SMALL-SCALE EFFECTS ON THE BUCKLING OF SKEW NANOPLATES BASED ON NON-LOCAL ELASTICITY AND SECOND-ORDER STRAIN GRADIENT THEORY

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ABSTRACT

In recent years, nanostructures have been used in a vast number of applications, making the study of the mechanical behaviour of such structures important. In this paper, two different constitutive equations including first-order strain gradient and simplified differential non-local are employed to model the buckling behaviour of skew nanoplates. The Galerkin method is used for solving the equations in order to obtain buckling load. Using this method, the influence of different parameters consisting of non-classical properties, boundary conditions, and geometrical parameters such as length and angle on the buckling load, are studied. The results showed that small-scale effects are very important in skew graphene sheets and their inclusion results in smaller buckling loads..

Keywords: Skew nanoplate, Non-local, Strain gradient, Galerkin method.

1. INTRODUCTION

Nowadays, small size structures such as micro and nanoplates are used in Micro and Nano Electro Mechanical Systems (MEMS/NEMS) as actuators and sensors. The more realistic modelling of such elements, the more reliable predictions of their performance is provided under different working conditions.

Due to the difficulty in fabrication of nanostructures at certain sizes, and the implementation of experimental setup and applying boundary conditions, more investigation on the theoretical approaches is required to understand both the concept and revealing possible correlations. In this regard, two categories of theoretical studies consisting of molecular dynamics and continuum-based approaches can be followed for the above-mentioned purposes. Although molecular dynamics methods can be employed in this field, it does not provide a parametric viewpoint for the problem, and to understand each parameter dependency, several computer runs should be done, which might take long time. On the other hand, continuum-based approaches are good candidates to model the nanostructures providing possibilities for parametric studies [1].

The literature review indicates that GS has a high number of potential applications in various fields of technology, so that it can be considered as a forehand nanostructure. In this regard, more scientific investigations are necessary in order to understand the properties of this element and predict its behaviour in different situations. Indeed, the more realistic the modelling of such elements, the more reliable the predictions of their performance under different working conditions. This is a strong motivation for researchers to investigate this subject.

Referring to the previously reported studies it can be concluded that a few theories considering size effects such as Eringen's integral non-local and differential non-local [2], strain gradient [3], modified strain gradient [4], couple stress [5], stress gradient [6, 7] and surface energy [8], have been developed for modeling purposes. Lu *et al.* [9] studied the effect of surface properties and size-dependent mechanical behaviour of nanoplates by using the theory of generalized Kirchhoff and Mindlin plate theory. Sakhaee-Pour [10] studied the buckling of single layer graphene sheets by use of the molecular dy-

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namic method. Pradhan *et al.* [11] analysed the buckling of rectangular single layer graphene sheets under biaxial compression by use of the non-local elasticity theory of Eringen. Murmu *et al.* [12] studied the elastic buckling behaviour of orthotropic small scale plates under biaxial compression by use of the non-local elasticity theory of Eringen. Sheng *et al.* [13] analysed the three dimensional elasticity of nanoplates, using the theory of laminated structures. Babaei *et al.* [14] analysed buckling of the quadrilateral nanoplates based on nonlocal plate theory. Malekzadeh *et al.* [15] investigated small-scale effects on the thermo-mechanical buckling characteristics of orthotropic arbitrary straight-sided nanoplates embedded in an elastic medium by use of the non-local elasticity theory of Eringen. Narendar [16] studied the buckling analysis of isotropic nanoplates by using the two variable refined plate theory and nonlocal parameter effects. Arash *et al.* [17] reported a study on wave propagations in single layer graphene sheets by a developed nonlocal finite element model and molecular dynamic simulations. Murmu *et al.* [18] reported an analytical study of the buckling of a double-nanoplate system by use of non-local elasticity theory.

Indeed, the size dependent constitutive equations (including those employed in the above mentioned references) provide extra coefficients as non-classical parameters, representing the size effects explicitly or implicitly in addition to the classical elastic constants. The main aim of the current research is the application of different constitutive equations to study buckling analysis of skew SLGS while exploring size dependency. Comparison of the obtained results of the employed formulations with those reported in the open literature, can help to reveal the potential of each constitutive equation for modelling of the nanoplates.

According to the literature, the buckling of nanoplates has been studied only for particular geometry and boundary conditions, because of limitations of the methods used to solve such problems. In this paper, governing differential equation is introduced before the solving of this equation using the Galerkin method. The results show that the Galerkin method might be used as a powerful method for solving such problems. In addition, higher convergence rate, analysis of different boundary conditions, and exploration of various structural models, are the original contributions of the present work.

2. CONSTITUTIVE EQUATIONS

Employing constitutive relations is the main part of developing governing equations of a mechanical problem, such as vibrational analysis focused on here. In fact, these relations represent the behaviour of the medium relevant to its physical properties. In the current research, in addition to the classical Hook's law as the most well-known constitutive equation, two more constitutive equations including non-local and strain gradient theory are used for this purpose.

2.1 Classical Theory

Based on the classical theory of elasticity, the elastic energy density, $\bar{u}$, for a Hookean solid depends on the symmetric part of the first-order deformation gradient (strain) tensor as follows:

$$\bar{u} = \bar{u}(\varepsilon_{ij}) = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \tag{1}$$

where, ε_{ij} is the symmetric part of the first-order deformation gradient tensor determined as below, named strain tensor, and σ_{ij} is its work-conjugate, or in other words the Cauchy stress tensor.

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2}$$

For conciseness, the symbol "," is used here instead of the symbol of partial differentiate. It is worth mentioning that the stress tensor can be determined by derivatives of elastic energy with respect to the strain via use of the following equation:

$$\sigma_{ij} = \frac{\partial \bar{u}}{\partial \varepsilon_{ij}} \tag{3}$$

On the other hand, based on the classical theory of elasticity, the relation between the stress and strain tensors at each point of the medium is as follow for a general anisotropic medium:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{4}$$

in which C_{ijkl} is fourth order elastic stiffness tensor. Although for fully anisotropic media, 21 elastic constants exist, the above equation is simplified as follows for an isotropic material so that only two independent elastic constants remain:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{5}$$

In this equation, λ and $\mu$are Lame's constants, and δ_{ij} is Kronocker delta. By definition of volumetric strain or dilation strain, ε_{kk}, and deviatoric strain tensor, $\varepsilon'_{ij} = \varepsilon_{ij} - 1/3\varepsilon_{kk}\delta_{ij}$, the above equation can be rewritten in the following form:

$$\sigma_{ij} = \kappa\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon'_{ij} \tag{6}$$

In this equation, κ is bulk modulus and μ is shear modulus of the medium. Substitution of Eqs. (4) and (6) into Eq. (1) gives the following alternatives for the density of elastic energy for anisotropic and isotropic media respectively.

$$\bar{u} = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \tag{7}$$

$$\bar{u} = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} = \frac{1}{2}\kappa\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon'_{ij}\varepsilon'_{ij} \tag{8}$$

From the above discussion, it is concluded that for an isotropic medium, two characteristics define the relation between stress and strain tensors in linear classical elasticity. As the elastic energy is known, applying the principle of virtual work means that the equilibrium equation is obtained as follows:

$$\sigma_{ij,j} + F_i^b = 0 \quad (9)$$

In this equation, F_i^b is the vector of body force per unit volume.

2.2 Second-Order Strain Gradient Theory (2ndSG)

Second-order gradient of the strain tensor is taken into account to construct the constitutive equation. In fact, the second-order deformation gradient (or in other words, the first-order strain gradient) is neglected, while the second-order gradient acts via operator ∇^2 on the symmetric strain tensor as follows, which is a type of formulation proposed by Aifantis [7]. In this regard, this version of second-order strain gradient elasticity is labelled by 2ndSG in the current text.

$$t_{ij} = C_{ijkl}\varepsilon_{kl} - C'_{ijkl}\nabla^2\varepsilon_{kl} \quad (10)$$

where t_{ij} is non-classical stress tensor, C_{ijkl} is classical elastic stiffness tensor, and C'_{ijkl} is anisotropic coefficient tensor of second-order strain gradients. As a special case, by redefining the last tensor as $C'_{ijkl} = \xi^2 C_{ijkl}$, the above equation reads to the following form:

$$t_{ij} = C_{ijkl}\varepsilon_{kl} - \xi^2 C_{ijkl}\nabla^2\varepsilon_{kl} \quad (11)$$

The parameter ξ is the extra coefficient for considering non-local effects. Afterwards, substitution of Hook's law into equation 11, gives the following result as constitutive equation in 2ndSG theory.

$$t_{ij} = (1 - \xi^2\nabla^2)\sigma_{ij} \quad (12)$$

For an isotropic medium, the above equation is simplified so that only two elastic stiffness and one length scale parameter remains, leading to the following form:

$$t_{ij} = \lambda\delta_{ij}\varepsilon_{mm} + 2\mu\varepsilon_{ij} - \xi^2\nabla^2(\lambda\delta_{ij}\varepsilon_{mm} + 2\mu\varepsilon_{ij}) \quad (13)$$

Therefore, in this version of strain gradient theory, in addition to the elastic constants in the classical theory of elasticity, one extra length scale parameter is exhibited. As a matter of fact, spatial derivatives of strains are related to the differences between the strains at the reference point and its neighbours. Therefore, the third term in the above equation represents the non-local effects.

2.3 Simplified Differential Non-Local Theory (SDNL)

Based on the classical elasticity, stresses are related to the strains at the same point. On the other hand, according to original version of Eringen's non-local theory, stress at a point is related to the strains at all points of the medium as follows:

$$t_{ij} = \int_V \beta_{ijkl}(|\mathbf{X}' - \mathbf{X}|)\varepsilon_{kl}(\mathbf{X}')dV(\mathbf{X}') \quad (14)$$

in which t_{ij} is non-classical stress tensor, β_{ijkl} is non-local stiffness kernel, X' is position vector of entire points in-

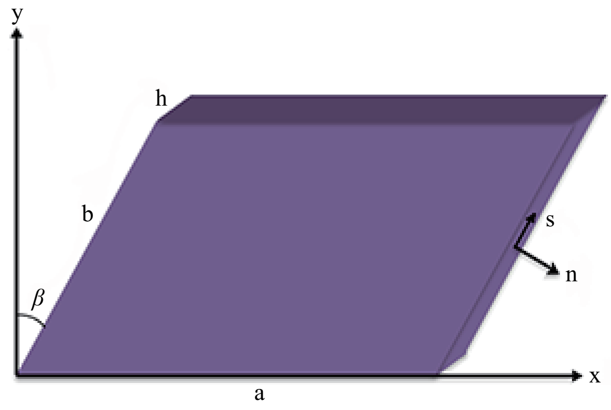

Fig. 1 Skew SLGS.

side the body, and X is the position vector of the reference point where the stress is determined. If the assumption of scalar non-local effects is applied, using the stress-strain relation in classical view, the above equation can be rewritten as follows:

$$\begin{aligned} t_{ij} &= \int_V \alpha(|\mathbf{X}' - \mathbf{X}|)C_{ijkl}\varepsilon_{kl}(\mathbf{X}')dV(\mathbf{X}') \\ &= \int_V \alpha(|\mathbf{X}' - \mathbf{X}|)\sigma_{ij}(\mathbf{X}')dV(\mathbf{X}') \end{aligned} \quad (15)$$

In the above, σ_{kl} is the classical stress tensor and $\alpha(|X'-X|)$ is the non-local modulus, having the dimension of L^{-3}, and can be a characteristic of the medium which has been described as the length scale function in the literature [2]. Applying Fourier transform, the above integral can be converted to differential form. After some manipulation, differential form of this equation becomes as follows [2]:

$$(1 - r_1^2\nabla^2 + r_2^4\nabla^4)t_{ij} = \sigma_{ij} \quad (16)$$

The parameters r_1 and r_2 are characteristics constants, representing the length scale effects in the differential version of Eringen's non-local. However, in the simplified form employed in most relevant studies (e.g., [19-22]) the last term of the left hand side of the equation (16) is neglected, leading to a formulation with only one length scale factor which is called simplified differential non-local (SDNL) theory, here. Therefore, in this version only one extra coefficient is added to the classical constants existed in the constitutive equation. The symbol r_1 is changed to ξ in order to make it similar to the other theories having one length scale, leading to the following constitutive equation:

$$(1 - \xi^2\nabla^2)t_{ij} = \sigma_{ij} \quad (17)$$

3. GOVERNING EQUATIONS

In the current section, the previously reported formulations are employed to develop the buckling formulation of skew SLGS. An initially flat, thin skew nanoplate of constant thickness h, length a, and width b is considered (Fig. 1).

To model a thin plate, Kirchhoff assumptions give the following displacement field:

$$\begin{aligned} u(x,y,z,t) &= u^0(x,y,t) - z\frac{\partial w(x,y,t)}{\partial x} \\ v(x,y,z,t) &= v^0(x,y,t) - z\frac{\partial w(x,y,t)}{\partial y} \\ w(x,y,z,t) &= w(x,y,t) \end{aligned} \tag{18}$$

where $u = u_1$, $v = u_2$ and $w = u_3$ are displacement vector components along x-, y-, and z-directions respectively, and superscript '0' indicates values at midplane of the plate. Substitution of these equations into the linear-strain-displacement relations yields:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u^0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \\ \varepsilon_{yy} &= \frac{\partial v^0}{\partial y} - z\frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} &= \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y} \end{aligned} \tag{19}$$

To complete the rest of the formulation, constitutive equations introduced previously are employed to develop the governing equations of the skew nanoplate, as illustrated in the following subsections. However, applying classical elasticity is omitted because it can be obtained easily from the other theories if the non-local parameter is set to zero.

3.1 Applying 2ndSG Theory

For the case of plane geometry, only three components of stress and strains play roles in the modelling procedure, and the constitutive equation relevant to 2ndSG theory for orthotropic medium becomes as follows:

$$\begin{Bmatrix} t_{xx} \\ t_{yy} \\ t_{xy} \end{Bmatrix} = \left(1-\xi^2\nabla^2\right)\begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{E_2\nu_{12}}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{E_2\nu_{12}}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \tag{20}$$

In the above, engineering elastic constants E_1 and E_2 are the Young's moduli of the orthotropic graphene sheet along directions 1 and 2 as shown in Fig. 2, ν_{12} and ν_{21} are the Poisson's ratios, and G_{12} is the shear modulus.

By substituting strain-displacement relations into the above, all stresses are obtained as functions of lateral displacement w and its derivatives. Applying the principle of virtual work, for a plate under in-plane loading and non-local elasticity illustrated above, the final governing equation is found as follows:

$$\frac{\partial^2 \bar{M}_{xx}}{\partial x^2} + 2\frac{\partial^2 \bar{M}_{xy}}{\partial x \partial y} + \frac{\partial^2 \bar{M}_{yy}}{\partial y^2} + N_{xx}\frac{\partial^2 w}{\partial x^2} + N_{yy}\frac{\partial^2 w}{\partial y^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} = 0 \tag{21}$$

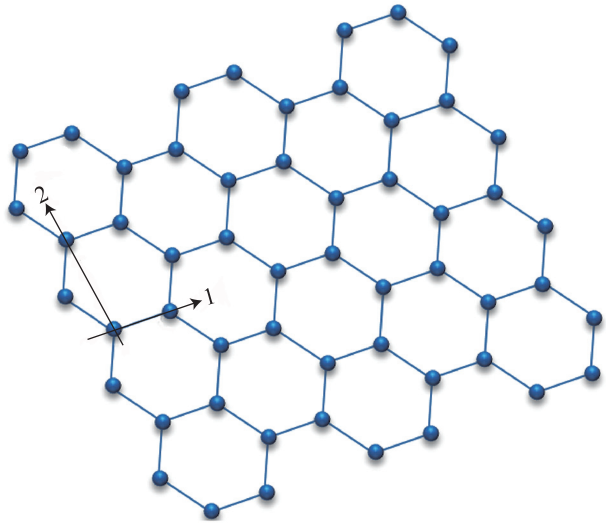

Fig. 2 Orthotropic directions of a graphene sheet.

Here, $\bar{M}_{ij}$'s are equivalent non-local moment components at each section of the nanoplate. They are calculated using the following integrations through the nanoplate thickness h:

$$\left\{\bar{M}_{xx}, \bar{M}_{xy}, \bar{M}_{yy}\right\} = \int_{-h/2}^{+h/2} \left\{t_{xx}z, t_{xy}z, t_{yy}z\right\} dz \tag{22}$$

As mentioned previously, the stress components are known functions of deflection and its derivatives. Therefore, the above moments can be determined in terms of displacement as follows:

$$\begin{aligned} \bar{M}_{xx} &= -(1-\xi^2\nabla^2)\left(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2}\right) \\ \bar{M}_{yy} &= -(1-\xi^2\nabla^2)\left(D_{22}\frac{\partial^2 w}{\partial y^2} + D_{12}\frac{\partial^2 w}{\partial x^2}\right) \\ \bar{M}_{xy} &= -2(1-\xi^2\nabla^2)\left(D_{66}\frac{\partial^2 w}{\partial x \partial y}\right) \end{aligned} \tag{23}$$

By these equations, moments are determined in terms of plate curvatures and the plate stiffness coefficients, D_{ij}, which are functions of the plate properties and its thickness via the following equations:

$$\begin{aligned} D_{11} &= \frac{E_1 h^3}{12(1-\nu_{12}\nu_{21})} \\ D_{22} &= \frac{E_2 h^3}{12(1-\nu_{12}\nu_{21})} \\ D_{12} &= \frac{\nu_{12}E_2 h^3}{12(1-\nu_{12}\nu_{21})} \\ D_{66} &= \frac{G_{12}h^3}{12} \end{aligned} \tag{24}$$

In the theory for a thin plate, however, we would like to work with force quantities which depend on x and y alone. This can be achieved by integrating the stresses in the z-direction, through the plate's thickness, in order to obtain the following stress resultant quantities:

$$N_{ij} = \int_{-z/2}^{z/2} \sigma_{ij} dz$$

and the moment resultant quantities:

$$M_{ij} = \int_{-z/2}^{z/2} \sigma_{ij} * zdz$$

By substitution of Eq. (23) into Eq. (21), the governing differential equation of the skew nanoplate, in terms of the lateral deflection, is obtained as follows:

$$\begin{aligned} & D_{11} \frac{\partial^4 w}{\partial x^4} + 2\left(D_{12} + 2D_{66}\right) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} \\ & -\xi^2 \left(D_{11} \left(\frac{\partial^6 w}{\partial x^6} + \frac{\partial^6 w}{\partial x^4 \partial y^2} \right) + 2(D_{12} + 2D_{66}) \left(\frac{\partial^6 w}{\partial x^4 \partial y^2} + \frac{\partial^6 w}{\partial x^2 \partial y^4} \right) + D_{22} \left(\frac{\partial^6 w}{\partial y^6} + \frac{\partial^6 w}{\partial x^2 \partial y^4} \right) \right) \\ & = \left[N_{xx} \frac{\partial^2 w}{\partial x^2} + N_{yy} \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right] \end{aligned} \quad (25)$$

3.2 Applying SDNL Theory

For the orthotropic nanoplate problem, the constitutive equation relevant to the SDNL theory can be rewritten as:

$$\begin{Bmatrix} t_{xx} \\ t_{yy} \\ t_{xy} \end{Bmatrix} - \xi^2 \nabla^2 \begin{Bmatrix} t_{xx} \\ t_{yy} \\ t_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{E_2 \nu_{12}}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{E_2 \nu_{12}}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} \quad (26)$$

Following similar details of subsection 2.1 for this case gives the following governing equation for the buckling of the skew nanoplate:

$$\begin{aligned} & D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial^2 x \partial^2 y} + D_{22} \frac{\partial^4 w}{\partial y^4} \\ & = \left(1 - \xi \nabla^2\right) \left[N_{xx} \frac{\partial^2 w}{\partial x^2} + N_{yy} \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right] \end{aligned} \quad (27)$$

The governing PDEs must be solved to get the buckling load of the skew nanoplate for each of these elasticity theories.

3.3 Boundary Conditions

From a classical point of view, four boundary conditions (BCs) including essential ones in which zeroth and first spatial derivatives of deflection function and natural BCs of forces and moments related to second and third derivatives of the deflection function are required to solve the fourth order governing equations. However, as the order of PDE in the case of applying 2^{nd}SG is six, two new types of BCs are required to solve the PDE which are termed *non-classical boundary conditions*. Two types of constraints, including simply supported (SSSS) and fully clamped (CCCC) edges, are considered here.

3.3.1 Forth Order PDE

For simply supported conditions and for the case of clamped edges, the BCs are as follows, respectively[23]:

$$\begin{aligned} & SSSS \begin{cases} w = 0 \\ M_{nn} = n_x^2 M_{xx} + 2 n_x n_y M_{xy} + n_y^2 M_{yy} = 0 \end{cases} \\ & CCCC \begin{cases} w = 0 \\ \frac{\partial w}{\partial n} = n_x \frac{\partial w}{\partial x} + n_y \frac{\partial w}{\partial y} = 0 \end{cases} \end{aligned} \quad (28)$$

which, for a rectangular plate are simplified to

$$\begin{aligned} & SSSS \begin{cases} w = w_{xx} = 0 \;\; @\, x = 0, a \\ w = w_{yy} = 0 \;\; @\, y = 0, b \end{cases} \\ & CCCC \begin{cases} w = w_x = 0 \;\; @\, x = 0, a \\ w = w_y = 0 \;\; @\, y = 0, b \end{cases} \end{aligned} \quad (29)$$

3.3.2 Sixth Order PDEs

For simply supported conditions and for the case of clamped edges, the BCs are as follows, respectively:

$$\begin{aligned} & SSSS \begin{cases} w = 0 \\ \bar{M}_{nn} = n_x^2 \bar{M}_{xx} + 2 n_x n_y \bar{M}_{xy} + n_y^2 \bar{M}_{yy} = 0 \end{cases} \\ & CCCC \begin{cases} w = 0 \\ \frac{\partial w}{\partial n} = \frac{\partial^3 w}{\partial n^3} = 0 \end{cases} \end{aligned} \quad (30)$$

which for a rectangular plate are simplified to [24].

$$\begin{aligned} & SSSS \begin{cases} w = w_{xx} - \xi^2 w_{xxxx} = w_{xxx} = 0 \;\; @\, x = 0, a \\ w = w_{yy} - \xi^2 w_{yyyy} = w_{yyy} = 0 \;\; @\, y = 0, b \end{cases} \\ & CCCC \begin{cases} w = w_x = w_{xxx} = 0 \;\; @\, x = 0, a \\ w = w_y = w_{yyy} = 0 \;\; @\, y = 0, b \end{cases} \end{aligned} \quad (31)$$

4. SOLUTION OF THE GOVERNING EQUATIONS

For non-rectangular plates, if it is not impossible to solve the governing equation to the related boundary conditions analytically, then it is very difficult to obtain such a solution. For this reason, most previous studies are limited to a rectangular geometry and analytical solution of the Navier.

In this paper, however, a robust solution with a higher convergence speed is used. In this study, Galerkin's method as a global technique is adopted for the case of skew SLGS. In this method, the out-of-plane buckling displacement is approximated as weighted summation of a few base functions, $\psi_k(x, y)$ in the following form:

$$w = \sum_{m'=1}^{M'} \sum_{n'=1}^{N'} A_{m'n'} \psi_{m'}(\xi) \psi_{n'}(\eta) \tag{32}$$

where M' and N' are the number of points in the ξ and η directions, respectively. If any point (m', n') is assigned an integer k' by the relationship [14]:

$$k' = (m'-1) \times N' + n' \tag{33}$$

then the flexural displacement may be defined by:

$$w = \sum_{k'=1}^{M' \times N'} A_{k'} \psi_{k'}(\xi, \eta) \tag{34}$$

Here, $\psi_{k'}(\xi, \eta)$ is the k'th base function weighted by $A_{k'}$. In this work, the following sets of trigonometric base functions are selected for simply supported and clamped boundary conditions, since they satisfy the essential boundary conditions for relevant support cases.

$$\psi_{k'} = \sin\left(\frac{m'\pi\zeta}{2}\right)\sin\left(\frac{n'\pi\eta}{2}\right) \quad \text{(Simply supported)} \qquad m' = 1,2,3,\ldots,M'$$

$$-1 \le \zeta, \eta \le +1 \; ; \tag{35}$$

$$\psi_{k'} = \cos^2\left(\frac{m'\pi\zeta}{2}\right)\cos^2\left(\frac{n'\pi\eta}{2}\right) \quad \text{(Clamped)} \qquad n' = 1,2,3,\ldots,N'$$

By substitution of the above approximation functions into equations (25) and (27), the following eigenvalue problems in matrix form are obtained:

$$\left\{ \left[K^S\right] - \bar{N}\left[K^L\right] \right\} \{H\} = 0 \tag{36}$$

Equation (36) is a standard eigenproblem that can be solved for the dimensionless buckling load parameter $(\bar{N} = Na^2/D_{11})$ by using standard eigenvalue extraction techniques. K^S and K^L are stiffness matrices. These matrices are given below for each of the theories:

$$\begin{aligned} {}^{2^{nd}\,SG}K_{ij}^S = \int_{-1}^{+1}\int_{-1}^{+1} \psi_i \Bigg\{ & \frac{\partial^4 \psi_j}{\partial \zeta^4} + \frac{2(D_{12}+2D_{66})}{D_{11}} \frac{\partial^4 \psi_j}{\partial \zeta^2 \partial \eta^2} + \frac{D_{22}}{D_{11}} \frac{\partial^4 \psi_j}{\partial \eta^4} \\ & - \xi^2 \left(\frac{\partial^6 \psi_j}{\partial \zeta^6} + \frac{\partial^6 \psi_j}{\partial \zeta^4 \partial \eta^2} + \frac{D_{22}}{D_{11}} \left(\frac{\partial^6 \psi_j}{\partial \eta^6} + \frac{\partial^6 \psi_j}{\partial \zeta^2 \partial \eta^4} \right) + \frac{2(D_{12}+2D_{66})}{D_{11}} \left(\frac{\partial^6 \psi_j}{\partial \zeta^2 \partial \eta^4} + \frac{\partial^6 \psi_j}{\partial \zeta^4 \partial \eta^2} \right) \right) \Bigg\} d\zeta\, d\eta \end{aligned} \tag{37}$$

$${}^{2^{nd}\,SG}K_{ij}^L = \int_{-1}^{+1}\int_{-1}^{+1} \psi_i \left[\frac{\partial^2 \psi_j}{\partial \zeta^2} + \Gamma \frac{\partial^2 \psi_j}{\partial \eta^2} \right] d\zeta\, d\eta \tag{38}$$

$${}^{SDNL}K_{ij}^S = \int_{-1}^{+1}\int_{-1}^{+1} \psi_i \left\{ \frac{\partial^4 \psi_j}{\partial \zeta^4} + \frac{2(D_{12}+2D_{66})}{D_{11}} \frac{\partial^4 \psi_j}{\partial \zeta^2 \partial \eta^2} + \frac{D_{22}}{D_{11}} \frac{\partial^4 \psi_j}{\partial \eta^4} \right\} d\zeta\, d\eta \tag{39}$$

$${}^{SDNL}K_{ij}^L = \int_{-1}^{+1}\int_{-1}^{+1} \psi_i \left\{ \left[1 - \xi^2 \nabla^2\right] \left[\frac{\partial^2 \psi_j}{\partial \zeta^2} + \Gamma \frac{\partial^2 \psi_j}{\partial \eta^2} \right] \right\} d\zeta\, d\eta \tag{40}$$

The compression ratio ($\Gamma = N_{yy}/N_{xx}$) is assumed to be equal to unity for all cases unless specified otherwise. In the case that the material properties are known, the above matrices are determined and the eigenvalue problem of Eq. (36) can be solved.

5. NUMERICAL RESULTS AND DISCUSSION

In terms of material properties, two types of material behaviours representing isotropic and orthotropic cases are considered for the medium, and their constants are

Table 1. Properties of SLGS used in this study [25]

material properties		
	Isotropic	Orthotropic
E_1 **(GPa)**	1060	1766
E_2 **(GPa)**	1060	1588
ν_{12}	0.3	0.3
ν_{21}	0.3	0.27

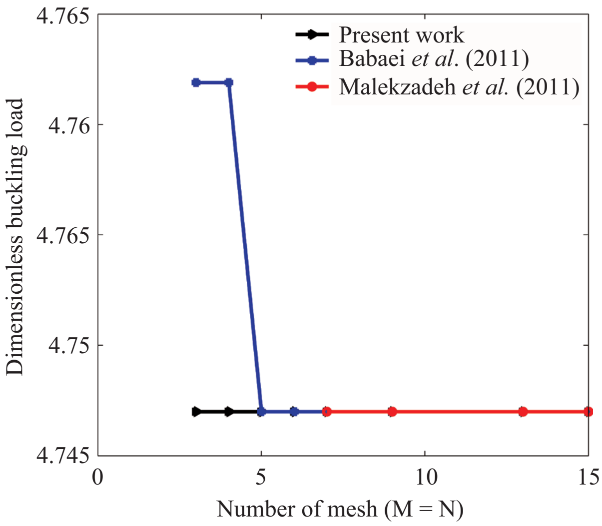

Fig. 3 Convergence study and comparison of dimensionless buckling load of simply supported isotropic square SLGS subjected to biaxial loading, under the various numbers of meshes, based on non-local theory. Note that (M = N) is chosen so that the rate of convergence of the proposed method can be compared with the other methods.

presented in Table 1 which is adopted from [25]. In addition to the well-known material constants such as elastic coefficients, the small size factor (or non-local parameter) is considered as an extra medium constant. To address the significance of the non-classical theories, the results here are presented as the buckling load ratio P_{NC}/P_C in which P_{NC} is the buckling load determined using each of the non-classical constitutive equations and P_C is the same buckling load obtained using classical theory.

5.1 Solution Convergence and Validation

In this section, the convergence and accuracy of the Galerkin method is investigated. Since there is no available result in the open literature for the buckling of the skew nanoplates based on second gradient theory, the verification of the method is conducted through examples for buckling analysis of square SLGS based on non-local theory.

As can be seen, a fast rate of convergence and excellent agreement exist between the results of this paper and other works. Moreover, Fig. 3 shows that the rate of convergence of the present solution is much faster than previous numerical solutions.

5.2 Effects of the Non-Local Parameter

There is a fundamental question about the role of non-local parameters in the skew nanoplate behaviour when the theories 2^{nd}SG and SDNL are employed as constitutive equations. In Fig. 4, dependency of the buckling load ratio to the non-local parameter is drawn for different sizes of the skew nanoplate. It is seen that when the nanoplate size is increased the dependency of buckling load to the non-local parameter is decreased. In other words, for large skew nanoplate sizes, all theories give almost identical results close to that of classical theory.

It is known that the buckling load is directly proportional to the stiffness of the structure. Therefore, SDNL theory predicts that the skew nanoplate becomes softer when its size ratio becomes smaller, whereas the other theory predicts that the stiffness increases when its size ratio decreases. Furthermore, as can be seen from the diagrams, the degree of softening and stiffening rises by increasing the non-local parameter.

5.3 Effects of Boundary Condition

Figure 5 indicates variation of dimensionless buckling load versus non-local parameter for two cases of simply supported and clamped skew nanoplate. It shows that the dimensionless buckling load of the clamped plate is higher than in the simply supported one for all values of non-local parameter, as expected. Moreover, it shows that the clamped condition is more affected by the non-local parameter in comparison to a simply supported condition.

5.4 Effects of Skew Angle

Figure 6 shows the variation of dimensionless buckling load with skew angle for various values for a non-local parameter. It is observed that the dimensionless buckling load increases when skew angle increases. This phenomenon is attributed to the stress concentration (due singularity in corners).

6. CONCLUSIONS

In this work, buckling governing equations of skew single layer graphene sheet employing non-conventional constitutive equations (including second order strain gradient and simplified differential non-local elasticity theories) are developed. In these theories, a non-classical property, called size effect or non-local parameter, appears in the equations in addition to the classical elastic constants. It was seen that the order of governing differential equation depended on the implemented constitutive equation. Accordingly, the forms of boundary conditions were also influenced. Based on the obtained numerical results, using SDNL theory, the dimensionless buckling load of the skew nanoplate is decreased by in-

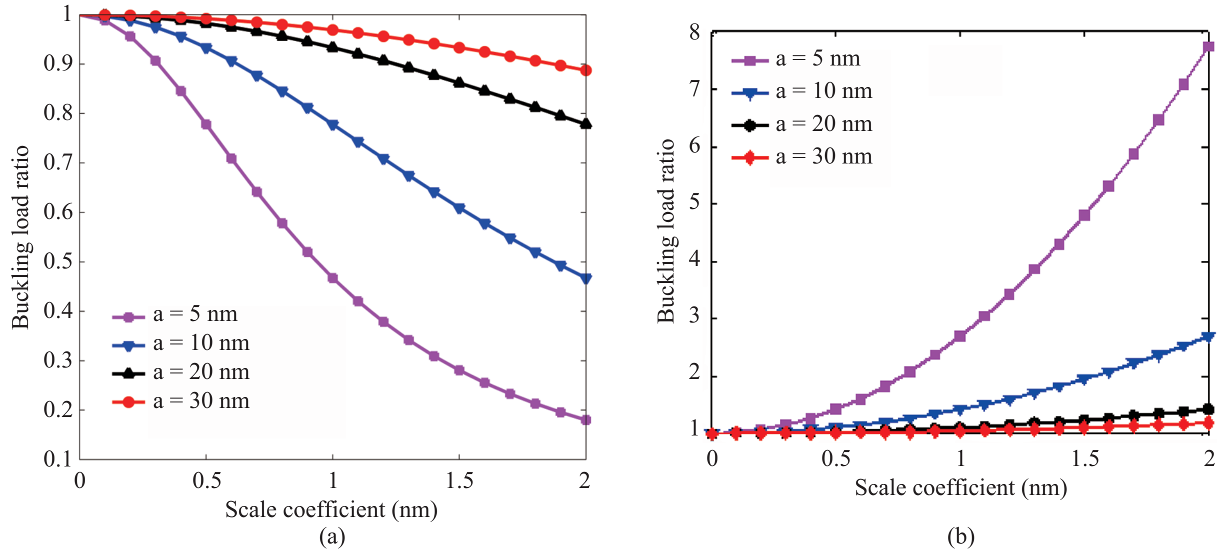

Fig. 4 Variation of buckling load ratio versus scale coefficient for different sizes of isotropic simply supported skew nanoplate obtained using (a) SDNL (b) 2ndSG theory.

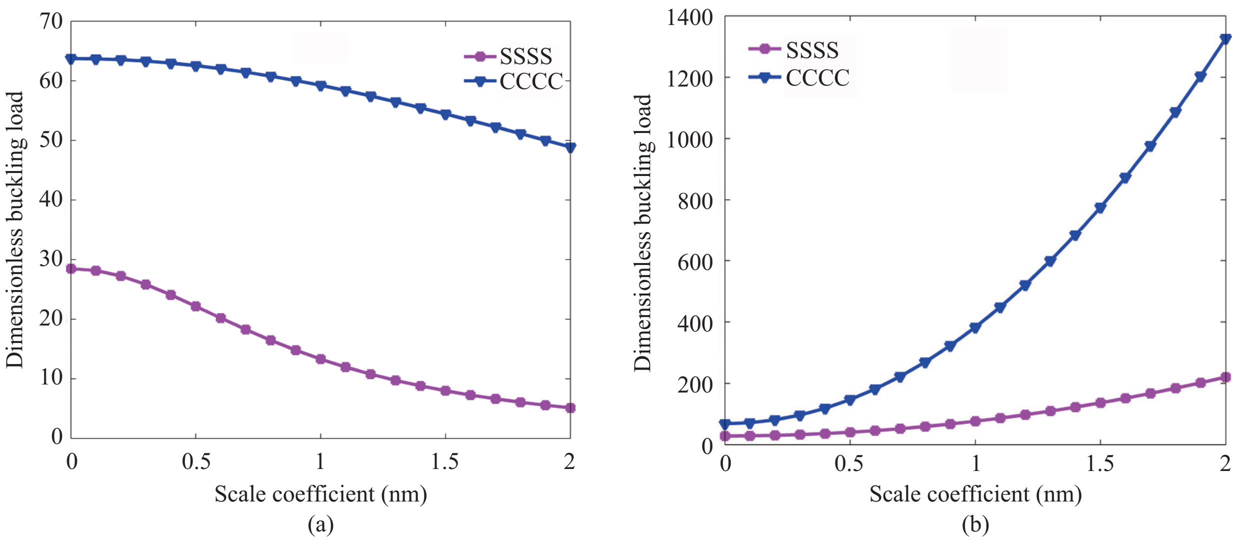

Fig. 5 Dependency of dimensionless buckling load to the boundary conditions and scale coefficient. (a) SDNL (b) 2ndSG theory.

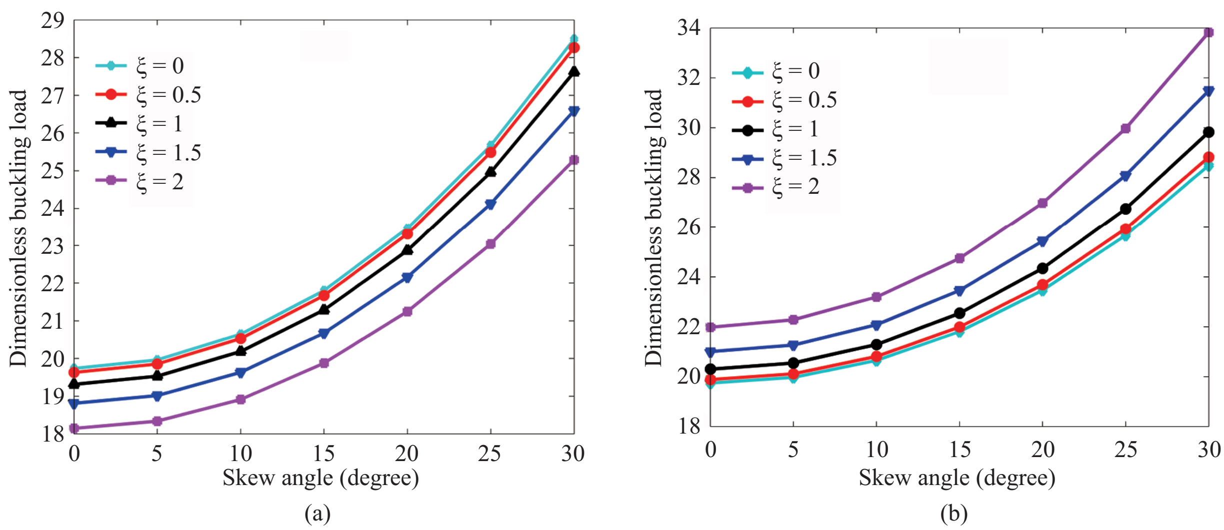

Fig. 6 Variation of dimensionless buckling load of skew SLGS with skew angles for different scale coefficient. (a) SDNL (b) 2ndSG theory.

creasing the non-local parameter, whereas an increasing trend is observed in the case of applying another theory. The literature review indicates that there is no unique agreement about modelling and understanding the size dependency of the nanoplate, and no perfect value can be assigned to the introduced non-classical property. As a matter of fact, there is a lack of reliable experimental data to validate the theoretical methods. However, no perfect judgment can be made about proposing a proper constitutive equation in the case of applying continuum-based approaches. The authors believe that a combination of constitutive equations proposed here can be employed to model nanoplates. Finally, it seems that much more investigation is required to develop adequate continuum-based theories that capture the behaviour of nanostructures properly.

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