P:A80a

تابع Γ (γ گامای کوچک ، Γ گاما بزرگ از الفبای یونانی)

برای هر $p>-1$ تعریف می‌کنیم

$$\Gamma(p+1)=\int_0^{\infty} e^{-x}\, x^{p}\, dx$$

خواص این تابع عبارتند از:

1) $\Gamma(p+1)=p\,\Gamma(p)$

2) $\Gamma(1)=1,\quad \Gamma(n+1)=n!\ \ (n=0,1,2,\dots)$

3) $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$

اثبات 1:

$$\Gamma(p+1)=\int_0^{\infty} e^{-x}\, x^{p}\, dx$$

$$\begin{cases} u=x^{p} \\ dv=e^{-x}dx \end{cases} \longrightarrow \begin{cases} du=p\,x^{p-1}dx \\ v=-e^{-x} \end{cases}$$

$$\Gamma(p+1)=-x^{p}e^{-x}\Big|_0^{\infty}+p\int_0^{\infty} e^{-x}\, x^{p-1}\, dx$$

$$=-\lim_{x\to\infty}\frac{x^{p}}{e^{x}}+0+p\,\Gamma(p)$$

$$=0+p\,\Gamma(p).$$

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اثبات ②:

$$\Gamma(1) = \int_0^{\infty} e^{-x} x^0 \, dx = -e^{-x} \Big|_0^{\infty} = 0 + 1$$

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \dots = n \times n-1 \times \dots \times 1 \times \Gamma(1)$$

$$= n!$$

اثبات ③:

$$\Gamma\Big(\overset{p+1=1/2}{\tfrac{1}{2}}\Big) = \int_0^{\infty} e^{-x} x^{\overset{\frac{1}{2}-1}{-\frac{1}{2}}} \, dx$$

$$\xrightarrow[dx = 2y\,dy]{x = y^2} \int_0^{\infty} e^{-y^2} y^{-1} \, 2y \, dy$$

$$= 2\int_0^{\infty} e^{-y^2} \, dy$$

$$= 2 \, \frac{\sqrt{\pi}}{2}$$

$$= \sqrt{\pi}$$

$$\boxed{\int_0^{\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{2}}$$

در ریاضی II با استفاده
مختصات قطبی

Ex $\Gamma\left(\frac{11}{2}\right) = ?$

$$\Gamma\left(\frac{11}{2}\right) = \Gamma\left(1 + \frac{9}{2}\right) = \frac{9}{2}\,\Gamma\Big(\overset{1+7/2}{9/2}\Big) = \frac{9}{2} \times \frac{7}{2}\,\Gamma\Big(\overset{1+5/2}{\tfrac{7}{2}}\Big)$$

$$= \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2}\,\Gamma\Big(\overset{1+3/2}{5/2}\Big) = \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \Gamma\Big(\overset{1+\frac{1}{2}}{\tfrac{3}{2}}\Big)$$

$$= \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \Gamma(1/2) = \frac{9 \times 7 \times 5 \times 3}{2^5} \sqrt{\pi}$$

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Ex) $\Gamma(-1/2) = ?$

$$\Gamma\left(\tfrac{1}{2}\right) = \Gamma\left(-\tfrac{1}{2}+1\right) = -\tfrac{1}{2}\,\Gamma\left(-\tfrac{1}{2}\right) \longrightarrow$$

$$\sqrt{\pi} = \frac{-1}{2}\,\Gamma(-1/2) \longrightarrow$$

$$-2\sqrt{\pi} = \Gamma(-1/2)$$

GA) مقدار $\Gamma(-1/2)$ کدام است

1- $\sqrt{\pi}$ 2- $2\sqrt{\pi}$ 3- $-\sqrt{\pi}$ 4- $-2\sqrt{\pi}$ ✓

GA) مقدار $I=\int_0^1 \frac{dx}{\sqrt{-\ln x}}$ کدام است؟

1- $-\sqrt{\pi}$ 2- $\sqrt{\pi}$ ✓ 3- $-\frac{\sqrt{\pi}}{2}$ 4- $\frac{\sqrt{\pi}}{2}$

حل:

$$I \xlongequal[\frac{dx}{x}=-dt]{\substack{x=e^{-t}\\ \ln x=-t}} \int_\infty^0 \frac{-e^{-t}\,dt}{\sqrt{t}} = \int_0^\infty e^{-t}\, t^{-1/2}\, dt$$

$$= \Gamma\left(-\tfrac{1}{2}+1\right) = \Gamma(1/2) = \sqrt{\pi}.$$

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تبدیل لاپلاس تابع $f(t)=t^p$ برای $p>-1$.

$$\mathcal{L}\{t^p\} = \int_0^\infty e^{-st}\, t^p\, dt \xrightarrow[dx=s\,dt]{x=st} \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^p \frac{dx}{s}$$

$$= \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p\, dx$$

$$= \frac{\Gamma(p+1)}{s^{p+1}}$$

	$f(t)$	$F(s)$
$(p>-1)$	t^p	$\frac{\Gamma(p+1)}{s^{p+1}}$
$(n=0,1,2,\dots)$	t^n	$\frac{n!}{s^{n+1}}$

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*مثال) تبدیل لاپلاس تابع $f(t)=\frac{-3}{\sqrt{t}}$ کدام است؟

1) $-\frac{6\pi}{\sqrt{s}}$ 2) $-3\frac{\pi}{\sqrt{s}}$ 3) $-6\sqrt{\frac{\pi}{s}}$ 4) $-3\sqrt{\frac{\pi}{s}}$

حل:

$$\mathcal{L}^{-1}\left\{\frac{-3}{t^{1/2}}\right\} = -3\,\mathcal{L}\{t^{-1/2}\} = -3\,\frac{\Gamma(1/2)}{s^{1/2}} = -3\,\frac{\sqrt{\pi}}{\sqrt{s}}$$

$$= -3\sqrt{\frac{\pi}{s}}.$$

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P:A80e

Ex) تبدیل لاپلاس معکوس تابع $F(s)=\frac{1}{\sqrt{9s-1}}$ کدام است؟

1- $\frac{1}{3}\sqrt{\frac{t}{\pi}}\,e^{t/9}$ 2- $-\frac{1}{3}\sqrt{\frac{t}{\pi}}\,e^{-t/9}$

3- $\frac{1}{3}\frac{1}{\sqrt{t\pi}}\,e^{t/9}$ ✓ 4- $\frac{-1}{3}\frac{1}{\sqrt{t\pi}}\,e^{-t/9}$

حل:

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{9(s-\frac{1}{9})}}\right\}=\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s-1/9}}\right\}\overset{*}{=}\frac{1}{3}e^{\frac{t}{9}}\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\}$$

$$=\frac{1}{3}e^{\frac{t}{9}}\frac{\mathcal{L}^{-1}}{\Gamma(\frac{1}{2})}\left\{\frac{\Gamma(\frac{1}{2})}{\sqrt{s}}\right\}=\frac{1}{3}e^{\frac{t}{9}}\frac{1}{\sqrt{\pi}}\mathcal{L}^{-1}\left\{\frac{\Gamma(\frac{1}{2})}{s^{1/2}}\right\}$$

$$=\frac{1}{3}e^{\frac{t}{9}}\frac{1}{\sqrt{\pi}}t^{-1/2}=\frac{1}{3}\frac{1}{\sqrt{\pi t}}e^{t/2}$$

$\mathcal{L}\{e^{at}f(t)\}=F(s-a)$, where $F(s)=\mathcal{L}\{f(t)\}$

یادآوری

$\to \mathcal{L}^{-1}\{F(s-a)\}=e^{at}f(t)\overset{*}{=}e^{at}\mathcal{L}^{-1}\{F(s)\}$

Ex) تمرین شما

$$\mathcal{L}^{-1}\left\{\frac{s-1}{(s+2)^{3/2}}\right\}=?$$

ج: $\frac{e^{-2t}}{\sqrt{\pi}}(\sqrt{t}-3t\sqrt{t})$