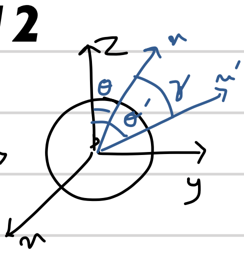
9/22/12

تابع گرین برای کره →



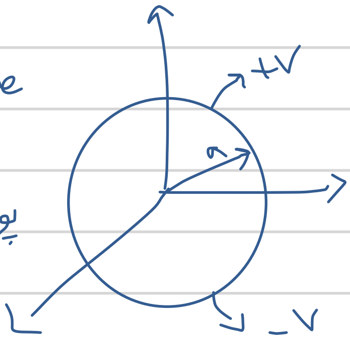
$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} - \frac{a}{r'\left|\vec{r}-\frac{a^2}{r'^2}\vec{r}'\right|}$$

$$G(\vec{r},\vec{r}') = \frac{1}{\sqrt{r^2+r'^2-2rr'\cdot\cos\gamma}} - \frac{1}{\sqrt{\frac{r^2r'^2}{a^2}+a^2-2rr'\cdot\cos\gamma}}$$

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

example

پوسته کروی



$$\Phi(\vec{r}') = \begin{cases} +V & 0<\theta<\pi/2 \\ -V & \pi/2<\theta<\pi \end{cases}$$

$$r<a \Rightarrow \Phi(r) = ?$$

$$\Phi(\vec{r}) = -\frac{1}{4\pi}\oint_s \Phi(\vec{r}')\frac{\partial}{\partial n'}G(\vec{r},\vec{r}')\,da'$$

$$= -\frac{1}{4\pi}\oint_s \Phi(\vec{r}')\boxed{\frac{\partial}{\partial n'}G(\vec{r},\vec{r}')}\,a^2\sin\theta'\,d\theta'\,d\phi'$$

$$\rightarrow \frac{r^2-a^2}{a(r^2+a^2-2ar\cos\gamma)^{3/2}}$$

$$\Rightarrow \Phi(\vec{r}) = -\frac{1}{4\pi}\left[\int_0^{2\pi}\int_0^{\pi/2} V\cdot\frac{a^2\sin\theta'd\theta'd\phi'\cdot(r^2-a^2)}{a(r^2+a^2-2ar\cos\gamma)^{3/2}} + \int_0^{2\pi}\int_0^{\pi/2}\frac{-V a^2\sin\theta' d\theta' d\phi'(r^2-a^2)}{a(r^2+a^2-2ar\cos\gamma)^{3/2}}\right.$$

$$\Rightarrow \Phi(\vec{r}) = -\frac{a(r^2-a^2)}{4\pi}\left[\int_0^{2\pi}\int_0^{\pi/2}\frac{V\sin\theta' d\theta' d\phi'}{(r^2+a^2-2ar\cos\gamma)^{3/2}} - \int_0^{2\pi}\int_0^{\pi/2}\frac{V\sin\theta'\cdot d\theta'\cdot d\phi'}{(r^2+a^2-2ar\cos\gamma)^{3/2}}\right]$$

$$\Phi(\vec{r}) = \frac{-aV(z^2-a^2)}{2}\left[\int_0^{\pi/2}\frac{\sin\theta'}{(z^2+a^2-2az\cos\theta')^{3/2}} - \int_{\pi/2}^{\pi}\frac{\sin\theta' d\theta'}{(z^2+a^2-2az\cos\theta')^{3/2}}\right]$$

$$\Longrightarrow \Phi(\vec{r}) = \frac{a.V(z^2-a^2)}{2}\left(-\frac{1}{a^2}\right)\left[\frac{1}{\sqrt{z^2+a^2}} - \frac{1}{a-z} - \frac{1}{a+z} + \frac{1}{\sqrt{z^2+a^2}}\right]$$

$$\Longrightarrow \Phi(\vec{r}) = \frac{V}{z}\left(a - \frac{a^2-z^2}{\sqrt{z^2+a^2}}\right) \quad z<a$$

برای $z>a$ $\longrightarrow$ $\Phi(\vec{r}) \frac{\partial}{\partial r'} G(\vec{r},\vec{r}') . da' \Longrightarrow \Phi(\vec{r}) = \frac{z^2-a^2}{4\pi a}\left[\int_0^{2\pi}\int_0^{\pi/2} \frac{V.\sin\theta' d\theta' . d\varphi' a^2}{(z^2+a^2-2az\cos\theta')^{3/2}} + \right.$

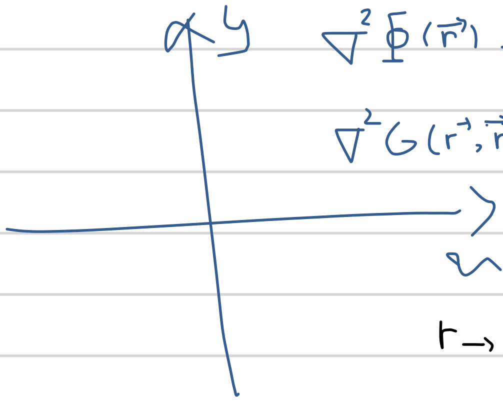
$$-\frac{\partial}{\partial r'} G(\vec{r},\vec{r}') = -\frac{(r^2-a^2)}{a(r^2+a^2-2ar\cos\gamma)^{3/2}}$$

$$\left.\int_0^{2\pi}\int_{\pi/2}^{\pi} \frac{-V.\sin\theta'.d\theta'.d\varphi'.a^2}{(z^2+a^2-2az\cos\theta')^{3/2}}\right]$$

$$\Longrightarrow \Phi(\vec{r}) = \frac{aV(z^2-a^2)}{2}\left(-\frac{1}{az}\right)\left(\frac{1}{\sqrt{z^2+a^2}} - \frac{1}{z-a} - \frac{1}{z+a} + \frac{1}{\sqrt{z^2+a^2}}\right)$$

$$\Longrightarrow \Phi(\vec{r}) = V\left(1 - \frac{z^2-a^2}{z(\sqrt{z^2+a^2})}\right)$$

محاسبه تابع گرین در دو بعد :



$$\nabla^2 \Phi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0}$$

$$\nabla^2 G(\vec{r},\vec{r}') = -2\pi\,\delta(\vec{r}-\vec{r}')$$

انتخاب دستگاه مختصات بر اساس $\vec{r}'=0$

$r \to$ قطبی

$\longrightarrow \nabla^2 G(\vec{r}) = -2\pi\,\delta(\vec{r}) = -2\pi\,\delta(x)\,\delta(y) \Rightarrow \nabla^2 G(r) =$

$$-2\pi\,\delta(r)\frac{1}{|J'|} \Longrightarrow \nabla^2 G(r) = -2\pi\frac{1}{2\pi r}\delta(r) \Longrightarrow$$

$$|J| = \int_0^{2\pi} |J|\,d\varphi = \int_0^{2\pi} R\,d\varphi = 2\pi r$$

$$\nabla^2 G(r) = \frac{-1}{r}\delta(r) \Longrightarrow \frac{1}{r}\frac{d}{dr}\left(r.\frac{dG(r)}{dr}\right) = \frac{-1}{r}\delta(r)$$

$$(r\neq 0) \Longrightarrow \frac{d}{dr}\left(r\frac{dG(r)}{dr}\right) = 0 \Longrightarrow r\frac{dG(r)}{dr} = C_1 \Longrightarrow \frac{dG(r)}{dr} = \frac{C_1}{r} \Rightarrow G(r) = C_1 \ln r + C_2$$

$r'=0$ ⚠

$$\nabla^2 G(r) = \frac{-1}{r}\delta(r) \Rightarrow \int_V \nabla^2 G(r).d^3\vec{r} = \int_V \frac{-1}{r}\delta(r).d^3\vec{r}$$

$$\Longrightarrow \int_V \vec{\nabla}.\vec{\nabla}G(r)\,d^3\vec{r} = -\int_V \frac{1}{r}\delta(r).d^3\vec{r} \Longrightarrow \oint_S \vec{\nabla}G(r).\hat{n}.da = -\int_V \frac{1}{r}\delta(r).d^3\vec{r} \Rightarrow$$

$$\oint_S \frac{C_1}{r}\hat{a}_r \cdot r\,d\varphi\,dz\,\hat{a}_z = -\int \frac{1}{r}\delta(r)\,r\,dr\,d\varphi\,dz$$

$$\Longrightarrow C_1\int_0^{2\pi} d\varphi \int_0^h dz = -2\pi h \Rightarrow C_1 2\pi h = -2\pi h \Rightarrow \boxed{C_1 = -1}$$

$\longrightarrow$ $G(r) = -\ln r + C_2$

$G(r)\big|_{r=R} = 0 \Longrightarrow -\ln R + C_2 = 0$

$\Longrightarrow C_2 = \ln R$

$$\Longrightarrow G(r) = -\ln r + \ln R$$

$$\nabla^2 G(\vec{r}) = -2\pi\,\delta(\vec{r})$$

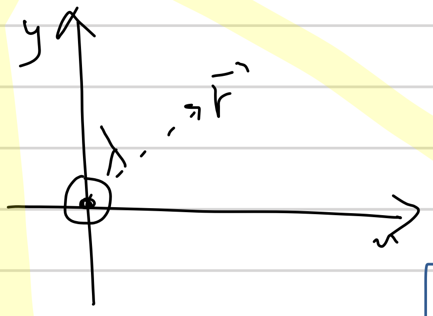
$$\nabla^2 \Phi(\vec{r}) = -\rho(\vec{r}')/\varepsilon_0$$

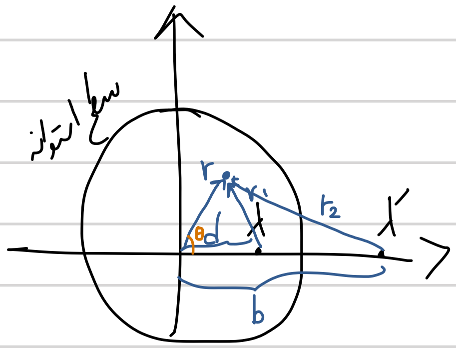
$$\vec{r}' = 0 \Longrightarrow \nabla^2 G(\vec{r}) = -2\pi\,\delta(\vec{r}) \Longrightarrow G(\vec{r}) = -\ln|\vec{r}| + \ln R$$

$$\vec{r}' \neq 0 \Longrightarrow G(\vec{r},\vec{r}') = -\ln|\vec{r}-\vec{r}'|$$

تابع گرین برای دو بعد
بدون سطح مرزی

● محاسبه تابع گرین برای یک دایره در صفحه xoy ⟵





$\boxed{\text{example}}$

$$\varepsilon_0 E\, 2\pi r l = \lambda l$$

$$\Longrightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r} \Longrightarrow \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{a}_r$$

$$\Phi(\vec{r}) = -\int_{\substack{Ref\\R_0}}^{r} \vec{E}\cdot d\vec{l} = \int_{R_0}^{r} \frac{\lambda}{2\pi\varepsilon_0 r}\hat{a}_r \cdot dr\,\hat{a}_r$$

$$\Longrightarrow \Phi(\vec{r}) = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{R_0}{r} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{r}{R_0} \xrightarrow{\lambda \to 2\pi\varepsilon_0} G(\vec{r},\vec{r}') = -\ln\frac{r}{R_0}$$

$$\vec{E}_1 = \frac{\lambda}{2\pi\varepsilon_0 R}\hat{a}_R \Longrightarrow \Phi_1 = -\int \vec{E}_1\cdot d\vec{L} = -\int_{R_0}^{R_1} \frac{\lambda}{2\pi\varepsilon_0 R}\hat{a}_R\cdot dR = \frac{-\lambda}{2\pi\varepsilon_0}\ln\frac{r_1}{R_0}$$

$$\vec{E}_2 = \frac{\lambda'}{2\pi\varepsilon_0 R'};\hat{a}_{R'} \Longrightarrow \Phi_2 = -\int E_2\cdot d\vec{L} = -\int_{R'_0}^{R_2} \frac{\lambda'}{2\pi\varepsilon_0 R'}\cdot dR' = \frac{-\lambda'}{2\pi\varepsilon_0}\ln\frac{r_2}{R'_0}$$

$$\Longrightarrow \Phi = \Phi_1 + \Phi_2 = \frac{-\lambda}{2\pi\varepsilon_0}\ln r_1 + \frac{\lambda}{2\pi\varepsilon_0}\ln R_0 - \frac{\lambda'}{2\pi\varepsilon_0}\ln r_2 + \frac{\lambda'}{2\pi\varepsilon_0}\ln R'_0$$

$$\Phi = \frac{-\lambda}{2\pi\varepsilon_0}\ln\sqrt{r^2+d^2-2rd\cos\theta} - \frac{\lambda'}{2\pi\varepsilon_0}\ln\sqrt{R^2+b^2-2rb\cos\theta} + \frac{\lambda}{2\pi\varepsilon_0}\ln R_0 + \frac{\lambda'}{2\pi\varepsilon_0}\ln R'_0$$

اعمال شرایط دیریکله و اتصال محیط ⚠

$$\Phi(\vec{r})\Big|_{|\vec{r}|=a} = 0 \Longrightarrow \frac{-\lambda}{2\pi\varepsilon_0}\ln\sqrt{a^2+d^2-2ad\cos\theta} - \frac{\lambda'}{2\pi\varepsilon_0}\ln\sqrt{a^2+b^2-2ab\cos\theta} + \frac{\lambda}{2\pi\varepsilon_0}\ln R_0 + \frac{\lambda'}{2\pi\varepsilon_0}\ln R'_0 = 0$$

$$\Longrightarrow \frac{-\lambda}{2\pi\varepsilon_0}\ln\sqrt{a^2+d^2-2ad\cos\theta} - \frac{\lambda'}{2\pi\varepsilon_0}\ln\frac{a}{d}\sqrt{d^2+\frac{b^2d^2}{a^2}-2\frac{bd^2}{a}\cos\theta} + \frac{\lambda}{2\pi\varepsilon_0}\ln R_0 + \frac{\lambda'}{2\pi\varepsilon_0}\ln R'_0 = 0 \Longrightarrow$$

$$a^2 = \frac{b^2 d^2}{a^2} \Rightarrow b^2 = \frac{a^4}{d^2} \Rightarrow b = \frac{a^2}{d}$$

$$ad = \frac{bd^2}{a} \longrightarrow b = \frac{a^2}{d}$$

$$\Rightarrow -\frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{a^2+d^2-2ad\cos\theta} - \frac{\lambda'}{2\pi\varepsilon_0}\ln\frac{a}{d} - \frac{\lambda'}{2\pi\varepsilon_0}\sqrt{a^2+d^2-2ad\cos\theta} + \frac{\lambda}{2\pi\varepsilon_0}\ln R_0 + \frac{\lambda'}{2\pi\varepsilon_0}\ln R'_0 = 0$$

$$\downarrow$$

$$-\frac{1}{2\pi\varepsilon_0}\ln\sqrt{a^2+d^2-2ad\cos\theta}\,(\lambda+\lambda') = 0$$

$$\boxed{\begin{matrix}\lambda+\lambda'=0\\ \lambda'=-\lambda\end{matrix}}\ !$$

$$\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{a}{d} + \frac{\lambda}{2\pi\varepsilon_0}\ln R_0 - \frac{\lambda}{2\pi\varepsilon_0}\ln R'_0 = 0$$

$$\ln\left(\frac{a}{d}\cdot R_0 \frac{1}{R'_0}\right) = 0 \Rightarrow \ln\frac{aR_0}{dR'_0} = 0 \Rightarrow \frac{aR_0}{dR'_0} = 1 \Rightarrow R'_0 = \frac{aR_0}{d}$$

$$\Phi = -\frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{r^2+d^2-2rd\cos\theta} - \frac{\lambda'}{2\pi\varepsilon_0}\ln\sqrt{r^2+b^2-2rb\cos\theta} + \frac{\lambda}{2\pi\varepsilon_0}\ln R_0 + \frac{\lambda'}{2\pi\varepsilon_0}\ln R'_0$$

$$\Phi(\vec{r}) = -\frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{r^2+d^2-2rd\cos\theta} + \frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{r^2+\frac{a^4}{d^2}-2r\frac{a^2}{d}\cos\theta} + \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{R_0}{R'_0} \Rightarrow$$

$$\Rightarrow \Phi(\vec{r}) = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{\sqrt{r^2+\frac{a^4}{d^2}-2r\frac{a^2}{d}\cos\theta}}{\sqrt{r^2+d^2-2rd\cos\theta}} + \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{d}{a} \Rightarrow G(\vec{r},\vec{r}') = \Phi(\vec{r})\Big|_{\lambda\to 2\pi\varepsilon_0} =$$

$$\ln\left(\frac{d\sqrt{r^2+\frac{a^4}{d^2}-2r\frac{a^2}{d}\cos\theta}}{a\sqrt{r^2+d^2-2rd\cos\theta}}\right)$$

$$G(\vec{r},\vec{r}') = \ln\left(\frac{d\sqrt{r^2+b^2-2rb\cos\theta}}{a\sqrt{r^2+d^2-2rd\cos\theta}}\right) \Rightarrow G(\vec{r},\vec{r}') = \ln\left(\frac{r'}{a}\frac{|\vec{r}-\vec{b}|}{|\vec{r}-\vec{d}|}\right) = \ln\left(\frac{r'\left|\vec{r}-\frac{a^2}{r'^2}\vec{r}'\right|}{a\,|\vec{r}-\vec{r}'|}\right)$$

روش دوم $\longmapsto G(\vec{r},\vec{r}') = -\ln|\vec{r}-\vec{r}'| + \ln|\vec{r}-\vec{r}''| + C \longrightarrow \vec{r}'' = \frac{a^2}{r'^2}\vec{r}'$

$$G(\vec{r},\vec{r}') = \ln\frac{\left|\vec{r}-\frac{a^2}{r'^2}\vec{r}'\right|}{|\vec{r}-\vec{r}'|} + C = \ln\sqrt{\frac{r^2+\frac{a^4}{r'^4}r'^2-\frac{2a^2}{r'^2}r\cos(\theta-\theta')}{r^2+r'^2-2rr'\cos(\theta-\theta')}} + C$$

$$G(\vec{r},\vec{r}')\Big|_{r=a} = 0 \Rightarrow \ln\frac{a}{r'}\sqrt{a} + C \longrightarrow C = -\ln\frac{a}{r'}$$

$$G(\vec{r},\vec{r}') = \ln\left(\frac{r'\left|\vec{r}-\frac{a^2}{r'^2}\vec{r}'\right|}{a\,|\vec{r}-\vec{r}'|}\right)$$

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$y'' - y' - 2y = 2x^2 - 3$ Homework حل تمرین

$y(-1) = y(0) = 0$

$$\Rightarrow G(x,s)\begin{cases} C_1 e^{x} + C_2 e^{-2x} & x < s \\ C_3 e^{x} + C_4 e^{-2x} & x > s \end{cases}$$

$G''(x,s) + G'(x,s) - 2G(x,s) = \delta(x-s)$

$G'' + G' - 2G = 0 \quad (s \neq x)$

$\longrightarrow G(-1,s) = 0 \longrightarrow C_1 e^{-1} + C_2 e^{2} = 0$

$G(0,s) = 0 \longrightarrow C_3 + C_4 = 0$

$G(x,s)\Big|_{s^+} = G(x,s)\Big|_{s^-} \longrightarrow C_1 e^{s} + C_2 e^{-2s} = C_3 e^{s} + C_4 e^{-2s}$

$G'(x,s)\Big|_{s^+} - G(x,s)\Big|_{s^-} = 1 \longrightarrow$

$C_3 e^{s} - 2C_4 e^{-2s} - (C_1 e^{s} - 2C_2 e^{-2s}) = 1$

$$y(x) = \int_{-1}^{0} G(x,s) f(s) ds = \int_{-1}^{x} (C_3 e^{x} + C_4 e^{-2x})(2s^2 - 3) ds + \int_{x}^{0} (C_1 e^{x} + C_2 e^{-2x})(2s^2 - 3) ds$$

Home work

$$\begin{cases} y'' = f(x) \\ y(0) = 1 \\ y(1) = 2 \end{cases}$$

تابع گرین را بدست آورید

پس $y(x)$ را بدست آورید!

~~$G'' = \delta(x,s)$~~

~~$G'' = 0 \quad (x \neq s) \Rightarrow \begin{cases} G = Ax + C & x > s \\ G = Bx + D & x < s \end{cases}$~~

~~$L[y] = f \quad \int f(s) [G(x,s)] ds$~~

~~$L[G] = f \quad = [y'']$~~

⚠ - تابع گرین برای شرط مرزی همگن صادق است فقط [تابع گرین دیریکله]

$\Longrightarrow$ تبدیل به شرایط مرزی همگن $\longrightarrow$ $y = u + v$, $y'' = f(x)$

$$\begin{cases} u'' + v'' = f(x) \\ u(0) + v(0) = 1 \\ u(1) + v(1) = 2 \end{cases} \Longrightarrow \begin{cases} u'' = f(x) \\ u(0) = 0 \\ u(1) = 0 \end{cases} \; ① \quad \& \quad \begin{cases} v'' = 0 \\ v(0) = 1 \\ v(1) = 2 \end{cases} \; ②$$

$$1 \longrightarrow G(x,s) = \begin{cases} (s-1)x & x<s \\ (s)(x-1) & x>s \end{cases}$$

$$2 \longrightarrow V = C_1 x + C_2$$

$$C_2 = 1$$

$$C_1 + C_2 = 2 \longrightarrow C_1 = 1$$

$$\longrightarrow V(x) = x+1$$

$$y(x) = \int_0^x s(x-1) f(s)\, ds + \int_x^1 (s-1) x f(s)\, ds + \underset{(x+1)}{\underbrace{V(x)}}$$

$$\Longrightarrow y(x) = (x-1)\int_0^x s f(s)\, ds + x\int_x^1 (s-1) f(s)\, ds + (x+1)$$

---

orthogonal توابع

$U_n(\xi)$ $(n = 1,2,3,\dots)$ orthonormal

در فاصله (a,b) راست به هنجار هستند

اگر توابع U_n به صورت مجذوری در این فاصله
انتگرال پذیر باشند. و رابطه زیر برقرار باشد

$$\int_a^b U_n^*(\xi)\, U_m(\xi)\, d\xi = \delta_{m,n}$$

voice ...

if $f(\xi)$ انتگرال پذیر مجذوری باشد در این فاصله $\longrightarrow f(\xi) \approx \sum_{m=1}^{N} A_m U_m(\xi) \Longrightarrow A = \int_a^b \left| f(\xi) - \sum_{m=1}^{N} A_m U_m(\xi) \right|^2 d\xi$

$A_{min} \Longrightarrow$ مشتق صفر

$$A_m = \int_a^b f(\xi)\, U_m^*(\xi)\, d\xi$$

$$\longrightarrow f(\xi) = \sum_{m=1}^{N} A_m U_m(\xi) \quad \boxed{00}$$

voice

اگر $U_n(\xi)$ [illegible] کامل باشند $\longrightarrow f(\xi) = \sum_{m=1}^{\infty} A_m U_m(\xi)$

به میانگین $f(\xi)$ میل می‌کند

$$\longrightarrow f(\xi) = \sum_{m=1}^{\infty} A_m U_m(\xi) \Longrightarrow f(\xi) = \sum_{m=1}^{\infty} \int_a^b f(\xi')\, U_m^*(\xi')\, d\xi'\, U_m(\xi) \longrightarrow$$

$$f(\xi)=\int_a^b \sum_{m=1}^{\infty} U_m^*(\xi')U_m(\xi)\, f(\xi')\, d\xi'$$ oo Voice

$$\delta(\xi-\xi')$$

$$\Rightarrow \boxed{\sum_{m=1}^{\infty} U_m^*(\xi')U_m(\xi)=\delta(\xi-\xi')}\ ✓$$

انتگرال پذیر مجذوری بودن $f(\xi)$ شرط ⟶ oo

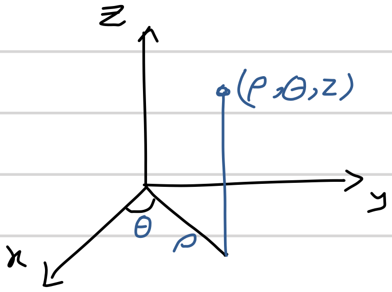
$$\left(-\tfrac{a}{2},\tfrac{a}{2}\right) \quad U_m=\frac{1}{\sqrt{a}}e^{\frac{i(2\pi m x)}{a}} \quad ; \ m=0,\pm1,\pm2$$

$$f(x)=\sum_m A_m U_m(x) \Rightarrow f(x)=\sum_m A_m \frac{1}{\sqrt{a}} e^{\frac{i(2\pi m x)}{a}}$$

$$A_m=\frac{1}{\sqrt{a}}\int_{-a/2}^{a/2} f(x')\, e^{-i\frac{(2\pi m x')}{a}}\, dx'$$

$$a\to\infty \qquad \frac{2\pi m}{a}\equiv K \longmapsto \sum_m \xrightarrow{\text{تبدیل}} \int_{-\infty}^{+\infty} dm \Longrightarrow dm=\frac{a}{2\pi}dK$$

$$f(x)=\int_{-\infty}^{+\infty}\frac{a\,dk}{2\pi}\frac{1}{\sqrt{a}}e^{ikx}\sqrt{\frac{2\pi}{a}}\,dK \Longrightarrow f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ikx}A(k)\,dK$$



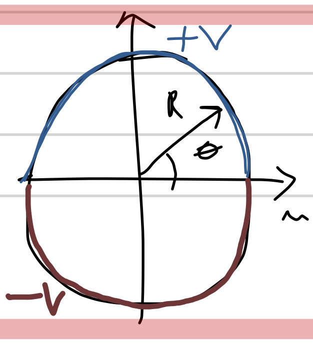
$z=x+iy$

$f'(z)=$ وجود دارد

⟵ [مشتق برای هر نقطه وجود داشته باشد] Conformal mapping نگاشت همدیس

متقارن بر حسب z $\Longrightarrow \Phi=\Phi(\rho,\theta)$

$$\nabla^2\Phi(\rho,\theta)=0 \Longrightarrow \Phi(\rho,\theta)=a_0+b_0\ln\rho+\sum_{n=1}^{\infty}\rho(a_n\sin n\theta+b_n\cos n\theta)+\rho^{-n}(c_n\sin n\theta+D_n\cos n\theta$$



مسأله ← یک پوسته استوانه‌ای طویل به شعاع R دارای پتانسیل به صورت زیر است

$$V(\theta)\begin{cases}+V & 0\leq\theta<\pi\\ -V & \pi\leq\theta<2\pi\end{cases}$$

محاسبه پتانسیل در نقاط داخل استوانه

حل ← پتانسیل الکتروستاتیکی در داخل استوانه (ρ,φ) $(\rho<R)$

$$\Phi(\rho,\varphi)=a_0+b_0\ln\rho+\sum_{n=1}^{\infty}\left[\rho^n(a_n\cos n\varphi+b_n\sin n\varphi)+\rho^{-n}(C_n\cos n\varphi+D_n\sin n\varphi)\right]$$

$$\Phi\Big|_{\rho=0}\ \text{باید مقدار داشته باشد} \Rightarrow \begin{matrix}b_0=0\\ C_n=d_n=0\end{matrix}$$

$$\Phi(\rho,\varphi)=a_0+\sum_{n=1}^{\infty}\rho^n(a_n\cos n\varphi+b_n\sin n\varphi)$$

$$\Phi(R,\varphi)=f(\varphi)=a_0+\sum_{n=1}^{\infty}R^n(a_n\cos n\varphi+b_n\sin n\varphi)\xrightarrow{\int_0^{2\pi}}\int_0^{2\pi}f(\varphi)\,d\varphi=a_0 2\pi\longrightarrow\boxed{a_0=\frac{1}{2\pi}\int_0^{2\pi}f(\varphi)\,d\varphi}$$

$$\int_0^{2\pi}f(\varphi)\cos m\varphi\,d\varphi=a_0\underbrace{\int_0^{2\pi}\cos m\varphi\,d\varphi}_{0}+\sum_{n=1}^{\infty}R^n\left[a_n\underbrace{\int_0^{2\pi}\cos m\varphi\cos n\varphi\,d\varphi}_{0}+b_n\underbrace{\int_0^{2\pi}\cos m\varphi\sin n\varphi\,d\varphi}_{0}\right]\Rightarrow$$

$$\Longrightarrow a_0\int_0^{2\pi}f(\varphi)\cos m\varphi\,d\varphi=R^m a_m\pi\longrightarrow$$

$$a_n=\frac{1}{R^n\pi}\int_0^{2\pi}f(\varphi)\cos n\varphi\,d\varphi$$

$$b_n=\frac{1}{R^n\pi}\int_0^{2\pi}f(\varphi)\sin n\varphi\,d\varphi$$

$$a_0=\frac{1}{2\pi}\left[\int_0^{\pi}V\,d\varphi+\int_{\pi}^{2\pi}(-V)\,d\varphi\right]=0$$

$$a_n=\frac{1}{\pi R^n}\left[\int_0^{\pi}V\cos n\varphi\,d\varphi+\int_{\pi}^{2\pi}(-V)\cos n\varphi\,d\varphi\right]=0$$

$$b_n=\frac{1}{\pi R^n}\left[\int_0^{\pi}V\sin n\varphi\,d\varphi+\int_{\pi}^{2\pi}(-V)\sin n\varphi\,d\varphi\right]\Rightarrow b_n=\begin{cases}0 & n\ \text{زوج}\\ \frac{4V}{\pi nR^n} & n\ \text{فرد}\end{cases}$$

$$\Phi(\rho,\varphi)=\frac{4V}{\pi}\sum_{n=1\ \text{فرد}}^{\infty}\rho^n\sin n\varphi\,\frac{1}{\pi R^n}$$

$$\Phi_1=\frac{4V}{\pi}\sum_{n=1\ \text{فرد}}^{\infty}\rho^n e^{in\varphi}\cdot\frac{1}{n\cdot R^n}\qquad (\rho e^{i\varphi})^n=z^n$$

$$\Phi=I_m(\Phi_1)\quad ,\ \Phi_1=\frac{4V}{\pi}\sum_{n=1\ \text{فرد}}^{\infty}\frac{1}{n}\left(\frac{z}{R}\right)^n\longrightarrow\frac{d\varphi_1}{dz}=\frac{4V}{\pi}\sum_{n=1\ \text{فرد}}^{\infty}\frac{1}{n}\cdot n\cdot\frac{1}{R}\left(\frac{z}{R}\right)^{n-1}$$

$$\Longrightarrow\frac{d\Phi_1}{dz}=\frac{4\pi}{\pi R}\sum_{n=1\ \text{فرد}}^{\infty}\left(\frac{z}{R}\right)^{n-1}=\frac{4V}{\pi R}\left[1+\frac{z^2}{R^2}+\frac{z^4}{R^4}+\cdots\right]=\frac{4V}{R\pi}\,\frac{1}{1-\frac{z^2}{R^2}}=\frac{4VR}{\pi}\,\frac{1}{R^2-z^2}\Longrightarrow$$

$$\left|\frac{z^2}{R^2}\right|=\frac{\rho^2}{R^2}<1$$

همگرا!

$$\frac{d\Phi_1}{dz} = \frac{4VR}{\pi}\left(\frac{1}{2R}\right)\left(\frac{1}{R-z}+\frac{1}{R+z}\right) = \frac{2V}{\pi}\left(\frac{1}{z+R}-\frac{1}{z-R}\right)$$

$$\Longrightarrow \Phi_1 = \frac{2V}{\pi}\ln\frac{z+R}{z-R} \rightarrow \Phi_1 = \frac{2V}{\pi}\ln\frac{x+iy+R}{x+iy-R} \longrightarrow$$

$$\Phi_1 = \frac{2V}{\pi}\ln\left(\frac{x+iy+R}{x+iy-R}\cdot\frac{x-R-iy}{x-R-iy}\right) = \frac{2V}{\pi}\ln\left[\frac{x^2+y^2-R^2-i2Ry}{(x-R)^2+y^2}\right] = \frac{2V}{\pi}\ln\left[\frac{\sqrt{(x^2+y^2-R^2)^2+4R^2y^2}}{(x-R)^2+y^2} \times e^{i\tan^{-1}\frac{-2Ry}{x^2+y^2-R^2}}\right]$$

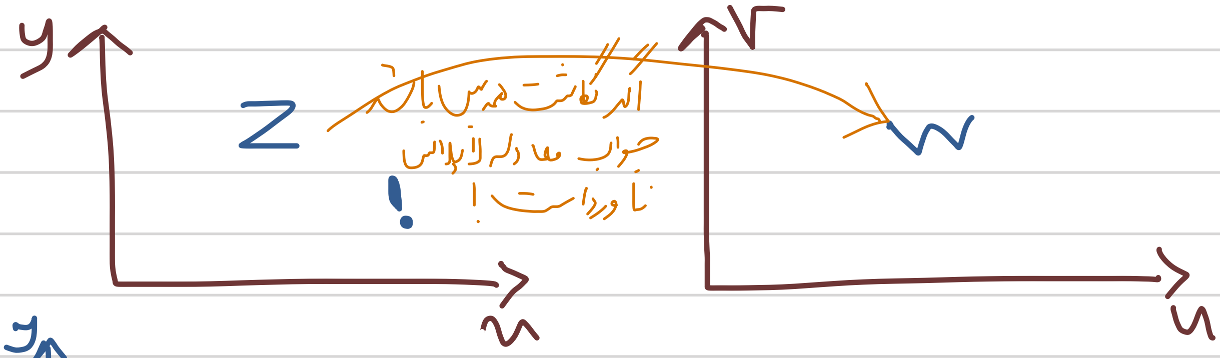
$$\Longrightarrow \Phi_1 = \frac{2V}{\pi}\left[\ln\frac{\sqrt{(x^2+y^2-R^2)^2+4R^2y^2}}{(x-R)^2+y^2} + i\tan^{-1}\frac{-2Ry}{x^2+y^2-R^2}\right]$$

$$\Longrightarrow \boxed{\Phi = \operatorname{Im}(\Phi_1) = \frac{2V}{\pi}\tan^{-1}\frac{2Ry}{R^2-x^2-y^2}}$$

$$R=1 \rightarrow \Phi(x,y) = \frac{2V}{\pi}\tan^{-1}\frac{2y}{1-x^2-y^2} \rightarrow$$

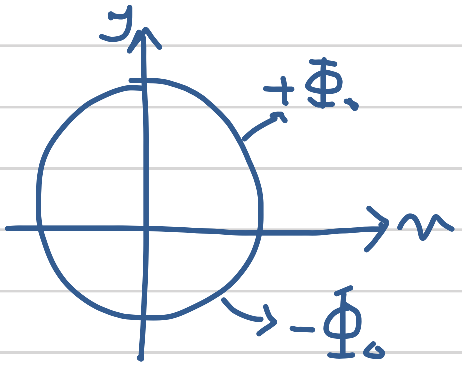
$$\Phi(\rho,\varphi) = \frac{2V}{\pi}\tan^{-1}\frac{2\rho\sin\varphi}{1-\varphi^2}$$

Conformal mapping ⟶



اگر نگاشت همدیس باشد
جواب معادله لاپلاس
ناورداست!

!



$$z = e^{i\theta} \text{ OR } |z| = 1$$

$$W(z) = \ln\frac{1+z}{1-z} = \ln(1+z) - \ln(1-z)$$

$$\frac{dW}{dz} = \frac{1}{1+z} - \frac{-1}{1-z} = \frac{1-z+1+z}{1-z^2} = \frac{2}{1-z^2}$$ در همه نقاط تحلیلی بجز در

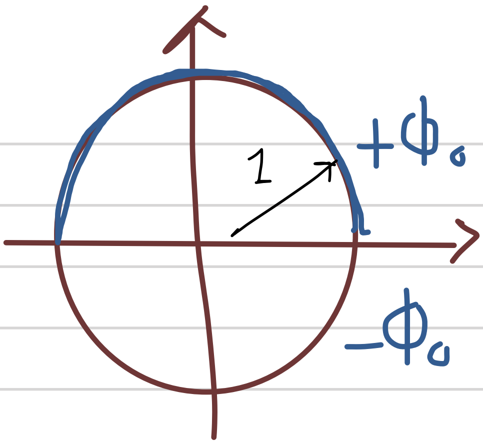
$$1-z^2=0 \rightarrow z=\pm 1$$

نگاشت ؛ یک نگاشت همدیس است ⇐

$$W(z) = U+iV = \ln\frac{1+z}{1-z} = \frac{1+x+iy}{1-x-iy} = \ln\left[\frac{1+x+iy}{1-x-iy}\cdot\frac{1-x+iy}{1-x+iy}\right] = \ln\left(\frac{1-x^2-y^2+2iy}{1-2x+x^2+y^2}\right)$$

$$W(z) = \ln\left[\frac{\sqrt{(1-x^2-y^2)^2+4y^2}}{1-2x+x^2+y^2}\cdot e^{i\tan^{-1}\frac{2y}{1-x^2-y^2}}\right] \longrightarrow W(z) = U(x,y)+iV(x,y) = \ln\underbrace{\frac{\sqrt{(1-x^2-y^2)^2+4y^2}}{1-2x+x^2+y^2}}_{U(x,y)} + i\tan^{-1}\underbrace{\frac{2y}{1-x^2-y^2}}_{V(x,y)}$$

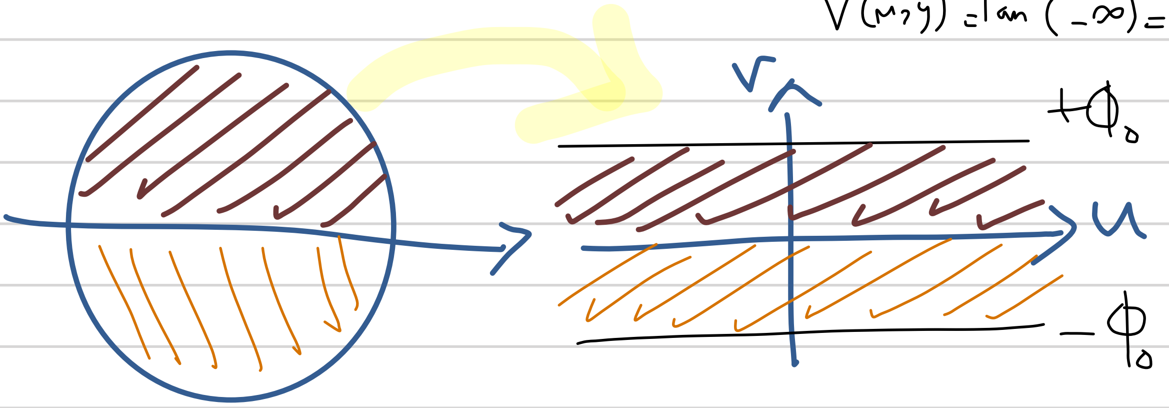
$$\begin{cases} u(x,y) = \ln \dfrac{\sqrt{(1-x^2-y^2)^2+4y^2}}{|1-2x+x^2+y^2|} \\ V(x,y) = \tan^{-1} \dfrac{2y}{1-x^2-y^2} \end{cases}$$



$\rho = \sqrt{x^2+y^2} < 1$ نیمه مثبت

نیمه بالایی دایره
$$\begin{cases} x^2+y^2 = 1 \\ y > 0 \end{cases} \Rightarrow u(x,y) = \ln \frac{2y}{2|1-x|} = \ln \frac{y}{|1-x|}$$
$$V(x,y) = \tan^{-1} \frac{2y}{1-x^2-y^2} = \tan^{-1}\infty = +\pi/2$$

نیمه پایینی
$$\begin{matrix} x^2+y^2=1 \\ y<0 \end{matrix} \rightarrow u(x,y) = \ln \left|\frac{y}{1-x}\right|$$
$$V(x,y) = \tan^{-1}(-\infty) = -\frac{\pi}{2}$$



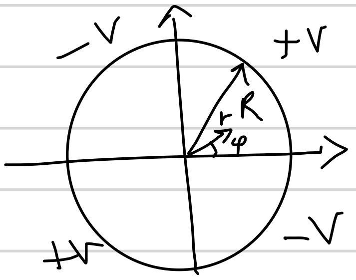
برای صفحه u,v: $\nabla^2\Phi = 0 \rightarrow \frac{d^2\Phi}{dV^2} = 0 \rightarrow \Phi = C_1V + C_2$

$$\begin{cases} \Phi_0 = C_1\frac{\pi}{2} + C_2 \\ -\Phi_0 = C_1\left(-\frac{\pi}{2}\right) + C_2 \end{cases} \rightarrow \begin{cases} C_1 = \frac{2\Phi_0}{\pi} \\ C_2 = 0 \end{cases}$$

$$\Phi = \frac{2\Phi_0}{\pi}\tan^{-1}\frac{2y}{1-x^2-y^2} \Rightarrow$$

$$\Phi = \frac{2\Phi_0}{\pi}\tan^{-1}\frac{2\rho\sin\varphi}{1-\rho^2}$$

استوانه رسانای
طویل به شعاع R مطابق
شکل زیر است.
پیدا کنید در نقاط داخل



جواب $\rightarrow \Phi(r,\varphi) = \frac{2V}{\pi}\tan^{-1}\frac{2R^2r^2\sin 2\varphi}{R^4-r^4} \quad r<R$

— با استفاده از هماهنگ‌های استوانه‌ای
— 〃 〃 روش نگاشت همدیس

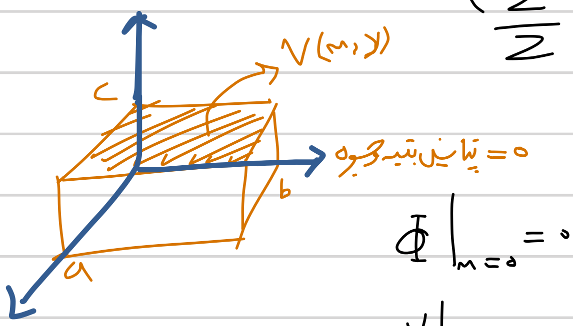
روش جداسازی متغیرها برای حل معادله لاپلاس در دستگاه مختصات دکارتی:

$$\nabla^2\Phi(x,y,z)=0 \longrightarrow \frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \qquad \Phi(x,y,z)=X(x)Y(y)Z(z)$$

$$X''YZ+XY''Z+XYZ''=0$$

$$\underset{-\alpha^2}{\frac{X''}{X}}+\underset{-\beta^2}{\frac{Y''}{Y}}+\underset{+\gamma^2}{\frac{Z''}{Z}}=0 \Longrightarrow \begin{cases}\frac{X''}{X}=-\alpha^2\\ \frac{Y''}{Y}=-\beta^2\\ \frac{Z''}{Z}=\gamma^2\end{cases} \Longrightarrow \begin{cases}X(x)=\lambda_1\sin\alpha x+\lambda_2\cos\alpha x\\ Y(y)=\lambda_3\sin\beta y+\lambda_4\cos\beta y\\ Z(z)=\lambda_5\sinh\gamma z+\lambda_6\cosh\gamma z\end{cases}$$

$$\gamma^2=\alpha^2+\beta^2$$



$$\Phi\Big|_{x=0}=0 \Longrightarrow X\Big|_{x=0}=0 \longrightarrow \lambda_2=0$$

$$Y\Big|_{y=0}=0 \Longrightarrow \lambda_4=0$$

$$Z\Big|_{z=0}=0 \Longrightarrow \lambda_6=0$$

$$\Phi\Big|_{x=a}=0 \longrightarrow X\Big|_{x=a}=0 \longrightarrow \lambda_1\sin\alpha a=0 \longrightarrow \alpha a=n\pi \longrightarrow \alpha=\frac{n\pi}{a} \qquad (n=1,2,3,\dots)$$

$$Y\Big|_{y=b}=0 \longrightarrow \sin\beta b=0 \longrightarrow \beta b=m\pi \longrightarrow \beta=\frac{m\pi}{b} \qquad (m=1,2,3,\dots)$$

$$\boxed{\gamma=\sqrt{\alpha^2+\beta^2}=\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\cdot\pi} \longrightarrow \Phi(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{mn}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sinh\left(\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\cdot\pi\cdot z\right)$$

$$z=c \longrightarrow V(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{mn}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sinh\left(\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\,\pi.c\right) \longrightarrow \alpha$$

$$\int_0^b\int_0^a V(x,y)\sin\frac{n\pi x}{a}\cdot\sin\frac{m\pi y}{b}\,dx\,dy=A_{mn}\sinh\left(\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\,\pi.c\right)\underbrace{\int_0^b\int_0^a\sin^2\frac{n\pi x}{a}\sin^2\frac{m\pi y}{b}\,dx\,dy}_{\frac{ab}{4}}$$

$$A_{mn}=\frac{4}{ab\sinh\left(\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\,\pi.c\right)}\int_0^b\int_0^a V(x,y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dx\,dy$$

$$\Longrightarrow \boxed{\Phi(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}A_{mn}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\sinh\left(\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\cdot\pi\cdot z\right)}$$

$$V(x,y)=\sum_{n=1}^{\infty}\alpha_n \sin\frac{n\pi x}{a}$$

$$\int_0^a V(x,y)\sin\frac{p\pi x}{a}\,dx=\sum_{n=1}^{\infty}\alpha_n\underbrace{\int_0^a \sin\frac{p\pi x}{a}\sin\frac{n\pi x}{a}\,dx}_{\frac{a}{2}\delta_{p,n}}$$

$$\int_0^a V(x,y)\sin\frac{p\pi x}{a}\,dx=\alpha_p\underbrace{\int_0^a \sin^2\frac{p\pi x}{a}\,dx}_{a/2}\longrightarrow \alpha_p=\frac{2}{a}\int_0^a V(x,y)\sin\frac{p\pi x}{a}\,dx$$

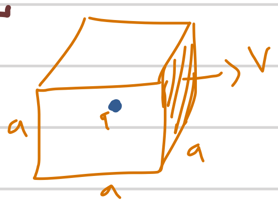
$$\alpha_n=\frac{2}{a}\int_0^a V(x,y)\sin\frac{n\pi x}{a}\,dx$$

$$\longrightarrow \sum_{m=1}^{\infty}A_{mn}\sin\frac{m\pi y}{b}\,\underset{!?]}{\sinh(\gamma c)}\Longrightarrow \frac{2}{a}\int_0^b\int_0^a V(x,y)\sin\frac{n\pi x}{a}\,dx\,\sin\frac{q\pi y}{b}\,dy=\sum_{m=1}^{\infty}A_{mn}\underbrace{\boxed{\int_0^b S\cdot\sin\frac{m\pi y}{b}\sin\frac{q\pi y}{y}}}_{\frac{b}{2}\delta_{q,m}}\sin(\gamma c)\,dy$$

$$\frac{2}{a}\int_0^b\int_0^a V(x,y)\sin\frac{n\pi x}{a}\sin\frac{q\pi y}{b}\,dx\,dy=A_{qn}\frac{b}{2}\rightarrow A_{qn}=\frac{4}{ab}\int_0^b\int_0^a V(x,y)\sin\frac{n\pi x}{a}\sin\frac{q\pi y}{b}\,dx\,dy\,\sinh(\gamma c)$$

$$\Longrightarrow \boxed{A_{m,n}=\frac{4}{ab\sinh\left(\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\cdot\pi c\right.}\int_0^a\int_0^b V(x,y)\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\,dx\,dy}$$

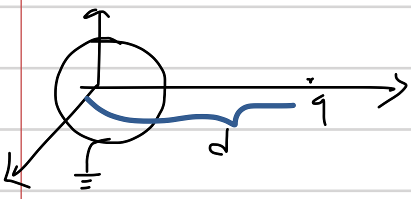
Homework



بقیه وجوه
پتانسیل = 0

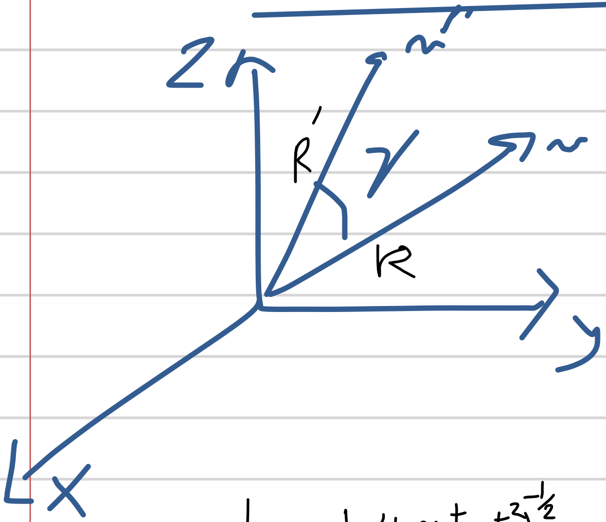
quiz!?

$$\nabla^2 \Phi(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(a_l r^l + b_l r^{-l-1}\right) Y_{l,m}(\theta,\varphi)$$



$$\Phi(\vec{r}) = \sum_{n=0}^{\infty} \left(a_n r^n + b_n r^{-n-1}\right) P_n(\cos\theta) + \frac{q}{4\pi\varepsilon_0 |\vec{r} - d\hat{a}_z|}$$

Singularity !need!



$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{\sqrt{R^2 + R'^2 - 2RR'\cos\gamma}} = \frac{1}{R}\left(1 + \frac{R'^2}{R^2} - \frac{2R'}{R}\cos\gamma\right)^{-1/2}$$

$R > R' \to \frac{R'}{R} < 1$

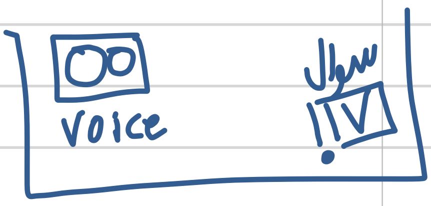
$\frac{R'}{R} = t < 1$

$\cos\gamma = x$

$$\Longrightarrow \frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{R}\left(1 - 2xt + t^2\right)^{-1/2} = \frac{1}{R}\sum_{l=0} P_l(x)\, t^l \Longrightarrow \frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{R}\sum_{l=0}^{\infty} P_l(\cos\gamma)\left(\frac{R'}{R}\right)^l \Longrightarrow \frac{1}{|\vec{r}-\vec{r}'|} \sum_{l=0}^{\infty} P_l(\cos\gamma)\frac{R'^l}{R^{l+1}}$$

$$\boxed{R < R' \Longrightarrow \frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l} P_l(\cos\gamma)\frac{R^l}{R'^{l+1}}}$$

$$\boxed{\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty} P_l(\cos\gamma)\frac{R_<^l}{R_>^{l+1}}}$$

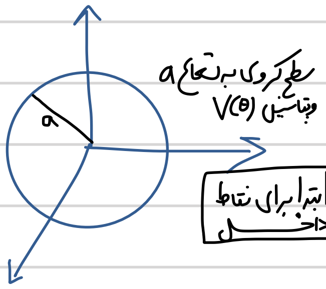


$$\nabla^2\Phi(r,\theta,\varphi)=0 \xrightarrow[\text{سمتی}]{\text{متقارن}} \Phi(r,\theta)=\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+B_\ell r^{-\ell-1}\right)P_\ell(\cos\theta)$$

متقارن سمتی

کشی اویلر

لژاندر



ابتدا برای نقاط داخل

$$\Phi\Big|_{r=0} \text{ باید معین باشد} \longrightarrow B_\ell=0 \longrightarrow \Phi(r,\theta)=\sum_{\ell=0}^{\infty}A_\ell r^\ell P_\ell(\cos\theta)$$

$$\Phi(r,\theta)\Big|_{r=a}=\Phi(a,\theta)=V(\theta)=\sum_{\ell=0}^{\infty}A_\ell a^\ell P_\ell(\cos\theta)\Longrightarrow \int_{-1}^{1}P_\ell(x)P_{\ell'}(x)\,dx=\frac{2}{2\ell+1}\delta_{\ell\ell'}$$

$$(x=\cos\theta)\,; \int_0^{\pi}P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta\,d\theta=\frac{2}{2\ell+1}\delta_{\ell\ell'}$$

$$\Rightarrow \int_0^{P}V(\theta)P_{\ell'}(\cos\theta)\sin\theta\,d\theta=\sum_{\ell=0}^{\infty}A_\ell a^\ell\underbrace{\int_0^{\pi}P_\ell(\cos\theta)P_{\ell'}(\cos\theta)\sin\theta\,d\theta}_{\frac{2}{2\ell+1}\delta_{\ell\ell'}}\Longrightarrow \int_0^{\pi}V(\theta)P_{\ell'}(\cos\theta)\sin\theta\,d\theta=A_{\ell'}a^{\ell}\frac{2}{2\ell+1}$$

$$\boxed{A_\ell=\ldots\frac{2\ell+1}{2a^\ell}\times\int_0^{\pi}V(\theta)P_\ell(\cos\theta)\sin\theta\,d\theta}$$

$$V(\theta)=\begin{cases}+V & 0\le\theta\le\pi/2\\ -V & \pi/2\le\theta\le\pi\end{cases}$$ حالت خاص

$$A_\ell=\frac{2\ell+1}{2a^\ell}V\left[\int_0^{\pi/2}P_\ell(\cos\theta)\sin\theta\,d\theta+\int_{\pi/2}^{\pi}-P_\ell(\cos\theta)\sin\theta\,d\theta\right]$$

$$(x=\cos\theta)\longrightarrow A_\ell=\frac{2\ell+1}{2a^\ell}V\left[\int_0^1P_\ell(x)\,dx-\int_{-1}^{0}P_\ell(x)\,dx\right]\Longrightarrow A_\ell=\frac{2\ell+1}{2a^\ell}V\int_0^1P_\ell(x)\,dx\left[1-(-1)^\ell\right]\Rightarrow\begin{cases}A_\ell=0 & \ell\text{ زوج}\\ A_\ell=\frac{2\ell+1}{a^\ell}V\int_0^1P_\ell(x)\,dx & \ell\text{ فرد}\end{cases}$$

$$x\to -x$$

$$\int_1^0P_\ell(-x)(-dx)$$

$$\int_0^1P_\ell(-x)\,dx \to \int_0^1(-1)^\ell P_\ell(x)\,dx$$

$$\boxed{\int_0^1P_{2s+1}(x)\,dx=\begin{cases}\frac{1}{2} & s=0\\ \frac{(-1)^s(2s-1)!!}{(2s+2)!!} & s=1,2,\ldots\end{cases}}\Longrightarrow \ell=2s+1\longrightarrow A_{2s+1}=(4s+3)a^{-2s-1}V\int_0^1P_{2s+1}(x)\,dx\quad(s=1,2,\ldots)$$

$$A_1=\frac{3V}{a}\,\frac{1}{2}=\frac{3V}{2a}$$

$$A_1=\frac{3V}{2a}$$

$$A_{2s+1}=\frac{4s+3}{a^{2s+1}}V\,\frac{(-1)^s(2s-1)!!}{(2s+2)!!}\quad(s=1,2\ldots)$$

$$\Phi(r,\theta)=\sum_{\substack{\ell=0\\ \text{فرد}}}^{\infty}A_\ell r^\ell P_\ell(\cos\theta)\Rightarrow\Phi(r,\theta)=\sum_{s=0}^{\infty}A_{2s+1}r^{2s+1}P_{2s+1}(\cos\theta)=A_1rP_1(\cos\theta)+\sum_{s=1}^{\infty}A_{2s+1}r^{2s+1}P_{2s+1}(\cos\theta)\Longrightarrow$$

$$\Phi(r,\theta)=\frac{3V}{2}\left(\frac{r}{a}\right)\cos\theta+V\sum_{s=1}^{\infty}\frac{(4s+3)(-1)^s(2s-1)!!}{(2s+2)!!}\left(\frac{r}{a}\right)^{2s+1}P_{2s+1}(\cos\theta)$$

برای نقاط داخلی

$$\boxed{\Phi(r,\theta) = \frac{-1}{4\pi} \oint_S \Phi(\vec{x}') \frac{\partial G(\vec{x},\vec{x}')}{\partial n'} da'}$$

$\boxed{\text{OG}}$ Voice

پتانسیل برای نقاط خارج ؟

$$(r>a): \ \Phi(r,\theta) = \sum_{\ell=0}^{\infty} (A_\ell r^{\ell} + B_\ell r^{-\ell-1}) P_\ell(\cos\theta) \longrightarrow \Phi\Big|_{r\to\infty} = 0 \longrightarrow A_\ell = 0$$

$$\Longrightarrow \Phi(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell r^{-\ell-1} P_\ell(\cos\theta) \Longrightarrow \Phi(a,\theta) = V(\theta) = \sum_{\ell=0}^{\infty} B_\ell a^{-\ell-1} P_\ell(\cos\theta) \Longrightarrow$$

$$\int_0^{\pi} V(\theta) P_\ell(\cos\theta) \sin\theta \, d\theta = B_\ell a^{-\ell-1} \frac{2}{2\ell+1} \Longrightarrow B_\ell = \frac{2\ell+1}{2} a^{\ell+1} \int_0^{\pi} V(\theta) P_\ell(\cos\theta) \sin\theta \, d\theta$$

$$\longrightarrow B_\ell = \frac{2\ell+1}{2} a^{\ell+1} V \left[\int_0^{\pi/2} P_\ell(\cos\theta) \sin\theta \, d\theta - \int_{\pi/2}^{\pi} P_\ell(\cos\theta) \sin\theta \, d\theta \right] \longrightarrow$$

$$\longrightarrow B_\ell = \frac{2\ell+1}{2} a^{\ell+1} V \left[\int_0^1 P_\ell(x) dx - \int_{-1}^{0} P_\ell(x) dx \right] \longrightarrow B_\ell = \begin{cases} 0 & \ell \text{ زوج} \\ (2\ell+1) a^{\ell+1} V \int_0^1 P_\ell(x) dx & \ell \text{ فرد} \end{cases}$$

$$\Longrightarrow \overset{\ell \to 2s+1}{B_{2s+1}} = \begin{cases} 0 \\ (4s+3) a^{2s+2} V \int_0^1 P_{2s+1}(x) dx \end{cases} \longrightarrow \begin{cases} (s=0) \ , \ B_1 = \frac{3}{2} a^2 V \\ (s=1,2,\dots) \ , \ B_{2s+1} = (4s+3) a^{2s+2} V \frac{(-1)^s (2s-1)!!}{(2s+2)!!} \end{cases}$$

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell r^{-\ell-1} P_\ell(\cos\theta) = B_1 r^{-2} P_1(\cos\theta) + \sum_{s=1}^{\infty} B_{2s+1} r^{-2s-2} P_{2s+1}(\cos\theta)$$

$$\longrightarrow \Phi(r,\theta) = \frac{3}{2} V \left(\frac{a}{r}\right)^2 \cos\theta + \sum_{s=1}^{\infty} (4s+3) a^{2s+2} V \frac{(-1)^s (2s-1)!!}{(2s+2)!!} r^{-2s-2} P_{2s+1}(\cos\theta)$$

$$\longrightarrow \Phi(r,\theta) = \frac{3}{2} V \left(\frac{a}{r}\right)^2 \cos\theta + V \cdot \sum_{s=1}^{\infty} \frac{(-1)^s (4s+3)(2s-1)!!}{(2s+2)!!} \left(\frac{a}{r}\right)^{2s+2} P_{2s+1}(\cos\theta)$$

$r>a$
خارج

observation Point : $\left.\begin{matrix} r = z > a \\ \theta = 0 \end{matrix}\right\}$ $\Phi(z)\Big|_{r=z} = \frac{3}{2} V \left(\frac{a}{z}\right)^2 + V \sum_{s=1}^{\infty} \frac{(-1)^s (4s+3)(2s-1)!!}{(2s+2)!!} \left(\frac{a}{z}\right)^{2s+2}$

!خارج است

by Green's function solved before!

$$\Phi(z) = V\left(1 - \frac{z^2 - a^2}{z\sqrt{z^2+a^2}}\right)$$

Homework

$$\Phi(r)\Big|_{r=z} = \frac{V}{\sqrt{\pi}} \frac{3}{2} \Gamma\left(\tfrac{1}{2}\right) \left(\frac{a}{r}\right)^2 + \frac{V}{\sqrt{\pi}} \sum_{s=1}^{\infty} \frac{(-1)^s \boxed{(4s+3)} \times 1 \times 3 \times \dots \times (2s-1)}{\boxed{2} \times 4 \times 6 \times \dots \times (2s)(2s+2)} \Gamma\left(\tfrac{1}{2}\right) \left(\frac{a}{r}\right)^{2s+2}$$

$2s+\frac{3}{2}$ 2×2 2×3

$$\longrightarrow = \frac{V}{\sqrt{\pi}} \frac{3}{2} \Gamma\left(\tfrac{1}{2}\right) \left(\frac{a}{r}\right)^2 + \frac{V}{\sqrt{\pi}} \sum_{s=1}^{\infty} (-1)^s \left(2s+\tfrac{3}{2}\right) \underbrace{\boxed{\left(\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdots \frac{2s-1}{2} \Gamma\left(\tfrac{1}{2}\right)\right)}}_{\Gamma(s+\frac{1}{2})} \times \frac{1}{(s+1)!} \left(\frac{a}{r}\right)^{2s+2}$$

$$\Longrightarrow \Phi(r,\theta) = \frac{V}{\sqrt{\pi}} \frac{3}{2} \Gamma\left(\tfrac{1}{2}\right) \left(\frac{a}{r}\right)^2 + \frac{V}{\sqrt{\pi}} (-1)^s \left(2s+\tfrac{3}{2}\right) \frac{\Gamma(s+\frac{1}{2})}{(s+1)!} \left(\frac{a}{r}\right)^{2s+2} \quad (s+1 \to j) \longrightarrow$$

$$\Phi(r,\theta) = \frac{V}{\sqrt{\pi}}\frac{3}{2}\Gamma(\tfrac{1}{2})\left(\frac{a}{r}\right)^2 + \frac{V}{\sqrt{\pi}}\sum_{j=2}^{\infty}(-1)^{j-1}(2j-\tfrac{1}{2})\frac{\Gamma(j-\frac{1}{2})}{j!}\left(\frac{a}{r}\right)^{2j}$$

$$\Longrightarrow \Phi(r) = \frac{V}{\sqrt{\pi}}\sum_{v=1}^{\infty}(-1)^{j-1}\frac{(2j-\frac{1}{2})\Gamma(j-\frac{1}{2})}{j!}\times\left(\frac{a}{r}\right)^{2j} \quad \boxed{z>a}$$

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell^{-\ell-1})P_\ell(\cos\theta)$$

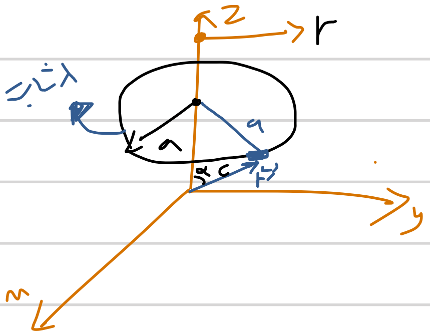
روی محور z (محور تقارن) $\longrightarrow \Phi(r,\theta) = \sum_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-\ell-1})$ → برای هر نقطه دیگر در فضا نقطه در $P_\ell(x)$ ضرب می‌کنیم

$\times P_\ell(\cos\theta)$ → پتانسیل در نقاط از فضا

Voice

---

example بار الکتریکی خطی با چگالی ثابت λ، روی حلقه نارسانا به شعاع a توزیع شده است. پتانسیل را در نقاط فضا بدست آورید!



$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{|\vec{r}'-\vec{r}|}$$

$$\vec{r} = r\hat{a}_z$$

$$\vec{r}' = c\sin\alpha\cos\varphi\,\hat{a}_x + c\sin\alpha\sin\varphi\,\hat{a}_r + c\cos\alpha\,\hat{a}_z$$

$$|\vec{r}'-\vec{r}| = \sqrt{c^2+r^2-2rc\cos\alpha}$$

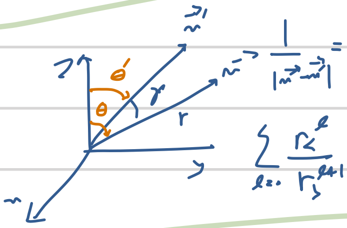
$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\lambda a\,d\phi}{\sqrt{r^2+c^2-2rc\cos\alpha}} = \frac{\lambda a2\pi}{\sqrt{r^2+c^2-2rc\cos\alpha}} = \frac{\frac{1}{4\pi\epsilon_0}q}{\sqrt{r^2+c^2-2rc\cos\alpha}} = \frac{q}{|\vec{r}-\vec{c}|}\frac{1}{4\pi\epsilon_0} \quad \Bigg| \begin{array}{l}\vec{c}=c\vec{a}_r' \\ \vec{r}=r\hat{a}_z\end{array}$$

$$\boxed{\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty}\frac{x_<^{\ell}}{x_>^{\ell+1}}P_\ell(\cos\underset{\text{زاویه بین}}{\alpha})} \longrightarrow = \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos\alpha)$$

$r_<$ و $r_>$ مربوط به کمیت‌ها کوچکتر و بزرگترین بین r و c است!

محاسبه پتانسیل روی محور z

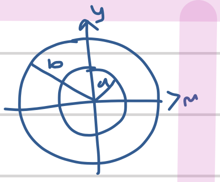
↳ پتانسیل در نقطه از فضا $\longrightarrow \frac{q}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\frac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos\alpha)P_\ell(\cos\theta)$



$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}\frac{r_<^{\ell}}{r_>^{\ell+1}}P_\ell(\cos\gamma)$$

Homework

تالیف گری → دو دایره هم مرکز



دو تا کره هم مرکز به شعاع‌های a و b مفروضند، نیمکره بالای کره داخلی به پتانسیل V
پیوسته رسانا
و نیمکره پایینی آن به زمین متصل است. نیم کره پایینی کره بیرونی
به پتانسیل V+ و نیم کره بالای آن به زمین وصل است. پتانسیل را در نقطه‌ای بین دو کره
بدست آورید.

حل → $(a<r<b) \longrightarrow \Phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + B_l r^{-l-1}\right) P_l(\cos\theta)$

متقارن سمتی

$$V_1(\theta) = \begin{cases} V & 0 \leq \theta < \pi/2 \\ 0 & \pi/2 \leq \theta \leq \pi \end{cases} \quad (R=a)$$

$$V_2(\theta) = \begin{cases} 0 & 0 < \theta < \pi/2 \\ V & \pi/2 < \theta \leq \pi \end{cases} \quad (r=b)$$

$$\Phi(a,\theta) = V_1(\theta) = \sum_{l=0}^{\infty}\left(A_l a^l + B_l a^{-l-1}\right) \times P_l(\cos\theta)$$

$$\Longrightarrow \int_0^{\pi} V_1(\theta) P_l(\cos\theta)\sin\theta\, d\theta =$$

$$\left(A_l a^l + B_l a^{-l-1}\right)\boxed{\int_0^{\pi}\left[P_l(\cos\theta)\right]^2 \sin\theta\, d\theta}$$

$$\frac{2}{2l+1}$$

$$A_l a^l + B_l a^{-l-1} = \frac{2l+1}{2}\boxed{\int_0^{\pi/2} V P_l(\cos\theta)\sin\theta\, d\theta}$$

$$V\int_0^1 P_l(x)\,dx$$

$\longrightarrow$

$$\boxed{\Longrightarrow A_l a^l + B_l a^{-l-1} = \frac{(2l+1)}{2} V \int_0^1 P_l(x)\,dx}$$

$$\int_0^1 P_{2s}(x)\,dx = \begin{cases} 1 & s=0 \\ 0 & s=1,2,.. \end{cases} \qquad \int_0^1 P_{2s+1}(x)\,dx = \begin{cases} \frac{1}{2} & s=0 \\ \frac{(-1)^s (2s-1)!!}{(2s+2)!!} & s=1,2,.. \end{cases}$$

برای کره خارجی ←

$$A_l b^l + B_l b^{-l-1} = \frac{(2l+1)}{2} V \boxed{\int_{\pi/2}^{\pi} P_l(\cos\theta)\sin\theta\, d\theta} \Longrightarrow \boxed{A_l b^l + B_l b^{-l-1} = \frac{(2l+1)}{2} V (-1)^l \int_0^1 P_l(x)\,dx}$$

$$\underbrace{\int_{-1} P_l(x)\,dx}_{x \to -x}$$

در تابع مزدوج
$x \to -x$
$t \to -t$
مجاز است

$$\int_0^1 P_l(-x)\,dx = (-1)^l \int_0^1 P_l(x)\,dx$$

$$l=0 \Longrightarrow \begin{cases} A_0 + B_0 a^{-1} = V/2 \\ A_0 + B_0 b^{-1} = \frac{V}{2} \end{cases} \longrightarrow \begin{cases} A_0 = V/2 \\ B_0 = 0 \end{cases}$$

$$l=1 \Longrightarrow \begin{cases} A_1 a + B_1 a^{-2} = \frac{3V}{4} \\ A_1 b + B_1 b^{-2} = -\frac{3V}{4} \end{cases} \longrightarrow \begin{cases} A_1 = \frac{3V}{4}\,\frac{a^2+b^2}{a^3-b^3} \\ B_1 = \frac{-3V}{4}\,\frac{a^2b^2(a+b)}{a^3-b^3} \end{cases}$$

برای $\to l=2,4,\dots \longrightarrow A_l = B_l = 0$

برای $\longrightarrow l=3,5,7,\dots \Longrightarrow$

$$\begin{cases} A_{2s+1} a^{2s+1} + B_{2s+1} a^{2s-2} = \frac{4s+3}{2} V \int_0^1 P_{2s+1}(x)\,dx \\ A_{2s+1} b^{2s+1} + B_{2s+1} b^{-2s-2} = \frac{4s+3}{2} V (-1)^l \boxed{\int_0^1 P_{2s+1}(x)\,dx} \end{cases} \longrightarrow \begin{cases} A_{2s+1} = \frac{4s+3}{2} V \frac{a^{2s+2} + b^{2s+2}}{a^{4s+3} - b^{4s+3}} \frac{(-1)^s (2s-1)!!}{(2s+2)!!} \quad (s=1,2,..) \\ B_{2s+1} = -\frac{(4s+3)}{2} V \frac{a^{2s+2} b^{2s+2}\left(a^{2s+1} + b^{2s+1}\right)}{a^{4s+3} - b^{4s+3}} \frac{(-1)^s (2s-1)!!}{(2s+2)!!} \end{cases}$$

$\boxed{00}$
Vbke

if $b \longrightarrow \infty$

$$A_1 = A_3 = \dots = 0$$

$$A_0 = \frac{V}{2}, \quad B_1 = \frac{3Va^2}{4}$$

$$B_{2s+1} = \frac{(-1)^s V (4s+3) a^{2s+2} (2s-1)!!}{(2s+2)!!}$$

$b \to \infty$
$\not\Downarrow$
نقاط خارجی
در واقع به
می‌ره

$$\Phi(r,\theta) = \frac{V}{2} + \frac{3Va^2}{4}\left(\frac{a}{r}\right)^2 \cos\theta + V\sum_{s=1}^{\infty}\frac{(-1)^s (4s+3)(2s-1)!!}{(2s+2)!!} \times \left(\frac{a}{r}\right)^{2s+2} P_{2s+1}(\cos\theta)$$

$$a \to 0 \longrightarrow A_0 = V/2 \ , A_1 = \frac{3V}{4b} \ , B_1 = B_3 = B_5 = \cdots = 0 \ , \ A_{2s+1} = \frac{-(4s+3)V}{2b^{2s+1}} \frac{(-1)^s (2s-1)!!}{(2s+2)!!}$$

$$\Phi(r,\theta) = \frac{V}{2} - \frac{3V}{4}\left(\frac{r}{b}\right)\cos\theta - \frac{V}{2}\sum_{s=1}^{\infty} \frac{(-1)^s (4s+3)(2s-1)!!}{(2s+2)!!}\left(\frac{r}{b}\right)^{2s+1} P_{2s+1}(\cos\theta) .$$

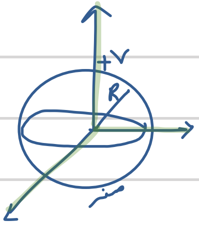
Homework

+V ⟵ نیمکره بالا

0 ⟵ نیمکره پایین

پتانسیل الکتریکی را برای نقاط داخل؟

میدان الکتریکی؟



$$V(\theta) = \begin{cases} V & 0 \le \theta \le \pi/2 \\ 0 & \pi/2 \le \theta \le \pi \end{cases}$$

بسط تابع گرین در دستگاه مختصات کروی

Completeness relation for (هماهنگ‌های کروی)

$$Y_{l,m}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta) e^{im\varphi}$$

$$\int_0^{2\pi}\int_0^{\pi} Y^*_{l'm'}(\theta,\varphi) Y_{l,m} \sin\theta \, d\theta \, d\varphi = \delta_{l,l'} \cdot \delta_{m,m'}$$

$$g(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm} Y_{l,m}(\theta,\varphi) \longrightarrow g(\theta,\varphi) Y^*_{l,m}(\theta,\varphi)\sin\theta \, d\theta \, d\varphi = A_{lm} \underbrace{\int_0^{2\pi}\int_0^{\pi} Y^*_{l,m}(\theta,\varphi) Y_{l,n}(\theta,\varphi) \sin\theta \, d\theta \, d\varphi}_{1} = >$$

$$A_{lm} = \int_0^{2\pi}\int_0^{\pi} g(\theta',\varphi') \cdot Y^*_{lm}(\theta',\varphi') \sin\theta' d\theta' \, d\varphi'$$

$$g(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{-l}^{l} \int_0^{2\pi}\int_0^{\pi} g(\theta',\varphi') Y^*_{l,m}(\theta',\varphi') Y_{l,m}(\theta',\varphi') \sin\theta' d\theta' d\varphi' \, Y_{l,m}(\theta,\varphi)$$

$$\longrightarrow g(\theta,\varphi) = \int_0^{2\pi}\int_0^{\pi} g(\theta',\varphi') \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{l,m}(\theta',\varphi') Y_{l,m}(\theta,\varphi) \sin\theta' d\theta' d\varphi'$$

$$g(\theta,\varphi) = \int_0^{2\pi}\int_{-1}^{1} g(x',\varphi') \boxed{\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{l,m}(x',\varphi') \cdot Y(x,\varphi)} \, dx' d\varphi'$$

$$\delta(\varphi-\varphi')\,\delta(x-x')$$

$$\boxed{\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{l,m}(\theta',\varphi') Y_{l,m}(\theta,\varphi) = \delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta')}$$

رابطه تمامیت برای هماهنگ‌های کروی

بسط تابع گرین در دستگاه مختصات کروی!

$$\nabla^2 G(\vec{x},\vec{x}') = 4\pi\,\delta(\vec{x}-\vec{x}')$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial G}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 G}{\partial\varphi^2} = -4\pi\,\delta(\vec{x}-\vec{x}')$$

$$G(\vec{x},\vec{x}') = G(r,\theta,\varphi,r',\theta',\varphi') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g(r,r',\theta',\varphi') Y_{lm}(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g(r,r') Y^*_{l,m}(\theta',\varphi') Y_{l,m}(\theta,\varphi)$$

$$\delta(\vec{x}-\vec{x}') = \frac{1}{|J|}\delta(\vec{r},\vec{r}')\,\delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta') \Longrightarrow \delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\delta(r-r')\,\delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta')$$

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{l,m}(\theta',\varphi') Y_{l,m}(\theta,\varphi) \Longrightarrow \delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\delta(r-r')\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{l,m}(\theta',\varphi') Y_{l,m}(\theta,\varphi)$$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \longrightarrow |J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial(\cos\theta)} & \frac{\partial x}{\partial\varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial(\cos\theta)} & \frac{\partial y}{\partial\varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial(\cos\theta)} & \frac{\partial z}{\partial\varphi} \end{vmatrix} = r^2$$

$\boxed{60}$

$$\frac{\partial^2 G}{\partial\varphi^2}=-m^2G$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial G}{\partial\theta}\right)+\frac{1}{r^2\sin^2}\left(-m^2G\right)=-4\pi\,\delta(\vec{x}-\vec{x}')$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial G}{\partial\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]G=0$$ لزا بر تقسیم یه فرم

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial G}{\partial\theta}\right)-\frac{m^2G}{\sin^2\theta}=-l(l+1)\,G$$

$$\Longrightarrow\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right)-\frac{1}{r^2}l(l+1)G=-4\pi\,\delta(\vec{x}-\vec{x}')$$

$$\longrightarrow\frac{1}{\cancel{r^2}}\frac{\partial}{\partial r}\left[r^2\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{\partial g(r,r')}{\partial r}Y^*_{l,m}(\theta',\varphi')Y_{l,m}(\theta,\varphi)\right]-\frac{1}{\cancel{r^2}}l(l+1)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}g(r,r')Y^*_{l,m}(\theta',\varphi')Y_{l,m}(\theta,\varphi)$$

$$=-4\pi\frac{1}{\cancel{r^2}}\delta(r-r')\sum_{l=0}^{\infty}\sum_{-l}Y^*_{l,m}(\theta',\varphi')Y_{l,m}(\theta,\varphi)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial g(r,r')}{\partial r}\right)-l(l+1)g(r,r')=-4\pi\,\delta(r-r')$$

$$r^2\frac{\partial^2 g(r,r')}{\partial^2 r}+2r\frac{\partial g(r,r')}{\partial r}-l(l+1)g(r,r')=-4\pi\,\delta(r-r')$$

Cauchy-Euler equation

$$G(\vec{x},\vec{x}')=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}g(\vec{r},\vec{r}')Y^*_{l,m}(\theta',\varphi')Y_{l,m}(\theta,\varphi)$$

+ کوشی اویلر

حالت خاص: سطح مرزی وجود ندارد ؟

$$r^2\frac{\partial^2 g(r,r')}{\partial^2 r}+2r\frac{\partial g(r,r')}{\partial r}-l(l+1)g(r,r')=-4\pi\,\delta(r-r')\Longrightarrow r^2\frac{\partial^2 g}{\partial r^2}+2r\frac{\partial g}{\partial r}-l(l+1)g=0\quad(r\neq r')$$

$$\Longrightarrow g(r,r')=\begin{cases}A_1r^{l}+A_2r^{-l-1} & r<r'\\ A_3r^{l}+A_4r^{-l-1}\end{cases}$$

$g(r,r')\Big|_{r=0}$ باید معین باشد $\longrightarrow A_2=0$

$g(r,r')\Big|_{r\to\infty}=0\longrightarrow A_3=0$

$$\longrightarrow g(r,r')=\begin{cases}A_1r^{l} & r<r'\\ A_4r^{-l-1} & r>r'\end{cases}\qquad\Bigg|\ g(r,r')\Big|_{r=r'^+}=g(r,r')\Big|_{r=r'^-}$$

$$\Longrightarrow\boxed{A_1r'^{l}=A_4r'^{-l-1}}$$

$\boxed{00}$

انتگرال از کرنش اینرو

$$\frac{\partial g}{\partial r}\Big|_{r=r'^+}-\frac{\partial g}{\partial r}\Big|_{r=r'^-}=-\frac{4\pi}{r'^2}\Longrightarrow\boxed{A_4(l-1)r'^{-l-2}-A_1l\,r'^{l-1}=\frac{4\pi}{r'^2}}$$

$$\Longrightarrow\begin{cases}A_1=\frac{4\pi}{2l+1}\frac{1}{r'^{l+1}}\\ A_4=\frac{4\pi}{2l+1}r'^{l}\end{cases}$$

$$g(r,r')=\begin{cases}\frac{4\pi}{2l+1}\frac{1}{r'^{l+1}}r^{l} & r<r'\\ \frac{4\pi}{2l+1}r'^{l}\frac{1}{r^{l+1}} & r>r'\end{cases}\longrightarrow g(r,r')=\frac{4\pi}{2l+1}\frac{r_<^{l}}{r_>^{l+1}}$$

$$G(\vec{r},\vec{r}')=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} g(r,r')\, Y^{*}_{l,m}(\theta',\varphi')\, Y_{l,m}(\theta,\varphi)$$

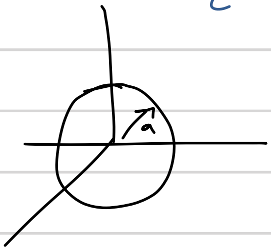
$$\begin{cases} G(\vec{r},\vec{r}')=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^{l}}{r_>^{l+1}}\, Y^{*}_{l,m}(\theta',\varphi')\, Y_{l,m}(\theta,\varphi) \\ G(\vec{r},\vec{r}')=\frac{1}{|\vec{r}-\vec{r}'|}=\sum_{l=0}^{\infty}\frac{r_<^{l}}{r_>^{l+1}}P_l(\cos\theta) \end{cases} \Longrightarrow \boxed{P_l(\cos\theta)=\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y^{*}_{l,m}(\theta',\varphi')\,Y_{l,m}(\theta,\varphi)}$$

- قضیه جمع هماهنگ‌های کروی
- قضیه جمع لژاندر

محاسبه تابع گرین شرط مرزی دیریکله برای معادله لاپلاس و پواسون، برای کره ای به شعاع a :

$$g(r,r')=\begin{cases} A_1 r^{l}+A_2 r^{-l-1} & r<r' \\ A_3 r^{l}+A_4 r^{-l-1} & r>r' \end{cases}$$

برای نقاط خارجی $\hookrightarrow (r,r')>a$



$$g(r,r')\Big|_{r\to\infty}=0\Longrightarrow A_3=0$$

شرط مرزی دیریکله: $g(r,r')\Big|_{r=a}=0\longrightarrow A_1 a^{l}+A_2 a^{-l-1}=0\Longrightarrow A_2=-A_1 a^{2l+1}$

$$g(r,r')=\begin{cases} A_1\left(r^{l}-\frac{a^{2l+1}}{a^{l+1}}\right) & r<r' \\ \frac{A_4}{r^{l+1}}, & r>r' \end{cases}\;\Bigg|\longrightarrow g(r,r')\Big|_{r=r'^{+}}=g\Big|_{r=r'^{-}}\Longrightarrow A\left(r'^{l}-\frac{a^{2l+1}}{r'^{l+1}}\right)=\frac{A_4}{r'^{l+1}}$$

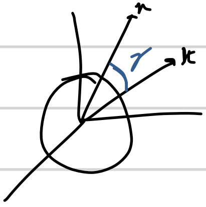
$$\frac{dg(r,r')}{dr}\Big|_{r=r'^{+}}-\frac{dg}{dr}\Big|_{r=r'^{-}}=\frac{4\pi}{r'^{2}}\Longrightarrow A_4(-l-1)\cdot r'^{-l-2}-A_1\left(l\cdot r'^{l-1}+a^{2l+1}(l+1)r'^{-l-2}\right)=\frac{4\pi}{r'^{2}}$$

$$\longrightarrow\begin{cases} A_1=\frac{4\pi r'^{l}}{r'^{2l+1}(2l+1)} \\ A_4=\frac{4\pi r'^{l}\left(r'^{2l+1}-a^{2l+1}\right)}{r'^{2l+1}(2l+1)} \end{cases}\longrightarrow g(r,r')=\begin{cases}\frac{4\pi}{2l+1}\frac{1}{r'^{l+1}}\left(r^{l}-\frac{a^{2l+1}}{r^{l+1}}\right) & r<r' \\ \frac{4\pi}{2l+1}\left(r'^{l}-\frac{a^{2l+1}}{r'^{l+1}}\right)\frac{1}{r^{l+1}} & r>r'\end{cases}\longrightarrow$$

$$G(\vec{r},\vec{r}')=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\begin{cases}\frac{1}{r'^{l+1}}\left(r^{l}-\frac{a^{2l+1}}{r^{l+1}}\right) & (r<r') \\ \dots\end{cases}\;Y^{*}_{l,m}(\theta',\varphi')\,Y_{l,m}(\theta,\varphi)$$

$$\longrightarrow G(\vec{r},\vec{r}')=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left(\frac{r_<^{l}}{r_>^{l+1}}-\frac{a^{2l+1}}{(rr')^{l+1}}\right)Y^{+}_{l,m}Y_{l,m}$$

$$\longrightarrow G(\vec{r},\vec{r}')=\frac{1}{|\vec{r}-\vec{r}'|}-\frac{a}{r'|\vec{r}-\vec{r}'^{*}|},\quad \vec{r}'^{*}=\frac{a^{2}}{r'^{2}}\vec{r}'$$



$$G(\vec{r},\vec{r}')=\frac{1}{\sqrt{r^{2}+r'^{2}-2rr'\cos\gamma}}-\frac{a}{r'\sqrt{r^{2}+\left(\frac{a^{2}}{r'^{2}}\right)^{2}r'^{2}-2r\frac{a^{2}}{r'^{2}}r'\cos\gamma}}$$

$$\boxed{\infty}\;(r<r')\longrightarrow G(\vec{r},\vec{r}')=\frac{1}{r'\sqrt{1+\frac{r^{2}}{r'^{2}}-\frac{2r}{r'}\cos\gamma}}-\frac{a}{rr'\sqrt{1+\frac{a^{4}}{r^{2}r'^{2}}-\frac{2ra^{2}}{rr'}\cos\gamma}}$$

چون نقاط خارج از کره است

$$\boxed{\frac{a^{2}}{rr'}<1}\longleftarrow rr'>a^{2}\Longleftarrow \begin{matrix} r>a \\ r'>a\end{matrix}$$

☐ بسط لژاندر؟ تابع مولد؟
$r<r'$؟ باید بشه

$$G(\vec{x},\vec{x}')=\frac{1}{r'}\sum_{l=0}^{\infty}P_l(\cos\gamma)\left(\frac{r}{r'}\right)^l-\frac{a}{rr'}\sum_{l=0}^{\infty}P_l(\cos\gamma)\left(\frac{a^2}{rr'}\right)^l \Longrightarrow G(\vec{x},\vec{x}')=\sum_{l=0}^{\infty}\frac{1}{r'^{l+1}}\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right)P_l(\cos\gamma)$$

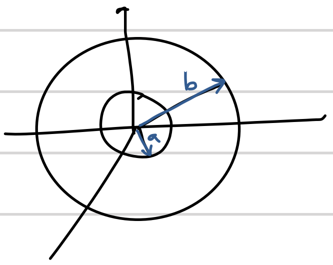
$$G(\vec{x},\vec{x}')=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{1}{r'^{l+1}}\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right)Y^*_{l,m}Y_{lm} \qquad r<r'$$

$$G(\vec{x},\vec{x}')=\sum\sum^{l}\frac{4\pi}{2l+1}\left(r'^l-\frac{a^{2l+1}}{r'^{l+1}}\right)\frac{1}{r^{l+1}}Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi) \qquad r>r'$$

Homework

تابع گرین شرط مرزی دیریکله را برای معادله پواسون و لاپلاس
و برای نقاط بین دو کره متحدالمرکز به شعاع های a و b (a<b)
به عنوان سطوح مرزی بدست آورید.

$a<r<b$
$a<r'<b$



$$g(r,r')=\begin{cases}A_1r^l+A_2r^{-l-1} & r<r'\\ A_3r^l+A_4r^{-l-1} & r>r'\end{cases}$$

$$g(r,r')\Big|_{r=a}=0\longrightarrow A_1a^l+A_2a^{-l-1}=0$$
$$\longrightarrow A_2=-A_1a^{2l+1}$$

$$g(r,r')\Big|_{r=b}=0\longrightarrow A_3=-\frac{A_4}{b^{2l+1}}$$

$$g(r,r')=\begin{cases}A_1\left(r^l-\frac{a^{2l+1}}{r^{l+1}}\right) & r<r'\\ A_4\left(\frac{1}{r^{l+1}}-\frac{r^l}{b^{2l+1}}\right) & r>r'\end{cases}$$

$$\begin{cases}g\Big|_{r=r'^+}=g\Big|_{r=r'^-} & \text{شرط پیوستگی } r=r'\\ \frac{dg}{dr}\Big|_{r=r'^+}-\frac{dg}{dr}\Big|_{r=r'^-}=\frac{-4\pi}{r'^2} & \text{ناپیوستگی}\end{cases}\longrightarrow\begin{cases}A_1=\frac{4\pi}{2l+1}\frac{1}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(\frac{1}{r'^{l+1}}-\frac{r'^l}{b^{2l+1}}\right)\\ A_4=\frac{4\pi}{2l+1}\frac{1}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(r'^l-\frac{a^{2l+1}}{r'^{l+1}}\right)\end{cases}$$

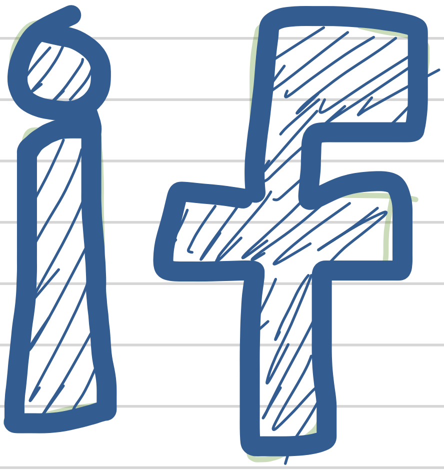
$$g(r,r')=\frac{4\pi}{2l+1}\cdot\frac{1}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(r_<^l-\frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right)$$

$$G(\vec{x},\vec{x}')=\sum_{l=0}^{\infty}\sum_{-l}^{l}\frac{4\pi}{2l+1}\frac{Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi)}{1-\left(\frac{a}{b}\right)^{2l+1}}\left(r_<^l-\frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right)$$

1 → if $a\to 0$ → باید تبدیل شود به تابع گرین نقاط داخل کره ای به شعاع b

2 if $b\to\infty$ → باید تبدیل شود به تابع گرین برای نقاط خارج کره به شعاع a

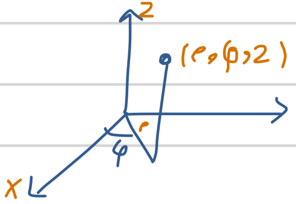
3 if $a\to 0$ & $b\to\infty$ → تابع گرین برای حالت بدون سطح مرزی (سطح مرزی در ∞) بدست می آید.



بسط تابع گرین در دستگاه مختصات استوانه‌ای ←

$$\nabla^2 G(\vec{r},\vec{r}\,') = -4\pi\,\delta(\vec{r}-\vec{r}\,') \Longrightarrow$$

$$\nabla^2 G(\vec{r},\vec{r}\,') = -4\pi\,\frac{1}{\rho}\,\delta(\rho-\rho')\,\delta(\varphi-\varphi')\,\delta(z-z')$$



$$\Longrightarrow \frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial G(\vec{r},\vec{r}\,')}{\partial\rho}\right)+\frac{1}{\rho^2}\,\frac{\partial^2}{\partial\varphi^2}(G)+\frac{\partial^2 G(\vec{r},\vec{r}\,')}{\partial z^2} = -\frac{4\pi}{\rho}\,\delta(\rho-\rho')\,\delta(\varphi-\varphi')\times\delta(z-z') \qquad \boxed{1}$$

$$U_m(\varphi) = \frac{1}{\sqrt{2\pi}}\,e^{im\varphi} \quad , m = 0, \pm 1, \pm 2, \dots$$

$$f(\varphi) = \sum_{m=-\infty}^{+\infty} A_m U_m(\varphi) \Longrightarrow \int_0^{2\pi} f(\varphi)\,U^*_{m'}(\varphi)\,d\varphi = \sum_{m=-\infty}^{+\infty} A_m \underbrace{\int_0^{2\pi} U^*_{m'}(\varphi)\,U_m(\varphi)\,d\varphi}_{\delta_{m,m'}}$$

$$\Longrightarrow A_n = \int_0^{2\pi} f(\varphi)\,U^*_{n}(\varphi)\,d\varphi \Longrightarrow A_m = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} f(\varphi)\,e^{-im\varphi}\,d\varphi$$

$$f(\varphi) = \sum_{m=-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} f(\varphi')\,e^{-im\varphi'}\,d\varphi'\cdot\frac{1}{\sqrt{2\pi}}\,e^{im\varphi} \Longrightarrow f(\varphi) = \int_0^{2\pi} f(\varphi')\,d\varphi'\cdot\underbrace{\boxed{\sum_{m=-\infty}^{+\infty}\frac{1}{2\pi}\,e^{im(\varphi-\varphi')}}}_{\delta(\varphi-\varphi')}$$

$$\Longrightarrow \delta(\varphi-\varphi') = \frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{+\infty} e^{im(\varphi-\varphi')}$$

$$\int_{-\infty}^{+\infty} e^{ikx}\cdot dk = \lim_{L\to\infty}\int_{-L}^{L} e^{ikx}\cdot dk = \lim_{L\to\infty}\left.\frac{e^{ikx}}{ix}\right|_{-L}^{L} = \lim_{L\to\infty}\frac{e^{iLx}-e^{-iLx}}{2i}\,\frac{2\pi}{\pi x} = \lim_{L\to\infty}\frac{\sin Lx}{\pi x}\,2\pi =$$

$$2\pi\,\delta(x)$$

$$\Longrightarrow \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\,dk \Longrightarrow \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\cos kx\,dx \xrightarrow{x\to z-z'} \delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\cos k(z-z')\,dk \Longrightarrow$$

$$\delta(z-z') = \frac{1}{\pi}\int_0^{\infty}\cos k(z-z')\,dk$$

$$\Downarrow$$

$$\delta(\varphi-\varphi')\cdot\delta(z-z') = \frac{1}{2\pi^2}\sum_{m=-\infty}^{+\infty}\int_0^{\infty} e^{im(\varphi-\varphi')}\cos k(z-z')\cdot dk$$

$$\nabla^2 G(\vec{r},\vec{r}\,') = -4\pi\,\frac{1}{\rho}\,\delta(\rho-\rho')\cdot\frac{1}{2\pi^2}\sum_{m=-\infty}^{+\infty}\int_0^{\infty} e^{im(\varphi-\varphi')}\cos k(z-z')\,dk \qquad \boxed{2}$$

Compare 1 &2 ⟶ $$\boxed{G(\vec{n},\vec{n}') = \boxed{\frac{1}{2\pi^2}\sum_{m=-\infty}^{+\infty}\int_0^\infty e^{im(\varphi-\varphi')}\cos k(z-z')\,dk}\; g(\rho,\rho')}$$

A

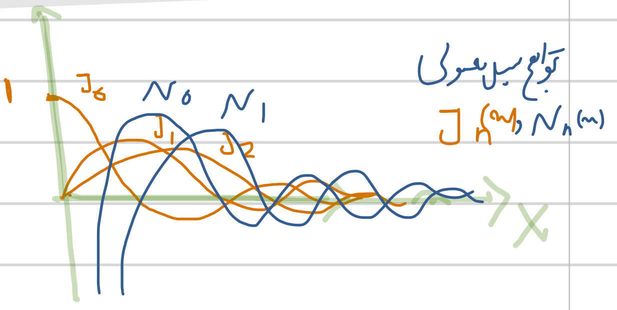
$$\Longrightarrow \frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}\left(\rho\cdot\frac{\partial g(\rho,\rho')}{\partial\rho}\right)A + \frac{1}{\rho^2}(-m^2)A\,g(\rho,\rho') - k^2 A\cdot g(\rho,\rho') = \frac{-4\pi}{\rho}\delta(\rho-\rho')A \Longrightarrow$$

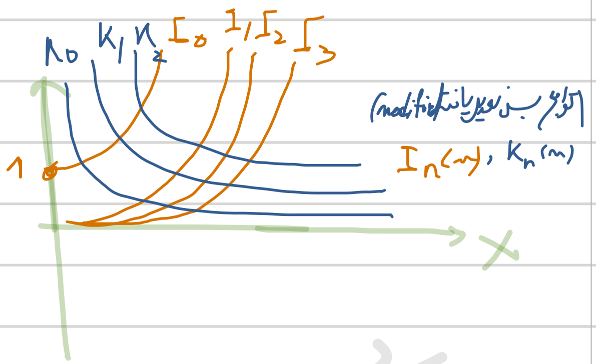
$$\frac{\partial^2 g(\rho,\rho')}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial g(\rho,\rho')}{\partial\rho} - \left(k^2 - \frac{m^2}{\rho^2}\right)g(\rho,\rho') = -\frac{4\pi}{\rho}\delta(\rho-\rho')$$

$$\longrightarrow \rho\neq\rho' \Longrightarrow \frac{\partial^2 g}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial g}{\partial\rho} - \left(k^2+\frac{m^2}{\rho^2}\right)g = 0 \xrightarrow[k\rho=w]{\times\rho^2} \boxed{\frac{\partial^2 g}{\partial w^2} + \frac{1}{w}\frac{\partial g}{\partial w} - \left(1+\frac{m^2}{w^2}\right)g = 0}$$

modified Bessel's equation

$$g(\rho,\rho') = \begin{cases} A_1 \overset{\text{Bessel}}{I_m}(k\rho) + A_2 K_m(k\rho) & (\rho<\rho') \\ A_3 I_m(k\rho) + A_4 K_m(k\rho) & (\rho>\rho') \end{cases}$$





Bessel's

example

محاسبه تابع گرین برای فضای بدون سطح مرزی (سطح مرزی در بی‌نهایت)

$$\Longrightarrow g(\rho,\rho')\Big|_{\rho=0} \text{ باید معین باشه } \Longrightarrow A_2=0$$

$$g(\rho,\rho')\Big|_{\rho\to\infty} = 0 \Longrightarrow A_3 = 0$$

$$\Longrightarrow g(\rho,\rho') = \begin{cases} A_1 I_m(k\rho) & ;(\rho<\rho') \\ A_4 K_m(k\rho) & ;(\rho>\rho') \end{cases}$$

- $g(\rho,\rho')\Big|_{\rho=\rho'^+} = g(\rho,\rho')\Big|_{\rho=\rho'^-} \longrightarrow A_1 I_m(k\rho') = A_4 K_m(k\rho')$
- $\frac{\partial g}{\partial\rho}\Big|_{\rho=\rho'^+} - \frac{\partial g}{\partial\rho}\Big|_{\rho=\rho'^-} = -\frac{4\pi}{\rho'} \longrightarrow A_4 k\frac{dK_m(k\rho)}{d(k\rho)}\Big|_{\rho=\rho'} - Ak\frac{dI_m(k\rho)}{d(k\rho)}\Big|_{\rho=\rho'} = \frac{-4\pi}{\rho'}$

$$A_1 = \frac{\frac{4\pi}{x}K_m(\overset{k\rho'}{x})}{I'_m(x)K_m(x)-I_m(x)K'_m}$$

$$A_4 = \frac{\frac{4\pi}{x}I_m(x)}{I'_m(x)K_m(x)-I_m(x)K'_m(x)}$$

$1/x$

اثبات

Homework

$$\Longrightarrow \begin{cases} A_1 = 4\pi K_m(k\rho') \\ A_4 = 4\pi I_m(k\rho') \end{cases} \longrightarrow g(\rho,\rho') = \begin{cases} 4\pi K_m(k\rho') I_m(k\rho) & (\rho<\rho') \\ 4\pi I_m(k\rho') K_m(k\rho) & (\rho>\rho') \end{cases}$$

$$\Longrightarrow \boxed{\boxed{g(\rho,\rho') = 4\pi I_m(k\rho_<) K_m(k\rho_>)}}$$

$$\Longrightarrow G(\vec{r};\vec{r}\,') = \frac{2}{\pi}\sum_{m=-\infty}^{+\infty}\int_0^\infty e^{im(\varphi-\varphi')}\cos k(z-z') I_m(k\rho_<)K_m(k\rho_>)\,dk$$

برای فضای بدون سطح مرزی

if m :: integer $(m=0, \pm 1, \pm 2, \cdots)$

$I_m(x) = I_{-m}(x)$

$K_m(x) = K_{-m}(x)$

Hint

$\xrightarrow{\quad}$ $\boxed{\text{GO}}$ Voice

$$G(\vec{r},\vec{r}\,') = \frac{2}{\pi}\int_0^\infty \cos k(z-z') I_0(k\rho_<)K_0(k\rho_>)\,dk + \frac{4}{\pi}\int_0^\infty \cos k(z-z')\,dk \cdot \sum_{m=1}^{\infty} \cos m(\varphi-\varphi') I_m(k\rho_<) \times K_m(k\rho_>)$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} \Longrightarrow \rho' = 0 \quad z'=0 \quad r'=0 \longrightarrow \frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{|\vec{r}|} = \frac{1}{\sqrt{\rho^2+z^2}}$$

$\hookrightarrow \rho_< = 0$

$\rho_> = \rho$

$$= \frac{2}{\pi}\int_0^\infty \cos kz\, I_0(0) K_0(k\rho) \times dk$$

زیرا $\boxed{I_0(0)=1}$ $\boxed{I_m(0)=0}$

$$\rho \longrightarrow \sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')}$$

$$\frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+z^2}} = \frac{2}{\pi}\int_0^\infty \cos kz.\, K_0\left(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')}\right) dk \quad ①$$

$$\left.\frac{1}{|\vec{r}-\vec{r}\,'|}\right|_{z'=0}$$

$$\left.\begin{aligned}\vec{r}\,' &= \rho'\hat{\rho}' + z'\hat{z} \\ \vec{r} &= \rho\hat{\rho} + z\hat{z}\end{aligned}\right\} \longrightarrow \vec{r}-\vec{r}\,' = \rho\hat{\rho} - \rho'\hat{\rho}' + (z-z')\hat{z}$$

$$z=0 \longrightarrow \frac{1}{\rho} = \frac{2}{\pi}\int_0^\pi K_0(k\rho)\,dk$$

$$\Longrightarrow \int_0^\infty K_0(k\rho)\,dk = \frac{\pi}{2\rho}$$

$$\int_0^\infty K_0(x)\frac{dx}{\rho} = \frac{\pi}{2\rho} \Longrightarrow \boxed{\int_0^\infty K_0(x)\,dx = \frac{\pi}{2}}$$

$z'=0 \longrightarrow$

$$\frac{1}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+z^2}} = 2\pi\int_0^\infty \cos kz\, I_0(k\rho_<)K_0(k\rho_>)\,dk + \frac{2}{\pi}\int_0^\infty \cos kz\,dk \cdot 2\sum_{m=1}^{\infty}\cos m(\varphi-\varphi') I_m(k\rho_<)K_m(k\rho_>) \quad ②$$

$$(1,2) \longrightarrow K_0\left(k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')}\right) = I_0(k\rho_<)K_0(k\rho_>) + 2\sum_{m=1}^{\infty}\cos m(\varphi-\varphi') I_m(k\rho_<)K_m(k\rho_>)$$

$$k \to 0 \qquad -\ln\frac{k\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')}}{2} = -\ln\frac{k\rho_>}{2} + 2\sum_{m=1}^{\infty}\cos m(\varphi-\varphi')\frac{1}{m!}\left(\frac{k\rho_<}{2}\right)^m \frac{(m-1)!}{2}\left(\frac{2}{k\rho_>}\right)^m \quad -!$$

$$-\ln\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')} = -\ln\rho_> + 2\sum_{m=1}^{\infty}\cos m(\varphi-\varphi')\,\frac{1}{2m}\left(\frac{\rho_<}{\rho_>}\right)^m \Longrightarrow$$

$$\boxed{\ln\frac{1}{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')} = 2\ln\frac{1}{\rho_>}+2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos m(\varphi-\varphi')}$$

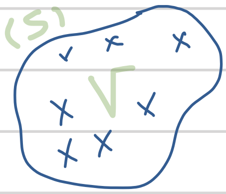
با کمی تمرین برای مختصات قطبی در دو بعد

→ ثابت کنید
تمرین مهمیه!

Homework

$$\nabla^2\psi(\vec r)+[f(\vec r)+\lambda]\,\psi(\vec r)=0 \longleftarrow$$

elliptic Differential equation



λ_n — باید مقدار داشته باشد تا شرط مرزی برآورده شود و سپس بتوان برای یک حجم حل شود $\boxed{\infty}$

$\lambda_n \rightarrow$ ویژه مقدار

$\psi_n \rightarrow$ ویژه تابع

$$\nabla^2\psi_n(\vec r)+[f(\vec r)+\lambda_n]\,\psi_n(\vec r)=0$$

$$\nabla^2 G(\vec r,\vec r')+[f(\vec r)+\lambda]\,G(\vec r,\vec r')=-4\pi\,\delta(\vec r-\vec r') \quad ②$$

$$G(\vec r,\vec r')=\sum_n \psi_n(\vec r)\,a_n(\vec r') \quad ③$$

$\Longrightarrow$ ③ into ②

$$\sum_n a_n(\vec r')\nabla^2\psi_n(\vec r)+[f(\vec r)+\lambda]\sum_n a_n(\vec r')\cdot\psi_n(\vec r)=-4\pi\,\delta(\vec r-\vec r')$$

$$\longrightarrow \sum_n a_n(\vec r')\Big[\underbrace{\nabla^2\psi_n(\vec r)+[f(\vec r)+\lambda_n]\psi_n(\vec r)}_{\text{zero}}\Big]+\sum_n(\lambda-\lambda_n)\,a_n(\vec r')\psi_n(\vec r)=-4\pi\,\delta(\vec r-\vec r')$$

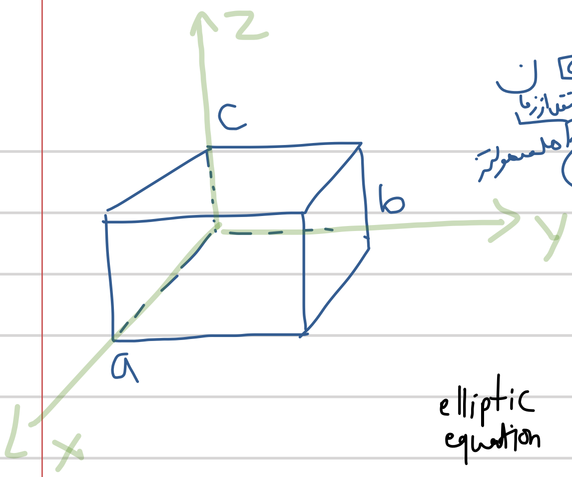
$$\longrightarrow \sum_n(\lambda-\lambda_n)\,a_n(\vec r')\,\psi_n(\vec r)=-4\pi\,\delta(\vec r-\vec r')$$

$$\sum_n(\lambda-\lambda_n)\,a_n(\vec r')\underbrace{\int_V\psi_n(\vec r)\cdot\psi_m^*(\vec r)\,d^3\vec r}_{\delta_{m,n}}=-4\pi\int_V\delta(\vec r-\vec r')\,\psi_m^*(\vec r)\,d^3\vec r$$

توابع ویژه $\psi_n(\vec r)$ را راست به هنجار در نظر می‌گیریم

$$\Longrightarrow(\lambda-\lambda_n)\,a_n(\vec r')=-4\pi\,\psi_n^*(\vec r')\Longrightarrow a_n(\vec r')=\frac{4\pi\,\psi_n^*(\vec r')}{\lambda_n-\lambda}$$

$$\Longrightarrow \boxed{G(\vec r,\vec r')=\sum_n a_n(\vec r')\,\psi_n(\vec r)=\sum_n\frac{4\pi\,\psi_n^*(\vec r')\,\psi_n(\vec r)}{\lambda_n-\lambda}}$$



ن $\boxed{OG}$

محاسبه تابع گرین شرط مرزی دیریکله برای معادله پواسون، لاپلاس (بسط تابع گرین برحسب ویژه توابع معادله موج هلمهولتز مستقل از زمان)

$$\nabla^2 \Phi(\vec{x}) = -\frac{\rho(\vec{x})}{\varepsilon_0}$$

$$\nabla^2 G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$$

$$\text{elliptic equation} \longleftarrow \nabla^2\psi + K^2\psi = 0$$

معادله مرتبط relevent!

$K^2>0$

معادله موج هلمهولتز مستقل از زمان $\xrightarrow{\text{می‌خواهیم}} \psi_n(x)$!?

آیا ψ خوش رفتار

$\boxed{00}$ [بی نهایت نیست و پیوسته باشد]

می‌خواهیم شرط مرزی ارضا شود. باید K^2 مقادیر خاصی اختیار کند.

$$\nabla^2\psi + k^2\psi = 0 \Longrightarrow \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+K^2\psi = 0$$

؛ $\psi = \psi(\vec{x}) = \psi(x,y,z) = X(x)\,Y(y)\,Z(z)$

$$\frac{d^2X}{dx^2}YZ + \frac{d^2Y}{dy^2}XZ + \frac{d^2Z}{dz^2}XY + k^2XYZ = 0 \xrightarrow[XYZ]{\text{تقسیم بر}} \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{-A^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-B^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{-C^2} + k^2 = 0$$

$$K^2 = A^2 + B^2 + C^2$$

$$X(x) = \lambda_1 \sin Ax + \lambda_2 \cos Ax \xrightarrow[\text{داریم } X=0]{\text{برای } x=0} \lambda_2 = 0 \longrightarrow \overset{\text{برای } x=a}{X(a)=0} \longrightarrow \sin Aa = 0 \longrightarrow A = \frac{l\pi}{a} \quad l = 0,\pm1,\dots$$

$$Y(x) = \lambda_3 \sin By + \lambda_4 \cos By \xrightarrow[Y=0]{y=0} \lambda_4 = 0 \longrightarrow Y(b) = 0 \longrightarrow B = \frac{m\pi}{b} \quad m = //$$

$$Z(z) = \lambda_5 \sin Cz + \lambda_6 \cos Cz \xrightarrow[Z=0]{z=0} \lambda_6 = 0 \longrightarrow \underset{\text{با ترکیب}}{Z(c) = 0} \longrightarrow C = \frac{n\pi}{c}, \; n = //$$

$$\psi(\vec{x}) = \psi(x,y,z) = \lambda \sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c} \quad (l,m,n = 1,2,\dots)$$

$\boxed{00}$

$$\int_V \psi^*(\vec{x})\,\psi(\vec{x})\,d^3\vec{x} = 1 \Longrightarrow \int_0^c\!\!\int_0^b\!\!\int_0^a \lambda^2 \sin^2\frac{l\pi x}{a}\sin^2\frac{m\pi y}{b}\sin^2\frac{n\pi z}{c}\,dx\,dy\,dz = 1$$

V حجم مکعب

$$\Longrightarrow \lambda^2\frac{abc}{8} = 1 \longrightarrow \lambda = \sqrt{\frac{8}{abc}}$$

$$\psi_{l,m,n}(\vec{x}) = \psi_{l,m,n}(x,y,z) = \sqrt{\frac{8}{abc}}\sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}$$

eignfunction

$$G(\vec{x},\vec{x}') = \sum_{l,m,n} A_{l,m,n}\,\psi_{l,m,n}(x,y,z) \longrightarrow \sum_{l,m,n} A_{l,m,n}\,\nabla^2\psi_{l,m,n}(x,y,z) = -4\pi\,\delta(\vec{x}-\vec{x}')$$

$$\Longrightarrow \sum_{l,m,n} A_{l,m,n}\left(-K^2\psi_{l,m,n}(\vec{x})\right) = -4\pi\,\delta(\vec{x}-\vec{x}')$$

Next Page

$$\longrightarrow \sum_{l,m,n} A_{l,m,n}\left(-K^2\right)\int_V \psi_{l,m,n}(\vec{r})\,\psi^*_{l',m',n'}(\vec{r})\,d^3\vec{r} = -4\pi\int_V \delta(\vec{r}-\vec{r}\,')\,\psi^*_{l',m',n'}(\vec{r})\,d^3\vec{r}$$

$$\Longrightarrow -K^2 A_{l,m,n} = -4\pi\,\psi^*_{l,m,n}(\vec{r}\,') \Longrightarrow A_{l,m,n} = \frac{4\pi}{K^2}\,\psi^*_{l,m,n}(\vec{r}\,')$$

$$K^2 = \pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)$$

$$G(\vec{r},\vec{r}\,') = \sum \frac{4\pi}{K^2}\,\psi^*_{l,m,n}(\vec{r}\,')\,\psi_{l,m,n}(\vec{r}) \Longrightarrow$$

$$G(\vec{r},\vec{r}\,') = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4\pi}{\pi^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)} \times \frac{8}{abc}$$

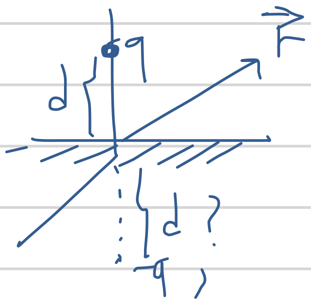
$$\times \sin\frac{m\pi y'}{b}\,\sin\frac{n\pi z'}{c}\,\sin\frac{l\pi x'}{a}$$

$$\times \sin\frac{m\pi y}{b}\,\sin\frac{n\pi z}{c}\,\sin\frac{l\pi x}{a}$$

$$\Longrightarrow \boxed{G(\vec{r},\vec{r}\,') = G(x,y,z,x',y',z') = \frac{3}{\pi abc}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin\frac{l\pi x'}{a}\,\sin\frac{l\pi x}{a}\,\sin\frac{m\pi y'}{b}\,\sin\frac{m\pi y}{b}\,\sin\frac{n\pi z'}{c}\,\sin\frac{n\pi z}{c}}{\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right) - \lambda}}$$

zero

Homework



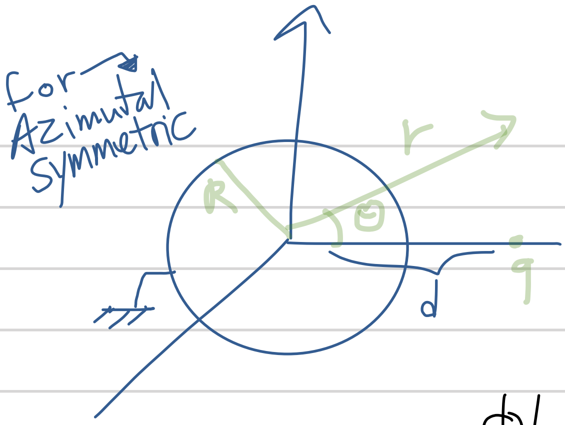
$$\phi(r) = \sum_{l=1}^{\infty}\left(a_l r^l + b_l r^{-l-1}\right)P_l(\cos\theta) + \frac{q}{4\pi\epsilon_0}\,\frac{1}{|\vec{r}-d\hat{a}_z|} \qquad (z>0)$$

$$= \qquad + \qquad \downarrow +$$

$\Longrightarrow -d \to -q$ [illegible]

$\boxed{G_0}$ $\to$

روش تصویر ـــــ پتانسیل برای نقاط خارج



$$\phi(\vec r)=\sum_{n=0}^{\infty} b_n r^{-n-1}P_n(\cos\theta)+\frac{q}{4\pi\varepsilon_0|\vec r-d\cdot\hat a_z|}$$

$$\longrightarrow \phi(\vec r)=\sum_{n=0}^{\infty} b_n r^{-n-1}P_n(\cos\theta)+\frac{q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{r_<^{\,n}}{r_>^{\,n+1}}P_n(\cos\theta)\quad (r>R)$$

$$\phi\Big|_{r=R}=0\quad \sum_{n=0}^{\infty} b_n R^{-n-1}P_n(\cos\theta)+\frac{q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{R^n}{d^{n+1}}P_n(\cos\theta)=0$$

$r\Longleftrightarrow r_<$

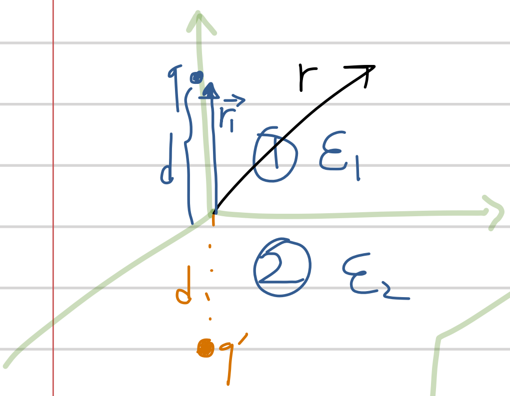
$d\Longleftrightarrow r_>$

$$b_nR^{-n-1}+\frac{q}{4\pi\varepsilon_0}\frac{R^n}{d^{n+1}}=0\longrightarrow b_n=\frac{-q}{4\pi\varepsilon_0}\frac{R^{2n+1}}{d^{n+1}}$$

$$\longrightarrow \phi(\vec r)=\sum_{n=0}^{\infty}\left(\frac{-q}{4\pi\varepsilon_0}\right)\frac{R^{2n+1}}{d^{n+1}}\frac{1}{r^{n+1}}P_n(\cos\theta)+\frac{q}{4\pi\varepsilon_0}\frac{1}{|\vec r-d\hat a_z|}$$

$$\Longrightarrow \phi(\vec r)=\sum_{n=0}^{\infty}\frac{\left(-\frac{R}{d}q\right)}{4\pi\varepsilon_0}\frac{\left(\frac{R^2}{d}\right)^n}{r^{n+1}}P_n(\cos\theta)+\frac{q}{4\pi\varepsilon_0|\vec r-d\hat a_z|}$$

$$\xrightarrow[b=\frac{R^2}{d}]{q'=\frac{-R}{d}q}\phi(\vec r)=\frac{q'}{4\pi\varepsilon_0|\vec r-b\hat a_z|}+\frac{q}{4\pi\varepsilon_0|\vec r-d\hat a_z|}$$



region ① $\longrightarrow z>0$

〃 ② $\longrightarrow z<0$

$$\vec D=\varepsilon_0\vec E+\vec P\Longrightarrow \vec P=(\varepsilon-\varepsilon_0)\vec E$$

$$\rho_p=-\vec\nabla\cdot\vec P$$

$$\sigma_p=\vec P\cdot\hat n\Big|_{z=0}$$

$$\boxed{\phi_1(\vec r)=\frac{q}{4\pi\varepsilon_1}\cdot\frac{1}{|\vec r-\vec r_1|}+\frac{q'}{4\pi\varepsilon_1}\frac{1}{|\vec r-\vec r_2|}}$$

$\vec r=\rho\,\hat a_\rho+z\,\hat a_z$

$\vec r_1=d\cdot\hat a_z$

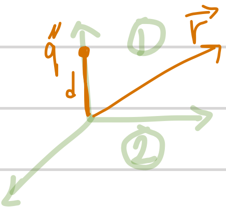
$\longrightarrow \vec r-\vec r_1=\rho\hat a_\rho+(z-d)\hat a_z$

$|\vec r-\vec r_1|=\sqrt{\rho^2+(z-d)^2}$

$\vec r_2=-d\cdot\hat a_z$

$$\phi_1(\vec r)=\frac{q}{4\pi\varepsilon_1\sqrt{\rho^2+(z-d)^2}}+\frac{q'}{4\pi\varepsilon_1\sqrt{\rho^2+(z+d)^2}}$$

& in region 2 $\longrightarrow \phi_2(\vec r)=\frac{q''}{4\pi\varepsilon_2}\cdot\frac{1}{\sqrt{\vec r-\vec r_3}}$



$\vec r=\rho\hat a_\rho+z\hat a_z$

$\vec r_3=d\cdot\hat a_z$

$$\longrightarrow \phi_2(\vec r)=\frac{q''}{4\pi\varepsilon_2\sqrt{\rho^2+(z-d)^2}}\quad (z<0)$$

مرز (z=0) $\longrightarrow E_1t = E_2t \Longrightarrow \phi_1 = \phi_2$ (z=0) مرز $\Longrightarrow \frac{q}{4\pi\varepsilon_1\sqrt{\rho^2+d^2}} + \frac{q'}{4\pi\varepsilon_1\sqrt{\rho^2+d^2}} = \frac{q''}{4\pi\varepsilon_2\sqrt{\rho^2+d^2}} \Longrightarrow \boxed{\frac{q'}{\varepsilon_1} - \frac{q''}{\varepsilon_2} = \frac{-q}{\varepsilon_1}}$

$(\vec{D}_1 - \vec{D}_2)\cdot\hat{n}_{21} = \sigma_{\rho_{free}} \Rightarrow D_{1n} = D_{2n}$ (z=0) $\Rightarrow \varepsilon_1\vec{E}_1\cdot(\hat{a}_z)\Big|_{z=0} = \varepsilon_0\vec{E}_2\cdot\hat{a}_z\Big|_{z=0} \Rightarrow -\varepsilon_1E_{1z} = \varepsilon_2E_{2z}$ (z=0)

zero

$\longrightarrow +\varepsilon_1\frac{\partial\phi_1}{\partial z} = \varepsilon_2 - \frac{\partial\phi_2}{\partial z} \Longrightarrow \boxed{q' + q = q''}$

دو معادله دو مجهول

$$q' = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1+\varepsilon_2}q$$

$$q'' = \frac{2\varepsilon_2}{\varepsilon_1+\varepsilon_2}q$$

Voice
singularity
real
imaginary

$$\phi_1(\vec{r}) = \frac{q}{4\pi\varepsilon_1}\left[\frac{1}{\sqrt{\rho^2+(z-d)^2}} + \frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\cdot\frac{1}{\sqrt{\rho^2+(z+d)^2}}\right] \quad (z>0)$$

$$\phi_2 = \frac{2q}{4\pi(\varepsilon_1+\varepsilon_2)\sqrt{\rho^2+(z-d)^2}} \quad (z<0)$$

$\vec{P}_1 = (\varepsilon_1 - \varepsilon_0)\vec{E}_1$

$\vec{P}_2 = (\varepsilon_2 - \varepsilon_0)\vec{E}_2$

$\vec{E}_1 = -\vec{\nabla}\phi_1 = -\frac{\partial\phi_1}{\partial\rho}\hat{a}_\rho - \frac{\partial\phi_1}{\partial z}\hat{a}_z$

$\vec{E}_2 = -\frac{\partial\phi_2}{\partial\rho}\hat{a}_\rho - \frac{\partial\phi_2}{\partial z}\hat{a}_z$

$\rho_{P_1} = -\vec{\nabla}\cdot\vec{P}_1$, $\rho_{P_2} = -\vec{\nabla}\cdot\vec{P}_2$

$\sigma_{P_1} = \vec{P}_1\cdot\hat{n}\Big|_{z=0} = \vec{P}_1\cdot(-\hat{a}_z)\Big|_{z=0} = -(\varepsilon_1-\varepsilon_0)\hat{E}_1\cdot\hat{a}_z\Big|_{z=0} = -(\varepsilon_1-\varepsilon_0)\cdot\frac{-\partial\phi_1}{\partial z}\Big|_{z=0}$

$\Longrightarrow \sigma_{1p} = \frac{2q\cdot d\cdot\varepsilon_2(\varepsilon_1-\varepsilon_0)}{4\pi\varepsilon_1(\rho^2+d^2)^{\frac{3}{2}}(\varepsilon_1+\varepsilon_2)}$

$\sigma_{2p} = \vec{P}_2\cdot\hat{a}_z \Rightarrow \sigma_{2p} = \frac{-2qd(\varepsilon_2-\varepsilon_0)}{4\pi(\varepsilon_1+\varepsilon_2)(\rho^2+d^2)^{\frac{3}{2}}}$

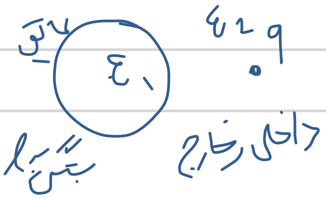
$$\boxed{\Longrightarrow z=0 \rightarrow \sigma_p = \sigma_{1p} + \sigma_{2p} = \frac{qd\varepsilon_0(\varepsilon_1-\varepsilon_0)}{2\pi\varepsilon_1(\varepsilon_1+\varepsilon_2)(\rho^2+d^2)^{\frac{3}{2}}}}$$

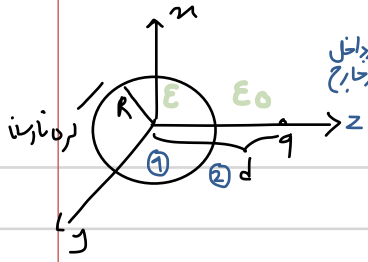
if region ② → empty ☐ $\varepsilon_2 = \infty$
$\varepsilon_1 = \varepsilon_0$

$\longrightarrow \sigma = \frac{-qd}{2\pi(\rho^2+d^2)^{\frac{3}{2}}}$

voice

Next Session





لاپلاس در داخل
و پواسون در خارج
کره سازاست

$$\phi_1(r,\theta)=\sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta)$$

$$\phi_2(r,\theta)=\sum_{l=0}^{\infty} b_l r^{-l-1} P_l(\cos\theta)+\frac{q}{4\pi\varepsilon_0|\vec{r}-d\hat{a}_z|}=\sum_{l=0}^{\infty} b_l r^{-l-1} P_l(\cos\theta)+\frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)$$

$$(r=R)\rightarrow \phi_1=\phi_2\Rightarrow \sum_{l=0}^{\infty} a_l R^l P_l(\cos\theta)=\sum_{l=0}^{\infty} b_l R^{-l-1}\cdot P_l(\cos\theta)+q/_{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{R^l}{d^{l+1}}P_l(\cos\theta)\Rightarrow \boxed{a_l R^l=b_l R^{-l-1}+\frac{q}{4\pi\varepsilon_0}\frac{R^l}{d^{l+1}}}$$

(r< و r> مربوط به d و r است) Hint

$$D_{1n}=D_{2n}\Rightarrow \varepsilon.E_{1r}\Big|_{r<R}=\varepsilon_0 E_{2r}\Big|_{r=R}\Rightarrow -\varepsilon\frac{d\phi_1}{dr}\Big|_{r=R}=-\varepsilon_0\frac{\partial\phi_2}{\partial r}\Big|_{r=R}$$

$$\Rightarrow \varepsilon\sum_{l=0}^{\infty} a_l l R^{l-1}\cdot P_l(\cos\theta)=\varepsilon_0\left[\sum_{l=0}^{\infty} b_l(-l-1)R^{-l-2}P_l(\cos\theta)+\frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{lR^{l-1}}{d^{l+1}}P_l(\cos\theta)\right]\Rightarrow$$

$$\boxed{\varepsilon_0 a_l\cdot l R^{l-1}=\varepsilon_0\left[b_l(-l-1)R^{-l-2}+\frac{1}{4\pi\varepsilon_0}\frac{lR^{l-1}}{d^{l+1}}\right]}$$

یکسان

$$\begin{cases} a_l=\dfrac{q(2l+1)}{4\pi\varepsilon_0 d^{l+1}[l+\frac{\varepsilon_0}{\varepsilon}(l+1)]} \\ b_l=\dfrac{(\frac{\varepsilon_0}{\varepsilon}-1)lR^{2l+1}q}{4\pi\varepsilon_0 d^{l+1}[l+\frac{\varepsilon_0}{\varepsilon}(l+1)]} \end{cases}\Longrightarrow \begin{cases} \phi_1(r,\theta)=\dfrac{q}{4\pi\varepsilon d}\displaystyle\sum_{l=0}^{\infty}\frac{2l+1}{l+\frac{\varepsilon_0}{\varepsilon}(l+1)}\left(\frac{r}{d}\right)^l P_l(\cos\theta) \\ \phi_2(r,\theta)=\dfrac{q}{4\pi\varepsilon_0}\displaystyle\sum_{l=0}^{\infty}\left[\frac{r_<^l}{r_>^{l+1}}+\frac{(\frac{\varepsilon_0}{\varepsilon}-1)l.R^{2l+1}}{l+\frac{\varepsilon_0}{\varepsilon}(l+1)}\cdot\frac{1}{(r.d)^{l+1}}\right]P_l(\cos\theta) \end{cases}$$

if $\varepsilon=\varepsilon_0\Longrightarrow$

$$\phi_1=\frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r^l}{d^{l+1}}P_l(\cos\theta)$$

$$\phi_2=\frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)\Longrightarrow \phi=\frac{q}{4\pi\varepsilon_0|\vec{r}-d\hat{a}_z|}$$

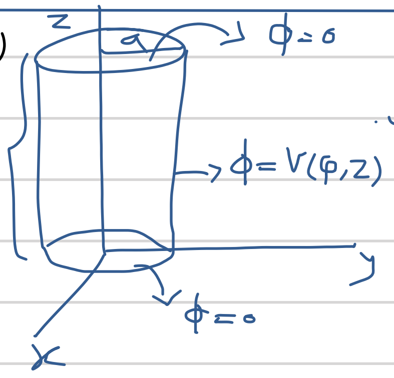
! as we expect

voice

$$\phi_2=\sum_{l=0}^{\infty}\frac{q}{4\pi\varepsilon_0}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)+\sum_{l=0}^{\infty}\boxed{\frac{(\frac{\varepsilon_0}{\varepsilon}-1)l\cdot q.R}{4\pi\varepsilon_0[l+\frac{\varepsilon_0}{\varepsilon}(l+1)]d}}\cdot\frac{(\frac{R^2}{d})^l}{r^{l+1}}$$

Δq?

$$\phi=\phi(\rho,\varphi,z)=R(\rho)Q(\varphi)Z(z)$$



پوسته استوانه‌ای به شعاع a و ارتفاع l، پتانسیل روی قاعده بالا و قاعده پایین صفر است. درحالیکه سطح جانبی دارای پتانسیل V(ρ,φ) است. پتانسیل را در هر نقطه داخل به دست آورید.

$$\nabla^2\phi=0$$

$$\frac{d^2Z}{dz^2}-k^2Z=0\Rightarrow Z(z)=\lambda_1\sinh kz+\lambda_2\cosh kz$$

$$\frac{d^2Q}{d\varphi^2}+\nu^2Q=0\Rightarrow Q(\varphi)=\lambda_3\sin\nu\varphi+\lambda_4\cos\nu\varphi$$

$$\rho^2\frac{d^2R}{d\rho^2}+\rho\frac{dR}{d\rho}+(k^2\rho^2-\nu^2)R=0\Rightarrow R(\rho)=\lambda_5 J_\nu(k\rho)+\lambda_6 N_\nu(k\rho)$$

Wrong

$$\phi\Big|_{z=0}=0 \Longrightarrow Z\Big|_{z=0}=0 \Longrightarrow \lambda_2=0 \Longrightarrow Z(z)=\lambda_1 \operatorname{Sinh} kz$$

$$\phi\Big|_{z=L}=0 \Longrightarrow Z\Big|_{z=L}=0 \Longrightarrow \lambda_2 \operatorname{Sinh} kL=0 \Longrightarrow kL=0 \Longrightarrow \cancel{k}=0$$

$$\operatorname{Sinh}(ix)=i\operatorname{Sin}x$$

$$x \to -ix \Longrightarrow \operatorname{Sinh}x=i\operatorname{Sin}(-ix) \Longrightarrow \operatorname{Sinh}x=-i\operatorname{Sin}(ix)$$

$$\lambda_1 \operatorname{Sin} kL=0 \Longrightarrow (\lambda_1)\cdot(-i)\operatorname{Sin}(ikL)=0$$

$$\Longrightarrow \lambda_7 \operatorname{Sin}(ikL)=0 \Longrightarrow ikL=n\pi$$

$$K=\frac{-n\pi i}{L} \quad (n=0,\pm1,\pm2,..)$$

$$Z(z)=\lambda_1 \operatorname{Sinh}(kz)=\lambda_1 \operatorname{Sinh}\left(\frac{-n\pi}{L}iz\right)=\lambda_1(-i)\operatorname{Sin}\frac{n\pi}{L}z \Longrightarrow \boxed{\lambda_7 \operatorname{Sin}\frac{n\pi z}{L}}$$

$$\phi \xrightarrow{\text{جواب یکتا دارد}} \nu=m \quad (m=0,\pm1,\pm2,\dots)$$

$$\rho^2\frac{d^2R}{d\rho^2}+\rho\frac{dR}{d\rho}+\left(-\frac{n^2\pi^2}{L^2}\rho^2-m^2\right)R=0$$

$$\Longrightarrow \rho^2\frac{d^2R}{d\rho^2}+\rho\frac{dR}{d\rho}-\left(\frac{n^2\pi^2}{L^2}\rho^2+m^2\right)R=0 \longrightarrow$$ modified Bessel's equation

$$\boxed{R(\rho)=\lambda_8\cdot I_m\left(\frac{n\pi}{L}\rho\right)+\lambda_9 K_m\left(\frac{n\pi}{L}\rho\right)}$$

$$\phi\Big|_{\rho=0} \text{ باید معین باشد} \Longrightarrow R\Big|_{\rho=0} \text{ باید معین باشد} \xRightarrow{\text{Voice}} \lambda_9=0 \Longrightarrow$$

$$Z(z)=\lambda_7 \operatorname{Sin}\frac{n\pi}{L}z$$

$$Q(\varphi)=\lambda_3 \operatorname{Sin}m\varphi+\lambda_4 \operatorname{Cos}m\varphi$$

$$R(\rho)=\lambda_8 I_m\left(\frac{n\pi}{L}\rho\right)$$

Voice

$m=0,1,2,3,\dots$

$n=1,2,3,\dots$

$$\boxed{\phi(\rho,\varphi,z)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\operatorname{Sin}\frac{n\pi z}{L}\, I_m\left(\frac{n\pi}{L}\rho\right)\left(A_{mn}\operatorname{Sin}m\varphi+B_{mn}\operatorname{Cos}m\varphi\right)}$$

$$\Longrightarrow =\sum_{n=1}^{\infty}\operatorname{Sin}\frac{n\pi}{L}z\, I_0\left(\frac{n\pi}{L}\rho\right).B_{0n}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\operatorname{Sin}\frac{n\pi}{L}z\cdot I_m\left(\frac{n\pi}{L}\rho\right)\left(A_{mn}\operatorname{Sin}m\varphi+B_{mn}\operatorname{Cos}m\varphi\right)$$

$$\Longrightarrow \phi\Big|_{\rho=a}=V(\varphi,z) \Longrightarrow V(\varphi,z)=\sum_{n=1}^{\infty}\operatorname{Sin}\frac{n\pi}{L}z.I_0\left(\frac{n\pi}{L}a\right)B_{0n}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\operatorname{Sin}\frac{n\pi}{L}z.I_m\left(\frac{n\pi}{L}a\right)\left(A_{mn}\operatorname{Sin}m\varphi+B_{mn}\operatorname{Cos}m\varphi\right)$$

$$\int_0^{2\pi}\int_0^{L}V(\varphi,z)\operatorname{Sin}\frac{n\pi}{L}z\,dz\,d\varphi=\int_0^{L}\operatorname{Sin}^2\frac{n\pi}{L}z\,dz\int_0^{2\pi}d\varphi\, I_0\left(\frac{n\pi}{L}a\right)B_{0n}+\overset{\boxed{\text{Zero}}}{0}=\frac{L}{2}2\pi I_0\left(\frac{n\pi}{L}a\right)B_{0n} \longleftarrow$$

طرفین ضرب در $\boxed{\operatorname{Sin}\frac{n\pi}{L}z}$ و انتگرال نسبت به z از 0 تا L سپس انتگرال نسبت به φ از 0 تا 2π

$$\Longrightarrow \boxed{B_{0n}=\frac{1}{\pi L I_0\left(\frac{n\pi}{L}a\right)}\int_0^{2\pi}\int_0^{L}V(\varphi,z)\operatorname{Sin}\frac{n\pi}{L}z\,dz\,d\varphi}$$

اینبار

$\times \sin m\varphi . \sin \frac{n\pi}{L} z$

و انتگرال از ٠ تا L و ٠ تا 2π

$$\int_0^L \int_0^{2\pi} V(\varphi,z) \sin\frac{n\pi}{L}z \sin m\varphi \, dz . d\varphi = 0 + \int_0^L \sin^2 \frac{n\pi}{L} z \, dz . I_m(\frac{n\pi}{L}a) A_{mn} \int_0^{2\pi} \sin^2 m\varphi \, d\varphi = \frac{L}{2} I_m(\frac{n\pi}{L}a) . A_{mn} \pi \Rightarrow$$

$$\boxed{A_{mn} = \frac{2}{\pi L I_m(\frac{n\pi}{L}a)} \int_0^{2\pi} \int_0^L V(\varphi,z) \sin\frac{n\pi}{L}z \sin m\varphi \, dz d\varphi}$$

$$\boxed{B_{mn} = \frac{2}{\pi L I_m(\frac{n\pi}{L}a)} \int_0^{2\pi} \int_0^L V(\varphi,z) \sin\frac{n\pi}{L}z \cos m\varphi \, dz d\varphi}$$

اینبار ضرب در $\sin\frac{n\pi}{L}z$ و $\cos m\varphi$ و ... ⟸

if $V(\varphi,z) = V = $ Constant

$$B_{0n} = \frac{V 2\pi}{\pi L I_0(\frac{n\pi}{L}a)} \times \frac{L(1-\cos n\pi)}{n\pi} = \frac{4V}{n\pi I_0(\frac{n\pi}{L}a)} \quad (\text{فرد } n)$$

$$B_{mn} = A_{mn} = 0 \quad \boxed{00}$$

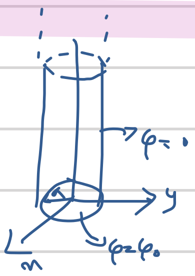
m مربوط به φ

انتظار داریم V که ثابت است جواب به φ مربوط نشه پس بسل درجه بالاتر باید صفر باشه

$$\Longrightarrow \phi(\rho,z) = \sum_{n=1}^{\infty} \frac{4V I_0(\frac{n\pi}{L}\rho)}{n\pi I_0(\frac{n\pi}{L}a)} \sin\frac{n\pi}{L}z$$

$$\boxed{\phi(\rho,z) = \frac{4V}{\pi} \sum_{n=1}^{\infty} \frac{I_0(\frac{n\pi}{L}\rho)}{I_0(\frac{n\pi}{L}a)} \times \frac{\sin\frac{n\pi}{L}z}{n}}$$

Quiz!



$$Z(z) = \lambda_1 e^{-kz} + \lambda_2 e^{+kz}$$

$$Q(\varphi) = \lambda_3 \sin m\varphi + \lambda_4 \cos m\varphi$$

$$R(\rho) = \lambda_5 J_m(k\rho) + \lambda_6 N_m(k\rho)$$

$\phi |_{z\to\infty}$ معین است $\rightarrow \lambda_2 = 0$

$\phi |_{\rho = 0}$ معین است $\rightarrow \lambda_6 = 0$

$$\phi(\rho,\varphi,z) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} e^{-kz} J_m(k\rho) . (A_{mk} \sin m\varphi + B_{mk} \cos m\varphi)$$

$$\rightarrow \phi(\rho,\varphi,z) = \sum_{k=0}^{\infty} e^{-kz} J_0(k\rho) . B_{0k} + \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} e^{-kz} J_m(k\rho)(A_{mk} \sin m\varphi + B_{mk} \cos m\varphi)$$

$$\rightarrow \phi |_{\rho=a} = 0 = \sum_{k=0}^{\infty} e^{-kz} J_0(ka) B_{0k} + \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} e^{-kz} J_m(ka) (A_{mk} \sin m\varphi + B_{mk} \cos m\varphi) = 0$$

$B_{00} = 0$

$A_{mk} = B_{mk} = 0$

$J_0(ka) = 0 \Rightarrow ka = x_{0n} \Rightarrow k = \frac{x_{0n}}{a} = k_n \quad (n = 1, 2, \cdots)$

$$\phi(\rho,\varphi,z)=\sum_{n=1}^{\infty} e^{-K_n z} J_0(K_n\rho)\, B_{0n}$$

$$\phi\Big|_{z=0}=\phi_0 \Rightarrow \phi_0=\sum_{n=1}^{\infty} J_0(K_n\rho).B_{0n} \xrightarrow{\int_0^a \times \rho J_0(K_n\rho)}$$

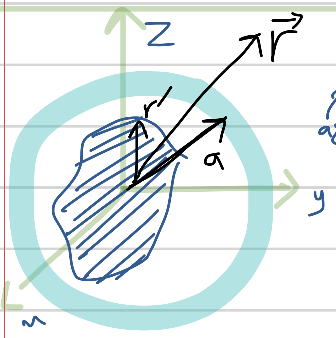
$$\int_0^a \phi_0 \rho J_0(K_n\rho)\, d\rho = \underset{\frac{a^2}{2}(J_1(K_n a))^2}{\boxed{\int_0^a \rho[J_0(K_n\rho)]^2 d\rho}}.B_{0n}$$

$$B_{0n}=\frac{2\phi_0}{a^2[J_1(K_n a)]^2}\int_0^a \rho(J_0(K_n\rho).d\rho \longrightarrow (K_n\rho)=u$$

$$B_{0n}=\frac{2\phi_0}{a^2[J_1(K_n a)]^2}\int_0^{K_n a}\left(\frac{1}{K_n}\right)^2 u J_0(u).du \Rightarrow B_{0n}=\frac{2\phi_0}{a^2[J_1(K_n a)]^2}\cdot\frac{1}{(K_n)^2}[u.J_1(u)]\Big|_0^{K_n a}=\frac{2\phi_0}{a^2[J_1(K_n a)]^2}\frac{a}{K_n}J_1(K_n a) \Longrightarrow$$

$$B_{0n}=\frac{2\phi_0}{a.K_n J_1(K_n a)},$$

$$\phi(\rho,\varphi,z)=\sum_{n=1}^{\infty} e^{-K_n z} J_0(k_n\rho)\, B_{0n} = 2\phi_0\sum_{n=1}^{\infty} e^{-\frac{x_{0n}}{a}} J_0(\ a)\frac{\qquad}{x_{0n}J_1(x_{0n})} \Rightarrow \boxed{\phi(\rho,\varphi,z)=2\phi_0\sum_{n=1}^{\infty}\frac{e^{-\frac{x_{0n}}{a}z}}{x_{0n}J_1(x_{0n})}J_0\left(\frac{x_{0n}}{a}\rho\right)}$$



محاسبه پتانسیل در نقاط خارج از کره به شعاع a

$$\phi(\vec{r})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}')\,d^3r'}{|\vec{r}-\vec{r}'|}$$

$$\frac{1}{|\vec{r}-\vec{r}'|}=\sum_{l=0}^{\infty}\frac{r_<^{l}}{r_>^{l+1}}P_l(\cos\gamma)$$

$$P_l(\cos\gamma)=\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{l,m}^{*}(\theta',\varphi').Y_{l,m}(\theta,\varphi)$$

$$\frac{1}{|\vec{r}-\vec{r}'|}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{l,m}^{*}(\theta',\varphi')Y_{l,m}(\theta,\varphi)\times\frac{r_<^{l}}{r_>^{l+1}}$$

$$\Longrightarrow$$

$$\phi(\vec{r})=\frac{1}{4\pi\varepsilon_0}\int\rho(\vec{r}')\,d^3r'\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{l,m}^{*}(\theta',\varphi')Y_{l,m}(\theta,\varphi).\frac{r_<^{l}}{r_>^{l+1}}$$

$\Longrightarrow$ برای نقاط خارج از کره به شعاع R

$r_< = r'$

$r_> = r$

$$\Rightarrow \phi(\vec{r})=\frac{1}{4\pi\varepsilon_0}\underbrace{\int\rho(\vec{r}')\,d^3r'\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{l,m}^{*}\,Y_{l,m}\frac{r'^{l}}{r^{l+1}}}\Rightarrow q_{l,m}$$

$$q_{l,m}=\int r'^{l}\,Y_{l,m}^{*}(\theta',\varphi')\rho(\vec{r}')\,d^3r' \longrightarrow \text{electric multipole moment}$$

گشتاور چند قطبی الکتریکی

$$\phi(\vec{r})=\frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{Y_{l,m}(\theta,\varphi)}{r^{l+1}}q_{l,m}$$

$$\boxed{q_{l,-m}=(-1)^m q_{l,m}^{*}}$$

$$Y_{l,m}(\theta,\varphi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)e^{im\varphi}$$

$$P_l^m(x)=(-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x)$$

$$P_l^{-m}=(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x)$$

$$Y_{l,-m}=(-1)^m Y_{l,m}^{*}(\theta,\varphi)$$

$$P_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

$l=0$
$m=0$

$$\Rightarrow q_{0,0} = \int \rho(\vec{r}\,') Y^{*}_{0,0}(\theta',\varphi') r'^{0} d^3r' = \int \frac{1}{\sqrt{4\pi}} \rho(\vec{r}\,') d^3r' \Rightarrow q_{0,0} = \frac{1}{\sqrt{4\pi}} q$$

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{4\pi}{2\times 0+1} \frac{Y_{0,0}(\theta,\varphi)}{r^{0+1}} q_{0,0} = \frac{q}{4\pi\varepsilon_0 r}$$

$l=1$
$m=-1,0,1$

$m=0$

$$q_{1,0} = \sqrt{\frac{3}{4\pi}} \int \underbrace{r'\cos\theta'}_{z'} \rho(\vec{r}\,') d^3r' = \sqrt{\frac{3}{4\pi}} \int z' \rho(\vec{r}\,') d^3r' = \sqrt{\frac{3}{4\pi}} \int z' . dq' = \sqrt{\frac{3}{4\pi}} P_z$$

$$\vec{P} = \int \vec{r}\,' \rho(\vec{r}\,') d^3r'$$

دوقطبی الکتریکی

$$\phi_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{4\pi}{2\times1+1} q_{1,0} \frac{Y_{1,0}(\theta,\varphi)}{r^2} \Rightarrow \phi_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0} . \frac{4\pi}{3} \underbrace{\sqrt{\frac{3}{4\pi}} P_z}_{q_{1,0}} \sqrt{\frac{3}{4\pi}} \cos\theta \frac{1}{r^2} \Rightarrow \phi_1(\vec{r}) = \frac{P_z \cos\theta}{4\pi\varepsilon_0 r^2}$$

$m=1$

$$q_{1,1} = \int \rho(\vec{r}\,') r' Y^{*}_{1,1}(\theta',\varphi') d^3r' = \int \rho(\vec{r}\,') d^3r' . r'\left(-\sqrt{\frac{3}{8\pi}} \sin\theta' e^{i\varphi'}\right)^{*} = -\sqrt{\frac{3}{8\pi}} \int r' \sin\theta' (\cos\varphi' - i\sin\varphi') . \rho(\vec{r}\,') . d^3r'$$

$$\Rightarrow q_{1,1} = -\sqrt{\frac{3}{8\pi}} \int (x' - iy') \underbrace{\rho(\vec{r}\,') d^3r'}_{dq'} = -\sqrt{\frac{3}{8\pi}} (P_x - iP_y)$$

$$\Rightarrow \phi_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{4\pi}{2\times1+1} q_{1,1} \frac{Y_{1,1}(\theta,\varphi)}{r^2} = \frac{1}{4\pi\varepsilon_0} \frac{4\pi}{3} \left(-\sqrt{\frac{3}{8\pi}} (P_x + iP_y) . \left(-\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\varphi}\right)\right) . \frac{1}{r^2} \Rightarrow$$

$$\phi_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{1}{2} (P_x - iP_y) \sin\theta (\cos\varphi + i\sin\varphi) \frac{1}{r^2}$$

$$\Rightarrow \phi_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sin\theta \frac{1}{2r^2} \left[P_x \cos\varphi + P_y \sin\varphi + i(P_x \sin\varphi - P_y \cos\varphi)\right]$$

$m=-1$

$$q_{l,-m} = (-1)^m q^{*}_{l,m} \Rightarrow q_{1,-1} = (-1)^{-1} q^{*}_{1,1} \Rightarrow q_{1,1} = \sqrt{\frac{3}{8\pi}} (P_x + iP_y)$$

$$\phi_3(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{4\pi}{2\times1+1} q_{1,-1} \frac{Y_{1,-1}(\theta,\varphi)}{r^2}$$

$$\Rightarrow \phi_3(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sin\theta \frac{1}{2r^2} \left[P_x \cos\varphi + P_y \sin\varphi + i(P_x \sin\varphi + P_y \cos\varphi)\right]$$

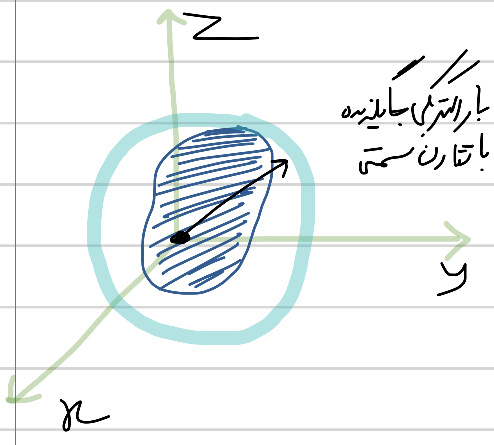
$$\vec{P} = P_x \hat{a}_x + P_y \hat{a}_y + P_z \hat{a}_z$$

$$\phi(\vec{r}) = \frac{\vec{P} . \hat{a}_r}{4\pi\varepsilon_0 r^2}$$

$l=2$

$m=0,\pm1,\pm2$

گشتاورهای چهارقطبی الکتریکی



$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^{l}}{r_>^{l+1}} P_l(\cos\gamma)$$

$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^{l}}{r_>^{l+1}} Y_{l,m}^*(\theta',\varphi') \cdot Y_{l,m}(\theta,\varphi)$$

تقارن سمتی $: m=0 \Rightarrow G(\vec{r},\vec{r}') = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \frac{r_<^{l}}{r_>^{l+1}} Y_{l,0}^*(\theta',\varphi') Y_{l,m}(\theta,\varphi)$

$$\phi(\vec{r}) = \int \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \frac{r_<^{l}}{r_>^{l+1}} \underbrace{Y_{l,0}^*(\theta',\varphi')}_{\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta')} \underbrace{Y_{l,0}(\theta,\varphi)}_{\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)} \rho(\vec{r}') d^3r' \Longrightarrow \phi(\vec{r}) = \frac{1}{2\varepsilon_0} \sum_{l=0}^{\infty} \iint \frac{r_<^{l}}{r_>^{l+1}} P_l(\cos\theta) P_l(\cos\theta') \rho(r',\varphi) r'^2 \sin\theta' dr' d\theta'$$

برای نقاط خارج از ناحیه R $\left| \begin{array}{l} r_< = r' \\ r_> = r \end{array}\right.$

$$\Longrightarrow \boxed{\phi(\vec{r}) = \frac{1}{2\varepsilon_0} \sum_{l=0}^{\infty} \iint \frac{r'^{l}}{r^{l+1}} P_l(\cos\theta) P_l(\cos\theta') \rho(r',\theta') r'^2 \sin\theta' dr' d\theta'}$$

$\rho(\vec{r}) = \frac{1}{64\pi} r^2 e^{-r} \sin^2\theta$

بسط چندقطبی را برای پتانسیل مربوط به توزیع بار الکتریکی حجمی زیر بنویسید.

توزیع بار تقارن سمتی است $\Rightarrow m=0 \Longrightarrow q_{l,0} = \int r'^{l} Y_{l,0}^*(\theta',\varphi') \rho(\vec{r}') d^3r' \Longrightarrow$

$$\boxed{\sqrt{\frac{2l+1}{4\pi}\frac{(l-0)!}{(l+0)!}} P_l^0(\cos\theta) e^{-i(0\times\varphi)} = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)}$$

$$q_{l,0} = \int r'^{l} \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) \rho(r',\theta') r'^2 \sin\theta' dr' d\theta' d\varphi' \Longrightarrow q_{l,0} = \sqrt{(2l+1)\pi} \int_0^{\pi}\int_0^{\infty} \frac{1}{64\pi} e^{-r'} r'^{l+4} \sin^3\theta' P_l(\cos\theta') dr' d\theta'$$

$$\Gamma(z) = \int_0^{\infty} z^{-t} t^{z-1} dt = (z-1)!$$

$$\Longrightarrow q_{l,0} = \sqrt{(2l+1)\pi}\,(l+4)! \int_{-1}^{1} (1-x^2) P_l(x) dx \cdot \frac{1}{64\pi} \Longrightarrow q_{l,0} = \frac{1}{64} \sqrt{\frac{2l+1}{\pi}}\,(l+4)! \int_{-1}^{1} (1-x^2) P_l(x) dx$$

$$\int_{-1}^{1} x^2 P_l(x) P_{l'}(x) dx = \begin{cases} \frac{2(l+1)(l+2)}{(2l+1)(2l+3)(2l+5)} & (l'=l+2) \\ \frac{2(2l^2+2l-1)}{(2l-1)(2l+1)(2l+3)} & (l'=l) \end{cases}$$

$$l=0 \Rightarrow \int_{-1}^{1} x^2 P_{l'}(x) dx = \begin{cases} \frac{4}{15} & l'=2 \\ \frac{2}{3} & l'=0 \end{cases}$$

$$l=0 \Rightarrow q_{0,0}=\frac{1}{2\sqrt{\pi}}$$

$$l=2 \Rightarrow q_{2,0}=-3\sqrt{\frac{5}{\pi}}$$

$$\phi(\vec{r})=\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{l,m}\frac{Y_{l,m}(\theta,\varphi)}{r^{l+1}}$$

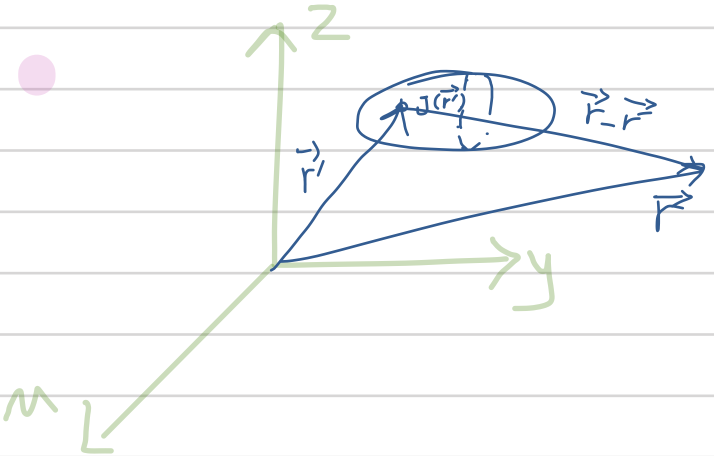
متقارن سمتی $\xrightarrow{}$ $m=0$ $\quad \phi(\vec{r})=\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}q_{l,0}\frac{Y_{l,0}(\theta,\varphi)}{r^{l+1}}$

$$\phi(\vec{r})=\frac{1}{4\pi\epsilon_0}\left[\sqrt{4\pi}\,q_{0,0}\cdot P_0(\cos\theta)\cdot\frac{1}{r}+\sqrt{\frac{4\pi}{5}}\,q_{2,0}\,P_2(\cos\theta)\cdot\frac{1}{r^3}\right]=\frac{1}{4\pi\epsilon_0 r}\left[1-\frac{3(3\cos^2\theta-1)}{r^2}\right]$$

بار الکتریکی با چگالی ثابت در درون کره‌ای به شعاع a
پتانسیل را در خارج از کره با استفاده از بسط چندقطبی های الکتریکی بدست آورید!

$\boxed{Q = \text{کل بار}}$

if $\boxed{\int_0 P_l(x)\,dx}$



$$B(\vec{r})=\frac{\mu_0}{4\pi}\int_V\frac{J(\vec{r}')\times(\vec{r}-\vec{r}')\,dV'}{|\vec{r}-\vec{r}'|^3}$$

$$\vec{E}(\vec{r})=\frac{1}{4\pi\epsilon_0}\int_{V'}\frac{\rho(\vec{r}')\cdot(\vec{r}-\vec{r}')\,dV'}{|\vec{r}-\vec{r}'|^3}=\frac{1}{4\pi\epsilon_0}\int\rho(\vec{r}')\,dV'\,\vec{\nabla}\frac{-1}{|\vec{r}-\vec{r}'|}$$

$$\Longrightarrow \vec{\nabla}\times\vec{E}=\frac{1}{4\pi\epsilon_0}\int_{V'}\rho(\vec{r}')\,dV'\,\underbrace{\nabla\times\nabla\frac{-1}{|\vec{r}-\vec{r}'|}}_{\text{zero}}$$

$$\Rightarrow \boxed{\nabla\times E=0}$$

$$\vec{\nabla}\cdot\vec{E}(\vec{r})=\frac{1}{4\pi\epsilon_0}\int\rho(\vec{r}')\cdot dV'\,\underbrace{\vec{\nabla}\cdot\vec{\nabla}\frac{-1}{|\vec{r}-\vec{r}'|}}_{\nabla^2\frac{-1}{|\vec{r}-\vec{r}'|}}=\frac{1}{4\pi\epsilon_0}\int\rho(\vec{r}')\cdot dV'\,4\pi\delta(\vec{r}-\vec{r}')=\frac{1}{\epsilon_0}\underbrace{\int\rho(\vec{r}')\cdot dV'\,\delta(\vec{r}-\vec{r}')}_{\rho(\vec{r})}\Rightarrow\boxed{\nabla\cdot E=\frac{\rho}{\epsilon_0}}$$

zero
∇ بدون پرایم است

zero

$$\vec{\nabla}\cdot\vec{B}=\frac{\mu_0}{4\pi}\int\vec{\nabla}\cdot\left[J(\vec{r}')\times\vec{\nabla}\frac{-1}{|\vec{r}-\vec{r}'|}\right]dV'=\frac{\mu_0}{4\pi}\int\vec{\nabla}\frac{-1}{|\vec{r}-\vec{r}'|}\cdot(\nabla\times\vec{J}(\vec{r}'))-\frac{\mu_0}{4\pi}\int J\cdot\left(\nabla\times\nabla\frac{-1}{|\vec{r}-\vec{r}'|}\right)dV'\Rightarrow\boxed{\vec{\nabla}\cdot\vec{B}=0}$$

$$\vec{\nabla}\cdot(\vec{F}\times\vec{G})=\vec{G}\cdot\vec{\nabla}\times\vec{F}-\vec{F}\cdot\vec{\nabla}\times\vec{G}$$

$$\vec{\nabla}\times\vec{B}=\frac{\mu_0}{4\pi}\int\vec{\nabla}\times\left(J(\vec{r}')\times\frac{\hat{a}_{\vec{r}-\vec{r}'}}{|\vec{r}-\vec{r}'|^2}\right)dV'$$

Next Page →

$$\vec{\nabla}\times(\vec{A}\times\vec{B})=(\nabla\cdot B)A-(\nabla\cdot A)B+(B\cdot\nabla)A-(A\cdot\nabla)B$$

$$\Rightarrow \vec{\nabla}\times\vec{B}=\frac{\mu_0}{4\pi}\left[\int\left(\vec{\nabla}\cdot\frac{\hat{a}_{\vec{r}-\vec{r}'}}{|\vec{r}-\vec{r}'|^2}\right)J(\vec{r}')\cdot dv-\underbrace{\int(\nabla\cdot J)\frac{\hat{a}_{\vec{r}-\vec{r}'}}{|\vec{r}-\vec{r}'|^2}dv'}_{zero}+\underbrace{\int\left(\frac{\hat{a}_{r..}}{|\vec{r}-\vec{r}'|^2}\cdot\vec{\nabla}\right)J\,dv'}_{zero}-\int(J\cdot\vec{\nabla})\frac{\hat{a}_{\vec{r}-\vec{r}'}}{|r-r'|^2}dv'\right]$$

$$\vec{\nabla}\times\vec{B}=\frac{\mu_0}{4\pi}\int\left(\nabla\cdot\frac{\hat{a}_{\vec{r}-\vec{r}'}}{|\vec{r}-\vec{r}'|^2}\right)\vec{J}(\vec{r}')dv'+\frac{\mu_0}{4\pi}\boxed{\int(\vec{J}(r')\cdot\vec{\nabla}')\frac{\hat{a}}{|r-r'|^2}dv'}$$

- $\vec{\nabla}f(|\vec{r}-\vec{r}'|)=-\vec{\nabla}'f(|\vec{r}-\vec{r}'|)$
- $\oint_s\vec{B}(\vec{A}\cdot d\vec{s})=\int\left[(\vec{\nabla}\cdot\vec{A})\vec{B}+(\vec{A}\cdot\vec{\nabla})\vec{B}\right]dv$

$$\Rightarrow \oint_s\frac{\hat{a}_{\vec{r}-\vec{r}'}}{|\vec{r}-\vec{r}'|^2}(\vec{J}(r')\cdot d\vec{s}')=\int\left[\vec{\nabla}'\cdot\vec{J}(\vec{r}')\right]\frac{\hat{a}}{|\vec{r}-\vec{r}'|^2}dv'+\boxed{\int\left[\vec{J}\cdot\vec{\nabla}'\right]\frac{\hat{a}}{|\vec{r}-\vec{r}'|^2}dv'}$$

$$\Rightarrow \vec{\nabla}\times\vec{B}=\frac{\mu_0}{4\pi}\int\left(\vec{\nabla}\cdot\frac{\hat{a}_{\vec{r}-\vec{r}'}}{|\vec{r}-\vec{r}'|^2}\right)\vec{J}dv'+\underbrace{\frac{\mu_0}{4\pi}\oint_s\frac{\hat{a}}{|\vec{r}-\vec{r}'|^2}(\vec{J}\cdot d\vec{s}')}_{zero}-\underbrace{\frac{\mu_0}{4\pi}\int(\vec{\nabla}'\cdot\vec{J}(\vec{r}'))\frac{\hat{a}}{|\vec{r}-\vec{r}'|^2}dv'}_{zero}$$

$\nabla\cdot\left(\frac{\hat{a}_r}{r^2}\right)=4\pi\delta(r)$

$\nabla^2\left(\frac{1}{r}\right)=-4\pi\delta(r)$

surface: a sphere with $r\to\infty$

Voice

$\nabla\cdot J=-\frac{\partial\rho}{\partial t}$

برای حالت جریان پایا
که میدان مغناطیسی ساکن است
تغییر زمانی نداریم
$\nabla'\cdot\vec{J}(r')=0$

$$\vec{\nabla}\times\vec{B}=\frac{\mu_0}{4\pi}\int 4\pi\delta(\vec{r}-\vec{r}')\vec{J}(r')dv'\Rightarrow\boxed{\nabla\times B=\mu_0\vec{J}}$$ Ok

$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0}\longrightarrow\vec{\nabla}\cdot\vec{D}=\rho_f$ Voice

$\vec{\nabla}\times\vec{E}=0$

$\vec{\nabla}\cdot\vec{B}=0$

$\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$

Maxwell

$$\vec{\nabla}\times\vec{B}=\mu_0(\vec{J}_f+\vec{J}_M)$$
$$=\mu_0(J_f+\vec{\nabla}\times\vec{M})$$
$$\nabla\times\left(\frac{\vec{B}}{\mu_0}-\vec{M}\right)=\vec{J}_f\Rightarrow\vec{\nabla}\times\vec{H}=J_f$$

$\vec{H}=\frac{\vec{B}}{\mu_0}-M$

$\vec{B}=\mu_0(\vec{H}+\vec{M})$

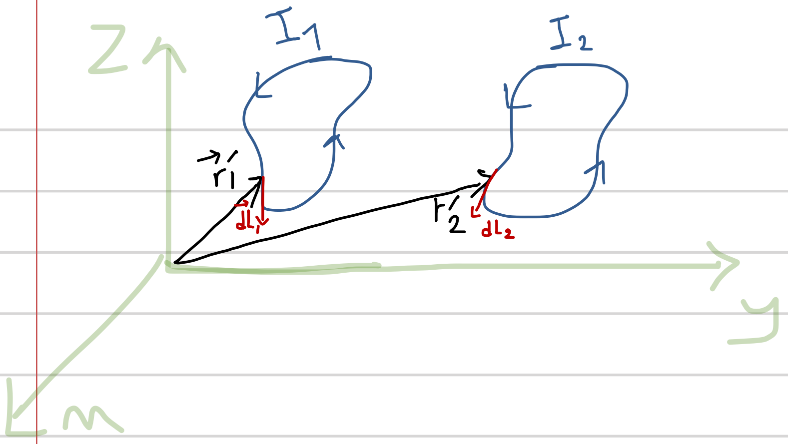
$\vec{M}=\chi_m\vec{H}\to$ خطی همگن همسانگرد

$$\vec{B}=\mu_0(\vec{H}+\chi_m\vec{H})=\underbrace{\mu_0(1+\chi_m)}_{\mu}\vec{H}\Rightarrow\vec{B}=\mu\vec{H}\Rightarrow\boxed{\vec{B}=\mu_0K_m\vec{H}}$$

$\mu=\mu_0K_m$

$K_m=1+\chi_m$

$$\boxed{\chi_m=\begin{cases}+10^5 & \text{پارامغناطیس}\\ -10^{-5} & \text{دیا //}\\ 50-10^5 & \text{فرو //}\end{cases}}$$



$$\vec{B}_1(\vec{r}_2') = \frac{\mu_0 I_1}{4\pi} \oint_1 \frac{d\vec{L}_1 \times (\vec{r}_2' - \vec{r}_1')}{|\vec{r}_2' - \vec{r}_1'|^3}$$

$$d\vec{F}_{21} = I_2 . d\vec{L}_2 \times \vec{B}_1(\vec{r}_2')$$

$$\longrightarrow \vec{F}_{21} = I_2 \oint_2 d\vec{L}_2 \times B_1(\vec{r}_2')$$

$$\Longrightarrow \vec{F}_{21} = I_2 \oint_2 d\vec{L}_2 \times \frac{\mu_0 I_1}{4\pi} \oint_1 d\vec{L}_1 \times \frac{(\vec{r}_2' - \vec{r}_1')}{|\vec{r}_2' - \vec{r}_1'|^3} \Rightarrow$$

$$\vec{A} \quad \vec{B} \quad \vec{C}$$

$$\Longrightarrow \vec{F}_{21} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_2 \oint_1 \frac{dL_2 \times (d\vec{L}_1 \times (\vec{r}_2' - \vec{r}_1'))}{|r_2' - r_1'|^3}$$

$$!! \; \text{3rd law} \longrightarrow \vec{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_1 \oint_2 d\vec{L}_1 \times \frac{[d\vec{L}_2 \times (\vec{r}_1' - \vec{r}_2')]}{|r_1' - r_2'|^3}$$

BAC – CAB ⟶

$$\vec{F}_{21} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_2 \oint_1 dL_1 \left[\frac{dL_2 . (\vec{r}_2' - \vec{r}_1')}{|r_2' - r_1'|^3} \right] - \frac{\mu_0 I_1 I_2}{4\pi} \oint_2 \oint_1 \frac{(d\vec{L}_2 . d\vec{L}_1)(\vec{r}_2' - \vec{r}_1')}{|r_2' - r_1'|^3}$$

$$\frac{\mu_0 I_1 I_2}{4\pi} \oint_1 d\vec{L}_1 \oint_2 d\vec{L}_2 . \vec{\nabla}_2 \frac{-1}{|r_2' - r_1'|} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_1 d\vec{l}_1 \int_{S_2} \vec{\nabla}_2 \times \underbrace{(\vec{\nabla}_2 \frac{-1}{|\vec{r}_2' - \vec{r}_1'|})}_{\text{zero}} . d\vec{s}_2 = 0$$

استوکس

$$\Longrightarrow \vec{F}_{21} = \frac{-\mu_0 I_1 I_2}{4\pi} \oint_2 \oint_1 \frac{(d\vec{L}_2 . d\vec{L}_1)(\vec{r}_2' - \vec{r}_1')}{|\vec{r}_2' - \vec{r}_1'|^3}$$

⟹ Newton's 3rd law is valid **OK**

1 notion [illegible]

$$\vec{\nabla} \times \frac{\vec{J}(r')}{|\vec{r} - \vec{r}'|} = \vec{\nabla} \times \left(\frac{1}{|\vec{r} - \vec{r}'|} \vec{J}(\vec{r}') \right) = \left(\vec{\nabla} \frac{1}{|\vec{r} - \vec{r}'|} \right) \times \vec{J}(\vec{r}') + \frac{1}{|\vec{r} - \vec{r}'|} \underbrace{\vec{\nabla} \times J(\vec{r}')}_{\text{zero}} \Longrightarrow \nabla \times \frac{J(\vec{r}')}{|\vec{r} - \vec{r}'|} = \frac{-(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \times J(\vec{r}') = \boxed{\frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}}$$

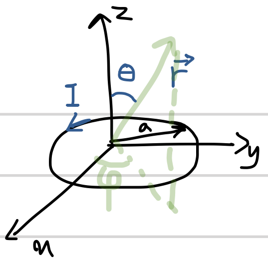
2

$$\vec{B} = \frac{\mu_0}{4\pi} \int \boxed{\frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}} dV$$

$$\Longrightarrow \vec{B} = \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} . dV' \Rightarrow \vec{B} = \vec{\nabla} \times \overset{\vec{A}}{\boxed{\frac{\mu_0}{4\pi} \int \frac{J . dV'}{|\vec{r} - \vec{r}'|}}} \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}') . dV'}{|\vec{r} - \vec{r}'|} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{L}}{|\vec{r} - \vec{r}'|} = \frac{\mu_0}{4\pi} \int \frac{\vec{V} dq}{|\vec{r} - \vec{r}'|}$$

محاسبه $\vec{B}$ و $\vec{A}$
برای یک حلقه جریان



$$\vec{A}=\frac{\mu_0 I}{4\pi}\int\frac{d\vec{L}'}{|\vec{r}-\vec{r}'|}$$

$$d\vec{L}'=a\,d\varphi'\,\hat{a}_{\varphi'}=a(-\sin\varphi'\hat{a}_x+\cos\varphi'\hat{a}_y).d\varphi'$$

$$\vec{r}=r(\sin\theta\cos\varphi\hat{a}_x+\sin\theta\sin\varphi\hat{a}_y+\cos\theta\hat{a}_z)$$

$$\vec{r}'=a(\cos\varphi'\hat{a}_x+\sin\varphi'\hat{a}_y)$$

$$\vec{r}-\vec{r}'=(r\sin\theta\cos\varphi-a\cos\varphi')\hat{a}_x+(r\sin\theta\sin\varphi-a\sin\varphi')\hat{a}_y+r\cos\theta\hat{a}_z$$

$$|\vec{r}-\vec{r}'|=(r^2+a^2-2ar\sin\theta\cos(\varphi-\varphi'))^{1/2}\equiv\alpha$$

$$\vec{A}=\frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{(-\sin\varphi'\hat{a}_x+\cos\varphi'\hat{a}_y).d\varphi'}{\alpha}\xrightarrow[\text{حول z متقارن است}]{(\varphi=0)}\vec{A}=\frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{-\sin\varphi'\hat{a}_x+\cos\varphi'\hat{a}_y}{\sqrt{r^2+a^2-2ar\sin\theta\cos\varphi'}}d\varphi'\Longrightarrow\boxed{\vec{A}=\frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{\cos\varphi'd\varphi'}{\sqrt{r^2+a^2-2ar\sin\theta\cos\varphi'}}\hat{a}_y}$$

$$\vec{J}(r',\theta',\varphi')=\delta(\theta-\underset{\pi/2}{\theta'})\delta(r-a)\hat{a}_\varphi.C$$

$$\int\vec{J}.d\vec{s}=I\Rightarrow\int\delta(\theta-\tfrac{\pi}{2})\delta(r-a).C\hat{a}_\varphi.r\,dr\,d\theta.\hat{a}_\varphi=\int_0^\pi\int_0^\infty\delta(\theta-\tfrac{\pi}{2})\delta(r-a).C.r.dr\,d\theta=I$$

$$C.a=I\Rightarrow\boxed{C=\frac{I}{a}}$$

$$\vec{J}=\frac{I}{a}\delta(\theta-\tfrac{\pi}{2})\delta(r-a)\hat{a}_\varphi$$

or

$$\Longrightarrow\vec{J}=\frac{I}{a}\sin\theta\,\delta(\cos\theta-\cos\tfrac{\pi}{2})\delta(r-a)\hat{a}_\varphi$$

$$\vec{A}=\frac{\mu_0}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{\overbrace{\frac{I}{a}\sin\theta'\delta(\cos\theta')\delta(r'-a)\hat{a}_{\varphi'}}^{\vec{J}(\vec{r}')}\,r'^2\sin\theta'dr'd\theta'd\varphi'}{\sqrt{a^2+r^2-2ar\sin\theta\cos(\varphi-\varphi')}}$$

$$\vec{A}=\frac{\mu_0 I}{4\pi a}\underbrace{\int_0^\pi\sin^2\theta'\delta(\cos\theta')d\theta'}_{1}\int_0^\infty\underbrace{r'^2\delta(r'-a).dr'}_{a^2}\int_0^{2\pi}\frac{\hat{a}_{\varphi'}.d\varphi'}{\sqrt{a^2+r^2-2ar\sin\theta\cos(\varphi-\varphi')}}\Longrightarrow\vec{A}\frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{(-\sin\varphi'\hat{a}_x+\cos\varphi'\hat{a}_y)d\varphi'}{\sqrt{r^2+a^2-2ar\sin\theta\cos(\varphi-\varphi')}}\xrightarrow{\varphi=0}$$

$$\Updownarrow$$

$$\int_{-1}^{1}\delta(x)dx\sqrt{1-x^2}=1$$

$$\vec{A}=\frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{\cos\varphi'd\varphi'}{\sqrt{r^2+a^2-2ar\sin\theta\cos\varphi'}}.\hat{a}_y$$

$$\Longrightarrow\boxed{\vec{A}=\frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{\cos\varphi'd\varphi'}{\sqrt{a^2+r^2-2ar\sin\theta\cos\varphi}}.\hat{a}_\varphi}$$

$$\varphi'=2u\Longrightarrow A_\varphi(r,\theta)=\frac{\mu_0 I a}{4\pi}\int_0^\pi\frac{\cos 2u\times 2du}{\sqrt{a^2+r^2-2ar\sin\theta\cos 2u}}=\frac{2\mu_0 I a}{4\pi}\Big[\int_0^{\pi/2}\frac{\cos 2u\,du}{}+\underbrace{\int_{\pi/2}^{\pi}\frac{}{}}_{u=\pi-v}\Big]\Longrightarrow A_\varphi(r,\theta)=\frac{4\mu_0 I a}{4\pi}\times\int_0^{\pi/2}\frac{\cos 2u\,du}{\sqrt{a^2+r^2-2ar\sin\theta\cos 2u}}$$

$$\Longrightarrow A_\varphi(r,\theta)=\frac{4\mu_0 I a}{4\pi}\int_0^{\pi/2}\frac{(2\cos^2u-1)\,du}{\sqrt{r^2+a^2-2ar\sin\theta\,(2\cos^2u-1)}}=\boxed{\frac{4\mu_0 I a}{4\pi}\times\frac{1}{(r^2+a^2-2ar\sin\theta)^{1/2}}\int_0^{\pi/2}\frac{(2\cos^2u-1)\,du}{\sqrt{1-k^2\cos^2u}}\ ,\ k=\frac{4ar\sin\theta}{r^2+a^2+2ar\sin\theta}}$$

$$\boxed{\frac{2\cos^2u-1}{\sqrt{1-k^2\cos^2u}}=\frac{\alpha}{\sqrt{1-k^2\cos^2u}}+\beta\sqrt{1-k^2\cos^2u}\longrightarrow\begin{cases}\alpha+\beta=-1\\-\beta k^2=2\end{cases}\Longrightarrow\begin{cases}\alpha=\frac{2-k^2}{k^2}\\\beta=\frac{-2}{k^2}\end{cases}}$$

$$A_\varphi(r,\theta)=\frac{a\mu_0 I a}{4\pi\sqrt{a^2+r^2-2ar\sin\theta}}\left[\frac{2-k^2}{k^2}\underbrace{\int_0^{\pi/2}\frac{du}{\sqrt{1-k^2\cos^2u}}}_{K(k)}-\frac{2}{k^2}\underbrace{\int_0^{\pi/2}\sqrt{1-k^2\cos^2u}\,.du}_{E(k)}\right]$$

elliptic Integration first

Second

$$A_\varphi(r,\theta)=\frac{\mu_0}{4\pi}\frac{4Ia}{\sqrt{r^2+a^2+2ar\sin\theta}}\left[\frac{(2-k^2)K(k)-2E(k)}{k^2}\right]$$

$$A_\varphi(r,\theta)=\frac{\mu_0 I a}{4\pi}\int_0^{\pi}\frac{\cos\varphi'.d\varphi'}{\sqrt{r^2+a^2-2ar\sin\theta\cos\varphi'}}=\frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\cos\varphi'.d\varphi'\frac{1}{\sqrt{r^2+a^2}}\left[1-\frac{2ar\sin\theta}{r^2+a^2}.\cos\varphi'\right]^{-1/2}$$

$$\Longrightarrow A_\varphi(r,\theta)=\frac{\mu_0 I a}{4\pi\sqrt{r^2+a^2}}\int_0^{2\pi}\cos\varphi'.d\varphi'\left[1+\left(-\frac{1}{2}\right)\left(\frac{-2ar\sin\theta}{r^2+a^2}\right)\cos\varphi'+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(\frac{2ar\sin\theta}{r^2+a^2}\right)^2\cos^2\varphi'+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(\frac{-2ar\sin\theta}{a^2+r^2}\right)^3.\cos^3\varphi'\ldots\right]$$

$$\boxed{r\gg a\longrightarrow\frac{2ar\sin\theta}{r^2+a^2}\cos\varphi'<1}$$

$$\Longrightarrow A_\varphi(r,\theta)=\frac{\mu_0 I a^2\sin\theta}{4(r^2+a^2)^{3/2}}\left[1+\frac{15}{8}\frac{a^2r^2\sin^2\theta}{(r^2+a^2)^2}+\cdots\right]\Longrightarrow A_\varphi(r,\theta)=\frac{\mu_0 I.a^2.r\sin\theta}{4(r^2+a^2)^{3/2}}\simeq\frac{\mu_0 I.a^2.r\sin\theta}{4r^3}=\frac{\mu_0 I.a^2.\pi\sin\theta}{4\pi r^2}$$

$$\Longrightarrow A_\varphi(r,\theta)=\frac{\mu_0 .m.\sin\theta}{4\pi r^2}$$

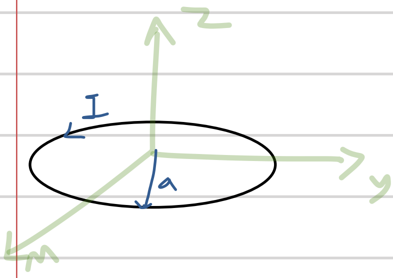
$$\boxed{A_\varphi(r,\theta)=\frac{\mu_0\ \vec{m}\times\hat{a}_r}{4\pi r^2}}\qquad \vec{m}=m\hat{a}_z=\pi a^2.I.\hat{a}_z$$

$$\boxed{\vec{B}=\vec{\nabla}\times\vec{A}=\frac{\mu_0 m}{4\pi r^2}\left(2\cos\theta\,\hat{a}_r+\sin\theta\,\hat{a}_\theta\right)}$$

$A_\varphi(r,\theta)$ برحسب هماهنگ‌های کروی

$$\vec{A}(r,\theta)=\frac{\mu_0 I}{4\pi}\int\frac{a\,d\varphi'\,\hat{a}_{\varphi'}}{|\vec{x}-\vec{x}'|}=\frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{(-\sin\varphi'\,\hat{a}_x+\cos\varphi'\,\hat{a}_y).d\varphi'}{|\vec{x}-\vec{x}'|}$$

$$\vec{A}(r,\theta)=\frac{\mu_0 I a}{4\pi}\int_0^{2\pi}(-\sin\varphi'\,\hat{a}_x+\cos\varphi'\,\hat{a}_y)\,d\varphi'\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\,Y^*_{l,m}(\theta',\varphi')\,Y_{l,m}(\theta,\varphi).\frac{r_<^l}{r_>^{l+1}}$$



$\theta'=\pi/2$

$\varphi=0$

$$Y^*_{l,m}(\underset{\pi/2}{\theta'},\varphi')=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\tfrac{\pi}{2})e^{-im\varphi'}=Y_{l,m}(\tfrac{\pi}{2},0)\,e^{-im\varphi'}$$

$$\vec{A}(r,\theta)=\frac{\mu_0 I a}{4\pi}\int_0^{2\pi}(-\sin\varphi'\,\hat{a}_x+\cos\varphi'\,\hat{a}_y)\,d\varphi'\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\,Y_{l,m}(\tfrac{\pi}{2},0)e^{-im\varphi'}\,Y_{l,m}(\theta,0)$$

$$=\frac{\mu_0 I a}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\,Y_{l,m}(\tfrac{\pi}{2},0)Y_{l,m}(\theta,0)\underbrace{\int_0^{2\pi}(-\sin\varphi'\,\hat{a}_x+\cos\varphi'\,\hat{a}_y)(\cos m\varphi'\ i\sin m\varphi')\,d\varphi'}_{I}$$

I مقدار انتگرال غیر صفر است $m=\pm1$

$$\int_0^{2\pi}\cos mx\cos nx\,dx=\pi\,\delta_{m,n}$$

$m=1\rightarrow I_1=i\pi\hat{a}_x+\pi\hat{a}_y$

$m=-1\rightarrow I_{-1}=i\pi\hat{a}_x+\pi\hat{a}_y$

$$\vec{A}(r,\theta)=\frac{\mu_0 I a}{4\pi}\sum_{l=1}^{\infty}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\left[Y_{l,1}(\tfrac{\pi}{2},0)Y_{l,1}(\theta,0)I_1+Y_{l,-1}(\tfrac{\pi}{2},0)Y_{l,-1}(\theta,0)I_{-1}\right]$$

$$Y_{l,-m}(\theta,\varphi)=(-1)^m.\,Y^*_{l,m}(\theta,\varphi)$$

$$\Rightarrow\left|\begin{array}{l}Y_{l,-1}(\theta,0)=-Y_{l,1}(\theta,0)\\ Y_{l,-1}(\tfrac{\pi}{2},0)=-Y_{l,1}(\tfrac{\pi}{2},0)\end{array}\right.$$

$$\Longrightarrow\vec{A}(r,\theta)=2\mu_0 I.a.\pi\sum_{l=1}^{\infty}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}\,Y_{l,1}(\theta,0).Y_{l,1}(\tfrac{\pi}{2},0)\,\hat{a}_y$$

$$\Longrightarrow A_\varphi(r,\theta)=2\mu_0 I a.\pi\sum_{l=1}^{\infty}\frac{\boxed{Y_{l,1}(\tfrac{\pi}{2},0)}Y_{l,1}(\theta,0)}{2l+1}\frac{r_<^l}{r_>^{l+1}}$$

$$\boxed{\sqrt{\frac{2l+1}{4\pi l(l+1)}}\,P_l^1(\cos\tfrac{\pi}{2}).e^{im\times0}=\sqrt{\frac{2l+1}{4\pi l(l+1)}}\,P_l^1(0)}$$

تابع مولد تابع لژاندر وابسته

$$\sum_{s=0}^{\infty}P_{s+m}^m(x)t^s=\frac{(-1)^m.(2m)!(1-x^2)^{m/2}}{2^m.m!(1-2xt+t^2)^{m+1/2}}$$

$$\begin{matrix} n=0 \\ m=1 \end{matrix} \Rightarrow \sum_{s=0}^{\infty} P_{s+1}^{1}(0)t^{s} = -(1+t^2)^{3/2}$$

$$\sum_{s=0}^{\infty} P_{s+1}^{1}(0)t^{s} = \sum_{s=0}^{\infty} \frac{(-1)^{s+1}(2s+1)!!\,t^{2s}}{2^{s}\cdot s!}$$

$$\Rightarrow \sum_{m=0}^{\infty} P_{2n+1}^{1}(0)t^{2n} + \sum_{n=0}^{\infty} P_{2n+2}^{1}(0)t^{2n+1} = \sum_{s=0}^{\infty} \frac{(-1)^{s+1}(2s+1)!!}{2^{s}\cdot s!}\, t^{2s}$$

$$\boxed{\Rightarrow P_{\ell}^{1}(0) = \begin{cases} 0 & \text{؟(زوج)}\ \ell = 2n \\ (-1)^{n+1}\dfrac{(2n+1)!!}{2^{n}\cdot n!} & \text{؟}\ \ell = 2n+1 \end{cases}}$$

$$\Longrightarrow A_{\varphi}(r,\theta) = -\frac{\mu_0 I a}{4} \sum_{n=0}^{\infty} \frac{(-1)^{n}(2n+1)!!}{2^{n}\cdot n!} \cdot \frac{1}{(2n+1)(n+1)} \cdot \frac{r_<^{2n+1}}{r_>^{2n+2}} P_{2n+1}^{1}(\cos\theta)$$

$$\longrightarrow A_{\varphi}(r,\theta) = \frac{-\mu_0 I a}{4} \sum_{n=0}^{\infty} \frac{(-1)^{n}\cdot(2n-1)!!}{2^{n}\cdot(n+1)!} \frac{r_<^{2n+1}}{r_>^{2n+2}} P_{2n+1}^{1}(\cos\theta)$$

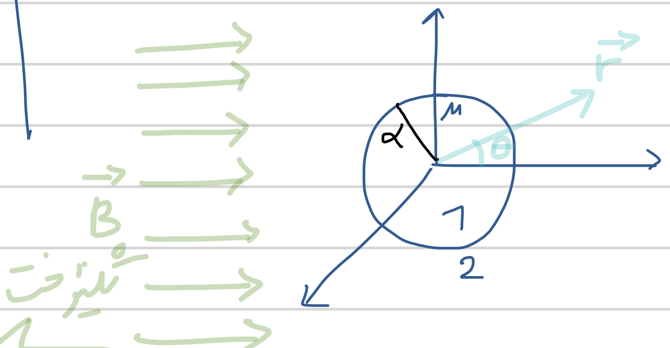
missed class

$$\left.\begin{array}{l}\vec\nabla\times\vec B=\mu_0\vec J\\ \vec J=0\Rightarrow\vec\nabla\times B=0\end{array}\right\}\longrightarrow\vec B=-\mu_0\vec\nabla\Phi_M$$

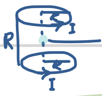
پتانسیل اسکالر مغناطیسی

$$\vec\nabla\cdot\vec B=0\Rightarrow\vec\nabla\cdot(-\mu_0\vec\nabla\phi_M)=0\xrightarrow[\text{برای پتانسیل مغناطیسی}]{\text{معادله لاپلاس}}\boxed{\nabla^2\phi_M=0}$$

کره ای از ماده ای با گذردهی مغناطیسی μ در محیطی که در آنجا میدان مغناطیسی یکنواخت B_0 قرار دارد، قرار گرفته است. میدان مغناطیسی را در داخل و خارج از این کره بدست آورید



ایجاد این میدان توسط پیچه هلمهولتز



در مرکز میدان یکنواخت است

ماده فرومغناطیسی $\vec B=\mu\vec H$ غیرخطی است

$\vec B=\mu_0K_m\vec H$

فرض اولیه اینکه ماده دیامغناطیسی است!

Voice

Let's First Solve For a paramagnetic matter !!

$$\phi_{M1}=\sum_{l=0}^{\infty}(A_lr^l+B_lr^{-l-1})P_l(\cos\theta)\quad ①$$

$$\phi_{M2}=\sum_{l=0}^{\infty}(C_lr^l+D_lr^{-l-1})P_l(\cos\theta)\quad ②$$

$$\phi_{M1}\Big|_{r=0}\ \text{معین است}\Rightarrow B_l=0\Rightarrow\phi_{M1}=\sum_{l=0}^{\infty}A_lr^lP_l(\cos\theta)\quad ③$$

$$\vec B\Big|_{r\to\infty}=B_0\hat a_z=\mu_0\vec H\Big|_{r\to\infty}=B_0\hat a_z\Rightarrow-\mu_0\frac{\phi_2}{\partial z}\Big|_{r\to\infty}=B_0\Rightarrow-\frac{d\phi_{M2}}{dz}\Big|_{r\to\infty}=\frac{B_0}{\mu_0}$$

Hint

$$\mu=\mu_0(1+\chi_m)$$

μ → تراوایی مطلق

χ_m → پذیرفتاری مغناطیسی

$$\chi_m=\begin{cases}-10^{-5} & \text{دیا}\\ +10^{-5} & \text{پارا}\\ 50-10^{5} & \text{فرو}\end{cases}\longrightarrow\mu\begin{cases}\mu_0\times\overbrace{0.99999}^{K_m} & \text{دیا}\\ \mu_0\times\overline{1.00001} & \text{پارامغناطیس}\\ \mu_0\times\overline{(50-10^{5})} & \text{فرو}\end{cases}$$

$$\Rightarrow-\phi_{M2}\Big|_{r\to\infty}=\frac{B_0}{\mu_0}z=\frac{B_0}{\mu_0}r\cos\theta\Rightarrow-\phi_{M2}\Big|_{r\to\infty}=\frac{B_0}{\mu_0}rP_1(\cos\theta)$$

$$\Longrightarrow-\frac{B_0}{\mu_0}rP_1(\cos\theta)=\sum_{l=0}^{\infty}C_lr^lP_l(\cos\theta)\Rightarrow\begin{cases}C_1=-\frac{B_0}{\mu_0}\\ C_0=C_2=C_3=\dots=0\end{cases}\quad ④$$

$$\phi_{M1}=\sum_{l=0}^{\infty}A_lr^lP_l(\cos\theta)\quad ⑤$$

$$\phi_{M2}=-\frac{B_0}{\mu_0}r\cos\theta+\sum_{l=0}^{\infty}D_lr^{-l-1}P_l(\cos\theta)\quad ⑥$$

$$\longrightarrow\phi_{M1}\Big|_{r=a}=\phi_{M2}\Big|_{r=a}\Rightarrow\sum_{l=0}^{\infty}A_la^lP_l(\cos\theta)=-\frac{B_0}{\mu_0}a\cos\theta+\sum_{l=0}^{\infty}D_la^{-l-1}P_l(\cos\theta)\longrightarrow$$

$$A_1a=-\frac{B_0}{\mu_0}a+D_1a^{-2}\quad ⑦$$

$$A_la^l=D_la^{-l-1}\quad(l=0,2,3,4,\dots)\quad ⑧$$

مرزی (r=a) $B_{1n} = B_{2n} \Rightarrow B_{1r}\big|_{r=a} = B_{2r} \Rightarrow \mu H_{1r}\Big|_{r=a} = \mu_0 H_{2r}\Big|_{r=a} \Rightarrow \mu\left(\frac{-\partial \phi_{M1}}{\partial r}\right)\Big|_{r=a} = \mu_0\left(\frac{-\partial \phi_{M2}}{\partial r}\right)\Big|_{r=a}$

$$\Longrightarrow \mu \sum_{\ell=0}^{\infty} A_\ell \cdot \ell \cdot a^{\ell-1} P_\ell(\cos\theta) = \mu_0\left[-\frac{B_0}{\mu_0}\cos\theta + \sum_{\ell=0}^{\infty} D_\ell \cdot (-\ell-1) \cdot a^{-\ell-2} P_\ell(\cos\theta)\right] \quad ⑨$$

$$\ell = 1 \Rightarrow \mu A_1 = -B_0 - 2D_1 \mu_0 a^{-3} \quad ⑩$$

$$⑦ \, \& \, ⑩ \longrightarrow \begin{cases} A_1 = \dfrac{-3B_0}{\mu+2\mu_0} \\ D_1 = \dfrac{B_0 a^3(\mu-\mu_0)}{\mu_0(\mu+2\mu_0)} \end{cases} \quad ⑪$$

$$⑨ \longrightarrow \mu A_\ell \cdot \ell \cdot a^{\ell-1} = D_\ell \cdot (-\ell-1)\, a^{-\ell-2} \mu_0 \quad (\ell = 0, 2, 3, \dots) \quad ⑫$$

$$⑧ \, \& \, ⑫ \Rightarrow \begin{cases} A_0 = A_2 = A_3 = \dots = 0 \\ D_0 = D_2 = \dots = 0 \end{cases} \quad ⑬$$

$$\Longrightarrow \begin{cases} \phi_{M1} = \dfrac{-3B_0}{\mu+2\mu_0} r\cos\theta = \dfrac{3B_0 Z}{\mu+2\mu_0} \\ \phi_{M2} = -\dfrac{B_0}{\mu_0}\left[r - \dfrac{a^3(\mu-\mu_0)}{(\mu+2\mu_0)r^2}\right]\cos\theta \end{cases}$$

$$\vec{B}_1 = \mu \vec{H}_1 \Rightarrow \vec{B}_1 = \mu(-\vec{\nabla}\phi_{M1}) = \mu\left(-\frac{d\phi_{M1}}{dz}\right)\hat{a}_z \Rightarrow \vec{B}_1 = \frac{-3\mu B_0}{\mu+2\mu_0}\hat{a}_z$$

$$\boxed{\text{داخل کره} \longrightarrow \vec{B}_1 = \frac{3\mu}{\mu+2\mu_0}\vec{B}_0} \quad ⑭$$

$$\vec{B}_2 = -\mu_0 \vec{\nabla}\phi_{M2} \Rightarrow \boxed{\vec{B}_2 = \vec{B}_0 - \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\, \hat{a}_r + \sin\theta\, \hat{a}_\theta\right)} \quad ⑮$$

$$\boxed{m = \frac{4\pi a^2 B_0(\mu_0-\mu)}{\mu(\mu+2\mu_0)}}$$

$\vec{M} = \chi_m \vec{H}_1$

$\mu = \mu_0(1+\chi_m)$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} = \mu_0(\vec{J}_f + \vec{J}_M) \Rightarrow \vec{\nabla}\times\vec{B} = \mu_0(\vec{J}_f + \vec{\nabla}\times\vec{M}) \Rightarrow \vec{\nabla}\times\underbrace{\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right)}_{\vec{H}} = \mu_0\vec{J}_f$$

$$\left.\begin{array}{l} \vec{H} = \dfrac{\vec{B}}{\mu_0} - \vec{M} \Rightarrow \vec{B} = \mu_0(\vec{H}+\vec{M}) \\ \vec{M} = \chi_m \vec{H} \end{array}\right\} \Rightarrow \vec{B} = \overbrace{\mu_0\underbrace{(1+\chi_m)}_{K_m}}^{\mu}\vec{H}$$

$$\vec{M} = \frac{\mu-\mu_0}{\mu_0}\vec{H}_1 \Rightarrow \vec{M} = \frac{\mu-\mu_0}{\mu_0}\frac{\vec{B}_1}{\mu} \Rightarrow \vec{M} = \frac{3}{\mu_0}\cdot\frac{\mu-\mu_0}{\mu+2\mu_0}\vec{B}_0 \quad ⑯$$

$$\chi_m = \frac{\mu}{\mu_0} - 1 = \frac{\mu-\mu_0}{\mu_0}$$

$$\vec{H}_1 = \frac{3}{\mu+2\mu_0}\vec{B}_0 \quad ⑰$$

$$\vec{B}_1 = \frac{3\mu}{\mu+2\mu_0}\vec{B}_0 \quad ⑱$$

$$\vec{M}_1 = \frac{3}{\mu_0}\frac{\mu-\mu_0}{\mu+2\mu_0}\vec{B}_0 \quad ⑲$$

یک کره ی

برای جا مغناطیس
یا دارا =

Voice

$$\vec{B}_1-\vec{B}_0 = \frac{2(\mu-\mu_0)}{\mu+2\mu_0}\vec{B}_0 \quad ⑳$$

حذف $\vec{B}_0$ بین 19 و 20 $\Rightarrow \vec{B}_1-\vec{B}_0 = \frac{2\mu_0}{3}\vec{M}_1$ ㉑

$$\vec{H}_1-\vec{H}_0 = \vec{H}_1-\frac{\vec{B}_0}{\mu_0} = \frac{3\vec{B}_0}{\mu+2\mu_0}-\frac{\vec{B}_0}{\mu_0} \Rightarrow$$

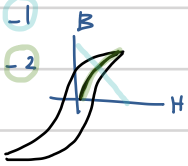
$$\vec{H}_1-\frac{\vec{B}_0}{\mu_0} = \frac{(\mu_0-\mu)}{\mu_0(\mu+2\mu_0)}\vec{B}_0 \quad ㉒$$

حذف $\vec{B}_0$ بین 19 و 22 $\longrightarrow \vec{H}_1-\frac{\vec{B}_0}{\mu_0} = -\frac{1}{3}\vec{M}_1 \Rightarrow \vec{H}_1 = \frac{\vec{B}_0}{\mu_0}-\frac{1}{3}\vec{M}_1$ ㉓

حذف $\vec{M}_1$ بین 21 و 23 $\longrightarrow \boxed{\vec{B}_1+2\mu_0\vec{H}_1 = 3B_0}$ ㉙ → برای کره

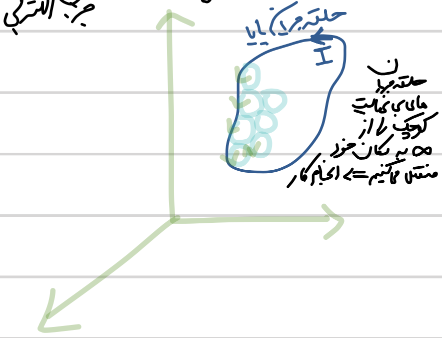
رابطه بین H و B
برای ماده‌ی غیر مغناطیسی را خطی در نظر گرفتیم
در این رابطه μ ندارد!

1- یک معادله خط در صفحه B-H
2- رابطه دیگر بین B₁ و H₁
همان منحنی هیسترزیس است



انرژی مغناطیسی

انرژی حاصل از یک جریان الکتریکی پایا → magnotostatic



حلقه جریان های بی نهایت کوچک را از ∞ به مکان خود منتقل میکنیم => انجام کار

استفاده از مرم
قانون ـ BAC-CAB
مداری آمپر

$$W = (W_{12}+W_{13}+W_{23}+W_{14}+W_{24}+W_{34}+\dots+W_{21}+W_{31}+W_{32}+\dots)/2$$

$$W = \frac{1}{2}\sum_{i,j} W_{ij} \Rightarrow W = \frac{1}{2}\int \vec{F}_{ij}\cdot d\vec{L}_{ij}$$

$$\vec{F}_{ij} = \frac{\mu_0}{4\pi} I_i \cdot I_j \oint\oint \frac{(d\vec{L}_i\cdot d\vec{L}_j)(\vec{r}_i-\vec{r}_j)}{|\vec{r}_i-\vec{r}_j|^3}$$

$$W = \frac{\mu_0}{8\pi} I_i\cdot I_j \oint\oint \int_{\infty}^{|\vec{r}_i-\vec{r}_j|} \frac{(d\vec{L}_i\cdot d\vec{L}_j)(\vec{r}_i-\vec{r}_j)}{|\vec{r}_i-\vec{r}_j|^3}\cdot d(\vec{r}_i-\vec{r}_j)$$

$|\vec{r}_i-\vec{r}_j|\hat{a}_{\vec{r}_i-\vec{r}_j}$

$$\Rightarrow W = \frac{+\mu_0 I_i I_j}{8\pi}\oint\oint \frac{d\vec{L}_i\cdot d\vec{L}_j}{|\vec{r}_i-\vec{r}_j|} \qquad ، \quad I d\vec{L} = \vec{J} dv$$

$$\Rightarrow W = \frac{1}{2}\underbrace{\frac{\mu_0}{4\pi}\iint \frac{\vec{J}_i\cdot\vec{J}_j\, dv_i dv_j}{|\vec{r}_i-\vec{r}_j|}}_{\vec{A}_i} \Rightarrow W = \frac{1}{2}\int \vec{J}_i\cdot\vec{A}_i\cdot dv_i \Rightarrow W = \frac{1}{2}\int \vec{J}\cdot\vec{A}\, dv = \frac{1}{2\mu_0}\int (\vec{\nabla}\times\vec{B})\cdot\vec{A}\, dv \Rightarrow$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \Rightarrow \vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B}$$

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$$W=\frac{1}{2\mu_0}\int \vec{B}\cdot\underbrace{\boxed{(\vec{\nabla}\times\vec{A})}}_{\vec{B}}\,dv-\frac{1}{2\mu_0}\int\vec{\nabla}\cdot(\vec{A}\times\vec{B})\,dv \Rightarrow W=\frac{1}{2\mu_0}\int B^2 dv-\frac{1}{2\mu_0}\oint_S \boxed{(\underset{A\propto\frac{1}{r}}{\vec{A}}\times\underset{B\propto\frac{1}{r^2}}{\vec{B}})\cdot\underset{\propto r^2}{\vec{ds}}} \Longrightarrow W=\frac{1}{2\mu_0}\int B^2\cdot dv$$

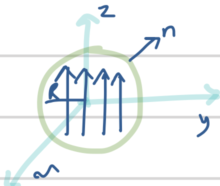
کره بی‌نهایت

zero

$$\boxed{\Rightarrow W=\frac{1}{2}\int\vec{B}\cdot H\,dv}$$

missed

$$\vec{A}=\frac{\mu_0}{4\pi}\int\frac{\vec{\nabla}\times\vec{M}}{|\vec{r}-\vec{r}'|}d^3r'+\frac{\mu_0}{4\pi}\oint_S\frac{\vec{M}\times\vec{n}}{|\vec{r}-\vec{r}'|}da'$$



$\vec{M}=M\hat{a}_z$

$\vec{J}_M=\vec{\nabla}\times\vec{M}=0$

$\vec{M}\times\vec{n}=\vec{M}\times\hat{a}_r=M\cdot\sin\theta\,\hat{a}_\varphi$

$$\Rightarrow \vec{A}=\frac{\mu_0}{4\pi}\oint_S\frac{M\sin\theta'\hat{a}_\varphi R^2\sin\theta' d\theta' d\varphi'}{|\vec{r}-\vec{r}'|}=\frac{\mu_0 M R^2}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin^2\theta'(-\sin\varphi'\hat{a}_x+\cos\varphi'\hat{a}_y)\,d\theta'\cdot d\varphi'.$$
$$\times\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\cdot Y^*_{l,m}(\theta',\varphi')Y_{l,m}(\theta,\varphi)$$

$$\vec{A}=\frac{\mu_0 M R^2}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\int_0^{2\pi}\int_0^{\pi}(-\sin\varphi'\hat{a}_x+\cos\varphi'\hat{a}_y)\cdot\sin^2\theta'\,d\theta' d\varphi'\,\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}\times$$
$$P_l^m(\cos\theta')e^{-im\varphi'}\cdot P_l^m(\cos\theta)e^{im\varphi}\Rightarrow$$

$$\vec{A}=\frac{\mu_0 M R^2}{4\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r^l}{r_>^{l+1}}\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)e^{im\varphi}\underbrace{\int_0^{2\pi}\int_0^{\pi}\sin^2\theta'\cdot(-\sin\varphi'\hat{a}_x+\cos\varphi'\hat{a}_y)P_l^m(\cos\theta')e^{-im\varphi'}d\theta' d\varphi'}_{I}$$

$$I=\underbrace{\int_0^{2\pi}(-\sin\varphi'\hat{a}_x+\cos\varphi'\hat{a}_y)(\cos m\varphi'-i\sin m\varphi')\,d\varphi'}_{I_m}\underbrace{\int_0^{\pi}\sin^2\theta'\,P_l^m(\cos\theta')\,d\theta'}_{C_m}$$

$$I_m = \begin{cases} i\Pi \hat{a}_x + \Pi \hat{a}_y = I_1 & m=1 \\ i\Pi \hat{a}_x + \Pi \hat{a}_y = I_{-1} & m=-1 \end{cases}$$

$$\vec{A} = \frac{\mu_0 M R^2}{4\pi} \sum_{l=1}^{\infty} \frac{r_<^l}{r_>^{l+1}} \left[\frac{(l-1)!}{(l+1)!} P_l^1(\cos\theta) e^{i\varphi} \cdot C_1 I_1 + \frac{(l+1)!}{(l-1)!} P_l^{-1}(\cos\theta) e^{-i\varphi} \cdot C_{-1} \cdot I_{-1} \right]$$

$$C_1 = \int_0^{\pi} \sin^2\theta' \, P_l^1(\cos\theta') \, d\theta'$$

$$C_{-1} = \int_0^{\pi} \sin^2\theta' \, \underbrace{P_l^{-1}(\cos\theta')}_{} \, d\theta' \longrightarrow C_{-1} = -\frac{(l-1)!}{(l+1)!} \overbrace{\int_0^{\pi} \sin^2\theta' P_l^1(\cos\theta') d\theta'}^{C_1} \Rightarrow C_{-1} = \frac{-(l-1)!}{(l+1)!} C_1$$

$$P_l^{-1}(\cos\theta) = (-1)^1 \times \frac{(l-1)!}{(l+1)!} P_l^1(\cos\theta')$$

$$\vec{A} = \frac{\mu_0 M R^2}{4\pi} \sum_{l=1}^{\infty} \frac{r_<^l}{r_>^{l+1}} \frac{(l-1)!}{(l+1)!} P_l^1(\cos\theta) \left[e^{i\varphi} I_1 + e^{-i\varphi} I_{-1} \right] C_1$$

$$C_1 = \int_0^{\pi} \sin^2\theta' \, P_l^1(\cos\theta') \, d\theta' \xrightarrow{x=\cos\theta'} C_1 = \int_{-1}^{1} \sqrt{1-x^2} \, P_l^1(x) \, dx \longrightarrow C_1 = \int_{-1}^{1} (-1)^1 (1-x^2) \frac{d}{dx} P_l(x) \, dx = -\int_{-1}^{1} \underbrace{(1-x^2)}_{u} \underbrace{\frac{d}{dx} P_l(x) dx}_{dv}$$

$$\boxed{P_l^1(x) = (-1)^1 \sqrt{1-x^2} \frac{d}{dx} \left[P_l(x) \right]}$$

$$\longrightarrow C_1 = -\underbrace{\left[(1-x^2) P_l(x) \right]_{-1}^{1}}_{zero} - \int_{-1}^{1} P_l(x)(-2x) \, dx = -2 \underbrace{\int_{-1}^{1} x P_l(x) dx}_{}$$

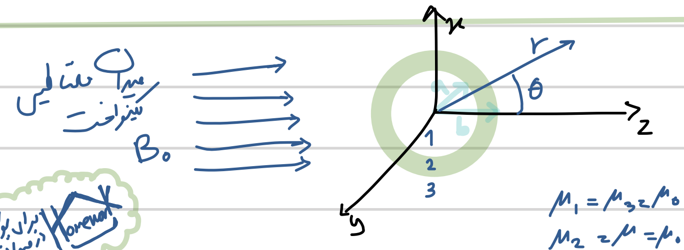
$$= \frac{2}{3}$$

فقط برای (l=1) مقدار دارد

$$l=1 \longrightarrow \vec{A} = \frac{\mu_0 M R^2}{3} \sin\theta \frac{r_<}{r_>^2} \hat{a}_\varphi \Rightarrow A_\varphi = \begin{cases} \frac{\mu_0 M}{3} r \sin\theta \; : \; 0<r<R \\ \frac{\mu_0 M R^3}{3R^2} \sin\theta \; : \; r>R \end{cases}$$

$$\boxed{P_1^1(x) = (-1)^1 (1-x^2)^{1/2} \frac{d}{dx} \underbrace{P_1(x)}_{x} = -\sqrt{1-x^2} \qquad P_1^1(\cos\theta) = -\sin\theta}$$

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A} = \begin{cases} \frac{2}{3} \mu_0 M \hat{a}_z & (0<r<R) \\ \frac{\mu_0 M R^3}{3R^3} (2\cos\theta \hat{a}_r + \sin\theta \hat{a}_\theta) & (r>R) \end{cases}}$$



برای پوسته کروی در میدان الکتریکی حل کنید Homework

$\mu_1 = \mu_3 = \mu_0$

$\mu_2 = \mu = \mu_0 K_m$

$$\Phi_{1m}(r,\theta) = \sum_{n=0}^{\infty} (a_n r^n + \overset{zero}{b_n r^{-n-1}}) P_n(\cos\theta)$$

$$\Phi_{2m}(r,\theta) = \sum_{n=0}^{\infty} (c_n r^n + d_n r^{-n-1}) P_n(\cos\theta)$$

$$\Phi_{3m}(r,\theta) = \sum_{n=1}^{\infty} (e_n r^n + F_n r^{-n-1}) P_n(\cos\theta)$$

حفاظ مغناطیسی SHield

Homework

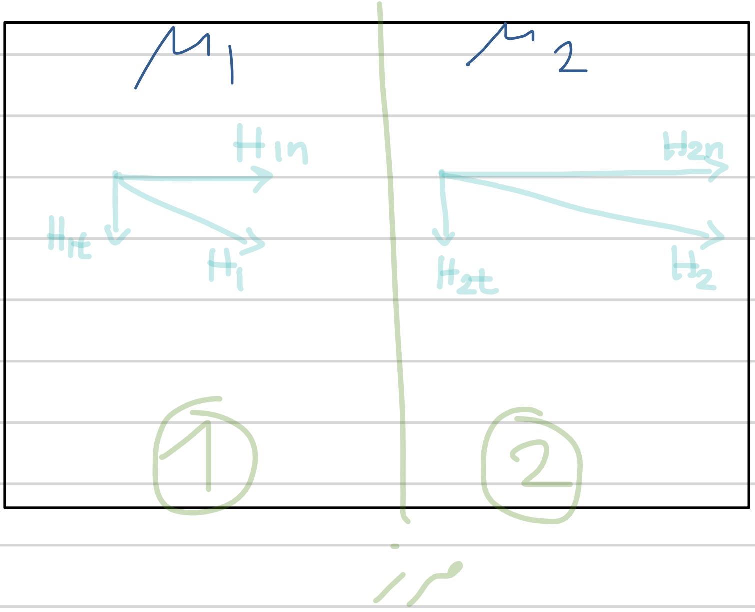
$$\vec{B}_3 \Big|_{r\to\infty} = B_0 \hat{a}_z \Rightarrow -\mu_0 \frac{\partial \phi_{3M}}{\partial z} \Big|_{r\to\infty} = B_0 \Rightarrow \phi_{3M} = \frac{-B_0}{\mu_0} z = \frac{-B_0}{\mu_0} r P_1(\cos\theta)$$

$$\phi_{1M} \Big|_{r=a} = \phi_{2m} \Big|_{r=a} , \quad \phi_{2M} \Big|_{r=b} = \phi_{3M} \Big|_{r=b}$$

$$B_{1n} = B_{2n} \Rightarrow \mu_0 \frac{-\partial \phi_{1M}}{\partial r} \Big|_{r=a} = \mu \frac{-\partial \phi_{2M}}{\partial r} \Big|_{r=a}$$

$$\mu \frac{-\partial \phi_{2M}}{\partial r} \Big|_{r=b} = \mu_0 \frac{\partial \phi_{3M}}{\partial r} \Big|_{r=b}$$

خطوط شدت مغناطیسی $\vec{H}$ که از محیطی غیر فرومغناطیسی به سطح فرومغناطیسی با μ خیلی زیاد برخورد کنند، بر آن عمود می‌شود ≡> شبیه رسانا در برابر الکتریسیته



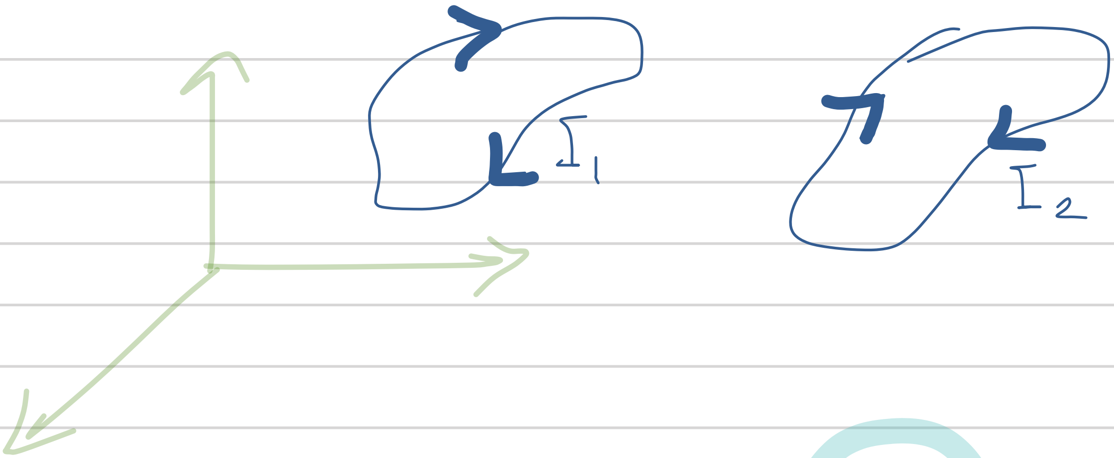
$$\underset{\text{در مرز}}{B_{1n}=B_{2n}} \longrightarrow \mu_1 H_{1n}=\mu_2 H_{2n} \Longrightarrow \frac{\mu_1}{\mu_2}=\frac{H_{2n}}{H_{1n}}$$

$$\text{if } \frac{\mu_1}{\mu_2} \longrightarrow \infty \Longrightarrow H_{2n} \gg H_{1n}$$

But still $\longrightarrow H_{1t}=H_{2t}$

$$\vec{n}_{12} \times (H_1-H_2)=\vec{K}$$

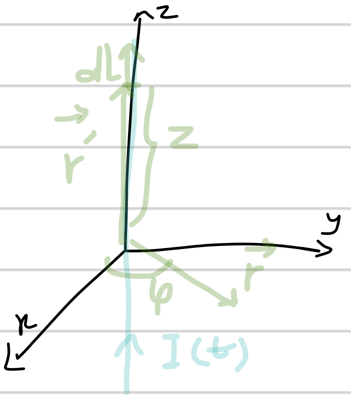
معمولاً $\longleftarrow \vec{K}=0 \Longrightarrow H_{1t}=H_{2t}$

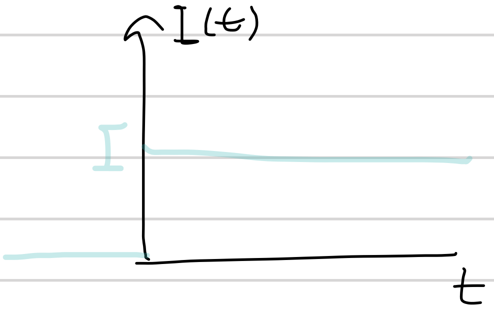


missed

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \Longrightarrow \phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}',t_r)\,dv'}{|\vec{r}-\vec{r}'|}$$

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} \Longrightarrow \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t_r)\,dv'}{|\vec{r}-\vec{r}'|}$$





$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{I(\vec{r}',t_r)\,\vec{dL}}{|\vec{r}-\vec{r}'|} \Longrightarrow \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{I\left(\vec{r}', t - \frac{\sqrt{r^2+z^2}}{c}\right) dz\,\hat{a}_z}{\sqrt{r^2+z^2}}$$

$t_r = t - \frac{|\vec{r}-\vec{r}'|}{c}$ $\quad \vec{r} = r\cos\varphi\,\hat{a}_x + r\sin\varphi\,\hat{a}_y$

$\vec{r}' = z\hat{a}_z \quad \Longrightarrow |\vec{r}-\vec{r}'| = \sqrt{r^2+z^2}$

$\vec{dL} = dz\,\hat{a}_z$

$$t_r \geqslant 0 \rightarrow t - \frac{\sqrt{r^2+z^2}}{c} \geqslant 0 \Longrightarrow \sqrt{r^2+z^2} \leq t.c \Longrightarrow |z| \leq \sqrt{t^2c^2-r^2}$$

$$\Longrightarrow -\sqrt{c^2t^2-r^2} \leq z \leq \sqrt{c^2t^2-r^2}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0 I}{4\pi}\int_{-\sqrt{c^2t^2-r^2}}^{\sqrt{c^2t^2-r^2}} \frac{dz\,\hat{a}_z}{\sqrt{r^2+z^2}} \Longrightarrow \vec{A}(\vec{r},t) = \frac{2\mu_0 I}{4\pi}\int_0^{\sqrt{c^2t^2-r^2}} \frac{dz}{\sqrt{r^2+z^2}}\hat{a}_z$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0 I}{2\pi}\hat{a}_z \ln\left(\sqrt{r^2+z^2}+z\right)\Big|_0^{\sqrt{c^2t^2-r^2}} \Longrightarrow \vec{A}(\vec{r},t) = \frac{\mu_0 I}{2\pi}\hat{a}_z \ln\left(\frac{ct+\sqrt{c^2t^2-r^2}}{r}\right)$$

$\phi(\vec{r},t) = 0$ $\boxed{\rho \text{ نداریم}}$

$$\vec{E} = -\underbrace{\nabla\phi}_{\text{zero}} - \frac{\partial\vec{A}}{\partial t} \Longrightarrow \vec{E} = \frac{-\mu_0 I c}{2\pi\sqrt{c^2t^2-r^2}}\hat{a}_z$$

و $\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0 I}{2\pi r}\frac{ct}{\sqrt{c^2t^2-r^2}}\hat{a}_\varphi$

برای حالت $t\rightarrow\infty$ $\longmapsto \vec{E} = 0$ و $\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{a}_\varphi$

Very familiar ✓

Homework اگر $I(t)=Kt$ باشد، آنگاه میدان‌های آخری $\vec{E}(\vec{r},t)$ و $\vec{B}(\vec{r},t)$ را بدست آورید ☺

- $$\vec E(\vec r,t) = \frac{-1}{4\pi\varepsilon_0}\int \frac{\vec\nabla \rho(\vec r', t_r)}{|\vec r-\vec r'|}\,dv' - \frac{\partial}{\partial t}\left(\frac{\mu_0}{4\pi}\int \frac{\vec J(\vec r', t_r)\,dv'}{|\vec r-\vec r'|}\right.$$

$$= \vec\nabla\left[\rho(\vec r', t_r)\cdot\frac{1}{|\vec r-\vec r'|}\right] = \vec\nabla\left[\underset{u}{\rho(\vec r', t-}\underset{v}{\frac{|\vec r-\vec r'|}{c}}\right)\cdot\frac{1}{|\vec r-\vec r'|} + \rho(\vec r', t-\frac{|\vec r-\vec r'|}{c})\cdot\vec\nabla\frac{1}{|\vec r-\vec r'|} =$$

$\dot\rho$ و $\dot{\vec J}$

مشتقات ρ و $\vec J$

نسبت به آرگومان

$t_r = t - \frac{|\vec r-\vec r'|}{c}$

هستند

$$t_r = t - \frac{(\vec r-\vec r')}{c}$$

$$\vec\nabla\phi(r) = \frac{d\phi(r)}{dr}\hat r$$

$$\vec\nabla\cdot\vec F(r) = \frac{d\vec F(r)}{dr}\hat r$$

$$\vec\nabla\times\vec F(r) = \hat a_r \times \frac{d\vec F(r)}{dr}$$

$$\frac{\partial\rho}{\partial v}\left(-\frac{1}{c}\right)\cdot\frac{\vec r-\vec r'}{|\vec r-\vec r'|^2} + \rho\cdot\frac{-(\vec r-\vec r')}{|\vec r-\vec r'|^3} \Rightarrow \vec\nabla\left(\frac{\rho(\vec r', t_r)}{|\vec r-\vec r'|}\right) = \frac{-1}{c}\dot\rho\frac{\vec r-\vec r'}{|\vec r-\vec r'|^2} - \frac{\rho(\vec r-\vec r')}{|\vec r-\vec r'|^3}$$

$$\frac{\mu_0}{4\pi}\int\frac{\dot{\vec j}}{|\vec r-\vec r'|}dv'$$

$$\Longrightarrow \boxed{\vec E(\vec r,t) = \frac{1}{4\pi\varepsilon_0}\int\left[\frac{\dot\rho(\vec r-\vec r')\,dv'}{c|\vec r-\vec r'|^2} + \frac{\rho(\vec r-\vec r')}{|\vec r-\vec r'|^3} - \frac{\dot{\vec j}}{c^2|\vec r-\vec r'|}\right]dv'}$$

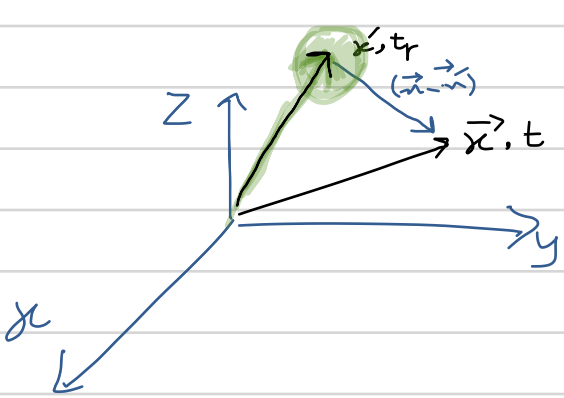
- $$\vec B(\vec r,t) = \vec\nabla\times\vec A(\vec r, t_r) = \frac{\mu_0}{4\pi}\int \vec\nabla\times\left[\vec J(\vec r', t_r)\cdot\frac{1}{|\vec r-\vec r'|}\right]dv' \Rightarrow$$

$$\vec B(\vec r,\vec t) = \frac{\mu_0}{4\pi}\int\left\{\left[\vec\nabla\times\vec J(\vec r', t_r)\right]\frac{1}{|\vec r-\vec r'|} + \left(\vec\nabla\frac{1}{|\vec r-\vec r'|}\right)\times\vec J(\vec r', t_r)\right\}dv' = \frac{\mu_0}{4\pi}\int\left\{\frac{-1}{c}\frac{\vec r-\vec r'}{|\vec r-\vec r'|^2}\times\dot{\vec J} + \left(-\frac{\vec r-\vec r'}{|\vec r-\vec r'|^3}\right)\times\vec J\right\}dv'$$

$$\Longrightarrow \boxed{\vec B(\vec r,t) = \frac{\mu_0}{4\pi}\int\left[\frac{\dot{\vec J}\times(\vec r-\vec r')}{c|\vec r-\vec r'|^2} + \frac{\vec J\times(\vec r-\vec r')}{|\vec r-\vec r'|^3}\right]dv'}$$

$$\vec{B}(\vec{r},t)=\frac{\mu_0}{4\pi}\int\left[\frac{\vec{J}(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|^3}+\frac{\dot{\vec{J}}(\vec{r}',t_r)}{c|\vec{r}-\vec{r}'|^2}\right]\times(\vec{r}-\vec{r}')\,d^3x' \longrightarrow$$

حالت مستقل از زمان $\longrightarrow$ $\vec{B}(\vec{x})=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')\times(\vec{r}-\vec{r}')\,d^3x'}{|\vec{r}-\vec{r}'|^3}$ قانون بیوساوار



تقریب نقاط نزدیک $|\vec{r}-\vec{r}'|<<\lambda$

$$\vec{J}(\vec{r}',t_r)=\vec{J}(\vec{r}',t)+\frac{(t_r-t)}{1!}\dot{\vec{J}}(\vec{r}',t)+\frac{(t_r-t)^2}{2!}\ddot{\vec{J}}(\vec{r}',t)+\dots\simeq$$

$$\simeq\vec{J}(\vec{r}',t)+(t_r-t)\dot{\vec{J}}(\vec{r}',t)$$

$$t_r=t-\frac{|\vec{r}-\vec{r}'|}{c}\Rightarrow t_r-t=-\frac{|\vec{r}-\vec{r}'|}{c}\longrightarrow\dot{\vec{J}}(\vec{r}',t_r)=\dot{\vec{J}}(\vec{r}',t)+\frac{(t_r-t)}{1!}\ddot{\vec{J}}(\vec{r}',t)+\dots\Rightarrow$$

$$-\frac{|\vec{r}-\vec{r}'|}{c}\qquad\dot{\vec{J}}(\vec{r}',t_r)\simeq\dot{\vec{J}}(\vec{r}',t)$$

$$\vec{B}(\vec{r},t)=\frac{\mu_0}{4\pi}\int\frac{\underbrace{c\vec{J}(\vec{r}',t)+c(t_r-t)\dot{\vec{J}}(\vec{r}',t)+\dot{\vec{J}}(\vec{r}',t)|\vec{r}-\vec{r}'|}}{c|\vec{r}-\vec{r}'|^3}\times(\vec{r}-\vec{r}')\,d^3r$$

$$\longrightarrow\vec{B}(\vec{r},t)=\frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}',t)\qquad !$$

بار $\boxed{dq}$

میدان الکترومغناطیسی $(\vec{B},\vec{E})$ روی بار dq کار انجام می‌دهد $\longrightarrow$ $dW=\vec{F}\cdot d\vec{L}=dq(\vec{E}+\vec{V}\times\vec{B})\cdot d\vec{L}=dq(\vec{E}+\vec{V}\times\vec{B})\cdot\vec{V}dt\Rightarrow dW=dq\,\vec{E}\cdot\vec{V}dt$

$$=\rho\,dv\,\vec{E}\cdot\vec{V}\,dt$$

حجم

dW: کار انجام شده بر بار dq از طرف میدان

$\frac{dW}{dt}=\vec{E}\cdot\vec{J}dV \Leftarrow$ کل کار انجام شده بر بارهای الکتریکی در حجم V در واحد زمان) $\qquad\frac{dW}{dt}=\int\vec{E}\cdot\vec{J}\,dV$

$\frac{dW}{dt}$: کار انجام شده بر واحد dq در واحد زمان از طرف میدان

$$\vec{\nabla}\times\vec{E}=-\partial\vec{B}/\partial t\longrightarrow\vec{H}\cdot(\vec{\nabla}\times\vec{E})=-\vec{H}\cdot\mu_0\frac{\partial\vec{H}}{\partial t}=-\mu_0\frac{\partial}{\partial t}(H^2)\cdot\tfrac{1}{2}\Rightarrow\vec{H}\cdot(\vec{\nabla}\times\vec{E})=-\tfrac{1}{2}\partial/\partial t(\vec{B}\cdot\vec{H})\quad①$$

$$\vec{\nabla}\times\vec{H}=\vec{J}+\partial\vec{D}/\partial t\longrightarrow\vec{E}\cdot(\vec{\nabla}\times\vec{H})=\vec{J}\cdot\vec{E}+\vec{E}\cdot\varepsilon_0\frac{\partial\vec{E}}{\partial t}\Rightarrow\vec{E}\cdot(\vec{\nabla}\times\vec{H})=\vec{J}\cdot\vec{E}+\varepsilon_0\tfrac{1}{2}\partial/\partial t(E^2)=\vec{J}\cdot\vec{E}+\tfrac{1}{2}\frac{\partial}{\partial t}(\vec{D}\cdot\vec{E})\quad②$$

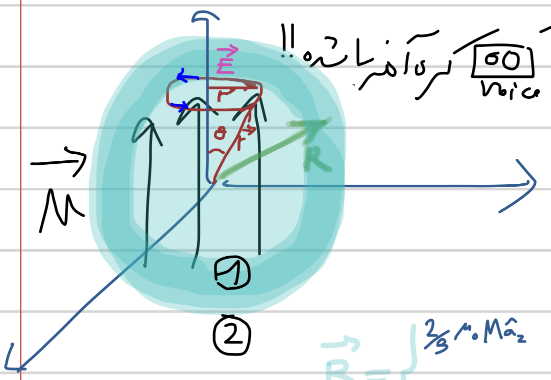
$$①-②\Rightarrow\underbrace{H\cdot(\vec{\nabla}\times\vec{E})-\vec{E}\cdot(\vec{\nabla}\times\vec{H})}_{\nabla\cdot(\vec{E}\times\vec{H})}=-\vec{J}\cdot\vec{E}-\frac{\partial}{\partial t}\Big[\underbrace{\tfrac{1}{2}(\vec{D}\cdot\vec{E}+\vec{B}\cdot\vec{H})}_{u}\Big]\Rightarrow\vec{\nabla}\cdot\vec{S}+\frac{\partial u}{\partial t}=-\vec{J}\cdot\vec{E}$$

(Poynting)

$$\Longrightarrow \oint_S (\vec{E}\times\vec{H})\cdot\vec{da} + \frac{d}{dt}\int_V u\cdot dv = \underbrace{-\int_V \vec{J}\cdot\vec{E}\, dv}$$

کل کار انجام شده در حجم V در واحد زمان

از طرف بارهای الکتریکی بر میدان الکترومغناطیسی



کره آهنربا شده !! $\boxed{60}$ ۱۴۰۱ ص

(ماده فرومغناطیس)

تا دمای کوری گرم می‌کنیم ، خاصیت فرومغناطیسی را از دست می‌دهد.

(حوزه‌های مغناطیسی تغییر می‌کنند و تبدیل به پارامغناطیسی می‌شود)

$$B \xrightarrow[\text{می‌کند تا صفر}]{\text{کاهش پیدا}}$$

i : قانون القای فارادی ←

این جریان القای مانع کاهش شار مغناطیسی است

$\vec{E}$ میدان الکتریکی القای روی کلیه بارهای روی سطح نیرو وارد می‌کند

← ایجاد گشتاور می‌کند (روی z)

روی سطح کره

که قبلاً به دست آورده بودیم

$$\vec{B} = \begin{cases} \frac{2}{3}\mu_0 M\hat{a}_z \\ \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta) \end{cases} \quad ; m = \frac{4}{3}\pi R^3 M$$

$$r' = r\sin\theta$$

$$\oint_C \vec{E}\cdot\vec{dL} = -\frac{d\Phi_B}{dt} \Rightarrow E.2\pi r' = -\frac{d}{dt}(B.\pi r'^2) \Rightarrow E.2\pi r\sin\theta = -\frac{dB}{dt}\pi(r\sin\theta)^2 \Longrightarrow E = \frac{-\mu_0 r\sin\theta}{3}\frac{dM}{dt}$$

$$\frac{dB}{dt} \to \frac{d(\frac{2}{3}\mu_0 M)}{dt}$$

$$\Longrightarrow \vec{E} = \frac{-\mu_0 r\sin\theta}{3}\frac{dM}{dt}\hat{a}_\varphi$$

$$\vec{dF} = \vec{E}\Big|_{r=R}\cdot dq \Rightarrow \vec{dF} = \frac{-\mu_0 R\sin\theta}{3}\frac{dM}{dt}\hat{a}_\varphi\cdot\sigma\cdot R^2\sin\theta\, d\theta\, d\varphi \Rightarrow \vec{dF} = \frac{-\mu_0 RQ}{12\pi}\sin^2\theta\frac{dM}{dt}\cdot d\theta\, d\varphi\cdot\hat{a}_\varphi$$

$$\sigma \to \frac{Q}{4\pi R^2}$$

$$\vec{\tau} = \int \vec{r}\times\vec{dF} \Rightarrow \vec{\tau} = \int R\hat{a}_r\times\left(\frac{-\mu_0 RQ}{12\pi}\right)\sin^2\theta\frac{dM}{dt}d\theta\, d\varphi\,\hat{a}_\varphi$$

$$-\boxed{\hat{a}_\theta = \cos\theta\cos\varphi\,\hat{a}_x + \cos\theta\sin\varphi\,\hat{a}_y} - \sin\theta\,\hat{a}_z$$

$$\Longrightarrow \vec{\tau} = \frac{-\mu_0 R^2 Q}{12\pi}\frac{dM}{dt}\hat{a}_z\int_0^{2\pi}d\varphi\int_0^{\pi}\sin^3\theta\, d\theta \Rightarrow \vec{\tau} = \frac{-2}{9}\mu_0 R^2 Q\frac{dM}{dt}\hat{a}_z$$

$$\vec{L} = \int \vec{\tau}\cdot dt \Rightarrow \vec{L} = \int_{t_i}^{t_f}\frac{-2}{9}\mu_0 R^2 Q\frac{dM}{dt}\hat{a}_z\cdot dt \Rightarrow \vec{L} = \int_M^0 \frac{-2}{9}\mu_0 R^2\cdot Q\cdot dM\cdot\hat{a}_z \Rightarrow \boxed{\vec{L} = \frac{2}{9}\mu_0 R^2 Q\cdot M\hat{a}_z}$$

← از کجا اومد؟ میدان الکترومغناطیسی ؟ هم تکانه داره هم انرژی

Hint

$$\boxed{\vec{B}\Big|_{r=R} = \frac{\vec{B}_1 + \vec{B}_2}{2}}$$

Homework

$\boxed{60}$ ۱۴۰۲

مسئله مشابه با این

جایگزین شدن $\vec{M}$

بار تخلیه شده از بالا

جریان سطحی ایجاد شد،

$$\vec{F} = I\int \vec{dl}\times B = \int \vec{J}\times\vec{B}\, dv = \int \vec{K}\times\vec{B}\, ds$$

چگالی جریان سطحی

اندازه حرکت یک میدان الکترومغناطیس $\longrightarrow d\vec{F} = dq\,(\vec{E}+\vec{V}\times\vec{B}) \Rightarrow d\vec{F} = \rho\,.dv(\vec{E}+\vec{V}\times\vec{B}) \Rightarrow d\vec{F} = (\rho\vec{E}+\vec{J}\times\vec{B})dv$

$$\downarrow$$

$$\frac{dP_{mechanical}}{dt} = \int(\rho\vec{E}+\vec{J}\times\vec{B})dv$$

$$\vec{\nabla}.\vec{D} = \rho \rightarrow \rho = \varepsilon_0\vec{\nabla}.\vec{E}$$

$$\vec{\nabla}\times\vec{H} = \vec{J}+\frac{\partial\vec{D}}{\partial t} \xrightarrow{\text{ضرب در } \mu_0} \vec{\nabla}\times\vec{B} = \mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \Rightarrow \vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B}-\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\boxed{\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \frac{\partial\vec{E}}{\partial t}\times\vec{B}+\vec{E}\times\frac{\partial B}{\partial t} \Rightarrow \vec{B}\times\frac{\partial\vec{E}}{\partial t} = \vec{E}\times\frac{\partial\vec{B}}{\partial t}-\frac{\partial}{\partial t}(\vec{E}\times\vec{B})}$$

$$\frac{d\vec{P}_m}{dt} = \int\varepsilon_0\Big[\vec{E}(\nabla.\vec{E})+\frac{1}{\mu_0\varepsilon_0}(\vec{\nabla}\times\vec{B})\times\vec{B}-\frac{\partial\vec{E}}{\partial t}\times\vec{B}\Big]dv \rightarrow \frac{d\vec{P}_m}{dt} = \int\varepsilon_0\Big[\vec{E}(\vec{\nabla}.\vec{E})-c^2\vec{B}\times(\vec{\nabla}\times\vec{B})+\vec{B}\times\frac{\partial\vec{E}}{\partial t}\Big]dv \Rightarrow \frac{dP_m}{dt}+\frac{d}{dt}\int\varepsilon_0(\vec{E}\times\vec{B})dv =$$

$$\varepsilon_0\int\Big[\vec{E}(\vec{\nabla}.\vec{E})-c^2\vec{B}\times(\vec{\nabla}\times\vec{B})+\vec{E}\times\underbrace{\boxed{\frac{\partial B}{\partial t}}}_{-\vec{\nabla}\times\vec{E}}\Big] \Rightarrow \frac{d\vec{P}_m}{dt}+\frac{d}{dt}\underbrace{\boxed{\int\varepsilon_0(\vec{E}\times\vec{B})dv}}_{\vec{P}_{em} = \varepsilon_0\int(\vec{E}\times\vec{B})dv} = \varepsilon_0\int\Big[\vec{E}(\vec{\nabla}.\vec{E})-c^2\vec{B}\times(\vec{\nabla}\times\vec{B})-\vec{E}\times(\vec{\nabla}\times\vec{E})\Big]dv$$

$$\vec{g} = \varepsilon_0\vec{E}\times\vec{B} = \mu_0\varepsilon_0\vec{E}\times\vec{H} = \frac{1}{c^2}\vec{S} \Rightarrow \boxed{\vec{g} = \frac{1}{c^2}\vec{S}}$$

اندازه حرکت میدان
الکترومغناطیس در واحد حجم

مورد کرهٔ مسألهٔ قبل

$$\vec{E} = \begin{cases} 0 & (r<R) \\ \dfrac{Q}{4\pi\varepsilon r^2} & (r>R) \end{cases} \quad , \quad \vec{B} = \begin{cases} \frac{2}{3}\mu_0 M\hat{a}_z & (r<R) \\ \dfrac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{a}_r+\sin\theta\hat{a}_\theta) & (r>R) \end{cases}$$

$$\vec{g} = \varepsilon_0\vec{E}\times\vec{B} = \frac{\mu_0}{(4\pi)^2}\cdot\frac{mQ}{r^5}\sin\theta\hat{a}_\varphi \quad (r>R)$$

$\vec{\mathcal{L}}$: تکانه زاویه‌ای میدان
الکترومغناطیس در واحد حجم

$$\vec{\mathcal{L}} = \vec{r}\times\vec{g} = r\hat{a}_r\times\vec{g} = -\frac{\mu_0}{(4\pi)^2}\cdot\frac{mQ}{r^4}\sin\theta\hat{a}_\theta \quad (r>R)$$

$$\vec{L} = \int\vec{\mathcal{L}}dv = -\frac{\mu_0 mQ}{(4\pi)^2}\int_0^{2\pi}\int_0^{\pi}\int_R^{\infty}\frac{\sin\theta}{r^4}\underbrace{\hat{a}_\theta}_{\cos\theta\cos\varphi\hat{a}_x+\cos\theta\sin\varphi\hat{a}_y-\sin\theta\hat{a}_z} r^2\sin\theta\,dr\,d\theta\,d\varphi \Rightarrow \vec{L} = \frac{\mu_0 mQ}{6\pi R}\hat{a}_z = \frac{\mu_0\frac{4}{3}\pi R^3 MQ}{6\pi R}\hat{a}_z \Rightarrow \vec{L} = \frac{2\mu_0 MQR^2}{9}\hat{a}_z$$