 & -1 & 2 \\ 2 & -1 & 10 \end{vmatrix}$$

on removing the factor 5 from the second row, using Theorem 15-4.

Now multiplying the elements in the second row by 5 and -1 and adding to the elements of the first and third rows respectively, we find

$$-5\begin{vmatrix} -3 & 0 & -7 \\ 1 & -1 & 2 \\ 1 & 0 & 8 \end{vmatrix}$$

which on expanding by the elements in the second column gives

$$(-5)(-1)\begin{vmatrix} -3 & -7 \\ 1 & 8 \end{vmatrix} = -85$$

15.14. Verify Theorem 15-8 if $A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 7 & 2 \\ -3 & 4 \end{pmatrix}$.

The theorem states that $\det(AB) = \det(A)\det(B)$. Then since

$$AB = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 17 & 0 \\ 15 & 14 \end{pmatrix}$$

it states that
$$\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}\begin{vmatrix} 7 & 2 \\ -3 & 4 \end{vmatrix} = \begin{vmatrix} 17 & 0 \\ 15 & 14 \end{vmatrix}$$

or
$$(7)(34) = (17)(14)$$

But since this is correct, the theorem is verified for this case.

15.15. Let $v_1 = (2\ {-1}\ \ 3)$, $v_2 = (1\ \ 2\ {-1})$, $v_3 = (-3\ \ 4\ {-7})$. (*a*) Show that v_1, v_2, v_3 are linearly dependent. (*b*) Illustrate Theorem 15-10, page 346, by showing that

$$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ -3 & 4 & -7 \end{vmatrix} = 0$$

(*a*) We must show that there exist constants $\lambda_1, \lambda_2, \lambda_3$ not all zero such that $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0 = (0\ 0\ 0)$. Now

$$\lambda_1(2\ \ -1\ \ \ 3) + \lambda_2(1\ \ \ 2\ \ -1) + \lambda_3(-3\ \ \ 4\ \ -7) = (0\ \ \ 0\ \ \ 0)$$

when
$$\begin{aligned} 2\lambda_1 + \lambda_2 - 3\lambda_3 &= 0 \\ -\lambda_1 + 2\lambda_2 + 4\lambda_3 &= 0 \\ 3\lambda_1 - \lambda_2 - 7\lambda_3 &= 0 \end{aligned}$$

Assuming that $\lambda_3 = 1$, for example, the equations become $2\lambda_1 + \lambda_2 = 3$, $\lambda_1 - 2\lambda_2 = 4$, $3\lambda_1 - \lambda_2 = 7$. Solving any two of these simultaneously, we find $\lambda_1 = 2$, $\lambda_2 = -1$. Thus $\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = 1$ provide the required constants.

(*b*) Multiplying the elements of the second row by $-2, 3$ and adding to the first and third rows respectively, the given determinant equals

$$\begin{vmatrix} 0 & -5 & 5 \\ 1 & 2 & -1 \\ 0 & 10 & -10 \end{vmatrix} = -(1)\begin{vmatrix} -5 & 5 \\ 10 & -10 \end{vmatrix} = 0$$

15.16. Prove Theorem 15-9, page 346.

By definition the determinant

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

when expanded according to the elements of the pth row has the value

$$\det(A) = a_{p1}A_{p1} + a_{p2}A_{p2} + \cdots + a_{pn}A_{pn} = \sum_{k=1}^{n} a_{pk}A_{pk} \tag{1}$$

Let us now replace the elements a_{pk} in the pth row of A by corresponding elements a_{qk} of the qth row where $p \neq q$. Then two rows will be identical and the new determinant thus obtained will be zero by Theorem 15-5. Since $a_{pk} = a_{qk}$, (*1*) is replaced by

$$0 = a_{q1}A_{p1} + a_{q2}A_{p2} + \cdots + a_{qn}A_{pn} = \sum_{k=1}^{n} a_{qk}A_{pk}$$

i.e.
$$\sum_{k=1}^{n} a_{qk}A_{pk} = 0 \qquad p \neq q \tag{2}$$

Similarly by using columns rather than rows we can show that

$$\sum_{k=1}^{n} a_{kq}A_{kp} = 0 \qquad p \neq q \tag{3}$$

If $p = q$, then (2) and (3) become respectively

$$\sum_{k=1}^{n} a_{pk}A_{pk} = \det(A) \tag{4}$$

$$\sum_{k=1}^{n} a_{kp}A_{kp} = \det(A) \tag{5}$$

INVERSE OF A MATRIX

15.17. Prove that $A^{-1} = \dfrac{(A_{jk})^T}{\det(A)} = \dfrac{(A_{kj})}{\det(A)}$.

We must show that $AA^{-1} = I$, the unit matrix. To do this consider the product

$$A(A_{jk})^T = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \begin{vmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{vmatrix}$$

Now by the rule for multiplying determinants [which is the same as that for multiplying matrices], the element c_{pq} in the resulting determinant is found by taking the sum of the products of elements in the qth row of the first determinant and the pth column of the second determinant. We thus have

$$c_{pq} = a_{q1}A_{p1} + a_{q2}A_{p2} + \cdots + a_{qn}A_{pn} = \sum_{k=1}^{n} a_{qk}A_{pk}$$

But by the results of Problem 15.16,

$$c_{pq} = \begin{cases} 0 & p \neq q \\ \det(A) & p = q \end{cases}$$

It follows that

$$A(A_{jk})^T = \begin{vmatrix} \det(A) & 0 & \dots & 0 \\ 0 & \det(A) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \det(A) \end{vmatrix}$$

Then if $\det(A) \neq 0$, this can be written

$$\frac{A(A_{jk})^T}{\det(A)} = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{vmatrix} = I$$

and it thus follows that $AB = I$ where

$$B = A^{-1} = \frac{(A_{jk})^T}{\det(A)}$$

15.18. (*a*) Find the inverse of the matrix $A = \begin{pmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{pmatrix}$ and (*b*) check the answer by direct multiplication.

(*a*) The matrix of cofactors of A is given by

$$(A_{jk}) = \begin{pmatrix} 7 & -14 & -7 \\ 6 & -2 & -11 \\ 2 & 11 & 8 \end{pmatrix}$$

The transpose of this matrix is

$$(A_{jk})^T = (A_{kj}) = \begin{pmatrix} 7 & 6 & 2 \\ -14 & -2 & 11 \\ -7 & -11 & 8 \end{pmatrix}$$

Since $\det(A) = 35$ [see Problem 15.8], we have

$$A^{-1} = \frac{(A_{jk})^T}{\det(A)} = \frac{1}{35}\begin{pmatrix} 7 & 6 & 2 \\ -14 & -2 & 11 \\ -7 & -11 & 8 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{6}{35} & \frac{2}{35} \\ -\frac{2}{5} & -\frac{2}{35} & \frac{11}{35} \\ -\frac{1}{5} & -\frac{11}{35} & \frac{8}{35} \end{pmatrix}$$

(b) $$AA^{-1} = \begin{pmatrix} 3 & -2 & 2 \\ 1 & 2 & -3 \\ 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{5} & \frac{6}{35} & \frac{2}{35} \\ -\frac{2}{5} & -\frac{2}{35} & \frac{11}{35} \\ -\frac{1}{5} & -\frac{11}{35} & \frac{8}{35} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

We can also show that $A^{-1}A = I$. This supplies the required check.

15.19. Prove that $(AB)^{-1} = B^{-1}A^{-1}$.

Let $X = (AB)^{-1}$. Then $(AB)X = I$ where I is the unit matrix. By the associative law this becomes $A(BX) = I$. Multiplying by A^{-1}, we have $A^{-1}[A(BX)] = A^{-1}I = A^{-1}$ which again using the associative law becomes $(A^{-1}A)(BX) = A^{-1}$ or $I(BX) = A^{-1}$, i.e. $BX = A^{-1}$. Multiplying by B^{-1} and using the associative law once more, we have $B^{-1}(BX) = B^{-1}A^{-1}$, $(B^{-1}B)X = B^{-1}A^{-1}$, $IX = B^{-1}A^{-1}$, i.e. $X = B^{-1}A^{-1}$, as required.

15.20. Prove that if A is a non-singular matrix, then $\det(A^{-1}) = \dfrac{1}{\det A}$.

Since $AA^{-1} = I$, $\det(AA^{-1}) = \det(I) = 1$. But by Theorem 15-8, $\det(AA^{-1}) = \det(A)\det(A^{-1})$. Thus $\det(A^{-1})\det(A) = 1$ and the required result follows.

ORTHOGONAL AND UNITARY MATRICES. ORTHOGONAL VECTORS

15.21. Show that $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is an orthogonal matrix.

We have, using the fact that A is real,

$$A^TA = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

since $\cos^2\theta + \sin^2\theta = 1$. Thus A is an orthogonal matrix.

15.22. Show that $A = \begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 & 0 \\ i\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a unitary matrix.

Since A is complex, we must show that $\bar{A}^TA = I$. We have

$$\bar{A}^TA = \begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 & 0 \\ i\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{2}/2 & -i\sqrt{2}/2 & 0 \\ i\sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

so that A is a unitary matrix.