Certified International Investment Analyst

Formulae

Final Examination

Fixed Income Valuation and Analysis

Derivatives Valuation and Analysis

Portfolio Management

Table of Contents

1. Fixed Income Valuation and Analysis

1.1 Time Value of Money

1.1.1 Time Value of Money

1.1.1.1 Present and Future Value

Simple Discounting and Compounding

$$Present\ value = \frac{Future\ value}{(1 + Interest\ rate\ p.a.)^{number\ of\ years}}$$

$$Future\ value = (Present\ value) \cdot (1 + Interest\ rate\ p.a.)^{number\ of\ years}$$

1.1.1.2 Annuities

The present value of an annuity is given by

$$Present\ value = \sum_{t=1}^{N} \frac{CF}{(1+R)^t} = \frac{CF}{R} \cdot \left(1 - \frac{1}{(1+R)^N}\right)$$

where

CF	constant cash flow
R	discount rate, assumed to be constant over time
N	number of cash flows

The future value of an annuity is given by

$$\text{Future value} = CF \cdot \left(\frac{(1+R)^N - 1}{R}\right)$$

where

CF	constant cash flow
R	discount rate, assumed to be constant over time
N	number of cash flows

1.1.1.3 Continuous Discounting and Compounding

$$Present\ value = \frac{Future\ value}{e^{number\ of\ years \cdot continuous\ interest\ rate\ p.a.}}$$

$$Future\ value = (Present\ value) \cdot e^{number\ of\ years \cdot continuous\ interest\ rate\ p.a.}$$

1.1.2 Bond Yield Measures

1.1.2.1 Current Yield

$$Current\ yield = \frac{Annual\ coupon}{Price}$$

1.1.2.2 Yield to maturity

The bond price as a function of the yield to maturity is given by

$$P_0 = \sum_{i=1}^{N} \frac{CF_i}{(1+Y)^{t_i}} = \frac{CF_1}{(1+Y)^{t_1}} + \frac{CF_2}{(1+Y)^{t_2}} + ... + \frac{CF_N}{(1+Y)^{t_N}}$$

where

Y	yield to maturity
P_0	current paid bond price (including accrued interest)
CF_i	cash flow (coupon) received at time t_i
CF_N	cash flow (coupon plus principal) received at repayment date t_N
N	number of cash flows

Between two coupon dates, for a bond paying coupons annually, the bond price is given by

$$P_{cum,f} = P_{ex,f} + f \cdot C = (1+Y)^f \left[\frac{CF_1}{(1+Y)^1} + \frac{CF_2}{(1+Y)^2} + ... + \frac{CF_N}{(1+Y)^N} \right]$$

where

$P_{cum,f}$	current paid bond price (including accrued interest)
$P_{ex,f}$	quoted price of the bond
Y	yield to maturity
f	time since last coupon date in years
CF_i	cash flow (coupon) received at time t_i
CF_N	final cash flow (coupon plus principal)
N	number of cash flows

1.1.2.3 Yield to Call

$$P_0 = \sum_{i=1}^{N} \frac{CF_i}{(1+Y_c)^{t_i}} = \frac{CF_1}{(1+Y_c)^{t_1}} + \frac{CF_2}{(1+Y_c)^{t_2}} + \ldots + \frac{CF_N}{(1+Y_c)^{t_N}}$$

where

P_0	current paid bond price (including accrued interest)
Y_c	yield to call
CF_i	cash flow (coupon) received at time t_i
CF_N	cash flow (coupon plus principal) received at call date t_N
N	number of cash flows until call date

1.1.2.4 Relation between Spot Rate and Forward Rate

$$(1+R_{0,t}) = \left[(1+R_{0,1})\cdot(1+F_{1,2})\cdot(1+F_{2,3})\ldots(1+F_{t-1,t})\right]^{\frac{1}{t}}$$

where

$R_{0,t}$	spot rate p.a. from 0 to t
$R_{0,1}$	spot rate p.a. from 0 to 1
$F_{t-1,t}$	forward rate p.a. from $t-1$ to t

$$(1+R_{0,t_1})^{t_1}\cdot(1+F_{t_1,t_2})^{t_2-t_1} = (1+R_{0,t_2})^{t_2} \Leftrightarrow F_{t_1,t_2} = \left(\frac{(1+R_{0,t_2})^{t_2}}{(1+R_{0,t_1})^{t_1}}\right)^{\frac{1}{t_2-t_1}} - 1$$

where

R_{0,t_1}	spot rate p.a. from 0 to t_1
R_{0,t_2}	spot rate p.a. from 0 to t_2
F_{t_1,t_2}	forward rate p.a. from t_1 to t_2

1.1.3 Term Structure of Interest Rates

1.1.3.1 Theories of Term Structures

Expectations hypothesis

$$F_{t_1,t_2} = E(\tilde{R}_{t_1,t_2})$$

where

F_{t_1,t_2}	forward rate from t_1 to t_2
$\tilde{R}_{t_1,t_2}$	random spot rate from t_1 to t_2
$E(.)$	expectation operator

Liquidity Preference Theory

$$F_{t_1,t_2} = \mathrm{E}(\tilde{R}_{t_1,t_2}) + L_{t_1,t_2}, \qquad L_{t_1,t_2} > 0$$

where

F_{t_1,t_2}	forward rate from t_1 to t_2
$\tilde{R}_{t_1,t_2}$	random spot rate from t_1 to t_2
L_{t_1,t_2}	liquidity premium for the time t_1 to t_2
$E(.)$	expectation operator

Market Segmentation Theory

$$F_{t_1,t_2} = \mathrm{E}(\tilde{R}_{t_1,t_2}) + \Pi_{t_1,t_2}, \qquad \Pi_{t_1,t_2} \gtreqless 0$$

where

F_{t_1,t_2}	forward rate from t_1 to t_2
$\tilde{R}_{t_1,t_2}$	random spot rate from t_1 to t_2
Π_{t_1,t_2}	risk premium for the time from t_1 to t_2
$E(.)$	expectation operator

1.1.4 Bond Price Analysis

1.1.4.1 Yield Spread Analysis

Relative Yield Spread

$$\text{Relative yield spread} = \frac{\text{Yield bond } B - \text{Yield bond } A}{\text{Yield bond } A}$$

Yield Ratio

$$\text{Yield ratio} = \frac{\text{Yield bond } B}{\text{Yield bond } A}$$

1.1.4.2 Valuation of Coupon Bonds using Zero-Coupon Prices

Valuation of Zero-Coupon Bonds

$$P_0 = \frac{CF_t}{(1+R_t)^t}$$

where

P_0	bond price at time 0
CF_t	cash flow (principal) received at repayment date t
R_t	spot rate from 0 to t

Valuation of Coupon-Bearing Bonds

$$P_0 = \sum_{i=1}^{N} \frac{CF_i}{(1+R_i)^{t_i}} = \frac{CF_1}{(1+R_1)^{t_1}} + \frac{CF_2}{(1+R_2)^{t_2}} + \ldots + \frac{CF_N}{(1+R_N)^{t_N}}$$

where

P_0	bond price at time 0
CF_i	cash flow (coupon) received at time t_i
CF_N	cash flow (coupon plus principal) received at repayment date t_N
R_i	spot rate from 0 to t_i
N	number of cash flows

Price with accrued interest of a bond paying yearly coupons

$$P_{cum,f} = P_{ex,f} + f \cdot C = \sum_{i=1}^{N} \frac{CF_i}{\left(1+R_{t_i}\right)^{t_i - f}}$$

where

$P_{cum,f}$	price of the bond including accrued interest
$P_{ex,f}$	quoted price of the bond
f	time since the last coupon date in fractions of a year
CF_i	cash flow at time t_i
R_{t_i}	spot rate from *f* to t_i
C	coupon

Valuation of Perpetual Bonds

$$P_0 = \frac{CF}{R}$$

where

P_0	current price of the perpetual bond
CF	perpetual cash flow (coupon)
R	discount rate, assumed to be constant over time

1.1.5 Risk Measurement

1.1.5.1 Duration and Modified Duration

Macaulay's Duration

$$D = \sum_{i=1}^{N} \frac{t_i \cdot PV(CF_i)}{P} = \frac{\sum_{i=1}^{N} \frac{t_i \cdot CF_i}{(1+Y)^{t_i}}}{\sum_{i=1}^{N} \frac{CF_i}{(1+Y)^{t_i}}} = \frac{\frac{t_1 \cdot CF_1}{(1+Y)^{t_1}} + \frac{t_2 \cdot CF_2}{(1+Y)^{t_2}} + \ldots + \frac{t_N \cdot CF_N}{(1+Y)^{t_N}}}{\frac{CF_1}{(1+Y)^{t_1}} + \frac{CF_2}{(1+Y)^{t_2}} + \ldots + \frac{CF_N}{(1+Y)^{t_N}}}$$

where

D	Macaulay's duration
P	current paid bond price (including accrued interest)
Y	bond's yield to maturity
CF_i	cash flow (coupon) received at time t_i
$PV(CF_i)$	present value of cash flow CF_i
CF_N	cash flow (coupon plus principal) received at repayment date t_N
N	number of cash flows

Modified and Price Duration

$$D^{\text{mod}} = -\frac{1}{P}\frac{\partial P}{\partial Y} = \frac{D}{1+Y}$$

$$D^{\mathbf{P}} = -\frac{\partial P}{\partial Y} = D^{\text{mod}} \cdot P = \frac{D}{1+Y} P$$

where

D^{mod}	modified duration
D^{P}	price or dollar duration
D	Macaulay's duration
P	current paid bond price (including accrued interest)
Y	bond's yield to maturity

Price Change Approximated with Duration

$$\Delta P \cong \frac{-D}{(1+Y)} \cdot P \cdot \Delta Y = -D^{mod} \cdot P \cdot \Delta Y = -D^{P} \cdot \Delta Y$$

$$\frac{\Delta P}{P} \cong \frac{-D}{(1+Y)} \cdot \Delta Y = -D^{mod} \cdot \Delta Y = \frac{-D^{P}}{P} \cdot \Delta Y$$

where

ΔP	price change of the bond
D^{mod}	modified duration
D^{P}	price or dollar duration
D	Macaulay's duration
P	current paid bond price (including accrued interest)
Y	bond's yield to maturity
ΔY	small change in the bond yield

Portfolio Duration

$$D_P = \sum_{i=1}^{N} x_i \cdot D_i$$

where

D_P	portfolio duration
x_i	proportion of wealth invested in bond i
D_i	duration of bond i
N	number of bonds in the portfolio

1.1.5.2 Convexity

$$C = \frac{1}{P} \cdot \frac{\partial^2 P}{\partial Y^2} = \frac{1}{\sum_{i=1}^{N} \frac{CF_i}{(1+Y)^{t_i}}} \cdot \frac{1}{(1+Y)^2} \cdot \sum_{i=1}^{N} \frac{t_i (t_i + 1) \cdot CF_i}{(1+Y)^{t_i}}$$

$$C^{\mathrm{P}} = C \cdot P$$

where

C	convexity
$C^{\mathbf{P}}$	price convexity
P	current paid bond price (including accrued interest)
Y	bond's yield to maturity
CF_i	cash flow (coupon) received at time t_i
CF_N	cash flow (coupon plus principal) received at repayment date t_N

Price change approximated with duration and convexity

$$\Delta P \cong -D^{\mathrm{P}} \cdot \Delta Y + \frac{1}{2} C^{\mathrm{P}} \cdot \Delta Y^2$$

$$\frac{\Delta P}{P} \cong \frac{-D}{(1+Y)} \cdot \Delta Y + \frac{1}{2} C \cdot \Delta Y^2 = -D^{mod} \cdot \Delta Y + \frac{1}{2} C \cdot \Delta Y^2$$

where

ΔP	price change of the bond
D^{mod}	modified duration
D^P	price or dollar duration
D	Macaulay's duration
C	convexity
$C^{\mathbf{P}}$	price convexity
P	current paid bond price (including accrued interest)
Y	bond's yield to maturity
ΔY	Small change in the bond yield

Portfolio Convexity

$$Portfolio\ convexity = \sum_{i=1}^{N} w_i \cdot C_i$$

where

w_i	weight (in market value terms) of bond *i* in the portfolio
C_i	convexity of bond *i*
N	number of bonds in the portfolio

1.2 Convertible Bonds

1.2.1 Investment Characteristics

$$\textit{Conversion ratio} = \textit{Number of shares if one bond is converted}$$

$$\textit{Conversion price} = \frac{\textit{Face value of the convertible bond}}{\textit{Number of shares per bond (if there is a conversion)}}$$

$$\textit{Conversion value} = \textit{Conversion ratio} \cdot \textit{Market price of stock}$$

$$\textit{Conversion premium (in \%)} = \frac{\textit{Market price of bond - Conversion value}}{\textit{Conversion value}}$$

1.2.1.1 Payback Analysis

$$PP = \frac{(MP - CV)/CV}{(CY - DY)} = \frac{\textit{Conversion premium}}{(CY - DY)}$$

where

PP	payback period in years
MP	market price of the convertible
CV	conversion value of the convertible
CY	current yield of the convertible = (coupon/*MP*)
DY	dividend yield on the common stock = dividend amount / stock price

1.2.1.2 Net Present Value Analysis

$$NPV = \frac{CV - FV}{(1 + Y_{nc})^n} - \sum \frac{FV \cdot (Y_{nc} - Y_c)}{(1 + Y_{nc})^t}$$

where

NPV	net present value
CV	conversion value
FV	face value
Y_{nc}	yield on non-convertible security of identical characteristics
Y_c	yield on the convertible security
n	years before the convertible is called

1.3 Callable Bonds

1.3.1 Valuation and Duration

1.3.1.1 Determining the Call Option Value

Callable bond price = Call-free equivalent bond price – Call option price

1.3.1.2 Effective Duration and Convexity

$$\begin{matrix}\text{Call adjusted}\\\text{duration}\end{matrix} = \frac{Price_{call\ free}}{Price_{callable}} \cdot \begin{pmatrix}\text{Duration of}\\\text{call - free bond}\end{pmatrix} \cdot (1-\delta)$$

$$\begin{matrix}\text{Call adjusted}\\\text{Convexity}\end{matrix} = \frac{Price_{call\ free}}{Price_{callable}} \cdot \left[\begin{pmatrix}\text{Convexity of}\\\text{call - free bond}\end{pmatrix} \cdot (1-\delta) - \begin{pmatrix}\text{Price of}\\\text{call - free}\\\text{bond}\end{pmatrix} \cdot \gamma \cdot \begin{pmatrix}\text{Duration of}\\\text{call - free}\\\text{bond}\end{pmatrix}^2\right]$$

where

δ	Delta of the call option embedded in the bond
γ	Gamma of the call option embedded in the bond

1.4 Fixed Income Portfolio Management Strategies

1.4.1 Passive Management

1.4.1.1 Immunisation

$$A = L$$

$$D_A = D_L$$

$$A \cdot D_A = L \cdot D_L$$

where

A	present value of the portfolio
L	present value of the debt
D_A	duration of the portfolio
D_L	duration of the debt

1.4.2 Computing the Hedge Ratio: The Modified Duration Method

$$HR = \rho_{\Delta S,\Delta F} \cdot \frac{\sigma_{\Delta S}}{\sigma_{\Delta F}} = \frac{S_t \cdot D_S^{mod}}{F_{t,T} \cdot D_F^{mod}}$$

$$N_F = -\frac{N_S \cdot S_t \cdot D_S^{mod}}{k \cdot F_{t,T} \cdot D_F^{mod}} = -\frac{N_S \cdot S_t \cdot D_S^{mod}}{k \cdot S_{CTD,t} \cdot D_F^{mod}} \cdot CF_{CTD,t}$$

where

HR	hedge ratio
S_t	spot price at time t
$F_{t,T}$	futures price at time t with maturity T
$\rho_{\Delta S,\Delta F}$	correlation coefficient between ΔS and ΔF
$\sigma_{\Delta S}$	standard deviation of ΔS
$\sigma_{\Delta F}$	standard deviation of ΔF
CTD	cheapest-to-deliver
D_S^{mod}	modified duration of the asset being hedged
D_F^{mod}	modified duration of the futures (i.e. of the CTD)
N_F	number of futures contracts
N_S	number of the spot asset to be hedged
k	contract size
$S_{CTD,t}$	spot price of the CTD
$CF_{CTD,t}$	conversion factor of the CTD

2. Derivative Valuation and Analysis

2.1 Financial Markets and Instruments

2.1.1 Related Markets

2.1.1.1 Swaps

Interest rate swap

The swap value for the party that receives fix may be expressed as

$$V = B_1 - B_2$$

where

- V value of the swap
- B_1 value of the fixed rate bond underlying the swap
- B_2 value of the floating rate bond underlying the swap

B_1 is the present value of the fixed bond cash flows

$$B_1 = \sum_{i=1}^{n} \frac{K}{(1+R_{0,t_i})^{t_i}} + \frac{Q}{(1+R_{0,t_n})^{t_n}}$$

where

- B_1 value of the fixed rate bond underlying the swap
- K fixed payment corresponding to the fixed interest to be paid at time t_i
- Q notional principal in the swap agreement
- R_{0,t_i} spot interest rate corresponding to maturity t_i

When entering in the swap and immediately after a coupon rate reset date, the value of bond B_2 is equal to the notional amount Q. Between reset dates, the value is

$$B_2 = \frac{K^*}{(1+R_{0,t_1})^{t_1}} + \frac{Q}{(1+R_{0,t_1})^{t_1}}$$

where

- B_2 value of the floating rate bond underlying the swap
- K^* floating amount (initially known) used for the payment at date t_1, the next reset date.
- Q notional principal in the swap agreement
- R_{0,t_1} spot interest rate corresponding to maturity t_1

Cross Currency Interest Rate Swap

The swap value may be expressed as

$$V = S \cdot B_F - B_D$$

where

V	value of the swap
S	spot rate in domestic per foreign currency units
B_F	value of the foreign currency bond in the swap, denoted in the foreign currency
B_D	value of the domestic bond in the swap, denoted in the domestic currency

2.2 Analysis of Derivatives and Other Products

2.2.1 Futures

2.2.1.1 Theoretical Price of Futures

Pricing Futures on Assets that Provide no Income

$$F_{t,T} = S_t (1 + R_{t,T})^{T-t}$$

where

$F_{t,T}$	futures price at date t of a contract for delivery at date T
S_t	spot price of the underlying at date t
$R_{t,T}$	risk-free interest rate for the period t to T

General Cost of Carry Relationship

$$F_{t,T} = S_t (1 + R_{t,T})^{T-t} + k(t,S) - FV(revenues)$$

where

$F_{t,T}$	forward or futures price at date t of a contract for delivery at date T
S_t	spot price of the underlying at date t
$R_{t,T}$	risk-free interest rate for the period t to T
$k(t,S)$	carrying costs, such as insurance costs, storage costs, etc.
$FV(revenues)$	future value of the revenues paid by the spot

Continuous Time Cost of Carry Relationship

$$F_{t,T} = S_t e^{(r_{t,T} - y)\cdot(T-t)}$$

where

- $F_{t,T}$ futures price at date t of a contract for delivery at date T
- S_t spot price of the underlying at date t
- y continuous net yield (revenues minus carrying costs) of the underlying asset or commodity
- $r_{t,T}$ continuously compounded risk-free interest rate

Stock Index Futures

$$F_{t,T} = I_t \cdot (1 + R_{t,T})^{T-t} - \sum_{i=1}^{N} \sum_{t_j=1}^{T} w_i \cdot D_{i,t_j} \cdot (1 + R_{t_j,T})^{T-t_j}$$

where

- $F_{t,T}$ futures price at date t of a contract that expires at date T
- I_t current spot price of the index
- D_{i,t_j} dividend paid by stock i at date t_j
- w_i weight of stock i in the index
- $R_{t,T}$ risk-free interest rate for the period t to T
- $R_{t_j,T}$ interest rate for the time period t_j until T
- N the number of securities in the index

Interest Rates Future Cost of Carry Relationship

$$F_{t,T} = \frac{(S_t + A_t)\cdot(1 + R_{t,T})^{T-t} - C_{t,T} - A_T}{Conversion\ Factor}$$

where

- $F_{t,T}$ quoted futures “fair” price at date t of a contract for delivery at date T
- $C_{t,T}$ future value of all coupons paid and reinvested between t and T
- S_t spot value of the underlying bond
- A_t accrued interest of the underlying at time t
- $R_{t,T}$ risk-free interest rate for the period t to T
- A_T accrued interest of the delivered bond at time T

Theoretical Futures at the Delivery Date

$$F_{T,T} = \frac{spot\ price\ of\ cheapest\ to\ deliver}{conversion\ factor}$$

Forward Exchange Rates

$$F_{t,T} = S_t \left(\frac{1 + R_{dom}}{1 + R_{for}} \right)^{T-t}$$

with continuous compounding

$$F_{t,T} = S_t e^{(r_{dom} - r_{for})(T-t)}$$

where

$F_{t,T}$	forward exchange rate (domestic per foreign currency)
S_t	spot exchange rate (domestic per foreign currency)
R_{dom}	domestic risk-free rate of interest for the period *t* to *T*
R_{for}	foreign risk-free rate of interest for the period *t* to *T*
r_{dom}	continuous domestic risk-free rate of interest for *t* to *T*
r_{for}	continuous foreign risk-free rate of interest for *t* to *T*

Commodity Futures

$$F_{t,T} = S_t \cdot (1 + R_{t,T}) + k(t,T) - Y_{t,T}$$

where

$F_{t,T}$	futures price at date *t* of a contract for delivery at date *T*
S_t	spot price of the underlying at date *t*
$R_{t,T}$	risk-free interest rate for the period (*T* – *t*)
$k(t,T)$	carrying costs, such as insurance costs, storage costs, etc.
$Y_{t,T}$	convenience yield

2.2.1.2 Hedging Strategies

The Hedge Ratio

$$HR = \frac{\Delta S}{\Delta F} = -\frac{N_F \cdot k}{N_S} \qquad N_F = -HR \cdot \frac{N_S}{k}$$

where

HR	hedge ratio
ΔS	change in spot price per unit
ΔF	change in futures price per unit
N_F	number of futures
N_S	number of spot assets
k	contract size

The Perfect (Naive) Hedge

$$\begin{cases} HR = \pm 1 \\ N_F = \mp \dfrac{N_S}{k} \end{cases}$$

where

HR	hedge ratio
N_F	number of futures
N_S	number of spot assets
k	contract size

Minimum Variance Hedge Ratio

- Hedged Profit

For a long position in the underlying asset

$$Hedged\ profit = (S_T - S_t) - (F_{T,T} - F_{t,T})$$

where

S_T	spot price at the maturity of the futures contract
S_t	spot price at time *t*
$F_{T,T}$	futures price at its maturity
$F_{t,T}$	futures price at time t with maturity *T*

- Minimum Variance Hedge Ratio

$$HR = \frac{\mathrm{Cov}(\Delta S, \Delta F)}{\mathrm{Var}(\Delta F)} = \rho_{\Delta S, \Delta F} \cdot \frac{\sigma_{\Delta S}}{\sigma_{\Delta F}}$$

where

HR	hedge ratio
$\mathrm{Cov}(\Delta S, \Delta F)$	covariance between the changes in spot price (ΔS) and the changes in futures price (ΔF)
$\mathrm{Var}(\Delta F)$	variance of changes in futures price (ΔF)
$\rho_{\Delta S, \Delta F}$	coefficient of correlation between ΔS and ΔF
$\sigma_{\Delta S}$	standard deviation of ΔS
$\sigma_{\Delta F}$	standard deviation of ΔF

2.2.2 Options

2.2.2.1 Determinants of Option Price

Put-Call Parity for European and American Options

$$P_E = C_E - S + D + Ke^{-r\tau}$$

$$C_{US} - S + Ke^{-r\tau} \le P_{US} \le C_{US} - S + K + D$$

where

τ	time until expiry of the option
K	strike or exercise price of the option
r	continuously compounded risk-free rate of interest
S	spot price of the underlying
C_E	value of European call option
P_E	value of European put option
C_{US}	value of American call option
P_{US}	value of American put option
D	present value of expected cash-dividends during the life of the option

2.2.2.2 Option Pricing Models

Black and Scholes Option Pricing Formula

The prices of European Options on Non-Dividend Paying Stocks

$$C_E = S \cdot N(d_1) - Ke^{-r\tau} \cdot N(d_2)$$

$$P_E = Ke^{-r\tau} \cdot N(-d_2) - S \cdot N(-d_1)$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

where

C_E	value of European call
P_E	value of European put
S	current stock price
τ	time in years until expiry of the option
K	strike price
σ	volatility p.a. of the underlying stock
r	continuously compounded risk-free rate p.a.
$N(\cdot)$	cumulative distribution function for a standardised normal random variable (see table in 2.2.3), and

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds$$

European Option on Stocks Paying Known Dividends

$$C_E = S^* \cdot N(d_1^*) - Ke^{-r\tau} \cdot N(d_2^*)$$

$$P_E = Ke^{-r\tau} \cdot N(-d_2^*) - S^* \cdot N(-d_1^*)$$

$$d_1^* = \frac{\ln\left(S^* / K\right) + (r + \sigma^2 / 2)\tau}{\sigma\sqrt{\tau}}, \quad d_2^* = d_1^* - \sigma\sqrt{\tau}, \quad S^* = S - \sum_{i=1}^{I} D_i \cdot e^{-r\tau_i}$$

where

C_E	value of European call
P_E	value of European put
τ_i	time in years until i^{th} dividend payment
D_i	dividend i
S	current stock price
τ	time in years until expiry of the option
K	strike price
σ	volatility p.a. of the underlying stock
r	continuously compounded risk-free rate p.a.
I	number of dividend payments
$N(\cdot)$	cumulative normal distribution function (see table in 2.2.3)

European Option on Stocks Paying Unknown Dividends

When dividends are unknown, a common practice is to assume a constant dividend yield *y.* Then

$$C_E = S \cdot e^{-y\tau} \cdot N(d_1) - Ke^{-r\tau} \cdot N(d_2)$$

$$P_E = Ke^{-r\tau} \cdot N(-d_2) - S \cdot e^{-y\tau} \cdot N(-d_1)$$

$$d_1 = \frac{\ln\left(S / K\right) + (r - y + \sigma^2 / 2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

where

C_E	value of European call
P_E	value of European put
y	continuous dividend yield
S	current stock price
τ	time in years until expiry of the option
K	strike price
σ	volatility p.a. of the underlying stock
r	continuously compounded risk-free rate p.a.
$N(\cdot)$	cumulative normal distribution function (see table in 2.2.3)

Options on Stock Indices

$$C_E = \left[S - \sum_{j=1}^{J} \sum_{i=1}^{I} D_{j,i} \cdot e^{-r \cdot \tau_{j,i}} \right] N(d_1) - K \cdot e^{-r \cdot \tau} \cdot N(d_2)$$

$$P_E = K \cdot e^{-r \cdot \tau} \cdot N(-d_2) - \left[S - \sum_{j=1}^{J} \sum_{i=1}^{I} D_{j,i} \cdot e^{-r \cdot \tau_{j,i}} \right] \cdot N(-d_1)$$

$$d_1 = \frac{ln\left(\frac{S_t - \sum_{j=1}^{J} \sum_{i=1}^{I} D_{j,i} \cdot e^{-r \cdot \tau_{j,i}}}{K \cdot e^{-r \cdot \tau}} \right)}{\sigma \cdot \sqrt{\tau}} + \frac{1}{2} \cdot \sigma \cdot \sqrt{\tau} \quad \text{and} \quad d_2 = d_1 - \sigma \cdot \sqrt{\tau}$$

where

C_E	price of the European call at date *t*
P_E	price of the European put at date *t*
S	price of the index at date *t*
K	strike price
r	continuously compounded risk-free rate p.a.
σ	standard deviation of the stock index instantaneous return
$D_{j,i}$	dividend paid at time t_i by company j weighted as the company in the index
τ	time in years until expiry of the option
$\tau_{j,i}$	time remaining until the dividend payment at time t_i by company *j*
$N(\cdot)$	cumulative normal distribution function (see table in 2.2.3)

Options on Futures

$$C_E = e^{-r\tau} \left[F \cdot N(d_1) - K \cdot N(d_2) \right]$$

$$P_E = e^{-r\tau} \left[K \cdot N(-d_2) - F \cdot N(-d_1) \right]$$

$$d_1 = \frac{\ln(F/K)}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

where

C_E	value of European call
P_E	value of European put
F	current futures price
τ	time in years until expiry of the option
K	strike price
σ	volatility p.a. of the futures returns
r	continuously compounded risk-free rate p.a.
$N(\cdot)$	cumulative normal distribution function (see table in 2.2.3)

Options on Currencies

$$C_E = S \cdot e^{-r_{for}\tau} \cdot N(d_1) - Ke^{-r\tau} \cdot N(d_2)$$

$$P_E = Ke^{-r\tau} \cdot N(-d_2) - S \cdot e^{-r_{for}\tau} \cdot N(-d_1)$$

$$d_1 = \frac{\ln(S/K) + (r - r_{for} + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

where

C_E	value of European call
P_E	value of European put
S	current exchange rate (domestic per foreign currency units)
τ	time in years until expiry of the option
K	strike price (domestic per foreign currency units)
σ	volatility p.a. of the underlying foreign currency
r	continuously compounded risk-free rate p.a.
r_{for}	continuously compounded risk-free rate p.a. of the foreign currency
$N(\cdot)$	cumulative normal distribution function (see table in 2.2.3)

Binomial Option Pricing Model

The option price at the beginning of the period is equal to the expected value of the option price at the end of the period under the probability measure π, discounted with the risk-free rate.

$$O = \frac{O_u \cdot \pi + O_d \cdot (1-\pi)}{1+R}$$

$$\pi = \frac{1+R-d}{u-d}, \quad u = e^{\sigma\sqrt{\tau/n}}, \quad d = \frac{1}{u}, \quad d < 1+R < u$$

where

O	value of the option at the beginning of the period
R	simple risk-free rate of interest for one period
O_u	value of the option in the up-state at the end of the period
O_d	value of the option in the down-state at the end of the period
σ	volatility of the underlying returns
τ	time until expiry of the option
n	number of periods τ is divided in
u	upward factor of the underlying
d	downward factor of the underlying
π	risk neutral probability

2.2.2.3 Sensitivity Analysis of Option Premiums

The Strike Price (κ)

$$\kappa_C = \frac{\partial C}{\partial K} = -e^{-r \cdot \tau} \cdot N(d_2) \qquad (\kappa_C \le 0)$$

$$\kappa_P = \frac{\partial P}{\partial K} = e^{-r \cdot \tau} \cdot N(-d_2) \qquad (\kappa_P \ge 0)$$

$$d_2 = \frac{ln(S/K) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

where

C	value of call option
P	value of put option
S	current price of the underlying
K	strike price
σ	volatility p.a. of the underlying returns
τ	time in years until expiry of the option
r	continuously compounded risk-free rate p.a.
$N(\cdot)$	cumulative distribution function (see table in 2.2.3)

Price of the Underlying Asset (delta (Δ) and gamma (Γ))

$$\Delta_c = \frac{\partial C}{\partial S} = N(d_1) \qquad (0 \le \Delta_c \le 1)$$

$$\Delta_P = \frac{\partial P}{\partial S} = N(d_1) - 1 \qquad (-1 \le \Delta_P \le 0)$$

$$\Gamma_C = \frac{\partial^2 C}{\partial S^2} = \frac{n(d_1)}{S \cdot \sigma \cdot \sqrt{\tau}} \qquad (\Gamma_C \ge 0)$$

$$\Gamma_P = \frac{\partial^2 P}{\partial S^2} = \frac{n(d_1)}{S \cdot \sigma \cdot \sqrt{\tau}} = \Gamma_C \qquad (\Gamma_P \ge 0)$$

$$d_1 = \frac{ln(S/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

where

C	value of call option
P	value of put option
S	current price of the underlying
τ	time in years until expiry of the option
σ	volatility p.a. of the underlying returns
r	continuously compounded risk-free rate p.a.
$N(\cdot)$	cumulative distribution function (see table in 2.2.3)
$n(x)$	probability density function:

$$n(x) = N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

The Leverage or Elasticity of the Option with respect to S (omega, Ω)

$$\Omega_C = \frac{\partial C}{\partial S} \cdot \frac{S}{C} \qquad \Omega_P = \frac{\partial P}{\partial S} \cdot \frac{S}{P}$$

where

- Ω_C elasticity of a call
- Ω_P elasticity of a put
- C current value of call option
- P current value of put option
- S current price of the underlying

The Time to Maturity (theta, θ)

$$\theta_C = \frac{\partial C}{\partial t} = -\frac{\partial C}{\partial \tau} = -\frac{S \cdot \sigma}{2 \cdot \sqrt{\tau}} \cdot n(d_1) - K \cdot r \cdot e^{-r\tau} N(d_2) \qquad (\theta_C \leq 0)$$

$$\theta_P = \frac{\partial P}{\partial t} = -\frac{\partial P}{\partial \tau} = -\frac{S \cdot \sigma}{2 \cdot \sqrt{\tau}} \cdot n(d_1) - K \cdot r \cdot e^{-r\tau} \left[N(d_2) - 1 \right]$$

$$d_1 = \frac{ln(S/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

where

- C value of call option
- P value of put option
- S current price of the underlying
- K strike price
- τ time in years until expiry of the option
- r continuously compounded risk-free rate p.a.
- σ volatility p.a. of the underlying returns
- $N(\cdot)$ cumulative distribution function (see table in 2.2.3)
- $n(x)$ probability density function (see formula 0 for definition)

Interest Rate (rho, ρ)

$$\rho_C = \frac{\partial C}{\partial r} = K \cdot \tau \cdot e^{-r \cdot \tau} \cdot N(d_2) \qquad (\rho_C \geq 0)$$

$$\rho_P = \frac{\partial P}{\partial r} = K \cdot \tau \cdot e^{-r \cdot \tau} \cdot (N(d_2) - 1) \qquad (\rho_P \leq 0)$$

$$d_2 = \frac{ln(S/K) + (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

where

- C value of call option
- P value of put option
- S current price of the underlying
- K strike price
- τ time in years until expiry of the option
- σ volatility p.a. of the underlying returns
- r continuously compounded risk-free rate p.a.
- $N(\cdot)$ cumulative distribution function (see table in 2.2.3)

Volatility of the Stock Returns (vega, ν)

$$\nu_C = \frac{\partial C}{\partial \sigma} = S \cdot \sqrt{\tau} \cdot n(d_1) \quad (\nu_C \geq 0)$$

$$\nu_P = \frac{\partial P}{\partial \sigma} = S \cdot \sqrt{\tau} \cdot n(d_1) = \nu_C \quad (\nu_P \geq 0)$$

$$d_1 = \frac{ln(S/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

where

C	value of call option
P	value of put option
S	current price of the underlying
K	strike price
τ	time in years until expiry of the option
σ	volatility p.a. of the underlying returns
r	continuously compounded risk-free rate p.a.
$n(x)$	probability density function (see formula 0 for definition)

2.2.3 Standard Normal Distribution: Table for CDF

Numerically defines function *N(x):* probability that a standard normal random variable is smaller than *x*. Property of *N(x):* *N(-x)=1-N(x).*

x	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

3. Portfolio Management

3.1 Modern Portfolio Theory

3.1.1 The Risk/Return Framework

3.1.1.1 Return

Holding Period Return

$$R_t = \frac{P_t - P_{t-1} + \sum_{j=1}^{J} D_{t_j} \cdot \left(1 + R^*_{t_j,t}\right)^{t-t_j}}{P_{t-1}}$$

where

R_t	simple (or discrete) return of the asset over period $t-1$ to t
P_t	price of the asset at date t
D_{t_j}	dividend or coupon paid at date t_j between $t-1$ and t
t_j	date of the j^{th} dividend or coupon payment
$R^*_{t_j,t}$	risk-free rate p.a. for the period t_j to t
J	number of intermediary payments

Arithmetic versus Geometric Average of Holding Period Returns

Arithmetic average of holding period returns

$$r_A = \frac{1}{N} \cdot \sum_{i=1}^{N} R_i$$

where

r_A	arithmetic average return over N sequential periods
R_i	holding period returns
N	number of compounding periods in the holding period

Geometric average return over a holding period using discrete compounding

$$R_A = \sqrt[N]{(1+R_1)\cdot(1+R_2)\cdot\ldots\cdot(1+R_N)} - 1$$

where

R_A	geometric average return over N sequential periods
R_i	discrete return for the period i
N	number of compounding periods in the holding period

Time Value of Money: Compounding and Discounting

Compounded returns

$$1 + R_{eff} = \left(1 + \frac{R_{nom}}{m}\right)^m$$

where

R_{eff}	effective rate of return over entire period
R_{nom}	nominal return
m	number of sub-periods

Continuously compounded versus simple (discrete) returns

In the case no dividends paid between time *t-1* and *t*

$$r_t = \ln \frac{P_t}{P_{t-1}} = \ln(1 + R_t)$$

$$R_t = e^{r_t} - 1$$

where

P_t	price of the asset at date *t*
r_t	continuously compounded return between time $t-1$ and t
R_t	simple (discrete) return between time $t-1$ and t

Annualisation of Returns

Annualising holding period returns (assuming 360 days per year)

Assuming reinvestment of interests at rate R_τ

$$R_{ann} = (1 + R_\tau)^{360/\tau} - 1$$

where

R_{ann}	annualised simple rate of return
R_τ	simple return for a time period of τ days
Note:	convention 360 days versus 365 or the effective number of days varies from one country to another.

Annualising continuously compounded returns (assuming 360 days per year)

$$r_{an} = \frac{360}{\tau} \times r_\tau$$

where

r_{an}	annualised rate of return
r_τ	continuously compounded rate of return earned over a period of τ days

Nominal versus Real Returns

With simple returns

$$R_t^{real} = R_t^{\text{nominal}} - I_t - R_t^{real} \cdot I_t \approx R_t^{\text{nominal}} - I_t$$

With continuously compounded returns

$$r_t^{real} = r_t^{\text{nominal}} - i_t$$

where

R_t^{real}	real rate of return on an asset over period *t* (simple)
R_t^{nominal}	nominal rate of return on an asset over period *t* (simple)
I_t	rate of inflation over period *t* (simple)
r_t^{real}	real rate of return on an asset over period *t* (cont. comp.)
r_t^{nominal}	nominal rate of return on an asset over period *t* (cont. comp.)
i_t	rate of inflation over period *t* (cont. comp.)

3.1.2 Measures of Risk

Probability Concepts

Expectation value E(.), **variance** Var(.), **covariance** Cov(.) and **correlation** Corr(.) of two random variables X and Y if the variables take values x_k, y_k in state k with probability p_k

$$\mathrm{E}(X) = \sum_{k=1}^{K} p_\mathrm{k} \cdot x_k, \quad \mathrm{E}(Y) = \sum_{k=1}^{K} p_\mathrm{k} \cdot y_k$$

$$\mathrm{Var}(X) = \sigma_X^2 = \mathrm{E}\left[(X - \mathrm{E}(X))^2\right] = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = \sum_{k=1}^{K} p_\mathrm{k}(x_k - \mathrm{E}(X))^2$$

$$\mathrm{Cov}(X,Y) = \sigma_{XY} = \mathrm{E}\left[(X - \mathrm{E}(X)) \cdot (Y - \mathrm{E}(Y))\right] = \sum_{k=1}^{K} p_\mathrm{k}(x_k - \mathrm{E}(X)) \cdot (y_k - \mathrm{E}(Y))$$

$$\mathrm{Corr}(X,Y) = \frac{\sigma_{XY}}{\sigma_X \cdot \sigma_Y}$$

where $\sum_{k=1}^{K} p_\mathrm{k} = 1$, and

p_k	probability of state *k*
x_k	value of X in state *k*
y_k	value of Y in state *k*
K	number of possible states

The **mean** E(.), **variance** Var(.), **covariance** Cov(.) of two random variables *X* and *Y*, in a sample of N observations of x_i and y_i, are given by

$$\mathrm{E}(X) = \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i\,, \quad \mathrm{Var}(X) = \sigma_X^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2$$

$$\mathrm{Cov}(X,Y) = \sigma_{XY} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})\cdot(y_i - \bar{y})$$

where

x_i, y_i	observation i
$\bar{x}, \bar{y}$	mean of X and *Y*
σ_X, σ_Y	standard deviations
σ_{XY}	covariance of *X* and *Y*
N	number of observations

Normal Distribution

Its probability density is given by the following function:

$$f(x) = \frac{1}{\sqrt{2\cdot\pi}\cdot\sigma}\cdot e^{-\frac{(x-\mu)^2}{2\cdot\sigma^2}}$$

where

x	the value of the variable,
μ	the mean of the distribution,
σ	standard deviation.

Computing and Annualising Volatility

<u>Computing volatility</u>

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{t=1}^{N}(r_t - \bar{r})^2}\,, \qquad \bar{r} = \frac{1}{N}\sum_{t=1}^{N} r_t$$

where

σ	standard deviation of the returns (the volatility)
N	number of observed returns
$r_t = \ln\frac{P_t}{P_{t-1}}$	continuously compounded return of asset P over period t

Annualising Volatility

Assuming that monthly returns are independent, then

$$\sigma_{ann} = \sqrt{12} \cdot \sigma_m = \frac{\sigma_\tau}{\sqrt{\tau}}$$

where

σ_{ann}	annualised volatility
σ_m	volatility of monthly returns
σ_τ	volatility of returns over periods of length τ
τ	length of one period in years

3.1.3 Portfolio Theory

3.1.3.1 Diversification and Portfolio Risk

Average and Expected Return on a Portfolio

- Ex-Post Return on a Portfolio *P* in period t

$$R_{P,t} = \sum_{i=1}^{N} x_i R_{i,t} = x_1 R_{1,t} + x_2 R_{2,t} + \cdots + x_N R_{N,t}$$

where $\sum x_i = 1$, and

$R_{P,t}$	return on the portfolio in period t
$R_{i,t}$	return on asset i in period t
x_i	initial (at beginning of period) proportion of the portfolio invested in asset i
N	number of assets in portfolio *P*

- Expectation of the Portfolio Return

$$\mathrm{E}(R_P) = \sum_{t=1}^{N} x_i \mathrm{E}(R_i) = x_1 \mathrm{E}(R_1) + x_2 \mathrm{E}(R_2) + \ldots + x_N \mathrm{E}(R_N)$$

where

$\mathrm{E}(R_P)$	expected return on the portfolio
$\mathrm{E}(R_i)$	expected return on asset *i*
x_i	relative weight of asset *i* in portfolio *P*
N	number of assets in portfolio *P*

Variance of the Portfolio Return

$$\mathrm{Var}(R_P) = \sigma_P^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \sigma_{ij} = \sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \rho_{ij}\sigma_i\sigma_j$$

where

σ_P^2	variance of the portfolio return
σ_{ij}	covariance between the returns on assets *i* and *j*
ρ_{ij}	correlation coefficient between the returns on assets *i* and *j*
σ_i , σ_j	standard deviations of the returns on assets *i* and *j*
x_i	initial proportion of the portfolio invested in asset i
x_j	initial proportion of the portfolio invested in asset j
N	number of assets in portfolio *P*

3.1.4 Capital Asset Pricing Model (CAPM)

3.1.4.1 Capital Market Line (CML)

$$E(R_P) = r_f + \frac{E(R_M) - r_f}{\sigma_M}\sigma_P$$

where

$\mathrm{E}(R_P)$	expected return of portfolio *P*
r_f	risk free rate
$\mathrm{E}(R_M)$	expected return of the market portfolio
σ_M	standard deviation of the return on the market portfolio
σ_p	standard deviation of the portfolio return

3.1.4.2 Security Market Line (SML)

$$E(R_i) = r_f + \lfloor E(R_M) - r_f \rfloor \cdot \beta_i$$

$$\beta_i = \frac{Cov(R_i, R_M)}{Var(R_M)}$$

where

$\mathrm{E}(R_i)$	expected return of asset *i*
$\mathrm{E}(R_M)$	expected return on the market portfolio
r_f	risk free rate
β_i	beta of asset i
$Cov(R_i, R_M)$	covariance between the returns on assets *i* and market portfolio
$Var(R_M)$	variance of returns on the market portfolio

Beta of a Portfolio

$$\beta_p = \sum_{i=1}^{N} x_i \beta_i$$

where

β_P	beta of the portfolio
β_i	beta of asset i
x_i	proportion of the portfolio invested in asset i
N	number of assets in the portfolio

3.1.4.3 International CAPM

$$E[r_i] - r_f = \beta_i \cdot \left(E[r_M] - r_f\right) + \sum_{k=1}^{K-1} \gamma_{i,k} \cdot \left(E[s_k] + r_f^k - r_f\right)$$

where

$E(r_i)$	expected return of asset i
$E(r_M)$	expected return on the market portfolio
β_i	beta of asset i
r_f	continuous compounded risk-free rate in the domestic country
s_k	exchange rate of country k
r_f^k	continuous compounded risk-free rate in country k
K	number of countries considered

and

$$\gamma_{i,k} = \frac{Cov[r_i, s_k]}{Var[s_k]}$$

3.1.5 Arbitrage Pricing Theory (APT)

$$\mathrm{E}(R_i) \approx R_f + \sum_{j=1}^{N} \lambda_j \beta_{ij}$$

where

$\mathrm{E}(R_i)$	expected return on asset i
R_f	risk free rate
λ_j	expected return premium per unit of sensitivity to the risk factor j
β_{ij}	sensitivity of asset i to factor j
N	number of risk factors

3.1.5.1 One Factor Models

Single-Index Model

$$R_{it} = \alpha_i + \beta_i \cdot R_{\text{index},t} + \varepsilon_{it}$$

where

R_{it}	return on asset or portfolio i over period t
α_i	intercept for asset or portfolio i
β_i	sensitivity of asset or portfolio i to the index return
$R_{\text{index},t}$	return on the index over period t
ε_{it}	random error term ($\mathrm{E}(\varepsilon_{it}) = 0$)

Market Model

$$R_{it} = \alpha_i + \beta_i \cdot R_{Mt} + \varepsilon_{it}$$

Market model in expectation terms

$$\mathrm{E}(R_{it}) = \alpha_i + \beta_i \cdot \mathrm{E}(R_{Mt})$$

where

R_{it}	return on asset or portfolio i over period t
α_i	intercept for asset or portfolio i
β_i	sensitivity of asset or portfolio i to the index return
R_{Mt}	return on the market portfolio
ε_{it}	random error term ($\mathrm{E}(\varepsilon_{it}) = 0$)

Covariance between two Assets in the Market Model or the CAPM Context

$$\sigma_{ij} = \beta_i \cdot \beta_j \cdot \sigma_M^2$$

where

σ_{ij}	covariance between the returns of assets i and j
β_i	beta of portfolio i
β_j	beta of portfolio j
σ_M^2	variance of the return on the market portfolio

Decomposing Variance into Systematic and Diversifiable Risk

In the case of a single security

$$\sigma_i^2 = \underbrace{\beta_i^2 \sigma_M^2}_{\text{market risk}} + \underbrace{\sigma_{\varepsilon_i}^2}_{\text{residual risk}}$$

where

σ_i^2 :	total variance of the return on asset or portfolio i
$\beta_i^2 \sigma_M^2$:	market or systematic risk (explained variance)
$\sigma_{\varepsilon_i}^2$:	idiosyncratic or residual or unsystematic risk (unexplained variance)

Quality of an index model: R^2 and ρ^2

$$R^2 = \frac{\beta_i^2 \cdot \sigma_I^2}{\sigma_i^2} = \frac{\beta_i^2 \cdot \sigma_I^2}{\beta_i^2 \cdot \sigma_I^2 + \sigma_{\varepsilon_i}^2} = 1 - \frac{\sigma_{\varepsilon_i}^2}{\sigma_i^2} = \rho_{iI}^2$$

where

R^2	coefficient of determination in a regression of R_i on R_I
σ_i^2 :	total variance of the returns on asset i
$\beta_i^2 \sigma_M^2$:	market or systematic risk (explained variance)
$\sigma_{\varepsilon_i}^2$:	idiosyncratic or residual or unsystematic risk (unexplained variance)
ρ_{iI}	correlation between asset i and the index *I*

3.1.5.2 Multi-Factor Models

Multi-Index Models

$$r_i = \alpha_i + \beta_{i1} I_1 + \beta_{i2} I_2 + ... + \beta_{in} I_n + \varepsilon_i$$

where

R_i	return on asset or portfolio i
β_{ij}	beta or sensitivity of the return of asset i to changes in index j
I_j	index j
ε_i	random error term
n	number of indices

Portfolio variance under a multi-index model (every index is assumed to be uncorrelated with each other)

$$\sigma_P^2 = \beta_{P,1}^2 \cdot \sigma_1^2 + \ldots + \beta_{P,n}^2 \cdot \sigma_n^2 + \sigma_{\varepsilon P}^2$$

where

σ_P^2	variance of the portfolio
σ_i^2	variance of the asset or portfolio *i*
$\beta_{P,j}^2 \sigma_j^2$	systematic risk due to index *j*
$\sigma_{\varepsilon P}^2$:	residual risk
n	number of indices

3.2 Practical Portfolio Management

3.2.1 Managing an Equity Portfolio

Active Return

$$R_{A,t}^{P,B} = R_t^P - R_t^B$$

where

$R_{A,t}^{P,B}$	active return in period *t*
R_t^P	return of the portfolio in period *t*
R_t^B	return of the benchmark in period *t*

Tracking Error

$$TE^{P,B} = \sqrt{V(R_A^{P,B})}$$

where

$TE^{P,B}$	tracking error
$V\left(R_A^{P,B}\right)$	variance of the active return

The Multi-Factor Model Approach

<u>Asset excess return</u>

$$R_i = \sum_{j=1}^{NF} x_{i,j} F_j + \varepsilon_i$$

where:

R_i	excess return of an asset *i* (*i* = 1, …, *N)*
$x_{i,j}$	exposure (factor-beta respectively factor-loading) of asset *i* to factor *j*
F_j	excess return of factor *j* (*j* = 1, … , *NF*)
ε_i	specific return of asset *i* (residual return)
NF	number of factors

Portfolio excess return

$$R_P = x_P' \cdot F + \varepsilon_P$$

where:

$$x_P' = (x_{P,1}, \ldots, x_{P,NF}), \quad x_{P,j} = \sum_{i=1}^{N} w_i^P \cdot x_{i,j} \quad (j=1,\ldots,NF)$$

and

$x_{i,j}$	exposure (factor-beta respectively factor-loading) of asset *i* to factor *j*
$x_{P,j}$	exposure of the portfolio to factor *j*
$F = (F_1,...,F_{NF})$	is the *NF* x 1 vector of factor returns,
w_i^P	is the weight of asset *i* in the portfolio
$\varepsilon_P = \sum_{i=1}^{N} w_i^P \cdot \varepsilon_i$	specific return of the portfolio, where ε_i is the specific return of asset *i*
NF	number of factors
N	number of assets in the portfolio

Volatility of the portfolio

$$V(R_P) = x_P' \cdot W \cdot x_P + s_P^2 \qquad s_P^2 = \sum_{i=1}^{N} (w_i^P)^2 \cdot s_i^2$$

where:

x_P'	1 x *NF* vector of portfolio exposures to factor returns
W	covariance matrix of vector *F*, i.e. of the factor returns
s_i^2	variance of asset *i* specific return
s_P^2	variance of the portfolio's specific return
N	number of assets in the portfolio

Tracking error

$$TE^{P,B} = \sqrt{(x_P' - x_B') \cdot W \cdot (x_P - x_B) + \sum_{i=1}^{N} (w_i^P - w_i^B)^2 \cdot s_i^2}$$

where:

$TE^{P,B}$	tracking error of the portfolio with respect to the benchmark
x_P'	1 x *NF* vector of portfolio exposures to factor returns
x_B'	1 x *NF* vector of benchmark exposures to factor returns
W	covariance matrix of vector *F*, i.e. of the factor returns
w_i^P	weight of asset *i* in the portfolio
w_i^B	weight of asset *i* in the benchmark
s_i^2	variance of asset *i* specific return
N	number of assets in the portfolio

Forecasting the tracking error

$$\tilde{TE}^{P,B} = \sqrt{(x'_P - x'_B) \cdot \tilde{W} \cdot (x_P - x_B) + \sum_{i=1}^{N} (w_i^P - w_i^B)^2 \cdot \tilde{s}_i^2}$$

where:

$\tilde{TE}^{P,B}$	forecasted tracking error of the portfolio with respect to the benchmark
x'_P	1 x *NF* vector of portfolio exposures to factor returns
x'_B	1 x *NF* vector of benchmark exposures to factor returns
$\tilde{W}$	is the forecast covariance matrix of vector *F*, i.e. of the factor returns
w_i^P	weight of asset *i* in the portfolio
w_i^B	weight of asset *i* in the benchmark
$\tilde{s}_i^2$	is the forecast of the variance of asset *i* specific return.
N	number of assets in the portfolio

3.2.1.1 Active management

Excess Return and Risk

Expected active return

$$\tilde{R}_A^{P,B} = \tilde{R}_P - \tilde{R}_B = \sum_{i=1}^{N} (w_i^P - w_i^B) \cdot (\tilde{R}_i - \tilde{R}_B)$$

where

w_i^P	weight of asset *i* in the portfolio
w_i^B	weight of asset *i* in the benchmark
$\tilde{R}_i$	expected return of asset *i*
$\tilde{R}_P$	expected return of the portfolio
$\tilde{R}_B$	expected return of the benchmark
N	number of assets in the portfolio

Expected tracking error

$$\tilde{TE}_A^{P,B} = \sqrt{\sum_{i=1;\,j=1}^{N} (w_i^P - w_i^B) \cdot \tilde{C}_{i,j} \cdot (w_j^P - w_j^B)}$$

where

$\tilde{C}_{i,j}$	is the forecast of the covariance of asset *i* and *j* returns
w_i^P	weight of asset *i* in the portfolio
w_i^B	weight of asset *i* in the benchmark
N	number of assets in the portfolio

Information Ratio

$$\widetilde{IR}_A^{P,B} = \frac{\widetilde{R}_A^{P,B}}{\widetilde{TE}_A^{P,B}}$$

where

$\widetilde{IR}_A^{P,B}$ information ratio for portfolio *P* with respect to benchmark *B*
$\widetilde{R}_A^{P,B}$ the expected active return of the portfolio
$\widetilde{TE}_A^{P,B}$ the expected tracking error

3.2.2 Derivatives in Portfolio Management

3.2.2.1 Portfolio Insurance

Static Portfolio Insurance

Portfolio Return

$$r_{PC} + r_{PD} = r_f + \beta(r_{MC} + r_{MD} - r_f)$$

where

r_{PC} capital gain of the portfolio
r_{PD} dividend yield of the portfolio
r_{MC} price index return
r_{MD} dividend yield of the index
r_f risk-free rate
β portfolio beta with respect to the index

The protective put strategy

$$N_P = \beta \cdot \frac{\text{Portfolio value}}{\text{Index level} \cdot \text{Option contract size}} = \beta \cdot \frac{S_0}{I_0 \cdot k}$$

where

N_P number of protective put options
S_0 initial value of the portfolio to be insured
I_0 initial level of the index
β portfolio beta with respect to the index
k option contract size

Initial Value of Insured Portfolio (per unit of option contract size)

$$V_0 = S_0 + \beta \cdot P(I_0, T, K) \cdot \frac{S_0}{I_0}$$

where

- V_0 initial total value of the insured portfolio
- S_0 initial value of the portfolio to be insured
- I_0 initial level of the index
- β portfolio beta with respect to the index
- $P(I_0,T,K)$ put premium for a spot I_0, a strike K and maturity T

Floor

$$f = \frac{\Phi}{V_0}$$

where

- f insured fraction of the initial total portfolio value
- Φ floor
- V_0 initial total value of the insured portfolio

Paying Insurance on Managed Funds

$$V_T = \left[(1-\beta)(1+r_f) + \beta \cdot r_{MD} + \beta \cdot \frac{K}{I_0}\right] \cdot S_0 = f \cdot \left(S_0 + \beta \cdot P(I_0, T, K)\frac{S_0}{I_0}\right)$$

Strike price

$$K = \frac{I_0}{\beta}\left[f \cdot \left(1 + \beta \cdot \frac{P(I_0, T, K)}{I_0}\right) - (1-\beta)(1+r_f) - \beta \cdot r_{MD}\right]$$

where

- V_T total final value of the insured portfolio
- S_0 initial value of the portfolio to be insured
- I_0 initial level of the index
- β portfolio beta with respect to the index
- f insured fraction of the initial total portfolio value
- r_{MD} dividend yield of the index
- r_f risk-free rate
- $P(I_0,T,K)$ put premium for a spot I_0, a strike K and maturity T

In Case of Insurance Paid Externally

$$V_T = \left[(1-\beta)(1+r_f) + \beta \cdot r_{MD} + \beta \cdot \frac{K}{I_0}\right] \cdot S_0 = f \cdot S_0$$

Strike price

$$K = \frac{I_0}{\beta}\left[f - (1-\beta)(1+r_f) - \beta \cdot r_{MD}\right]$$

where

V_T	total final value of the insured portfolio
S_0	initial value of the portfolio to be insured
I_0	initial level of the index
β	portfolio beta with respect to the index
f	insured fraction of the initial total portfolio value
r_{MD}	dividend yield of the index
r_f	risk-free rate

Dynamic Portfolio Insurance

Price of a European Put on an Index Paying a Continuous Dividend Yield *y*

Black&Scholes Model

$$P(S_t, T, K) = K \cdot e^{-r_f \cdot (T-t)} \cdot N(-d_2) - S_t \cdot \left(e^{-y \cdot (T-t)} \cdot N(-d_1)\right)$$

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r_f - y) \cdot (T-t)}{\sigma \cdot \sqrt{T-t}} + \frac{1}{2} \cdot \sigma \cdot \sqrt{T-t} \qquad d_2 = d_1 - \sigma \cdot \sqrt{T-t}$$

where

$P(S_t, T, K)$	put premium for a spot S_t, a strike K and maturity T
S_t	index spot price at time t
K	strike price
r_f	risk-free rate (continuously compounded, p.a.)
y	dividend yield (continuously compounded, p.a.)
σ	volatility of index returns (p.a.)
$T-t$	time to maturity (in years)
$N(.)$	cumulative normal distribution function

Delta of a European Put on an Index Paying a Continuous Dividend Yield *y*

$$\Delta_P = e^{-y \cdot (T-t)} \cdot \left[N(d_1) - 1\right]$$

where

Δ_P	delta of a put
y	dividend yield (continuously compounded, p.a.)
$T-t$	time to maturity (in years)

Dynamic Insurance with Futures

$$N_F = -e^{y\cdot(T^*-T)} \cdot e^{-r_f\cdot(T^*-t)} \cdot [1 - N(d_1)] \cdot \beta \cdot \frac{N_S}{k}$$

where

- N_F number of futures
- T^* maturity of the futures contract
- T maturity of the replicated put
- β risky asset beta with respect to the index
- N_S number of units of the risky assets
- k futures contract size

Constant Proportion Portfolio Insurance (CPPI)

Cushion

$$c_t = V_t - \Phi_t$$

where

- c_t cushion
- V_t value of the portfolio
- Φ_t floor

Amount Invested in Risky Assets

$$A_t = N_{S,t} \cdot S_t = m \cdot c_t$$

where

- A_t total amount invested in the risky assets at time t
- $N_{S,t}$ number of units of the risky assets
- S_t unit price of the risky assets
- m multiplier
- c_t cushion

Amount Invested in Risk-Free Assets

$$B_t = V_t - A_t$$

where

- B_t value of the risk-free portfolio at time t
- V_t value of the total portfolio at time t
- A_t value of the risky portfolio at time t

3.2.2.2 Hedging with Stock Index Futures

Hedging when Returns are Normally Distributed (OLS Regression)

$$\frac{\Delta S_t}{S_t} = \alpha + \beta \cdot \frac{\Delta F_t}{F_{t,T}} + \varepsilon_t$$

$$HR = \beta \cdot \frac{S_t}{F_{t,T}}$$

where

- ΔS_t changes in spot price at time t
- S_t spot price at time t
- α intercept of the regression line
- β slope of the regression line
- ΔF_t changes in the futures price at time t
- $F_{t,T}$ futures price at time t with maturity T
- ε_t residual term
- HR hedge ratio

And the number of futures contracts to use:

$$N_F = -\beta \cdot \frac{N_S \cdot S_t}{k \cdot F_{t,T}}$$

where

- β slope of the regression line
- N_F number of futures
- N_S number of spot assets
- S_t spot price at time t
- $F_{t,T}$ futures price at time t with maturity T
- k contract size

Adjusting the Beta of a Stock Portfolio

$$HR_{adj} = (\beta^{\text{actual}} - \beta^{\text{target}}) \cdot \frac{S_t}{F_{t,T}}$$

$$N_F = (\beta^{\text{target}} - \beta^{\text{actual}}) \cdot \frac{N_S \cdot S_t}{k \cdot F_{t,T}}$$

where

- HR_{adj} hedge ratio to adjust the beta to the target beta
- β^{actual} actual beta of the portfolio
- β^{target} target beta of the portfolio
- S_t spot price at time t
- $F_{t,T}$ futures price at time t with maturity T
- N_F number of futures contracts
- N_S number of the spot asset to be hedged
- k contract size

3.2.2.3 Hedging with Interest Rate Futures

Hedge Ratio

$$HR = \rho_{\Delta B, \Delta F} \frac{\sigma_{\Delta B}}{\sigma_{\Delta F}} = \frac{B_0 \cdot D_B^{mod}}{F_{0,T} \cdot D_F^{mod}}$$

where

- HR hedge ratio
- $\rho_{\Delta B, \Delta F}$ is correlation between bond portfolio and futures value
- σ is volatility of portfolio and futures returns respectively
- D^{mod} is modified duration of bond portfolio and futures respectively
- B_0 is the value of the bond portfolio at time 0
- $F_{0,T}$ is the value of the futures at time 0

Adjusting the Target Duration

$$HR = \frac{S_0 \cdot (D_S^{target} - D_S^{actual})}{F_{0,T} \cdot D_F}$$

where

HR	hedge ratio
S_0	is the value of the spot price at time 0
D_S^{target}	is the target duration
D_S^{actual}	is actual duration
$F_{0,T}$	is the futures price at time 0
D_F	is the duration of the futures (i.e. of the CTD)

And the number of futures contracts to use:

$$\begin{matrix}\text{Number of}\\\text{futures}\\\text{contracts}\end{matrix} = \frac{\begin{matrix}\text{Market value}\\\text{of portfolio}\end{matrix}}{\begin{matrix}\text{Market value}\\\text{of futures}\end{matrix}} \cdot \frac{\begin{pmatrix}\text{Target}\\\text{duration}\end{pmatrix} - \begin{pmatrix}\text{Portfolio}\\\text{duration}\end{pmatrix}}{\begin{pmatrix}\text{Duration}\\\text{of CTD}\end{pmatrix}}$$

3.3 Performance Measurement

3.3.1 Performance Measurement and Evaluation

3.3.1.1 Risk-Return Measurement

Internal Rate of Return (IRR)

$$CF_0 = \sum_{t=1}^{N} \frac{CF_t}{(1+IRR)^t}$$

where

CF_0	initial net cash flow
CF_t	net cash flow at the end of period *t*
IRR	internal rate of return (per period)
N	number of periods

Time Weighted Return (TWR)

Simple Return

$$TWR_{t/t-1} = \frac{MV_{end,t} - MV_{begin,t}}{MV_{begin,t}} = \frac{MV_{end,t}}{MV_{begin,t}} - 1$$

where

$TWR_{t/t-1}$	simple time weighted return for sub-period *t*
$MV_{begin,t}$	market value at the beginning of sub-period *t*
$MV_{end,t}$	market value at the end of sub-period *t*

Continuously Compounded Return

$$twr_{t/t-1} = ln\left(\frac{MV_{end,t}}{MV_{begin,t}}\right)$$

where

$twr_{t/t-1}$	continuously compounded time weighted return for sub-period *t*
$MV_{begin,t}$	market value at the beginning of sub-period *t*
$MV_{end,t}$	market value at the end of sub-period *t*

Total period simple return

$$1 + TWR_{tot} = \prod_{t=1}^{N}(1 + TWR_{t/t-1})$$

where

TWR_{tot}	simple time weighted return for the total period
$TWR_{t/t-1}$	simple time weighted return for sub-period *t*

Total Period Continuously Compounded Return

$$twr_{tot} = \sum_{t=1}^{N} twr_{t/t-1}$$

where

twr_{tot} continuously compounded time weighted return for the total period

$twr_{t/t-1}$ continuously compounded time weighted return for sub-period *t*

Money Weighted Return (MWR)

Gain or Loss Incurred on a Portfolio

Gain = (Ending Market Value – Beginning Market Value) – Net Cash Flow

Net Cash Flow (NCF)

$$NCF = (\sum C_t + \sum P_t + \sum E_t) - (\sum W_t + \sum S_t + \sum D_t + \sum R_t)$$

where

NCF net cash flow
C_t effective contributions
P_t purchases
E_t immaterial contributions measured by the expenses they generate
W_t effective withdrawals
S_t sales
D_t net dividend and other net income
R_t reclaimable taxes

Average Invested Capital (AIC)

Average Invested Capital = Beginning Market Value + Weighted Cash Flow

Dietz Formula

$$AIC = MV_{begin} + \frac{1}{2} \cdot NCF$$

where

AIC average invested capital
MV_{begin} market value at the beginning of the period
NCF net cash flow

Dietz formula

$$MWR = \frac{(MV_{end} - MV_{begin}) - NCF}{MV_{begin} + \frac{1}{2} \cdot NCF}$$

where

MWR money weighted return
MV_{begin} market value at the beginning of the period
MV_{end} market value at the end of the period
NCF net cash flow

Value Weighted Day (VWD)

$$VWD_j = \frac{\sum CF_{j,i} \cdot t_i}{\sum CF_{j,i}}$$

where

VWD_j value weighted day of the total cash flow of type *j* (contributions, purchases, sales, ...)
$CF_{j,i}$ i-th cash flow of type *j* (contributions, purchases, sales, ...)
t_i day when the *i*-th cash flow takes place

Day Weighted Return

$$AIC = MV_{begin} + \sum_{\substack{cash \\ flow\, i}} \frac{t_{end} - t_i}{t_{end} - t_{begin}} \cdot CF_i$$

$$= MV_{begin} + \underbrace{\left(p_C \cdot \sum C_i + p_P \cdot \sum P_i + p_E \cdot \sum E_i\right) - \left(p_W \cdot \sum W_i + p_S \cdot \sum S_i + p_D \cdot \sum D_i + p_R \cdot \sum R_i\right)}_{=WCF}$$

where

AIC average invested capital
MV_{begin} market value at the beginning of the period
t_i time of cash flow *I*
t_{begin} time corresponding to the beginning of the period
t_{end} time corresponding to the end of the period
CF_i cash flow
C_i effective contributions
P_i purchases
E_i immaterial contributions measured by the expenses they generate
W_i effective withdrawals
S_i sales
D_i net dividend and other net income
R_i reclaimable taxes
WCF weighted cash flow

p_C, p_P, p_E, p_W, p_S, p_D and p_R are the weights calculated as follows:

$$\text{weight} = p_j = \frac{\sum (t_{end} - t_i) \cdot CF_{j,i}}{(t_{end} - t_{begin}) \cdot \sum CF_{j,i}} = \frac{t_{end} - VWD_j}{t_{end} - t_{begin}}$$

where

j various cash flow types (contributions, purchases, expenses etc.)
$CF_{j,i}$ *i*-th cash flow of type *j*
VWD_j value weighted day

Day weighted return

$$MWR = \frac{(MV_{end} - MV_{begin}) - \left((\sum C_i + \sum P_i + \sum E_i) - (\sum W_i + \sum S_i + \sum D_i + \sum R_i)\right)}{MV_{begin} + \left((p_C \cdot \sum C_i + p_P \cdot \sum P_i + p_E \cdot \sum E_i) - (p_W \cdot \sum W_i + p_S \cdot \sum S_i + p_D \cdot \sum D_i + p_R \cdot \sum R_i)\right)}$$

where

MV_{end} market value at the end of the period

3.3.1.2 Risk Adjusted Performance Measures

Sharpe Ratio or *Reward-to-Variability Ratio*	$RVAR_P = \frac{\bar{r}_P - \bar{r}_f}{\sigma_P}$
Treynor Ratio or *Reward-to-Volatility Ratio*	$RVOL_P = \frac{\bar{r}_P - \bar{r}_f}{\beta_P}$
Jensen's α	$\alpha_P = (\bar{r}_P - \bar{r}_f) - \beta_P \cdot (\bar{r}_M - \bar{r}_f)$
Information Ratio (*Appraisal Ratio)*	$AR_P = \frac{\alpha}{\sigma_\varepsilon}$

where

$\bar{r}_P$ average portfolio return
$\bar{r}_M$ average market return
$\bar{r}_f$ average risk-free rate
α_P Jensen's alpha
α Active Return (Excess return)
β_P portfolio beta
σ_P portfolio volatility
σ_ε Active Risk (standard deviation of the tracking error)

3.3.1.3 Relative Investment Performance

Elementary Price Indices

$$P_{t/0} = \frac{p_t}{p_0} \cdot B = (1 + R_{t/0}) \cdot B$$

where

$P_{t/0}$ elementary price index at time *t* with basis at time 0
p_0 price of the original good at time 0
p_t price of the unchanged good at time *t*
$R_{t/0}$ index return for the period starting at 0 and ending at *t*
B index level at the reference time

Price-Weighted Indices

$$U_{t/0} = \frac{\sum_{j=1}^{n} p_t^j}{\sum_{j=1}^{n} p_0^j} \cdot B = \frac{p_t^1 + p_t^2 + \cdots + p_t^n}{\underbrace{p_0^1 + p_0^2 + \cdots + p_0^n}_{D_{t/0} = \sum_{j=1}^{n} c_{t/0}^j \cdot p_0^j}} \cdot B = \frac{p_t^1 + p_t^2 + \cdots + p_t^n}{D_{t/0}} \cdot B$$

where

$U_{t/0}$	price-weighted index at time *t* with basis 0
n	number of securities
t	actual time
0	reference time (the index basis)
j	security identifier
$D_{t/0}$	divisor
p_t^j , p_0^j	price of security *j* at time *t*, respectively at time 0
$c_{t/0}^j$	adjustment coefficient for a corporate action on security *j* (=1 at time 0)
B	index level at the reference time

Equally Weighted Price Indices

Arithmetic average of the elementary price indices:

$$\overline{P}_{t/0} = \frac{1}{n} \cdot \sum_{j=1}^{n} P_{t/0}^j = \frac{P_{t/0}^1 + P_{t/0}^2 + \cdots + P_{t/0}^n}{n}$$

where

$\overline{P}_{t/0}$	arithmetic average of the elementary price indices
n	number of elementary price indices
$P_{t/0}^j = \frac{p_t^j}{c_{t/0}^j \cdot p_0^j}$	elementary index for security *j* where:
p_t^j , p_0^j	price of security *j* at time *t*, respectively at time 0
$c_{t/0}^j$	adjustment coefficient for a corporate action on security *j* (=1 at time 0)

Geometric average of the elementary price indices:

$$\overline{P}_{t/0,g} = \left(\prod_{j=1}^{n} P_{t/0}^j \right)^{1/n} = \left(P_{t/0}^1 \cdot P_{t/0}^2 \cdot \cdots \cdot P_{t/0}^n \right)^{1/n} = \sqrt[n]{P_{t/0}^1 \cdot P_{t/0}^2 \cdot \cdots \cdot P_{t/0}^n}$$

where

$\overline{P}_{t/0,g}$	geometric average of the elementary price indices
n	number of elementary price indices
$P_{t/0}^j = \frac{p_t^j}{c_{t/0}^j \cdot p_0^j}$	elementary index for security *j*
p_t^j , p_0^j	price of security *j* at time *t*, respectively at time 0
$c_{t/0}^j$	adjustment coefficient for a corporate action on security *j* (=1 at time 0)

Capital Weighted Price Indices (Laspeyres Indices)

$$PIL_{t/0} = \frac{\sum_{j=1}^{n} p_t^j \cdot q_0^j}{\sum_{j=1}^{n} p_0^j \cdot q_0^j} = \sum_{j=1}^{n} w_0^j \cdot P_{t/0}^j = \sum_{j=1}^{n} \frac{p_0^j \cdot q_0^j}{\sum_{i=1}^{n} p_0^i \cdot q_0^i} \cdot P_{t/0}^j$$

$$P_{t/0}^j = \frac{p_t^j}{p_0^j} = 1 + R_{t/0}^j$$

where

$PIL_{t/0}$	Laspeyres capital-weighted index
p_t^j	actual price of security *j*
q_0^j	number of outstanding securities *j* at the basis
p_0^j	price of security *j* at the basis
w_0^j	weight of security j in the index at the basis i.e. relative market capitalization of security *j* at the basis
$R_{t/0}^j$	return in security *j* between the basis and time *t*

Index Scaling

$$I_{t/0}^{scaled} = \frac{I_{t/0}^{original}}{I_{t_k/0}^{original}} \cdot B_{t_k} = I_{t/0}^{original} \cdot \frac{B_{t_k}}{I_{t_k/0}^{original}}$$

where

$I_{t/0}^{scaled}$	scaled index at time t with base 0 and level B_{t_k} at scaling time t_k
$I_{t/0}^{original}$	unscaled original index at time *t* with basis 0
$I_{t_k/0}^{original}$	unscaled original index at scaling time t_k with basis 0
B_{t_k}	scaling level (typically 100 or 1000)

Index Chain-Linking

$$I_{t/0}^{chained} = \frac{I_{t/0}^{new}}{I_{t_k/0}^{new}} \cdot I_{t_k/0}^{old} = f_{t/t_k} \cdot I_{t_k/0}^{old}$$

where

$I_{t/0}^{chained}$	chained index level at time *t*, with original basis in 0
$I_{t/0}^{new}$	new index computed at time *t*
$I_{t_k/0}^{new}$	new index computed at chain-linking time t_k
$I_{t_k/0}^{old}$	old index level at chain-linking time t_k
f_{t/t_k}	chaining factor at time *t*, with chain-linking time t_k = 1 + R_{t/t_k}

$$I_{t/0}^{chained} = f_{t/t-1} \cdot f_{t-1/t-2} \cdot \ldots \cdot f_{2/1} \cdot f_{1/0} \cdot B_0$$
$$= (1+R_{t/t-1}) \cdot (1+R_{t-1/t-2}) \cdot \ldots \cdot (1+R_{2/1}) \cdot (1+R_{1/0}) \cdot B_0$$

where

$R_{t/t-1}$ elementary index return for period *t*
B_0 index level at reference time 0

Sub-Indices

General Index

$$I_{t/0}^{general} = \sum_{\text{segments } K_t^i} w_{t/0}^{K_t^i} \cdot I_{t/0}^{K_t^i}$$

where

$I_{t/0}^{general}$	index level for the general index at time *t* with reference time 0
$I_{t/0}^{K_t^i}$	index level for the segment K^i at time *t* with reference time 0
$w_{t/0}^{K_t^i}$	the relative market capitalization of the segment K^i at time *t*

Sub-Index Weight

$$w_0^K = \sum_{j \in K} w_0^j = \sum_{j \in K} \frac{p_0^j \cdot q_0^j}{\sum_{i=1}^{n} p_0^i \cdot q_0^i}$$

where

w_0^K	the relative market capitalisation of the segment *K*
w_0^j	the relative market capitalisation of security *j*
$\sum_{j \in K}$	the sum over all securities in segment *K*
$\sum_{i=1}^{n}$	the sum over all n securities in the general aggregated index

Performance Indices

Elementary Performance Index

$$I_{t/0}^{perf} = f_{t/s} \cdot I_{s/0}^{perf} = (1 + R_{t/s}) \cdot I_{s/0}^{perf}$$
$$= (1 + R_{t/s}) \cdot (1 + R_{s/s-1}) \cdot (1 + R_{s-1/s-2}) \cdot \ldots \cdot (1 + R_{2/1}) \cdot (1 + R_{1/0}) \cdot B_0$$

where

$I_{t/0}^{perf}$	index level at time *t*
$I_{s/0}^{perf}$	index level at time *s*
$f_{t/s}$	compounding or chain-linking factor
$R_{t/s}$	elementary index performance from time *s* until time *t*
B_0	index level at reference time 0

Compounding Factor in the Presence of Income

$$f_{t/s}^{i} = \frac{\sum_{j \in K_s^i} (p_t^j + d_{s+1}^j) \cdot q_s^j}{\sum_{j \in K_s^i} p_s^j \cdot q_s^j}$$

where

j	security identifier in segment *i*
$f_{t/s}^{i}$	compounding factor for segment *i*
q_s^j	number of outstanding shares of security *j* at previous closing/chaining time *s*
p_s^j	cum-dividend price of security *j* at previous closing time *s*
p_t^j	ex-dividend price of security *j* at actual time *t*
d_{s+1}^j	dividend detached from security *j* on day *s*+1

Compounding Factor in the Presence of Subscription Rights

$$f_{t/s}^{i} = \frac{\sum_{j \in K_s^i} (p_t^j + r_{s+1}^j) \cdot q_s^j}{\sum_{j \in K_s^i} p_s^j \cdot q_s^j}$$

where

$f_{t/s}^{i}$	compounding factor for segment *i*
q_s^j	number of outstanding shares of security *j* at previous closing/chaining time *s*
p_s^j	cum-right price of security *j* at previous closing time *s*
p_t^j	ex-right price of security *j* at actual time *t*
r_{s+1}^j	price quoted for the right detached from security *j* on day *s*+1

Multi-Currency Investments and Interest Rate Differentials

Simple Currency Return $C_{BC/LC}$

$$1 + C_{BC/LC} = \frac{S_{BC/LC,t}}{S_{BC/LC,t-1}}$$

where

$C_{BC/LC}$	simple currency return
$S_{BC/LC,t}$	spot exchange rate at the end of the period
$S_{BC/LC,t-1}$	spot exchange rate at the beginning of the period

Forward Exchange Rate Return $C_{F,BC/LC}$

$$1 + C_{F,BC/LC} = \frac{F_{BC/LC,T}}{S_{BC/LC,t}} = \frac{1 + R_{f,BC} \cdot (T - t)}{1 + R_{f,LC} \cdot (T - t)}$$

where

$C_{F,BC/LC}$	simple currency forward return
$S_{BC/LC,t}$	spot exchange rate at time *t*
$F_{BC/LC,T}$	frward exchange rate at time t with expiration in *T*
$R_{f,BC}$	risk-free rate on the base currency
$R_{f,LC}$	risk-free rate on the local currency

Unexpected Currency Return EBC/LC

$$1 + C_{BC/LC} = \frac{S_t}{F_{t-1,1}} \cdot \frac{1 + R_{f,BC}}{1 + R_{f,LC}} = \left(1 + E_{BC/LC}\right) \cdot \left(1 + C_{F,BC/LC}\right)$$

where

$C_{BC/LC}$	simple currency return
$E_{BC/LC}$	unexpected currency return
$C_{F,BC/LC}$	simple currency forward return
S_t	spot price at the end of the period
$F_{t-1,1}$	forward rate at the beginning of the period for one period
$R_{f,BC}$	risk-free rate on the base currency
$R_{f,LC}$	risk-free rate on the local currency

3.3.1.4 Performance attribution analysis

Attribution Methods Based on Simple Linear Regression

Jensen's α

$$\alpha_P = R_P - R_E = \underbrace{(R_P - R_B)}_{Net\ selectivity} + \underbrace{(R_B - R_E)}_{Diversification}$$

where

$$R_B = R_f + \frac{\sigma_P}{\sigma_M} \cdot (R_M - R_f)$$
$$R_E = R_f + \beta_P \cdot (R_M - R_f)$$

.

and

- α_P Jensen's alpha
- R_P actual portfolio return
- R_M actual market return
- R_f risk-free rate
- R_E portfolio return at equilibrium with beta equal to β_P
- R_B portfolio return at equilibrium with volatility equal to σ_P
- σ_P portfolio volatility
- σ_M market volatility

<u>Fama's Break-up of Excess Return</u>

$$R_P - R_f = \underbrace{(R_P - R_B)}_{Net\ selectivity} + \underbrace{(R_B - R_E)}_{Diversification} + \underbrace{(R_E - R_M)}_{\substack{return\ premium \\ above\ market\ risk}} + \underbrace{(R_M - R_f)}_{\substack{return\ premium \\ for\ market\ risk}}$$

where

- R_P actual portfolio return
- R_f risk-free rate
- R_B portfolio return at equilibrium with volatility equal to σ_P
- R_E portfolio return at equilibrium with beta equal to β_P
- R_M actual market return

3.3.1.5 Special Issues

Performance Evaluation of International Investments

Simple Return

$$1+R_{Dt}=(1+R_{Ft})\cdot(1+s_t)$$

where

R_{Dt}	simple rate of return denominated in domestic currency
R_{Ft}	simple rate of return denominated in foreign currency
s_t	relative change (depreciation or appreciation) in the value of the domestic currency

Continuously Compounded Return

$$r_{Dt}=r_{Ft}+s_t^{cc}$$

where

r_{Dt}	continuously compounded rate of return denominated in domestic currency
r_{Ft}	continuously compounded rate of return denominated in foreign currency
s_t^{cc}	continuously compounded relative change (depreciation or appreciation) in the value of the domestic currency

Variance of Continuously Compounded Returns

$$Var[r_{Dt}]=Var[r_{Ft}]+Var[s_t^{cc}]+2\cdot Cov[r_{Ft},s_t^{cc}]$$

where

$Var[\cdot]$	the covariance operator.
$Cov[\cdot]$	the covariance operator.
r_{Dt}	continuously compounded rate of return denominated in domestic currency
r_{Ft}	continuously compounded rate of return denominated in foreign currency
s_t^{cc}	continuously compounded relative change (depreciation or appreciation) in the value of the domestic currency

A Single Currency Attribution Model by Brinson & al.

$$VA = R - I = \sum w_{P,j} \cdot R_{P,j} - \sum w_{I,j} \cdot R_{I,j}$$

$$= \sum \left((w_{P,j} - w_{I,j}) \cdot R_{I,j} + w_{I,j} \cdot (R_{P,j} - R_{I,j}) + (w_{P,j} - w_{I,j}) \cdot (R_{P,j} - R_{I,j}) \right)$$

where

VA	value added
R	portfolio return
I	benchmark return
$\sum$	sum over every market in the portfolio and in the benchmark
$R_{P,j}$	portfolio return in each market
$R_{I,j}$	index return in each market
$w_{P,j}$	portfolio weight in each market at the beginning of the period
$w_{I,j}$	index weight in each market at the beginning of the period

<u>Practitioners' Break-up of Value Added</u>

$$VA = R - I = \sum (w_{P,j} - w_{I,j}) \cdot (R_{I,j} - I) + \sum w_{P,j} \cdot (R_{P,j} - R_{I,j})$$

where

VA	value added
R	portfolio return
I	benchmark return
$\sum$	sum over every market in the portfolio and in the benchmark
$R_{P,j}$	portfolio return in each market
$R_{I,j}$	index return in each market
$w_{P,j}$	portfolio weight in each market at the beginning of the period
$w_{I,j}$	index weight in each market at the beginning of the period

Multi-Currency Attribution and Interest Rate Differentials

<u>Value Added in Base Currency</u>

$$va_{BC} = r_{BC} - i_{BC}$$

where

va_{BC}	value added in base currency
r_{BC}	portfolio return in base currency(continuously compounded)
i_{BC}	benchmark return in base currency(continuously compounded)

<u>Base Currency Adjusted Market Return</u>

$$r_{BC,adj} = r_{LC} + r_{f,BC} - r_{f,LC}$$

where

$r_{BC,adj}$	adjusted local market return in base currency
r_{LC}	local market return in local currency
$r_{f,BC}$	risk-free rate in base currency at the beginning of the period
$r_{f,LC}$	risk-free rate in local currency at the beginning of the period

Unexpected Currency Return

$$e_{BC/LC} = c_{BC/LC} - r_{f,BC} + r_{f,LC}$$

where

$e_{BC/LC}$	unexpected currency return
$c_{BC/LC}$	actual currency return
$r_{f,BC}$	risk-free rate in base currency at the beginning of the period
$r_{f,LC}$	risk-free rate in local currency at the beginning of the period

Break-up of Value Added in Base Currency

$$va_{BC} = (\sum w_P \cdot r_{BC,P,adj} - \sum w_I \cdot r_{BC,I,adj}) + (\sum w_P \cdot e_P - \sum w_I \cdot e_I)$$

where

va_{BC}	value added in base currency
$\sum$	sum over every market in the portfolio and in the benchmark
$r_{BC,P,adj}$	adjusted portfolio return of the local market in base currency
$r_{BC,I,adj}$	adjusted index return of the local market in base currency
e_P	unexpected portfolio currency return
e_I	unexpected passive currency return
w_P	portfolio weight in each market at the beginning of the period
w_I	index weight in each market at the beginning of the period

