CHAPTER 3

Transcendental Functions

> It is well known that the central problem of the whole of modern mathematics is the study of the transcendental functions defined by differential equations.

Felix Klein 1849–1925
Lectures on Mathematics (1911)

Introduction

With the exception of the trigonometric functions, all the functions we have encountered so far have been of three main types: *polynomials, rational functions* (quotients of polynomials), and *algebraic functions* (fractional powers of rational functions). On an interval in its domain, each of these functions can be constructed from real numbers and a single real variable x by using finitely many arithmetic operations (addition, subtraction, multiplication, and division) and by taking finitely many roots (fractional powers). Functions that cannot be so constructed are called **transcendental functions**. The only examples of these that we have seen so far are the trigonometric functions.

Much of the importance of calculus and many of its most useful applications result from its ability to illuminate the behaviour of transcendental functions that arise naturally when we try to model concrete problems in mathematical terms. This chapter is devoted to developing other transcendental functions, including exponential and logarithmic functions and the inverse trigonometric functions.

Some of these functions "undo" what other ones "do" and vice versa. When a pair of functions behaves this way, we call each one the inverse of the other. We begin the chapter by studying inverse functions in general.

3.1 Inverse Functions

Consider the function $f(x) = x^3$ whose graph is shown in Figure 3.1. Like any function, $f(x)$ has only one value for each x in its domain (for x^3 this is the whole real line $\mathbb{R}$). In geometric terms, this means that any *vertical* line meets the graph of f at only one point. However, for this function f, any *horizontal* line also meets the graph at only one point. This means that different values of x always give different values $f(x)$. Such a function is said to be *one-to-one*.

DEFINITION 1

A function f is **one-to-one** if $f(x_1) \neq f(x_2)$ whenever x_1 and x_2 belong to the domain of f and $x_1 \neq x_2$, or, equivalently, if

$$f(x_1) = f(x_2) \quad \Longrightarrow \quad x_1 = x_2.$$

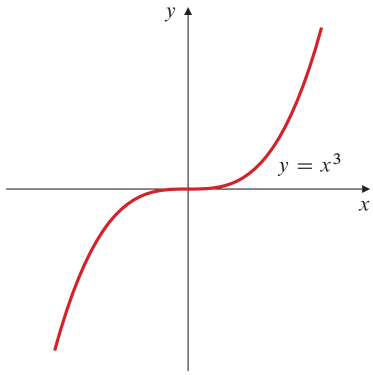


Figure 3.1 The graph of $f(x) = x^3$

A function is one-to-one if any horizontal line that intersects its graph does so at only one point. If a function defined on a single interval is increasing (or decreasing), then it is one-to-one. (See Section 2.6 for more discussion of this.)

Reconsider the one-to-one function $f(x) = x^3$ (Figure 3.1). Since the equation

$$y = x^3$$

has a unique solution x for every given value of y in the range of f, f is one-to-one. Specifically, this solution is given by

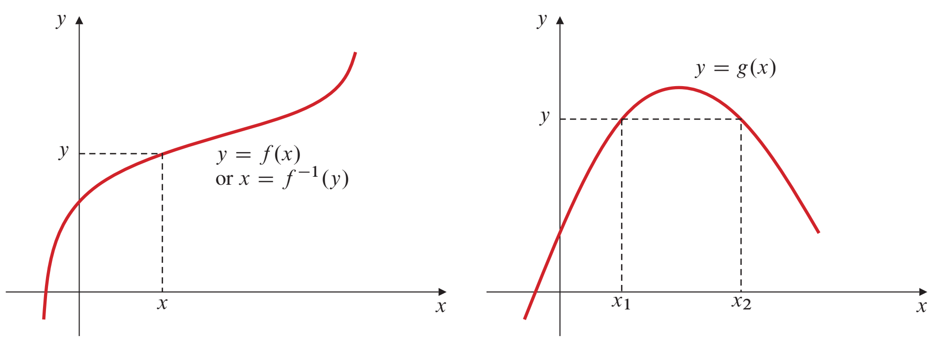
$$x = y^{1/3};$$

it defines x as a function of y. We call this new function the *inverse of f* and denote it f^{-1}. Thus,

$$f^{-1}(y) = y^{1/3}.$$

Do not confuse the -1 in f^{-1} with an exponent. The inverse f^{-1} is *not* the reciprocal $1/f$. If we want to denote the reciprocal $1/f(x)$ with an exponent we can write it as $\big(f(x)\big)^{-1}$.

In general, if a function f is one-to-one, then for any number y in its range there will always exist a single number x in its domain such that $y = f(x)$. Since x is determined uniquely by y, it is a function of y. We write $x = f^{-1}(y)$ and call f^{-1} the inverse of f. The function f whose graph is shown in Figure 3.2(a) is one-to-one and has an inverse. The function g whose graph is shown in Figure 3.2(b) is not one-to-one (some horizontal lines meet the graph twice) and so does not have an inverse.



(a) (b)

Figure 3.2

(a) f is one-to-one and has an inverse: $y = f(x)$ means the same thing as $x = f^{-1}(y)$

(b) g is not one-to-one

We usually like to write functions with the domain variable called x rather than y, so we reverse the roles of x and y and reformulate the above definition as follows.

DEFINITION 2

If f is one-to-one, then it has an **inverse function** f^{-1}. The value of $f^{-1}(x)$ is the unique number y in the domain of f for which $f(y) = x$. Thus,

$$y = f^{-1}(x) \quad \Longleftrightarrow \quad x = f(y).$$

As seen above, $y = f(x) = x^3$ is equivalent to $x = f^{-1}(y) = y^{1/3}$, or, reversing the roles of x and y, $y = f^{-1}(x) = x^{1/3}$ is equivalent to $x = f(y) = y^3$.

EXAMPLE 1 Show that $f(x) = 2x - 1$ is one-to-one, and find its inverse $f^{-1}(x)$.

Solution Since $f'(x) = 2 > 0$ on $\mathbb{R}$, f is increasing and therefore one-to-one there. Let $y = f^{-1}(x)$. Then

$$x = f(y) = 2y - 1.$$

Solving this equation for y gives $y = \dfrac{x+1}{2}$. Thus, $f^{-1}(x) = \dfrac{x+1}{2}$.

There are several things you should remember about the relationship between a function f and its inverse f^{-1}. The most important one is that the two equations

$$y = f^{-1}(x) \qquad \text{and} \qquad x = f(y)$$

say the same thing. They are equivalent just as, for example, $y = x + 1$ and $x = y - 1$ are equivalent. Either of the equations can be replaced by the other. This implies that the domain of f^{-1} is the range of f and vice versa.

The inverse of a one-to-one function is itself one-to-one and so also has an inverse. Not surprisingly, the inverse of f^{-1} is f:

$$y = (f^{-1})^{-1}(x) \iff x = f^{-1}(y) \iff y = f(x).$$

We can substitute either of the equations $y = f^{-1}(x)$ or $x = f(y)$ into the other and obtain the **cancellation identities**:

$$f\big(f^{-1}(x)\big) = x, \qquad f^{-1}\big(f(y)\big) = y.$$

The first of these identities holds for all x in the domain of f^{-1} and the second for all y in the domain of f. If S is any set of real numbers and I_S denotes the **identity function** on S, defined by

$$I_S(x) = x \quad \text{for all } x \text{ in } S,$$

then the cancellation identities say that if $\mathcal{D}(f)$ is the domain of f, then

$$f \circ f^{-1} = I_{\mathcal{D}(f^{-1})} \qquad \text{and} \qquad f^{-1} \circ f = I_{\mathcal{D}(f)},$$

where $f \circ g(x)$ denotes the composition $f\big(g(x)\big)$.

If the coordinates of a point $P = (a, b)$ are exchanged to give those of a new point $Q = (b, a)$, then each point is the reflection of the other in the line $x = y$. (To see this, note that the line PQ has slope -1, so it is perpendicular to $y = x$. Also, the midpoint of PQ is $\left(\frac{a+b}{2}, \frac{b+a}{2}\right)$, which lies on $y = x$.) It follows that the graphs of the equations $x = f(y)$ and $y = f(x)$ are reflections of each other in the line $x = y$. Since the equation $x = f(y)$ is equivalent to $y = f^{-1}(x)$, the graphs of the functions f^{-1} and f are reflections of each other in $y = x$. See Figure 3.3.

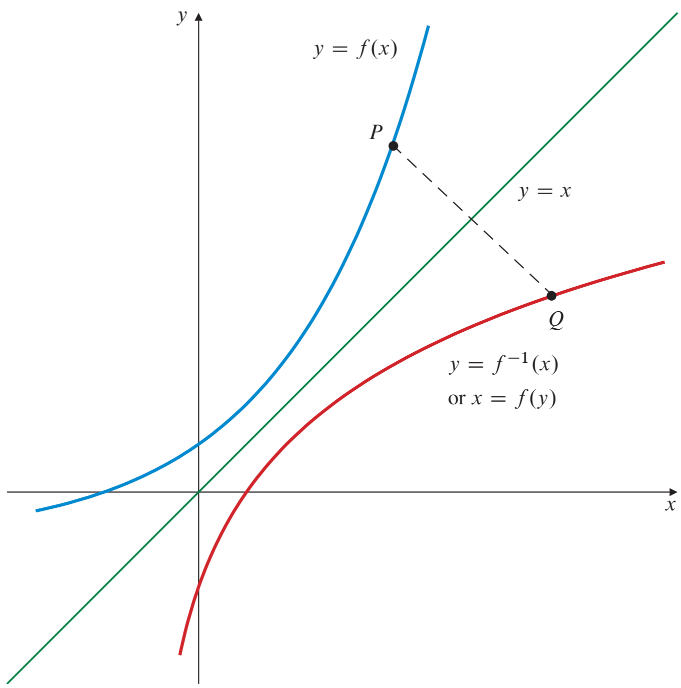


Figure 3.3 The graph of $y = f^{-1}(x)$ (red) is the reflection of the graph of $y = f(x)$ (blue) in the line $y = x$ (green)

Here is a summary of the properties of inverse functions discussed above:

Properties of inverse functions

1. $y = f^{-1}(x) \iff x = f(y)$.
2. The domain of f^{-1} is the range of f.
3. The range of f^{-1} is the domain of f.
4. $f^{-1}\big(f(x)\big) = x$ for all x in the domain of f.
5. $f\big(f^{-1}(x)\big) = x$ for all x in the domain of f^{-1}.
6. $(f^{-1})^{-1}(x) = f(x)$ for all x in the domain of f.
7. The graph of f^{-1} is the reflection of the graph of f in the line $x = y$.

EXAMPLE 2 Show that $g(x) = \sqrt{2x+1}$ is invertible, and find its inverse.

Solution If $g(x_1) = g(x_2)$, then $\sqrt{2x_1+1} = \sqrt{2x_2+1}$. Squaring both sides we get $2x_1 + 1 = 2x_2 + 1$, which implies that $x_1 = x_2$. Thus, g is one-to-one and invertible. Let $y = g^{-1}(x)$; then

$$x = g(y) = \sqrt{2y+1}.$$

It follows that $x \ge 0$ and $x^2 = 2y + 1$. Therefore, $y = \dfrac{x^2-1}{2}$ and

$$g^{-1}(x) = \frac{x^2-1}{2} \qquad \text{for } x \ge 0.$$

(The restriction $x \ge 0$ applies since the range of g is $[0, \infty)$.) See Figure 3.4(a) for the graphs of g and g^{-1}.

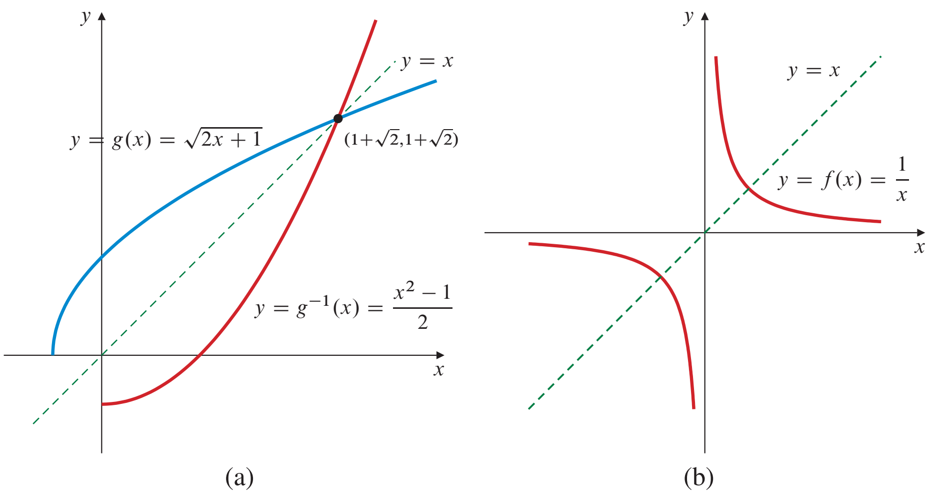


Figure 3.4
(a) The graphs of $g(x) = \sqrt{2x+1}$ and its inverse
(b) The graph of the self-inverse function $f(x) = 1/x$

DEFINITION 3

A function f is **self-inverse** if $f^{-1} = f$, that is, if $f\big(f(x)\big) = x$ for every x in the domain of f.

EXAMPLE 3 The function $f(x) = 1/x$ is self-inverse. If $y = f^{-1}(x)$, then $x = f(y) = \dfrac{1}{y}$. Therefore, $y = \dfrac{1}{x}$, so $f^{-1}(x) = \dfrac{1}{x} = f(x)$. See Figure 3.4(b). The graph of any self-inverse function must be its own reflection in the line $x = y$ and must therefore be symmetric about that line.

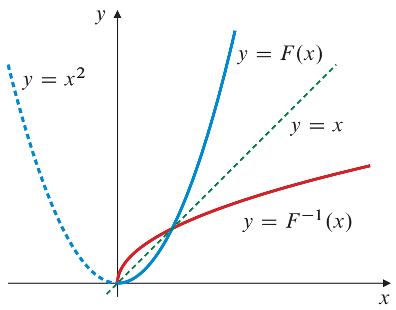


Figure 3.5 The restriction F of x^2 (blue) to $[0, \infty)$ and its inverse F^{-1} (red)

Inverting Non–One-to-One Functions

Many important functions, such as the trigonometric functions, are not one-to-one on their whole domains. It is still possible to define an inverse for such a function, but we have to restrict the domain of the function artificially so that the restricted function is one-to-one.

As an example, consider the function $f(x) = x^2$. Unrestricted, its domain is the whole real line and it is not one-to-one since $f(-a) = f(a)$ for any a. Let us define a new function $F(x)$ equal to $f(x)$ but having a smaller domain, so that it is one-to-one. We can use the interval $[0, \infty)$ as the domain of F:

$$F(x) = x^2 \qquad \text{for} \quad 0 \le x < \infty.$$

The graph of F is shown in Figure 3.5; it is the right half of the parabola $y = x^2$, the graph of f. Evidently F is one-to-one, so it has an inverse F^{-1}, which we calculate as follows:

Let $y = F^{-1}(x)$, then $x = F(y) = y^2$ and $y \ge 0$. Thus, $y = \sqrt{x}$. Hence $F^{-1}(x) = \sqrt{x}$.

This method of restricting the domain of a non–one-to-one function to make it invertible will be used when we invert the trigonometric functions in Section 3.5.

Derivatives of Inverse Functions

Suppose that the function f is differentiable on an interval (a, b) and that either $f'(x) > 0$ for $a < x < b$, so that f is increasing on (a, b), or $f'(x) < 0$ for $a < x < b$, so that f is decreasing on (a, b). In either case f is one-to-one on (a, b) and has an inverse, f^{-1} there. Differentiating the cancellation identity

$$f\big(f^{-1}(x)\big) = x$$

with respect to x, using the Chain Rule, we obtain

$$f'\big(f^{-1}(x)\big)\,\frac{d}{dx}\,f^{-1}(x) = \frac{d}{dx}\,x = 1.$$

Thus,

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'\big(f^{-1}(x)\big)}.$$

In Leibniz notation, if $y = f^{-1}(x)$, we have $\left.\dfrac{dy}{dx}\right|_{x} = \dfrac{1}{\left.\dfrac{dx}{dy}\right|_{y=f^{-1}(x)}}$.

The slope of the graph of f^{-1} at (x, y) is the reciprocal of the slope of the graph of f at (y, x). (See Figure 3.6.)

EXAMPLE 4 Show that $f(x) = x^3 + x$ is one-to-one on the whole real line, and, noting that $f(2) = 10$, find $\big(f^{-1}\big)'(10)$.

Solution Since $f'(x) = 3x^2 + 1 > 0$ for all real numbers x, f is increasing and therefore one-to-one and invertible. If $y = f^{-1}(x)$, then

$$\begin{aligned} x = f(y) = y^3 + y \quad &\Longrightarrow \quad 1 = (3y^2 + 1)y' \\ &\Longrightarrow \quad y' = \frac{1}{3y^2 + 1}. \end{aligned}$$

Now $x = f(2) = 10$ implies $y = f^{-1}(10) = 2$. Thus,

$$\big(f^{-1}\big)'(10) = \left.\frac{1}{3y^2 + 1}\right|_{y=2} = \frac{1}{13}.$$

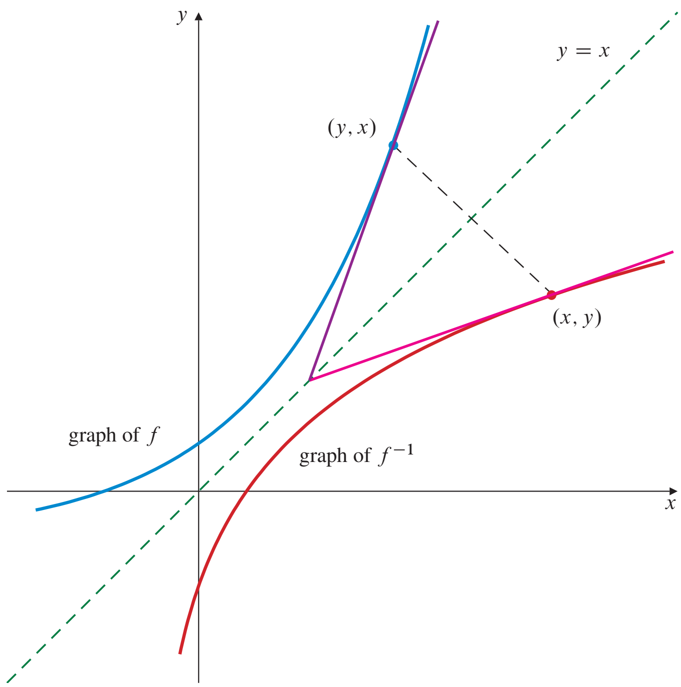


Figure 3.6 Tangents to the graphs of f and f^{-1}

EXERCISES 3.1

Show that the functions f in Exercises 1–12 are one-to-one, and calculate the inverse functions f^{-1}. Specify the domains and ranges of f and f^{-1}.

1. $f(x) = x - 1$

2. $f(x) = 2x - 1$

3. $f(x) = \sqrt{x - 1}$

4. $f(x) = -\sqrt{x - 1}$

5. $f(x) = x^3$

6. $f(x) = 1 + \sqrt[3]{x}$

7. $f(x) = x^2, \quad x \le 0$

8. $f(x) = (1 - 2x)^3$

9. $f(x) = \dfrac{1}{x + 1}$

10. $f(x) = \dfrac{x}{1 + x}$

11. $f(x) = \dfrac{1 - 2x}{1 + x}$

12. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$

In Exercises 13–20, f is a one-to-one function with inverse f^{-1}. Calculate the inverses of the given functions in terms of f^{-1}.

13. $g(x) = f(x) - 2$

14. $h(x) = f(2x)$

15. $k(x) = -3f(x)$

16. $m(x) = f(x - 2)$

17. $p(x) = \dfrac{1}{1 + f(x)}$

18. $q(x) = \dfrac{f(x) - 3}{2}$

19. $r(x) = 1 - 2f(3 - 4x)$

20. $s(x) = \dfrac{1 + f(x)}{1 - f(x)}$

In Exercises 21–23, show that the given function is one-to-one and find its inverse.

21. $f(x) = \begin{cases} x^2 + 1 & \text{if } x \ge 0 \\ x + 1 & \text{if } x < 0 \end{cases}$

22. $g(x) = \begin{cases} x^3 & \text{if } x \ge 0 \\ x^{1/3} & \text{if } x < 0 \end{cases}$

23. $h(x) = x|x| + 1$

24. Find $f^{-1}(2)$ if $f(x) = x^3 + x$.

25. Find $g^{-1}(1)$ if $g(x) = x^3 + x - 9$.

26. Find $h^{-1}(-3)$ if $h(x) = x|x| + 1$.

27. Assume that the function $f(x)$ satisfies $f'(x) = \dfrac{1}{x}$ and that f is one-to-one. If $y = f^{-1}(x)$, show that $dy/dx = y$.

28. Find $\left(f^{-1}\right)'(x)$ if $f(x) = 1 + 2x^3$.

29. Show that $f(x) = \dfrac{4x^3}{x^2 + 1}$ has an inverse and find $\left(f^{-1}\right)'(2)$.

30. Find $\left(f^{-1}\right)'(-2)$ if $f(x) = x\sqrt{3 + x^2}$.

31. If $f(x) = x^2/(1 + \sqrt{x})$, find $f^{-1}(2)$ correct to 5 decimal places.

32. If $g(x) = 2x + \sin x$, show that g is invertible, and find $g^{-1}(2)$ and $(g^{-1})'(2)$ correct to 5 decimal places.

33. Show that $f(x) = x \sec x$ is one-to-one on $(-\pi/2, \pi/2)$. What is the domain of $f^{-1}(x)$? Find $(f^{-1})'(0)$.

34. If functions f and g have respective inverses f^{-1} and g^{-1}, show that the composite function $f \circ g$ has inverse $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

35. For what values of the constants a, b, and c is the function $f(x) = (x - a)/(bx - c)$ self-inverse?

36. Can an even function be self-inverse? an odd function?

37. In this section it was claimed that an increasing (or decreasing) function defined on a single interval is necessarily one-to-one. Is the converse of this statement true? Explain.

38. Repeat Exercise 37 with the added assumption that f is continuous on the interval where it is defined.

3.2 Exponential and Logarithmic Functions

To begin we review exponential and logarithmic functions as you may have encountered them in your previous mathematical studies. In the following sections we will approach these functions from a different point of view and learn how to find their derivatives.

Exponentials

An **exponential function** is a function of the form $f(x) = a^x$, where the **base** a is a positive constant and the **exponent** x is the variable. Do not confuse such functions with **power** functions such as $f(x) = x^a$, where the base is variable and the exponent is constant. The exponential function a^x can be defined for integer and rational exponents x as follows:

DEFINITION 4

Exponential functions

If $a > 0$, then

$$a^0 = 1$$
$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}} \qquad \text{if } n = 1, 2, 3, \ldots$$
$$a^{-n} = \frac{1}{a^n} \qquad \text{if } n = 1, 2, 3, \ldots$$
$$a^{m/n} = \sqrt[n]{a^m} \qquad \text{if } n = 1, 2, 3, \ldots \quad \text{and } m = \pm 1, \pm 2, \pm 3, \ldots.$$

In this definition, $\sqrt[n]{a}$ is the number $b > 0$ that satisfies $b^n = a$.

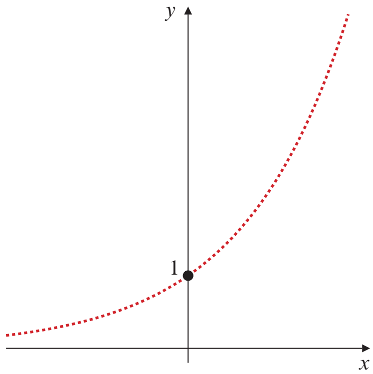


Figure 3.7 $y = 2^x$ for rational x

How should we define a^x if x is not rational? For example, what does 2^π mean? In order to calculate a derivative of a^x, we will want the function to be defined for all real numbers x, not just rational ones.

In Figure 3.7 we plot points with coordinates $(x, 2^x)$ for many closely spaced rational values of x. They appear to lie on a smooth curve. The definition of a^x can be extended to irrational x in such a way that a^x becomes a differentiable function of x on the whole real line. We will do so in the next section. For the moment, if x is irrational we can regard a^x as being the limit of values a^r for rational numbers r approaching x:

$$a^x = \lim_{\substack{r \to x \\ r \text{ rational}}} a^r.$$

EXAMPLE 1 Since the irrational number $\pi = 3.141\,592\,653\,59\ldots$ is the limit of the sequence of rational numbers

$$r_1 = 3, \quad r_2 = 3.1, \quad r_3 = 3.14, \quad r_4 = 3.141, \quad r_5 = 3.1415, \quad \ldots,$$

we can calculate 2^π as the limit of the corresponding sequence

$$2^3 = 8, \quad 2^{3.1} = 8.574\,187\,7\ldots, \quad 2^{3.14} = 8.815\,240\,9\ldots.$$

This gives $2^\pi = \lim_{n\to\infty} 2^{r_n} = 8.824\,977\,827\ldots.$

Exponential functions satisfy several identities called *laws of exponents*:

Laws of exponents

If $a > 0$ and $b > 0$, and x and y are any real numbers, then

(i) $a^0 = 1$ (ii) $a^{x+y} = a^x a^y$

(iii) $a^{-x} = \dfrac{1}{a^x}$ (iv) $a^{x-y} = \dfrac{a^x}{a^y}$

(v) $(a^x)^y = a^{xy}$ (vi) $(ab)^x = a^x b^x$

These identities can be proved for rational exponents using the definitions above. They remain true for irrational exponents, but we can't show that until the next section.

If $a = 1$, then $a^x = 1^x = 1$ for every x. If $a > 1$, then a^x is an increasing function of x; if $0 < a < 1$, then a^x is decreasing. The graphs of some typical exponential functions are shown in Figure 3.8(a). They all pass through the point (0,1) since $a^0 = 1$ for every $a > 0$. Observe that $a^x > 0$ for all $a > 0$ and all real x and that:

If $a > 1$, then $\lim_{x\to-\infty} a^x = 0$ and $\lim_{x\to\infty} a^x = \infty$.

If $0 < a < 1$, then $\lim_{x\to-\infty} a^x = \infty$ and $\lim_{x\to\infty} a^x = 0$.

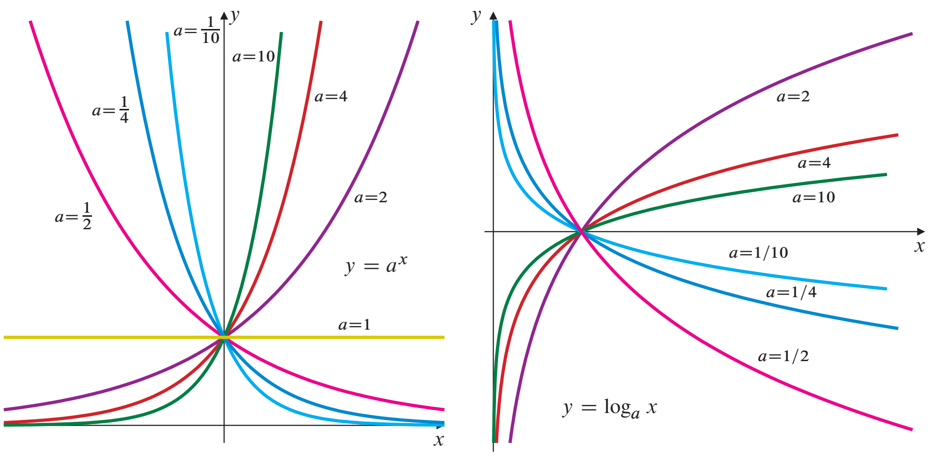


Figure 3.8

(a) Graphs of some exponential functions $y = a^x$

(b) Graphs of some logarithmic functions $y = \log_a(x)$

The graph of $y = a^x$ has the x-axis as a horizontal asymptote if $a \neq 1$. It is asymptotic on the left (as $x \to -\infty$) if $a > 1$ and on the right (as $x \to \infty$) if $0 < a < 1$.

Logarithms

The function $f(x) = a^x$ is a one-to-one function provided that $a > 0$ and $a \neq 1$. Therefore, f has an inverse which we call a *logarithmic function*.

DEFINITION 5

If $a > 0$ and $a \neq 1$, the function $\log_a x$, called **the logarithm of x to the base a**, is the inverse of the one-to-one function a^x:

$$y = \log_a x \iff x = a^y, \qquad (a > 0, \quad a \neq 1).$$

Since a^x has domain $(-\infty, \infty)$, $\log_a x$ has range $(-\infty, \infty)$. Since a^x has range $(0, \infty)$, $\log_a x$ has domain $(0, \infty)$. Since a^x and $\log_a x$ are inverse functions, the following **cancellation identities** hold:

$$\log_a(a^x) = x \quad \text{for all real } x \qquad \text{and} \qquad a^{\log_a x} = x \quad \text{for all } x > 0.$$

The graphs of some typical logarithmic functions are shown in Figure 3.8(b). They all pass through the point $(1, 0)$. Each graph is the reflection in the line $y = x$ of the corresponding exponential graph in Figure 3.8(a).

From the laws of exponents we can derive the following laws of logarithms:

Laws of logarithms

If $x > 0$, $y > 0$, $a > 0$, $b > 0$, $a \neq 1$, and $b \neq 1$, then

(i) $\log_a 1 = 0$ (ii) $\log_a(xy) = \log_a x + \log_a y$

(iii) $\log_a \left(\dfrac{1}{x}\right) = -\log_a x$ (iv) $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

(v) $\log_a (x^y) = y \log_a x$ (vi) $\log_a x = \dfrac{\log_b x}{\log_b a}$

EXAMPLE 2 If $a > 0$, $x > 0$, and $y > 0$, verify that $\log_a(xy) = \log_a x + \log_a y$, using laws of exponents.

Solution Let $u = \log_a x$ and $v = \log_a y$. By the defining property of inverse functions, $x = a^u$ and $y = a^v$. Thus, $xy = a^u a^v = a^{u+v}$. Inverting again, we get $\log_a(xy) = u + v = \log_a x + \log_a y$.

Logarithm law (vi) presented above shows that if you know logarithms to a particular base b, you can calculate logarithms to any other base a. Scientific calculators usually have built-in programs for calculating logarithms to base 10 and to base e, a special number that we will discover in Section 3.3. Logarithms to any base can be calculated using either of these functions. For example, computer scientists sometimes need to use logarithms to base 2. Using a scientific calculator, you can readily calculate

$$\log_2 13 = \frac{\log_{10} 13}{\log_{10} 2} = \frac{1.113\,943\,352\,31\ldots}{0.301\,029\,995\,664\ldots} = 3.700\,439\,718\,14\ldots\,.$$

The laws of logarithms can sometimes be used to simplify complicated expressions.

EXAMPLE 3 Simplify
(a) $\log_2 10 + \log_2 12 - \log_2 15$, (b) $\log_{a^2} a^3$, and (c) $3^{\log_9 4}$.

Solution

(a)
$$\begin{aligned}\log_2 10 + \log_2 12 - \log_2 15 &= \log_2 \frac{10 \times 12}{15} && \text{(laws (ii) and (iv))}\\ &= \log_2 8\\ &= \log_2 2^3 = 3. && \text{(cancellation identity)}\end{aligned}$$

(b)
$$\begin{aligned}\log_{a^2} a^3 &= 3 \log_{a^2} a && \text{(law (v))}\\ &= \frac{3}{2} \log_{a^2} a^2 && \text{(law (v) again)}\\ &= \frac{3}{2}. && \text{(cancellation identity)}\end{aligned}$$

(c)
$$\begin{aligned}3^{\log_9 4} &= 3^{(\log_3 4)/(\log_3 9)} && \text{(law (vi))}\\ &= \left(3^{\log_3 4}\right)^{1/\log_3 9}\\ &= 4^{1/\log_3 3^2} = 4^{1/2} = 2. && \text{(cancellation identity)}\end{aligned}$$

EXAMPLE 4 Solve the equation $3^{x-1} = 2^x$.

Solution We can take logarithms of both sides of the equation to any base a and get

$$\begin{aligned}(x-1)\log_a 3 &= x\log_a 2\\ (\log_a 3-\log_a 2)x &= \log_a 3\\ x &= \frac{\log_a 3}{\log_a 3-\log_a 2}=\frac{\log_a 3}{\log_a(3/2)}.\end{aligned}$$

The numerical value of x can be found using the "log" function on a scientific calculator. (This function is $\log_{10}$.) The value is $x = 2.7095\ldots$.

Corresponding to the asymptotic behaviour of the exponential functions, the logarithmic functions also exhibit asymptotic behaviour. Their graphs are all asymptotic to the y-axis as $x \to 0$ from the right:

If $a > 1$, then $\lim_{x\to 0+} \log_a x = -\infty$ and $\lim_{x\to\infty} \log_a x = \infty$.

If $0 < a < 1$, then $\lim_{x\to 0+} \log_a x = \infty$ and $\lim_{x\to\infty} \log_a x = -\infty$.

EXERCISES 3.2

Simplify the expressions in Exercises 1–18.

1. $\dfrac{3^3}{\sqrt{3^5}}$

2. $2^{1/2}8^{1/2}$

3. $(x^{-3})^{-2}$

4. $\left(\dfrac{1}{2}\right)^x 4^{x/2}$

5. $\log_5 125$

6. $\log_4\left(\dfrac{1}{8}\right)$

7. $\log_{1/3} 3^{2x}$

8. $2^{\log_4 8}$

9. $10^{-\log_{10}(1/x)}$

10. $x^{1/(\log_a x)}$

11. $(\log_a b)(\log_b a)$

12. $\log_x\left(x(\log_y y^2)\right)$

13. $(\log_4 16)(\log_4 2)$

14. $\log_{15} 75 + \log_{15} 3$

15. $\log_6 9 + \log_6 4$

16. $2\log_3 12 - 4\log_3 6$

17. $\log_a(x^4+3x^2+2)+\log_a(x^4+5x^2+6) - 4\log_a\sqrt{x^2+2}$

18. $\log_\pi(1-\cos x)+\log_\pi(1+\cos x)-2\log_\pi \sin x$

Use the base 10 exponential and logarithm functions 10^x and $\log x$ (that is, $\log_{10} x$) on a scientific calculator to evaluate the expressions or solve the equations in Exercises 19–24.

19. $3^{\sqrt{2}}$

20. $\log_3 5$

21. $2^{2x} = 5^{x+1}$

22. $x^{\sqrt{2}} = 3$

23. $\log_x 3 = 5$

24. $\log_3 x = 5$

Use the laws of exponents to prove the laws of logarithms in Exercises 25–28.

25. $\log_a\left(\dfrac{1}{x}\right) = -\log_a x$

26. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

27. $\log_a(x^y) = y\log_a x$

28. $\log_a x = (\log_b x)/(\log_b a)$

29. Solve $\log_4(x+4) - 2\log_4(x+1) = \dfrac{1}{2}$ for x.

30. Solve $2\log_3 x + \log_9 x = 10$ for x.

Evaluate the limits in Exercises 31–34.

31. $\lim_{x\to\infty} \log_x 2$

32. $\lim_{x\to 0+} \log_x(1/2)$

33. $\lim_{x\to 1+} \log_x 2$

34. $\lim_{x\to 1-} \log_x 2$

35. Suppose that $f(x) = a^x$ is differentiable at $x = 0$ and that $f'(0) = k$, where $k \neq 0$. Prove that f is differentiable at any real number x and that

$$f'(x) = k\,a^x = k\,f(x).$$

36. Continuing Exercise 35, prove that $f^{-1}(x) = \log_a x$ is differentiable at any $x > 0$ and that

$$(f^{-1})'(x) = \frac{1}{kx}.$$

3.3 The Natural Logarithm and Exponential Functions

Regard this paragraph as describing a game we are going to play in this section. The result of the game will be that we will acquire two new classes of functions, logarithms, and exponentials, to which the rules of calculus will apply.

In this section we are going to define a function $\ln x$, called the *natural* logarithm of x, in a way that does not at first seem to have anything to do with the logarithms considered in Section 3.2. We will show, however, that it has the same properties as those logarithms, and in the end we will see that $\ln x = \log_e x$, the logarithm of x to a certain specific base e. We will show that $\ln x$ is a one-to-one function, defined for all positive real numbers. It must therefore have an inverse, e^x, that we will call *the* exponential function. Our final goal is to arrive at a definition of the exponential functions a^x (for any $a > 0$) that is valid for any real number x instead of just rational numbers, and that is known to be continuous and even differentiable without our having to assume those properties as we did in Section 3.2.

The Natural Logarithm

Table 1. Derivatives of integer powers

$f(x)$	$f'(x)$
$\vdots$	$\vdots$
x^4	$4x^3$
x^3	$3x^2$
x^2	$2x$
x^1	$1x^0 = 1$
x^0	0
x^{-1}	$-x^{-2}$
x^{-2}	$-2x^{-3}$
x^{-3}	$-3x^{-4}$
$\vdots$	$\vdots$

Table 1 lists the derivatives of integer powers of x. Those derivatives are multiples of integer powers of x, but one integer power, x^{-1}, is conspicuously absent from the list of derivatives; we do not yet know a function whose derivative is $x^{-1} = 1/x$. We are going to remedy this situation by defining a function $\ln x$ in such a way that it will have derivative $1/x$.

To get a hint as to how this can be done, review Example 1 of Section 2.11. In that example we showed that the area under the graph of the velocity of a moving object in a time interval is equal to the distance travelled by the object in that time interval. Since the derivative of distance is velocity, measuring the area provided a way of finding a function (the distance) that had a given derivative (the velocity). This relationship between area and derivatives is one of the most important ideas in calculus. It is called the **Fundamental Theorem of Calculus**. We will explore it fully in Chapter 5, but we will make use of the idea now to define $\ln x$, which we want to have derivative $1/x$.

DEFINITION 6

The natural logarithm

For $x > 0$, let A_x be the area of the plane region bounded by the curve $y = 1/t$, the t-axis, and the vertical lines $t = 1$ and $t = x$. The function $\ln x$ is defined by

$$\ln x = \begin{cases} A_x & \text{if } x \ge 1, \\ -A_x & \text{if } 0 < x < 1, \end{cases}$$

as shown in Figure 3.9.

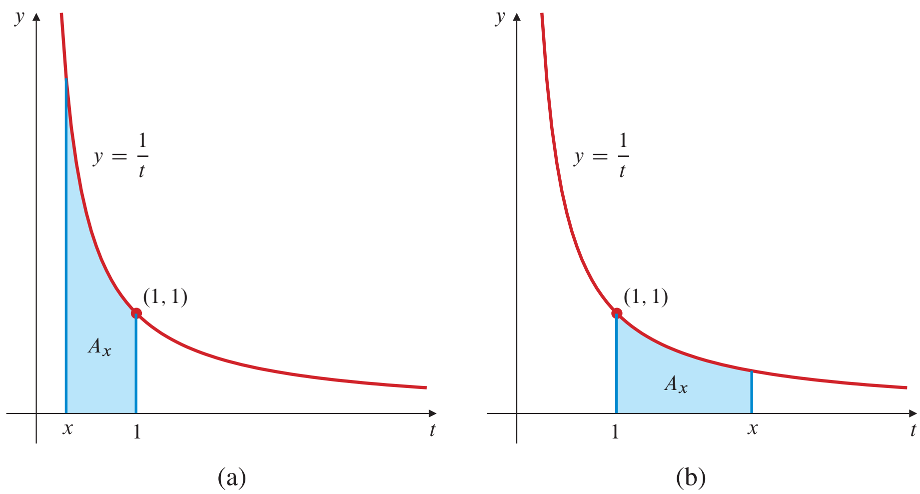


Figure 3.9
(a) $\ln x = -\text{area } A_x$ if $0 < x < 1$
(b) $\ln x = \text{area } A_x$ if $x \ge 1$

The definition implies that $\ln 1 = 0$, that $\ln x > 0$ if $x > 1$, that $\ln x < 0$ if $0 < x < 1$, and that ln is a one-to-one function. We now show that if $y = \ln x$, then $y' = 1/x$. The proof of this result is similar to the proof we will give for the Fundamental Theorem of Calculus in Section 5.5.

THEOREM 1

If $x > 0$, then

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

PROOF For $x > 0$ and $h > 0$, $\ln(x+h) - \ln x$ is the area of the plane region bounded by $y = 1/t$, $y = 0$, and the vertical lines $t = x$ and $t = x + h$; it is the shaded area in Figure 3.10. Comparing this area with that of two rectangles, we see that

$$\frac{h}{x+h} < \text{shaded area} = \ln(x+h) - \ln x < \frac{h}{x}.$$

Hence, the Newton quotient for $\ln x$ satisfies

$$\frac{1}{x+h} < \frac{\ln(x+h) - \ln x}{h} < \frac{1}{x}.$$

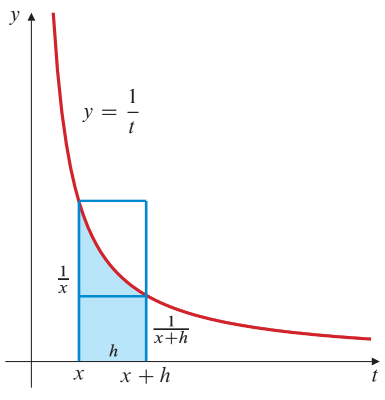


Figure 3.10

Letting h approach 0 from the right, we obtain (by the Squeeze Theorem applied to one-sided limits)

$$\lim_{h\to 0+}\frac{\ln(x+h) - \ln x}{h} = \frac{1}{x}.$$

A similar argument shows that if $0 < x + h < x$, then

$$\frac{1}{x} < \frac{\ln(x+h) - \ln x}{h} < \frac{1}{x+h},$$

so that

$$\lim_{h\to 0-}\frac{\ln(x+h) - \ln x}{h} = \frac{1}{x}.$$

Combining these two one-sided limits we get the desired result:

$$\frac{d}{dx}\ln x = \lim_{h\to 0}\frac{\ln(x+h) - \ln x}{h} = \frac{1}{x}.$$

The two properties $(d/dx)\ln x = 1/x$ and $\ln 1 = 0$ are sufficient to determine the function $\ln x$ completely. (This follows from Theorem 13 in Section 2.8.) We can deduce from these two properties that $\ln x$ satisfies the appropriate laws of logarithms:

THEOREM 2

Properties of the natural logarithm

(i) $\ln(xy) = \ln x + \ln y$ (ii) $\ln\left(\frac{1}{x}\right) = -\ln x$

(iii) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$ (iv) $\ln(x^r) = r\,\ln x$

Because we do not want to *assume* that exponentials are continuous (as we did in Section 3.2), we should regard (iv) for the moment as only valid for exponents r that are rational numbers.

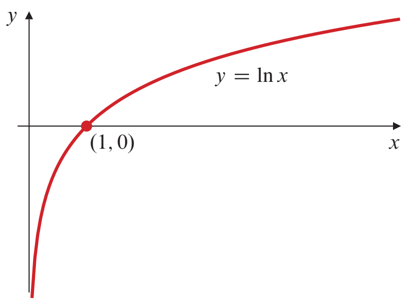


Figure 3.11 The graph of $\ln x$

PROOF We will only prove part (i) because the other parts are proved by the same method. If $y > 0$ is a constant, then by the Chain Rule,

$$\frac{d}{dx}\big(\ln(xy) - \ln x\big) = \frac{y}{xy} - \frac{1}{x} = 0 \quad \text{for all } x > 0.$$

Theorem 13 of Section 2.8 now tells us that $\ln(xy) - \ln x = C$ (a constant) for $x > 0$. Putting $x = 1$ we get $C = \ln y$ and identity (i) follows.

Part (iv) of Theorem 2 shows that $\ln(2^n) = n \ln 2 \to \infty$ as $n \to \infty$. Therefore, we also have $\ln(1/2)^n = -n \ln 2 \to -\infty$ as $n \to \infty$. Since $(d/dx)\ln x = 1/x > 0$ for $x > 0$, it follows that $\ln x$ is increasing, so we must have (see Figure 3.11)

$$\lim_{x\to\infty} \ln x = \infty, \qquad \lim_{x\to 0+} \ln x = -\infty.$$

EXAMPLE 1 Show that $\dfrac{d}{dx} \ln|x| = \dfrac{1}{x}$ for any $x \neq 0$. Hence find $\displaystyle\int \frac{1}{x}\,dx$.

Solution If $x > 0$, then

$$\frac{d}{dx} \ln|x| = \frac{d}{dx} \ln x = \frac{1}{x}$$

by Theorem 1. If $x < 0$, then, using the Chain Rule,

$$\frac{d}{dx} \ln|x| = \frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Therefore, $\dfrac{d}{dx} \ln|x| = \dfrac{1}{x}$, and on any interval not containing $x = 0$,

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

EXAMPLE 2 Find the derivatives of (a) $\ln|\cos x|$ and (b) $\ln\big(x + \sqrt{x^2+1}\big)$. Simplify your answers as much as possible.

Solution

(a) Using the result of Example 1 and the Chain Rule, we have

$$\frac{d}{dx} \ln|\cos x| = \frac{1}{\cos x}(-\sin x) = -\tan x.$$

(b)
$$\begin{aligned}
\frac{d}{dx} \ln\big(x + \sqrt{x^2+1}\big) &= \frac{1}{x + \sqrt{x^2+1}}\left(1 + \frac{2x}{2\sqrt{x^2+1}}\right) \\
&= \frac{1}{x + \sqrt{x^2+1}}\,\frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}} \\
&= \frac{1}{\sqrt{x^2+1}}.
\end{aligned}$$

The Exponential Function

The function $\ln x$ is one-to-one on its domain, the interval $(0, \infty)$, so it has an inverse there. For the moment, let us call this inverse $\exp x$. Thus,

$$y = \exp x \iff x = \ln y \quad (y > 0).$$

Since $\ln 1 = 0$, we have $\exp 0 = 1$. The domain of exp is $(-\infty, \infty)$, the range of ln. The range of exp is $(0, \infty)$, the domain of ln. We have cancellation identities

$$\ln(\exp x) = x \quad \text{for all real } x \qquad \text{and} \qquad \exp(\ln x) = x \quad \text{for } x > 0.$$

We can deduce various properties of exp from corresponding properties of ln. Not surprisingly, they are properties we would expect an exponential function to have.

THEOREM 3

Properties of the exponential function

(i) $(\exp x)^r = \exp(rx)$ (ii) $\exp(x+y) = (\exp x)(\exp y)$

(iii) $\exp(-x) = \dfrac{1}{\exp(x)}$ (iv) $\exp(x-y) = \dfrac{\exp x}{\exp y}$

For the moment, identity (i) is asserted only for rational numbers r.

PROOF We prove only identity (i); the rest are done similarly. If $u = (\exp x)^r$, then, by Theorem 2(iv), $\ln u = r \ln(\exp x) = rx$. Therefore, $u = \exp(rx)$.

Now we make an important definition!

$$\text{Let} \quad e = \exp(1).$$

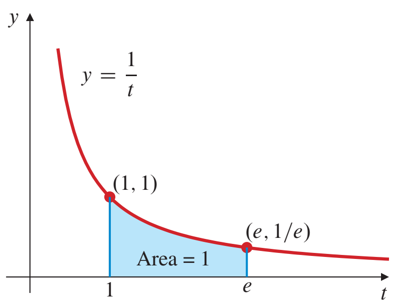


Figure 3.12 The definition of e

The number e satisfies $\ln e = 1$, so the area bounded by the curve $y = 1/t$, the t-axis, and the vertical lines $t = 1$ and $t = e$ must be equal to 1 square unit. See Figure 3.12. The number e is one of the most important numbers in mathematics. Like π, it is irrational and not a zero of any polynomial with rational coefficients. (Such numbers are called **transcendental**.) Its value is between 2 and 3 and begins

$$e = 2.7\,1828\,1828\,45\,90\,45\ldots.$$

Later on we will learn that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots,$$

a formula from which the value of e can be calculated to any desired precision.

Theorem 3(i) shows that $\exp r = \exp(1r) = (\exp 1)^r = e^r$ holds for any rational number r. Now here is a crucial observation. We only know what e^r means if r is a rational number (if $r = m/n$, then $e^r = \sqrt[n]{e^m}$). But $\exp x$ is defined for all *real* x, rational or not. Since $e^r = \exp r$ when r is rational, we can use $\exp x$ as a *definition* of what e^x means for any real number x, and there will be no contradiction if x happens to be rational.

$$e^x = \exp x \qquad \text{for all real } x.$$

Theorem 3 can now be restated in terms of e^x:

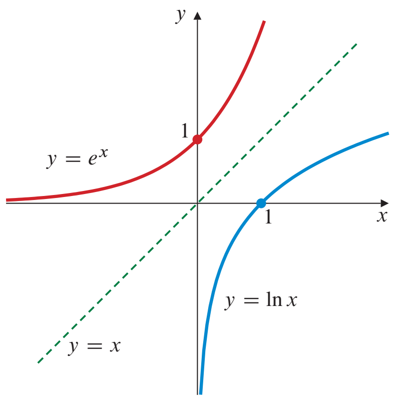


Figure 3.13 The graphs of e^x and $\ln x$

(i) $(e^x)^y = e^{xy}$ (ii) $e^{x+y} = e^x e^y$

(iii) $e^{-x} = \dfrac{1}{e^x}$ (iv) $e^{x-y} = \dfrac{e^x}{e^y}$

The graph of e^x is the reflection of the graph of its inverse, $\ln x$, in the line $y = x$. Both graphs are shown for comparison in Figure 3.13. Observe that the x-axis is a horizontal asymptote of the graph of $y = e^x$ as $x \to -\infty$. We have

$$\lim_{x\to-\infty} e^x = 0, \qquad \lim_{x\to\infty} e^x = \infty.$$

Since $\exp x = e^x$ actually *is* an exponential function, its inverse must actually *be* a logarithm:

$$\ln x = \log_e x.$$

The derivative of $y = e^x$ is calculated by implicit differentiation:

$$\begin{aligned} y = e^x \quad &\Longrightarrow \quad x = \ln y \\ &\Longrightarrow \quad 1 = \frac{1}{y}\frac{dy}{dx} \\ &\Longrightarrow \quad \frac{dy}{dx} = y = e^x. \end{aligned}$$

Thus, the exponential function has the remarkable property that it is its own derivative and, therefore, also its own antiderivative:

$$\frac{d}{dx} e^x = e^x, \qquad \int e^x\,dx = e^x + C.$$

EXAMPLE 3 Find the derivatives of

(a) e^{x^2-3x}, (b) $\sqrt{1+e^{2x}}$, and (c) $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

Solution

(a) $\dfrac{d}{dx} e^{x^2-3x} = e^{x^2-3x}(2x-3) = (2x-3)e^{x^2-3x}$.

(b) $\dfrac{d}{dx}\sqrt{1+e^{2x}} = \dfrac{1}{2\sqrt{1+e^{2x}}}\left(e^{2x}(2)\right) = \dfrac{e^{2x}}{\sqrt{1+e^{2x}}}$.

(c)
$$\begin{aligned} \frac{d}{dx}\frac{e^x - e^{-x}}{e^x + e^{-x}} &= \frac{(e^x + e^{-x})(e^x - (-e^{-x})) - (e^x - e^{-x})(e^x + (-e^{-x}))}{(e^x + e^{-x})^2} \\ &= \frac{(e^x)^2 + 2e^x e^{-x} + (e^{-x})^2 - [(e^x)^2 - 2e^x e^{-x} + (e^{-x})^2]}{(e^x + e^{-x})^2} \\ &= \frac{4e^{x-x}}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}. \end{aligned}$$

EXAMPLE 4 Let $f(t) = e^{at}$. Find (a) $f^{(n)}(t)$ and (b) $\int f(t)\,dt$.

Solution (a) We have

$$\begin{aligned} f'(t) &= a\,e^{at} \\ f''(t) &= a^2\,e^{at} \\ f'''(t) &= a^3\,e^{at} \\ &\ \vdots \\ f^{(n)}(t) &= a^n\,e^{at}. \end{aligned}$$

(b) Also, $\displaystyle\int f(t)\,dt = \int e^{at}\,dt = \frac{1}{a}e^{at} + C$, since $\displaystyle\frac{d}{dt}\frac{1}{a}e^{at} = e^{at}$.

General Exponentials and Logarithms

We can use the fact that e^x is now defined for *all* real x to define the arbitrary exponential a^x (where $a > 0$) for all real x. If r is rational, then $\ln(a^r) = r\ln a$; therefore, $a^r = e^{r\ln a}$. However, $e^{x\ln a}$ is defined for all real x, so we can use it as a definition of a^x with no possibility of contradiction arising if x is rational.

DEFINITION 7

The general exponential a^x

$$a^x = e^{x\ln a}, \qquad (a > 0, \quad x \text{ real}).$$

EXAMPLE 5 Evaluate 2^π, using the natural logarithm (ln) and exponential (exp or e^x) keys on a scientific calculator, but not using the y^x or ^ keys.

Solution $2^\pi = e^{\pi\ln 2} = 8.824\,977\,8\cdots$. If your calculator has a ^ key, or an x^y or y^x key, chances are that it is implemented in terms of the exp and ln functions.

The laws of exponents for a^x as presented in Section 3.2 can now be obtained from those for e^x, as can the derivative:

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\ln a = a^x\ln a.$$

We can also verify the General Power Rule for x^a, where a is any real number, provided $x > 0$:

$$\frac{d}{dx}x^a = \frac{d}{dx}e^{a\ln x} = e^{a\ln x}\frac{a}{x} = \frac{a\,x^a}{x} = a\,x^{a-1}.$$

EXAMPLE 6 Show that the graph of $f(x) = x^\pi - \pi^x$ has a negative slope at $x = \pi$.

Do not confuse x^π, which is a power function of x, and π^x, which is an exponential function of x.

Solution

$$\begin{aligned} f'(x) &= \pi\,x^{\pi-1} - \pi^x\ \ln\pi \\ f'(\pi) &= \pi\,\pi^{\pi-1} - \pi^\pi\ \ln\pi = \pi^\pi(1 - \ln\pi). \end{aligned}$$

Since $\pi > 3 > e$, we have $\ln\pi > \ln e = 1$, so $1 - \ln\pi < 0$. Since $\pi^\pi = e^{\pi\ln\pi} > 0$, we have $f'(\pi) < 0$. Thus, the graph $y = f(x)$ has negative slope at $x = \pi$.

EXAMPLE 7 Find the critical point of $y = x^x$.

Solution We can't differentiate x^x by treating it as a power (like x^a) because the exponent varies. We can't treat it as an exponential (like a^x) because the base varies. We can differentiate it if we first write it in terms of the exponential function,

$x^x = e^{x \ln x}$, and then use the Chain Rule and the Product Rule:

$$\frac{dy}{dx} = \frac{d}{dx} e^{x \ln x} = e^{x \ln x} \left(\ln x + x \left(\frac{1}{x} \right) \right) = x^x (1 + \ln x).$$

Now x^x is defined only for $x > 0$ and is itself never 0. (Why?) Therefore, the critical point occurs where $1 + \ln x = 0$; that is, $\ln x = -1$, or $x = 1/e$.

Finally, observe that $(d/dx)a^x = a^x \ln a$ is negative for all x if $0 < a < 1$ and is positive for all x if $a > 1$. Thus, a^x is one-to-one and has an inverse function, $\log_a x$, provided $a > 0$ and $a \neq 1$. Its properties follow in the same way as in Section 3.2. If $y = \log_a x$, then $x = a^y$ and, differentiating implicitly with respect to x, we get

$$1 = a^y \ln a \frac{dy}{dx} = x \ln a \frac{dy}{dx}.$$

Thus, the derivative of $\log_a x$ is given by

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}.$$

Since $\log_a x$ can be expressed in terms of logarithms to any other base, say e,

$$\log_a x = \frac{\ln x}{\ln a},$$

we normally use only natural logarithms. Exceptions are found in chemistry, acoustics, and other sciences where "logarithmic scales" are used to measure quantities for which a one-unit increase in the measure corresponds to a tenfold increase in the quantity. Logarithms to base 10 are used in defining such scales. In computer science, where powers of 2 play a central role, logarithms to base 2 are often encountered.

Logarithmic Differentiation

Suppose we want to differentiate a function of the form

$$y = (f(x))^{g(x)} \qquad (\text{for } f(x) > 0).$$

Since the variable appears in both the base and the exponent, neither the general power rule, $(d/dx)x^a = ax^{a-1}$, nor the exponential rule, $(d/dx)a^x = a^x \ln a$, can be directly applied. One method for finding the derivative of such a function is to express it in the form

$$y = e^{g(x) \ln f(x)}$$

and then differentiate, using the Product Rule to handle the exponent. This is the method used in Example 7.

The derivative in Example 7 can also be obtained by taking natural logarithms of both sides of the equation $y = x^x$ and differentiating implicitly:

$$\begin{aligned} \ln y &= x \ln x \\ \frac{1}{y} \frac{dy}{dx} &= \ln x + \frac{x}{x} = 1 + \ln x \\ \frac{dy}{dx} &= y(1 + \ln x) = x^x (1 + \ln x). \end{aligned}$$

This latter technique is called **logarithmic differentiation**.

EXAMPLE 8 Find dy/dt if $y = (\sin t)^{\ln t}$, where $0 < t < \pi$.

Solution We have $\ln y = \ln t \ \ln \sin t$. Thus,

$$\frac{1}{y}\frac{dy}{dt} = \frac{1}{t}\ln \sin t + \ln t\,\frac{\cos t}{\sin t}$$
$$\frac{dy}{dt} = y\left(\frac{\ln \sin t}{t} + \ln t\ \cot t\right) = (\sin t)^{\ln t}\left(\frac{\ln \sin t}{t} + \ln t\ \cot t\right).$$

Logarithmic differentiation is also useful for finding the derivatives of functions expressed as products and quotients of many factors. Taking logarithms reduces these products and quotients to sums and differences. This usually makes the calculation easier than it would be using the Product and Quotient Rules, especially if the derivative is to be evaluated at a specific point.

EXAMPLE 9 Differentiate $y = [(x+1)(x+2)(x+3)]/(x+4)$.

Solution $\ln|y| = \ln|x+1| + \ln|x+2| + \ln|x+3| - \ln|x+4|$. Thus,

$$\frac{1}{y}y' = \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}$$
$$y' = \frac{(x+1)(x+2)(x+3)}{x+4}\left(\frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4}\right)$$
$$= \frac{(x+2)(x+3)}{x+4} + \frac{(x+1)(x+3)}{x+4} + \frac{(x+1)(x+2)}{x+4} - \frac{(x+1)(x+2)(x+3)}{(x+4)^2}.$$

EXAMPLE 10 Find $\left.\dfrac{du}{dx}\right|_{x=1}$ if $u = \sqrt{(x+1)(x^2+1)(x^3+1)}$.

Solution

$$\ln u = \frac{1}{2}\Big(\ln(x+1) + \ln(x^2+1) + \ln(x^3+1)\Big)$$
$$\frac{1}{u}\frac{du}{dx} = \frac{1}{2}\left(\frac{1}{x+1} + \frac{2x}{x^2+1} + \frac{3x^2}{x^3+1}\right).$$

At $x = 1$ we have $u = \sqrt{8} = 2\sqrt{2}$. Hence,

$$\left.\frac{du}{dx}\right|_{x=1} = \sqrt{2}\left(\frac{1}{2} + 1 + \frac{3}{2}\right) = 3\sqrt{2}.$$

EXERCISES 3.3

Simplify the expressions given in Exercises 1–10.

1. $e^3/\sqrt{e^5}$
2. $\ln\left(e^{1/2}e^{2/3}\right)$
3. $e^{5\ln x}$
4. $e^{(3\ln 9)/2}$
5. $\ln\dfrac{1}{e^{3x}}$
6. $e^{2\ln\cos x} + \left(\ln e^{\sin x}\right)^2$
7. $3\ln 4 - 4\ln 3$
8. $4\ln\sqrt{x} + 6\ln(x^{1/3})$
9. $2\ln x + 5\ln(x-2)$
10. $\ln(x^2 + 6x + 9)$

Solve the equations in Exercises 11–14 for x.

11. $2^{x+1} = 3^x$

12. $3^x = 9^{1-x}$

13. $\dfrac{1}{2^x} = \dfrac{5}{8^{x+3}}$

14. $2^{x^2-3} = 4^x$

Find the domains of the functions in Exercises 15–16.

15. $\ln \dfrac{x}{2-x}$

16. $\ln(x^2 - x - 2)$

Solve the inequalities in Exercises 17–18.

17. $\ln(2x-5) > \ln(7-2x)$

18. $\ln(x^2-2) \le \ln x$

In Exercises 19–48, differentiate the given functions. If possible, simplify your answers.

19. $y = e^{5x}$

20. $y = xe^x - x$

21. $y = \dfrac{x}{e^{2x}}$

22. $y = x^2 e^{x/2}$

23. $y = \ln(3x-2)$

24. $y = \ln|3x-2|$

25. $y = \ln(1+e^x)$

26. $f(x) = e^{(x^2)}$

27. $y = \dfrac{e^x + e^{-x}}{2}$

28. $x = e^{3t} \ln t$

29. $y = e^{(e^x)}$

30. $y = \dfrac{e^x}{1+e^x}$

31. $y = e^x \sin x$

32. $y = e^{-x} \cos x$

33. $y = \ln \ln x$

34. $y = x \ln x - x$

35. $y = x^2 \ln x - \dfrac{x^2}{2}$

36. $y = \ln|\sin x|$

37. $y = 5^{2x+1}$

38. $y = 2^{(x^2-3x+8)}$

39. $g(x) = t^x x^t$

40. $h(t) = t^x - x^t$

41. $f(s) = \log_a(bs + c)$

42. $g(x) = \log_x(2x+3)$

43. $y = x^{\sqrt{x}}$

44. $y = (1/x)^{\ln x}$

45. $y = \ln|\sec x + \tan x|$

46. $y = \ln|x + \sqrt{x^2 - a^2}|$

47. $y = \ln\left(\sqrt{x^2+a^2} - x\right)$

48. $y = (\cos x)^x - x^{\cos x}$

49. Find the nth derivative of $f(x) = xe^{ax}$.

50. Show that the nth derivative of $(ax^2 + bx + c)e^x$ is a function of the same form but with different constants.

51. Find the first four derivatives of e^{x^2}.

52. Find the nth derivative of $\ln(2x+1)$.

53. Differentiate (a) $f(x) = (x^x)^x$ and (b) $g(x) = x^{(x^x)}$. Which function grows more rapidly as x grows large?

54. Solve the equation $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = a$, where $a > 0$. The exponent tower goes on forever.

Use logarithmic differentiation to find the required derivatives in Exercises 55–57.

55. $f(x) = (x-1)(x-2)(x-3)(x-4)$. Find $f'(x)$.

56. $F(x) = \dfrac{\sqrt{1+x}(1-x)^{1/3}}{(1+5x)^{4/5}}$. Find $F'(0)$.

57. $f(x) = \dfrac{(x^2-1)(x^2-2)(x^2-3)}{(x^2+1)(x^2+2)(x^2+3)}$. Find $f'(2)$. Also find $f'(1)$.

58. At what points does the graph $y = x^2 e^{-x^2}$ have a horizontal tangent line?

59. Let $f(x) = xe^{-x}$. Determine where f is increasing and where it is decreasing. Sketch the graph of f.

60. Find the equation of a straight line of slope 4 that is tangent to the graph of $y = \ln x$.

61. Find an equation of the straight line tangent to the curve $y = e^x$ and passing through the origin.

62. Find an equation of the straight line tangent to the curve $y = \ln x$ and passing through the origin.

63. Find an equation of the straight line that is tangent to $y = 2^x$ and that passes through the point $(1, 0)$.

64. For what values of $a > 0$ does the curve $y = a^x$ intersect the straight line $y = x$?

65. Find the slope of the curve $e^{xy} \ln \dfrac{x}{y} = x + \dfrac{1}{y}$ at $(e, 1/e)$.

66. Find an equation of the straight line tangent to the curve $xe^y + y - 2x = \ln 2$ at the point $(1, \ln 2)$.

67. Find the derivative of $f(x) = Ax \cos \ln x + Bx \sin \ln x$. Use the result to help you find the indefinite integrals

$$\int \cos \ln x \, dx \text{ and } \int \sin \ln x \, dx.$$

68. Let $F_{A,B}(x) = Ae^x \cos x + Be^x \sin x$. Show that $(d/dx)F_{A,B}(x) = F_{A+B,B-A}(x)$.

69. Using the results of Exercise 68, find (a) $(d^2/dx^2)F_{A,B}(x)$ and (b) $(d^3/dx^3)e^x \cos x$.

70. Find $\dfrac{d}{dx}(Ae^{ax} \cos bx + Be^{ax} \sin bx)$ and use the answer to help you evaluate

(a) $\displaystyle\int e^{ax} \cos bx \, dx$ and (b) $\displaystyle\int e^{ax} \sin bx \, dx$.

71. Prove identity (ii) of Theorem 2 by examining the derivative of the left side minus the right side, as was done in the proof of identity (i).

72. Deduce identity (iii) of Theorem 2 from identities (i) and (ii).

73. Prove identity (iv) of Theorem 2 for rational exponents r by the same method used for Exercise 71.

74. Let $x > 0$, and let $F(x)$ be the area bounded by the curve $y = t^2$, the t-axis, and the vertical lines $t = 0$ and $t = x$. Using the method of the proof of Theorem 1, show that $F'(x) = x^2$. Hence, find an explicit formula for $F(x)$. What is the area of the region bounded by $y = t^2$, $y = 0$, $t = 0$, and $t = 2$?

75. Carry out the following steps to show that $2 < e < 3$. Let $f(t) = 1/t$ for $t > 0$.

(a) Show that the area under $y = f(t)$, above $y = 0$, and between $t = 1$ and $t = 2$ is less than 1 square unit. Deduce that $e > 2$.

(b) Show that all tangent lines to the graph of f lie below the graph. *Hint:* $f''(t) = 2/t^3 > 0$.

(c) Find the lines T_2 and T_3 that are tangent to $y = f(t)$ at $t = 2$ and $t = 3$, respectively.

(d) Find the area A_2 under T_2, above $y = 0$, and between $t = 1$ and $t = 2$. Also find the area A_3 under T_3, above $y = 0$, and between $t = 2$ and $t = 3$.

(e) Show that $A_2 + A_3 > 1$ square unit. Deduce that $e < 3$.

3.4 Growth and Decay

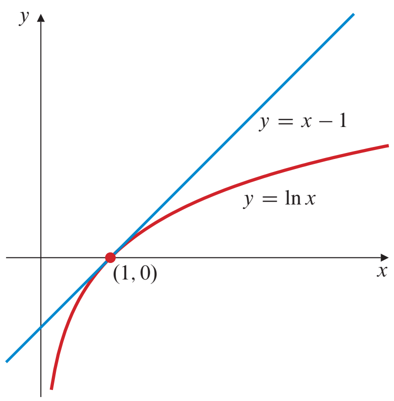


Figure 3.14 $\ln x \le x - 1$ for $x > 0$

In this section we will study the use of exponential functions to model the growth rates of quantities whose rate of growth is directly related to their size. The growth of such quantities is typically governed by differential equations whose solutions involve exponential functions. Before delving into this topic, we prepare the way by examining the growth behaviour of exponential and logarithmic functions.

The Growth of Exponentials and Logarithms

In Section 3.3 we showed that both e^x and $\ln x$ grow large (approach infinity) as x grows large. However, e^x increases very rapidly as x increases, and $\ln x$ increases very slowly. In fact, e^x increases faster than any positive power of x (no matter how large the power), while $\ln x$ increases more slowly than any positive power of x (no matter how small the power). To verify this behaviour we start with an inequality satisfied by $\ln x$. The straight line $y = x - 1$ is tangent to the curve $y = \ln x$ at the point $(1, 0)$. The following theorem asserts that the curve lies below that line. (See Figure 3.14.)

THEOREM 4

If $x > 0$, then $\ln x \le x - 1$.

PROOF Let $g(x) = \ln x - (x - 1)$ for $x > 0$. Then $g(1) = 0$ and

$$g'(x) = \frac{1}{x} - 1 \quad \begin{cases} > 0 & \text{if } 0 < x < 1 \\ < 0 & \text{if } x > 1. \end{cases}$$

As observed in Section 2.8, these inequalities imply that g is increasing on $(0, 1)$ and decreasing on $(1, \infty)$. Thus, $g(x) \le g(1) = 0$ for all $x > 0$ and $\ln x \le x - 1$ for all such x.

THEOREM 5

The growth properties of exp and ln

If $a > 0$, then

(a) $\displaystyle\lim_{x\to\infty} \frac{x^a}{e^x} = 0,$ (b) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x^a} = 0,$

(c) $\displaystyle\lim_{x\to-\infty} |x|^a e^x = 0,$ (d) $\displaystyle\lim_{x\to 0+} x^a \ln x = 0.$

Each of these limits makes a statement about who "wins" in a contest between an exponential or logarithm and a power. For example, in part (a), the denominator e^x grows large as $x \to \infty$, so it tries to make the fraction x^a/e^x approach 0. On the other hand, if a is a large positive number, the numerator x^a also grows large and tries to make the fraction approach infinity. The assertion of (a) is that in this contest between the exponential and the power, the exponential is stronger and wins; the fraction approaches 0. The content of Theorem 5 can be paraphrased as follows:

In a struggle between a power and an exponential, the exponential wins.
In a struggle between a power and a logarithm, the power wins.

PROOF First, we prove part (b). Let $x > 1$, $a > 0$, and let $s = a/2$. Since $\ln(x^s) = s \ln x$, we have, using Theorem 4,

$$0 < s \ln x = \ln(x^s) \le x^s - 1 < x^s.$$

Thus, $0 < \ln x < \frac{1}{s} x^s$ and, dividing by $x^a = x^{2s}$,

$$0 < \frac{\ln x}{x^a} < \frac{1}{s} \frac{x^s}{x^{2s}} = \frac{1}{s\,x^s}.$$

Now $1/(s\,x^s) \to 0$ as $x \to \infty$ (since $s > 0$); therefore, by the Squeeze Theorem,

$$\lim_{x\to\infty} \frac{\ln x}{x^a} = 0.$$

Next, we deduce part (d) from part (b) by substituting $x = 1/t$. As $x \to 0+$, we have $t \to \infty$, so

$$\lim_{x\to 0+} x^a \ln x = \lim_{t\to\infty} \frac{\ln(1/t)}{t^a} = \lim_{t\to\infty} \frac{-\ln t}{t^a} = -0 = 0.$$

Now we deduce (a) from (b). If $x = \ln t$, then $t \to \infty$ as $x \to \infty$, so

$$\lim_{x\to\infty} \frac{x^a}{e^x} = \lim_{t\to\infty} \frac{(\ln t)^a}{t} = \lim_{t\to\infty} \left(\frac{\ln t}{t^{1/a}}\right)^a = 0^a = 0.$$

Finally, (c) follows from (a) via the substitution $x = -t$:

$$\lim_{x\to-\infty} |x|^a\, e^x = \lim_{t\to\infty} |-t|^a\, e^{-t} = \lim_{t\to\infty} \frac{t^a}{e^t} = 0.$$

Exponential Growth and Decay Models

Many natural processes involve quantities that increase or decrease at a rate proportional to their size. For example, the mass of a culture of bacteria growing in a medium supplying adequate nourishment will increase at a rate proportional to that mass. The value of an investment bearing interest that is continuously compounding increases at a rate proportional to that value. The mass of undecayed radioactive material in a sample decreases at a rate proportional to that mass.

All of these phenomena, and others exhibiting similar behaviour, can be modelled mathematically in the same way. If $y = y(t)$ denotes the value of a quantity y at time t, and if y changes at a rate proportional to its size, then

$$\frac{dy}{dt} = ky,$$

where k is the constant of proportionality. The above equation is called the **differential equation of exponential growth or decay** because, for any value of the constant C, the function $y = Ce^{kt}$ satisfies the equation. In fact, if $y(t)$ is any solution of the differential equation $y' = ky$, then

$$\frac{d}{dt}\left(\frac{y(t)}{e^{kt}}\right) = \frac{e^{kt}y'(t) - ke^{kt}y(t)}{e^{2kt}} = \frac{y'(t) - ky(t)}{e^{kt}} = 0 \quad \text{for all } t.$$

Thus, $y(t)/e^{kt} = C$, a constant, and $y(t) = Ce^{kt}$. Since $y(0) = Ce^0 = C$,

The initial-value problem $\begin{cases} \dfrac{dy}{dt} = ky \\ y(0) = y_0 \end{cases}$ has unique solution $y = y_0 e^{kt}$.

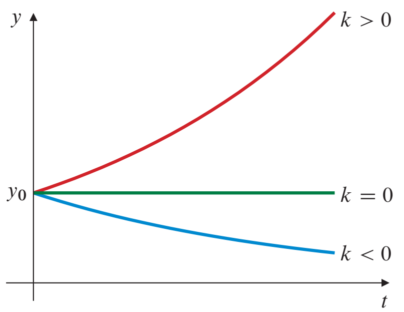


Figure 3.15 Solutions of the initial-value problem $dy/dt = ky$, $y(0) = y_0$, for $k > 0$, $k = 0$, and $k < 0$

If $y_0 > 0$, then $y(t)$ is an increasing function of t if $k > 0$ and a decreasing function of t if $k < 0$. We say that the quantity y exhibits **exponential growth** if $k > 0$ and **exponential decay** if $k < 0$. (See Figure 3.15.)

EXAMPLE 1 **(Growth of a cell culture)** A certain cell culture grows at a rate proportional to the number of cells present. If the culture contains 500 cells initially and 800 after 24 h, how many cells will there be after a further 12 h?

Solution Let $y(t)$ be the number of cells present t hours after there were 500 cells. Thus, $y(0) = 500$ and $y(24) = 800$. Because $dy/dt = ky$, we have

$$y(t) = y(0)e^{kt} = 500e^{kt}.$$

Therefore, $800 = y(24) = 500e^{24k}$, so $24k = \ln \frac{800}{500} = \ln(1.6)$. It follows that $k = (1/24)\ln(1.6)$ and

$$y(t) = 500e^{(t/24)\ln(1.6)} = 500(1.6)^{t/24}.$$

We want to know y when $t = 36$: $y(36) = 500e^{(36/24)\ln(1.6)} = 500(1.6)^{3/2} \approx 1012$. The cell count grew to about 1,012 in the 12 h after it was 800.

Exponential growth is characterized by a **fixed doubling time**. If T is the time at which y has doubled from its size at $t = 0$, then $2y(0) = y(T) = y(0)e^{kT}$. Therefore, $e^{kT} = 2$. Since $y(t) = y(0)e^{kt}$, we have

$$y(t + T) = y(0)e^{k(t+T)} = e^{kT}y(0)e^{kt} = 2y(t);$$

that is, T units of time are required for y to double from any value. Similarly, exponential decay involves a fixed halving time (usually called the **half-life**). If $y(T) = \frac{1}{2}y(0)$, then $e^{kT} = \frac{1}{2}$ and

$$y(t + T) = y(0)e^{k(t+T)} = \frac{1}{2}y(t).$$

EXAMPLE 2 **(Radioactive decay)** A radioactive material has a half-life of 1,200 years. What percentage of the original radioactivity of a sample is left after 10 years? How many years are required to reduce the radioactivity by 10%?

Solution Let $p(t)$ be the percentage of the original radioactivity left after t years. Thus $p(0) = 100$ and $p(1{,}200) = 50$. Since the radioactivity decreases at a rate proportional to itself, $dp/dt = kp$ and

$$p(t) = 100e^{kt}.$$

Now $50 = p(1{,}200) = 100e^{1{,}200k}$, so

$$k = \frac{1}{1{,}200}\ln\frac{50}{100} = -\frac{\ln 2}{1{,}200}.$$

The percentage left after 10 years is

$$p(10) = 100e^{10k} = 100e^{-10(\ln 2)/1{,}200} \approx 99.424.$$

If after t years 90% of the radioactivity is left, then

$$\begin{aligned} 90 &= 100e^{kt}, \\ kt &= \ln\frac{90}{100}, \\ t &= \frac{1}{k}\ln(0.9) = -\frac{1{,}200}{\ln 2}\ln(0.9) \approx 182.4, \end{aligned}$$

so it will take a little over 182 years to reduce the radioactivity by 10%.

Sometimes an exponential growth or decay problem will involve a quantity that changes at a rate proportional to the difference between itself and a fixed value:

$$\frac{dy}{dt} = k(y - a).$$

In this case, the change of dependent variable $u(t) = y(t) - a$ should be used to convert the differential equation to the standard form. Observe that $u(t)$ changes at the same rate as $y(t)$ (i.e., $du/dt = dy/dt$), so it satisfies

$$\frac{du}{dt} = ku.$$

EXAMPLE 3 **(Newton's law of cooling)** A hot object introduced into a cooler environment will cool at a rate proportional to the excess of its temperature above that of its environment. If a cup of coffee sitting in a room maintained at a temperature of 20 °C cools from 80 °C to 50 °C in 5 minutes, how much longer will it take to cool to 40 °C?

Solution Let $y(t)$ be the temperature of the coffee t min after it was 80 °C. Thus, $y(0) = 80$ and $y(5) = 50$. Newton's law says that $dy/dt = k(y - 20)$ in this case, so let $u(t) = y(t) - 20$. Thus, $u(0) = 60$ and $u(5) = 30$. We have

$$\frac{du}{dt} = \frac{dy}{dt} = k(y - 20) = ku.$$

Thus,

$$\begin{aligned} u(t) &= 60e^{kt}, \\ 30 = u(5) &= 60e^{5k}, \\ 5k &= \ln \tfrac{1}{2} = -\ln 2. \end{aligned}$$

We want to know t such that $y(t) = 40$, that is, $u(t) = 20$:

$$\begin{aligned} 20 = u(t) &= 60e^{-(t/5)\ln 2} \\ -\frac{t}{5}\ln 2 &= \ln\frac{20}{60} = -\ln 3, \\ t &= 5\frac{\ln 3}{\ln 2} \approx 7.92. \end{aligned}$$

The coffee will take about $7.92 - 5 = 2.92$ min to cool from 50 °C to 40 °C.

Interest on Investments

Suppose that \$10,000 is invested at an annual rate of interest of 8%. Thus, the value of the investment at the end of one year will be \$10,000(1.08) = \$10,800. If this amount remains invested for a second year at the same rate, it will grow to $\$10{,}000(1.08)^2 = \$11{,}664$; in general, n years after the original investment was made, it will be worth $\$10{,}000(1.08)^n$.

Now suppose that the 8% rate is *compounded semiannually* so that the interest is actually paid at a rate of 4% per 6-month period. After one year (two interest periods) the \$10,000 will grow to $\$10{,}000(1.04)^2 = \$10{,}816$. This is \$16 more than was obtained when the 8% was compounded only once per year. The extra \$16 is the interest paid in the second 6-month period on the \$400 interest earned in the first 6-month period. Continuing in this way, if the 8% interest is compounded *monthly* (12 periods per year and $\frac{8}{12}$% paid per period) or *daily* (365 periods per year and $\frac{8}{365}$% paid per period), then the original \$10,000 would grow in one year to $\$10{,}000\left(1 + \frac{8}{1{,}200}\right)^{12} = \$10{,}830$ or $\$10{,}000\left(1 + \frac{8}{36{,}500}\right)^{365} = \$10{,}832.78$, respectively.

For any given *nominal* interest rate, the investment grows more if the compounding period is shorter. In general, an original investment of $\$A$ invested at $r\%$ per annum compounded n times per year grows in one year to

$$\$A\left(1+\frac{r}{100n}\right)^n.$$

It is natural to ask how well we can do with our investment if we let the number of periods in a year approach infinity, that is, we compound the interest *continuously*. The answer is that in 1 year the $\$A$ will grow to

$$\$A\lim_{n\to\infty}\left(1+\frac{r}{100n}\right)^n = \$Ae^{r/100}.$$

For example, at 8% per annum compounded continuously, our \$10,000 will grow in one year to $\$10{,}000e^{0.08} \approx \$10,832.87$. (Note that this is just a few cents more than we get by compounding daily.) To justify this result we need the following theorem.

THEOREM 6

For every real number x,

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

PROOF If $x = 0$, there is nothing to prove; both sides of the identity are 1. If $x \neq 0$, let $h = x/n$. As n tends to infinity, h approaches 0. Thus,

$$\begin{aligned}
\lim_{n\to\infty}\ln\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty} n\ln\left(1+\frac{x}{n}\right)\\
&= \lim_{n\to\infty} x\,\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{x}{n}}\\
&= x\lim_{h\to 0}\frac{\ln(1+h)}{h} && \text{(where } h = x/n\text{)}\\
&= x\lim_{h\to 0}\frac{\ln(1+h)-\ln 1}{h} && \text{(since } \ln 1 = 0\text{)}\\
&= x\left(\frac{d}{dt}\ln t\right)\Bigg|_{t=1} && \text{(by the definition of derivative)}\\
&= x\,\frac{1}{t}\Bigg|_{t=1} = x.
\end{aligned}$$

Since ln is differentiable, it is continuous. Hence, by Theorem 7 of Section 1.4,

$$\ln\left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right) = \lim_{n\to\infty}\ln\left(1+\frac{x}{n}\right)^n = x.$$

Taking exponentials of both sides gives the required formula. ■

In the case $x = 1$, the formula given in Theorem 6 takes the following form:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

We can use this formula to compute approximations to e, as shown in Table 2. In a sense we have cheated in obtaining the numbers in this table; they were produced using the y^x function on a scientific calculator. However, this function is actually computed as $e^{x\ln y}$. In any event, the formula in this table is not a very efficient way to calculate e to any great accuracy. Only 4 decimal places are correct for $n = 100{,}000$. A much better way is to use the series

Table 2.

n	$\left(1+\frac{1}{n}\right)^n$
1	2
10	2.593 74 ···
100	2.704 81 ···
1,000	2.716 92 ···
10,000	2.718 15 ···
100,000	2.718 27 ···

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots,$$

which we will establish in Section 4.10.

A final word about interest rates. Financial institutions sometimes quote *effective* rates of interest rather than *nominal* rates. The effective rate tells you what the actual effect of the interest rate will be after one year. Thus, \$10,000 invested at an effective rate of 8% will grow to \$10,800.00 in one year regardless of the compounding period. A nominal rate of 8% per annum compounded daily is equivalent to an effective rate of about 8.3278%.

Logistic Growth

Few quantities in nature can sustain exponential growth over extended periods of time; the growth is usually limited by external constraints. For example, suppose a small number of rabbits (of both sexes) is introduced to a small island where there were no rabbits previously, and where there are no predators who eat rabbits. By virtue of natural fertility, the number of rabbits might be expected to grow exponentially, but this growth will eventually be limited by the food supply available to the rabbits. Suppose the island can grow enough food to supply a population of L rabbits indefinitely. If there are $y(t)$ rabbits in the population at time t, we would expect $y(t)$ to grow at a rate proportional to $y(t)$ provided $y(t)$ is quite small (much less than L). But as the numbers increase, it will be harder for the rabbits to find enough food, and we would expect the rate of increase to approach 0 as $y(t)$ gets closer and closer to L. One possible model for such behaviour is the differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right),$$

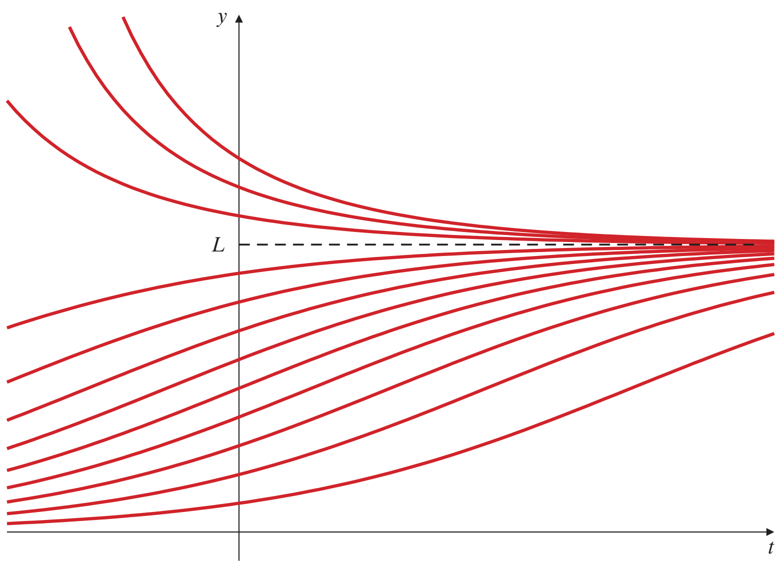


Figure 3.16 Some logistic curves

which is called the **logistic equation** since it models growth that is limited by the *supply* of necessary resources. Observe that $dy/dt > 0$ if $0 < y < L$ and that this rate is small if y is small (there are few rabbits to reproduce) or if y is close to L (there are almost as many rabbits as the available resources can feed). Observe also that $dy/dt < 0$ if $y > L$; there being more animals than the resources can feed, the rabbits die at a greater rate than they are born. Of course, the steady-state populations $y = 0$ and $y = L$ are solutions of the logistic equation; for both of these $dy/dt = 0$. We will examine techniques for solving differential equations like the logistic equation in

Section 7.9. For now, we invite the reader to verify by differentiation that the solution satisfying $y(0) = y_0$ is

$$y = \frac{Ly_0}{y_0 + (L - y_0)e^{-kt}}.$$

Observe that, as expected, if $0 < y_0 < L$, then

$$\lim_{t\to\infty} y(t) = L, \qquad \lim_{t\to-\infty} y(t) = 0.$$

The solution given above also holds for $y_0 > L$. However, the solution does not approach 0 as t approaches $-\infty$ in this case. It has a vertical asymptote at a certain negative value of t. (See Exercise 30 below.) The graphs of solutions of the logistic equation for various positive values of y_0 are given in Figure 3.16.

EXERCISES 3.4

Evaluate the limits in Exercises 1–8.

1. $\lim_{x\to\infty} x^3 e^{-x}$

2. $\lim_{x\to\infty} x^{-3} e^{x}$

3. $\lim_{x\to\infty} \dfrac{2e^x - 3}{e^x + 5}$

4. $\lim_{x\to\infty} \dfrac{x - 2e^{-x}}{x + 3e^{-x}}$

5. $\lim_{x\to 0+} x \ln x$

6. $\lim_{x\to 0+} \dfrac{\ln x}{x}$

7. $\lim_{x\to 0} x\big(\ln|x|\big)^2$

8. $\lim_{x\to\infty} \dfrac{(\ln x)^3}{\sqrt{x}}$

9. **(Bacterial growth)** Bacteria grow in a certain culture at a rate proportional to the amount present. If there are 100 bacteria present initially and the amount doubles in 1 h, how many will there be after a further $1\frac{1}{2}$ h?

10. **(Dissolving sugar)** Sugar dissolves in water at a rate proportional to the amount still undissolved. If there were 50 kg of sugar present initially, and at the end of 5 h only 20 kg are left, how much longer will it take until 90% of the sugar is dissolved?

11. **(Radioactive decay)** A radioactive substance decays at a rate proportional to the amount present. If 30% of such a substance decays in 15 years, what is the half-life of the substance?

12. **(Half-life of radium)** If the half-life of radium is 1,690 years, what percentage of the amount present now will be remaining after (a) 100 years, (b) 1,000 years?

13. Find the half-life of a radioactive substance if after 1 year 99.57% of an initial amount still remains.

14. **(Bacterial growth)** In a certain culture where the rate of growth of bacteria is proportional to the number present, the number triples in 3 days. If at the end of 7 days there are 10 million bacteria present in the culture, how many were present initially?

15. **(Weight of a newborn)** In the first few weeks after birth, babies gain weight at a rate proportional to their weight. A baby weighing 4 kg at birth weighs 4.4 kg after 2 weeks. How much did the baby weigh 5 days after birth?

16. **(Electric current)** When a simple electrical circuit containing inductance and resistance but no capacitance has the electromotive force removed, the rate of decrease of the current is proportional to the current. If the current is $I(t)$ amperes t s after cutoff, and if $I = 40$ when $t = 0$, and $I = 15$ when $t = 0.01$, find a formula for $I(t)$.

17. **(Continuously compounding interest)** How much money needs to be invested today at a nominal rate of 4% compounded continuously, in order that it should grow to $10,000 in 7 years?

18. **(Continuously compounding interest)** Money invested at compound interest (with instantaneous compounding) accumulates at a rate proportional to the amount present. If an initial investment of $1,000 grows to $1,500 in exactly 5 years, find (a) the doubling time for the investment and (b) the effective annual rate of interest being paid.

19. **(Purchasing power)** If the purchasing power of the dollar is decreasing at an effective rate of 9% annually, how long will it take for the purchasing power to be reduced to 25 cents?

! **20.** **(Effective interest rate)** A bank claims to pay interest at an effective rate of 9.5% on an investment account. If the interest is actually being compounded monthly, what is the nominal rate of interest being paid on the account?

! **21.** Suppose that 1,000 rabbits were introduced onto an island where they had no natural predators. During the next five years, the rabbit population grew exponentially. After the first two years the population was 3,500 rabbits. After the first five years a rabbit virus was sprayed on the island, and after that the rabbit population decayed exponentially. Two years after the virus was introduced (so seven years after rabbits were introduced to the island), the rabbit population had dropped to 3,000 rabbits. How many rabbits will there be on the island 10 years after they were introduced?

22. Lab rats are to be used in experiments on an isolated island. Initially R rats are brought to the island and released. Having a plentiful food supply and no natural predators on the island, the rat population grows exponentially and doubles in three months. At the end of the fifth month, and at the end of every five months thereafter, 1,000 of the rats are captured and killed. What is the minimum value of R that ensures that the scientists will never run out of rats?

Differential equations of the form $y' = a + by$

23. Suppose that $f(x)$ satisfies the differential equation

$$f'(x) = a + bf(x),$$

where a and b are constants.

(a) Solve the differential equation by substituting $u(x) = a + bf(x)$ and solving the simpler differential equation that results for $u(x)$.

(b) Solve the initial-value problem:

$$\begin{cases} \dfrac{dy}{dx} = a + by \\ y(0) = y_0 \end{cases}$$

24. (Drug concentrations in the blood) A drug is introduced into the bloodstream intravenously at a constant rate and breaks down and is eliminated from the body at a rate proportional to its concentration in the blood. The concentration $x(t)$ of the drug in the blood satisfies the differential equation

$$\frac{dx}{dt} = a - bx,$$

where a and b are positive constants.

(a) What is the limiting concentration $\lim_{t\to\infty} x(t)$ of the drug in the blood?

(b) Find the concentration of the drug in the blood at time t, given that the concentration was zero at $t = 0$.

(c) How long after $t = 0$ will it take for the concentration to rise to half its limiting value?

25. (Cooling) Use Newton's law of cooling to determine the reading on a thermometer five minutes after it is taken from an oven at 72 °C to the outdoors where the temperature is 20 °C, if the reading dropped to 48 °C after one minute.

26. (Cooling) An object is placed in a freezer maintained at a temperature of −5 °C. If the object cools from 45 °C to 20 °C in 40 minutes, how many more minutes will it take to cool to 0 °C?

27. (Warming) If an object in a room warms up from 5 °C to 10 °C in 4 minutes, and if the room is being maintained at 20 °C, how much longer will the object take to warm up to 15 °C? Assume the object warms at a rate proportional to the difference between its temperature and room temperature.

The logistic equation

28. Suppose the quantity $y(t)$ exhibits logistic growth. If the values of $y(t)$ at times $t = 0$, $t = 1$, and $t = 2$ are y_0, y_1, and y_2, respectively, find an equation satisfied by the limiting value L of $y(t)$, and solve it for L. If $y_0 = 3$, $y_1 = 5$, and $y_2 = 6$, find L.

29. Show that a solution $y(t)$ of the logistic equation having $0 < y(0) < L$ is increasing most rapidly when its value is $L/2$. (*Hint:* You do not need to use the formula for the solution to see this.)

30. If $y_0 > L$, find the interval on which the given solution of the logistic equation is valid. What happens to the solution as t approaches the left endpoint of this interval?

31. If $y_0 < 0$, find the interval on which the given solution of the logistic equation is valid. What happens to the solution as t approaches the right endpoint of this interval?

32. (Modelling an epidemic) The number y of persons infected by a highly contagious virus is modelled by a logistic curve

$$y = \frac{L}{1 + Me^{-kt}},$$

where t is measured in months from the time the outbreak was discovered. At that time there were 200 infected persons, and the number grew to 1,000 after 1 month. Eventually, the number levelled out at 10,000. Find the values of the parameters L, M, and k of the model.

33. Continuing Exercise 32, how many people were infected 3 months after the outbreak was discovered, and how fast was the number growing at that time?

3.5 The Inverse Trigonometric Functions

The six trigonometric functions are periodic and, hence, not one-to-one. However, as we did with the function x^2 in Section 3.1, we can restrict their domains in such a way that the restricted functions are one-to-one and invertible.

The Inverse Sine (or Arcsine) Function

Let us define a function $\operatorname{Sin} x$ (note the capital letter) to be $\sin x$, restricted so that its domain is the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$:

DEFINITION 8

The restricted sine function Sin *x*

$$\operatorname{Sin} x = \sin x \qquad \text{if } -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

Since its derivative $\cos x$ is positive on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the function $\operatorname{Sin} x$ is increasing on its domain, so it is a one-to-one function. It has domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and range $[-1, 1]$. (See Figure 3.17.)

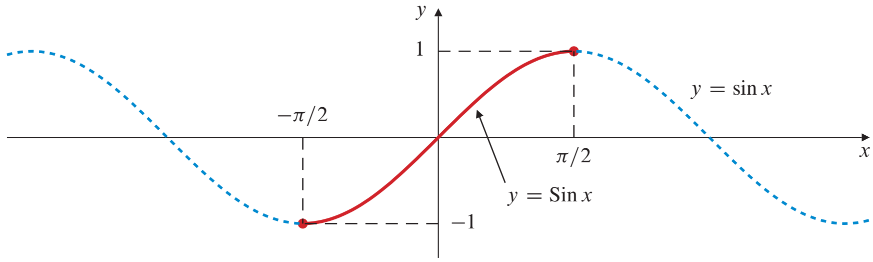


Figure 3.17 The graph of Sin x forms part of the graph of sin x

Being one-to-one, Sin has an inverse function which is denoted $\sin^{-1}$ (or, in some books and computer programs, by arcsin, Arcsin, or asin) and which is called the **inverse sine** or **arcsine** function.

DEFINITION 9

The inverse sine function $\sin^{-1} x$ or Arcsin x

$$y = \sin^{-1} x \iff x = \operatorname{Sin} y$$
$$\iff x = \sin y \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

The graph of $\sin^{-1}$ is shown in Figure 3.18; it is the reflection of the graph of Sin in the line $y = x$. The domain of $\sin^{-1}$ is $[-1, 1]$ (the range of Sin), and the range of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (the domain of Sin). The **cancellation identities** for Sin and $\sin^{-1}$ are

$$\sin^{-1}(\operatorname{Sin} x) = \arcsin(\operatorname{Sin} x) = x \qquad \text{for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$
$$\operatorname{Sin}(\sin^{-1} x) = \operatorname{Sin}(\arcsin x) = x \qquad \text{for } -1 \le x \le 1$$

Since the intervals where they apply are specified, Sin can be replaced by sin in both identities above.

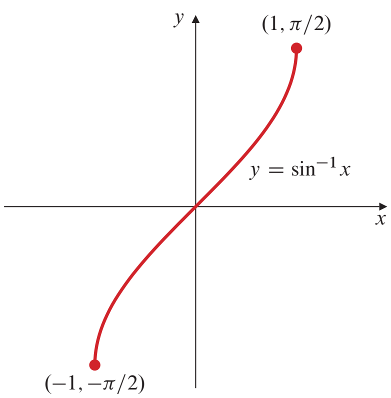


Figure 3.18 The arcsine function

Remark As for the general inverse function f^{-1}, be aware that $\sin^{-1} x$ does *not* represent the *reciprocal* $1/\sin x$. (We already have a perfectly good name for the reciprocal of $\sin x$; we call it $\csc x$.) We should think of $\sin^{-1} x$ as "the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x."

EXAMPLE 1

(a) $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ (because $\sin\frac{\pi}{6} = \frac{1}{2}$ and $-\frac{\pi}{2} < \frac{\pi}{6} < \frac{\pi}{2}$).

(b) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$ (because $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and $-\frac{\pi}{2} < -\frac{\pi}{4} < \frac{\pi}{2}$).

(c) $\sin^{-1}(-1) = -\frac{\pi}{2}$ (because $\sin\left(-\frac{\pi}{2}\right) = -1$).

(d) $\sin^{-1} 2$ is not defined. (2 is not in the range of sine.)

EXAMPLE 2 Find (a) $\sin\left(\sin^{-1} 0.7\right)$, (b) $\sin^{-1}(\sin 0.3)$, (c) $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)$, and (d) $\cos\left(\sin^{-1} 0.6\right)$.

Solution

(a) $\sin\left(\sin^{-1} 0.7\right) = 0.7$ (cancellation identity).

(b) $\sin^{-1}(\sin 0.3) = 0.3$ (cancellation identity).

(c) The number $\frac{4\pi}{5}$ does not lie in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so we can't apply the cancellation identity directly. However, $\sin\frac{4\pi}{5} = \sin\left(\pi - \frac{\pi}{5}\right) = \sin\frac{\pi}{5}$ by the supplementary angle identity. Therefore, $\sin^{-1}\left(\sin\frac{4\pi}{5}\right) = \sin^{-1}\left(\sin\frac{\pi}{5}\right) = \frac{\pi}{5}$ (by cancellation).

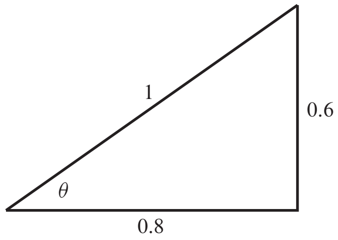


Figure 3.19

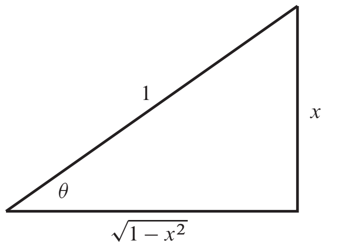


Figure 3.20

(d) Let $\theta = \sin^{-1} 0.6$, as shown in the right triangle in Figure 3.19, which has hypotenuse 1 and side opposite θ equal to 0.6. By the Pythagorean Theorem, the side adjacent θ is $\sqrt{1-(0.6)^2} = 0.8$. Thus, $\cos\left(\sin^{-1} 0.6\right) = \cos\theta = 0.8$.

EXAMPLE 3 Simplify the expression $\tan(\sin^{-1} x)$.

Solution We want the tangent of an angle whose sine is x. Suppose first that $0 \le x < 1$. As in Example 2, we draw a right triangle (Figure 3.20) with one angle θ, and label the sides so that $\theta = \sin^{-1} x$. The side opposite θ is x, and the hypotenuse is 1. The remaining side is $\sqrt{1-x^2}$, and we have

$$\tan(\sin^{-1} x) = \tan\theta = \frac{x}{\sqrt{1-x^2}}.$$

Because both sides of the above equation are odd functions of x, the same result holds for $-1 < x < 0$.

Now let us use implicit differentiation to find the derivative of the inverse sine function. If $y = \sin^{-1} x$, then $x = \sin y$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Differentiating with respect to x, we obtain

$$1 = (\cos y)\frac{dy}{dx}.$$

Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, we know that $\cos y \ge 0$. Therefore,

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2},$$

and $dy/dx = 1/\cos y = 1/\sqrt{1-x^2}$;

$$\frac{d}{dx}\sin^{-1} x = \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}.$$

Note that the inverse sine function is differentiable only on the *open* interval $(-1, 1)$; the slope of its graph approaches infinity as $x \to -1+$ or as $x \to 1-$. (See Figure 3.18.)

EXAMPLE 4 Find the derivative of $\sin^{-1}\left(\frac{x}{a}\right)$ and hence evaluate $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}}$, where $a > 0$.

Solution By the Chain Rule,

$$\frac{d}{dx}\sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{1-\dfrac{x^2}{a^2}}}\,\frac{1}{a} = \frac{1}{\sqrt{\dfrac{a^2-x^2}{a^2}}}\,\frac{1}{a} = \frac{1}{\sqrt{a^2-x^2}} \qquad \text{if } a > 0.$$

Hence,

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C \qquad (a > 0).$$

EXAMPLE 5 Find the solution y of the following initial-value problem:

$$\begin{cases} y' = \dfrac{4}{\sqrt{2-x^2}} & (-\sqrt{2} < x < \sqrt{2}) \\ y(1) = 2\pi. \end{cases}$$

Solution Using the integral from the previous example, we have

$$y = \int \frac{4}{\sqrt{2-x^2}}\,dx = 4\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

for some constant C. Also $2\pi = y(1) = 4\sin^{-1}(1/\sqrt{2}) + C = 4\left(\frac{\pi}{4}\right) + C = \pi + C$. Thus, $C = \pi$ and $y = 4\sin^{-1}(x/\sqrt{2}) + \pi$.

EXAMPLE 6 **(A sawtooth curve)** Let $f(x) = \sin^{-1}(\sin x)$ for all real numbers x.

(a) Calculate and simplify $f'(x)$.

(b) Where is f differentiable? Where is f continuous?

(c) Use your results from (a) and (b) to sketch the graph of f.

Solution (a) Using the Chain Rule and the Pythagorean identity we calculate

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-(\sin x)^2}}(\cos x) \\ &= \frac{\cos x}{\sqrt{\cos^2 x}} = \frac{\cos x}{|\cos x|} = \begin{cases} 1 & \text{if } \cos x > 0 \\ -1 & \text{if } \cos x < 0. \end{cases} \end{aligned}$$

(b) f is differentiable at all points where $\cos x \neq 0$, that is, everywhere except at odd multiples of $\pi/2$, namely, $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$.
Since sin is continuous everywhere and has values in $[-1, 1]$, and since $\sin^{-1}$ is continuous on $[-1, 1]$, we have that f is continuous on the whole real line.

(c) Since f is continuous, its graph has no breaks. The graph consists of straight line segments of slopes alternating between 1 and -1 on intervals between consecutive odd multiples of $\pi/2$. Since $f'(x) = 1$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (where $\cos x \geq 0$), the graph must be as shown in Figure 3.21.

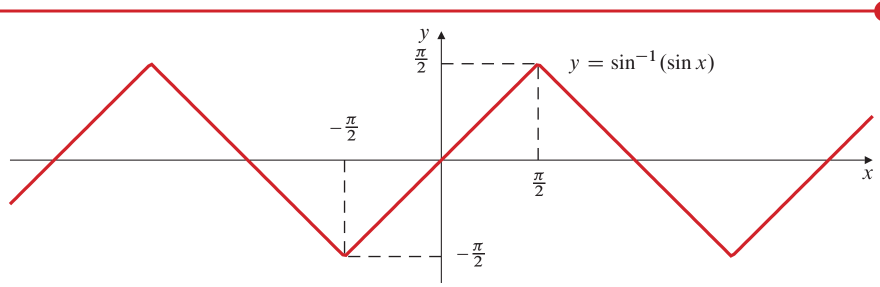


Figure 3.21 A sawtooth graph

The Inverse Tangent (or Arctangent) Function

The inverse tangent function is defined in a manner similar to the inverse sine. We begin by restricting the tangent function to an interval where it is one-to-one; in this case we use the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. See Figure 3.22(a).

DEFINITION 10

The restricted tangent function Tan x

$$\text{Tan}\, x = \tan x \qquad \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

The inverse of the function Tan is called the **inverse tangent** function and is denoted $\tan^{-1}$ (or arctan, Arctan, or atan). The domain of $\tan^{-1}$ is the whole real line (the range of Tan). Its range is the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

DEFINITION 11

The inverse tangent function $\tan^{-1} x$ or Arctan x

$$y = \tan^{-1} x \iff x = \text{Tan}\, y$$
$$\iff x = \tan y \quad \text{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

The graph of $\tan^{-1}$ is shown in Figure 3.22(b); it is the reflection of the graph of Tan in the line $y = x$.

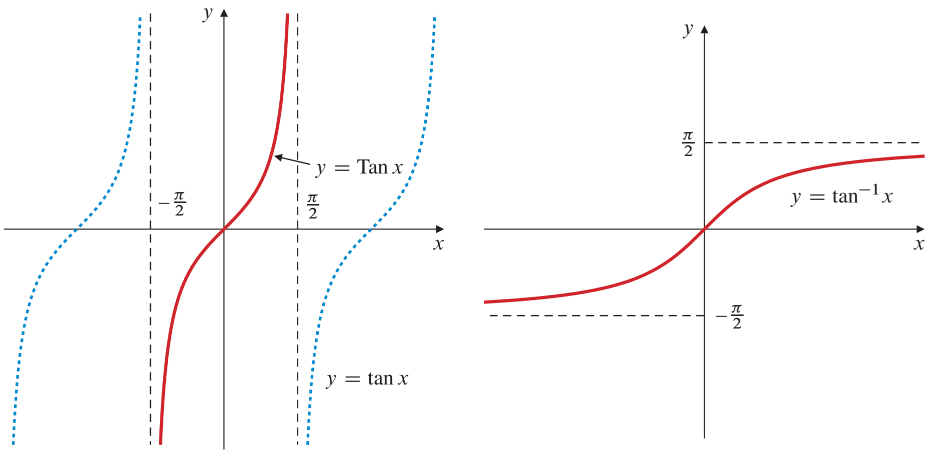


Figure 3.22
(a) The graph of Tan x
(b) The graph of $\tan^{-1} x$

The cancellation identities for Tan and $\tan^{-1}$ are

$$\tan^{-1}(\text{Tan}\, x) = \arctan(\text{Tan}\, x) = x \qquad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$\text{Tan}(\tan^{-1} x) = \text{Tan}(\arctan x) = x \qquad \text{for } -\infty < x < \infty$$

Again, we can replace Tan with tan above since the intervals are specified.

EXAMPLE 7 Evaluate: (a) $\tan(\tan^{-1} 3)$, (b) $\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$, and (c) $\cos(\tan^{-1} 2)$.

Solution

(a) $\tan(\tan^{-1} 3) = 3$ by cancellation.

(b) $\tan^{-1}\left(\tan \frac{3\pi}{4}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$.

(c) $\cos(\tan^{-1} 2) = \cos\theta = \frac{1}{\sqrt{5}}$ via the triangle in Figure 3.23. Alternatively, we have $\tan(\tan^{-1} 2) = 2$, so $\sec^2(\tan^{-1} 2) = 1 + 2^2 = 5$. Thus, $\cos^2(\tan^{-1} 2) = \frac{1}{5}$. Since cosine is positive on the range of $\tan^{-1}$, we have $\cos(\tan^{-1} 2) = \frac{1}{\sqrt{5}}$.

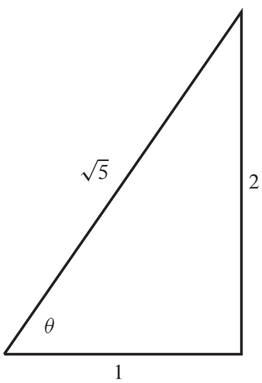


Figure 3.23

The derivative of the inverse tangent function is also found by implicit differentiation: if $y = \tan^{-1} x$, then $x = \tan y$ and

$$1 = (\sec^2 y)\frac{dy}{dx} = (1 + \tan^2 y)\frac{dy}{dx} = (1 + x^2)\frac{dy}{dx}.$$

Thus,

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1 + x^2}.$$

EXAMPLE 8 Find $\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right)$, and hence evaluate $\int \frac{1}{x^2+a^2}\,dx$.

Solution We have

$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{1}{1+\frac{x^2}{a^2}}\frac{1}{a} = \frac{a}{a^2+x^2};$$

hence,

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

EXAMPLE 9 Prove that $\tan^{-1}\left(\frac{x-1}{x+1}\right) = \tan^{-1}x - \frac{\pi}{4}$ for $x > -1$.

Solution Let $f(x) = \tan^{-1}\left(\frac{x-1}{x+1}\right) - \tan^{-1}x$. On the interval $(-1, \infty)$ we have, by the Chain Rule and the Quotient Rule,

$$\begin{aligned} f'(x) &= \frac{1}{1+\left(\frac{x-1}{x+1}\right)^2}\frac{(x+1)-(x-1)}{(x+1)^2} - \frac{1}{1+x^2} \\ &= \frac{(x+1)^2}{(x^2+2x+1)+(x^2-2x+1)}\frac{2}{(x+1)^2} - \frac{1}{1+x^2} \\ &= \frac{2}{2+2x^2} - \frac{1}{1+x^2} = 0. \end{aligned}$$

Hence, $f(x) = C$ (constant) on that interval. We can find C by finding $f(0)$:

$$C = f(0) = \tan^{-1}(-1) - \tan^{-1}0 = -\frac{\pi}{4}.$$

Hence, the given identity holds on $(-1, \infty)$.

Remark Some computer programs, especially spreadsheets, implement two versions of the arctangent function, usually called "atan" and "atan2." The function atan is just the function $\tan^{-1}$ that we have defined; $\text{atan}(y/x)$ gives the angle in radians, between the line from the origin to the point (x, y) and the positive x-axis, provided (x, y) lies in quadrants I or IV of the plane. The function atan2 is a function of two variables: $\text{atan2}(x, y)$ gives that angle for any point (x, y) not on the y-axis. See Figure 3.24. Some programs, for instance MATLAB, reverse the order of the variables x and y in their atan2 function. Maple uses `arctan(x)` and `arctan(y,x)` for the one- and two-variable versions of arctangent.

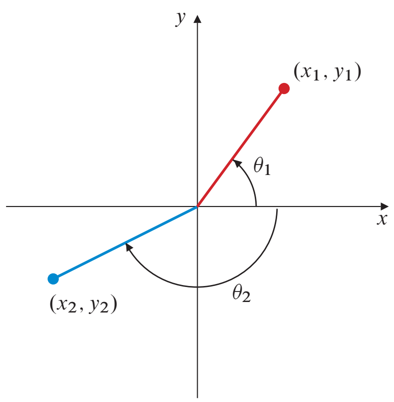


Figure 3.24

$\theta_1 = \tan^{-1}(y_1/x_1)$
$= \text{atan}(y_1/x_1)$
$= \text{atan2}(x_1, y_1)$
$= \text{arctan}(y_1/x_1)$ (Maple)
$= \text{arctan}(y_1, x_1)$ (Maple)
$\theta_2 = \text{atan2}(x_2, y_2)$
$= \text{arctan}(y_2, x_2)$ (Maple)

Other Inverse Trigonometric Functions

The function $\cos x$ is one-to-one on the interval $[0, \pi]$, so we could define the **inverse cosine function**, $\cos^{-1}x$ (or $\arccos x$, or $\text{Arccos}\,x$, or $\text{acos}\,x$), so that

$$y = \cos^{-1}x \iff x = \cos y \quad \text{and} \quad 0 \le y \le \pi.$$

However, $\cos y = \sin\left(\frac{\pi}{2} - y\right)$ (the complementary angle identity), and $\frac{\pi}{2} - y$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ when $0 \le y \le \pi$. Thus, the definition above would lead to

$$y = \cos^{-1}x \iff x = \sin\left(\frac{\pi}{2} - y\right) \iff \sin^{-1}x = \frac{\pi}{2} - y = \frac{\pi}{2} - \cos^{-1}x.$$

It is easier to use this result to define $\cos^{-1} x$ directly:

DEFINITION 12

The inverse cosine function $\cos^{-1} x$ or Arccos x

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \qquad \text{for} \quad -1 \le x \le 1.$$

The cancellation identities for $\cos^{-1} x$ are

$$\cos^{-1}(\cos x) = \arccos(\cos x) = x \qquad \text{for } 0 \le x \le \pi$$
$$\cos(\cos^{-1} x) = \cos(\arccos x) = x \qquad \text{for } -1 \le x \le 1$$

The derivative of $\cos^{-1} x$ is the negative of that of $\sin^{-1} x$ (why?):

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}.$$

The graph of $\cos^{-1}$ is shown in Figure 3.25(a).

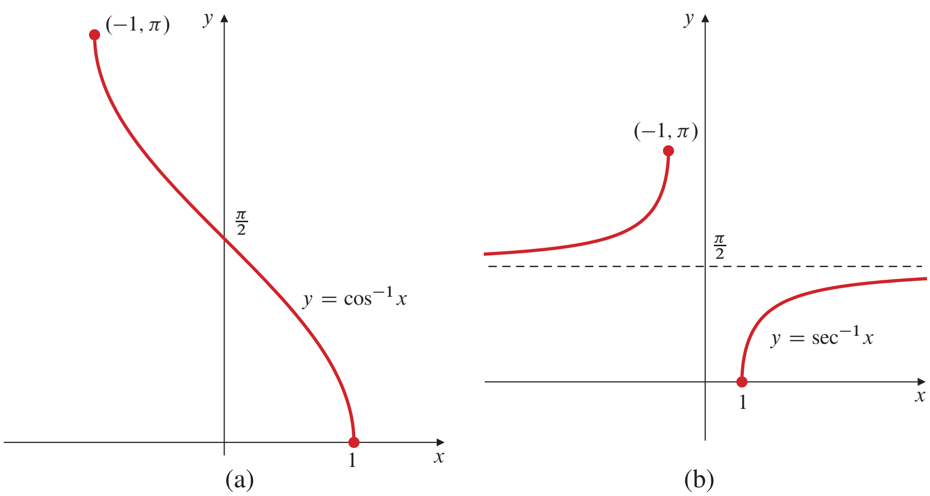


Figure 3.25 The graphs of $\cos^{-1}$ and $\sec^{-1}$

Scientific calculators usually implement only the primary trigonometric functions—sine, cosine, and tangent—and the inverses of these three. The secondary functions—secant, cosecant, and cotangent—are calculated using the reciprocal key; to calculate $\sec x$ you calculate $\cos x$ and take the reciprocal of the answer. The inverses of the secondary trigonometric functions are also easily expressed in terms of those of their reciprocal functions. For example, we define:

DEFINITION 13

The inverse secant function $\sec^{-1} x$ (or Arcsec x)

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) \qquad \text{for} \quad |x| \ge 1.$$

The domain of $\sec^{-1}$ is the union of intervals $(-\infty, -1] \cup [1, \infty)$, and its range is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. The graph of $y = \sec^{-1} x$ is shown in Figure 3.25(b). It is the reflection in the line $y = x$ of that part of the graph of $\sec x$ for x between 0 and π.

Observe that

$$\begin{aligned}\sec(\sec^{-1} x) &= \sec\left(\cos^{-1}\left(\frac{1}{x}\right)\right)\\ &= \frac{1}{\cos\left(\cos^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\frac{1}{x}} = x \qquad \text{for } |x| \ge 1,\\ \sec^{-1}(\sec x) &= \cos^{-1}\left(\frac{1}{\sec x}\right)\\ &= \cos^{-1}(\cos x) = x \qquad \text{for } x \text{ in } [0, \pi],\ x \ne \frac{\pi}{2}.\end{aligned}$$

We calculate the derivative of $\sec^{-1}$ from that of $\cos^{-1}$:

Some authors prefer to define $\sec^{-1}$ as the inverse of the restriction of $\sec x$ to the separated intervals $[0, \pi/2)$ and $[\pi, 3\pi/2)$ because this prevents the absolute value from appearing in the formula for the derivative. However, it is much harder to calculate values with that definition. Our definition makes it easy to obtain a value such as $\sec^{-1}(-3)$ from a calculator. Scientific calculators usually have just the inverses of sine, cosine, and tangent built in.

$$\begin{aligned}\frac{d}{dx}\sec^{-1} x &= \frac{d}{dx}\cos^{-1}\left(\frac{1}{x}\right) = \frac{-1}{\sqrt{1-\frac{1}{x^2}}}\left(-\frac{1}{x^2}\right)\\ &= \frac{1}{x^2}\sqrt{\frac{x^2}{x^2-1}} = \frac{1}{x^2}\,\frac{|x|}{\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}.\end{aligned}$$

Note that we had to use $\sqrt{x^2} = |x|$ in the last line. There are negative values of x in the domain of $\sec^{-1}$. Observe in Figure 3.25(b) that the slope of $y = \sec^{-1}(x)$ is always positive.

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}.$$

The corresponding integration formula takes different forms on intervals where $x \ge 1$ or $x \le -1$:

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \begin{cases}\sec^{-1}x + C & \text{on intervals where } x \ge 1\\ -\sec^{-1}x + C & \text{on intervals where } x \le -1\end{cases}$$

Finally, note that $\csc^{-1}$ and $\cot^{-1}$ are defined similarly to $\sec^{-1}$. They are seldom encountered.

DEFINITION 14

The inverse cosecant and inverse cotangent functions

$$\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right), \quad (|x| \ge 1); \qquad \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right), \quad (x \ne 0)$$

EXERCISES 3.5

In Exercises 1–12, evaluate the given expression.

1. $\sin^{-1} \frac{\sqrt{3}}{2}$

2. $\cos^{-1}\left(\frac{-1}{2}\right)$

3. $\tan^{-1}(-1)$

4. $\sec^{-1}\sqrt{2}$

5. $\sin(\sin^{-1} 0.7)$

6. $\cos(\sin^{-1} 0.7)$

7. $\tan^{-1}\left(\tan \frac{2\pi}{3}\right)$

8. $\sin^{-1}(\cos 40°)$

9. $\cos^{-1}(\sin(-0.2))$

10. $\sin\left(\cos^{-1}\left(\frac{-1}{3}\right)\right)$

11. $\cos\left(\tan^{-1}\frac{1}{2}\right)$

12. $\tan(\tan^{-1} 200)$

In Exercises 13–18, simplify the given expression.

13. $\sin(\cos^{-1} x)$

14. $\cos(\sin^{-1} x)$

15. $\cos(\tan^{-1} x)$

16. $\sin(\tan^{-1} x)$

17. $\tan(\cos^{-1} x)$

18. $\tan(\sec^{-1} x)$

In Exercises 19–32, differentiate the given function and simplify the answer whenever possible.

19. $y = \sin^{-1}\left(\dfrac{2x-1}{3}\right)$

20. $y = \tan^{-1}(ax+b)$

21. $y = \cos^{-1}\left(\dfrac{x-b}{a}\right)$

22. $f(x) = x\sin^{-1}x$

23. $f(t) = t\tan^{-1}t$

24. $u = z^2\sec^{-1}(1+z^2)$

25. $F(x) = (1+x^2)\tan^{-1}x$

26. $y = \sin^{-1}\dfrac{a}{x}$

27. $G(x) = \dfrac{\sin^{-1}x}{\sin^{-1}2x}$

28. $H(t) = \dfrac{\sin^{-1}t}{\sin t}$

29. $f(x) = (\sin^{-1}x^2)^{1/2}$

30. $y = \cos^{-1}\dfrac{a}{\sqrt{a^2+x^2}}$

31. $y = \sqrt{a^2-x^2} + a\sin^{-1}\dfrac{x}{a} \quad (a>0)$

32. $y = a\cos^{-1}\left(1-\dfrac{x}{a}\right) - \sqrt{2ax-x^2} \quad (a>0)$

33. Find the slope of the curve $\tan^{-1}\left(\dfrac{2x}{y}\right) = \dfrac{\pi x}{y^2}$ at the point $(1,2)$.

34. Find equations of two straight lines tangent to the graph of $y = \sin^{-1}x$ and having slope 2.

35. Show that, on their respective domains, $\sin^{-1}$ and $\tan^{-1}$ are increasing functions and $\cos^{-1}$ is a decreasing function.

36. The derivative of $\sec^{-1}x$ is positive for every x in the domain of $\sec^{-1}$. Does this imply that $\sec^{-1}$ is increasing on its domain? Why?

37. Sketch the graph of $\csc^{-1}x$ and find its derivative.

38. Sketch the graph of $\cot^{-1}x$ and find its derivative.

39. Show that $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ for $x>0$. What is the sum if $x<0$?

40. Find the derivative of $g(x) = \tan(\tan^{-1}x)$ and sketch the graph of g.

In Exercises 41–44, plot the graphs of the given functions by first calculating and simplifying the derivative of the function. Where is each function continuous? Where is it differentiable?

41. $\cos^{-1}(\cos x)$

42. $\sin^{-1}(\cos x)$

43. $\tan^{-1}(\tan x)$

44. $\tan^{-1}(\cot x)$

45. Show that $\sin^{-1}x = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$ if $|x|<1$.

46. Show that $\sec^{-1}x = \begin{cases} \tan^{-1}\sqrt{x^2-1} & \text{if } x \ge 1 \\ \pi - \tan^{-1}\sqrt{x^2-1} & \text{if } x \le -1 \end{cases}$

47. Show that $\tan^{-1}x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right)$ for all x.

48. Show that $\sec^{-1}x = \begin{cases} \sin^{-1}\dfrac{\sqrt{x^2-1}}{x} & \text{if } x \ge 1 \\ \pi + \sin^{-1}\dfrac{\sqrt{x^2-1}}{x} & \text{if } x \le -1 \end{cases}$

49. Show that the function $f(x)$ of Example 9 is also constant on the interval $(-\infty,-1)$. Find the value of the constant. *Hint:* Find $\lim_{x\to-\infty} f(x)$.

50. Find the derivative of $f(x) = x - \tan^{-1}(\tan x)$. What does your answer imply about $f(x)$? Calculate $f(0)$ and $f(\pi)$. Is there a contradiction here?

51. Find the derivative of $f(x) = x - \sin^{-1}(\sin x)$ for $-\pi \le x \le \pi$ and sketch the graph of f on that interval.

In Exercises 52–55, solve the initial-value problems.

52. $\begin{cases} y' = \dfrac{1}{1+x^2} \\ y(0) = 1 \end{cases}$

53. $\begin{cases} y' = \dfrac{1}{9+x^2} \\ y(3) = 2 \end{cases}$

54. $\begin{cases} y' = \dfrac{1}{\sqrt{1-x^2}} \\ y(1/2) = 1 \end{cases}$

55. $\begin{cases} y' = \dfrac{4}{\sqrt{25-x^2}} \\ y(0) = 0 \end{cases}$

3.6 Hyperbolic Functions

Any function defined on the real line can be expressed (in a unique way) as the sum of an even function and an odd function. (See Exercise 35 of Section P.5.) The **hyperbolic functions** $\cosh x$ and $\sinh x$ are, respectively, the even and odd functions whose sum is the exponential function e^x.

DEFINITION 15

The hyperbolic cosine and hyperbolic sine functions

For any real x the **hyperbolic cosine**, $\cosh x$, and the **hyperbolic sine**, $\sinh x$, are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

(The symbol "sinh" is somewhat hard to pronounce as written. Some people say "shine," and others say "sinch.") Recall that cosine and sine are called *circular functions* because, for any t, the point $(\cos t, \sin t)$ lies on the circle with equation $x^2 + y^2 = 1$. Similarly, cosh and sinh are called *hyperbolic functions* because the point $(\cosh t, \sinh t)$ lies on the rectangular hyperbola with equation $x^2 - y^2 = 1$,

$$\cosh^2 t - \sinh^2 t = 1 \quad \text{for any real } t.$$

To see this, observe that

$$\begin{aligned}\cosh^2 t - \sinh^2 t &= \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 \\ &= \frac{1}{4}\left(e^{2t} + 2 + e^{-2t} - (e^{2t} - 2 + e^{-2t})\right) \\ &= \frac{1}{4}(2 + 2) = 1.\end{aligned}$$

There is no interpretation of t as an arc length or angle as there was in the circular case; however, the *area* of the *hyperbolic sector* bounded by $y = 0$, the hyperbola $x^2 - y^2 = 1$, and the ray from the origin to $(\cosh t, \sinh t)$ is $t/2$ square units (see Exercise 21 of Section 8.4), just as is the area of the circular sector bounded by $y = 0$, the circle $x^2 + y^2 = 1$, and the ray from the origin to $(\cos t, \sin t)$. (See Figure 3.26.)

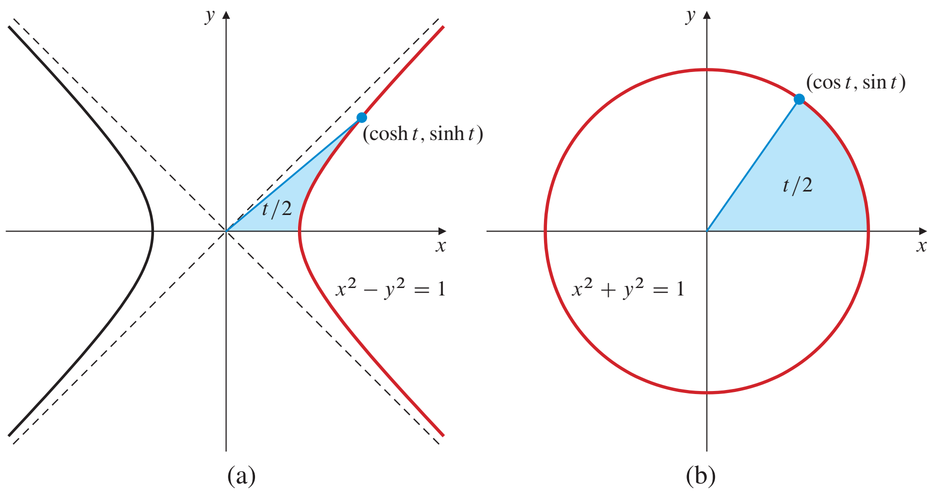


Figure 3.26 Both shaded areas are $t/2$ square units

Observe that, similar to the corresponding values of $\cos x$ and $\sin x$, we have

$$\cosh 0 = 1 \quad \text{and} \quad \sinh 0 = 0,$$

and $\cosh x$, like $\cos x$, is an even function, and $\sinh x$, like $\sin x$, is an odd function:

$$\cosh(-x) = \cosh x, \qquad \sinh(-x) = -\sinh x.$$

The graphs of cosh and sinh are shown in Figure 3.27. The graph $y = \cosh x$ is called a **catenary**. A chain hanging by its ends will assume the shape of a catenary.

Many other properties of the hyperbolic functions resemble those of the corresponding circular functions, sometimes with signs changed.

EXAMPLE 1 Show that

$$\frac{d}{dx}\cosh x = \sinh x \quad \text{and} \quad \frac{d}{dx}\sinh x = \cosh x.$$

Solution We have

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\frac{e^x + e^{-x}}{2} = \frac{e^x + e^{-x}(-1)}{2} = \sinh x$$
$$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{e^x - e^{-x}(-1)}{2} = \cosh x.$$

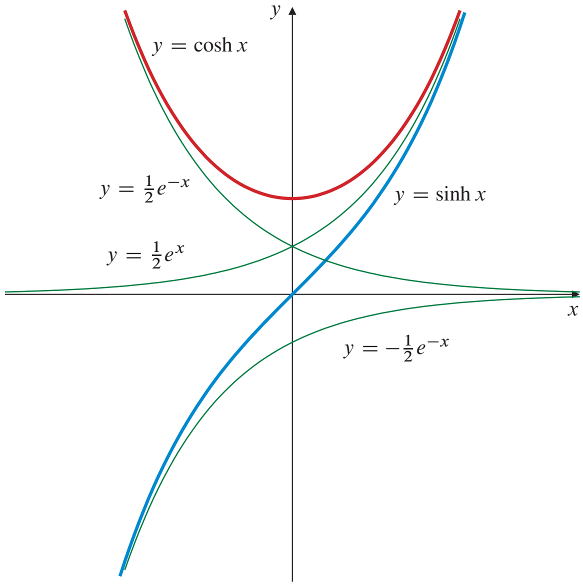


Figure 3.27 The graphs of cosh (red) and sinh (blue), and some exponential graphs (green) to which they are asymptotic

The following addition formulas and double-angle formulas can be checked algebraically by using the definition of cosh and sinh and the laws of exponents:

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y,$$
$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y,$$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x = 2\cosh^2 x - 1,$$
$$\sinh(2x) = 2\sinh x \cosh x.$$

By analogy with the trigonometric functions, four other hyperbolic functions can be defined in terms of cosh and sinh.

DEFINITION 16

Other hyperbolic functions

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Multiplying the numerator and denominator of the fraction defining $\tanh x$ by e^{-x} and e^x, respectively, we obtain

$$\lim_{x\to\infty}\tanh x = \lim_{x\to\infty}\frac{1 - e^{-2x}}{1 + e^{-2x}} = 1 \qquad \text{and}$$
$$\lim_{x\to-\infty}\tanh x = \lim_{x\to-\infty}\frac{e^{2x} - 1}{e^{2x} + 1} = -1,$$

so that the graph of $y = \tanh x$ has two horizontal asymptotes. The graph of $\tanh x$ (Figure 3.28) resembles those of $x/\sqrt{1 + x^2}$ and $(2/\pi)\tan^{-1} x$ in shape, but, of course, they are not identical.

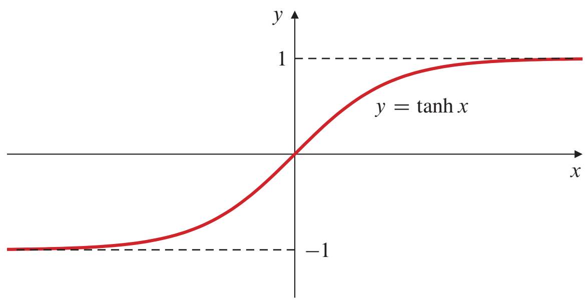


Figure 3.28 The graph of tanh x

The derivatives of the remaining hyperbolic functions

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \qquad \frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$$
$$\frac{d}{dx}\coth x = -\operatorname{csch}^2 x \qquad \frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$$

are easily calculated from those of cosh x and sinh x using the Reciprocal and Quotient Rules. For example,

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\frac{\sinh x}{\cosh x} = \frac{(\cosh x)(\cosh x) - (\sinh x)(\sinh x)}{\cosh^2 x}$$
$$= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x.$$

Remark The distinction between trigonometric and hyperbolic functions largely disappears if we allow complex numbers instead of just real numbers as variables. If i is the imaginary unit (so that $i^2 = -1$), then

$$e^{ix} = \cos x + i \sin x \qquad \text{and} \qquad e^{-ix} = \cos x - i \sin x.$$

(See Appendix I.) Therefore,

$$\cosh(ix) = \frac{e^{ix} + e^{-ix}}{2} = \cos x, \qquad \cos(ix) = \cosh(-x) = \cosh x,$$
$$\sinh(ix) = \frac{e^{ix} - e^{-ix}}{2} = i \sin x, \qquad \sin(ix) = \frac{1}{i}\sinh(-x) = i \sinh x.$$

Inverse Hyperbolic Functions

The functions sinh and tanh are increasing and therefore one-to-one and invertible on the whole real line. Their inverses are denoted $\sinh^{-1}$ and $\tanh^{-1}$, respectively:

$$y = \sinh^{-1} x \iff x = \sinh y,$$
$$y = \tanh^{-1} x \iff x = \tanh y.$$

Since the hyperbolic functions are defined in terms of exponentials, it is not surprising that their inverses can be expressed in terms of logarithms.

EXAMPLE 2 Express the functions $\sinh^{-1} x$ and $\tanh^{-1} x$ in terms of natural logarithms.

Solution Let $y = \sinh^{-1} x$. Then

$$x = \sinh y = \frac{e^y - e^{-y}}{2} = \frac{(e^y)^2 - 1}{2e^y}.$$

(We multiplied the numerator and denominator of the first fraction by e^y to get the second fraction.) Therefore,

$$(e^y)^2 - 2xe^y - 1 = 0.$$

This is a quadratic equation in e^y, and it can be solved by the quadratic formula:

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}.$$

Note that $\sqrt{x^2+1} > x$. Since e^y cannot be negative, we need to use the positive sign:

$$e^y = x + \sqrt{x^2+1}.$$

Hence, $y = \ln\left(x + \sqrt{x^2+1}\right)$, and we have

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right).$$

Now let $y = \tanh^{-1} x$. Then

$$\begin{aligned} x = \tanh y &= \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1} \qquad (-1 < x < 1), \\ xe^{2y} + x &= e^{2y} - 1, \\ e^{2y} &= \frac{1+x}{1-x}, \qquad y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right). \end{aligned}$$

Thus,

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \qquad (-1 < x < 1).$$

Since cosh is not one-to-one, its domain must be restricted before an inverse can be defined. Let us define the principal value of cosh to be

$$\operatorname{Cosh} x = \cosh x \qquad (x \ge 0).$$

The inverse, $\cosh^{-1}$, is then defined by

$$\begin{aligned} y = \cosh^{-1} x \quad &\Longleftrightarrow \quad x = \operatorname{Cosh} y \\ &\Longleftrightarrow \quad x = \cosh y \qquad (y \ge 0). \end{aligned}$$

As we did for $\sinh^{-1}$, we can obtain the formula

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right), \qquad (x \ge 1).$$

As was the case for the inverses of the reciprocal trigonometric functions, the inverses of the remaining three hyperbolic functions, coth, sech, and csch, are best defined using the inverses of their reciprocals.

$$\coth^{-1} x = \tanh^{-1}\left(\frac{1}{x}\right) = \frac{1}{2}\ln\left(\frac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}}\right) \qquad \text{for} \quad \left|\frac{1}{x}\right| < 1$$

$$= \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \qquad \text{for } x > 1 \text{ or } x < 1$$

$$\operatorname{sech}^{-1} x = \cosh^{-1}\left(\frac{1}{x}\right) = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right) \qquad \text{for} \quad \frac{1}{x} \geq 1$$

$$= \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad \text{for } 0 < x \leq 1$$

$$\operatorname{csch}^{-1} x = \sinh^{-1}\left(\frac{1}{x}\right) = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right)$$

$$= \begin{cases} \ln\left(\dfrac{1+\sqrt{1+x^2}}{x}\right) & \text{if } x > 0 \\[2ex] \ln\left(\dfrac{1-\sqrt{1+x^2}}{x}\right) & \text{if } x < 0. \end{cases}$$

The derivatives of all six inverse hyperbolic functions are left as exercises for the reader. See Exercise 5 and Exercises 8–10 below.

EXERCISES 3.6

1. Verify the formulas for the derivatives of $\operatorname{sech} x$, $\operatorname{csch} x$, and $\coth x$ given in this section.

2. Verify the addition formulas

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y.$$

Proceed by expanding the right-hand side of each identity in terms of exponentials. Find similar formulas for $\cosh(x-y)$ and $\sinh(x-y)$.

3. Obtain addition formulas for $\tanh(x+y)$ and $\tanh(x-y)$ from those for sinh and cosh.

4. Sketch the graphs of $y = \coth x$, $y = \operatorname{sech} x$, and $y = \operatorname{csch} x$, showing any asymptotes.

5. Calculate the derivatives of $\sinh^{-1} x$, $\cosh^{-1} x$, and $\tanh^{-1} x$. Hence, express each of the indefinite integrals

$$\int \frac{dx}{\sqrt{x^2+1}}, \qquad \int \frac{dx}{\sqrt{x^2-1}}, \qquad \int \frac{dx}{1-x^2}$$

in terms of inverse hyperbolic functions.

6. Calculate the derivatives of the functions $\sinh^{-1}(x/a)$, $\cosh^{-1}(x/a)$, and $\tanh^{-1}(x/a)$ (where $a > 0$), and use your answers to provide formulas for certain indefinite integrals.

7. Simplify the following expressions: (a) $\sinh \ln x$, (b) $\cosh \ln x$, (c) $\tanh \ln x$, (d) $\dfrac{\cosh \ln x + \sinh \ln x}{\cosh \ln x - \sinh \ln x}$.

8. Find the domain, range, and derivative of $\coth^{-1} x$ and sketch the graph of $y = \coth^{-1} x$.

9. Find the domain, range, and derivative of $\operatorname{sech}^{-1} x$ and sketch the graph of $y = \operatorname{sech}^{-1} x$.

10. Find the domain, range, and derivative of $\operatorname{csch}^{-1} x$, and sketch the graph of $y = \operatorname{csch}^{-1} x$.

✻ **11.** Show that the functions $f_{A,B}(x) = Ae^{kx} + Be^{-kx}$ and $g_{C,D}(x) = C \cosh kx + D \sinh kx$ are both solutions of the differential equation $y'' - k^2 y = 0$. (They are both general solutions.) Express $f_{A,B}$ in terms of $g_{C,D}$, and express $g_{C,D}$ in terms of $f_{A,B}$.

✻ **12.** Show that $h_{L,M}(x) = L \cosh k(x-a) + M \sinh k(x-a)$ is also a solution of the differential equation in the previous exercise. Express $h_{L,M}$ in terms of the function $f_{A,B}$ above.

✻ **13.** Solve the initial-value problem $y'' - k^2 y = 0$, $y(a) = y_0$, $y'(a) = v_0$. Express the solution in terms of the function $h_{L,M}$ of Exercise 12.

3.7 Second-Order Linear DEs with Constant Coefficients

A differential equation of the form

$$a\,y'' + b\,y' + cy = 0, \qquad (*)$$

where a, b, and c are constants and $a \neq 0$, is called a **second-order, linear, homogeneous** differential equation with constant coefficients. The *second-order* refers to the highest order derivative present; the terms *linear* and *homogeneous* refer to the fact that if $y_1(t)$ and $y_2(t)$ are two solutions of the equation, then so is $y(t) = Ay_1(t) + By_2(t)$ for any constants A and B:

$$\text{If } ay_1''(t) + by_1'(t) + cy_1(t) = 0 \text{ and } ay_2''(t) + by_2'(t) + cy_2(t) = 0,$$
$$\text{and if } y(t) = Ay_1(t) + By_2(t), \text{ then } ay''(t) + by'(t) + cy(t) = 0.$$

(See Section 18.1 for more details on this terminology.) Throughout this section we will assume that the independent variable in our functions is t rather than x, so the prime ($'$) refers to the derivative d/dt. This is because in most applications of such equations the independent variable is time.

Equations of type $(*)$ arise in many applications of mathematics. In particular, they can model mechanical vibrations such as the motion of a mass suspended from an elastic spring or the current in certain electrical circuits. In most such applications the three constants a, b, and c are positive, although sometimes we may have $b = 0$.

Recipe for Solving $ay'' + by' + cy = 0$

In Section 3.4 we observed that the first-order, constant-coefficient equation $y' = ky$ has solution $y = Ce^{kt}$. Let us try to find a solution of equation $(*)$ having the form $y = e^{rt}$. Substituting this expression into equation $(*)$, we obtain

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0.$$

Since e^{rt} is never zero, $y = e^{rt}$ will be a solution of the differential equation $(*)$ if and only if r satisfies the quadratic **auxiliary equation**

$$ar^2 + br + c = 0, \qquad (**)$$

which has roots given by the quadratic formula,

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{D}}{2a},$$

where $D = b^2 - 4ac$ is called the **discriminant** of the auxiliary equation $(**)$.

There are three cases to consider, depending on whether the discriminant D is positive, zero, or negative.

CASE I Suppose $D = b^2 - 4ac > 0$. Then the auxiliary equation has two different real roots, r_1 and r_2, given by

$$r_1 = \frac{-b - \sqrt{D}}{2a}, \qquad r_2 = \frac{-b + \sqrt{D}}{2a}.$$

(Sometimes these roots can be found easily by factoring the left side of the auxiliary equation.) In this case both $y = y_1(t) = e^{r_1 t}$ and $y = y_2(t) = e^{r_2 t}$ are solutions of the differential equation $(*)$, and neither is a multiple of the other. As noted above, the function

$$y = A\,e^{r_1 t} + B\,e^{r_2 t}$$

is also a solution for any choice of the constants A and B. Since the differential equation is of second order and this solution involves two arbitrary constants, we suspect it is the **general solution**, that is, that every solution of the differential equation can be written in this form. Exercise 18 at the end of this section outlines a way to prove this.

CASE II Suppose $D = b^2 - 4ac = 0$. Then the auxiliary equation has two equal roots, $r_1 = r_2 = -b/(2a) = r$, say. Certainly, $y = e^{rt}$ is a solution of $(*)$. We can find the general solution by letting $y = e^{rt}u(t)$ and calculating:

$$\begin{aligned} y' &= e^{rt}\big(u'(t) + ru(t)\big), \\ y'' &= e^{rt}\big(u''(t) + 2ru'(t) + r^2u(t)\big)\,. \end{aligned}$$

Substituting these expressions into $(*)$, we obtain

$$e^{rt}\big(au''(t) + (2ar + b)u'(t) + (ar^2 + br + c)u(t)\big) = 0.$$

Since $e^{rt} \neq 0$, $2ar + b = 0$ and r satisfies $(**)$, this equation reduces to $u''(t) = 0$, which has general solution $u(t) = A + Bt$ for arbitrary constants A and B. Thus, the general solution of $(*)$ in this case is

$$y = A\,e^{rt} + Bt\,e^{rt}.$$

CASE III Suppose $D = b^2 - 4ac < 0$. Then the auxiliary equation $(**)$ has complex conjugate roots given by

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = k \pm i\omega,$$

where $k = -b/(2a)$, $\omega = \sqrt{4ac - b^2}/(2a)$, and i is the imaginary unit ($i^2 = -1$; see Appendix I). As in Case I, the functions $y_1^*(t) = e^{(k+i\omega)t}$ and $y_2^*(t) = e^{(k-i\omega)t}$ are two independent solutions of (*), but they are not real-valued. However, since

$$e^{ix} = \cos x + i\sin x \qquad \text{and} \qquad e^{-ix} = \cos x - i\sin x$$

(as noted in the previous section and in Appendix II), we can find two real-valued functions that are solutions of (*) by suitably combining y_1^* and y_2^*:

$$\begin{aligned} y_1(t) &= \frac{1}{2}y_1^*(t) + \frac{1}{2}y_2^*(t) = e^{kt}\cos(\omega t), \\ y_2(t) &= \frac{1}{2i}y_1^*(t) - \frac{1}{2i}y_2^*(t) = e^{kt}\sin(\omega t). \end{aligned}$$

Therefore, the general solution of $(*)$ in this case is

$$y = A\,e^{kt}\cos(\omega t) + B\,e^{kt}\sin(\omega t).$$

The following examples illustrate the recipe for solving $(*)$ in each of the three cases.

EXAMPLE 1 Find the general solution of $y'' + y' - 2y = 0$.

Solution The auxiliary equation is $r^2 + r - 2 = 0$, or $(r + 2)(r - 1) = 0$. The auxiliary roots are $r_1 = -2$ and $r_2 = 1$, which are real and unequal. According to Case I, the general solution of the differential equation is

$$y = A\,e^{-2t} + Be^{t}.$$

EXAMPLE 2 Find the general solution of $y'' + 6y' + 9y = 0$.

Solution The auxiliary equation is $r^2 + 6r + 9 = 0$, or $(r+3)^2 = 0$, which has equal roots $r = -3$. According to Case II, the general solution of the differential equation is

$$y = A e^{-3t} + Bt\, e^{-3t}.$$

EXAMPLE 3 Find the general solution of $y'' + 4y' + 13y = 0$.

Solution The auxiliary equation is $r^2 + 4r + 13 = 0$, which has solutions

$$r = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = -2 \pm 3i.$$

Thus, $k = -2$ and $\omega = 3$. According to Case III, the general solution of the given differential equation is

$$y = A e^{-2t} \cos(3t) + B e^{-2t} \sin(3t).$$

Initial-value problems for $ay'' + by' + cy = 0$ specify values for y and y' at an initial point. These values can be used to determine the values of the constants A and B in the general solution, so the initial-value problem has a unique solution.

EXAMPLE 4 Solve the initial-value problem

$$\begin{cases} y'' + 2y' + 2y = 0 \\ y(0) = 2 \\ y'(0) = -3. \end{cases}$$

Solution The auxiliary equation is $r^2 + 2r + 2 = 0$, which has roots

$$r = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i.$$

Thus, Case III applies, with $k = -1$ and $\omega = 1$. Therefore, the differential equation has the general solution

$$y = A e^{-t} \cos t + B e^{-t} \sin t.$$

Also,

$$\begin{aligned} y' &= e^{-t}\big(-A\cos t - B\sin t - A\sin t + B\cos t\big) \\ &= (B - A)\, e^{-t} \cos t - (A + B)\, e^{-t} \sin t. \end{aligned}$$

Applying the initial conditions $y(0) = 2$ and $y'(0) = -3$, we obtain $A = 2$ and $B - A = -3$. Hence, $B = -1$ and the initial-value problem has the solution

$$y = 2e^{-t} \cos t - e^{-t} \sin t.$$

Simple Harmonic Motion

Many natural phenomena exhibit periodic behaviour. The swinging of a clock pendulum, the vibrating of a guitar string or drum membrane, the altitude of a rider on a rotating ferris wheel, the motion of an object floating in wavy seas, and the voltage produced by an alternating current generator are but a few examples where quantities depend on time in a periodic way. Being periodic, the circular functions sine and cosine provide a useful model for such behaviour.

It often happens that a quantity displaced from an equilibrium value experiences a restoring force that tends to move it back in the direction of its equilibrium. Besides the obvious examples of elastic motions in physics, one can imagine such a model applying, say, to a biological population in equilibrium with its food supply or the price of a commodity in an elastic economy where increasing price causes decreasing demand and hence decreasing price. In the simplest models, the restoring force is proportional to the amount of displacement from equilibrium. Such a force causes the quantity to oscillate sinusoidally; we say that it executes *simple harmonic motion*.

As a specific example, suppose a mass m is suspended by an elastic spring so that it hangs unmoving in its equilibrium position with the upward spring tension force balancing the downward gravitational force on the mass. If the mass is displaced vertically by an amount y from this position, the spring tension changes; the extra force exerted by the spring is directed to restore the mass to its equilibrium position. (See Figure 3.29.) This extra force is proportional to the displacement (Hooke's Law); its magnitude is $-ky$, where k is a positive constant called the **spring constant**. Assuming the spring is weightless, this force imparts to the mass m an acceleration d^2y/dt^2 that satisfies, by Newton's Second Law, $m(d^2y/dt^2) = -ky$ (mass × acceleration = force). Dividing this equation by m, we obtain the equation

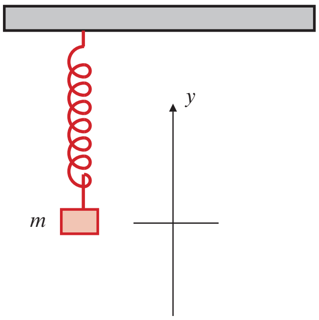


Figure 3.29

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \qquad \text{where} \quad \omega^2 = \frac{k}{m}.$$

The second-order differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

is called the **equation of simple harmonic motion**. Its auxiliary equation, $r^2 + \omega^2 = 0$, has complex roots $r = \pm i\omega$, so it has general solution

$$y = A\cos\omega t + B\sin\omega t,$$

where A and B are arbitrary constants.

For any values of the constants R and t_0, the function

$$y = R\cos\big(\omega(t - t_0)\big)$$

is also a general solution of the differential equation of simple harmonic motion. If we expand this formula using the addition formula for cosine, we get

$$\begin{aligned} y &= R\cos\omega t_0 \cos\omega t + R\sin\omega t_0 \sin\omega t \\ &= A\cos\omega t + B\sin\omega t, \end{aligned}$$

where

$$\begin{aligned} A &= R\cos(\omega t_0), & B &= R\sin(\omega t_0), \\ R^2 &= A^2 + B^2, & \tan(\omega t_0) &= B/A. \end{aligned}$$

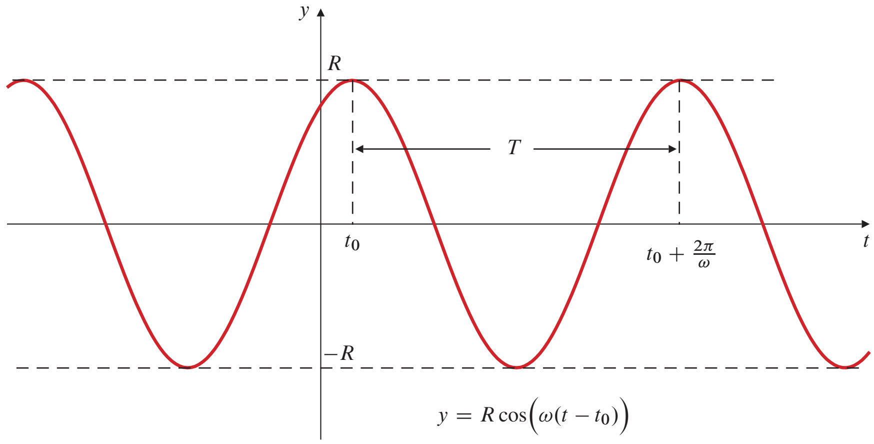


Figure 3.30 Simple harmonic motion

The constants A and B are related to the position y_0 and the velocity v_0 of the mass m at time $t = 0$:

$$y_0 = y(0) = A\cos 0 + B\sin 0 = A,$$
$$v_0 = y'(0) = -A\omega\sin 0 + B\omega\cos 0 = B\omega.$$

The constant $R = \sqrt{A^2 + B^2}$ is called the **amplitude** of the motion. Because $\cos x$ oscillates between -1 and 1, the displacement y varies between $-R$ and R. Note in Figure 3.30 that the graph of the displacement as a function of time is the curve $y = R\cos\omega t$ shifted t_0 units to the right. The number t_0 is called the **time-shift**. (The related quantity ωt_0 is called a **phase-shift**.) The **period** of this curve is $T = 2\pi/\omega$; it is the time interval between consecutive instants when the mass is at the same height moving in the same direction. The reciprocal $1/T$ of the period is called the **frequency** of the motion. It is usually measured in Hertz (Hz), that is, cycles per second. The quantity $\omega = 2\pi/T$ is called the **circular frequency**. It is measured in radians per second since 1 cycle = 1 revolution = 2π radians.

EXAMPLE 5 Solve the initial-value problem

$$\begin{cases} y'' + 16y = 0 \\ y(0) = -6 \\ y'(0) = 32. \end{cases}$$

Find the amplitude, frequency, and period of the solution.

Solution Here, $\omega^2 = 16$, so $\omega = 4$. The solution is of the form

$$y = A\cos(4t) + B\sin(4t).$$

Since $y(0) = -6$, we have $A = -6$. Also, $y'(t) = -4A\sin(4t) + 4B\cos(4t)$. Since $y'(0) = 32$, we have $4B = 32$, or $B = 8$. Thus, the solution is

$$y = -6\cos(4t) + 8\sin(4t).$$

The amplitude is $\sqrt{(-6)^2 + 8^2} = 10$, the frequency is $\omega/(2\pi) \approx 0.637$ Hz, and the period is $2\pi/\omega \approx 1.57$ s.

EXAMPLE 6 **(Spring-mass problem)** Suppose that a 100 g mass is suspended from a spring and that a force of 3×10^4 dynes (3×10^4 g-cm/s^2) is required to produce a displacement from equilibrium of 1/3 cm. At time $t = 0$ the mass is pulled down 2 cm below equilibrium and flicked upward with a velocity of 60 cm/s. Find its subsequent displacement at any time $t > 0$. Find the frequency, period, amplitude, and time-shift of the motion. Express the position of the mass at time t in terms of the amplitude and the time-shift.

Solution The spring constant k is determined from Hooke's Law, $F = -ky$. Here $F = -3 \times 10^4$ g-cm/s^2 is the force of the spring on the mass displaced 1/3 cm:

$$-3 \times 10^4 = -\frac{1}{3}k,$$

so $k = 9 \times 10^4$ g/s^2. Hence, the circular frequency is $\omega = \sqrt{k/m} = 30$ rad/s, the frequency is $\omega/2\pi = 15/\pi \approx 4.77$ Hz, and the period is $2\pi/\omega \approx 0.209$ s.

Since the displacement at time $t = 0$ is $y_0 = -2$ and the velocity at that time is $v_0 = 60$, the subsequent displacement is $y = A\cos(30t) + B\sin(30t)$, where $A = y_0 = -2$ and $B = v_0/\omega = 60/30 = 2$. Thus,

$$y = -2\cos(30t) + 2\sin(30t), \qquad (y \text{ in cm, } t \text{ in seconds}).$$

The amplitude of the motion is $R = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2} \approx 2.83$ cm. The time-shift t_0 must satisfy

$$\begin{aligned} -2 = A &= R\cos(\omega t_0) = 2\sqrt{2}\cos(30t_0), \\ 2 = B &= R\sin(\omega t_0) = 2\sqrt{2}\sin(30t_0), \end{aligned}$$

so $\sin(30t_0) = 1/\sqrt{2} = -\cos(30t_0)$. Hence the phase-shift is $30t_0 = 3\pi/4$ radians, and the time-shift is $t_0 = \pi/40 \approx 0.0785$ s. The position of the mass at time $t > 0$ is also given by

$$y = 2\sqrt{2}\cos\left[30\left(t - \frac{\pi}{40}\right)\right].$$

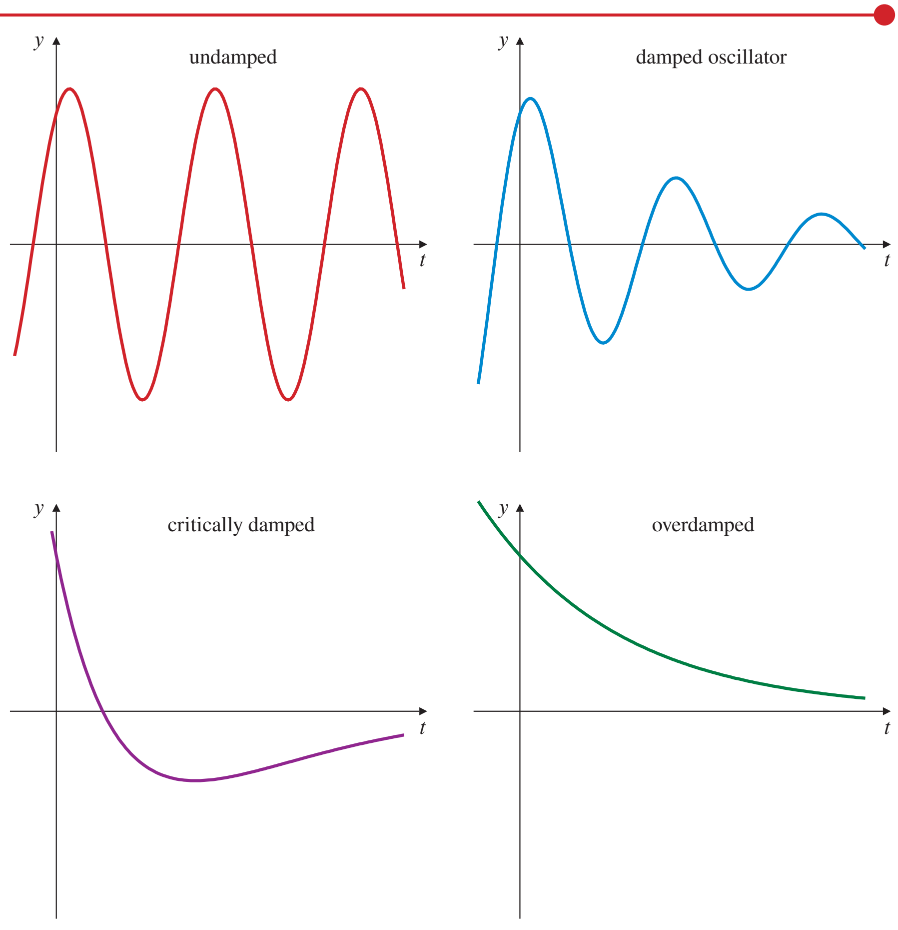


Figure 3.31
Undamped oscillator ($b = 0$)
Damped oscillator ($b > 0$, $b^2 < 4ac$)
Critically damped case ($b > 0$, $b^2 = 4ac$)
Overdamped case ($b > 0$, $b^2 > 4ac$)

Damped Harmonic Motion

If a and c are positive and $b = 0$, then equation

$$ay'' + by' + cy = 0$$

is the differential equation of simple harmonic motion and has oscillatory solutions of fixed amplitude as shown above. If $a > 0$, $b > 0$, and $c > 0$, then the roots of the auxiliary equation are either negative real numbers or, if $b^2 < 4ac$, complex numbers $k \pm i\omega$ with negative real parts $k = -b/(2a)$ (Case III). In this latter case the solutions still oscillate, but the amplitude diminishes exponentially as $t \to \infty$ because of the factor $e^{kt} = e^{-(b/2a)t}$. (See Exercise 17 below.) A system whose behaviour is modelled by such an equation is said to exhibit **damped harmonic motion**. If $b^2 = 4ac$ (Case II), the system is said to be **critically damped**, and if $b^2 > 4ac$ (Case I), it is **overdamped**. In these cases the behaviour is no longer oscillatory. (See Figure 3.31. Imagine a mass suspended by a spring in a jar of oil.)

EXERCISES 3.7

In Exercises 1–12, find the general solutions for the given second-order equations.

1. $y'' + 7y' + 10y = 0$ **2.** $y'' - 2y' - 3y = 0$

3. $y'' + 2y' = 0$ **4.** $4y'' - 4y' - 3y = 0$

5. $y'' + 8y' + 16y = 0$ **6.** $y'' - 2y' + y = 0$

7. $y'' - 6y' + 10y = 0$ **8.** $9y'' + 6y' + y = 0$

9. $y'' + 2y' + 5y = 0$ **10.** $y'' - 4y' + 5y = 0$

11. $y'' + 2y' + 3y = 0$ **12.** $y'' + y' + y = 0$

In Exercises 13–15, solve the given initial-value problems.

13. $\begin{cases} 2y'' + 5y' - 3y = 0 \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$ **14.** $\begin{cases} y'' + 10y' + 25y = 0 \\ y(1) = 0 \\ y'(1) = 2. \end{cases}$

15. $\begin{cases} y'' + 4y' + 5y = 0 \\ y(0) = 2 \\ y'(0) = 2. \end{cases}$

16. Show that if $\epsilon \ne 0$, the function $y_\epsilon(t) = \dfrac{e^{(1+\epsilon)t} - e^t}{\epsilon}$ satisfies the equation $y'' - (2+\epsilon)y' + (1+\epsilon)y = 0$. Calculate $y(t) = \lim_{\epsilon \to 0} y_\epsilon(t)$ and verify that, as expected, it is a solution of $y'' - 2y' + y = 0$.

17. If $a > 0$, $b > 0$, and $c > 0$, prove that all solutions of the differential equation $ay'' + by' + cy = 0$ satisfy $\lim_{t\to\infty} y(t) = 0$.

18. Prove that the solution given in the discussion of Case I, namely, $y = A\,e^{r_1 t} + B\,e^{r_2 t}$, is the general solution for that case as follows: First, let $y = e^{r_1 t}u$ and show that u satisfies the equation

$$u'' - (r_2 - r_1)u' = 0.$$

Then let $v = u'$, so that v must satisfy $v' = (r_2 - r_1)v$. The general solution of this equation is $v = C\,e^{(r_2 - r_1)t}$, as shown in the discussion of the equation $y' = ky$ in Section 3.4. Hence, find u and y.

Simple harmonic motion

Exercises 19–22 all refer to the differential equation of simple harmonic motion:

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \qquad (\omega \ne 0). \qquad (\dagger)$$

Together they show that $y = A\cos\omega t + B\sin\omega t$ is a *general solution* of this equation, that is, every solution is of this form for some choice of the constants A and B.

19. Show that $y = A\cos\omega t + B\sin\omega t$ is a solution of (†).

20. If $f(t)$ is any solution of (†), show that $\omega^2(f(t))^2 + (f'(t))^2$ is constant.

21. If $g(t)$ is a solution of (†) satisfying $g(0) = g'(0) = 0$, show that $g(t) = 0$ for all t.

22. Suppose that $f(t)$ is any solution of the differential equation (†). Show that $f(t) = A\cos\omega t + B\sin\omega t$, where $A = f(0)$ and $B\omega = f'(0)$.
(*Hint:* Let $g(t) = f(t) - A\cos\omega t - B\sin\omega t$.)

23. If $b^2 - 4ac < 0$, show that the substitution $y = e^{kt}u(t)$, where $k = -b/(2a)$, transforms $ay'' + by' + cy = 0$ into the equation $u'' + \omega^2 u = 0$, where $\omega^2 = (4ac - b^2)/(4a^2)$. Together with the result of Exercise 22, this confirms the recipe for Case III, in case you didn't feel comfortable with the complex number argument given in the text.

In Exercises 24–25, solve the given initial-value problems. For each problem determine the circular frequency, the frequency, the period, and the amplitude of the solution.

24. $\begin{cases} y'' + 4y = 0 \\ y(0) = 2 \\ y'(0) = -5. \end{cases}$ **25.** $\begin{cases} y'' + 100y = 0 \\ y(0) = 0 \\ y'(0) = 3. \end{cases}$

26. Show that $y = \alpha\cos(\omega(t-c)) + \beta\sin(\omega(t-c))$ is a solution of the differential equation $y'' + \omega^2 y = 0$, and that it satisfies $y(c) = \alpha$ and $y'(c) = \beta\omega$. Express the solution in the form $y = A\cos(\omega t) + B\sin(\omega t)$ for certain values of the constants A and B depending on α, β, c, and ω.

27. Solve $\begin{cases} y'' + y = 0 \\ y(2) = 3 \\ y'(2) = -4. \end{cases}$

28. Solve $\begin{cases} y'' + \omega^2 y = 0 \\ y(a) = A \\ y'(a) = B. \end{cases}$

29. What mass should be suspended from the spring in Example 6 to provide a system whose natural frequency of oscillation is 10 Hz? Find the displacement of such a mass from its equilibrium position t s after it is pulled down 1 cm from equilibrium and flicked upward with a speed of 2 cm/s. What is the amplitude of this motion?

30. A mass of 400 g suspended from a certain elastic spring will oscillate with a frequency of 24 Hz. What would be the frequency if the 400 g mass were replaced with a 900 g mass? a 100 g mass?

31. Show that if t_0, A, and B are constants and $k = -b/(2a)$ and $\omega = \sqrt{4ac - b^2}/(2a)$, then

$$y = e^{kt}\left[A\cos\left(\omega(t - t_0)\right) + B\sin\left(\omega(t - t_0)\right)\right]$$

is an alternative to the general solution of the equation $ay'' + by' + cy = 0$ for Case III ($b^2 - 4ac < 0$). This form of the general solution is useful for solving initial-value problems where $y(t_0)$ and $y'(t_0)$ are specified.

32. Show that if t_0, A, and B are constants and $k = -b/(2a)$ and $\omega = \sqrt{b^2 - 4ac}/(2a)$, then

$$y = e^{kt}\left[A\cosh\left(\omega(t - t_0)\right) + B\sinh\left(\omega(t - t_0)\right)\right]$$

is an alternative to the general solution of the equation $ay'' + by' + cy = 0$ for Case I ($b^2 - 4ac > 0$). This form of the general solution is useful for solving initial-value problems where $y(t_0)$ and $y'(t_0)$ are specified.

Use the forms of solution provided by the previous two exercises to solve the initial-value problems in Exercises 33–34.

33. $\begin{cases} y'' + 2y' + 5y = 0 \\ y(3) = 2 \\ y'(3) = 0. \end{cases}$

34. $\begin{cases} y'' + 4y' + 3y = 0 \\ y(3) = 1 \\ y'(3) = 0. \end{cases}$

35. By using the change of dependent variable $u(x) = c - k^2 y(x)$, solve the initial-value problem

$$\begin{cases} y''(x) = c - k^2 y(x) \\ y(0) = a \\ y'(0) = b. \end{cases}$$

36. A mass is attached to a spring mounted horizontally so the mass can slide along the top of a table. With a suitable choice of units, the position $x(t)$ of the mass at time t is governed by the differential equation

$$x'' = -x + F,$$

where the $-x$ term is due to the elasticity of the spring, and the F is due to the friction of the mass with the table. The frictional force should be constant in magnitude and directed opposite to the velocity of the mass when the mass is moving. When the mass is stopped, the friction should be constant and opposed to the spring force unless the spring force has the smaller magnitude, in which case the friction force should just cancel the spring force and the mass should remain at rest thereafter. For this problem, let the magnitude of the friction force be 1/5. Accordingly,

$$F = \begin{cases} -\dfrac{1}{5} & \text{if } x' > 0 \text{ or if } x' = 0 \text{ and } x < -\dfrac{1}{5} \\ \dfrac{1}{5} & \text{if } x' < 0 \text{ or if } x' = 0 \text{ and } x > \dfrac{1}{5} \\ x & \text{if } x' = 0 \text{ and } |x| \le \dfrac{1}{5}. \end{cases}$$

Find the position $x(t)$ of the mass at all times $t > 0$ if $x(0) = 1$ and $x'(0) = 0$.

CHAPTER REVIEW

Key Ideas

- **State the laws of exponents.**
- **State the laws of logarithms.**
- **What is the significance of the number e?**
- **What do the following statements and phrases mean?**
  - ◇ f is one-to-one. ◇ f is invertible.
  - ◇ Function f^{-1} is the inverse of function f.
  - ◇ $a^b = c$ ◇ $\log_a b = c$
  - ◇ the natural logarithm of x
  - ◇ logarithmic differentiation
  - ◇ the half-life of a varying quantity
  - ◇ The quantity y exhibits exponential growth.
  - ◇ The quantity y exhibits logistic growth.
  - ◇ $y = \sin^{-1} x$ ◇ $y = \tan^{-1} x$
  - ◇ The quantity y exhibits simple harmonic motion.
  - ◇ The quantity y exhibits damped harmonic motion.
- **Define the functions** $\sinh x$**,** $\cosh x$**, and** $\tanh x$**.**
- **What kinds of functions satisfy second-order differential equations with constant coefficients?**

Review Exercises

1. If $f(x) = 3x + x^3$, show that f has an inverse and find the slope of $y = f^{-1}(x)$ at $x = 0$.

2. Let $f(x) = \sec^2 x \tan x$. Show that f is increasing on the interval $(-\pi/2, \pi/2)$ and, hence, one-to-one and invertible there. What is the domain of f^{-1}? Find $(f^{-1})'(2)$. *Hint:* $f(\pi/4) = 2$.

Exercises 3–5 refer to the function $f(x) = x\,e^{-x^2}$.

3. Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

4. On what intervals is f increasing? decreasing?

5. What are the maximum and minimum values of $f(x)$?

6. Find the points on the graph of $y = e^{-x}\sin x$, $(0 \le x \le 2\pi)$, where the graph has a horizontal tangent line.

7. Suppose that a function $f(x)$ satisfies $f'(x) = x\,f(x)$ for all real x, and $f(2) = 3$. Calculate the derivative of $f(x)/e^{x^2/2}$, and use the result to help you find $f(x)$ explicitly.

8. A lump of modelling clay is being rolled out so that it maintains the shape of a circular cylinder. If the length is increasing at a rate proportional to itself, show that the radius is decreasing at a rate proportional to itself.

9. (a) What nominal interest rate, compounded continuously, will cause an investment to double in 5 years?

(b) By about how many days will the doubling time in part (a) increase if the nominal interest rate drops by 0.5%?

10. (A poor man's natural logarithm)

(a) Show that if $a > 0$, then

$$\lim_{h\to 0} \frac{a^h - 1}{h} = \ln a.$$

Hence, show that

$$\lim_{n\to\infty} n(a^{1/n} - 1) = \ln a.$$

(b) Most calculators, even nonscientific ones, have a square root key. If n is a power of 2, say $n = 2^k$, then $a^{1/n}$ can be calculated by entering a and hitting the square root key k times:

$$a^{1/2^k} = \sqrt{\sqrt{\cdots\sqrt{a}}} \qquad (k \text{ square roots}).$$

Then you can subtract 1 and multiply by n to get an approximation for $\ln a$. Use $n = 2^{10} = 1024$ and $n = 2^{11} = 2048$ to find approximations for $\ln 2$. Based on the agreement of these two approximations, quote a value of $\ln 2$ to as many decimal places as you feel justified.

11. A nonconstant function f satisfies

$$\frac{d}{dx}\big(f(x)\big)^2 = \big(f'(x)\big)^2$$

for all x. If $f(0) = 1$, find $f(x)$.

12. If $f(x) = (\ln x)/x$, show that $f'(x) > 0$ for $0 < x < e$ and $f'(x) < 0$ for $x > e$, so that $f(x)$ has a maximum value at $x = e$. Use this to show that $e^\pi > \pi^e$.

13. Find an equation of a straight line that passes through the origin and is tangent to the curve $y = x^x$.

14. (a) Find $x \ne 2$ such that $\dfrac{\ln x}{x} = \dfrac{\ln 2}{2}$.

(b) Find $b > 1$ such that there is *no* $x \ne b$ with $\dfrac{\ln x}{x} = \dfrac{\ln b}{b}$.

15. Investment account A bears simple interest at a certain rate. Investment account B bears interest at the same nominal rate but compounded instantaneously. If $1,000 is invested in each account, B produces $10 more in interest after one year than does A. Find the nominal rate both accounts use.

16. Express each of the functions $\cos^{-1} x$, $\cot^{-1} x$, and $\csc^{-1} x$ in terms of $\tan^{-1}$.

17. Express each of the functions $\cos^{-1} x$, $\cot^{-1} x$, and $\csc^{-1} x$ in terms of $\sin^{-1}$.

18. (A warming problem) A bottle of milk at 5 °C is removed from a refrigerator into a room maintained at 20 °C. After 12 min the temperature of the milk is 12 °C. How much longer will it take for the milk to warm up to 18 °C?

19. (A cooling problem) A kettle of hot water at 96 °C is allowed to sit in an air-conditioned room. The water cools to 60 °C in 10 min and then to 40 °C in another 10 min. What is the temperature of the room?

20. Show that $e^x > 1 + x$ if $x \ne 0$.

21. Use mathematical induction to show that

$$e^x > 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

if $x > 0$ and n is any positive integer.

Challenging Problems

1. (a) Show that the function $f(x) = x^x$ is strictly increasing on $[e^{-1}, \infty)$.

(b) If g is the inverse function to f of part (a), show that

$$\lim_{y\to\infty} \frac{g(y)\ln(\ln y)}{\ln y} = 1$$

Hint: Start with the equation $y = x^x$ and take the ln of both sides twice.

Two models for incorporating air resistance into the analysis of the motion of a falling body

2. (Air resistance proportional to speed) An object falls under gravity near the surface of the earth, and its motion is impeded by air resistance proportional to its speed. Its velocity v therefore satisfies the equation

$$\frac{dv}{dt} = -g - kv, \qquad (*)$$

where k is a positive constant depending on such factors as the shape and density of the object and the density of the air.

(a) Find the velocity of the object as a function of time t, given that it was v_0 at $t = 0$.

(b) Find the limiting velocity $\lim_{t\to\infty} v(t)$. Observe that this can be done either directly from $(*)$ or from the solution found in (a).

(c) If the object was at height y_0 at time $t = 0$, find its height $y(t)$ at any time during its fall.

3. (Air resistance proportional to the square of speed) Under certain conditions a better model for the effect of air resistance on a moving object is one where the resistance is proportional to the square of the speed. For an object falling under constant gravitational acceleration g, the equation of motion is

$$\frac{dv}{dt} = -g - kv|v|,$$

where $k > 0$. Note that $v|v|$ is used instead of v^2 to ensure that the resistance is always in the opposite direction to the velocity. For an object falling from rest at time $t = 0$, we have $v(0) = 0$ and $v(t) < 0$ for $t > 0$, so the equation of motion becomes

$$\frac{dv}{dt} = -g + kv^2.$$

We are not (yet) in a position to solve this equation. However, we can verify its solution.

(a) Verify that the velocity is given for $t \geq 0$ by

$$v(t) = \sqrt{\frac{g}{k}}\,\frac{1 - e^{2t\sqrt{gk}}}{1 + e^{2t\sqrt{gk}}}.$$

(b) What is the limiting velocity $\lim_{t\to\infty} v(t)$?

(c) Also verify that if the falling object was at height y_0 at time $t = 0$, then its height at subsequent times during its fall is given by

$$y(t) = y_0 + \sqrt{\frac{g}{k}}\,t - \frac{1}{k}\ln\left(\frac{1 + e^{2t\sqrt{gk}}}{2}\right).$$

✻ **4. (A model for the spread of a new technology)** When a new and superior technology is introduced, the percentage p of potential clients that adopt it might be expected to increase logistically with time. However, even newer technologies are continually being introduced, so adoption of a particular one will fall off exponentially over time. The following model exhibits this behaviour:

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{e^{-bt}M}\right).$$

This DE suggests that the growth in p is logistic but that the asymptotic limit is not a constant but rather $e^{-bt}M$, which decreases exponentially with time.

(a) Show that the change of variable $p = e^{-bt}y(t)$ transforms the equation above into a standard logistic equation, and hence find an explicit formula for $p(t)$ given that $p(0) = p_0$. It will be necessary to assume that $M < 100k/(b+k)$ to ensure that $p(t) < 100$.

(b) If $k = 10$, $b = 1$, $M = 90$, and $p_0 = 1$, how large will $p(t)$ become before it starts to decrease?