CHAPTER 9

# Sequences, Series, and Power Series

> 'Then you should say what you mean,' the March Hare went on.
> 'I do,' Alice hastily replied; 'at least—at least I mean what I say—that's the same thing, you know.'
> 'Not the same thing a bit!' said the Hatter. 'Why, you might just as well say that "I see what I eat" is the same thing as "I eat what I see!"'

**Lewis Carroll (Charles Lutwidge Dodgson) 1832–1898**
**from *Alice's Adventures in Wonderland***

## Introduction

An infinite series is a sum that involves infinitely many terms. Since addition is carried out on two numbers at a time, the evaluation of the sum of an infinite series necessarily involves finding a limit. Complicated functions $f(x)$ can frequently be expressed as series of simpler functions. For example, many of the transcendental functions we have encountered can be expressed as series of powers of $x$ so that they resemble polynomials of infinite degree. Such series can be differentiated and integrated term by term, and they play a very important role in the study of calculus.

## 9.1 Sequences and Convergence

By a **sequence** (or an **infinite sequence**) we mean an ordered list having a first element but no last element. For our purposes, the elements (called **terms**) of a sequence will always be real numbers, although much of our discussion could be applied to complex numbers as well. Examples of sequences are:

$\{1, 2, 3, 4, 5, \ldots\}$ the sequence of positive integers,

$\left\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots\right\}$ the sequence of positive integer powers of $-\frac{1}{2}$.

The terms of a sequence are usually listed in braces as shown. The ellipsis points (...) should be read "and so on."

An infinite sequence is a special kind of function, one whose domain is a set of integers extending from some starting integer to infinity. The starting integer is usually 1, so the domain is the set of positive integers. The sequence $\{a_1, a_2, a_3, a_4, \ldots\}$ is the function $f$ that takes the value $f(n) = a_n$ at each positive integer $n$. A sequence can be specified in three ways:

(i) We can list the first few terms followed by ... *if the pattern is obvious*.

(ii) We can provide a formula for the **general term** $a_n$ as a function of $n$.

(iii) We can provide a formula for calculating the term $a_n$ as a function of earlier terms $a_1, a_2, \ldots, a_{n-1}$ and specify enough of the beginning terms so the process of computing higher terms can begin.

In each case it must be possible to determine any term of the sequence, although it may be necessary to calculate all the preceding terms first.

**EXAMPLE 1** **(Some examples of sequences)**

(a) $\{n\} = \{1, 2, 3, 4, 5, \ldots\}$

(b) $\left\{\left(-\dfrac{1}{2}\right)^n\right\} = \left\{-\dfrac{1}{2}, \dfrac{1}{4}, -\dfrac{1}{8}, \dfrac{1}{16}, \ldots\right\}$

(c) $\left\{\dfrac{n-1}{n}\right\} = \left\{0, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots\right\}$

(d) $\{(-1)^{n-1}\} = \{\cos((n-1)\pi)\} = \{1, -1, 1, -1, 1, \ldots\}$

(e) $\left\{\dfrac{n^2}{2^n}\right\} = \left\{\dfrac{1}{2}, 1, \dfrac{9}{8}, 1, \dfrac{25}{32}, \dfrac{36}{64}, \dfrac{49}{128}, \ldots\right\}$

(f) $\left\{\left(1+\dfrac{1}{n}\right)^n\right\} = \left\{2, \left(\dfrac{3}{2}\right)^2, \left(\dfrac{4}{3}\right)^3, \left(\dfrac{5}{4}\right)^4, \ldots\right\}$

(g) $\left\{\dfrac{\cos(n\pi/2)}{n}\right\} = \left\{0, -\dfrac{1}{2}, 0, \dfrac{1}{4}, 0, -\dfrac{1}{6}, 0, \dfrac{1}{8}, 0, \ldots\right\}$

(h) $a_1 = 1, \quad a_{n+1} = \sqrt{6 + a_n}, \quad (n = 1, 2, 3, \ldots)$
In this case $\{a_n\} = \{1, \sqrt{7}, \sqrt{6+\sqrt{7}}, \ldots\}$. Note that there is no *obvious* formula for $a_n$ as an explicit function of $n$ here, but we can still calculate $a_n$ for any desired value of $n$ provided we first calculate all the earlier values $a_2, a_3, \ldots, a_{n-1}$.

(i) $a_1 = 1, a_2 = 1, a_{n+2} = a_n + a_{n+1}, \quad (n = 1, 2, 3, \ldots)$
Here $\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$. This is called the **Fibonacci sequence**. Each term after the second is the sum of the previous two terms.

In parts (a)–(g) of Example 1, the formulas on the left sides define the general term of each sequence $\{a_n\}$ as an explicit function of $n$. In parts (h) and (i) we say the sequence $\{a_n\}$ is defined **recursively** or **inductively**; each term must be calculated from previous ones rather than directly as a function of $n$. We now introduce terminology used to describe various properties of sequences.

**DEFINITION 1**

**Terms for describing sequences**

(a) The sequence $\{a_n\}$ is **bounded below** by $L$, and $L$ is a **lower bound** for $\{a_n\}$, if $a_n \geq L$ for every $n = 1, 2, 3, \ldots$. The sequence is **bounded above** by $M$, and $M$ is an **upper bound**, if $a_n \leq M$ for every such $n$.

The sequence $\{a_n\}$ is **bounded** if it is both bounded above and bounded below. In this case there is a constant $K$ such that $|a_n| \leq K$ for every $n = 1, 2, 3, \ldots$. (We can take $K$ to be the larger of $|L|$ and $|M|$.)

(b) The sequence $\{a_n\}$ is **positive** if it is bounded below by zero, that is, if $a_n \geq 0$ for every $n = 1, 2, 3, \ldots$; it is **negative** if $a_n \leq 0$ for every $n$.

(c) The sequence $\{a_n\}$ is **increasing** if $a_{n+1} \geq a_n$ for every $n = 1, 2, 3, \ldots$; it is **decreasing** if $a_{n+1} \leq a_n$ for every such $n$. The sequence is said to be **monotonic** if it is either increasing or decreasing. (The terminology here is looser than that used for functions, where we would have used *nondecreasing* and *nonincreasing* to describe this behaviour. The distinction between $a_{n+1} > a_n$ and $a_{n+1} \geq a_n$ is not as important for sequences as it is for functions defined on intervals.)

(d) The sequence $\{a_n\}$ is **alternating** if $a_n a_{n+1} < 0$ for every $n = 1, 2, \ldots$, that is, if any two consecutive terms have opposite signs. Note that this definition requires $a_n \neq 0$ for each $n$.

**EXAMPLE 2** **(Describing some sequences)**

(a) The sequence $\{n\} = \{1, 2, 3, \ldots\}$ is positive, increasing, and bounded below. A lower bound for the sequence is 1 or any smaller number. The sequence is not bounded above.

(b) $\left\{\dfrac{n-1}{n}\right\} = \left\{0, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \ldots\right\}$ is positive, bounded, and increasing. Here, 0 is a lower bound and 1 is an upper bound.

(c) $\left\{\left(-\dfrac{1}{2}\right)^n\right\} = \left\{-\dfrac{1}{2}, \dfrac{1}{4}, -\dfrac{1}{8}, \dfrac{1}{16}, \ldots\right\}$ is bounded and alternating. Here, $-1/2$ is a lower bound and $1/4$ is an upper bound.

(d) $\{(-1)^n n\} = \{-1, 2, -3, 4, -5, \ldots\}$ is alternating but not bounded either above or below.

When you want to show that a sequence is increasing, you can try to show that the inequality $a_{n+1} - a_n \ge 0$ holds for $n \ge 1$. Alternatively, if $a_n = f(n)$ for a differentiable function $f(x)$, you can show that $f$ is a nondecreasing function on $[1, \infty)$ by showing that $f'(x) \ge 0$ there. Similar approaches are useful for showing that a sequence is decreasing.

**EXAMPLE 3** If $a_n = \dfrac{n}{n^2+1}$, show that the sequence $\{a_n\}$ is decreasing.

***Solution*** Since $a_n = f(n)$, where $f(x) = \dfrac{x}{x^2+1}$ and

$$f'(x) = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \le 0 \quad \text{for } x \ge 1,$$

the function $f(x)$ is decreasing on $[1, \infty)$; therefore, $\{a_n\}$ is a decreasing sequence.

The sequence $\left\{\dfrac{n^2}{2^n}\right\} = \left\{\dfrac{1}{2}, 1, \dfrac{9}{8}, 1, \dfrac{25}{32}, \dfrac{36}{64}, \dfrac{49}{128}, \ldots\right\}$ is positive and, therefore, bounded below. It seems clear that from the fourth term on, all the terms are getting smaller. However, $a_2 > a_1$ and $a_3 > a_2$. Since $a_{n+1} \le a_n$ only if $n \ge 3$, we say that this sequence is **ultimately decreasing**. The adverb *ultimately* is used to describe any termwise property of a sequence that the terms have from some point on, but not necessarily at the beginning of the sequence. Thus, the sequence

$$\{n - 100\} = \{-99, -98, \ldots, -2, -1, 0, 1, 2, 3, \ldots\}$$

is *ultimately positive* even though the first 99 terms are negative, and the sequence

$$\left\{(-1)^n + \frac{4}{n}\right\} = \left\{3, 3, \frac{1}{3}, 2, -\frac{1}{5}, \frac{5}{3}, -\frac{3}{7}, \frac{3}{2}, \ldots\right\}$$

is *ultimately alternating* even though the first few terms do not alternate.

## Convergence of Sequences

Central to the study of sequences is the notion of convergence. The concept of the limit of a sequence is a special case of the concept of the limit of a function $f(x)$ as $x \to \infty$. We say that the sequence $\{a_n\}$ **converges to the limit** $L$, and we write $\lim_{n\to\infty} a_n = L$, provided the distance from $a_n$ to $L$ on the real line approaches 0 as $n$ increases toward $\infty$. We state this definition more formally as follows:

**DEFINITION 2**

**Limit of a sequence**

We say that sequence $\{a_n\}$ converges to the limit $L$, and we write $\lim_{n\to\infty} a_n = L$, if for every positive real number $\epsilon$ there exists an integer $N$ (which may depend on $\epsilon$) such that if $n \ge N$, then $|a_n - L| < \epsilon$.

This definition is illustrated in Figure 9.1.

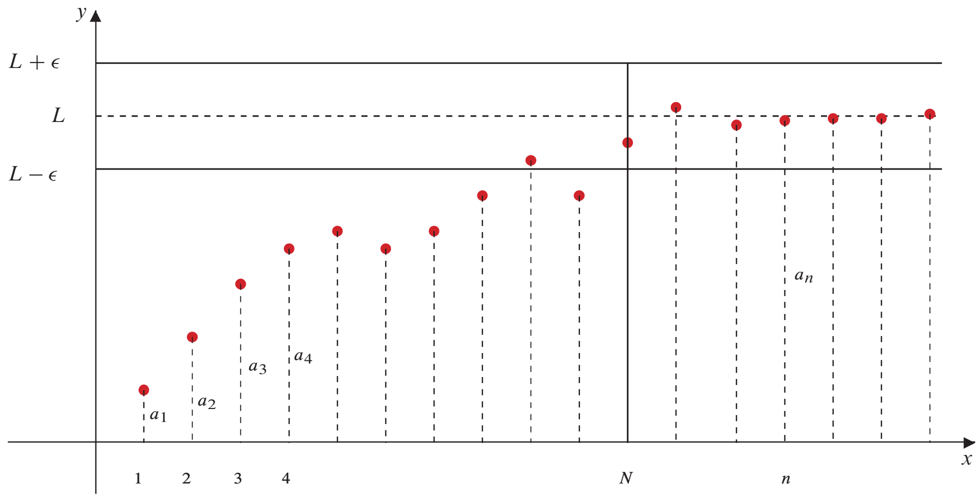



**Figure 9.1** A convergent sequence

**EXAMPLE 4** Show that $\lim_{n\to\infty} \dfrac{c}{n^p} = 0$ for any real number $c$ and any $p > 0$.

***Solution*** Let $\epsilon > 0$ be given. Then

$$\left|\frac{c}{n^p}\right| < \epsilon \quad \text{if} \quad n^p > \frac{|c|}{\epsilon},$$

that is, if $n \ge N$, the least integer greater than $(|c|/\epsilon)^{1/p}$. Therefore, by Definition 2, $\lim_{n\to\infty} \dfrac{c}{n^p} = 0$.

Every sequence $\{a_n\}$ must either **converge** to a finite limit $L$ or **diverge**. That is, either $\lim_{n\to\infty} a_n = L$ exists (is a real number) or $\lim_{n\to\infty} a_n$ does not exist. If $\lim_{n\to\infty} a_n = \infty$, we can say that the sequence diverges to $\infty$; if $\lim_{n\to\infty} a_n = -\infty$, we can say that it diverges to $-\infty$. If $\lim_{n\to\infty} a_n$ simply does not exist (but is not $\infty$ or $-\infty$), we can only say that the sequence diverges.

**EXAMPLE 5** **(Examples of convergent and divergent sequences)**

(a) $\{(n-1)/n\}$ converges to 1; $\lim_{n\to\infty}(n-1)/n = \lim_{n\to\infty}\big(1-(1/n)\big) = 1$.

(b) $\{n\} = \{1, 2, 3, 4, \dots\}$ diverges to $\infty$.

(c) $\{-n\} = \{-1, -2, -3, -4, \dots\}$ diverges to $-\infty$.

(d) $\{(-1)^n\} = \{-1, 1, -1, 1, -1, \dots\}$ simply diverges.

(e) $\{(-1)^n n\} = \{-1, 2, -3, 4, -5, \dots\}$ diverges (but not to $\infty$ or $-\infty$ even though $\lim_{n\to\infty} |a_n| = \infty$).

The limit of a sequence is equivalent to the limit of a function as its argument approaches infinity:

If $\displaystyle\lim_{x\to\infty} f(x) = L$ and $a_n = f(n)$, then $\displaystyle\lim_{n\to\infty} a_n = L$.

Because of this, the standard rules for limits of functions (Theorems 2 and 4 of Section 1.2) also hold for limits of sequences, with the appropriate changes of notation. Thus, if $\{a_n\}$ and $\{b_n\}$ converge, then

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n,$$

$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n,$$

$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right),$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\displaystyle\lim_{n\to\infty} a_n}{\displaystyle\lim_{n\to\infty} b_n} \qquad \text{assuming } \lim_{n\to\infty} b_n \neq 0.$$

If $a_n \le b_n$ ultimately, then $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$.

If $a_n \le b_n \le c_n$ ultimately, and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = L$.

The limits of many explicitly defined sequences can be evaluated using these properties in a manner similar to the methods used for limits of the form $\lim_{x\to\infty} f(x)$ in Section 1.3.

**EXAMPLE 6** Calculate the limits of the sequences

$$\text{(a)}\ \left\{\frac{2n^2 - n - 1}{5n^2 + n - 3}\right\}, \quad \text{(b)}\ \left\{\frac{\cos n}{n}\right\}, \quad \text{and} \quad \text{(c)}\ \{\sqrt{n^2 + 2n} - n\}.$$

### Solution

(a) We divide the numerator and denominator of the expression for $a_n$ by the highest power of $n$ in the denominator, that is, by $n^2$:

$$\lim_{n\to\infty}\frac{2n^2 - n - 1}{5n^2 + n - 3} = \lim_{n\to\infty}\frac{2 - (1/n) - (1/n^2)}{5 + (1/n) - (3/n^2)} = \frac{2 - 0 - 0}{5 + 0 - 0} = \frac{2}{5},$$

since $\lim_{n\to\infty} 1/n = 0$ and $\lim_{n\to\infty} 1/n^2 = 0$. The sequence converges and its limit is 2/5.

(b) Since $|\cos n| \le 1$ for every $n$, we have

$$-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n} \quad \text{for} \quad n \ge 1.$$

Now, $\lim_{n\to\infty} -1/n = 0$ and $\lim_{n\to\infty} 1/n = 0$. Therefore, by the sequence version of the Squeeze Theorem, $\lim_{n\to\infty}(\cos n)/n = 0$. The given sequence converges to 0.

(c) For this sequence we multiply the numerator and the denominator (which is 1) by the conjugate of the expression in the numerator:

$$\begin{aligned}\lim_{n\to\infty}(\sqrt{n^2 + 2n} - n) &= \lim_{n\to\infty}\frac{(\sqrt{n^2 + 2n} - n)(\sqrt{n^2 + 2n} + n)}{\sqrt{n^2 + 2n} + n} \\ &= \lim_{n\to\infty}\frac{2n}{\sqrt{n^2 + 2n} + n} = \lim_{n\to\infty}\frac{2}{\sqrt{1 + (2/n)} + 1} = 1.\end{aligned}$$

The sequence converges to 1.

**EXAMPLE 7** Evaluate $\lim_{n\to\infty} n \tan^{-1}\left(\frac{1}{n}\right)$.

***Solution*** For this example it is best to replace the $n$th term of the sequence by the corresponding function of a real variable $x$ and take the limit as $x \to \infty$. We use l'Hôpital's Rule:

$$\begin{aligned}\lim_{n\to\infty} n \tan^{-1}\left(\frac{1}{n}\right) &= \lim_{x\to\infty} x \tan^{-1}\left(\frac{1}{x}\right)\\ &= \lim_{x\to\infty} \frac{\tan^{-1}\left(\frac{1}{x}\right)}{\frac{1}{x}} \qquad \left[\frac{0}{0}\right]\\ &= \lim_{x\to\infty} \frac{\frac{1}{1+(1/x^2)}\left(-\frac{1}{x^2}\right)}{-\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = 1.\end{aligned}$$

**THEOREM 1**

If $\{a_n\}$ converges, then $\{a_n\}$ is bounded.

***PROOF*** Suppose $\lim_{n\to\infty} a_n = L$. According to Definition 2, for $\epsilon = 1$ there exists a number $N$ such that if $n > N$, then $|a_n - L| < 1$; therefore $|a_n| < 1 + |L|$ for such $n$. (Why is this true?) If $K$ denotes the largest of the numbers $|a_1|, |a_2|, \ldots, |a_N|$, and $1 + |L|$, then $|a_n| \le K$ for every $n = 1, 2, 3, \ldots$. Hence, $\{a_n\}$ is bounded.

The converse of Theorem 1 is false; the sequence $\{(-1)^n\}$ is bounded but does not converge.

The *completeness property* of the real number system (see Section P.1) can be reformulated in terms of sequences to read as follows:

**Bounded monotonic sequences converge**

If the sequence $\{a_n\}$ is bounded above and is (ultimately) increasing, then it converges. The same conclusion holds if $\{a_n\}$ is bounded below and is (ultimately) decreasing.

Thus, a bounded, ultimately monotonic sequence is convergent. (See Figure 9.2.)

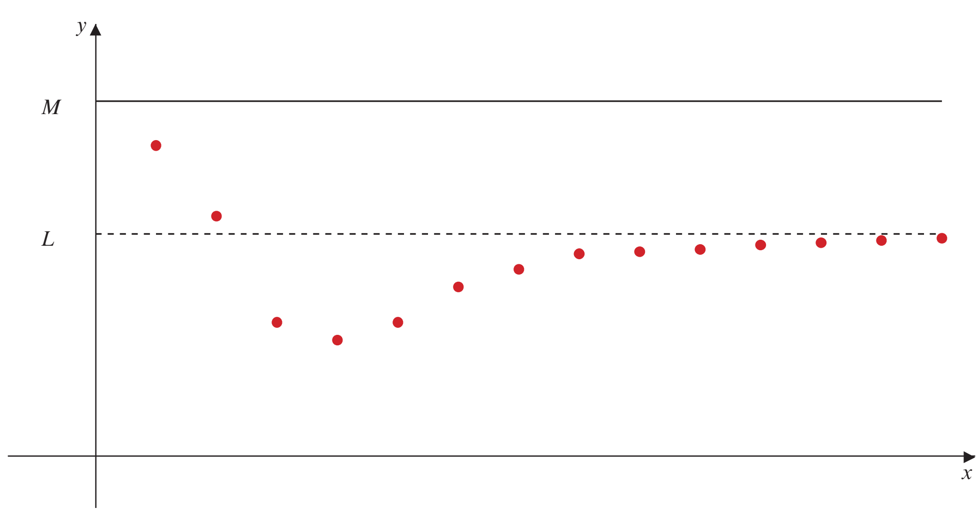



**Figure 9.2** An ultimately increasing sequence that is bounded above

There is a subtle point to note in this solution. Showing that $\{a_n\}$ is increasing is pretty obvious, but how did we know to try and show that 3 (rather than some other number) was an upper bound? The answer is that we actually did the last part first and showed that *if* $\lim a_n = a$ exists, *then* $a = 3$. It then makes sense to try and show that $a_n < 3$ for all $n$.

Of course, we can easily show that any number greater than 3 is an upper bound.

**EXAMPLE 8** Let $a_n$ be defined recursively by

$$a_1 = 1, \qquad a_{n+1} = \sqrt{6 + a_n} \qquad (n = 1, 2, 3, \ldots).$$

Show that $\lim_{n\to\infty} a_n$ exists and find its value.

***Solution*** Observe that $a_2 = \sqrt{6+1} = \sqrt{7} > a_1$. If $a_{k+1} > a_k$, then we have $a_{k+2} = \sqrt{6 + a_{k+1}} > \sqrt{6 + a_k} = a_{k+1}$, so $\{a_n\}$ is increasing, by induction. Now observe that $a_1 = 1 < 3$. If $a_k < 3$, then $a_{k+1} = \sqrt{6 + a_k} < \sqrt{6+3} = 3$, so $a_n < 3$ for every $n$ by induction. Since $\{a_n\}$ is increasing and bounded above, $\lim_{n\to\infty} a_n = a$ exists, by completeness. Since $\sqrt{6+x}$ is a continuous function of $x$, we have

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{6 + a_n} = \sqrt{6 + \lim_{n\to\infty} a_n} = \sqrt{6 + a}.$$

Thus, $a^2 = 6 + a$, or $a^2 - a - 6 = 0$, or $(a-3)(a+2) = 0$. This quadratic has roots $a = 3$ and $a = -2$. Since $a_n \ge 1$ for every $n$, we must have $a \ge 1$. Therefore, $a = 3$ and $\lim_{n\to\infty} a_n = 3$.

**EXAMPLE 9** Does $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ converge or diverge?

***Solution*** We could make an effort to show that the given sequence is, in fact, increasing and bounded above. (See Exercise 32 at the end of this section.) However, we already know the answer. The sequence converges by Theorem 6 of Section 3.4:

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e^1 = e.$$

**THEOREM 2**

If $\{a_n\}$ is (ultimately) increasing, then either it is bounded above, and therefore convergent, or it is not bounded above and diverges to infinity.

The proof of this theorem is left as an exercise. A corresponding result holds for (ultimately) decreasing sequences.

The following theorem evaluates two important limits that find frequent application in the study of series.

**THEOREM 3**

(a) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

(b) If $x$ is any real number, then $\lim_{n\to\infty} \dfrac{x^n}{n!} = 0$.

***PROOF*** For part (a) observe that

$$\lim_{n\to\infty} \ln |x|^n = \lim_{n\to\infty} n \ln |x| = -\infty,$$

since $\ln |x| < 0$ when $|x| < 1$. Accordingly, since $e^x$ is continuous,

$$\lim_{n\to\infty} |x|^n = \lim_{n\to\infty} e^{\ln |x|^n} = e^{\lim_{n\to\infty} \ln |x|^n} = 0.$$

Since $-|x|^n \le x^n \le |x|^n$, we have $\lim_{n\to\infty} x^n = 0$ by the Squeeze Theorem.

For part (b), pick any $x$ and let $N$ be an integer such that $N > |x|$. If $n > N$ we have

$$\begin{aligned}\left|\frac{x^n}{n!}\right| &= \frac{|x|}{1}\frac{|x|}{2}\frac{|x|}{3}\cdots\frac{|x|}{N-1}\frac{|x|}{N}\frac{|x|}{N+1}\cdots\frac{|x|}{n}\\ &< \frac{|x|^{N-1}}{(N-1)!}\frac{|x|}{N}\frac{|x|}{N}\frac{|x|}{N}\cdots\frac{|x|}{N}\\ &= \frac{|x|^{N-1}}{(N-1)!}\left(\frac{|x|}{N}\right)^{n-N+1} = K\left(\frac{|x|}{N}\right)^n,\end{aligned}$$

where $K = \dfrac{|x|^{N-1}}{(N-1)!}\left(\dfrac{|x|}{N}\right)^{1-N}$ is a constant that is independent of $n$. Since $|x|/N < 1$, we have $\lim_{n\to\infty}(|x|/N)^n = 0$ by part (a). Thus, $\lim_{n\to\infty}|x^n/n!| = 0$, so $\lim_{n\to\infty} x^n/n! = 0$.

**EXAMPLE 10** Find $\lim_{n\to\infty}\dfrac{3^n+4^n+5^n}{5^n}$.

***Solution*** $\displaystyle\lim_{n\to\infty}\frac{3^n+4^n+5^n}{5^n} = \lim_{n\to\infty}\left[\left(\frac{3}{5}\right)^n + \left(\frac{4}{5}\right)^n + 1\right] = 0+0+1 = 1$, by Theorem 3(a).

## EXERCISES 9.1

In Exercises 1–13, determine whether the given sequence is (a) bounded (above or below), (b) positive or negative (ultimately), (c) increasing, decreasing, or alternating, and (d) convergent, divergent, divergent to $\infty$ or $-\infty$.

**1.** $\left\{\dfrac{2n^2}{n^2+1}\right\}$

**2.** $\left\{\dfrac{2n}{n^2+1}\right\}$

**3.** $\left\{4-\dfrac{(-1)^n}{n}\right\}$

**4.** $\left\{\sin\dfrac{1}{n}\right\}$

**5.** $\left\{\dfrac{n^2-1}{n}\right\}$

**6.** $\left\{\dfrac{e^n}{\pi^n}\right\}$

**7.** $\left\{\dfrac{e^n}{\pi^{n/2}}\right\}$

**8.** $\left\{\dfrac{(-1)^n n}{e^n}\right\}$

**9.** $\left\{\dfrac{2^n}{n^n}\right\}$

**10.** $\left\{\dfrac{(n!)^2}{(2n)!}\right\}$

**11.** $\left\{n\cos\left(\dfrac{n\pi}{2}\right)\right\}$

**12.** $\left\{\dfrac{\sin n}{n}\right\}$

**13.** $\{1, 1, -2, 3, 3, -4, 5, 5, -6, \ldots\}$

In Exercises 14–29, evaluate, wherever possible, the limit of the sequence $\{a_n\}$.

**14.** $a_n = \dfrac{5-2n}{3n-7}$

**15.** $a_n = \dfrac{n^2-4}{n+5}$

**16.** $a_n = \dfrac{n^2}{n^3+1}$

**17.** $a_n = (-1)^n\dfrac{n}{n^3+1}$

**18.** $a_n = \dfrac{n^2-2\sqrt{n}+1}{1-n-3n^2}$

**19.** $a_n = \dfrac{e^n-e^{-n}}{e^n+e^{-n}}$

**20.** $a_n = n\sin\dfrac{1}{n}$

**21.** $a_n = \left(\dfrac{n-3}{n}\right)^n$

**22.** $a_n = \dfrac{n}{\ln(n+1)}$

**23.** $a_n = \sqrt{n+1}-\sqrt{n}$

**24.** $a_n = n-\sqrt{n^2-4n}$

**25.** $a_n = \sqrt{n^2+n}-\sqrt{n^2-1}$

**26.** $a_n = \left(\dfrac{n-1}{n+1}\right)^n$

**27.** $a_n = \dfrac{(n!)^2}{(2n)!}$

**28.** $a_n = \dfrac{n^2 2^n}{n!}$

**29.** $a_n = \dfrac{\pi^n}{1+2^{2n}}$

**30.** Let $a_1 = 1$ and $a_{n+1} = \sqrt{1+2a_n}$ $(n = 1, 2, 3, \ldots)$. Show that $\{a_n\}$ is increasing and bounded above. (*Hint:* Show that 3 is an upper bound.) Hence, conclude that the sequence converges, and find its limit.

**31.** Repeat Exercise 30 for the sequence defined by $a_1 = 3$, $a_{n+1} = \sqrt{15+2a_n}$, $n = 1, 2, 3, \ldots$. This time you will have to guess an upper bound.

**32.** Let $a_n = \left(1+\dfrac{1}{n}\right)^n$ so that $\ln a_n = n\ln\left(1+\dfrac{1}{n}\right)$. Use properties of the logarithm function to show that (a) $\{a_n\}$ is increasing and (b) $e$ is an upper bound for $\{a_n\}$.

**33.** Prove Theorem 2. Also, state an analogous theorem pertaining to ultimately decreasing sequences.

**34.** If $\{|a_n|\}$ is bounded, prove that $\{a_n\}$ is bounded.

**35.** If $\lim_{n\to\infty}|a_n| = 0$, prove that $\lim_{n\to\infty} a_n = 0$.

**36.** Which of the following statements are TRUE and which are FALSE? Justify your answers.

(a) If $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = L > 0$, then $\lim_{n\to\infty} a_n b_n = \infty$.

(b) If $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty}(a_n + b_n) = 0$.

(c) If $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = -\infty$, then $\lim_{n\to\infty} a_n b_n = -\infty$.

(d) If neither $\{a_n\}$ nor $\{b_n\}$ converges, then $\{a_n b_n\}$ does not converge.

(e) If $\{|a_n|\}$ converges, then $\{a_n\}$ converges.

## 9.2 Infinite Series

An **infinite series**, usually just called a **series**, is a formal sum of infinitely many terms; for instance, $a_1 + a_2 + a_3 + a_4 + \cdots$ is a series formed by adding the terms of the sequence $\{a_n\}$. This series is also denoted $\sum_{n=1}^{\infty} a_n$:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots .$$

For example,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots .$$

It is sometimes necessary or useful to start the sum from some index other than 1:

$$\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + a^3 + \cdots$$

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \cdots .$$

Note that the latter series would make no sense if we had started the sum from $n = 1$; the first term would have been undefined.

When necessary, we can change the index of summation to start at a different value. This is accomplished by a substitution, as illustrated in Example 3 of Section 5.1. For instance, using the substitution $n = m - 2$, we can rewrite $\sum_{n=1}^{\infty} a_n$ in the form $\sum_{m=3}^{\infty} a_{m-2}$. Both sums give rise to the same expansion

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots = \sum_{m=3}^{\infty} a_{m-2}.$$

Addition is an operation that is carried out on two numbers at a time. If we want to calculate the finite sum $a_1 + a_2 + a_3$, we could proceed by adding $a_1 + a_2$ and then adding $a_3$ to this sum, or else we might first add $a_2 + a_3$ and then add $a_1$ to the sum. Of course, the associative law for addition assures us we will get the same answer both ways. This is the reason the symbol $a_1 + a_2 + a_3$ makes sense; we would otherwise have to write $(a_1 + a_2) + a_3$ or $a_1 + (a_2 + a_3)$. This reasoning extends to any sum $a_1 + a_2 + \cdots + a_n$ of finitely many terms, but it is not obvious what should be meant by a sum with infinitely many terms:

$$a_1 + a_2 + a_3 + a_4 + \cdots .$$

We no longer have any assurance that the terms can be added up in any order to yield the same sum. In fact, we will see in Section 9.4 that in certain circumstances, changing the order of terms in a series can actually change the sum of the series. The interpretation we place on the infinite sum is that of adding from left to right, as suggested by the grouping

$$\cdots((((a_1 + a_2) + a_3) + a_4) + a_5) + \cdots.$$

We accomplish this by defining a new sequence $\{s_n\}$, called the **sequence of partial sums** of the series $\sum_{n=1}^{\infty} a_n$, so that $s_n$ is the sum of the first $n$ terms of the series:

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= s_1 + a_2 = a_1 + a_2 \\
s_3 &= s_2 + a_3 = a_1 + a_2 + a_3 \\
&\vdots \\
s_n &= s_{n-1} + a_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{j=1}^{n} a_j \\
&\vdots
\end{aligned}$$

We then define the sum of the infinite series to be the limit of this sequence of partial sums.

DEFINITION 3

**Convergence of a series**

We say that the series $\sum_{n=1}^{\infty} a_n$ **converges to the sum** $s$, and we write

$$\sum_{n=1}^{\infty} a_n = s,$$

if $\lim_{n\to\infty} s_n = s$, where $s_n$ is the $n$th partial sum of $\sum_{n=1}^{\infty} a_n$:

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{j=1}^{n} a_j.$$

Thus, a *series* converges if and only if the *sequence* of its partial sums converges.

Similarly, a series is said to diverge to infinity, diverge to negative infinity, or simply diverge if its sequence of partial sums does so. It must be stressed that the convergence of the series $\sum_{n=1}^{\infty} a_n$ depends on the convergence of the sequence $\{s_n\} = \{\sum_{j=1}^{n} a_j\}$, *not* the sequence $\{a_n\}$.

## Geometric Series

DEFINITION 4

**Geometric series**

A series of the form $\sum_{n=1}^{\infty} a\,r^{n-1} = a + ar + ar^2 + ar^3 + \cdots$, whose $n$th term is $a_n = a\,r^{n-1}$, is called a **geometric series**. The number $a$ is the first term. The number $r$ is called the **common ratio** of the series, since it is the value of the ratio of the $(n+1)$st term to the $n$th term for any $n \geq 1$:

$$\frac{a_{n+1}}{a_n} = \frac{ar^n}{ar^{n-1}} = r, \qquad n = 1, 2, 3, \ldots.$$

The $n$th partial sum $s_n$ of a geometric series is calculated as follows:

$$\begin{aligned} s_n &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \\ rs_n &= \phantom{a + {}} ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \end{aligned}$$

The second equation is obtained by multiplying the first by $r$. Subtracting these two equations (note the cancellations), we get $(1-r)s_n = a - ar^n$. If $r \neq 1$, we can divide by $1 - r$ and get a formula for $s_n$.

**Partial sums of geometric series**

If $r = 1$, then the $n$th partial sum of a geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ is $s_n = a + a + \cdots + a = na$. If $r \neq 1$, then

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

If $a = 0$, then $s_n = 0$ for every $n$, and $\lim_{n\to\infty} s_n = 0$. Now suppose $a \neq 0$. If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$, so $\lim_{n\to\infty} s_n = a/(1-r)$. If $r > 1$, then $\lim_{n\to\infty} r^n = \infty$, and $\lim_{n\to\infty} s_n = \infty$ if $a > 0$, or $\lim_{n\to\infty} s_n = -\infty$ if $a < 0$. The same conclusion holds if $r = 1$, since $s_n = na$ in this case. If $r \leq -1$, $\lim_{n\to\infty} r^n$ does not exist and neither does $\lim_{n\to\infty} s_n$. Hence, we conclude that

$$\sum_{n=1}^{\infty} ar^{n-1} \begin{cases} \text{converges to } 0 & \text{if } a = 0 \\ \text{converges to } \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{diverges to } \infty & \text{if } r \geq 1 \text{ and } a > 0 \\ \text{diverges to } -\infty & \text{if } r \geq 1 \text{ and } a < 0 \\ \text{diverges} & \text{if } r \leq -1 \text{ and } a \neq 0. \end{cases}$$

The representation of the function $1/(1-x)$ as the sum of a geometric series,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad \text{for } -1 < x < 1,$$

will be important in our discussion of power series later in this chapter.

**EXAMPLE 1** **(Examples of geometric series and their sums)**

(a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = \frac{1}{1 - \frac{1}{2}} = 2$. Here $a = 1$ and $r = \frac{1}{2}$.

Since $|r| < 1$, the series converges.

(b) $\pi - e + \frac{e^2}{\pi} - \frac{e^3}{\pi^2} + \cdots = \sum_{n=1}^{\infty} \pi \left(-\frac{e}{\pi}\right)^{n-1}$ Here $a = \pi$ and $r = -\frac{e}{\pi}$.

$$= \frac{\pi}{1 - \left(-\frac{e}{\pi}\right)} = \frac{\pi^2}{\pi + e}.$$

The series converges since $\left|-\frac{e}{\pi}\right| < 1$.

(c) $1 + 2^{1/2} + 2 + 2^{3/2} + \cdots = \sum_{n=1}^{\infty} (\sqrt{2})^{n-1}$. This series diverges to $\infty$ since $a = 1 > 0$ and $r = \sqrt{2} > 1$.

(d) $1 - 1 + 1 - 1 + 1 - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}$. This series diverges since $r = -1$.

(e) Let $x = 0.32\,32\,32\cdots = 0.\overline{32}$; then

$$x = \frac{32}{100} + \frac{32}{100^2} + \frac{32}{100^3} + \cdots = \sum_{n=1}^{\infty} \frac{32}{100}\left(\frac{1}{100}\right)^{n-1} = \frac{32}{100}\,\frac{1}{1 - \dfrac{1}{100}} = \frac{32}{99}.$$

This is an alternative to the method of Example 1 of Section P.1 for representing repeating decimals as quotients of integers.

**EXAMPLE 2** If money earns interest at a constant effective rate of 5% per year, how much should you pay today for an annuity that will pay you (a) \$1,000 at the end of each of the next 10 years and (b) \$1,000 at the end of every year forever?

***Solution*** A payment of \$1,000 that is due to be received $n$ years from now has present value $\$1{,}000 \times \left(\frac{1}{1.05}\right)^n$ (since $\$A$ would grow to $\$A(1.05)^n$ in $n$ years). Thus, \$1,000 payments at the end of each of the next $n$ years are worth $\$s_n$ at the present time, where

$$\begin{aligned} s_n &= 1{,}000\left[\frac{1}{1.05} + \left(\frac{1}{1.05}\right)^2 + \cdots + \left(\frac{1}{1.05}\right)^n\right] \\ &= \frac{1{,}000}{1.05}\left[1 + \frac{1}{1.05} + \left(\frac{1}{1.05}\right)^2 + \cdots + \left(\frac{1}{1.05}\right)^{n-1}\right] \\ &= \frac{1{,}000}{1.05}\,\frac{1 - \left(\dfrac{1}{1.05}\right)^n}{1 - \dfrac{1}{1.05}} = \frac{1{,}000}{0.05}\left[1 - \left(\frac{1}{1.05}\right)^n\right]. \end{aligned}$$

(a) The present value of 10 future payments is $\$s_{10} = \$7{,}721.73$.

(b) The present value of future payments continuing forever is

$$\$\lim_{n\to\infty} s_n = \frac{\$1{,}000}{0.05} = \$20{,}000.$$

## Telescoping Series and Harmonic Series

**EXAMPLE 3** Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} + \cdots$$

converges and find its sum.

***Solution*** Since $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$, we can write the partial sum $s_n$ in the form

$$\begin{aligned} s_n &= \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) \\ &\qquad + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \cdots - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Therefore, $\lim_{n\to\infty} s_n = 1$ and the series converges to 1:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

This is an example of a **telescoping series**, so called because the partial sums *fold up* into a simple form when the terms are expanded in partial fractions. Other examples can be found in the exercises at the end of this section. As these examples show, the method of partial fractions can be a useful tool for series as well as for integrals.

**EXAMPLE 4** Show that the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges to infinity.

***Solution*** If $s_n$ is the $n$th partial sum of the harmonic series, then

$$\begin{aligned} s_n &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\ &= \text{sum of areas of rectangles shaded in blue in Figure 9.3} \\ &> \text{area under } y = \frac{1}{x} \text{ from } x = 1 \text{ to } x = n+1 \\ &= \int_1^{n+1} \frac{dx}{x} = \ln(n+1). \end{aligned}$$

Now $\lim_{n\to\infty} \ln(n+1) = \infty$. Therefore, $\lim_{n\to\infty} s_n = \infty$ and

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \qquad \text{diverges to infinity.}$$

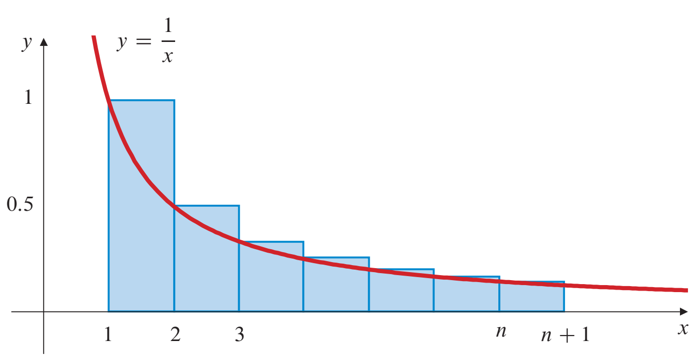



**Figure 9.3** A partial sum of the harmonic series

Like geometric series, the harmonic series will often be encountered in subsequent sections.

## Some Theorems About Series

**THEOREM 4**

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Therefore, if $\lim_{n\to\infty} a_n$ does not exist, or exists but is not zero, then the series $\sum_{n=1}^{\infty} a_n$ is divergent. (This amounts to an ***n*th term test for divergence of a series.**)

**PROOF** If $s_n = a_1 + a_2 + \cdots + a_n$, then $s_n - s_{n-1} = a_n$. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} s_n = s$ exists, and $\lim_{n\to\infty} s_{n-1} = s$. Hence, $\lim_{n\to\infty} a_n = s - s = 0$.

**Remark** Theorem 4 is *very important* for the understanding of infinite series. Students often err either in forgeting that *a series cannot converge if its terms do not approach zero* or in confusing this result with its *converse*, which is false. The converse would say that if $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ must converge. The harmonic series is a counterexample showing the falsehood of this assertion:

$$\lim_{n\to\infty} \frac{1}{n} = 0 \qquad \text{but} \qquad \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges to infinity.}$$

When considering whether a given series converges, the first question you should ask yourself is: "Does the $n$th term approach 0 as $n$ approaches $\infty$?" If the answer is *no*, then the series does *not* converge. If the answer is *yes*, then the series *may or may not* converge. If the sequence of terms $\{a_n\}$ tends to a nonzero limit $L$, then $\sum_{n=1}^{\infty} a_n$ diverges to infinity if $L > 0$ and diverges to negative infinity if $L < 0$.

## EXAMPLE 5

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n-1}$ diverges to infinity since $\displaystyle\lim_{n\to\infty} \frac{n}{2n-1} = 1/2 > 0$.

(b) $\sum_{n=1}^{\infty} (-1)^n n \sin(1/n)$ diverges since

$$\lim_{n\to\infty} \left|(-1)^n n \sin\frac{1}{n}\right| = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\to 0+} \frac{\sin x}{x} = 1 \neq 0.$$

The following theorem asserts that it is only the *ultimate* behaviour of $\{a_n\}$ that determines whether $\sum_{n=1}^{\infty} a_n$ converges. Any finite number of terms can be dropped from the beginning of a series without affecting the convergence; the convergence depends only on the *tail* of the series. Of course, the actual sum of the series depends on *all* the terms.

**THEOREM 5**

$\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=N}^{\infty} a_n$ converges for any integer $N \geq 1$.

**THEOREM 6**

If $\{a_n\}$ is ultimately positive, then the series $\sum_{n=1}^{\infty} a_n$ must either converge (if its partial sums are bounded above) or diverge to infinity (if its partial sums are not bounded above).

The proofs of these two theorems are posed as exercises at the end of this section. The following theorem is just a reformulation of standard laws of limits.

**THEOREM 7**

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge to $A$ and $B$, respectively, then

(a) $\sum_{n=1}^{\infty} ca_n$ converges to $cA$ (where $c$ is any constant);

(b) $\sum_{n=1}^{\infty} (a_n \pm b_n)$ converges to $A \pm B$;

(c) if $a_n \leq b_n$ for all $n = 1, 2, 3, \ldots$, then $A \leq B$.

## EXAMPLE 6

Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^{n+1}}{3^n}$.

**Solution** The given series is the sum of two geometric series,

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} = \frac{1/3}{1-(1/3)} = \frac{1}{2} \quad \text{and}$$
$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^n} = \sum_{n=1}^{\infty} \frac{4}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{4/3}{1-(2/3)} = 4.$$

Thus, its sum is $\frac{1}{2} + 4 = \frac{9}{2}$ by Theorem 7(b).

## EXERCISES 9.2

In Exercises 1–18, find the sum of the given series, or show that the series diverges (possibly to infinity or negative infinity). Exercises 11–14 are telescoping series and should be done by partial fractions as suggested in Example 3 in this section.

**1.** $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \sum_{n=1}^{\infty} \frac{1}{3^n}$

**2.** $3 - \frac{3}{4} + \frac{3}{16} - \frac{3}{64} + \cdots = \sum_{n=1}^{\infty} 3\left(-\frac{1}{4}\right)^{n-1}$

**3.** $\sum_{n=5}^{\infty} \frac{1}{(2+\pi)^{2n}}$

**4.** $\sum_{n=0}^{\infty} \frac{5}{10^{3n}}$

**5.** $\sum_{n=2}^{\infty} \frac{(-5)^n}{8^{2n}}$

**6.** $\sum_{n=0}^{\infty} \frac{1}{e^n}$

**7.** $\sum_{k=0}^{\infty} \frac{2^{k+3}}{e^{k-3}}$

**8.** $\sum_{j=1}^{\infty} \pi^{j/2} \cos(j\pi)$

**9.** $\sum_{n=1}^{\infty} \frac{3+2^n}{2^{n+2}}$

**10.** $\sum_{n=0}^{\infty} \frac{3+2^n}{3^{n+2}}$

**11.** $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{1\times 3} + \frac{1}{2\times 4} + \frac{1}{3\times 5} + \cdots$

**12.** $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots$

**13.** $\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)} = \frac{1}{1\times 4} + \frac{1}{4\times 7} + \frac{1}{7\times 10} + \cdots$

**14.** $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$
$= \frac{1}{1\times 2\times 3} + \frac{1}{2\times 3\times 4} + \frac{1}{3\times 4\times 5} + \cdots$

**15.** $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

**16.** $\sum_{n=1}^{\infty} \frac{n}{n+2}$

**17.** $\sum_{n=1}^{\infty} n^{-1/2}$

**18.** $\sum_{n=1}^{\infty} \frac{2}{n+1}$

**19.** Obtain a simple expression for the partial sum $s_n$ of the series $\sum_{n=1}^{\infty}(-1)^n$, and use it to show that the series diverges.

**20.** Find the sum of the series

$$\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \cdots.$$

**21.** When dropped, an elastic ball bounces back up to a height three-quarters of that from which it fell. If the ball is dropped from a height of 2 m and allowed to bounce up and down indefinitely, what is the total distance it travels before coming to rest?

**22.** If a bank account pays 10% simple interest into an account once a year, what is the balance in the account at the end of 8 years if \$1,000 is deposited into the account at the beginning of each of the 8 years? (Assume there was no balance in the account initially.)

**23.** Prove Theorem 5.

**24.** Prove Theorem 6.

**25.** State a theorem analogous to Theorem 6 but for a negative sequence.

In Exercises 26–31, decide whether the given statement is TRUE or FALSE. If it is true, prove it. If it is false, give a counterexample showing the falsehood.

**26.** If $a_n = 0$ for every $n$, then $\sum a_n$ converges.

**27.** If $\sum a_n$ converges, then $\sum(1/a_n)$ diverges to infinity.

**28.** If $\sum a_n$ and $\sum b_n$ both diverge, then so does $\sum(a_n + b_n)$.

**29.** If $a_n \ge c > 0$ for every $n$, then $\sum a_n$ diverges to infinity.

**30.** If $\sum a_n$ diverges and $\{b_n\}$ is bounded, then $\sum a_n b_n$ diverges.

**31.** If $a_n > 0$ and $\sum a_n$ converges, then $\sum(a_n)^2$ converges.

# 9.3 Convergence Tests for Positive Series

In the previous section we saw a few examples of convergent series (geometric and telescoping series) whose sums could be determined exactly because the partial sums $s_n$ could be expressed in closed form as explicit functions of $n$ whose limits as $n \to \infty$ could be evaluated. It is not usually possible to do this with a given series, and therefore it is not usually possible to determine the sum of the series exactly. However, there are many techniques for determining whether a given series converges and, if it does, for approximating the sum to any desired degree of accuracy.

In this section we deal exclusively with *positive series*, that is, series of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots,$$

where $a_n \geq 0$ for all $n \geq 1$. As noted in Theorem 6, such a series will converge if its partial sums are bounded above and will diverge to infinity otherwise. All our results apply equally well to *ultimately* positive series since convergence or divergence depends only on the *tail* of a series.

## The Integral Test

The integral test provides a means for determining whether an ultimately positive series converges or diverges by comparing it with an improper integral that behaves similarly. Example 4 in Section 9.2 is an example of the use of this technique. We formalize the method in the following theorem.

**THEOREM 8**

**The integral test**

Suppose that $a_n = f(n)$, where $f$ is positive, continuous, and nonincreasing on an interval $[N, \infty)$ for some positive integer $N$. Then

$$\sum_{n=1}^{\infty} a_n \qquad \text{and} \qquad \int_N^{\infty} f(t)\,dt$$

either both converge or both diverge to infinity.

***PROOF*** Let $s_n = a_1 + a_2 + \cdots + a_n$. If $n > N$, we have

$$\begin{aligned} s_n &= s_N + a_{N+1} + a_{N+2} + \cdots + a_n \\ &= s_N + f(N+1) + f(N+2) + \cdots + f(n) \\ &= s_N + \text{ sum of areas of rectangles shaded in Figure 9.4(a)} \\ &\leq s_N + \int_N^{\infty} f(t)\,dt. \end{aligned}$$

If the improper integral $\int_N^{\infty} f(t)\,dt$ converges, then the sequence $\{s_n\}$ is bounded above and $\sum_{n=1}^{\infty} a_n$ converges.

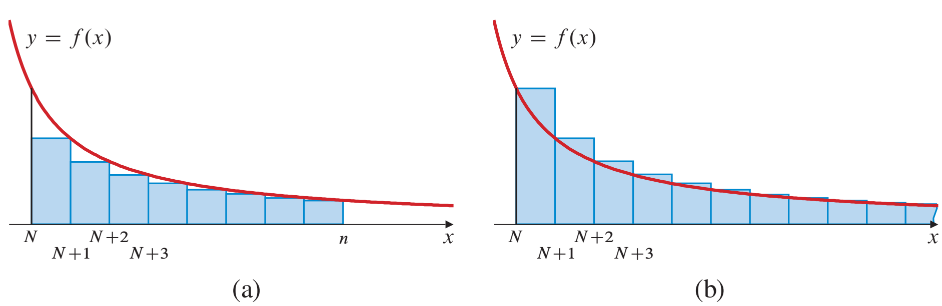



**Figure 9.4** Comparing integrals and series

Conversely, suppose that $\sum_{n=1}^{\infty} a_n$ converges to the sum $s$. Then

$$\begin{aligned}\int_N^{\infty} f(t)\,dt &= \text{area under } y = f(t) \text{ above } y = 0 \text{ from } t = N \text{ to } t = \infty\\ &\le \text{sum of areas of shaded rectangles in Figure 9.4(b)}\\ &= a_N + a_{N+1} + a_{N+2} + \cdots\\ &= s - s_{N-1} < \infty,\end{aligned}$$

so the improper integral represents a finite area and is thus convergent. (We omit the remaining details showing that $\lim_{R\to\infty}\int_N^R f(t)\,dt$ exists; like the series case, the argument depends on the completeness of the real numbers.)

**Remark** If $a_n = f(n)$, where $f$ is positive, continuous, and nonincreasing on $[1, \infty)$, then Theorem 8 assures us that $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ both converge or both diverge to infinity. It does *not* tell us that the sum of the series is equal to the value of the integral. The two are not likely to be equal in the case of convergence. However, as we see below, integrals can help us approximate the sum of a series.

The principal use of the integral test is to establish the result of the following example concerning the series $\sum_{n=1}^{\infty} n^{-p}$, which is called a ***p*-series**. This result should be memorized; we will frequently compare the behaviour of other series with $p$-series later in this and subsequent sections.

**EXAMPLE 1** **(*p*-series)** Show that

$$\sum_{n=1}^{\infty} n^{-p} = \sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases}\text{converges if } p > 1\\ \text{diverges to infinity if } p \le 1.\end{cases}$$

**Solution** Observe that if $p > 0$, then $f(x) = x^{-p}$ is positive, continuous, and decreasing on $[1, \infty)$. By the integral test, the $p$-series converges for $p > 1$ and diverges for $0 < p \le 1$ by comparison with $\int_1^{\infty} x^{-p}\,dx$. (See Theorem 2(a) of Section 6.5.) If $p \le 0$, then $\lim_{n\to\infty}(1/n^p) \ne 0$, so the series cannot converge in this case. Being a positive series, it must diverge to infinity.

**Remark** The harmonic series $\sum_{n=1}^{\infty} n^{-1}$ (the case $p = 1$ of the $p$-series) is on the borderline between convergence and divergence, although it diverges. While its terms decrease toward 0 as $n$ increases, they do not decrease *fast enough* to allow the sum of the series to be finite. If $p > 1$, the terms of $\sum_{n=1}^{\infty} n^{-p}$ decrease toward zero fast enough that their sum is finite. We can refine the distinction between convergence and divergence at $p = 1$ by using terms that decrease faster than $1/n$, but not as fast as $1/n^q$ for any $q > 1$. If $p > 0$, the terms $1/\big(n(\ln n)^p\big)$ have this property since $\ln n$ grows more slowly than any positive power of $n$ as $n$ increases. The question now arises whether $\sum_{n=2}^{\infty} 1/(n(\ln n)^p)$ converges. It does, provided again that $p > 1$; you can use the substitution $u = \ln x$ to check that

$$\int_2^{\infty} \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^{\infty} \frac{du}{u^p},$$

which converges if $p > 1$ and diverges if $0 < p \le 1$. This process of fine-tuning Example 1 can be extended even further. (See Exercise 36 below.)

## Using Integral Bounds to Estimate the Sum of a Series

Suppose that $a_k = f(k)$ for $k = n+1, n+2, n+3, \ldots$, where $f$ is a positive, continuous function, decreasing at least on the interval $[n, \infty)$. We have:

$$\begin{aligned} s - s_n &= \sum_{k=n+1}^{\infty} f(k) \\ &= \text{sum of areas of rectangles shaded in Figure 9.5(a)} \\ &\le \int_n^{\infty} f(x)\,dx. \end{aligned}$$

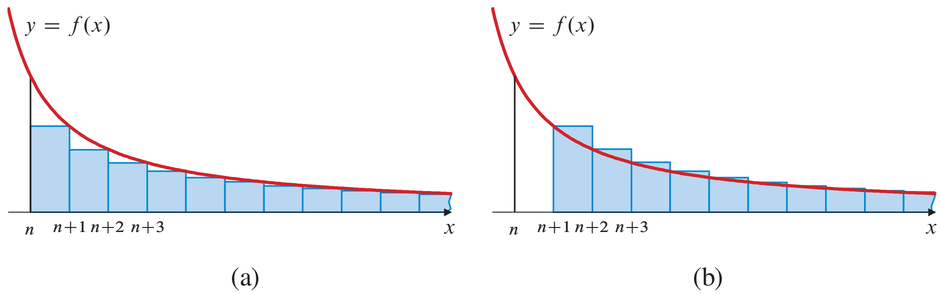



**Figure 9.5** Using integrals to estimate the tail of a series

Similarly,

$$\begin{aligned} s - s_n &= \text{sum of areas of rectangles in Figure 9.5(b)} \\ &\ge \int_{n+1}^{\infty} f(x)\,dx. \end{aligned}$$

If we define

$$A_n = \int_n^{\infty} f(x)\,dx,$$

then we can combine the above inequalities to obtain

$$A_{n+1} \le s - s_n \le A_n,$$

or, equivalently:

$$s_n + A_{n+1} \le s \le s_n + A_n.$$

The error in the approximation $s \approx s_n$ satisfies $0 \le s - s_n \le A_n$. However, since $s$ must lie in the interval $[s_n + A_{n+1}, s_n + A_n]$, we can do better by using the midpoint $s_n^*$ of this interval as an approximation for $s$. The error is then less than half the length $A_n - A_{n+1}$ of the interval:

**A better integral approximation**

The error $|s - s_n^*|$ in the approximation

$$s \approx s_n^* = s_n + \frac{A_{n+1} + A_n}{2}, \qquad \text{where} \quad A_n = \int_n^{\infty} f(x)\,dx,$$

satisfies $\quad |s - s_n^*| \le \dfrac{A_n - A_{n+1}}{2}.$

(Whenever a quantity is known to lie in a certain interval, the midpoint of that interval can be used to approximate the quantity, and the absolute value of the error in that approximation does not exceed half the length of the interval.)

**EXAMPLE 2** Find the best approximation $s_n^*$ to the sum $s$ of the series $\sum_{n=1}^{\infty} 1/n^2$, making use of the partial sum $s_n$ of the first $n$ terms. How large would $n$ have to be to ensure that the approximation $s \approx s_n^*$ has error less than 0.001 in absolute value? How large would $n$ have to be to ensure that the approximation $s \approx s_n$ has error less than 0.001 in absolute value?

**Solution** Since $f(x) = 1/x^2$ is positive, continuous, and decreasing on $[1, \infty)$ for any $n = 1, 2, 3, \ldots$, we have

$$s_n + A_{n+1} \le s \le s_n + A_n,$$

where

$$A_n = \int_n^\infty \frac{dx}{x^2} = \lim_{R\to\infty} \left(-\frac{1}{x}\right)\bigg|_n^R = \frac{1}{n}.$$

The best approximation to $s$ using $s_n$ is

$$\begin{aligned} s_n^* &= s_n + \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n}\right) = s_n + \frac{2n+1}{2n(n+1)} \\ &= 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} + \frac{2n+1}{2n(n+1)}. \end{aligned}$$

The error in this approximation satisfies

$$|s - s_n^*| \le \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2n(n+1)} \le 0.001,$$

provided $2n(n+1) \ge 1/0.001 = 1{,}000$. It is easily checked that this condition is satisfied if $n \ge 22$; the approximation

$$s \approx s_{22}^* = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{22^2} + \frac{45}{44 \times 23}$$

will have error with absolute value not exceeding 0.001. Had we used the approximation $s \approx s_n$ we could only have concluded that

$$0 \le s - s_n \le A_n = \frac{1}{n} < 0.001,$$

provided $n > 1{,}000$; we would need 1,000 terms of the series to get the desired accuracy.

## Comparison Tests

The next test we consider for positive series is analogous to the comparison theorem for improper integrals. (See Theorem 3 of Section 6.5.) It enables us to determine the convergence or divergence of one series by comparing it with another series that is known to converge or diverge.

**THEOREM 9**

**A comparison test**

Let $\{a_n\}$ and $\{b_n\}$ be sequences for which there exists a positive constant $K$ such that, ultimately, $0 \le a_n \le K b_n$.

(a) If the series $\sum_{n=1}^{\infty} b_n$ converges, then so does the series $\sum_{n=1}^{\infty} a_n$.

(b) If the series $\sum_{n=1}^{\infty} a_n$ diverges to infinity, then so does the series $\sum_{n=1}^{\infty} b_n$.

**PROOF** Since a series converges if and only if its tail converges (Theorem 5), we can assume, without loss of generality, that the condition $0 \le a_n \le Kb_n$ holds for all $n \ge 1$. Let $s_n = a_1 + a_2 + \cdots + a_n$ and $S_n = b_1 + b_2 + \cdots + b_n$. Then $s_n \le KS_n$. If $\sum b_n$ converges, then $\{S_n\}$ is convergent and hence is bounded by Theorem 1. Hence $\{s_n\}$ is bounded above. By Theorem 6, $\sum a_n$ converges. Since the convergence of $\sum b_n$ guarantees that of $\sum a_n$, if the latter series diverges to infinity, then the former cannot converge either, so it must diverge to infinity too.

BEWARE! Theorem 9 does *not* say that if $\sum a_n$ converges then $\sum b_n$ converges. It is possible that the *smaller* sum may be finite while the *larger* one is infinite. (Do not confuse a theorem with its converse.)

**EXAMPLE 3** Which of the following series converge? Give reasons for your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+1}{n^3+1}$, (c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$.

**Solution** In each case we must find a suitable comparison series that we already know converges or diverges.

(a) Since $0 < \dfrac{1}{2^n+1} < \dfrac{1}{2^n}$ for $n = 1, 2, 3, \ldots$, and since $\sum_{n=1}^{\infty} \dfrac{1}{2^n}$ is a convergent geometric series, the series $\sum_{n=1}^{\infty} \dfrac{1}{2^n+1}$ also converges by comparison.

(b) Observe that $\dfrac{3n+1}{n^3+1}$ behaves like $\dfrac{3}{n^2}$ for large $n$, so we would expect to compare the series with the convergent $p$-series $\sum_{n=1}^{\infty} n^{-2}$. We have, for $n \ge 1$,

$$\frac{3n+1}{n^3+1} = \frac{3n}{n^3+1} + \frac{1}{n^3+1} < \frac{3n}{n^3} + \frac{1}{n^3} < \frac{3}{n^2} + \frac{1}{n^2} = \frac{4}{n^2}.$$

Thus, the given series converges by Theorem 9.

(c) For $n = 2, 3, 4, \ldots$, we have $0 < \ln n < n$. Thus $\dfrac{1}{\ln n} > \dfrac{1}{n}$. Since $\sum_{n=2}^{\infty} \dfrac{1}{n}$ diverges to infinity (it is a harmonic series), so does $\sum_{n=2}^{\infty} \dfrac{1}{\ln n}$ by comparison.

The following theorem provides a version of the comparison test that is not quite as general as Theorem 9 but is often easier to apply in specific cases.

**THEOREM 10**

**A limit comparison test**

Suppose that $\{a_n\}$ and $\{b_n\}$ are positive sequences and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L,$$

where $L$ is either a nonnegative finite number or $+\infty$.

(a) If $L < \infty$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

(b) If $L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges to infinity, then so does $\sum_{n=1}^{\infty} a_n$.

**PROOF** If $L < \infty$, then for $n$ sufficiently large, we have $b_n > 0$ and

$$0 \le \frac{a_n}{b_n} \le L + 1,$$

so $0 \le a_n \le (L+1)b_n$. Hence $\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=1}^{\infty} b_n$ converges, by Theorem 9(a).

If $L > 0$, then for $n$ sufficiently large

$$\frac{a_n}{b_n} \geq \frac{L}{2}.$$

Therefore, $0 < b_n \leq (2/L)a_n$, and $\sum_{n=1}^{\infty} a_n$ diverges to infinity if $\sum_{n=1}^{\infty} b_n$ does, by Theorem 9(b).

**EXAMPLE 4** Which of the following series converge? Give reasons for your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$.

**Solution** Again we must make appropriate choices for comparison series.

(a) The terms of this series decrease like $1/\sqrt{n}$. Observe that

$$L = \lim_{n\to\infty} \frac{\dfrac{1}{1+\sqrt{n}}}{\dfrac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{(1/\sqrt{n})+1} = 1.$$

Since the $p$-series $\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ diverges to infinity $(p = 1/2)$, so does the series $\sum_{n=1}^{\infty} \dfrac{1}{1+\sqrt{n}}$, by the limit comparison test.

(b) For large $n$, the terms behave like $n/n^3$, so let us compare the series with the $p$-series $\sum_{n=1}^{\infty} 1/n^2$, which we know converges.

$$L = \lim_{n\to\infty} \frac{\dfrac{n+5}{n^3-2n+3}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+5n^2}{n^3-2n+3} = 1.$$

Since $L < \infty$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$ also converges by the limit comparison test.

In order to apply the original version of the comparison test (Theorem 9) successfully, it is important to have an intuitive feeling for whether the given series converges or diverges. The form of the comparison will depend on whether you are trying to prove convergence or divergence. For instance, if you did not know intuitively that

$$\sum_{n=1}^{\infty} \frac{1}{100n + 20{,}000}$$

would have to diverge to infinity, you might try to argue that

$$\frac{1}{100n + 20{,}000} < \frac{1}{n} \qquad \text{for } n = 1, 2, 3, \ldots.$$

While true, this doesn't help at all. $\sum_{n=1}^{\infty} 1/n$ diverges to infinity; therefore Theorem 9 yields no information from this comparison. We could, of course, argue instead that

$$\frac{1}{100n + 20{,}000} \geq \frac{1}{20{,}100n} \qquad \text{if } n \geq 1,$$

and conclude by Theorem 9 that $\sum_{n=1}^{\infty}(1/(100n+20{,}000))$ diverges to infinity by comparison with the divergent series $\sum_{n=1}^{\infty} 1/n$. An easier way is to use Theorem 10 and the fact that

$$L = \lim_{n\to\infty} \frac{\dfrac{1}{100n+20{,}000}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n}{100n+20{,}000} = \frac{1}{100} > 0.$$

However, the limit comparison test Theorem 10 has a disadvantage when compared to the ordinary comparison test Theorem 9. It can fail in certain cases because the limit $L$ does not exist. In such cases it is possible that the ordinary comparison test may still work.

**EXAMPLE 5** Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sin n}{n^2}$ for convergence.

***Solution*** Since

$$\lim_{n\to\infty} \frac{\dfrac{1+\sin n}{n^2}}{\dfrac{1}{n^2}} = \lim_{n\to\infty}(1+\sin n)$$

does not exist, the limit comparison test gives us no information. However, since $\sin n \le 1$, we have

$$0 \le \frac{1+\sin n}{n^2} \le \frac{2}{n^2} \qquad \text{for } n = 1, 2, 3, \ldots.$$

The given series does, in fact, converge by comparison with $\sum_{n=1}^{\infty} 1/n^2$, using the ordinary comparison test.

## The Ratio and Root Tests

**THEOREM 11**

**The ratio test**

Suppose that $a_n > 0$ (ultimately) and that $\rho = \displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists or is $+\infty$.

(a) If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $1 < \rho \le \infty$, then $\lim_{n\to\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

(c) If $\rho = 1$, this test gives no information; the series may either converge or diverge to infinity.

***PROOF*** Here $\rho$ is the lowercase Greek letter "rho" (pronounced "roh").

(a) Suppose $\rho < 1$. Pick a number $r$ such that $\rho < r < 1$. Since we are given that $\lim_{n\to\infty} a_{n+1}/a_n = \rho$, we have $a_{n+1}/a_n \le r$ for $n$ sufficiently large; that is, $a_{n+1} \le r a_n$ for $n \ge N$, say. In particular,

$$\begin{aligned}
a_{N+1} &\le r a_N \\
a_{N+2} &\le r a_{N+1} \le r^2 a_N \\
a_{N+3} &\le r a_{N+2} \le r^3 a_N \\
&\ \vdots \\
a_{N+k} &\le r^k a_N \qquad (k = 0, 1, 2, 3, \ldots).
\end{aligned}$$

Hence, $\sum_{n=N}^{\infty} a_n$ converges by comparison with the convergent geometric series $\sum_{k=0}^{\infty} r^k$. It follows that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n$ must also converge.

(b) Now suppose that $\rho > 1$. Pick a number $r$ such that $1 < r < \rho$. Since $\lim_{n\to\infty} a_{n+1}/a_n = \rho$, we have $a_{n+1}/a_n \geq r$ for $n$ sufficiently large, say for $n \geq N$. We assume $N$ is chosen large enough that $a_N > 0$. It follows by an argument similar to that used in part (a) that $a_{N+k} \geq r^k a_N$ for $k = 0, 1, 2, \ldots,$ and since $r > 1$, $\lim_{n\to\infty} a_n = \infty$. Therefore, $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

(c) If $\rho$ is computed for the series $\sum_{n=1}^{\infty} 1/n$ and $\sum_{n=1}^{\infty} 1/n^2$, we get $\rho = 1$ for each. Since the first series diverges to infinity and the second converges, the ratio test cannot distinguish between convergence and divergence if $\rho = 1$.

All $p$-series fall into the indecisive category where $\rho = 1$, as does $\sum_{n=1}^{\infty} a_n$, where $a_n$ is any rational function of $n$. The ratio test is most useful for series whose terms decrease at least exponentially fast. The presence of factorials in a term also suggests that the ratio test might be useful.

**EXAMPLE 6** Test the following series for convergence:

$$\text{(a)}\ \sum_{n=1}^{\infty} \frac{99^n}{n!}, \qquad \text{(b)}\ \sum_{n=1}^{\infty} \frac{n^5}{2^n}, \qquad \text{(c)}\ \sum_{n=1}^{\infty} \frac{n!}{n^n}, \qquad \text{(d)}\ \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}.$$

**Solution** We use the ratio test for each of these series.

$$\text{(a)}\ \rho = \lim_{n\to\infty} \frac{99^{n+1}}{(n+1)!} \bigg/ \frac{99^n}{n!} = \lim_{n\to\infty} \frac{99}{n+1} = 0 < 1.$$

Thus, $\sum_{n=1}^{\infty} (99^n/n!)$ converges.

$$\text{(b)}\ \rho = \lim_{n\to\infty} \frac{(n+1)^5}{2^{n+1}} \bigg/ \frac{n^5}{2^n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^5 = \frac{1}{2} < 1.$$

Hence, $\sum_{n=1}^{\infty} (n^5/2^n)$ converges.

$$\text{(c)}\ \rho = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \bigg/ \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)!\,n^n}{(n+1)^{n+1}\,n!} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\dfrac{1}{n}\right)^n} = \frac{1}{e} < 1.$$

Thus, $\sum_{n=1}^{\infty} (n!/n^n)$ converges.

$$\text{(d)}\ \rho = \lim_{n\to\infty} \frac{(2(n+1))!}{((n+1)!)^2} \bigg/ \frac{(2n)!}{(n!)^2} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2} = 4 > 1.$$

Thus, $\sum_{n=1}^{\infty} (2n)!/(n!)^2$ diverges to infinity.

The following theorem is very similar to the ratio test but is less frequently used. Its proof is left as an exercise. (See Exercise 37.) For examples of series to which it can be applied, see Exercises 38 and 39.

**THEOREM 12**

**The root test**

Suppose that $a_n > 0$ (ultimately) and that $\sigma = \lim_{n\to\infty} (a_n)^{1/n}$ exists or is $+\infty$.

(a) If $0 \leq \sigma < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $1 < \sigma \leq \infty$, then $\lim_{n\to\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

(c) If $\sigma = 1$, this test gives no information; the series may either converge or diverge to infinity.

## Using Geometric Bounds to Estimate the Sum of a Series

Suppose that an inequality of the form

$$0 \le a_k \le K r^k$$

holds for $k = n+1, n+2, n+3, \ldots$, where $K$ and $r$ are constants and $r < 1$. We can then use a geometric series to bound the tail of $\sum_{n=1}^{\infty} a_n$.

$$\begin{aligned} 0 \le s - s_n &= \sum_{k=n+1}^{\infty} a_k \le \sum_{k=n+1}^{\infty} K r^k \\ &= K r^{n+1}(1 + r + r^2 + \cdots) \\ &= \frac{K r^{n+1}}{1-r}. \end{aligned}$$

Since $r < 1$, the series converges and the error approaches 0 at an exponential rate as $n$ increases.

**EXAMPLE 7** In Section 9.6 we will show that

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

(Recall that $0! = 1$.) Estimate the error if the sum $s_n$ of the first $n$ terms of the series is used to approximate $e$. Find $e$ to 3-decimal-place accuracy using the series.

***Solution*** We have

$$\begin{aligned} s_n &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots + \frac{1}{(n-1)!}. \end{aligned}$$

(Since the series starts with the term for $n = 0$, the $n$th term is $1/(n-1)!$.) We can estimate the error in the approximation $s \approx s_n$ as follows:

$$\begin{aligned} 0 < s - s_n &= \frac{1}{n!} + \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots \\ &= \frac{1}{n!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\right) \\ &< \frac{1}{n!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots\right) \end{aligned}$$

since $n+2 > n+1$, $n+3 > n+1$, and so on. The latter series is geometric, so

$$0 < s - s_n < \frac{1}{n!}\,\frac{1}{1 - \dfrac{1}{n+1}} = \frac{n+1}{n!\,n}.$$

If we want to evaluate $e$ accurately to 3 decimal places, then we must ensure that the error is less than 5 in the fourth decimal place, that is, that the error is less than 0.0005. Hence, we want

$$\frac{n+1}{n}\,\frac{1}{n!} < 0.0005 = \frac{1}{2{,}000}.$$

Since $7! = 5{,}040$ but $6! = 720$, we can use $n = 7$ but no smaller. We have

$$\begin{aligned} e \approx s_7 &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} \\ &= 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} \approx 2.718 \quad \text{to 3 decimal places.} \end{aligned}$$

It is appropriate to use geometric series to bound the tails of positive series whose convergence would be demonstrated by the ratio test. Such series converge ultimately faster than any $p$-series $\sum_{n=1}^{\infty} n^{-p}$, for which the limit ratio is $\rho = 1$.

## EXERCISES 9.3

In Exercises 1–26, determine whether the given series converges or diverges by using any appropriate test. The $p$-series can be used for comparison, as can geometric series. Be alert for series whose terms do not approach 0.

**1.** $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

**2.** $\sum_{n=1}^{\infty} \frac{n}{n^4-2}$

**3.** $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

**4.** $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n+1}$

**5.** $\sum_{n=1}^{\infty} \left|\sin \frac{1}{n^2}\right|$

**6.** $\sum_{n=8}^{\infty} \frac{1}{\pi^n+5}$

**7.** $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^3}$

**8.** $\sum_{n=1}^{\infty} \frac{1}{\ln(3n)}$

**9.** $\sum_{n=1}^{\infty} \frac{1}{\pi^n - n^\pi}$

**10.** $\sum_{n=0}^{\infty} \frac{1+n}{2+n}$

**11.** $\sum_{n=1}^{\infty} \frac{1+n^{4/3}}{2+n^{5/3}}$

**12.** $\sum_{n=1}^{\infty} \frac{n^2}{1+n\sqrt{n}}$

**13.** $\sum_{n=3}^{\infty} \frac{1}{n \ln n \sqrt{\ln \ln n}}$

**14.** $\sum_{n=2}^{\infty} \frac{1}{n \ln n (\ln \ln n)^2}$

**15.** $\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^4}$

**16.** $\sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}}$

**17.** $\sum_{n=1}^{\infty} \frac{1}{2^n(n+1)}$

**18.** $\sum_{n=1}^{\infty} \frac{n^4}{n!}$

**19.** $\sum_{n=1}^{\infty} \frac{n!}{n^2 e^n}$

**20.** $\sum_{n=1}^{\infty} \frac{(2n)!6^n}{(3n)!}$

**21.** $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{3^n \ln n}$

**22.** $\sum_{n=0}^{\infty} \frac{n^{100}2^n}{\sqrt{n!}}$

**23.** $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^3}$

**24.** $\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$

**25.** $\sum_{n=4}^{\infty} \frac{2^n}{3^n - n^3}$

**26.** $\sum_{n=1}^{\infty} \frac{n^n}{\pi^n n!}$

In Exercises 27–30, use $s_n$ and integral bounds to find the smallest interval that you can be sure contains the sum $s$ of the series. If the midpoint $s_n^*$ of this interval is used to approximate $s$, how large should $n$ be chosen to ensure that the error is less than 0.001?

**27.** $\sum_{k=1}^{\infty} \frac{1}{k^4}$

**28.** $\sum_{k=1}^{\infty} \frac{1}{k^3}$

**29.** $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$

**30.** $\sum_{k=1}^{\infty} \frac{1}{k^2+4}$

For each positive series in Exercises 31–34, find the best upper bound you can for the error $s - s_n$ encountered if the partial sum $s_n$ is used to approximate the sum $s$ of the series. How many terms of each series do you need to be sure that the approximation has error less than 0.001?

**31.** $\sum_{k=1}^{\infty} \frac{1}{2^k k!}$

**32.** $\sum_{n=1}^{\infty} \frac{1}{(2n-1)!}$

**33.** $\sum_{n=0}^{\infty} \frac{2^n}{(2n)!}$

**34.** $\sum_{n=1}^{\infty} \frac{1}{n^n}$

**35.** Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges. Show that the sum $s$ of the series is less than $\pi/2$.

**36.** Show that $\sum_{n=3}^{\infty} (1/(n \ln n (\ln \ln n)^p))$ converges if and only if $p > 1$. Generalize this result to series of the form

$$\sum_{n=N}^{\infty} \frac{1}{n(\ln n)(\ln \ln n) \cdots (\ln_j n)(\ln_{j+1} n)^p},$$

where $\ln_j n = \underbrace{\ln \ln \ln \ln \cdots \ln}_{j \ \ln' s} n$.

**37.** Prove the root test. *Hint:* Mimic the proof of the ratio test.

**38.** Use the root test to show that $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$ converges.

**39.** Use the root test to test the following series for convergence:

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}.$$

**40.** Repeat Exercise 38, but use the ratio test instead of the root test.

**41.** Try to use the ratio test to determine whether $\sum_{n=1}^{\infty} \frac{2^{2n}(n!)^2}{(2n)!}$ converges. What happens? Now observe that

$$\frac{2^{2n}(n!)^2}{(2n)!} = \frac{[2n(2n-2)(2n-4)\cdots 6 \times 4 \times 2]^2}{2n(2n-1)(2n-2)\cdots 4 \times 3 \times 2 \times 1}$$
$$= \frac{2n}{2n-1} \times \frac{2n-2}{2n-3} \times \cdots \times \frac{4}{3} \times \frac{2}{1}.$$

Does the given series converge? Why or why not?

**42.** Determine whether the series $\sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}$ converges. *Hint:* Proceed as in Exercise 41. Show that $a_n \geq 1/(2n)$.

**43.** (a) Show that if $k > 0$ and $n$ is a positive integer, then $n < \frac{1}{k}(1+k)^n$.

(b) Use the estimate in (a) with $0 < k < 1$ to obtain an upper bound for the sum of the series $\sum_{n=0}^{\infty} n/2^n$. For what value of $k$ is this bound lowest?

(c) If we use the sum $s_n$ of the first $n$ terms to approximate the sum $s$ of the series in (b), obtain an upper bound for the error $s - s_n$ using the inequality from (a). For given $n$, find $k$ to minimize this upper bound.

**44.** **(Improving the convergence of a series)** We know that $\sum_{n=1}^{\infty} 1/\big(n(n+1)\big) = 1$. (See Example 3 of Section 9.2.) Since $\frac{1}{n^2} = \frac{1}{n(n+1)} + c_n$, where $c_n = \frac{1}{n^2(n+1)}$, we have $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} c_n$.

The series $\sum_{n=1}^{\infty} c_n$ converges more rapidly than does $\sum_{n=1}^{\infty} 1/n^2$ because its terms decrease like $1/n^3$. Hence, fewer terms of that series will be needed to compute $\sum_{n=1}^{\infty} 1/n^2$ to any desired degree of accuracy than would be needed if we calculated with $\sum_{n=1}^{\infty} 1/n^2$ directly. Using integral upper and lower bounds, determine a value of $n$ for which the modified partial sum $s_n^*$ for the series $\sum_{n=1}^{\infty} c_n$ approximates the sum of that series with error less than 0.001 in absolute value. Hence, determine $\sum_{n=1}^{\infty} 1/n^2$ to within 0.001 of its true value. (The technique exibited in this exercise is known as **improving the convergence** of a series. It can be applied to estimating the sum $\sum a_n$ if we know the sum $\sum b_n$ and if $a_n - b_n = c_n$, where $|c_n|$ decreases faster than $|a_n|$ as $n$ tends to infinity.)

**45.** Consider the series $s = \sum_{n=1}^{\infty} 1/(2^n + 1)$, and the partial sum $s_n$ of its first $n$ terms.

(a) How large need $n$ be taken to ensure that the error in the approximation $s \approx s_n$ is less than 0.001 in absolute value?

(b) The geometric series $\sum_{n=1}^{\infty} 1/2^n$ converges to 1. If

$$b_n = \frac{1}{2^n} - \frac{1}{2^n + 1}$$

for $n = 1, 2, 3, \ldots$, how many terms of the series $\sum_{n=1}^{\infty} b_n$ are needed to calculate its sum to within 0.001?

(c) Use the result of part (b) to calculate the $\sum_{n=1}^{\infty} 1/(2^n + 1)$ to within 0.001.

## 9.4 Absolute and Conditional Convergence

All of the series $\sum_{n=1}^{\infty} a_n$ considered in the previous section were ultimately positive; that is, $a_n \ge 0$ for $n$ sufficiently large. We now drop this restriction and allow arbitrary real terms $a_n$. We can, however, always obtain a positive series from any given series by replacing all the terms with their absolute values.

**DEFINITION 5**

**Absolute convergence**

The series $\sum_{n=1}^{\infty} a_n$ is said to be **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

The series

$$s = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \cdots$$

converges absolutely since

$$S = \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

converges. It seems reasonable that the first series must converge, and its sum $s$ should satisfy $-S \le s \le S$. In general, the cancellation that occurs because some terms are negative and others positive makes it *easier* for a series to converge than if all the terms are of one sign. We verify this insight in the following theorem.

**THEOREM 13**

If a series converges absolutely, then it converges.

***PROOF*** Let $\sum_{n=1}^{\infty} a_n$ be absolutely convergent, and let $b_n = a_n + |a_n|$ for each $n$. Since $-|a_n| \le a_n \le |a_n|$, we have $0 \le b_n \le 2|a_n|$ for each $n$. Thus, $\sum_{n=1}^{\infty} b_n$ converges by the comparison test. Therefore, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} |a_n|$ also converges.

Again you are cautioned not to confuse the statement of Theorem 13 with the converse statement, which is false. We will show later in this section that the **alternating harmonic series**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

converges, although it does not converge absolutely. If we replace all the terms by their absolute values, we get the divergent harmonic series

**BEWARE!** Although absolute convergence implies convergence, convergence does *not* imply absolute convergence.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty.$$

**DEFINITION 6**

**Conditional convergence**

If $\sum_{n=1}^{\infty} a_n$ is convergent, but not absolutely convergent, then we say that it is **conditionally convergent** or that it **converges conditionally**.

The alternating harmonic series is an example of a conditionally convergent series.

The comparison tests, the integral test, and the ratio test can each be used to test for absolute convergence. They should be applied to the series $\sum_{n=1}^{\infty} |a_n|$. For the ratio test we calculate $\rho = \lim_{n\to\infty} |a_{n+1}/a_n|$. If $\rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\rho > 1$, then $\lim_{n\to\infty} |a_n| = \infty$, so both $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} a_n$ must diverge. If $\rho = 1$, we get no information; the series $\sum_{n=1}^{\infty} a_n$ may converge absolutely, it may converge conditionally, or it may diverge.

**EXAMPLE 1** Test the following series for absolute convergence:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n}$.

*Solution*

(a) $\displaystyle\lim_{n\to\infty} \left|\frac{(-1)^{n-1}}{2n-1}\right| \bigg/ \frac{1}{n} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2} > 0.$

Since the harmonic series $\sum_{n=1}^{\infty}(1/n)$ diverges to infinity, the comparison test assures us that $\sum_{n=1}^{\infty}((-1)^{n-1}/(2n-1))$ does not converge absolutely.

(b) $\displaystyle\rho = \lim_{n\to\infty} \left|\frac{(n+1)\cos((n+1)\pi)}{2^{n+1}} \bigg/ \frac{n\cos(n\pi)}{2^n}\right| = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1.$

(Note that $\cos(n\pi)$ is just a fancy way of writing $(-1)^n$.) Therefore, by the ratio test, $\sum_{n=1}^{\infty}((n\cos(n\pi))/2^n)$ converges absolutely.

## The Alternating Series Test

We cannot use any of the previously developed tests to show that the alternating harmonic series converges; all of those tests apply only to (ultimately) positive series, so they can test only for absolute convergence. Demonstrating convergence that is not absolute is generally more difficult to do. We present only one test that can establish such convergence; this test can only be used on a very special kind of series.

**THEOREM 14**

**The alternating series test**

Suppose $\{a_n\}$ is a sequence whose terms satisfy, for some positive integer $N$,

(i) $a_n a_{n+1} < 0$ for $n \geq N$,

(ii) $|a_{n+1}| \leq |a_n|$ for $n \geq N$, and

(iii) $\lim_{n\to\infty} a_n = 0$,

that is, the terms are ultimately alternating in sign and decreasing in size, and the sequence has limit zero. Then the series $\sum_{n=1}^{\infty} a_n$ converges.

***PROOF*** Without loss of generality we can assume $N = 1$ because convergence only depends on the tail of a series. We also assume $a_1 > 0$; the proof if $a_1 < 0$ is similar. If $s_n = a_1 + a_2 + \cdots + a_n$ is the $n$th partial sum of the series, it follows from the alternation of $\{a_n\}$ that $a_{2n+1} > 0$ and $a_{2n} < 0$ for each $n$. Since the terms decrease in size, $a_{2n+1} \geq -a_{2n+2}$. Therefore, $s_{2n+2} = s_{2n} + a_{2n+1} + a_{2n+2} \geq s_{2n}$ for $n = 1, 2, 3, \ldots$; the even partial sums $\{s_{2n}\}$ form an increasing sequence. Similarly, $s_{2n+1} = s_{2n-1} + a_{2n} + a_{2n+1} \leq s_{2n-1}$, so the odd partial sums $\{s_{2n-1}\}$ form a decreasing sequence. Since $s_{2n} = s_{2n-1} + a_{2n} \leq s_{2n-1}$, we can say, for any $n$, that

BEWARE! Read this proof slowly and think about why each statement is true.

$$s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_{2n} \leq s_{2n-1} \leq s_{2n-3} \leq \cdots \leq s_5 \leq s_3 \leq s_1.$$

Hence, $s_2$ is a lower bound for the decreasing sequence $\{s_{2n-1}\}$, and $s_1$ is an upper bound for the increasing sequence $\{s_{2n}\}$. Both of these sequences therefore converge by the completeness of the real numbers:

$$\lim_{n\to\infty} s_{2n-1} = s_{\text{odd}}, \qquad \lim_{n\to\infty} s_{2n} = s_{\text{even}}.$$

Now $a_{2n} = s_{2n} - s_{2n-1}$, so $0 = \lim_{n\to\infty} a_{2n} = \lim_{n\to\infty}(s_{2n} - s_{2n-1}) = s_{\text{even}} - s_{\text{odd}}$. Therefore $s_{\text{odd}} = s_{\text{even}} = s$, say. Every partial sum $s_n$ is either of the form $s_{2n-1}$ or of the form $s_{2n}$. Thus, $\lim_{n\to\infty} s_n = s$ exists and the series $\sum(-1)^{n-1}a_n$ converges to this sum $s$.

***Remark*** The proof of Theorem 14 shows that the sum $s$ of the series always lies between any two consecutive partial sums of the series:

$$\text{either} \quad s_n < s < s_{n+1} \quad \text{or} \quad s_{n+1} < s < s_n.$$

This proves the following theorem.

**THEOREM 15**

**Error estimate for alternating series**

If the sequence $\{a_n\}$ satisfies the conditions of the alternating series test (Theorem 14), so that the series $\sum_{n=1}^{\infty} a_n$ converges to the sum $s$, then the error in the approximation $s \approx s_n$ (where $n \geq N$) has the same sign as the first omitted term $a_{n+1} = s_{n+1} - s_n$, and its size is no greater than the size of that term:

$$|s - s_n| \leq |s_{n+1} - s_n| = |a_{n+1}|.$$

**EXAMPLE 2** How many terms of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{1+2^n}$ are needed to compute the sum of the series with error less than 0.001?

***Solution*** This series satisfies the hypotheses for Theorem 15. If we use the partial sum of the first $n$ terms of the series to approximate the sum of the series, the error will satisfy

$$|\text{error}| \leq |\text{first omitted term}| = \frac{1}{1+2^{n+1}}.$$

This error is less than 0.001 if $1 + 2^{n+1} > 1{,}000$. Since $2^{10} = 1{,}024$, $n + 1 = 10$ will do; we need 9 terms of the series to compute the sum to within 0.001 of its actual value.

When determining the convergence of a given series, it is best to consider first whether the series converges absolutely. If it does not, then there remains the possibility of conditional convergence.

**EXAMPLE 3** Test the following series for absolute and conditional convergence:

$$\text{(a)}\quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}, \qquad \text{(b)}\quad \sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\ln n}, \qquad \text{(c)}\quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4}.$$

***Solution*** The absolute values of the terms in series (a) and (b) are $1/n$ and $1/(\ln n)$, respectively. Since $1/(\ln n) > 1/n$, and $\sum_{n=1}^{\infty} 1/n$ diverges to infinity, neither series (a) nor (b) converges absolutely. However, both series satisfy the requirements of Theorem 14 and so both converge. Each of these series is conditionally convergent.

Series (c) is absolutely convergent because $|(-1)^{n-1}/n^4| = 1/n^4$, and $\sum_{n=1}^{\infty} 1/n^4$ is a convergent $p$-series ($p = 4 > 1$). We could establish its convergence using Theorem 14, but there is no need to do that since every absolutely convergent series is convergent (Theorem 13).

**EXAMPLE 4** For what values of $x$ does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n\,2^n}$ converge absolutely? converge conditionally? diverge?

***Solution*** For such series whose terms involve functions of a variable $x$, it is usually wisest to begin testing for absolute convergence with the ratio test. We have

$$\rho = \lim_{n\to\infty} \left| \frac{(x-5)^{n+1}}{(n+1)2^{n+1}} \Big/ \frac{(x-5)^n}{n\,2^n} \right| = \lim_{n\to\infty} \frac{n}{n+1} \left| \frac{x-5}{2} \right| = \left| \frac{x-5}{2} \right|.$$

The series converges absolutely if $|(x-5)/2| < 1$. This inequality is equivalent to $|x-5| < 2$ (the distance from $x$ to 5 is less than 2), that is, $3 < x < 7$. If $x < 3$ or $x > 7$, then $|(x-5)/2| > 1$. The series diverges; its terms do not approach zero.

If $x = 3$, the series is $\sum_{n=1}^{\infty} \big((-1)^n/n\big)$, which converges conditionally (it is an alternating harmonic series); if $x = 7$, the series is the harmonic series $\sum_{n=1}^{\infty} 1/n$, which diverges to infinity. Hence, the given series converges absolutely on the open interval $(3, 7)$, converges conditionally at $x = 3$, and diverges everywhere else.

**EXAMPLE 5** For what values of $x$ does the series $\displaystyle\sum_{n=0}^{\infty} (n+1)^2 \left(\frac{x}{x+2}\right)^n$ converge absolutely? converge conditionally? diverge?

***Solution*** Again we begin with the ratio test.

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| (n+2)^2 \left(\frac{x}{x+2}\right)^{n+1} \Big/ (n+1)^2 \left(\frac{x}{x+2}\right)^n \right| \\ &= \lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)^2 \left| \frac{x}{x+2} \right| = \left| \frac{x}{x+2} \right| = \frac{|x|}{|x+2|}\end{aligned}$$

The series converges absolutely if $|x|/|x+2| < 1$. This condition says that the distance from $x$ to 0 is less than the distance from $x$ to $-2$. Hence, $x > -1$. The series diverges if $|x|/|x+2| > 1$, that is, if $x < -1$. If $x = -1$, the series is $\sum_{n=0}^{\infty}(-1)^n(n+1)^2$, which diverges. We conclude that the series converges absolutely for $x > -1$, converges conditionally nowhere, and diverges for $x \le -1$.

When using the alternating series test, it is important to verify (at least mentally) that *all three conditions* (i)–(iii) are satisfied.

**EXAMPLE 6** Test the following series for convergence:

(a) $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+1}{n},$$

(b) $$1-\frac{1}{4}+\frac{1}{3}-\frac{1}{16}+\frac{1}{5}-\cdots=\sum_{n=1}^{\infty}a_n, \quad \text{where}$$

$$a_n = \begin{cases} 1/n & \text{if } n \text{ is odd,} \\ -1/n^2 & \text{if } n \text{ is even.} \end{cases}$$

### *Solution*

(a) Certainly, the terms $a_n$ alternate and decrease in size as $n$ increases. However, $\lim_{n\to\infty}|a_n| = 1 \ne 0$. The alternating series test does not apply. In fact, the given series diverges because its terms do not approach 0.

(b) This series alternates and its terms have limit zero. However, the terms are not decreasing in size (even ultimately). Once again, the alternating series test cannot be applied. In fact, since

$$-\frac{1}{4}-\frac{1}{16}-\cdots-\frac{1}{(2n)^2}-\cdots \qquad \text{converges, and}$$

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}+\cdots \qquad \text{diverges to infinity,}$$

it is readily seen that the given series diverges to infinity.

## Rearranging the Terms in a Series

The basic difference between absolute and conditional convergence is that when a series $\sum_{n=1}^{\infty}a_n$ converges absolutely, it does so because its terms $\{a_n\}$ decrease in size fast enough that their sum can be finite even if no cancellation occurs due to terms of opposite sign. If cancellation is required to make the series converge (because the terms decrease slowly), then the series can only converge conditionally.

Consider the alternating harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots.$$

This series converges, but only conditionally. If we take the subseries containing only the positive terms, we get the series

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots,$$

which diverges to infinity. Similarly, the subseries of negative terms

$$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}-\cdots$$

diverges to negative infinity.

If a series converges absolutely, the subseries consisting of positive terms and the subseries consisting of negative terms must each converge to a finite sum. If a series converges conditionally, the positive and negative subseries will both diverge, to $\infty$ and $-\infty$, respectively.

Using these facts we can answer a question raised at the beginning of Section 9.2. If we rearrange the terms of a convergent series so that they are added in a different order, must the rearranged series converge, and if it does will it converge to the same sum as the original series? The answer depends on whether the original series was absolutely convergent or merely conditionally convergent.

**THEOREM 16**

**Convergence of rearrangements of a series**

(a) If the terms of an absolutely convergent series are rearranged so that addition occurs in a different order, the rearranged series still converges to the same sum as the original series.

(b) If a series is conditionally convergent, and $L$ is any real number, then the terms of the series can be rearranged so as to make the series converge (conditionally) to the sum $L$. It can also be rearranged so as to diverge to $\infty$ or to $-\infty$, or just to diverge.

Part (b) shows that conditional convergence is a rather suspect kind of convergence, being dependent on the order in which the terms are added. We will not present a formal proof of the theorem but will give an example suggesting what is involved. (See also Exercise 30 below.)

**EXAMPLE 7** In Section 9.5 we will show that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots$$

converges (conditionally) to the sum $\ln 2$. Describe how to rearrange its terms so that it converges to 8 instead.

***Solution*** Start adding terms of the positive subseries

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots,$$

and keep going until the partial sum exceeds 8. (It will, eventually, because the positive subseries diverges to infinity.) Then add the first term $-1/2$ of the negative subseries

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \cdots.$$

This will reduce the partial sum below 8 again. Now resume adding terms of the positive subseries until the partial sum climbs above 8 once more. Then add the second term of the negative subseries and the partial sum will drop below 8. Keep repeating this procedure, alternately adding terms of the positive subseries to force the sum above 8 and then terms of the negative subseries to force it below 8. Since both subseries have infinitely many terms and diverge to $\infty$ and $-\infty$, respectively, eventually every term of the original series will be included, and the partial sums of the new series will oscillate back and forth around 8, converging to that number. Of course, any number other than 8 could also be used in place of 8.

## EXERCISES 9.4

Determine whether the series in Exercises 1–12 converge absolutely, converge conditionally, or diverge.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+\ln n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{(n+1)\ln(n+1)}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{2^n}$

5. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(n^2-1)}{n^2+1}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi^n}$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{-n}{n^2+1}$

9. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{20n^2-n-1}{n^3+n^2+33}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{100\cos(n\pi)}{2n+3}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(-100)^n}$

12. $\displaystyle\sum_{n=10}^{\infty} \frac{\sin(n+1/2)\pi}{\ln\ln n}$

For the series in Exercises 13–16, find the smallest integer $n$ that ensures that the partial sum $s_n$ approximates the sum $s$ of the series with error less than 0.001 in absolute value.

13. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$

14. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$

15. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{2^n}$

16. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{n!}$

Determine the values of $x$ for which the series in Exercises 17–24 converge absolutely, converge conditionally, or diverge.

17. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n+1}}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 2^{2n}}$

19. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{2n+3}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}\left(\frac{3x+2}{-5}\right)^n$

21. $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{2^n \ln n}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^3}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+3)^n}{n^{1/3}4^n}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(1+\frac{1}{x}\right)^n$

25. Does the alternating series test apply directly to the series $\sum_{n=1}^{\infty}(1/n)\,\sin(n\pi/2)$? Determine whether the series converges.

26. Show that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $a_n = 10/n^2$ for even $n$ and $a_n = -1/10n^3$ for odd $n$.

27. Which of the following statements are TRUE and which are FALSE? Justify your assertion of truth, or give a counter-example to show falsehood.
   (a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.
   (b) If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty}(-1)^n a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
   (c) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges absolutely.

28. (a) Use a Riemann sum argument to show that

$$\ln n! \ge \int_1^n \ln t\,dt = n\ln n - n + 1.$$

   (b) For what values of $x$ does the series $\sum_{n=1}^{\infty} \dfrac{n!x^n}{n^n}$ converge absolutely? converge conditionally? diverge? (*Hint:* First use the ratio test. To test the cases where $\rho = 1$, you may find the inequality in part (a) useful.)

29. For what values of $x$ does the series $\sum_{n=1}^{\infty} \dfrac{(2n)!x^n}{2^{2n}(n!)^2}$ converge absolutely? converge conditionally? diverge? *Hint:* See Exercise 42 of Section 9.3.

30. Devise procedures for rearranging the terms of the alternating harmonic series so that the rearranged series
   (a) diverges to $\infty$, (b) converges to $-2$.

## 9.5 Power Series

This section is concerned with a special kind of infinite series called a *power series*, which may be thought of as a polynomial of infinite degree.

**DEFINITION 7**

**Power series**

A series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

is called a **power series in powers of $x - c$** or a **power series about $c$**. The constants $a_0, a_1, a_2, \ldots$ are called the **coefficients** of the power series.

Since the terms of a power series are functions of a variable $x$, the series may or may not converge for each value of $x$. For those values of $x$ for which the series does converge, the sum defines a function of $x$. For example, if $-1 < x < 1$, then

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

The geometric series on the left side is a power series *representation* of the function $1/(1-x)$ in powers of $x$ (or about 0). Note that the representation is valid only in the open interval $(-1, 1)$ even though $1/(1-x)$ is defined for all real $x$ except $x = 1$. For $x = -1$ and for $|x| > 1$ the series does not converge, so it cannot represent $1/(1-x)$ at these points.

The point $c$ is the **centre of convergence** of the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$. The series certainly converges (to $a_0$) at $x = c$. (All the terms except possibly the first are 0.) Theorem 17 below shows that if the series converges anywhere else, then it converges on an interval (possibly infinite) centred at $x = c$, and it converges absolutely everywhere on that interval except possibly at one or both of the endpoints if the interval is finite. The geometric series

$$1 + x + x^2 + x^3 + \cdots$$

is an example of this behaviour. It has centre of convergence $c = 0$, and converges only on the interval $(-1, 1)$, centred at 0. The convergence is absolute at every point of the interval. Another example is the series

$$\sum_{n=1}^{\infty} \frac{1}{n\,2^n}(x-5)^n = \frac{x-5}{2} + \frac{(x-5)^2}{2\times 2^2} + \frac{(x-5)^3}{3\times 2^3} + \cdots,$$

which we discussed in Example 4 of Section 9.4. We showed that this series converges on the interval $[3, 7)$, an interval with centre $x = 5$, and that the convergence is absolute on the open interval $(3, 7)$ but is only conditional at the endpoint $x = 3$.

**THEOREM 17**

For any power series $\sum_{n=0}^{\infty} a_n\,(x-c)^n$ one of the following alternatives must hold:

(i) the series may converge only at $x = c$,

(ii) the series may converge at every real number $x$, or

(iii) there may exist a positive real number $R$ such that the series converges at every $x$ satisfying $|x-c| < R$ and diverges at every $x$ satisfying $|x-c| > R$. In this case the series may or may not converge at either of the two *endpoints* $x = c - R$ and $x = c + R$.

In each of these cases the convergence is absolute except possibly at the endpoints $x = c - R$ and $x = c + R$ in case (iii).

***PROOF*** We observed above that every power series converges at its centre of convergence; only the first term can be nonzero, so the convergence is absolute. To prove the rest of this theorem, it suffices to show that if the series converges at any number $x_0 \neq c$, then it converges absolutely at every number $x$ closer to $c$ than $x_0$ is, that is, at every $x$ satisfying $|x-c| < |x_0 - c|$. This means that convergence at any $x_0 \neq c$ implies absolute convergence on $(c - x_0, c + x_0)$, so the set of points $x$ where the series converges must be an interval centred at $c$.

Suppose, therefore, that $\sum_{n=0}^{\infty} a_n(x_0-c)^n$ converges. Then $\lim a_n(x_0-c)^n = 0$, so $|a_n(x_0-c)^n| \le K$ for all $n$, where $K$ is some constant (Theorem 1 of Section 9.1). If $r = |x-c|/|x_0-c| < 1$, then

$$\sum_{n=0}^{\infty} |a_n(x-c)^n| = \sum_{n=0}^{\infty} |a_n(x_0-c)^n| \left|\frac{x-c}{x_0-c}\right|^n \le K \sum_{n=0}^{\infty} r^n = \frac{K}{1-r} < \infty.$$

Thus, $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges absolutely.

By Theorem 17, the set of values $x$ for which the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges is an interval centred at $x = c$. We call this interval the **interval of convergence** of the power series. It must have one of the following forms:

(i) the isolated point $x = c$ (a degenerate closed interval $[c, c]$),

(ii) the entire line $(-\infty, \infty)$,

(iii) a finite interval centred at $c$:
$[c-R, c+R]$, or $[c-R, c+R)$, or $(c-R, c+R]$, or $(c-R, c+R)$.

The number $R$ in (iii) is called the **radius of convergence** of the power series. In case (i) we say the radius of convergence is $R = 0$; in case (ii) it is $R = \infty$.

The radius of convergence, $R$, can often be found by using the ratio test on the power series: if

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n}\right| = \left(\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\right)|x-c|$$

exists, then the series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges absolutely where $\rho < 1$, that is, where

$$|x-c| < R = 1 \Big/ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

The series diverges if $|x-c| > R$.

**Radius of convergence**

Suppose that $L = \lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|$ exists or is $\infty$. Then the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ has radius of convergence $R = 1/L$. (If $L = 0$, then $R = \infty$; if $L = \infty$, then $R = 0$.)

**EXAMPLE 1** Determine the centre, radius, and interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(2x+5)^n}{(n^2+1)3^n}.$$

***Solution*** The series can be rewritten

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \frac{1}{n^2+1}\left(x+\frac{5}{2}\right)^n.$$

The centre of convergence is $x = -5/2$. The radius of convergence, $R$, is given by

$$\frac{1}{R} = L = \lim \left|\frac{\left(\frac{2}{3}\right)^{n+1}\dfrac{1}{(n+1)^2+1}}{\left(\frac{2}{3}\right)^n \dfrac{1}{n^2+1}}\right| = \lim \frac{2}{3}\,\frac{n^2+1}{(n+1)^2+1} = \frac{2}{3}.$$

Thus, $R = 3/2$. The series converges absolutely on $(-5/2 - 3/2, -5/2 + 3/2) = (-4, -1)$, and it diverges on $(-\infty, -4)$ and on $(-1, \infty)$. At $x = -1$ the series is $\sum_{n=0}^{\infty} 1/(n^2+1)$; at $x = -4$ it is $\sum_{n=0}^{\infty} (-1)^n/(n^2+1)$. Both series converge (absolutely). The interval of convergence of the given power series is therefore $[-4, -1]$.

**EXAMPLE 2** Determine the radii of convergence of the series

$$\text{(a)} \quad \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{and} \qquad \text{(b)} \quad \sum_{n=0}^{\infty} n!\,x^n.$$

*Solution*

(a) $L = \left| \lim \dfrac{1}{(n+1)!} \bigg/ \dfrac{1}{n!} \right| = \lim \dfrac{n!}{(n+1)!} = \lim \dfrac{1}{n+1} = 0$. Thus, $R = \infty$.

This series converges (absolutely) for all $x$. The sum is $e^x$, as will be shown in Example 1 in the next section.

(b) $L = \left| \lim \dfrac{(n+1)!}{n!} \right| = \lim(n+1) = \infty$. Thus, $R = 0$.

This series converges only at its centre of convergence, $x = 0$.

## Algebraic Operations on Power Series

To simplify the following discussion, we will consider only power series with centre of convergence 0, that is, series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

Any properties we demonstrate for such series extend automatically to power series of the form $\sum_{n=0}^{\infty} a_n (y-c)^n$ via the change of variable $x = y - c$.

First, we observe that series having the same centre of convergence can be added or subtracted on whatever interval is common to their intervals of convergence. The following theorem is a simple consequence of Theorem 7 of Section 9.2 and does not require a proof.

**THEOREM 18**

Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be two power series with radii of convergence $R_a$ and $R_b$, respectively, and let $c$ be a constant. Then

(i) $\sum_{n=0}^{\infty} (ca_n) x^n$ has radius of convergence $R_a$, and

$$\sum_{n=0}^{\infty} (ca_n) x^n = c \sum_{n=0}^{\infty} a_n x^n$$

wherever the series on the right converges.

(ii) $\sum_{n=0}^{\infty} (a_n + b_n) x^n$ has radius of convergence $R$ at least as large as the smaller of $R_a$ and $R_b$ ($R \ge \min\{R_a, R_b\}$), and

$$\sum_{n=0}^{\infty} (a_n + b_n) x^n = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n$$

wherever both series on the right converge.

The situation regarding multiplication and division of power series is more complicated. We will mention only the results and will not attempt any proofs of our assertions. A textbook in mathematical analysis will provide more details.

Long multiplication of the form

$$\begin{aligned}&(a_0 + a_1x + a_2x^2 + \cdots)(b_0 + b_1x + b_2x^2 + \cdots)\\&\qquad = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots\end{aligned}$$

leads us to conjecture the formula

$$\left(\sum_{n=0}^{\infty} a_nx^n\right)\left(\sum_{n=0}^{\infty} b_nx^n\right) = \sum_{n=0}^{\infty} c_nx^n,$$

where

$$c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0 = \sum_{j=0}^{n} a_jb_{n-j}.$$

The series $\sum_{n=0}^{\infty} c_nx^n$ is called the **Cauchy product** of the series $\sum_{n=0}^{\infty} a_nx^n$ and $\sum_{n=0}^{\infty} b_nx^n$. Like the sum, the Cauchy product also has radius of convergence at least equal to the lesser of those of the factor series.

**EXAMPLE 3** Since $\dfrac{1}{1-x} = 1+x+x^2+x^3+\cdots = \displaystyle\sum_{n=0}^{\infty} x^n$ holds for $-1 < x < 1$, we can determine a power series representation for $1/(1-x)^2$ by taking the Cauchy product of this series with itself. Since $a_n = b_n = 1$ for $n = 0, 1, 2, \ldots$, we have

$$c_n = \sum_{j=0}^{n} 1 = n + 1 \qquad \text{and}$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty}(n+1)x^n,$$

which must also hold for $-1 < x < 1$. The same series can be obtained by direct long multiplication of the series:

$$\begin{array}{rccccccccc} & 1 & + & x & + & x^2 & + & x^3 & + & \cdots\\ \times & 1 & + & x & + & x^2 & + & x^3 & + & \cdots\\ \hline & 1 & + & x & + & x^2 & + & x^3 & + & \cdots\\ & & & x & + & x^2 & + & x^3 & + & \cdots\\ & & & & & x^2 & + & x^3 & + & \cdots\\ & & & & & & & x^3 & + & \cdots\\ & & & & & & & & & \cdots\\ \hline & 1 & + & 2x & + & 3x^2 & + & 4x^3 & + & \cdots\end{array}$$

Long division can also be performed on power series, but there is no simple rule for determining the coefficients of the quotient series. The radius of convergence of the quotient series is not less than the least of the three numbers $R_1$, $R_2$, and $R_3$, where $R_1$ and $R_2$ are the radii of convergence of the numerator and denominator series and $R_3$ is the distance from the centre of convergence to the nearest *complex number* where the denominator series has sum equal to 0. To illustrate this point, observe that 1 and $1 - x$ are both power series with infinite radii of convergence:

$$\begin{aligned}1 &= 1 + 0x + 0x^2 + 0x^3 + \cdots &&\text{for all } x,\\ 1 - x &= 1 - \phantom{0}x + 0x^2 + 0x^3 + \cdots &&\text{for all } x.\end{aligned}$$

Their quotient, $1/(1-x)$, however, only has radius of convergence 1, the distance from the centre of convergence $x = 0$ to the point $x = 1$ where the denominator vanishes:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad \text{for} \quad |x| < 1.$$

## Differentiation and Integration of Power Series

If a power series has a positive radius of convergence, it can be differentiated or integrated term by term. The resulting series will converge to the appropriate derivative or integral of the sum of the original series everywhere except possibly at the endpoints of the interval of convergence of the original series. This very important fact ensures that, for purposes of calculation, power series behave just like polynomials, the easiest functions to differentiate and integrate. We formalize the differentiation and integration properties of power series in the following theorem.

**THEOREM 19**

**Term-by-term differentiation and integration of power series**

If the series $\sum_{n=0}^{\infty} a_n x^n$ converges to the sum $f(x)$ on an interval $(-R, R)$, where $R > 0$, that is,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots, \qquad (-R < x < R),$$

then $f$ is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots, \qquad (-R < x < R).$$

Also, $f$ is integrable over any closed subinterval of $(-R, R)$, and if $|x| < R$, then

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots.$$

While understanding the statement of this theorem is very important for what follows, understanding the proof is not. Feel free to skip the proof and go on to the applications.

***PROOF*** Let $x$ satisfy $-R < x < R$ and choose $H > 0$ such that $|x| + H < R$. By Theorem 17 we then have[1]

$$\sum_{n=1}^{\infty} |a_n|(|x| + H)^n = K < \infty.$$

The Binomial Theorem (see Section 9.8) shows that if $n \ge 1$, then

$$(x+h)^n = x^n + nx^{n-1}h + \sum_{k=2}^{n} \binom{n}{k} x^{n-k} h^k.$$

[1] This proof is due to R. Vyborny, *American Mathematical Monthly*, April 1987.

Therefore, if $|h| \le H$, we have

$$
\begin{aligned}
|(x+h)^n - x^n - nx^{n-1}h| &= \left| \sum_{k=2}^{n} \binom{n}{k} x^{n-k} h^k \right| \\
&\le \sum_{k=2}^{n} \binom{n}{k} |x|^{n-k} \frac{|h|^k}{H^k} H^k \\
&\le \frac{|h|^2}{H^2} \sum_{k=0}^{n} \binom{n}{k} |x|^{n-k} H^k \\
&= \frac{|h|^2}{H^2} \big(|x| + H\big)^n .
\end{aligned}
$$

Also,

$$|nx^{n-1}| = \frac{n|x|^{n-1}H}{H} \le \frac{1}{H} \big(|x| + H\big)^n .$$

Thus,

$$\sum_{n=1}^{\infty} |na_n x^{n-1}| \le \frac{1}{H} \sum_{n=1}^{\infty} |a_n|(|x| + H)^n = \frac{K}{H} < \infty,$$

so the series $\sum_{n=1}^{\infty} na_n x^{n-1}$ converges (absolutely) to $g(x)$, say. Now

$$
\begin{aligned}
\left| \frac{f(x+h) - f(x)}{h} - g(x) \right| &= \left| \sum_{n=1}^{\infty} \frac{a_n(x+h)^n - a_n x^n - na_n x^{n-1}h}{h} \right| \\
&\le \frac{1}{|h|} \sum_{n=1}^{\infty} |a_n| |(x+h)^n - x^n - nx^{n-1}h| \\
&\le \frac{|h|}{H^2} \sum_{n=1}^{\infty} |a_n| \big(|x| + H\big)^n \le \frac{K|h|}{H^2}.
\end{aligned}
$$

Letting $h$ approach zero, we obtain $|f'(x) - g(x)| \le 0$, so $f'(x) = g(x)$, as required.

Now observe that since $|a_n/(n+1)| \le |a_n|$, the series

$$h(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

converges (absolutely) at least on the interval $(-R, R)$. Using the differentiation result proved above, we obtain

$$h'(x) = \sum_{n=0}^{\infty} a_n x^n = f(x).$$

Since $h(0) = 0$, we have

$$\int_0^x f(t)\, dt = \int_0^x h'(t)\, dt = h(t)\Big|_0^x = h(x),$$

as required.

Together, these results imply that the termwise differentiated or integrated series have the same radius of convergence as the given series. In fact, as the following examples illustrate, the interval of convergence of the differentiated series is the same as that of the original series except for the *possible* loss of one or both endpoints if the original series converges at endpoints of its interval of convergence. Similarly, the integrated series will converge everywhere on the interval of convergence of the original series and possibly at one or both endpoints of that interval, even if the original series does not converge at the endpoints.

Being differentiable on $(-R, R)$, where $R$ is the radius of convergence, the sum $f(x)$ of a power series is necessarily continuous on that open interval. If the series happens to converge at either or both of the endpoints $-R$ and $R$, then $f$ is also continuous (on one side) up to these endpoints. This result is stated formally in the following theorem. We will not prove it here; the interested reader is referred to textbooks on mathematical analysis for a proof.

**THEOREM 20**

**Abel's Theorem**

The sum of a power series is a continuous function everywhere on the interval of convergence of the series. In particular, if $\sum_{n=0}^{\infty} a_n R^n$ converges for some $R > 0$, then

$$\lim_{x \to R-} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n,$$

and if $\sum_{n=0}^{\infty} a_n (-R)^n$ converges, then

$$\lim_{x \to -R+} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n (-R)^n.$$

The following examples show how the above theorems are applied to obtain power series representations for functions.

**EXAMPLE 4** Find power series representations for the functions

$$\text{(a)}\quad \frac{1}{(1-x)^2}, \qquad \text{(b)}\quad \frac{1}{(1-x)^3}, \qquad \text{and} \qquad \text{(c)}\quad \ln(1+x)$$

by starting with the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (-1 < x < 1)$$

and using differentiation, integration, and substitution. Where is each series valid?

***Solution***

(a) Differentiate the geometric series term by term to obtain

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots \qquad (-1 < x < 1).$$

This is the same result obtained by multiplication of series in Example 3 above.

(b) Differentiate again to get, for $-1 < x < 1$,

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)\, x^{n-2} = (1 \times 2) + (2 \times 3)x + (3 \times 4)x^2 + \cdots.$$

Now divide by 2:

$$\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2} = 1 + 3x + 6x^2 + 10x^3 + \cdots \qquad (-1 < x < 1).$$

(c) Substitute $-t$ in place of $x$ in the original geometric series:

$$\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n = 1 - t + t^2 - t^3 + t^4 - \cdots \qquad (-1 < t < 1).$$

Integrate from 0 to $x$, where $|x| < 1$, to get

$$\begin{aligned} \ln(1+x) &= \int_0^x \frac{dt}{1+t} = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n \, dt \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad (-1 < x \le 1). \end{aligned}$$

Note that the latter series converges (conditionally) at the endpoint $x = 1$ as well as on the interval $-1 < x < 1$. Since $\ln(1+x)$ is continuous at $x = 1$, Theorem 20 assures us that the series must converge to that function at $x = 1$ also. In particular, therefore, the alternating harmonic series converges to $\ln 2$:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.$$

This would not, however, be a very useful formula for calculating the value of $\ln 2$. (Why not?)

**EXAMPLE 5** Use the geometric series of the previous example to find a power series representation for $\tan^{-1} x$.

***Solution*** Substitute $-t^2$ for $x$ in the geometric series. Since $0 \le t^2 < 1$ whenever $-1 < t < 1$, we obtain

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 - \cdots \qquad (-1 < t < 1).$$

Now integrate from 0 to $x$, where $|x| < 1$:

$$\begin{aligned} \tan^{-1} x &= \int_0^x \frac{dt}{1+t^2} = \int_0^x (1 - t^2 + t^4 - t^6 + t^8 - \cdots) \, dt \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad (-1 < x < 1). \end{aligned}$$

However, note that the series also converges (conditionally) at $x = -1$ and 1. Since $\tan^{-1}$ is continuous at $\pm 1$, the above series representation for $\tan^{-1} x$ also holds for these values, by Theorem 20. Letting $x = 1$ we get another interesting result:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots.$$

Again, however, this would not be a good formula with which to calculate a numerical value of $\pi$. (Why not?)

**EXAMPLE 6** Find the sum of the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ by first finding the sum of the power series

$$\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots.$$

**Solution** Observe in Example 4(a) how the process of differentiating the geometric series produces a series with coefficients 1, 2, 3, .... Start with the series obtained for $1/(1-x)^2$ and multiply it by $x$ to obtain

$$\sum_{n=1}^{\infty} n x^n = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \frac{x}{(1-x)^2}.$$

Now differentiate again to get a series with coefficients $1^2, 2^2, 3^2, \ldots$:

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = 1 + 4x + 9x^2 + 16x^3 + \cdots = \frac{d}{dx}\frac{x}{(x-1)^2} = \frac{1+x}{(1-x)^3}.$$

Multiplication by $x$ again gives the desired power series:

$$\sum_{n=1}^{\infty} n^2 x^n = x + 4x^2 + 9x^3 + 16x^4 + \cdots = \frac{x(1+x)}{(1-x)^3}.$$

Differentiation and multiplication by $x$ do not change the radius of convergence, so this series converges to the indicated function for $-1 < x < 1$. Putting $x = 1/2$, we get

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{\frac{1}{2} \times \frac{3}{2}}{\frac{1}{8}} = 6.$$

The following example illustrates how substitution can be used to obtain power series representations of functions with centres of convergence different from 0.

**EXAMPLE 7** Find a series representation of $f(x) = 1/(2+x)$ in powers of $x - 1$. What is the interval of convergence of this series?

**Solution** Let $t = x - 1$ so that $x = t + 1$. We have

$$\begin{aligned}
\frac{1}{2+x} &= \frac{1}{3+t} = \frac{1}{3}\,\frac{1}{1+\frac{t}{3}} \\
&= \frac{1}{3}\left(1 - \frac{t}{3} + \frac{t^2}{3^2} - \frac{t^3}{3^3} + \cdots\right) && (-1 < t/3 < 1) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{3^{n+1}} && (-3 < t < 3) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{3^{n+1}} && (-2 < x < 4).
\end{aligned}$$

Note that the radius of convergence of this series is 3, the distance from the centre of convergence, 1, to the point $-2$ where the denominator is 0. We could have predicted this in advance.

## Maple Calculations

Maple can find the sums of many kinds of series, including absolutely and conditionally convergent numerical series and many power series. Even when Maple can't find the formal sum of a (convergent) series, it can provide a decimal approximation to the precision indicated by the current value of its variable `Digits`, which defaults to 10. Here are some examples.

```
> sum(n^4/2^n, n=1..infinity);
```

$$150$$

```
> sum(1/n^2, n=1..infinity);
```

$$\frac{1}{6}\pi^2$$

```
> sum(exp(-n^2), n=0..infinity);
```

$$\sum_{n=0}^{\infty} e^{(-n^2)}$$

```
> evalf(%);
```

$$1.386\,318\,602$$

```
> f := x -> sum(x^(n-1)/n, n=1..infinity);
```

$$f := x \to \sum_{n=1}^{\infty}\left(\frac{x^{(n-1)}}{n}\right)$$

```
> f(1); f(-1); f(1/2);
```

$$\infty$$
$$\ln(2)$$
$$2\ \ln(2)$$

## EXERCISES 9.5

Determine the centre, radius, and interval of convergence of each of the power series in Exercises 1–8.

**1.** $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{\sqrt{n+1}}$

**2.** $\displaystyle\sum_{n=0}^{\infty}3n\,(x+1)^n$

**3.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x+2}{2}\right)^n$

**4.** $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4 2^{2n}}x^n$

**5.** $\displaystyle\sum_{n=0}^{\infty}n^3(2x-3)^n$

**6.** $\displaystyle\sum_{n=1}^{\infty}\frac{e^n}{n^3}(4-x)^n$

**7.** $\displaystyle\sum_{n=0}^{\infty}\frac{(1+5^n)}{n!}x^n$

**8.** $\displaystyle\sum_{n=1}^{\infty}\frac{(4x-1)^n}{n^n}$

**9.** Use multiplication of series to find a power series representation of $1/(1-x)^3$ valid in the interval $(-1,1)$.

**10.** Determine the Cauchy product of the series $1+x+x^2+x^3+\cdots$ and $1-x+x^2-x^3+\cdots$. On what interval and to what function does the product series converge?

**11.** Determine the power series expansion of $1/(1-x)^2$ by formally dividing $1-2x+x^2$ into 1.

Starting with the power series representation

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots, \qquad (-1<x<1),$$

determine power series representations for the functions indicated in Exercises 12–20. On what interval is each representation valid?

**12.** $\dfrac{1}{2-x}$ in powers of $x$

**13.** $\dfrac{1}{(2-x)^2}$ in powers of $x$

**14.** $\dfrac{1}{1+2x}$ in powers of $x$

**15.** $\ln(2-x)$ in powers of $x$

**16.** $\dfrac{1}{x}$ in powers of $x-1$

**17.** $\dfrac{1}{x^2}$ in powers of $x+2$

**18.** $\dfrac{1-x}{1+x}$ in powers of $x$

**19.** $\dfrac{x^3}{1-2x^2}$ in powers of $x$

**20.** $\ln x$ in powers of $x-4$

Determine the interval of convergence and the sum of each of the series in Exercises 21–26.

**21.** $\displaystyle 1-4x+16x^2-64x^3+\cdots=\sum_{n=0}^{\infty}(-1)^n(4x)^n$

**22.** $3 + 4x + 5x^2 + 6x^3 + \cdots = \sum_{n=0}^{\infty} (n+3)x^n$

**23.** $\frac{1}{3} + \frac{x}{4} + \frac{x^2}{5} + \frac{x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n+3}$

**24.** $1 \times 3 - 2 \times 4x + 3 \times 5x^2 - 4 \times 6x^3 + \cdots$
$= \sum_{n=0}^{\infty} (-1)^n (n+1)(n+3)\, x^n$

**25.** $2 + 4x^2 + 6x^4 + 8x^6 + 10x^8 + \cdots = \sum_{n=0}^{\infty} 2(n+1)\, x^{2n}$

**26.** $1 - \frac{x^2}{2} + \frac{x^4}{3} - \frac{x^6}{4} + \frac{x^8}{5} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n+1}$

Use the technique (or the result) of Example 6 to find the sums of the numerical series in Exercises 27–32.

**27.** $\sum_{n=1}^{\infty} \frac{n}{3^n}$

**28.** $\sum_{n=0}^{\infty} \frac{n+1}{2^n}$

**29.** $\sum_{n=0}^{\infty} \frac{(n+1)^2}{\pi^n}$

**30.** $\sum_{n=1}^{\infty} \frac{(-1)^n n(n+1)}{2^n}$

**31.** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n}$

**32.** $\sum_{n=3}^{\infty} \frac{1}{n2^n}$

## 9.6 Taylor and Maclaurin Series

If a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ has a positive radius of convergence $R$, then the sum of the series defines a function $f(x)$ on the interval $(c-R, c+R)$. We say that the power series is a **representation** of $f(x)$ on that interval. What relationship exists between the function $f(x)$ and the coefficients $a_0, a_1, a_2, \ldots$ of the power series? The following theorem answers this question.

### THEOREM 21

Suppose the series

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

converges to $f(x)$ for $c - R < x < c + R$, where $R > 0$. Then

$$a_k = \frac{f^{(k)}(c)}{k!} \qquad \text{for } k = 0, 1, 2, 3, \ldots.$$

***PROOF*** This proof requires that we differentiate the series for $f(x)$ term by term several times, a process justified by Theorem 19 (suitably reformulated for powers of $x - c$):

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n(x-c)^{n-2} = 2a_2 + 6a_3(x-c) + 12a_4(x-c)^2 + \cdots$$

$$\vdots$$

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)a_n(x-c)^{n-k} \\ &= k!a_k + \frac{(k+1)!}{1!}a_{k+1}(x-c) + \frac{(k+2)!}{2!}a_{k+2}(x-c)^2 + \cdots. \end{aligned}$$

Each series converges for $c - R < x < c + R$. Setting $x = c$, we obtain $f^{(k)}(c) = k!a_k$, which proves the theorem. ∎

Theorem 21 shows that a function $f(x)$ that has a power series representation with centre at $c$ and positive radius of convergence must have derivatives of all orders in an interval around $x = c$, and it can have only one representation as a power series in powers of $x - c$, namely

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots.$$

Such a series is called a Taylor series or, if $c = 0$, a Maclaurin series.

**DEFINITION 8**

**Taylor and Maclaurin series**

If $f(x)$ has derivatives of all orders at $x = c$ (i.e., if $f^{(k)}(c)$ exists for $k = 0, 1, 2, 3, \ldots$), then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$
$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \cdots$$

is called the **Taylor series of $f$ about $c$** (or the **Taylor series of $f$ in powers of $x - c$**). If $c = 0$, the term **Maclaurin series** is usually used in place of Taylor series.

Note that the partial sums of such Taylor (or Maclaurin) series are just the Taylor (or Maclaurin) polynomials studied in Section 4.10.

The Taylor series is a power series as defined in the previous section. Theorem 17 implies that $c$ must be the centre of any interval on which such a series converges, but the definition of Taylor series makes no requirement that the series should converge anywhere except at the point $x = c$, where the series is just $f(c) + 0 + 0 + \cdots$. The series exists provided all the derivatives of $f$ exist at $x = c$; in practice this means that each derivative must exist in an open interval containing $x = c$. (Why?) However, the series may converge nowhere except at $x = c$, and if it does converge elsewhere, it may converge to something other than $f(x)$. (See Exercise 40 at the end of this section for an example where this happens.) If the Taylor series does converge to $f(x)$ in an open interval containing $c$, then we will say that $f$ is analytic at $c$.

**DEFINITION 9**

**Analytic functions**

A function $f$ is **analytic at $c$** if $f$ has a Taylor series at $c$ and that series converges to $f(x)$ in an open interval containing $c$. If $f$ is analytic at each point of an open interval, then we say it is analytic on that interval.

Most, but not all, of the elementary functions encountered in calculus are analytic wherever they have derivatives of all orders. On the other hand, whenever a power series in powers of $x - c$ converges for all $x$ in an open interval containing $c$, then its sum $f(x)$ is analytic at $c$, and the given series is the Taylor series of $f$ about $c$.

## Maclaurin Series for Some Elementary Functions

Calculating Taylor and Maclaurin series for a function $f$ directly from Definition 8 is practical only when we can find a formula for the $n$th derivative of $f$. Examples of such functions include $(ax+b)^r$, $e^{ax+b}$, $\ln(ax+b)$, $\sin(ax+b)$, $\cos(ax+b)$, and sums of such functions.

**EXAMPLE 1** Find the Taylor series for $e^x$ about $x = c$. Where does the series converge to $e^x$? Where is $e^x$ analytic? What is the Maclaurin series for $e^x$?

**Solution** Since all the derivatives of $f(x) = e^x$ are $e^x$, we have $f^{(n)}(c) = e^c$ for every integer $n \ge 0$. Thus, the Taylor series for $e^x$ about $x = c$ is

$$\sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n = e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + \cdots.$$

The radius of convergence $R$ of this series is given by

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{e^c/(n+1)!}{e^c/n!} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Thus, the radius of convergence is $R = \infty$ and the series converges for all $x$. Suppose the sum is $g(x)$:

$$g(x) = e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + \cdots.$$

By Theorem 19, we have

$$\begin{aligned} g'(x) &= 0 + e^c + \frac{e^c}{2!}2(x-c) + \frac{e^c}{3!}3(x-c)^2 + \cdots \\ &= e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \cdots = g(x). \end{aligned}$$

Also, $g(c) = e^c + 0 + 0 + \cdots = e^c$. Since $g(x)$ satisfies the differential equation $g'(x) = g(x)$ of exponential growth, we have $g(x) = Ce^x$. Substituting $x = c$ gives $e^c = g(c) = Ce^c$, so $C = 1$. Thus, the Taylor series for $e^x$ in powers of $x - c$ converges to $e^x$ for every real number $x$:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n \\ &= e^c + e^c(x-c) + \frac{e^c}{2!}(x-c)^2 + \frac{e^c}{3!}(x-c)^3 + \cdots \qquad \text{(for all } x\text{)}. \end{aligned}$$

In particular, $e^x$ is analytic on the whole real line $\mathbb{R}$. Setting $c = 0$ we obtain the Maclaurin series for $e^x$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \text{(for all } x\text{)}.$$

**EXAMPLE 2** Find the Maclaurin series for (a) $\sin x$ and (b) $\cos x$. Where does each series converge?

**Solution** Let $f(x) = \sin x$. Then we have $f(0) = 0$ and

$$\begin{aligned} f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1 \\ &\vdots & &\vdots \end{aligned}$$

Thus, the Maclaurin series for $\sin x$ is

$$\begin{aligned} g(x) &= 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \end{aligned}$$

We have denoted the sum by $g(x)$ since we don't yet know whether the series converges to $\sin x$. The series does converge for all $x$ by the ratio test:

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}}{(2(n+1)+1)!} x^{2(n+1)+1}}{\dfrac{(-1)^n}{(2n+1)!} x^{2n+1}} \right| &= \lim_{n\to\infty} \frac{(2n+1)!}{(2n+3)!} |x|^2 \\ &= \lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0. \end{aligned}$$

Now we can differentiate the function $g(x)$ twice to get

$$\begin{aligned} g'(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ g''(x) &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots = -g(x). \end{aligned}$$

Thus, $g(x)$ satisfies the differential equation $g''(x) + g(x) = 0$ of simple harmonic motion. The general solution of this equation, as observed in Section 3.7, is

$$g(x) = A \cos x + B \sin x.$$

Observe, from the series, that $g(0) = 0$ and $g'(0) = 1$. These values determine that $A = 0$ and $B = 1$. Thus, $g(x) = \sin x$ and $g'(x) = \cos x$ for all $x$.

We have therefore demonstrated that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{(for all } x\text{)},$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{(for all } x\text{)}.$$

Theorem 21 shows that we can use any available means to find a power series converging to a given function on an interval, and the series obtained will turn out to be the Taylor series. In Section 9.5 several series were constructed by manipulating a geometric series. These include:

**Some Maclaurin series**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad (-1 < x < 1)$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots \quad (-1 < x < 1)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (-1 < x \le 1)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad (-1 \le x \le 1)$$

These series, together with the intervals on which they converge, are frequently used hereafter and should be memorized.

## Other Maclaurin and Taylor Series

Series can be combined in various ways to generate new series. For example, we can find the Maclaurin series for $e^{-x}$ by replacing $x$ with $-x$ in the series for $e^x$:

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \quad \text{(for all } x\text{)}.$$

The series for $e^x$ and $e^{-x}$ can then be subtracted or added and the results divided by 2 to obtain Maclaurin series for the hyperbolic functions $\sinh x$ and $\cosh x$:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \quad \text{(for all } x\text{)}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \quad \text{(for all } x\text{)}.$$

**Remark** Observe the similarity between the series for $\sin x$ and $\sinh x$ and between those for $\cos x$ and $\cosh x$. If we were to allow complex numbers (numbers of the form $z = x + iy$, where $i^2 = -1$ and $x$ and $y$ are real; see Appendix I) as arguments for our functions, and if we were to demonstrate that our operations on series could be extended to series of complex numbers, we would see that $\cos x = \cosh(ix)$ and $\sin x = -i \sinh(ix)$. In fact, $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$, so

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Such formulas are encountered in the study of functions of a complex variable (see Appendix II); from the complex point of view the trigonometric and exponential functions are just different manifestations of the same basic function, a complex exponential $e^z = e^{x+iy}$. We content ourselves here with having mentioned the interesting relationships above and invite the reader to verify them formally by calculating with series. (Such formal calculations do not, of course, constitute a proof, since we have not established the various rules covering series of complex numbers.)

**EXAMPLE 3** Obtain Maclaurin series for the following functions:

(a) $e^{-x^2/3}$, (b) $\dfrac{\sin(x^2)}{x}$, (c) $\sin^2 x$.

**Solution**

(a) We substitute $-x^2/3$ for $x$ in the Maclaurin series for $e^x$:

$$\begin{aligned} e^{-x^2/3} &= 1 - \frac{x^2}{3} + \frac{1}{2!}\left(\frac{x^2}{3}\right)^2 - \frac{1}{3!}\left(\frac{x^2}{3}\right)^3 + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^n n!} x^{2n} \qquad \text{(for all real } x\text{)}. \end{aligned}$$

(b) For all $x \neq 0$ we have

$$\begin{aligned} \frac{\sin(x^2)}{x} &= \frac{1}{x}\left(x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \cdots\right) \\ &= x - \frac{x^5}{3!} + \frac{x^9}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n+1)!}. \end{aligned}$$

Note that $f(x) = (\sin(x^2))/x$ is not defined at $x = 0$ but does have a limit (namely 0) as $x$ approaches 0. If we define $f(0) = 0$ (the continuous extension of $f(x)$ to $x = 0$), then the series converges to $f(x)$ for all $x$.

(c) We use a trigonometric identity to express $\sin^2 x$ in terms of $\cos 2x$ and then use the Maclaurin series for $\cos x$ with $x$ replaced by $2x$:

$$\begin{aligned}\sin^2 x &= \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right)\\ &= \frac{1}{2}\left(\frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} + \frac{(2x)^6}{6!} - \cdots\right)\\ &= \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n+1}}{(2n+2)!}x^{2n+2} \qquad \text{(for all real } x).\end{aligned}$$

Taylor series about points other than 0 can often be obtained from known Maclaurin series by a change of variable.

**EXAMPLE 4** Find the Taylor series for $\ln x$ in powers of $x - 2$. Where does the series converge to $\ln x$?

***Solution*** Note that if $t = (x-2)/2$, then

$$\ln x = \ln\big(2 + (x-2)\big) = \ln\left[2\left(1 + \frac{x-2}{2}\right)\right] = \ln 2 + \ln(1+t).$$

We use the known Maclaurin series for $\ln(1+t)$:

$$\begin{aligned}\ln x &= \ln 2 + \ln(1+t)\\ &= \ln 2 + t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots\\ &= \ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{2\times 2^2} + \frac{(x-2)^3}{3\times 2^3} - \frac{(x-2)^4}{4\times 2^4} + \cdots\\ &= \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\,2^n}(x-2)^n.\end{aligned}$$

Since the series for $\ln(1+t)$ is valid for $-1 < t \le 1$, this series for $\ln x$ is valid for $-1 < (x-2)/2 \le 1$, that is, for $0 < x \le 4$.

**EXAMPLE 5** Find the Taylor series for $\cos x$ about $\pi/3$. Where is the series valid?

***Solution*** We use the addition formula for cosine:

$$\begin{aligned}\cos x &= \cos\left(x - \frac{\pi}{3} + \frac{\pi}{3}\right) = \cos\left(x - \frac{\pi}{3}\right)\cos\frac{\pi}{3} - \sin\left(x - \frac{\pi}{3}\right)\sin\frac{\pi}{3}\\ &= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{3}\right)^4 - \cdots\right]\\ &\quad - \frac{\sqrt{3}}{2}\left[\left(x - \frac{\pi}{3}\right) - \frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3 + \cdots\right]\\ &= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2}\frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{2}\frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3\\ &\quad + \frac{1}{2}\frac{1}{4!}\left(x - \frac{\pi}{3}\right)^4 - \cdots.\end{aligned}$$

This series representation is valid for all $x$. A similar calculation would enable us to expand $\cos x$ or $\sin x$ in powers of $x - c$ for any real $c$; both functions are analytic at every point of the real line.

Sometimes it is quite difficult, if not impossible, to find a formula for the general term of a Maclaurin or Taylor series. In such cases it is usually possible to obtain the first few terms before the calculations get too cumbersome. Had we attempted to solve Example 3(c) by multiplying the series for $\sin x$ by itself we might have found ourselves in this bind. Other examples occur when it is necessary to substitute one series into another or to divide one by another.

**EXAMPLE 6** Obtain the first three nonzero terms of the Maclaurin series for (a) $\tan x$ and (b) $\ln \cos x$.

## Solution

(a) $\tan x = (\sin x)/(\cos x)$. We can obtain the first three terms of the Maclaurin series for $\tan x$ by long division of the series for $\cos x$ into that for $\sin x$:

$$
\begin{array}{rc|ccccccc}
 & & x & + & \frac{x^3}{3} & + & \frac{2}{15}x^5 & + & \cdots \\ \cline{3-9}
1 - \frac{x^2}{2} + \frac{x^4}{24} & & x & - & \frac{x^3}{6} & + & \frac{x^5}{120} & - & \cdots \\
 & & x & - & \frac{x^3}{2} & + & \frac{x^5}{24} & - & \cdots \\ \cline{3-9}
 & & & & \frac{x^3}{3} & - & \frac{x^5}{30} & + & \cdots \\
 & & & & \frac{x^3}{3} & - & \frac{x^5}{6} & + & \cdots \\ \cline{5-9}
 & & & & & & \frac{2x^5}{15} & - & \cdots \\
 & & & & & & \frac{2x^5}{15} & - & \cdots \\ \cline{7-9}
\end{array}
$$

Thus, $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$.

We cannot easily find all the terms of the series; only with considerable computational effort can we find many more terms than we have already found. This Maclaurin series for $\tan x$ converges for $|x| < \pi/2$, but we cannot demonstrate this fact by the techniques we have at our disposal now. (It is true because the complex number $z = x + iy$ closest to 0 where the "denominator" of $\tan z$, that is, $\cos z$, is zero, is, in fact, the real value $z = \pi/2$.)

(b)
$$
\begin{aligned}
\ln \cos x &= \ln\left(1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\right) \\
&= \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - \frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)^2 \\
&\quad + \frac{1}{3}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)^3 - \cdots \\
&= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots - \frac{1}{2}\left(\frac{x^4}{4} - \frac{x^6}{24} + \cdots\right) \\
&\quad + \frac{1}{3}\left(-\frac{x^6}{8} + \cdots\right) - \cdots
\end{aligned}
$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \cdots.$$

Note that at each stage of the calculation we kept only enough terms to ensure that we could get all the terms with powers up to $x^6$. Being an even function, $\ln\cos x$ has only even powers in its Maclaurin series. Again, we cannot find the general term of this series. We could try to calculate terms by using the formula $a_k = f^{(k)}(0)/k!$, but even this becomes difficult after the first few values of $k$.

Observe that the series for $\tan x$ could also have been derived from that of $\ln\cos x$ because we have $\tan x = -\dfrac{d}{dx}\ln\cos x$.

## Taylor's Formula Revisited

In the examples above we have used a variety of techniques to obtain Taylor series for functions and verify that functions are analytic. As shown in Section 4.10, Taylor's Theorem provides a means for estimating the size of the error $E_n(x) = f(x) - P_n(x)$ involved when the Taylor polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

is used to approximate the value of $f(x)$ for $x \neq c$. Since the Taylor polynomials are partial sums of the Taylor series for $f$ at $c$ (if the latter exists), another technique for verifying the convergence of a Taylor series is to use the formula for $E_n(x)$ provided by Taylor's Theorem to show, at least for an interval of values of $x$ containing $c$, that $\lim_{n\to\infty} E_n(x) = 0$. This implies that $\lim_{n\to\infty} P_n(x) = f(x)$ so that $f$ is indeed the sum of its Taylor series about $c$ on that inverval, and $f$ is analytic at $c$. Here is a somewhat more general version of Taylor's Theroem.

**THEOREM 22**

**Taylor's Theorem**

If the $(n+1)$st derivative of $f$ exists on an interval containing $c$ and $x$, and if $P_n(x)$ is the Taylor polynomial of degree $n$ for $f$ about the point $x = c$, then

$$f(x) = P_n(x) + E_n(x) \qquad \textbf{Taylor's Formula}$$

holds, where the error term $E_n(x)$ is given by *either* of the following formulas:

**Lagrange remainder** $\quad E_n(x) = \dfrac{f^{(n+1)}(s)}{(n+1)!}(x-c)^{n+1},$

for some $s$ between $c$ and $x$

**Integral remainder** $\quad E_n(x) = \dfrac{1}{n!}\displaystyle\int_c^x (x-t)^n f^{(n+1)}(t)\,dt.$

Taylor's Theorem with Lagrange remainder was proved in Section 4.10 (Theorem 12) by using the Mean-Value Theorem and induction on $n$. The integral remainder version is also proved by induction on $n$. See Exercise 42 for hints on how to carry out the proof. We will not make any use of the integral form of the remainder here.

Our final example in this section re-establishes the Maclaurin series for $e^x$ by finding the limit of the Lagrange remainder as suggested above.

**EXAMPLE 7** Use Taylor's Theorem to find the Maclaurin series for $f(x) = e^x$. Where does the series converge to $f(x)$?

**Solution** Since $e^x$ is positive and increasing, $e^s \le e^{|x|}$ for any $s \le |x|$. Since $f^{(k)}(x) = e^x$ for any $k$ we have, taking $c = 0$ in the Lagrange remainder in Taylor's Formula,

$$|E_n(x)| = \left|\frac{f^{(n+1)}(s)}{(n+1)!}\,x^{n+1}\right| \quad \text{for some } s \text{ between 0 and } x$$
$$= \frac{e^s}{(n+1)!}\,|x|^{n+1} \le e^{|x|}\,\frac{|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty$$

for any real $x$, as shown in Theorem 3(b) of Section 9.1. Thus, $\lim_{n\to\infty} E_n(x) = 0$. Since the $n$th-order Maclaurin polynomial for $e^x$ is $\sum_{k=0}^{n}(x^k/k!)$,

$$e^x = \lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{x^k}{k!} + E_n(x)\right) = \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

and the series converges to $e^x$ for all real numbers $x$.

## EXERCISES 9.6

Find Maclaurin series representations for the functions in Exercises 1–14. For what values of $x$ is each representation valid?

**1.** $e^{3x+1}$

**2.** $\cos(2x^3)$

**3.** $\sin(x - \pi/4)$

**4.** $\cos(2x - \pi)$

**5.** $x^2 \sin(x/3)$

**6.** $\cos^2(x/2)$

**7.** $\sin x \cos x$

**8.** $\tan^{-1}(5x^2)$

**9.** $\dfrac{1+x^3}{1+x^2}$

**10.** $\ln(2 + x^2)$

**11.** $\ln\dfrac{1+x}{1-x}$

**12.** $(e^{2x^2} - 1)/x^2$

**13.** $\cosh x - \cos x$

**14.** $\sinh x - \sin x$

Find the required Taylor series representations of the functions in Exercises 15–26. Where is each series representation valid?

**15.** $f(x) = e^{-2x}$ about $-1$

**16.** $f(x) = \sin x$ about $\pi/2$

**17.** $f(x) = \cos x$ in powers of $x - \pi$

**18.** $f(x) = \ln x$ in powers of $x - 3$

**19.** $f(x) = \ln(2 + x)$ in powers of $x - 2$

**20.** $f(x) = e^{2x+3}$ in powers of $x + 1$

**21.** $f(x) = \sin x - \cos x$ about $\dfrac{\pi}{4}$

**22.** $f(x) = \cos^2 x$ about $\dfrac{\pi}{8}$

**23.** $f(x) = 1/x^2$ in powers of $x + 2$

**24.** $f(x) = \dfrac{x}{1+x}$ in powers of $x - 1$

**25.** $f(x) = x \ln x$ in powers of $x - 1$

**26.** $f(x) = xe^x$ in powers of $x + 2$

Find the first three nonzero terms in the Maclaurin series for the functions in Exercises 27–30.

**27.** $\sec x$

**28.** $\sec x \tan x$

**29.** $\tan^{-1}(e^x - 1)$

**30.** $e^{\tan^{-1} x} - 1$

! **31.** Use the fact that $(\sqrt{1+x})^2 = 1 + x$ to find the first three nonzero terms of the Maclaurin series for $\sqrt{1+x}$.

**32.** Does $\csc x$ have a Maclaurin series? Why? Find the first three nonzero terms of the Taylor series for $\csc x$ about the point $x = \pi/2$.

Find the sums of the series in Exercises 33–36.

**33.** $1 + x^2 + \dfrac{x^4}{2!} + \dfrac{x^6}{3!} + \dfrac{x^8}{4!} + \cdots$

! **34.** $x^3 - \dfrac{x^9}{3! \times 4} + \dfrac{x^{15}}{5! \times 16} - \dfrac{x^{21}}{7! \times 64} + \dfrac{x^{27}}{9! \times 256} - \cdots$

**35.** $1 + \dfrac{x^2}{3!} + \dfrac{x^4}{5!} + \dfrac{x^6}{7!} + \dfrac{x^8}{9!} + \cdots$

! **36.** $1 + \dfrac{1}{2 \times 2!} + \dfrac{1}{4 \times 3!} + \dfrac{1}{8 \times 4!} + \cdots$

**37.** Let $P(x) = 1 + x + x^2$. Find (a) the Maclaurin series for $P(x)$ and (b) the Taylor series for $P(x)$ about 1.

! **38.** Verify by direct calculation that $f(x) = 1/x$ is analytic at $a$ for every $a \ne 0$.

! **39.** Verify by direct calculation that $\ln x$ is analytic at $a$ for every $a > 0$.

! **40.** Review Exercise 41 of Section 4.5. It shows that the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has derivatives of all orders at every point of the real line, and $f^{(k)}(0) = 0$ for every positive integer $k$. What is the Maclaurin series for $f(x)$? What is the interval of convergence of this Maclaurin series? On what interval does the series converge to $f(x)$? Is $f$ analytic at 0?

! **41.** By direct multiplication of the Maclaurin series for $e^x$ and $e^y$ show that $e^x e^y = e^{x+y}$.

**42.** **(Taylor's Formula with integral remainder)** Verify that if $f^{(n+1)}$ exists on an interval containing $c$ and $x$, and if $P_n(x)$ is the $n$th-order Taylor polynomial for $f$ about $c$, then $f(x) = P_n(x) + E_n(x)$, where

$$E_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt.$$

Proceed as follows:

(a) First observe that the case $n = 0$ is just the Fundamental Theorem of Calculus:

$$f(x) = f(c) + \int_c^x f'(t)\,dt.$$

Now integrate by parts in this formula, taking $U = f'(t)$ and $dV = dt$. Contrary to our usual policy of not including a constant of integration in $V$, here write $V = -(x-t)$ rather than just $V = t$. Observe that the result of the integration by parts is the case $n = 1$ of the formula.

(b) Use induction argument (and integration by parts again) to show that if the formula is valid for $n = k$, then it is also valid for $n = k + 1$.

**43.** Use Taylor's formula with integral remainder to re-prove that the Maclaurin series for $\ln(1+x)$ converges to $\ln(1+x)$ for $-1 < x \le 1$.

**44.** **(Stirling's Formula)** The limit

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi}\,n^{n+1/2}e^{-n}} = 1$$

says that the *relative error* in the approximation

$$n! \approx \sqrt{2\pi}\,n^{n+1/2}e^{-n}$$

approaches zero as $n$ increases. That is, $n!$ grows at a rate comparable to $\sqrt{2\pi}\,n^{n+1/2}e^{-n}$. This result, known as Stirling's Formula, is often very useful in applied mathematics and statistics. Prove it by carrying out the following steps:

(a) Use the identity $\ln(n!) = \sum_{j=1}^{n} \ln j$ and the increasing nature of ln to show that if $n \ge 1$,

$$\int_0^n \ln x\,dx < \ln(n!) < \int_1^{n+1} \ln x\,dx$$

and hence that

$$n\ln n - n < \ln(n!) < (n+1)\ln(n+1) - n.$$

(b) If $c_n = \ln(n!) - \left(n + \frac{1}{2}\right)\ln n + n$, show that

$$\begin{aligned} c_n - c_{n+1} &= \left(n + \frac{1}{2}\right)\ln\frac{n+1}{n} - 1 \\ &= \left(n + \frac{1}{2}\right)\ln\frac{1 + 1/(2n+1)}{1 - 1/(2n+1)} - 1. \end{aligned}$$

(c) Use the Maclaurin series for $\ln\dfrac{1+t}{1-t}$ (see Exercise 11) to show that

$$\begin{aligned} 0 < c_n - c_{n+1} &< \frac{1}{3}\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \cdots\right) \\ &= \frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right), \end{aligned}$$

and therefore that $\{c_n\}$ is decreasing and $\left\{c_n - \frac{1}{12n}\right\}$ is increasing. Hence conclude that $\lim_{n\to\infty} c_n = c$ exists, and that

$$\lim_{n\to\infty} \frac{n!}{n^{n+1/2}e^{-n}} = \lim_{n\to\infty} e^{c_n} = e^c.$$

(d) Now use the Wallis Product from Exercise 38 of Section 6.1 to show that

$$\lim_{n\to\infty} \frac{(2^n n!)^2}{(2n)!\sqrt{2n}} = \sqrt{\frac{\pi}{2}},$$

and hence deduce that $e^c = \sqrt{2\pi}$, which completes the proof.

**45.** **(A Modified Stirling Formula)** A simpler approximation to $n!$ for large $n$ is given by

$$n! \approx n^n\,e^{-n} \quad \text{or, equivalently,} \quad \ln(n!) \approx n\ln n - n.$$

While not as accurate as Stirling's Formula, this modified version still has relative error approaching zero as $n \to \infty$ and can be useful in many applications.

(a) Prove this assertion about the relative error by using the conclusion of part (a) of the previous exercise.

(b) Compare the relative errors in the approximations for $\ln(10!)$ and $\ln(20!)$ using Stirling's Formula and the Modified Stirling Formula.

# 9.7 Applications of Taylor and Maclaurin Series

## Approximating the Values of Functions

We saw in Section 4.10 how Taylor and Maclaurin polynomials (the partial sums of Taylor and Maclaurin series) can be used as polynomial approximations to more complicated functions. In Example 5 of that section we used the Lagrange remainder in

Taylor's Formula to determine how many terms of the Maclaurin series for $e^x$ are needed to calculate $e^1 = e$ correct to 3 decimal places. For comparison, we obtained the same result in Example 7 in Section 9.3 by using a geometric series to bound the tail of the series for $e$.

The following example shows how the error bound associated with the alternating series test (see Theorem 15 in Section 9.4) can also be used for such approximations. When the terms $a_n$ of a series (i) alternate in sign, (ii) decrease steadily in size, and (iii) approach zero as $n \to \infty$, then the error involved in using a partial sum of the series as an approximation to the sum of the series has the same sign as, and is no greater in absolute value than, the first omitted term.

**EXAMPLE 1** Find $\cos 43°$ with error less than $1/10{,}000$.

***Solution*** We give two alternative solutions:

**METHOD I.** We can use the Maclaurin series for cosine:

$$\cos 43° = \cos\frac{43\pi}{180} = 1 - \frac{1}{2!}\left(\frac{43\pi}{180}\right)^2 + \frac{1}{4!}\left(\frac{43\pi}{180}\right)^4 - \cdots.$$

Now $43\pi/180 \approx 0.750\,49\cdots < 1$, so the series above must satisfy the conditions (i)–(iii) mentioned above. If we truncate the series after the $n$th term

$$(-1)^{n-1}\frac{1}{(2n-2)!}\left(\frac{43\pi}{180}\right)^{2n-2},$$

then the error $E$ will be bounded by the size of the first omitted term:

$$|E| \le \frac{1}{(2n)!}\left(\frac{43\pi}{180}\right)^{2n} < \frac{1}{(2n)!}.$$

The error will not exceed $1/10{,}000$ if $(2n)! > 10{,}000$, so $n = 4$ will do $(8! = 40{,}320)$.

$$\cos 43° \approx 1 - \frac{1}{2!}\left(\frac{43\pi}{180}\right)^2 + \frac{1}{4!}\left(\frac{43\pi}{180}\right)^4 - \frac{1}{6!}\left(\frac{43\pi}{180}\right)^6 \approx 0.731\,35\cdots$$

**METHOD II.** Since $43°$ is close to $45° = \pi/4$ rad, we can do a bit better by using the Taylor series about $\pi/4$ instead of the Maclaurin series:

$$\begin{aligned}\cos 43° &= \cos\left(\frac{\pi}{4} - \frac{\pi}{90}\right)\\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{90} + \sin\frac{\pi}{4}\sin\frac{\pi}{90}\\ &= \frac{1}{\sqrt{2}}\left[\left(1 - \frac{1}{2!}\left(\frac{\pi}{90}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{90}\right)^4 - \cdots\right)\right.\\ &\qquad \left. + \left(\frac{\pi}{90} - \frac{1}{3!}\left(\frac{\pi}{90}\right)^3 + \cdots\right)\right].\end{aligned}$$

Since

$$\frac{1}{4!}\left(\frac{\pi}{90}\right)^4 < \frac{1}{3!}\left(\frac{\pi}{90}\right)^3 < \frac{1}{20{,}000},$$

we need only the first two terms of the first series and the first term of the second series:

$$\cos 43° \approx \frac{1}{\sqrt{2}}\left(1 + \frac{\pi}{90} - \frac{1}{2}\left(\frac{\pi}{90}\right)^2\right) \approx 0.731\,358\cdots.$$

(In fact, $\cos 43° = 0.731\,353\,7\cdots$.)

When finding approximate values of functions, it is best, whenever possible, to use a power series about a point as close as possible to the point where the approximation is desired.

## Functions Defined by Integrals

Many functions that can be expressed as simple combinations of elementary functions cannot be antidifferentiated by elementary techniques; their antiderivatives are not simple combinations of elementary functions. We can, however, often find the Taylor series for the antiderivatives of such functions and hence approximate their definite integrals.

**EXAMPLE 2** Find the Maclaurin series for

$$E(x) = \int_0^x e^{-t^2}\,dt,$$

and use it to evaluate $E(1)$ correct to 3 decimal places.

***Solution*** The Maclaurin series for $E(x)$ is given by

$$\begin{aligned} E(x) &= \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \cdots\right) dt \\ &= \left(t - \frac{t^3}{3} + \frac{t^5}{5\times 2!} - \frac{t^7}{7\times 3!} + \frac{t^9}{9\times 4!} - \cdots\right)\bigg|_0^x \\ &= x - \frac{x^3}{3} + \frac{x^5}{5\times 2!} - \frac{x^7}{7\times 3!} + \frac{x^9}{9\times 4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!}, \end{aligned}$$

and is valid for all $x$ because the series for $e^{-t^2}$ is valid for all $t$. Therefore,

$$\begin{aligned} E(1) &= 1 - \frac{1}{3} + \frac{1}{5\times 2!} - \frac{1}{7\times 3!} + \cdots \\ &\approx 1 - \frac{1}{3} + \frac{1}{5\times 2!} - \frac{1}{7\times 3!} + \cdots + \frac{(-1)^{n-1}}{(2n-1)(n-1)!}. \end{aligned}$$

We stopped with the $n$th term. Again, the alternating series test assures us that the error in this approximation does not exceed the first omitted term, so it will be less than 0.0005, provided $(2n+1)n! > 2{,}000$. Since $13 \times 6! = 9{,}360$, $n = 6$ will do. Thus,

$$E(1) \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \frac{1}{1{,}320} \approx 0.747,$$

rounded to 3 decimal places.

## Indeterminate Forms

Examples 9 and 10 of Section 4.10 showed how Maclaurin polynomials could be used for evaluating the limits of indeterminate forms. Here are two more examples, this time using the series directly and keeping enough terms to allow cancellation of the [0/0] factors.

**EXAMPLE 3** Evaluate (a) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3}$ and (b) $\displaystyle\lim_{x\to 0} \frac{(e^{2x}-1)\ln(1+x^3)}{(1-\cos 3x)^2}$.

**Solution**

(a) $\lim_{x\to 0}\frac{x-\sin x}{x^3}\quad\left[\frac{0}{0}\right]$

$$= \lim_{x\to 0}\frac{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)}{x^3}$$

$$= \lim_{x\to 0}\frac{\frac{x^3}{3!}-\frac{x^5}{5!}+\cdots}{x^3}$$

$$= \lim_{x\to 0}\left(\frac{1}{3!}-\frac{x^2}{5!}+\cdots\right)=\frac{1}{3!}=\frac{1}{6}.$$

(b) $\lim_{x\to 0}\frac{(e^{2x}-1)\ln(1+x^3)}{(1-\cos 3x)^2}\quad\left[\frac{0}{0}\right]$

$$= \lim_{x\to 0}\frac{\left(1+(2x)+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots-1\right)\left(x^3-\frac{x^6}{2}+\cdots\right)}{\left(1-\left(1-\frac{(3x)^2}{2!}+\frac{(3x)^4}{4!}-\cdots\right)\right)^2}$$

$$= \lim_{x\to 0}\frac{2x^4+2x^5+\cdots}{\left(\frac{9}{2}x^2-\frac{3^4}{4!}x^4+\cdots\right)^2}$$

$$= \lim_{x\to 0}\frac{2+2x+\cdots}{\left(\frac{9}{2}-\frac{3^4}{4!}x^2+\cdots\right)^2}=\frac{2}{\left(\frac{9}{2}\right)^2}=\frac{8}{81}.$$

You can check that the second of these examples is much more difficult if attempted using l'Hôpital's Rule.

## EXERCISES 9.7

1. Estimate the error if the Maclaurin polynomial of degree 5 for $\sin x$ is used to approximate $\sin(0.2)$.
2. Estimate the error if the Taylor polynomial of degree 4 for $\ln x$ in powers of $x-2$ is used to approximate $\ln(1.95)$.

Use Maclaurin or Taylor series to calculate the function values indicated in Exercises 3–14, with error less than $5\times 10^{-5}$ in absolute value.

3. $e^{0.2}$
4. $1/e$
5. $e^{1.2}$
6. $\sin(0.1)$
7. $\cos 5°$
8. $\ln(6/5)$
9. $\ln(0.9)$
10. $\sin 80°$
11. $\cos 65°$
12. $\tan^{-1} 0.2$
13. $\cosh(1)$
14. $\ln(3/2)$

Find Maclaurin series for the functions in Exercises 15–19.

15. $I(x)=\int_0^x \frac{\sin t}{t}\,dt$
16. $J(x)=\int_0^x \frac{e^t-1}{t}\,dt$
17. $K(x)=\int_1^{1+x} \frac{\ln t}{t-1}\,dt$
18. $L(x)=\int_0^x \cos(t^2)\,dt$
19. $M(x)=\int_0^x \frac{\tan^{-1} t^2}{t^2}\,dt$
20. Find $L(0.5)$ correct to 3 decimal places, with $L$ defined as in Exercise 18.
21. Find $I(1)$ correct to 3 decimal places, with $I$ defined as in Exercise 15.

Evaluate the limits in Exercises 22–27.

22. $\lim_{x\to 0}\frac{\sin(x^2)}{\sinh x}$
23. $\lim_{x\to 0}\frac{1-\cos(x^2)}{(1-\cos x)^2}$
24. $\lim_{x\to 0}\frac{(e^x-1-x)^2}{x^2-\ln(1+x^2)}$
25. $\lim_{x\to 0}\frac{2\sin 3x-3\sin 2x}{5x-\tan^{-1} 5x}$
26. $\lim_{x\to 0}\frac{\sin(\sin x)-x}{x(\cos(\sin x)-1)}$
27. $\lim_{x\to 0}\frac{\sinh x-\sin x}{\cosh x-\cos x}$

# 9.8 The Binomial Theorem and Binomial Series

**EXAMPLE 1** Use Taylor's Formula to prove the Binomial Theorem: if $n$ is a positive integer, then

$$\begin{aligned}(a+x)^n &= a^n + n\,a^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \cdots + n\,ax^{n-1} + x^n \\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k}x^k,\end{aligned}$$

where $\displaystyle\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

***Solution*** Let $f(x) = (a+x)^n$. Then

$$\begin{aligned}f'(x) &= n(a+x)^{n-1} = \frac{n!}{(n-1)!}(a+x)^{n-1} \\ f''(x) &= \frac{n!}{(n-1)!}(n-1)(a+x)^{n-2} = \frac{n!}{(n-2)!}(a+x)^{n-2} \\ &\vdots \\ f^{(k)}(x) &= \frac{n!}{(n-k)!}(a+x)^{n-k} \qquad (0 \le k \le n).\end{aligned}$$

In particular, $f^{(n)}(x) = \dfrac{n!}{0!}(a+x)^{n-n} = n!$, a constant, and so

$$f^{(k)}(x) = 0 \qquad \text{for all } x, \text{ if } k > n.$$

For $0 \le k \le n$ we have $f^{(k)}(0) = \dfrac{n!}{(n-k)!}a^{n-k}$. Thus, by Taylor's Theorem with Lagrange remainder, for some $s$ between $a$ and $x$,

$$\begin{aligned}(a+x)^n = f(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(n+1)}(s)}{(n+1)!}x^{n+1} \\ &= \sum_{k=0}^{n} \frac{n!}{(n-k)!k!}a^{n-k}x^k + 0 = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}x^k.\end{aligned}$$

This is, in fact, the Maclaurin *series* for $(a+x)^n$, not just the Maclaurin polynomial of degree $n$. Since all higher-degree terms are zero, the series has only finitely many nonzero terms and so converges for all $x$.

***Remark*** If $f(x) = (a+x)^r$, where $a > 0$ and $r$ is any real number, then calculations similar to those above show that the Maclaurin polynomial of degree $n$ for $f$ is

$$P_n(x) = a^r + \sum_{k=1}^{n} \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}a^{r-k}x^k.$$

However, if $r$ is not a positive integer, then there will be no positive integer $n$ for which the remainder $E_n(x) = f(x) - P_n(x)$ vanishes identically, and the corresponding Maclaurin series will not be a polynomial.

## The Binomial Series

To simplify the discussion of the function $(a+x)^r$ when $r$ is not a positive integer, we take $a = 1$ and consider the function $(1+x)^r$. Results for the general case follow via the identity

$$(a+x)^r = a^r\left(1+\frac{x}{a}\right)^r,$$

valid for any $a > 0$.

If $r$ is any real number and $x > -1$, then the $k$th derivative of $(1+x)^r$ is

$$r(r-1)(r-2)\cdots(r-k+1)\,(1+x)^{r-k}, \qquad (k = 1, 2, \ldots).$$

Thus, the Maclaurin series for $(1+x)^r$ is

$$1 + \sum_{k=1}^{\infty} \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}\,x^k,$$

which is called the **binomial series**. The following theorem shows that the binomial series does, in fact, converge to $(1+x)^r$ if $|x| < 1$. We could accomplish this by writing Taylor's Formula for $(1+x)^r$ with $c = 0$ and showing that the remainder $E_n(x) \to 0$ as $n \to \infty$. (We would need to use the integral form of the remainder to prove this for all $|x| < 1$.) However, we will use an easier method, similar to the one used for the exponential and trigonometric functions in Section 9.6.

**THEOREM 23**

**The binomial series**

If $|x| < 1$, then

$$\begin{aligned}(1+x)^r &= 1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \cdots \\ &= 1 + \sum_{n=1}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}\,x^n \qquad (-1 < x < 1).\end{aligned}$$

***PROOF*** If $|x| < 1$, then the series

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}\,x^n$$

converges by the ratio test, since

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| \frac{\dfrac{r(r-1)(r-2)\cdots(r-n+1)(r-n)}{(n+1)!}x^{n+1}}{\dfrac{r(r-1)(r-2)\cdots(r-n+1)}{n!}x^n} \right| \\ &= \lim_{n\to\infty} \left|\frac{r-n}{n+1}\right| |x| = |x| < 1.\end{aligned}$$

Note that $f(0) = 1$. We need to show that $f(x) = (1+x)^r$ for $|x| < 1$.

By Theorem 19, we can differentiate the series for $f(x)$ termwise on $|x| < 1$ to obtain

$$\begin{aligned}f'(x) &= \sum_{n=1}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n+1)}{(n-1)!}\,x^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n)}{n!}\,x^n.\end{aligned}$$

We have replaced $n$ with $n+1$ to get the second version of the sum from the first version. Adding the second version to $x$ times the first version, we get

$$\begin{aligned}(1+x)f'(x) &= \sum_{n=0}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n)}{n!}x^n \\ &\quad + \sum_{n=1}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n+1)}{(n-1)!}x^n \\ &= r + \sum_{n=1}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}x^n\big[(r-n)+n\big] \\ &= r\,f(x).\end{aligned}$$

The differential equation $(1+x)f'(x) = rf(x)$ implies that

$$\frac{d}{dx}\frac{f(x)}{(1+x)^r} = \frac{(1+x)^r f'(x) - r(1+x)^{r-1}f(x)}{(1+x)^{2r}} = 0$$

for all $x$ satisfying $|x|<1$. Thus, $f(x)/(1+x)^r$ is constant on that interval, and since $f(0)=1$, the constant must be 1. Thus, $f(x) = (1+x)^r$.

**Remark** For some values of $r$ the binomial series may converge at the endpoints $x=1$ or $x=-1$. As observed above, if $r$ is a positive integer, the series has only finitely many nonzero terms, and so converges for all $x$.

**EXAMPLE 2** Find the Maclaurin series for $\dfrac{1}{\sqrt{1+x}}$.

**Solution** Here $r = -(1/2)$:

$$\begin{aligned}\frac{1}{\sqrt{1+x}} &= (1+x)^{-1/2} \\ &= 1 - \frac{1}{2}x + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^2 + \frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^3 + \cdots \\ &= 1 - \frac{1}{2}x + \frac{1\times 3}{2^2 2!}x^2 - \frac{1\times 3\times 5}{2^3 3!}x^3 + \cdots \\ &= 1 + \sum_{n=1}^{\infty}(-1)^n \frac{1\times 3\times 5\times\cdots\times(2n-1)}{2^n n!}x^n.\end{aligned}$$

This series converges for $-1 < x \le 1$. (Use the alternating series test to get the endpoint $x=1$.)

**EXAMPLE 3** Find the Maclaurin series for $\sin^{-1}x$.

**Solution** Replace $x$ with $-t^2$ in the series obtained in the previous example to get

$$\frac{1}{\sqrt{1-t^2}} = 1 + \sum_{n=1}^{\infty}\frac{1\times 3\times 5\times\cdots\times(2n-1)}{2^n n!}t^{2n} \qquad (-1<t<1).$$

Now integrate $t$ from 0 to $x$:

$$\begin{aligned}\sin^{-1}x &= \int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x \left(1 + \sum_{n=1}^{\infty}\frac{1\times 3\times 5\times\cdots\times(2n-1)}{2^n n!}t^{2n}\right)dt \\ &= x + \sum_{n=1}^{\infty}\frac{1\times 3\times 5\times\cdots\times(2n-1)}{2^n n!(2n+1)}x^{2n+1} \\ &= x + \frac{x^3}{6} + \frac{3}{40}x^5 + \cdots \qquad (-1<x<1).\end{aligned}$$

## The Multinomial Theorem

The Binomial Theorem can be extended to provide for expansions of positive integer powers of sums of more than two quantities. Before stating this **Multinomial Theorem**, we require some new notation.

For an integer $n \geq 2$, let $m = (m_1, m_2, \dots, m_n)$ be an $n$-tuple of nonnegative integers. We call $m$ a **multiindex of order** $n$, and the number $|m| = m_1 + m_2 + \cdots + m_n$ the **degree** of the multiindex. In terms of multiindices, the Binomial Theorem can be restated in the form

$$(x_1 + x_2)^k = \sum_{|m|=k} \left( \frac{k!}{m_1!\, m_2!} \right) x_1^{m_1} x_2^{m_2} = \sum_{|m|=k} \frac{k!}{m_1!\, m_2!} x_1^{m_1} x_2^{m_2},$$

the sum being taken over all multiindices of order 2 having degree $k$. Here the binomial coefficients have been rewritten in the form

$$\binom{k}{m_1\, m_2} = \frac{k!}{m_1!\, m_2!},$$

which is correct since $m_2 = k - m_1$.

**THEOREM 24**

**The Multinomial Theorem**

If $m$ and $k$ are integers satisfying $n \geq 2$ and $k \geq 1$, then

$$(x_1 + x_2 + \cdots + x_n)^k = \sum_{|m|=k} \frac{k!}{m_1!\, m_2! \cdots m_n!} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n},$$

the sum being taken over all multiindices $m$ of order $n$ and degree $k$.

Evidently, the Binomial Theorem is the special case $n = 2$. The proof of the Multinomial Theorem can be carried out by induction on $n$. See Exercise 12 below.

The coefficients of the various products of powers of the variables $x_i$ in the Multinomial Theorem are called **multinomial coefficients**. By analogy with the notation used for binomial coefficients, if $m_1 + \cdots + m_n = k$, the multinomial coefficients can be denoted

$$\binom{k}{m_1, m_2, \dots, m_n} = \binom{m_1 + m_2 + \cdots + m_n}{m_1, m_2, \dots, m_n} = \frac{k!}{m_1! m_2! \cdots m_n!}. \qquad (*)$$

They are useful for counting distinct arrangements of objects where not all of the objects appear to be different.

**EXAMPLE 4** The number of ways that $k$ distinct objects can be arranged in a sequence of positions 1, 2, ..., $k$ is $k!$ because there are $k$ choices for the object to go in position 1, then $k - 1$ choices for the object to go into position 2, and so on, until there is only 1 choice for the object to go into position $k$. But what if the objects are not all distinct, but instead there are several objects of each of $n$ different types, say type 1, type 2, ..., type $n$ such that objects of the same type are indistinguishable from one another. If you just look at positions in the sequence containing objects of type $j$, and rearrange only those objects, you can't tell the difference. If there are $m_j$ objects of type $j$, $(1 \leq j \leq n)$, then the number of distinct rearrangements of the $k$ objects is given by the multinomial coefficient $(*)$. For example, the number of visually different arrangements of 9 balls, 2 of which are red, 3 green, and 4 blue is

$$\binom{9}{2, 3, 4} = \frac{9!}{2!3!4!} = \frac{362{,}880}{288} = 1{,}260.$$

***Remark*** A direct proof of the Multinomial Theorem can be based on the above example. When calculating the $k$th power of $(x_1 + x_2 + \cdots + x_n)$ by long multiplication, we obtain a sum of monomials of degree $k$ having the form $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$, where $m_1 + m_2 + \cdots + m_n = k$. The number of ways you can arrange $m_1$ factors $x_1$, $m_2$ factors $x_2, \ldots,$ and $m_n$ factors $x_n$ to form that monomial is the multinomial coefficient $(*)$. Since $(x_1 + x_2 + \cdots + x_n)^k$ is the sum of all such monomials, we must have

$$(x_1 + x_2 + \cdots + x_n)^k = \sum_{|m|=k} \frac{k!}{m_1!\, m_2! \cdots m_n!} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}.$$

## EXERCISES 9.8

Find Maclaurin series representations for the functions in Exercises 1–8. Use the binomial series to calculate the answers.

**1.** $\sqrt{1+x}$

**2.** $x\sqrt{1-x}$

**3.** $\sqrt{4+x}$

**4.** $\dfrac{1}{\sqrt{4+x^2}}$

**5.** $(1-x)^{-2}$

**6.** $(1+x)^{-3}$

**7.** $\cos^{-1} x$

**8.** $\sinh^{-1} x$

**9. (Binomial coefficients)** Show that the binomial coefficients $\dbinom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ satisfy

(i) $\dbinom{n}{0} = \dbinom{n}{n} = 1$ for every $n$, and

(ii) if $0 \le k \le n$, then $\dbinom{n}{k-1} + \dbinom{n}{k} = \dbinom{n+1}{k}$.

It follows that, for fixed $n \ge 1$, the binomial coefficients

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$$

are the elements of the $n$th row of **Pascal's triangle** below, where each element with value greater than 1 is the sum of the two diagonally above it.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & \cdot & & & & & & & & \cdot & \\
\cdot & & & & & & & & & & \cdot
\end{array}$$

**10. (An inductive proof of the Binomial Theorem)** Use mathematical induction and the results of Exercise 9 to prove the Binomial Theorem:

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= a^n + n a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \cdots + b^n.\end{aligned}$$

**11. (The Leibniz Rule)** Use mathematical induction, the Product Rule, and Exercise 9 to verify the Leibniz Rule for the $n$th derivative of a product of two functions:

$$\begin{aligned}(fg)^{(n)} &= \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)} \\ &= f^{(n)} g + n f^{(n-1)} g' + \binom{n}{2} f^{(n-2)} g'' \\ &\quad + \binom{n}{3} f^{(n-3)} g^{(3)} + \cdots + f g^{(n)}.\end{aligned}$$

**12. (Proof of the Multinomial Theorem)** Use the Binomial Theorem and induction on $n$ to prove Theorem 24. *Hint:* Assume the theorem holds for specific $n$ and all $k$. Apply the Binomial Theorem to

$(x_1 + \cdots + x_n + x_{n+1})^k = \big((x_1 + \cdots + x_n) + x_{n+1}\big)^k.$

**13. (A Multifunction Leibniz Rule)** Use the technique of Exercise 12 to generalize the Leibniz Rule of Exercise 11 to calculate the $k$th derivative of a product of $n$ functions $f_1 f_2 \cdots f_n$.

# 9.9 Fourier Series

As we have seen, power series representations of functions make it possible to approximate those functions as closely as we want in intervals near a particular point of interest by using partial sums of the series, that is, polynomials. However, in many important applications of mathematics, the functions involved are required to be periodic. For example, much of electrical engineering is concerned with the analysis and manipulation of *waveforms*, which are periodic functions of time. Polynomials are not periodic functions, and for this reason power series are not well suited to representing such functions.

Much more appropriate for the representations of periodic functions over extended intervals are certain infinite series of periodic functions called Fourier series.

## Periodic Functions

Recall that a function $f$ defined on the real line is **periodic** with period $T$ if

$$f(t+T) = f(t) \quad \text{for all real } t. \tag{$*$}$$

This implies that $f(t+mT) = f(t)$ for any integer $m$, so that if $T$ is a period of $f$, then so is any multiple $mT$ of $T$. The smallest positive number $T$ for which $(*)$ holds is called the **fundamental period**, or simply **the period** of $f$.

The entire graph of a function with period $T$ can be obtained by shifting the part of the graph in any half-open interval of length $T$ (e.g., the interval $[0, T)$) to the left or right by integer multiples of the period $T$. Figure 9.6 shows the graph of a function of period 2.

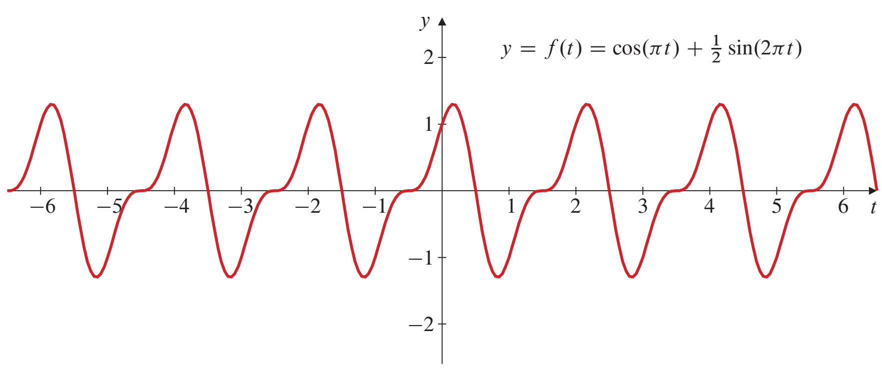



**Figure 9.6** This function has period 2. Observe how the graph repeats the part in the interval $[0, 2)$ over and over to the left and right

**EXAMPLE 1** The functions $g(t) = \cos(\pi t)$ and $h(t) = \sin(\pi t)$ are both periodic with period 2:

$$g(t+2) = \cos(\pi t + 2\pi) = \cos(\pi t) = g(t).$$

The function $k(t) = \sin(2\pi t)$ also has period 2, but this is not its fundamental period. The fundamental period is 1:

$$k(t+1) = \sin(2\pi t + 2\pi) = \sin(2\pi t) = k(t).$$

The sum $f(t) = g(t) + \frac{1}{2}k(t) = \cos(\pi t) + \frac{1}{2}\sin(2\pi t)$, graphed in Figure 9.6, has period 2, the least common multiple of the periods of its two terms.

**EXAMPLE 2** For any positive integer $n$, the functions

$$f_n(t) = \cos(n\omega t) \qquad \text{and} \qquad g_n(t) = \sin(n\omega t)$$

both have fundamental period $T = 2\pi/(n\omega)$. The collection of all such functions corresponding to all positive integers $n$ have common period $T = 2\pi/\omega$, the fundamental period of $f_1$ and $g_1$. $T$ is an integer multiple of the fundamental periods of all the functions $f_n$ and $g_n$. The subject of Fourier series is concerned with expressing general functions with period $T$ as series whose terms are real multiples of these functions.

## Fourier Series

It can be shown (but we won't do it here) that if $f(t)$ is periodic with fundamental period $T$, is continuous, and has a piecewise continuous derivative on the real line, then $f(t)$ is everywhere the sum of a series of the form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n\cos(n\omega t) + b_n\sin(n\omega t)\big), \qquad (**)$$

called the **Fourier series** of $f$, where $\omega = 2\pi/T$ and the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are the **Fourier coefficients** of $f$. Determining the values of these coefficients for a given such function $f$ is made possible by the following identities, valid for integers $m$ and $n$, which are easily proved by using the addition formulas for sine and cosine. (See Exercises 49–51 in Section 5.6.)

$$\int_0^T \cos(n\omega t)\,dt = \begin{cases} 0 & \text{if } n \neq 0 \\ T & \text{if } n = 0 \end{cases}$$

$$\int_0^T \sin(n\omega t)\,dt = 0$$

$$\int_0^T \cos(m\omega t)\cos(n\omega t)\,dt = \begin{cases} 0 & \text{if } m \neq n \\ T/2 & \text{if } m = n \end{cases}$$

$$\int_0^T \sin(m\omega t)\sin(n\omega t)\,dt = \begin{cases} 0 & \text{if } m \neq n \\ T/2 & \text{if } m = n \end{cases}$$

$$\int_0^T \cos(m\omega t)\sin(n\omega t)\,dt = 0.$$

If we multiply equation $(**)$ by $\cos(m\omega t)$ (or by $\sin(m\omega t)$) and integrate the resulting equation over $[0, T]$ term by term, all the terms on the right except the one involving $a_m$ (or $b_m$) will be 0. (The term-by-term integration requires justification, but we won't try to do that here either.) The integration results in

$$\int_0^T f(t)\cos(m\omega t)\,dt = \frac{1}{2}Ta_m$$

$$\int_0^T f(t)\sin(m\omega t)\,dt = \frac{1}{2}Tb_m.$$

(Note that the first of these formulas is even valid for $m = 0$ because we chose to call the constant term in the Fourier series $a_0/2$ instead of $a_0$.) Since the integrands are all periodic with period $T$, the integrals can be taken over any interval of length $T$; it is often convenient to use $[-T/2, T/2]$ instead of $[0, T]$. The Fourier coefficients of $f$ are therefore given by

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt \qquad (n = 0, 1, 2, \ldots)$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,dt \qquad (n = 1, 2, 3, \ldots),$$

where $\omega = 2\pi/T$.

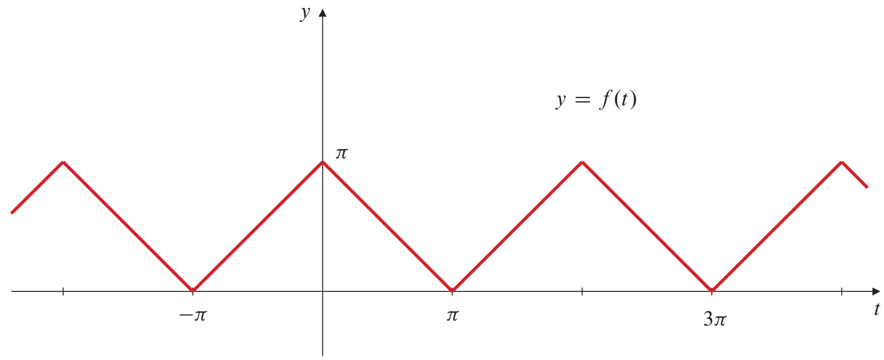



**Figure 9.7** A sawtooth function of period $2\pi$

**EXAMPLE 3** Find the Fourier series of the sawtooth function $f(t)$ of period $2\pi$ whose values in the interval $[-\pi, \pi]$ are given by $f(t) = \pi - |t|$. (See Figure 9.7.)

**Solution** Here $T = 2\pi$ and $\omega = 2\pi/(2\pi) = 1$. Since $f(t)$ is an even function, $f(t)\sin(nt)$ is odd, so all the Fourier sine coefficients $b_n$ are zero:

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = 0.$$

Also, $f(t)\cos(nt)$ is an even function, so

$$\begin{aligned} a_n &= \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{4}{2\pi}\int_0^{\pi} f(t)\cos(nt)\,dt \\ &= \frac{2}{\pi}\int_0^{\pi}(\pi - t)\cos(nt)\,dt \\ &= \begin{cases} \pi & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \text{ and } n \text{ is even} \\ 4/(\pi n^2) & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Since odd positive integers $n$ are of the form $n = 2k - 1$, where $k$ is a positive integer, the Fourier series of $f$ is given by

$$f(t) = \frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^2}\cos\big((2k-1)t\big).$$

## Convergence of Fourier Series

The partial sums of a Fourier series are called Fourier polynomials because they can be expressed as polynomials in $\sin(\omega t)$ and $\cos(\omega t)$, although we will not actually try to write them that way. The Fourier polynomial of order $m$ of the periodic function $f$ having period $T$ is

$$f_m(t) = \frac{a_0}{2} + \sum_{n=1}^{m}\big(a_n\cos(n\omega t) + b_n\sin(n\omega t)\big),$$

where $\omega = 2\pi/T$ and the coefficients $a_n$ $(0 \le n \le m)$ and $b_n$ $(1 \le n \le m)$ are given by the integral formulas developed earlier.

**EXAMPLE 4** The Fourier polynomial of order 3 of the sawtooth function of Example 3 is

$$f_3(t) = \frac{\pi}{2} + \frac{4}{\pi}\cos t + \frac{4}{9\pi}\cos(3t).$$

The graph of this function is shown in Figure 9.8. Observe that it appears to be a reasonable approximation to the graph of $f$ in Figure 9.7, but, being a finite sum of differentiable functions, $f_3(t)$ is itself differentiable everywhere, even at the integer multiples of $\pi$ where $f$ is not differentiable.

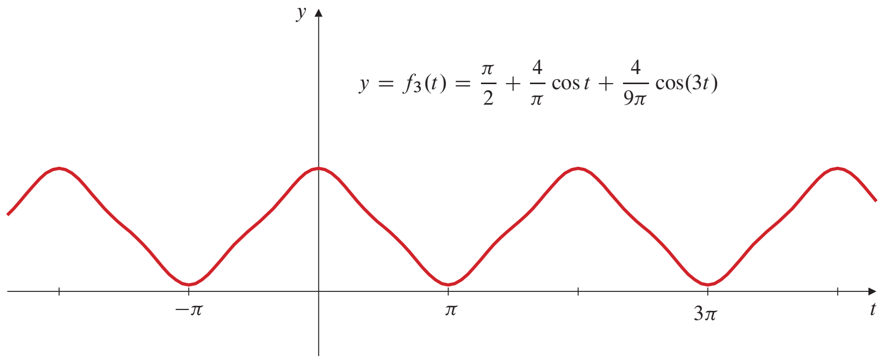



**Figure 9.8** The Fourier polynomial approximation $f_3(t)$ to the sawtooth function of Example 3

As noted earlier, the Fourier series of a function $f(t)$ that is periodic, continuous, and has a piecewise continuous derivative on the real line converges to $f(t)$ at each real number $t$. However, the Fourier coefficients (and hence the Fourier series) can be calculated (by the formulas given above) for periodic functions with piecewise continuous derivative even if the functions are not themselves continuous, but only piecewise continuous.

Recall that $f(t)$ is piecewise continuous on the interval $[a, b]$ if there exists a partition $\{a = x_0 < x_1 < x_2 < \cdots < x_k = b\}$ of $[a, b]$ and functions $F_1, F_2, \ldots, F_k$, such that

(i) $F_i$ is continuous on $[x_{i-1}, x_i]$, and

(ii) $f(t) = F_i(t)$ on $(x_{i-1}, x_i)$.

The integral of such a function $f$ is the sum of integrals of the functions $F_i$:

$$\int_a^b f(t)\,dt = \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} F_i(t)\,dt.$$

Since $f(t)\cos(n\omega t)$ and $f(t)\sin(n\omega t)$ are piecewise continuous if $f$ is, the Fourier coefficients of a piecewise continuous, periodic function can be calculated by the same formulas given for a continuous periodic function. The question of where and to what the Fourier series converges in this case is answered by the following theorem, proved in textbooks on Fourier analysis.

**THEOREM 25**

The Fourier series of a piecewise continuous, periodic function $f$ with piecewise continuous derivative converges to that function at every point $t$ where $f$ is continuous. Moreover, if $f$ is discontinuous at $t = c$, then $f$ has different, but finite, left and right limits at $c$:

$$\lim_{t \to c-} f(t) = f(c-), \qquad \text{and} \qquad \lim_{t \to c+} f(t) = f(c+).$$

The Fourier series of $f$ converges at $t = c$ to the average of these left and right limits:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(n\omega c) + b_n \sin(n\omega c)\big) = \frac{f(c-) + f(c+)}{2},$$

where $\omega = 2\pi/T$.

**EXAMPLE 5** Calculate the Fourier series for the periodic function $f$ with period 2 satisfying

$$f(t) = \begin{cases} -1 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x < 1. \end{cases}$$

Where does $f$ fail to be continuous? To what does the Fourier series of $f$ converge at these points?

**Solution** Here $T = 2$ and $\omega = 2\pi/2 = \pi$. Since $f$ is an odd function, its cosine coefficients are all zero:

$$a_n = \int_{-1}^{1} f(t)\cos(n\pi t)\,dt = 0. \qquad \text{(The integrand is odd.)}$$

The same symmetry implies that

$$\begin{aligned} b_n &= \int_{-1}^{1} f(t)\sin(n\pi t)\,dt \\ &= 2\int_0^1 \sin(n\pi t)\,dt = -\frac{2\cos(n\pi t)}{n\pi}\bigg|_0^1 \\ &= -\frac{2}{n\pi}\big((-1)^n - 1\big) = \begin{cases} 4/(n\pi) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Odd integers $n$ are of the form $n = 2k - 1$ for $k = 1, 2, 3, \ldots$. Therefore, the Fourier series of $f$ is

$$\begin{aligned} &\frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{2k-1}\sin\big((2k-1)\pi t\big) \\ =&\frac{4}{\pi}\left(\sin(\pi t) + \frac{1}{3}\sin(3\pi t) + \frac{1}{5}\sin(5\pi t) + \cdots\right). \end{aligned}$$

Note that $f$ is continuous except at the points where $t$ is an integer. At each of these points $f$ jumps from $-1$ to 1 or from 1 to $-1$, so the average of the left and right limits of $f$ at these points is 0. Observe that the sum of the Fourier series is 0 at integer values of $t$, in accordance with Theorem 25. See Figure 9.9.

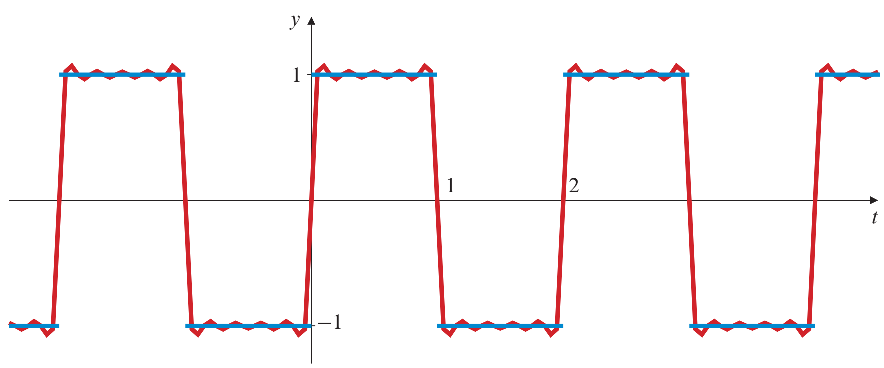



**Figure 9.9** The piecewise continuous function $f$ (blue) of Example 5 and its Fourier polynomial $f_{15}$ (red)

$$f_{15}(t) = \sum_{k=1}^{8} \frac{4\sin\big((2k-1)\pi t\big)}{(2k-1)\pi}$$

## Fourier Cosine and Sine Series

As observed in Example 3 and Example 5, even functions have no sine terms in their Fourier series, and odd functions have no cosine terms (including the constant term $a_0/2$). It is often necessary in applications to find a Fourier series representation of a given function defined on a finite interval $[0, a]$ having either no sine terms (a **Fourier cosine series**) or no cosine terms (a **Fourier sine series**). This is accomplished by extending the domain of $f$ to $[-a, 0)$ so as to make $f$ either even or odd on $[-a, a]$,

$$\begin{aligned} f(-t) &= f(t) \text{ if } -a \le t < 0 \text{ for the even extension} \\ f(-t) &= -f(t) \text{ if } -a \le t < 0 \text{ for the odd extension,} \end{aligned}$$

and then calculating its Fourier series considering the extended $f$ to have period $2a$. (If we want the odd extension, we may have to redefine $f(0)$ to be 0.)

**EXAMPLE 6** Find the Fourier cosine series of $g(t) = \pi - t$ defined on $[0, \pi]$.

**Solution** The even extension of $g(t)$ to $[-\pi, \pi]$ is the function $f$ of Example 3. Thus, the Fourier cosine series of $g$ is

$$\frac{\pi}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi(2k-1)^2} \cos\big((2k-1)t\big).$$

**EXAMPLE 7** Find the Fourier sine series of $h(t) = 1$ defined on $[0, 1]$.

**Solution** If we redefine $h(0) = 0$, then the odd extension of $h$ to $[-1, 1]$ coincides with the function $f(t)$ of Example 5 except that the latter function is undefined at $t = 0$. The Fourier sine series of $h$ is the series obtained in Example 5, namely,

$$\frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin\big((2k-1)\pi t\big).$$

**Remark** Fourier cosine and sine series are treated from a different perspective in Section 13.5.

## EXERCISES 9.9

In Exercises 1–4, what is the fundamental period of the given function?

1. $f(t) = \sin(3t)$
2. $g(t) = \cos(3 + \pi t)$
3. $h(t) = \cos^2 t$
4. $k(t) = \sin(2t) + \cos(3t)$

In Exercises 5–8, find the Fourier series of the given function.

5. $f(t) = t$, $-\pi < t \le \pi$, $f$ has period $2\pi$.
6. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ 1 & \text{if } 1 \le t < 2, \end{cases}$ $f$ has period 2.
7. $f(t) = \begin{cases} 0 & \text{if } -1 \le t < 0 \\ t & \text{if } 0 \le t < 1, \end{cases}$ $f$ has period 2.
8. $f(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 1 & \text{if } 1 \le t < 2 \\ 3 - t & \text{if } 2 \le t < 3, \end{cases}$ $f$ has period 3.
9. What is the Fourier cosine series of the function $h(t)$ of Example 7?
10. Calculate the Fourier sine series of the function $g(t)$ of Example 6.
11. Find the Fourier sine series of $f(t) = t$ on $[0, 1]$.
12. Find the Fourier cosine series of $f(t) = t$ on $[0, 1]$.
13. Use the result of Example 3 to evaluate

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

14. Verify that if $f$ is an even function of period $T$, then the Fourier sine coefficients $b_n$ of $f$ are all zero and the Fourier cosine coefficients $a_n$ of $f$ are given by

$$a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos(n\omega t)\, dt, \qquad n = 0, 1, 2, \ldots,$$

where $\omega = 2\pi/T$. State and verify the corresponding result for odd functions $f$.

# CHAPTER REVIEW

## Key Ideas

- **What does it mean to say that the sequence $\{a_n\}$**
  - is bounded above?
  - is ultimately positive?
  - is alternating?
  - is increasing?
  - converges?
  - diverges to infinity?
- **What does it mean to say that the series $\sum_{n=1}^{\infty} a_n$**
  - converges?
  - diverges?
  - is geometric?
  - is telescoping?
  - is a $p$-series?
  - is positive?
  - converges absolutely?
  - converges conditionally?

- **State the following convergence tests for series:**
  - ◇ the integral test
  - ◇ the comparison test
  - ◇ the limit comparison test
  - ◇ the ratio test
  - ◇ the alternating series test
- **How can you find bounds for the tail of a series?**
- **What is a bound for the tail of an alternating series?**
- **What do the following terms and phrases mean?**
  - ◇ a power series
  - ◇ interval of convergence
  - ◇ radius of convergence
  - ◇ centre of convergence
  - ◇ a Taylor series
  - ◇ a Maclaurin series
  - ◇ a Taylor polynomial
  - ◇ a binomial series
  - ◇ an analytic function
- **Where is the sum of a power series differentiable?**
- **Where does the integral of a power series converge?**
- **Where is the sum of a power series continuous?**
- **State Taylor's Theorem with Lagrange remainder.**
- **State Taylor's Theorem with integral remainder.**
- **What is the Binomial Theorem?**
- **What is a Fourier series?**
- **What is a Fourier cosine series? a Fourier sine series?**

## Review Exercises

In Exercises 1–4, determine whether the given sequence converges, and find its limit if it does converge.

**1.** $\left\{\dfrac{(-1)^n e^n}{n!}\right\}$ **2.** $\left\{\dfrac{n^{100}+2^n\pi}{2^n}\right\}$

**3.** $\left\{\dfrac{\ln n}{\tan^{-1} n}\right\}$ **4.** $\left\{\dfrac{(-1)^n n^2}{\pi n(n-\pi)}\right\}$

**5.** Let $a_1 > \sqrt{2}$, and let

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \quad \text{for} \quad n = 1, 2, 3, \ldots$$

Show that $\{a_n\}$ is decreasing and that $a_n > \sqrt{2}$ for $n \ge 1$. Why must $\{a_n\}$ converge? Find $\lim_{n\to\infty} a_n$.

**6.** Find the limit of the sequence $\{\ln\ln(n+1) - \ln\ln n\}$.

Evaluate the sums of the series in Exercises 7–10.

**7.** $\displaystyle\sum_{n=1}^{\infty} 2^{-(n-5)/2}$ **8.** $\displaystyle\sum_{n=0}^{\infty} \frac{4^{n-1}}{(\pi-1)^{2n}}$

**9.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-\frac{1}{4}}$ **10.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-\frac{9}{4}}$

Determine whether the series in Exercises 11–16 converge or diverge. Give reasons for your answers.

**11.** $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3}$ **12.** $\displaystyle\sum_{n=1}^{\infty} \frac{n+2^n}{1+3^n}$

**13.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(1+n)(1+n\sqrt{n})}$ **14.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(1+2^n)(1+n\sqrt{n})}$

**15.** $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n+1}}{n!}$ **16.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n+2)!+1}$

Do the series in Exercises 17–20 converge absolutely, converge conditionally, or diverge?

**17.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{1+n^3}$ **18.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n-n}$

**19.** $\displaystyle\sum_{n=10}^{\infty} \frac{(-1)^{n-1}}{\ln\ln n}$ **20.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2\cos(n\pi)}{1+n^3}$

For what values of $x$ do the series in Exercises 21–22 converge absolutely? converge conditionally? diverge?

**21.** $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n\sqrt{n}}$ **22.** $\displaystyle\sum_{n=1}^{\infty} \frac{(5-2x)^n}{n}$

Determine the sums of the series in Exercises 23–24 to within 0.001.

**23.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ **24.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4+n^2}$

In Exercises 25–32, find Maclaurin series for the given functions. State where each series converges to the function.

**25.** $\dfrac{1}{3-x}$ **26.** $\dfrac{x}{3-x^2}$

**27.** $\ln(e+x^2)$ **28.** $\dfrac{1-e^{-2x}}{x}$

**29.** $x\cos^2 x$ **30.** $\sin(x+(\pi/3))$

**31.** $(8+x)^{-1/3}$ **32.** $(1+x)^{1/3}$

Find Taylor series for the functions in Exercises 33–34 about the indicated points $x = c$.

**33.** $1/x, \quad c = \pi$ **34.** $\sin x + \cos x, \quad c = \pi/4$

Find the Maclaurin polynomial of the indicated degree for the functions in Exercises 35–38.

**35.** $e^{x^2+2x}$, degree 3 **36.** $\sin(1+x)$, degree 3

**37.** $\cos(\sin x)$, degree 4 **38.** $\sqrt{1+\sin x}$, degree 4

**39.** What function has Maclaurin series

$$1 - \frac{x}{2!} + \frac{x^2}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}?$$

**40.** A function $f(x)$ has Maclaurin series

$$1 + x^2 + \frac{x^4}{2^2} + \frac{x^6}{3^2} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n^2}.$$

Find $f^{(k)}(0)$ for all positive integers $k$.

Find the sums of the series in Exercises 41–44.

**41.** $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{\pi^n}$ **! 42.** $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{\pi^n}$

**43.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{ne^n}$ **! 44.** $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \pi^{2n-4}}{(2n-1)!}$

**45.** If $S(x) = \displaystyle\int_0^x \sin(t^2)\,dt$, find $\displaystyle\lim_{x\to 0} \frac{x^3 - 3S(x)}{x^7}$.

**46.** Use series to evaluate $\displaystyle\lim_{x\to 0} \frac{(x-\tan^{-1}x)(e^{2x}-1)}{2x^2-1+\cos(2x)}$.

**47.** How many nonzero terms in the Maclaurin series for $e^{-x^4}$ are needed to evaluate $\int_0^{1/2} e^{-x^4}\,dx$ correct to 5 decimal places? Evaluate the integral to that accuracy.

**48.** Estimate the size of the error if the Taylor polynomial of degree 4 about $x = \pi/2$ for $f(x) = \ln\sin x$ is used to approximate $\ln\sin(1.5)$.

**49.** Find the Fourier sine series for $f(t) = \pi - t$ on $[0, \pi]$.

**50.** Find the Fourier series for $f(t) = \begin{cases} 1 & \text{if } -\pi < t \le 0 \\ t & \text{if } 0 < t \le \pi. \end{cases}$

## Challenging Problems

**1.** **(A refinement of the ratio test)** Suppose $a_n > 0$ and $a_{n+1}/a_n \ge n/(n+1)$ for all $n$. Show that $\sum_{n=1}^{\infty} a_n$ diverges. *Hint:* $a_n \ge K/n$ for some constant $K$.

**2.** **(Summation by parts)** Let $\{u_n\}$ and $\{v_n\}$ be two sequences, and let $s_n = \sum_{k=1}^{n} v_k$.

(a) Show that $\sum_{k=1}^{n} u_k v_k = u_{n+1}s_n + \sum_{k=1}^{n}(u_k - u_{k+1})s_k$. (*Hint:* Write $v_n = s_n - s_{n-1}$, with $s_0 = 0$, and rearrange the sum.)

(b) If $\{u_n\}$ is positive, decreasing, and convergent to 0, and if $\{v_n\}$ has bounded partial sums, $|s_n| \le K$ for all $n$, where $K$ is a constant, show that $\sum_{n=1}^{\infty} u_n v_n$ converges. (*Hint:* Show that the series $\sum_{n=1}^{\infty}(u_n - u_{n+1})s_n$ converges by comparing it to the telescoping series $\sum_{n=1}^{\infty}(u_n - u_{n+1})$.)

**3.** Show that $\sum_{n=1}^{\infty}(1/n)\sin(nx)$ converges for every $x$. *Hint:* If $x$ is an integer multiple of $\pi$, all the terms in the series are 0, so there is nothing to prove. Otherwise, $\sin(x/2) \ne 0$. In this case show that

$$\sum_{n=1}^{N} \sin(nx) = \frac{\cos(x/2) - \cos((N+1/2)x)}{2\sin(x/2)}$$

using the identity

$$\sin a \sin b = \frac{\cos(a-b) - \cos(a+b)}{2}$$

to make the sum telescope. Then apply the result of Problem 2(b) with $u_n = 1/n$ and $v_n = \sin(nx)$.

**4.** Let $a_1, a_2, a_3, \ldots$ be those positive integers that do not contain the digit 0 in their decimal representations. Thus, $a_1 = 1$, $a_2 = 2, \ldots, a_9 = 9$, $a_{10} = 11, \ldots, a_{18} = 19$, $a_{19} = 21$, $\ldots, a_{90} = 99$, $a_{91} = 111$, etc. Show that the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{a_n}$ converges and that the sum is less than 90. (*Hint:* How many of these integers have $m$ digits? Each term $1/a_n$, where $a_n$ has $m$ digits, is less than $10^{-m+1}$.)

**5.** **(Using an integral to improve convergence)** Recall the error formula for the Midpoint Rule, according to which

$$\int_{k-1/2}^{k+1/2} f(x)\,dx - f(k) = \frac{f''(c)}{24},$$

where $k - (1/2) \le c \le k + (1/2)$.

(a) If $f''(x)$ is a decreasing function of $x$, show that

$$f'(k+\tfrac{3}{2}) - f'(k+\tfrac{1}{2}) \le f''(c) \le f'(k-\tfrac{1}{2}) - f'(k-\tfrac{3}{2}).$$

(b) If (i) $f''(x)$ is a decreasing function of $x$, (ii) $\int_{N+1/2}^{\infty} f(x)\,dx$ converges, and (iii) $f'(x) \to 0$ as $x \to \infty$, show that

$$\frac{f'(N-\frac{1}{2})}{24} \le \sum_{n=N+1}^{\infty} f(n) - \int_{N+1/2}^{\infty} f(x)\,dx \le \frac{f'(N+\frac{3}{2})}{24}.$$

(c) Use the result of part (b) to approximate $\sum_{n=1}^{\infty} 1/n^2$ to within 0.001.

**6.** **(The number $e$ is irrational)** Start with $e = \sum_{n=0}^{\infty} 1/n!$.

(a) Use the technique of Example 7 in Section 9.3 to show that for any $n > 0$,

$$0 < e - \sum_{j=0}^{n}\frac{1}{j!} < \frac{1}{n!n}.$$

(Note that the sum here has $n+1$ terms, not $n$ terms.)

(b) Suppose that $e$ is a rational number, say $e = M/N$ for certain positive integers $M$ and $N$. Show that $N!\left(e - \sum_{j=0}^{N}(1/j!)\right)$ is an integer.

(c) Combine parts (a) and (b) to show that there is an integer between 0 and $1/N$. Why is this not possible? Conclude that $e$ cannot be a rational number.

**7.** Let

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty}\frac{2^{2k}k!}{(2k+1)!}x^{2k+1} \\ &= x + \frac{2}{3}x^3 + \frac{4}{3\times 5}x^5 + \frac{8}{3\times 5\times 7}x^7 + \cdots. \end{aligned}$$

(a) Find the radius of convergence of this power series.

(b) Show that $f'(x) = 1 + 2xf(x)$.

(c) What is $\displaystyle\frac{d}{dx}\left(e^{-x^2}f(x)\right)$?

(d) Express $f(x)$ in terms of an integral.

**8.** **(The number $\pi$ is irrational)** Problem 6 above shows how to prove that $e$ is irrational by assuming the contrary and deducing a contradiction. In this problem you will show that $\pi$ is also irrational. The proof for $\pi$ is also by contradiction but is rather more complicated, so it will be broken down into several parts.

(a) Let $f(x)$ be a polynomial, and let

$$\begin{aligned} g(x) &= f(x) - f''(x) + f^{(4)}(x) - f^{(6)}(x) + \cdots \\ &= \sum_{j=0}^{\infty}(-1)^j f^{(2j)}(x). \end{aligned}$$

(Since $f$ is a polynomial, all but a finite number of terms in the above sum are identically zero, so there are no convergence problems.) Verify that

$$\frac{d}{dx}\left(g'(x)\sin x - g(x)\cos x\right) = f(x)\sin x,$$

and hence that $\displaystyle\int_0^{\pi} f(x)\sin x\,dx = g(\pi) + g(0)$.

(b) Suppose that $\pi$ is rational, say, $\pi = m/n$, where $m$ and $n$ are positive integers. You will show that this leads to a contradiction and thus cannot be true. Choose a positive integer $k$ such that $(\pi m)^k/k! < 1/2$. (Why is this possible?) Consider the polynomial

$$f(x) = \frac{x^k(m - nx)^k}{k!} = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} m^{k-j}(-n)^j x^{j+k}.$$

Show that $0 < f(x) < 1/2$ for $0 < x < \pi$, and hence that $0 < \int_0^{\pi} f(x)\sin x\,dx < 1$. Thus, $0 < g(\pi) + g(0) < 1$, where $g(x)$ is defined as in part (a).

(c) Show that the $i$th derivative of $f(x)$ is given by

$$f^{(i)}(x) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} m^{k-j}(-n)^j \frac{(j+k)!}{(j+k-i)!}x^{j+k-i}.$$

(d) Show that $f^{(i)}(0)$ is an integer for $i = 0, 1, 2, \ldots$. (*Hint:* Observe for $i < k$ that $f^{(i)}(0) = 0$, and for $i > 2k$ that $f^{(i)}(x) = 0$ for all $x$. For $k \le i \le 2k$, show that only one term in the sum for $f^{(i)}(0)$ is not 0, and that this term is an integer. You will need the fact that the binomial coefficients $\binom{k}{j}$ are integers.)

(e) Show that $f(\pi - x) = f(x)$ for all $x$, and hence that $f^{(i)}(\pi)$ is also an integer for each $i = 0, 1, 2, \ldots$. Therefore, if $g(x)$ is defined as in (a), then $g(\pi) + g(0)$ is an integer. This contradicts the conclusion of part (b) and so shows that $\pi$ cannot be rational.

**9.** **(An asymptotic series)** Integrate by parts to show that

$$\int_0^x e^{-1/t}\,dt = e^{-1/x}\sum_{n=2}^{N}(-1)^n(n-1)!x^n + (-1)^{N+1}N!\int_0^x t^{N-1}e^{-1/t}\,dt.$$

Why can't you just use a Maclaurin series to approximate this integral? Using $N = 5$, find an approximate value for $\int_0^{0.1} e^{-1/t}\,dt$, and estimate the error. Estimate the error for $N = 10$ and $N = 20$.

Note that the series $\sum_{n=2}^{\infty}(-1)^n(n-1)!x^n$ **diverges** for any $x \neq 0$. This is an example of what is called an **asymptotic series**. Even though it diverges, a properly chosen partial sum gives a good approximation to our function when $x$ is small.

CHAPTER 10

# Vectors and Coordinate Geometry in 3-Space

> "Lord Ronald said nothing; he flung himself from the room, flung himself upon his horse and rode madly off in all directions. . . .
>
> And who is this tall young man who draws nearer to Gertrude with every revolution of the horse? . . .
>
> The two were destined to meet. Nearer and nearer they came. And then still nearer. Then for one brief moment they met. As they passed Gertrude raised her head and directed towards the young nobleman two eyes so eye-like in their expression as to be absolutely circular, while Lord Ronald directed towards the occupant of the dogcart a gaze so gaze-like that nothing but a gazelle, or a gas-pipe, could have emulated its intensity."

**Stephen Leacock 1869–1944**
**from *Gertrude the Governess: or, Simple Seventeen***

## Introduction

A complete real-variable calculus program involves the study of

(i) real-valued functions of a single real variable,

(ii) vector-valued functions of a single real variable,

(iii) real-valued functions of a real vector variable, and

(iv) vector-valued functions of a real vector variable.

Chapters 1–9 are concerned with item (i). The remaining chapters deal with items (ii), (iii), and (iv). Specifically, Chapter 11 deals with vector-valued functions of a single real variable. Chapters 12–14 are concerned with the differentiation and integration of real-valued functions of several real variables, that is, of a real vector variable. Chapters 15–17 present aspects of the calculus of functions whose domains and ranges both have dimension greater than one, that is, vector-valued functions of a vector variable. Most of the time we will limit our attention to vector functions with domains and ranges in the plane, or in three-dimensional space.

In this chapter we will lay the foundation for multivariable and vector calculus by extending the concepts of analytic geometry to three or more dimensions and by introducing vectors as a convenient way of dealing with several variables as a single entity. We also introduce matrices, because these will prove useful for formulating some of the concepts of calculus. This chapter is not intended to be a full course in linear algebra; we develop only those aspects that we will use in later chapters, and we omit most proofs.

# 10.1 Analytic Geometry in Three Dimensions

We say that the physical world in which we live is three-dimensional because through any point there can pass three, and no more, straight lines that are **mutually perpendicular**; that is, each of them is perpendicular to the other two. This is equivalent to the fact that we require three numbers to locate a point in space with respect to some reference point (the **origin**). One way to use three numbers to locate a point is by having them represent (signed) distances from the origin, measured in the directions of three mutually perpendicular lines passing through the origin. We call such a set of lines a Cartesian coordinate system, and each of the lines is called a coordinate axis. We usually call these axes the $x$-axis, the $y$-axis, and the $z$-axis, regarding the $x$- and $y$-axes as lying in a horizontal plane and the $z$-axis as vertical. Moreover, the coordinate system should have a **right-handed orientation**. This means that the thumb, forefinger, and middle finger of the right hand can be extended so as to point, respectively, in the directions of the positive $x$-axis, the positive $y$-axis, and the positive $z$-axis. For the more mechanically minded, a right-handed screw will advance in the positive $z$ direction if twisted in the direction of rotation from the positive $x$-axis toward the positive $y$-axis. (See Figure 10.1(a).)

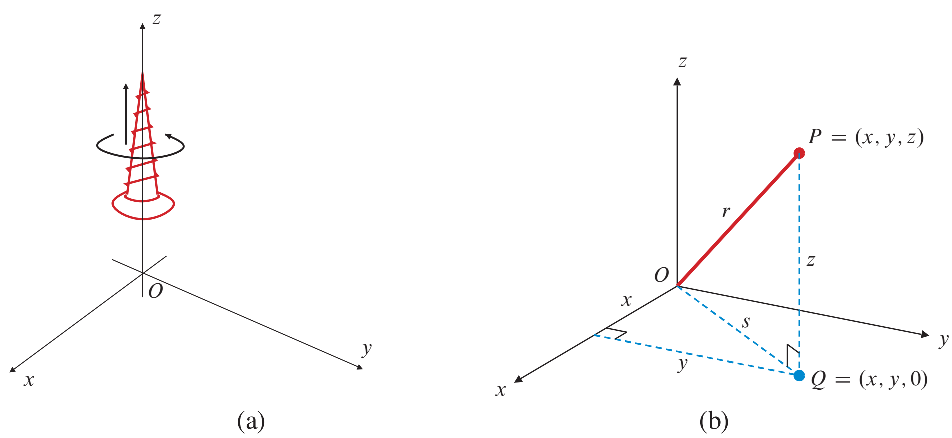



**Figure 10.1**

(a) The screw moves upward when twisted counterclockwise as seen from above

(b) The three coordinates of a point in 3-space

With respect to such a Cartesian coordinate system, the **coordinates** of a point $P$ in 3-space constitute an ordered triple of real numbers, $(x, y, z)$. The numbers $x$, $y$, and $z$ are, respectively, the signed distances of $P$ from the origin, measured in the directions of the $x$-axis, the $y$-axis, and the $z$-axis. (See Figure 10.1(b).)

Let $Q$ be the point with coordinates $(x, y, 0)$. Then $Q$ lies in the $xy$-plane (the plane containing the $x$- and $y$-axes) directly under (or over) $P$. We say that $Q$ is the vertical projection of $P$ onto the $xy$-plane. If $r$ is the distance from the origin $O$ to $P$ and $s$ is the distance from $O$ to $Q$, then, using two right-angled triangles, we have

$$s^2 = x^2 + y^2 \qquad \text{and} \qquad r^2 = s^2 + z^2 = x^2 + y^2 + z^2.$$

Thus, the distance from $P$ to the origin is given by

$$r = \sqrt{x^2 + y^2 + z^2}.$$

Similarly, the distance $r$ between points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ (see Figure 10.2) is

$$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

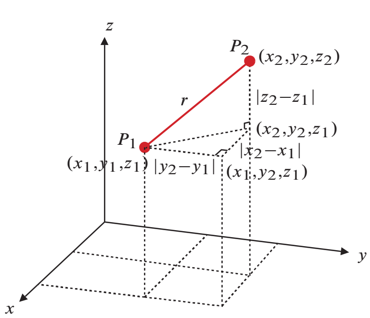



**Figure 10.2** Distance between points

**EXAMPLE 1** Show that the triangle with vertices $A = (1, -1, 2)$, $B = (3, 3, 8)$, and $C = (2, 0, 1)$ has a right angle.

**Solution** We calculate the lengths of the three sides of the triangle:

$$a = |BC| = \sqrt{(2-3)^2 + (0-3)^2 + (1-8)^2} = \sqrt{59}$$
$$b = |AC| = \sqrt{(2-1)^2 + (0+1)^2 + (1-2)^2} = \sqrt{3}$$
$$c = |AB| = \sqrt{(3-1)^2 + (3+1)^2 + (8-2)^2} = \sqrt{56}$$

By the cosine law, $a^2 = b^2+c^2-2bc\cos A$. In this case $a^2 = 59 = 3+56 = b^2+c^2$, so that $2bc\cos A$ must be 0. Therefore, $\cos A = 0$ and $A = 90°$.

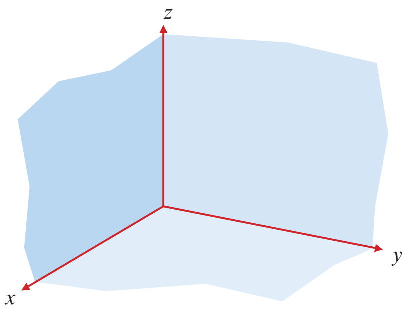


**Figure 10.3** The first octant

Just as the $x$- and $y$-axes divide the $xy$-plane into four quadrants, so also the three **coordinate planes** in 3-space (the $xy$-plane, the $xz$-plane, and the $yz$-plane) divide 3-space into eight **octants**. We call the octant in which $x \ge 0$, $y \ge 0$, and $z \ge 0$ the **first octant**. When drawing graphs in 3-space it is sometimes easier to draw only the part lying in the first octant (Figure 10.3).

An equation or inequality involving the three variables $x$, $y$, and $z$ defines a subset of points in 3-space whose coordinates satisfy the equation or inequality. A single equation usually represents a surface (a two-dimensional object) in 3-space.

**EXAMPLE 2** **(Some equations and the surfaces they represent)**

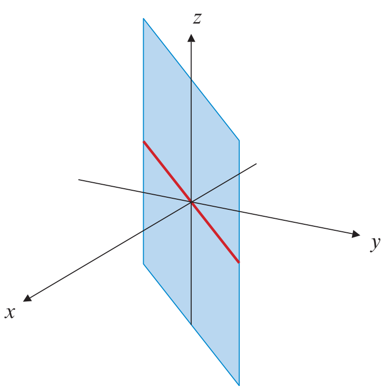


**Figure 10.4** Equation $x = y$ defines a vertical plane

(a) The equation $z = 0$ represents all points with coordinates $(x, y, 0)$, that is, the $xy$-plane. The equation $z = -2$ represents all points with coordinates $(x, y, -2)$, that is, the horizontal plane passing through the point $(0, 0, -2)$ on the $z$-axis.

(b) The equation $x = y$ represents all points with coordinates $(x, x, z)$. This is a vertical plane containing the straight line with equation $x = y$ in the $xy$-plane. The plane also contains the $z$-axis. (See Figure 10.4.)

(c) The equation $x + y + z = 1$ represents all points the sum of whose coordinates is 1. This set is a plane that passes through the three points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. These points are not collinear (they do not lie on a straight line), so there is only one plane passing through all three. (See Figure 10.5.) The equation $x + y + z = 0$ represents a plane parallel to the one with equation $x + y + z = 1$ but passing through the origin.

(d) The equation $x^2 + y^2 = 4$ represents all points on the vertical circular cylinder containing the circle with equation $x^2 + y^2 = 4$ in the $xy$-plane. This cylinder has radius 2 and axis along the $z$-axis. (See Figure 10.6.)

Here the term "cylinder" is used to describe a surface ruled by parallel straight lines, not a solid as it was in Section 7.1.

(e) The equation $z = x^2$ represents all points with coordinates $(x, y, x^2)$. This surface is a parabolic cylinder tangent to the $xy$-plane along the $y$-axis. (See Figure 10.7.)

(f) The equation $x^2 + y^2 + z^2 = 25$ represents all points $(x, y, z)$ at distance 5 from the origin. This set of points is a *sphere* of radius 5 centred at the origin.

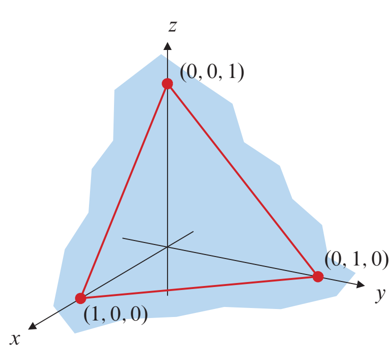


**Figure 10.5** The plane with equation $x + y + z = 1$

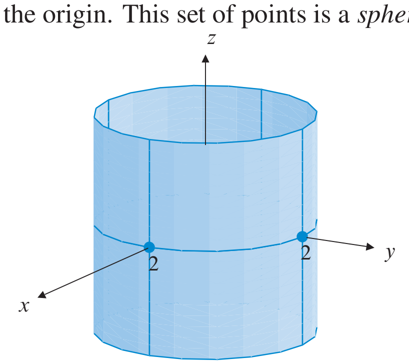


**Figure 10.6** The circular cylinder with equation $x^2 + y^2 = 4$

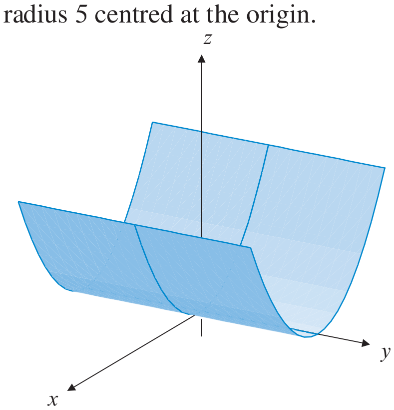


**Figure 10.7** The parabolic cylinder with equation $z = x^2$

Observe that equations in $x$, $y$, and $z$ need not involve each variable explicitly. When one of the variables is missing from the equation, the equation represents a surface *parallel to* the axis of the missing variable. Such a surface may be a plane or a cylinder. For example, if $z$ is absent from the equation, the equation represents in 3-space a vertical (i.e., parallel to the $z$-axis) surface containing the curve with the same equation in the $xy$-plane.

Occasionally, a single equation may not represent a two-dimensional object (a surface). It can represent a one-dimensional object (a line or curve), a zero-dimensional object (one or more points), or even nothing at all.

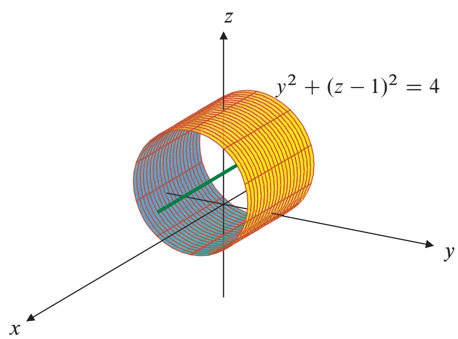



**Figure 10.8** The cylinder $y^2 + (z-1)^2 = 4$ and its axial line $y = 0$, $z = 1$, or $y^2 + (z-1)^2 = 0$

**EXAMPLE 3** Identify the graphs of (a) $y^2 + (z-1)^2 = 4$, (b) $y^2 + (z-1)^2 = 0$, (c) $x^2 + y^2 + z^2 = 0$, and (d) $x^2 + y^2 + z^2 = -1$.

## Solution

(a) Since $x$ is absent, the equation $y^2 + (z-1)^2 = 4$ represents an object parallel to the $x$-axis. In the $yz$-plane the equation represents a circle of radius 2 centred at $(y, z) = (0, 1)$. In 3-space it represents a horizontal circular cylinder, parallel to the $x$-axis, with axis one unit above the $x$-axis. (See Figure 10.8.)

(b) Since squares cannot be negative, the equation $y^2 + (z-1)^2 = 0$ implies that $y = 0$ and $z = 1$, so it represents points $(x, 0, 1)$. All these points lie on the line parallel to the $x$-axis and one unit above it. (See Figure 10.8.)

(c) As in part (b), $x^2 + y^2 + z^2 = 0$ implies that $x = 0$, $y = 0$, and $z = 0$. The equation represents only one point, the origin.

(d) The equation $x^2 + y^2 + z^2 = -1$ is not satisfied by any real numbers $x$, $y$, and $z$, so it represents no points at all.

A single inequality in $x$, $y$, and $z$ typically represents points lying on one side of the surface represented by the corresponding equation (together with points on the surface if the inequality is not strict).

**EXAMPLE 4** (a) The inequality $z > 0$ represents all points above the $xy$-plane.

(b) The inequality $x^2 + y^2 \geq 4$ says that the square of the distance from $(x, y, z)$ to the nearest point $(0, 0, z)$ on the $z$-axis is at least 4. This inequality represents all points lying on or outside the cylinder of Example 2(d).

(c) The inequality $x^2 + y^2 + z^2 \leq 25$ says that the square of the distance from $(x, y, z)$ to the origin is no greater than 25. It represents the solid ball of radius 5 centred at the origin, which consists of all points lying inside or on the sphere of Example 2(f).

Two equations in $x$, $y$, and $z$ normally represent a one-dimensional object, the line or curve along which the two surfaces represented by the two equations intersect. Any point whose coordinates satisfy both equations must lie on both the surfaces, so must lie on their intersection.

**EXAMPLE 5** What sets of points in 3-space are represented by the following pairs of equations?

(a) $\begin{cases} x + y + z = 1 \\ y - 2x = 0 \end{cases}$ (b) $\begin{cases} x^2 + y^2 + z^2 = 1 \\ x + y = 1 \end{cases}$

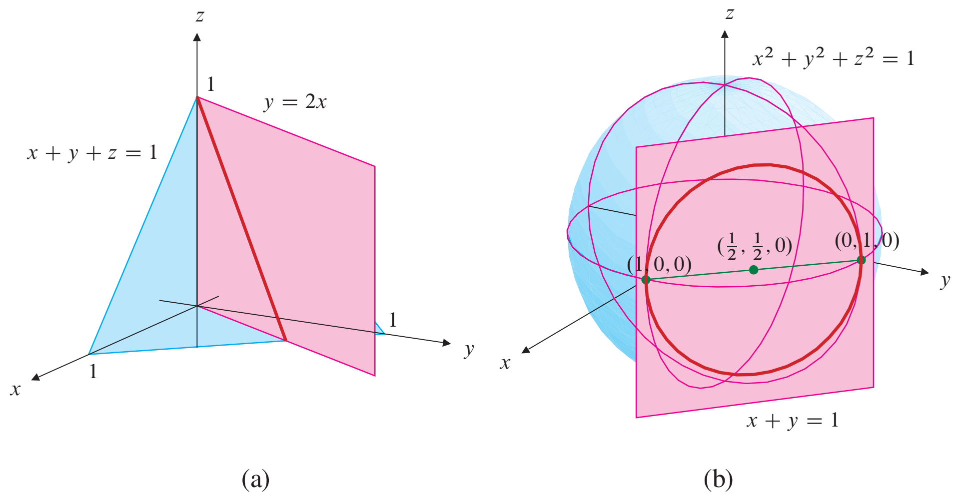



**Figure 10.9**

(a) The two planes intersect in a straight line

(b) The plane intersects the sphere in a circle

**Solution**

(a) The equation $x + y + z = 1$ represents the oblique plane of Example 2(c), and the equation $y - 2x = 0$ represents a vertical plane through the origin and the point $(1, 2, 0)$. Together these two equations represent the line of intersection of the two planes. This line passes through, for example, the points $(0, 0, 1)$ and $(\frac{1}{3}, \frac{2}{3}, 0)$. (See Figure 10.9(a).)

(b) The equation $x^2 + y^2 + z^2 = 1$ represents a sphere of radius 1 with centre at the origin, and $x + y = 1$ represents a vertical plane through the points $(1, 0, 0)$ and $(0, 1, 0)$. The two surfaces intersect in a circle, as shown in Figure 10.9(b). The line from $(1, 0, 0)$ to $(0, 1, 0)$ is a diameter of the circle, so the centre of the circle is $(\frac{1}{2}, \frac{1}{2}, 0)$, and its radius is $\sqrt{2}/2$.

In Sections 10.4 and 10.5 we will see many more examples of geometric objects in 3-space represented by simple equations.

## Euclidean $n$-Space

Mathematicians and users of mathematics frequently need to consider ***n*-dimensional space**, where $n$ is greater than 3 and may even be infinite. It is difficult to visualize a space of dimension 4 or higher geometrically. The secret to dealing with these spaces is to regard the points in $n$-space as *being* ordered $n$-tuples of real numbers; that is, $(x_1, x_2, \ldots, x_n)$ is a point in $n$-space instead of just being the coordinates of such a point. We stop thinking of points as existing in physical space and start thinking of them as algebraic objects. We usually denote $n$-space by the symbol $\mathbb{R}^n$ to show that its points are $n$-tuples of *real* numbers. Thus $\mathbb{R}^2$ and $\mathbb{R}^3$ denote the plane and 3-space, respectively. Note that in passing from $\mathbb{R}^3$ to $\mathbb{R}^n$ we have altered the notation a bit: in $\mathbb{R}^3$ we called the coordinates $x$, $y$, and $z$, while in $\mathbb{R}^n$ we called them $x_1, x_2, \ldots$ and $x_n$ so as not to run out of letters. We could, of course, talk about coordinates $(x_1, x_2, x_3)$ in $\mathbb{R}^3$ and $(x_1, x_2)$ in the plane $\mathbb{R}^2$, but $(x, y, z)$ and $(x, y)$ are traditionally used there.

Although we think of points in $\mathbb{R}^n$ as $n$-tuples rather than geometric objects, we do not want to lose all sight of the underlying geometry. By analogy with the two- and three-dimensional cases, we still consider the quantity

$$\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$$

as representing the *distance* between the points with coordinates $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$. Also, we call the $(n-1)$-dimensional set of points in $\mathbb{R}^n$ that satisfy the equation $x_n = 0$ a **hyperplane**, by analogy with the plane $z = 0$ in $\mathbb{R}^3$.

## Describing Sets in the Plane, 3-Space, and $n$-Space

We conclude this section by collecting some definitions of terms used to describe sets of points in $\mathbb{R}^n$ for $n \geq 2$. These terms belong to the branch of mathematics called

**topology**, and they generalize the notions of open and closed intervals and endpoints used to describe sets on the real line $\mathbb{R}$. We state the definitions for $\mathbb{R}^n$, but we are most interested in the cases where $n = 2$ or $n = 3$.

A **neighbourhood** of a point $P$ in $\mathbb{R}^n$ is a set of the form

$$B_r(P) = \{Q \in \mathbb{R}^n : \text{distance from } Q \text{ to } P < r\}$$

for some $r > 0$.

For $n = 1$, if $p \in \mathbb{R}$, then $B_r(p)$ is the **open interval** $(p - r, p + r)$ centred at $p$.

For $n = 2$, $B_r(P)$ is the **open disk** of radius $r$ centred at point $P$.

For $n = 3$, $B_r(P)$ is the **open ball** of radius $r$ centred at point $P$.

A set $S$ is **open** in $\mathbb{R}^n$ if every point of $S$ has a neighbourhood contained in $S$. Every neighbourhood is itself an open set. Other examples of open sets in $\mathbb{R}^2$ include the sets of points $(x, y)$ such that $x > 0$, or such that $y > x^2$, or even such that $y \neq x^2$. Typically, sets defined by strict inequalities (using $>$ and $<$) are open. Examples in $\mathbb{R}^3$ include the sets of points $(x, y, z)$ satisfying $x + y + z > 2$, or $1 < x < 3$.

The whole space $\mathbb{R}^n$ is an open set in itself. For technical reasons, the empty set (containing no points) is also considered to be open. (No point in the empty set fails to have a neighbourhood contained in the empty set.)

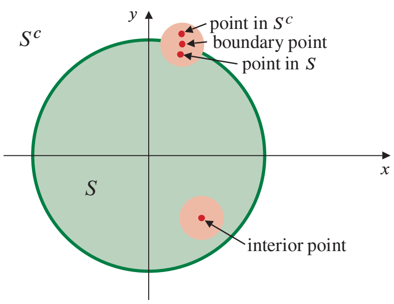



**Figure 10.10** The closed disk $S$ consisting of points $(x, y) \in \mathbb{R}^2$ that satisfy $x^2 + y^2 \le 1$. Note the shaded neighbourhoods of the boundary point and the interior point.
bdry$(S)$ is the circle $x^2 + y^2 = 1$
int$(S)$ is the open disk $x^2 + y^2 < 1$
ext$(S)$ is the open set $x^2 + y^2 > 1$

The **complement**, $S^c$, of a set $S$ in $\mathbb{R}^n$ is the set of all points in $\mathbb{R}^n$ that do not belong to $S$. For example, the complement of the set of points $(x, y)$ in $\mathbb{R}^2$ such that $x > 0$ is the set of points for which $x \le 0$. A set is said to be **closed** if its complement is open. Typically, sets defined by nonstrict inequalities (using $\ge$ and $\le$) are closed. Closed intervals are closed sets in $\mathbb{R}$. Since the whole space and the empty set are both open in $\mathbb{R}^n$ and are complements of each other, they are also both closed. They are the only sets that are both open and closed.

A point $P$ is called a **boundary point** of a set $S$ if every neighbourhood of $P$ contains both points in $S$ and points in $S^c$. The **boundary**, bdry$(S)$, of a set $S$ is the set of all boundary points of $S$. For example, the boundary of the closed disk $x^2 + y^2 \le 1$ in $\mathbb{R}^2$ is the circle $x^2 + y^2 = 1$. A closed set contains all its boundary points. An open set contains none of its boundary points.

A point $P$ is an **interior point** of a set $S$ if it belongs to $S$ but not to the boundary of $S$. $P$ is an **exterior point** of $S$ if it belongs to the complement of $S$ but not to the boundary of $S$. The **interior**, int$(S)$, and **exterior**, ext$(S)$, of $S$ consist of all the interior points and exterior points of $S$, respectively. Both int$(S)$ and ext$(S)$ are open sets. $S$ is open if and only if int$(S) = S$. $S$ is closed if and only if ext$(S) = S^c$. See Figure 10.10.

## EXERCISES 10.1

Find the distance between the pairs of points in Exercises 1–4.

**1.** $(0, 0, 0)$ and $(2, -1, -2)$

**2.** $(-1, -1, -1)$ and $(1, 1, 1)$

**3.** $(1, 1, 0)$ and $(0, 2, -2)$

**4.** $(3, 8, -1)$ and $(-2, 3, -6)$

**5.** What is the shortest distance from the point $(x, y, z)$ to (a) the $xy$-plane? (b) the $x$-axis?

**6.** Show that the triangle with vertices $(1, 2, 3)$, $(4, 0, 5)$, and $(3, 6, 4)$ has a right angle.

**7.** Find the angle $A$ in the triangle with vertices $A = (2, -1, -1)$, $B = (0, 1, -2)$, and $C = (1, -3, 1)$.

**8.** Show that the triangle with vertices $(1, 2, 3)$, $(1, 3, 4)$, and $(0, 3, 3)$ is equilateral.

**9.** Find the area of the triangle with vertices $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$.

**10.** What is the distance from the origin to the point $(1, 1, \ldots, 1)$ in $\mathbb{R}^n$?

**11.** What is the distance from the point $(1, 1, \ldots, 1)$ in $n$-space to the closest point on the $x_1$-axis?

In Exercises 12–23, describe (and sketch if possible) the set of points in $\mathbb{R}^3$ that satisfy the given equation or inequality.

**12.** $z = 2$

**13.** $y \ge -1$

**14.** $z = x$

**15.** $x + y = 1$

**16.** $x^2 + y^2 + z^2 = 4$

**17.** $(x-1)^2+(y+2)^2+(z-3)^2=4$

**18.** $x^2+y^2+z^2=2z$

**19.** $y^2+z^2\le 4$

**20.** $x^2+z^2=4$

**21.** $z=y^2$

**22.** $z\ge\sqrt{x^2+y^2}$

**23.** $x+2y+3z=6$

In Exercises 24–32, describe (and sketch if possible) the set of points in $\mathbb{R}^3$ that satisfy the given pair of equations or inequalities.

**24.** $\begin{cases} x=1 \\ y=2 \end{cases}$

**25.** $\begin{cases} x=1 \\ y=z \end{cases}$

**26.** $\begin{cases} x^2+y^2+z^2=4 \\ z=1 \end{cases}$

**27.** $\begin{cases} x^2+y^2+z^2=4 \\ x^2+y^2+z^2=4x \end{cases}$

**28.** $\begin{cases} x^2+y^2+z^2=4 \\ x^2+z^2=1 \end{cases}$

**29.** $\begin{cases} x^2+y^2=1 \\ z=x \end{cases}$

**30.** $\begin{cases} y\ge x \\ z\le y \end{cases}$

**31.** $\begin{cases} x^2+y^2\le 1 \\ z\ge y \end{cases}$

**32.** $\begin{cases} x^2+y^2+z^2\le 1 \\ \sqrt{x^2+y^2}\le z \end{cases}$

In Exercises 33–36, specify the boundary and the interior of the plane sets $S$ whose points $(x,y)$ satisfy the given conditions. Is $S$ open, closed, or neither?

**33.** $0<x^2+y^2<1$

**34.** $x\ge 0,\quad y<0$

**35.** $x+y=1$

**36.** $|x|+|y|\le 1$

In Exercises 37–40, specify the boundary and the interior of the sets $S$ in 3-space whose points $(x,y,z)$ satisfy the given conditions. Is $S$ open, closed, or neither?

**37.** $1\le x^2+y^2+z^2\le 4$

**38.** $x\ge 0,\quad y>1,\quad z<2$

**39.** $(x-z)^2+(y-z)^2=0$

**40.** $x^2+y^2<1,\ y+z>2$

## 10.2 Vectors

A **vector** is a quantity that involves both **magnitude** (size or length) and **direction**. For instance, the *velocity* of a moving object involves its speed and direction of motion, and so is a vector. Such quantities are represented geometrically by arrows (directed line segments) and are often actually identified with these arrows. For instance, the vector $\overrightarrow{AB}$ is an arrow with tail at the point $A$ and head at the point $B$. In print, such a vector is usually denoted by a single letter in boldface type,

$$\mathbf{v}=\overrightarrow{AB}.$$

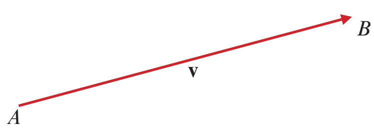



Figure 10.11 The vector $\mathbf{v}=\overrightarrow{AB}$

(See Figure 10.11.) In handwriting, an arrow over a letter ($\vec{v}=\overrightarrow{AB}$) can be used to denote a vector. The *magnitude* of the vector $\mathbf{v}$ is the length of the arrow and is denoted $|\mathbf{v}|$ or $|\overrightarrow{AB}|$.

While vectors have magnitude and direction, they do not generally have *position*; that is, they are not regarded as being in a particular place. Two vectors, $\mathbf{u}$ and $\mathbf{v}$, are considered *equal* if they have *the same length and the same direction*, even if their representative arrows do not coincide. The arrows must be parallel, have the same length, and point in the same direction. In Figure 10.12, for example, if $ABYX$ is a parallelogram, then $\overrightarrow{AB}=\overrightarrow{XY}$.

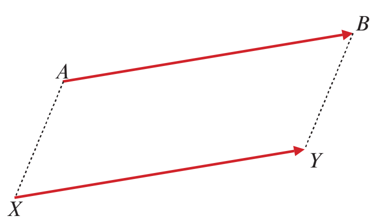



Figure 10.12 $\overrightarrow{AB}=\overrightarrow{XY}$

For the moment we consider plane vectors, that is, vectors whose representative arrows lie in a plane. If we introduce a Cartesian coordinate system into the plane, we can talk about the $x$ and $y$ components of any vector. If $A=(a,b)$ and $P=(p,q)$, as shown in Figure 10.13, then the $x$ and $y$ components of $\overrightarrow{AP}$ are, respectively, $p-a$ and $q-b$. Note that if $O$ is the origin and $X$ is the point $(p-a,q-b)$, then

$$|\overrightarrow{AP}|=\sqrt{(p-a)^2+(q-b)^2}=|\overrightarrow{OX}|$$

$$\text{slope of }\overrightarrow{AP}=\frac{q-b}{p-a}=\text{slope of }\overrightarrow{OX}.$$

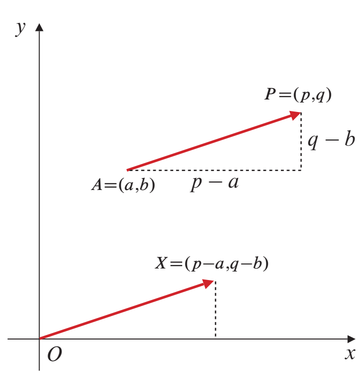



Figure 10.13 Components of a vector

Hence, $\overrightarrow{AP}=\overrightarrow{OX}$. In general, two vectors are equal if and only if they have the same $x$ components and $y$ components.

There are two important algebraic operations defined for vectors: addition and scalar multiplication.

**DEFINITION 1**

**Vector addition**

Given two vectors **u** and **v**, their **sum u + v** is defined as follows. If an arrow representing **v** is placed with its tail at the head of an arrow representing **u**, then an arrow from the tail of **u** to the head of **v** represents **u** + **v**. Equivalently, if **u** and **v** have tails at the same point, then **u** + **v** is represented by an arrow with its tail at that point and its head at the opposite vertex of the parallelogram spanned by **u** and **v**. This is shown in Figure 10.14(a).

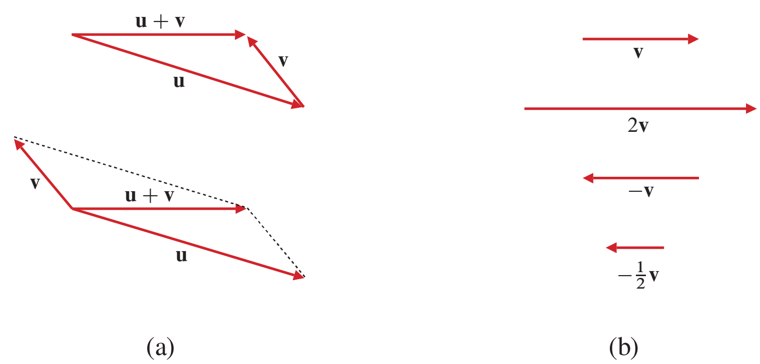



**Figure 10.14**
(a) Vector addition
(b) Scalar multiplication

**DEFINITION 2**

**Scalar multiplication**

If **v** is a vector and $t$ is a real number (also called a **scalar**), then the **scalar multiple** $t$**v** is a vector with magnitude $|t|$ times that of **v** and direction the same as **v** if $t > 0$, or opposite to that of **v** if $t < 0$. See Figure 10.14(b). If $t = 0$, then $t$**v** has zero length and therefore no particular direction. It is the **zero vector**, denoted **0**.

Suppose that **u** has components $a$ and $b$ and that **v** has components $x$ and $y$. Then the components of **u** + **v** are $a + x$ and $b + y$, and those of $t$**v** are $tx$ and $ty$. See Figure 10.15.

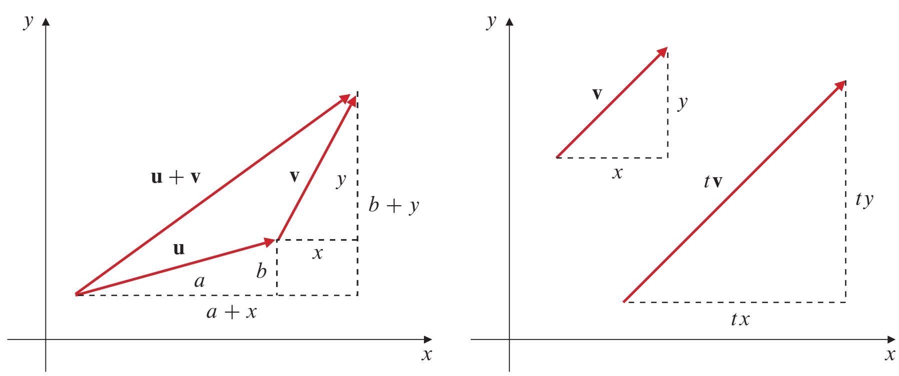



**Figure 10.15** The components of a sum of vectors or a scalar multiple of a vector is the same sum or multiple of the corresponding components of the vectors

In $\mathbb{R}^2$ we single out two particular vectors for special attention. They are

(i) the vector **i** from the origin to the point $(1, 0)$, and

(ii) the vector **j** from the origin to the point $(0, 1)$.

Thus, **i** has components 1 and 0, and **j** has components 0 and 1. These vectors are called the **standard basis vectors** in the plane. The vector **r** from the origin to the point $(x, y)$ has components $x$ and $y$ and can be expressed in the form

$$\mathbf{r} = \langle x, y\rangle = x\mathbf{i} + y\mathbf{j}.$$

In the first form we specify the vector by listing its components between angle brackets; in the second we write **r** as a **linear combination** of the standard basis vectors **i** and **j**. (See Figure 10.16.) The vector **r** is called the **position vector** of the point $(x, y)$. A

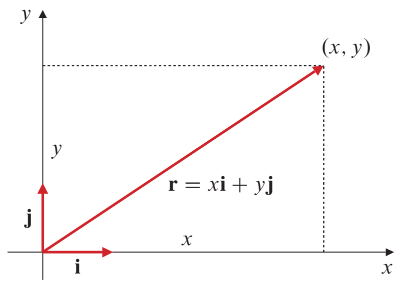



**Figure 10.16** Any vector is a linear combination of the basis vectors

position vector has its tail at the origin and its head at the point whose position it is specifying. The length of $\mathbf{r}$ is $|\mathbf{r}| = \sqrt{x^2 + y^2}$.

More generally, the vector $\overrightarrow{AP}$ from $A = (a, b)$ to $P = (p, q)$ in Figure 10.13 can also be written as a list of components or as a linear combination of the standard basis vectors:

$$\overrightarrow{AP} = \langle p - a, q - b \rangle = (p - a)\mathbf{i} + (q - b)\mathbf{j}.$$

Sums and scalar multiples of vectors are easily expressed in terms of components. If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$, and if $t$ is a scalar (i.e., a real number), then

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (u_1 + v_1)\mathbf{i} + (u_2 + v_2)\mathbf{j}, \\ t\mathbf{u} &= (tu_1)\mathbf{i} + (tu_2)\mathbf{j}. \end{aligned}$$

The zero vector is $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j}$. It has length zero and no specific direction. For any vector $\mathbf{u}$ we have $0\mathbf{u} = \mathbf{0}$. A **unit vector** is a vector of length 1. The standard basis vectors $\mathbf{i}$ and $\mathbf{j}$ are unit vectors. Given any nonzero vector $\mathbf{v}$, we can form a unit vector $\hat{\mathbf{v}}$ in the same direction as $\mathbf{v}$ by multiplying $\mathbf{v}$ by the reciprocal of its length (a scalar):

$$\hat{\mathbf{v}} = \left(\frac{1}{|\mathbf{v}|}\right)\mathbf{v}.$$

**EXAMPLE 1** If $A = (2, -1)$, $B = (-1, 3)$, and $C = (0, 1)$, express each of the following vectors as a linear combination of the standard basis vectors:

(a) $\overrightarrow{AB}$ (b) $\overrightarrow{BC}$ (c) $\overrightarrow{AC}$ (d) $\overrightarrow{AB} + \overrightarrow{BC}$ (e) $2\overrightarrow{AC} - 3\overrightarrow{CB}$

(f) a unit vector in the direction of $\overrightarrow{AB}$.

***Solution***

(a) $\overrightarrow{AB} = (-1 - 2)\mathbf{i} + (3 - (-1))\mathbf{j} = -3\mathbf{i} + 4\mathbf{j}$

(b) $\overrightarrow{BC} = (0 - (-1))\mathbf{i} + (1 - 3)\mathbf{j} = \mathbf{i} - 2\mathbf{j}$

(c) $\overrightarrow{AC} = (0 - 2)\mathbf{i} + (1 - (-1))\mathbf{j} = -2\mathbf{i} + 2\mathbf{j}$

(d) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = -2i + 2\mathbf{j}$

(e) $2\overrightarrow{AC} - 3\overrightarrow{CB} = 2(-2\mathbf{i} + 2\mathbf{j}) - 3(-\mathbf{i} + 2\mathbf{j}) = -\mathbf{i} - 2\mathbf{j}$

(f) A unit vector in the direction of $\overrightarrow{AB}$ is $\dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = -\dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j}$.

Implicit in the above example is the fact that the operations of addition and scalar multiplication obey appropriate algebraic rules, such as

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \mathbf{v} + \mathbf{u}, \\ (\mathbf{u} + \mathbf{v}) + \mathbf{w} &= \mathbf{u} + (\mathbf{v} + \mathbf{w}), \\ \mathbf{u} - \mathbf{v} &= \mathbf{u} + (-1)\mathbf{v}, \\ t(\mathbf{u} + \mathbf{v}) &= t\mathbf{u} + t\mathbf{v}. \end{aligned}$$

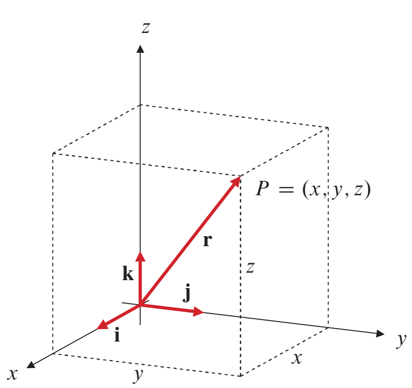



**Figure 10.17** The standard basis vectors **i**, **j**, and **k**

## Vectors in 3-Space

The algebra and geometry of vectors described here extends to spaces of any number of dimensions; we can still think of vectors as represented by arrows, and sums and scalar multiples are formed just as for plane vectors.

Given a Cartesian coordinate system in 3-space, we define three **standard basis vectors**, **i**, **j**, and **k**, represented by arrows from the origin to the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively. (See Figure 10.17.) Any vector in 3-space can be written as a *linear combination* of these basis vectors; for instance, the position vector of the point $(x, y, z)$ is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

We say that $\mathbf{r}$ has **components** $x$, $y$, and $z$. The length of $\mathbf{r}$ is

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}.$$

If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are two points in 3-space, then the vector $\mathbf{v} = \overrightarrow{P_1P_2}$ from $P_1$ to $P_2$ has components $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$ and is therefore represented in terms of the standard basis vectors by

$$\mathbf{v} = \overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

**EXAMPLE 2** If $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, find $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $3\mathbf{u} - 2\mathbf{v}$, $|\mathbf{u}|$, $|\mathbf{v}|$, and a unit vector $\hat{\mathbf{u}}$ in the direction of $\mathbf{u}$.

***Solution***

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= (2+3)\mathbf{i} + (1-2)\mathbf{j} + (-2-1)\mathbf{k} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k} \\
\mathbf{u} - \mathbf{v} &= (2-3)\mathbf{i} + (1+2)\mathbf{j} + (-2+1)\mathbf{k} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k} \\
3\mathbf{u} - 2\mathbf{v} &= (6-6)\mathbf{i} + (3+4)\mathbf{j} + (-6+2)\mathbf{k} = 7\mathbf{j} - 4\mathbf{k} \\
|\mathbf{u}| &= \sqrt{4+1+4} = 3, \qquad |\mathbf{v}| = \sqrt{9+4+1} = \sqrt{14} \\
\hat{\mathbf{u}} &= \left(\frac{1}{|\mathbf{u}|}\right)\mathbf{u} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.
\end{aligned}$$

The following example illustrates the way vectors can be used to solve problems involving relative velocities. If $A$ moves with velocity $\mathbf{v}_{A \text{ rel } B}$ relative to $B$, and $B$ moves with velocity $\mathbf{v}_{B \text{ rel } C}$ relative to $C$, then $A$ moves with velocity $\mathbf{v}_{A \text{ rel } C}$ relative to $C$, where

$$\mathbf{v}_{A \text{ rel } C} = \mathbf{v}_{A \text{ rel } B} + \mathbf{v}_{B \text{ rel } C}.$$

**EXAMPLE 3** An aircraft cruises at a speed of 300 km/h in still air. If the wind is blowing from the east at 100 km/h, in what direction should the aircraft head in order to fly in a straight line from city $P$ to city $Q$, 400 km north-northeast of $P$? How long will the trip take?

***Solution*** The problem is two-dimensional, so we use plane vectors. Let us choose our coordinate system so that the $x$- and $y$-axes point east and north, respectively. Figure 10.18 illustrates the three velocities that must be considered. The velocity of the air relative to the ground is

$$\mathbf{v}_{\text{air rel ground}} = -100\,\mathbf{i}.$$

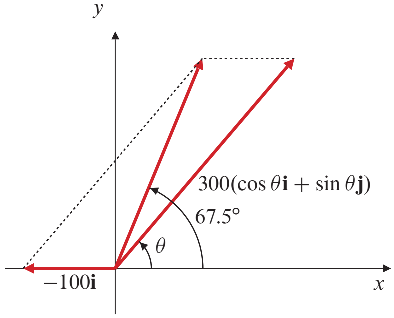


**Figure 10.18** Velocity diagram for the aircraft in Example 3

If the aircraft heads in a direction making angle $\theta$ with the positive direction of the $x$-axis, then the velocity of the aircraft relative to the air is

$$\mathbf{v}_{\text{aircraft rel air}} = 300\cos\theta\,\mathbf{i} + 300\sin\theta\,\mathbf{j}.$$

Thus, the velocity of the aircraft relative to the ground is

$$\begin{aligned}
\mathbf{v}_{\text{aircraft rel ground}} &= \mathbf{v}_{\text{aircraft rel air}} + \mathbf{v}_{\text{air rel ground}} \\
&= (300\cos\theta - 100)\,\mathbf{i} + 300\sin\theta\,\mathbf{j}.
\end{aligned}$$

We want this latter velocity to be in a north-northeasterly direction, that is, in the direction making angle $3\pi/8 = 67.5°$ with the positive direction of the $x$-axis. Thus, we will have

$$\mathbf{v}_{\text{aircraft rel ground}} = v\left[\left(\cos 67.5°\right)\mathbf{i} + \left(\sin 67.5°\right)\mathbf{j}\right],$$

where $v$ is the actual groundspeed of the aircraft. Comparing the two expressions for $\mathbf{v}_{\text{aircraft rel ground}}$ we obtain

$$\begin{aligned} 300\cos\theta - 100 &= v\ \cos 67.5^\circ \\ 300\sin\theta &= v\ \sin 67.5^\circ. \end{aligned}$$

Eliminating $v$ between these two equations we get

$$300\ \cos\theta\ \sin 67.5^\circ - 300\ \sin\theta\ \cos 67.5^\circ = 100\ \sin 67.5^\circ,$$

or

$$3\ \sin(67.5^\circ - \theta) = \sin 67.5^\circ.$$

Therefore, the aircraft should head in direction $\theta$ given by

$$\theta = 67.5^\circ - \arcsin\left(\frac{1}{3}\ \sin 67.5^\circ\right) \approx 49.56^\circ,$$

that is, $49.56^\circ$ north of east. The groundspeed is now seen to be

$$v = 300\sin\theta/\sin 67.5^\circ \approx 247.15\ \text{km/h}.$$

Thus, the 400 km trip will take about $400/247.15 \approx 1.618$ hours, or about 1 hour and 37 minutes.

## Hanging Cables and Chains

When it is suspended from both ends and allowed to hang under gravity, a heavy cable or chain assumes the shape of a **catenary** curve, which is the graph of the hyperbolic cosine function. We will demonstrate this now, using vectors to keep track of the various forces acting on the cable.

Suppose that the cable has line density $\delta$ (units of mass per unit length) and hangs as shown in Figure 10.19. Let us choose a coordinate system so that the lowest point $L$ on the cable is at $(0, y_0)$; we will specify the value of $y_0$ later. If $P = (x, y)$ is another point on the cable, there are three forces acting on the arc $LP$ of the cable between $L$ and $P$. These are all forces that we can represent using horizontal and vertical components.

(i) The horizontal tension $\mathbf{H} = -H\mathbf{i}$ at $L$. This is the force that the part of the cable to the left of $L$ exerts on the arc $LP$ at $L$.

(ii) The tangential tension $\mathbf{T} = T_h\mathbf{i} + T_v\mathbf{j}$. This is the force the part of the cable to the right of $P$ exerts on arc $LP$ at $P$.

(iii) The weight $\mathbf{W} = -\delta g s\mathbf{j}$ of arc $LP$, where $g$ is the acceleration of gravity and $s$ is the length of the arc $LP$.

Since the cable is not moving, these three forces must balance; their vector sum must be zero:

$$\begin{aligned} &\mathbf{T} + \mathbf{H} + \mathbf{W} = \mathbf{0} \\ &(T_h - H)\mathbf{i} + (T_v - \delta g s)\mathbf{j} = \mathbf{0} \end{aligned}$$

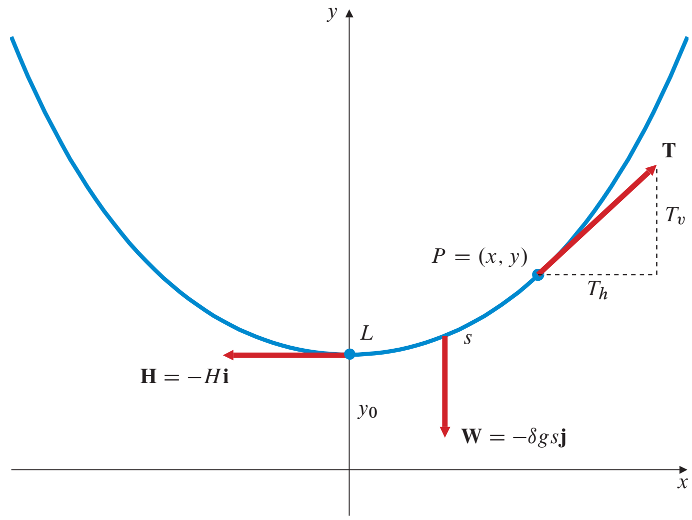



**Figure 10.19** A hanging cable and the forces acting on arc $LP$

Thus, $T_h = H$ and $T_v = \delta g s$. Since $\mathbf{T}$ is tangent to the cable at $P$, the slope of the cable there is

$$\frac{dy}{dx} = \frac{T_v}{T_h} = \frac{\delta g s}{H} = as,$$

where $a = \delta g/H$ is a constant for the given cable. Differentiating with respect to $x$ and using the fact, from our study of arc length, that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

we obtain a second-order differential equation,

$$\frac{d^2y}{dx^2} = a\frac{ds}{dx} = a\sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

to be solved for the equation of the curve along which the hanging cable lies. The appropriate initial conditions are $y = y_0$ and $dy/dx = 0$ at $x = 0$.

Since the differential equation depends on $dy/dx$ rather than $y$, we substitute $m(x) = dy/dx$ and obtain a first-order equation for $m$:

$$\frac{dm}{dx} = a\sqrt{1 + m^2}.$$

This equation is separable; we integrate it using the substitution $m = \sinh u$:

$$\int \frac{1}{\sqrt{1 + m^2}}\, dm = \int a\, dx$$
$$\int du = \int \frac{\cosh u}{\sqrt{1 + \sinh^2 u}}\, du = ax + C_1$$
$$\sinh^{-1} m = u = ax + C_1$$
$$m = \sinh(ax + C_1).$$

Since $m = dy/dx = 0$ at $x = 0$, we have $0 = \sinh C_1$, so $C_1 = 0$ and

$$\frac{dy}{dx} = m = \sinh(ax).$$

This equation is easily integrated to find $y$. (Had we used a tangent substitution instead of the hyperbolic sine substitution for $m$ we would have had more trouble here.)

$$y = \frac{1}{a}\cosh(ax) + C_2.$$

If we choose $y_0 = y(0) = 1/a$, then, substituting $x = 0$, we will get $C_2 = 0$. With this choice of $y_0$, we therefore find that the equation of the curve along which the hanging cable lies is the catenary

$$y = \frac{1}{a}\cosh(ax).$$

***Remark*** If a hanging cable bears loads other than its own weight, it will assume a different shape. For example, a cable supporting a level suspension bridge whose weight per unit length is much greater than that of the cable will assume the shape of a parabola. See Exercise 34 below.

## The Dot Product and Projections

There is another operation on vectors in any dimension by which two vectors are combined to produce a number called their *dot product*.

**DEFINITION 3**

**The dot product of two vectors**

Given two vectors, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ in $\mathbb{R}^2$, we define their **dot product** $\mathbf{u} \bullet \mathbf{v}$ to be the sum of the products of their corresponding components:

$$\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2.$$

The terms **scalar product** and **inner product** are also used in place of dot product. Similarly, for vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ in $\mathbb{R}^3$,

$$\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

The dot product has the following algebraic properties, easily checked using the definition above:

$$\begin{aligned}
&\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u} && \text{(commutative law),}\\
&\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w} && \text{(distributive law),}\\
&(t\mathbf{u}) \bullet \mathbf{v} = \mathbf{u} \bullet (t\mathbf{v}) = t(\mathbf{u} \bullet \mathbf{v}) && \text{(for real } t\text{),}\\
&\mathbf{u} \bullet \mathbf{u} = |\mathbf{u}|^2.
\end{aligned}$$

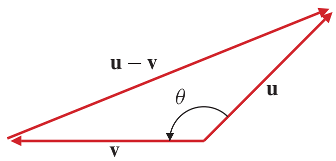
The real significance of the dot product is shown by the following result, which could have been used as the definition of dot product:

**THEOREM 1**

If $\theta$ is the angle between the directions of $\mathbf{u}$ and $\mathbf{v}$ $(0 \le \theta \le \pi)$, then

$$\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta.$$

In particular, $\mathbf{u} \bullet \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular. (Of course, the zero vector is perpendicular to every vector.)



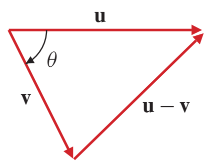




**Figure 10.20** Applying the Cosine Law to a triangle reveals the relationship between dot the product and angle between vectors

**PROOF** Refer to Figure 10.20 and apply the Cosine Law to the triangle with the arrows $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}-\mathbf{v}$ as sides:

$$\begin{aligned}|\mathbf{u}|^2+|\mathbf{v}|^2-2|\mathbf{u}|\,|\mathbf{v}|\cos\theta &= |\mathbf{u}-\mathbf{v}|^2=(\mathbf{u}-\mathbf{v})\bullet(\mathbf{u}-\mathbf{v})\\ &= \mathbf{u}\bullet(\mathbf{u}-\mathbf{v})-\mathbf{v}\bullet(\mathbf{u}-\mathbf{v})\\ &= \mathbf{u}\bullet\mathbf{u}-\mathbf{u}\bullet\mathbf{v}-\mathbf{v}\bullet\mathbf{u}+\mathbf{v}\bullet\mathbf{v}\\ &= |\mathbf{u}|^2+|\mathbf{v}|^2-2\mathbf{u}\bullet\mathbf{v}\end{aligned}$$

Hence, $|\mathbf{u}||\mathbf{v}|\cos\theta=\mathbf{u}\bullet\mathbf{v}$, as claimed.

**EXAMPLE 4** Find the angle $\theta$ between the vectors $\mathbf{u}=2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $\mathbf{v}=3\mathbf{i}-2\mathbf{j}-\mathbf{k}$.

**Solution** Solving the formula $\mathbf{u}\bullet\mathbf{v}=|\mathbf{u}||\mathbf{v}|\cos\theta$ for $\theta$, we obtain

$$\begin{aligned}\theta=\cos^{-1}\frac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} &= \cos^{-1}\left(\frac{(2)(3)+(1)(-2)+(-2)(-1)}{3\sqrt{14}}\right)\\ &= \cos^{-1}\frac{2}{\sqrt{14}}\approx 57.69^\circ.\end{aligned}$$

It is sometimes useful to project one vector along another. We define both scalar and vector projections of $\mathbf{u}$ in the direction of $\mathbf{v}$:

**DEFINITION 4**

**Scalar and vector projections**

The **scalar projection** $s$ of any vector $\mathbf{u}$ in the direction of a nonzero vector $\mathbf{v}$ is the dot product of $\mathbf{u}$ with a unit vector in the direction of $\mathbf{v}$. Thus, it is the *number*

$$s=\frac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{v}|}=|\mathbf{u}|\cos\theta,$$

where $\theta$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

The **vector projection**, $\mathbf{u}_\mathbf{v}$, of $\mathbf{u}$ in the direction of $\mathbf{v}$ (see Figure 10.21) is the scalar multiple of a unit vector $\hat{\mathbf{v}}$ in the direction of $\mathbf{v}$, by the scalar projection of $\mathbf{u}$ in the direction of $\mathbf{v}$; that is,

$$\text{vector projection of } \mathbf{u} \text{ along } \mathbf{v}=\mathbf{u}_\mathbf{v}=\frac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{v}|}\hat{\mathbf{v}}=\frac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}.$$

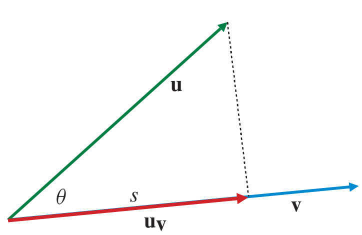



**Figure 10.21** The scalar projection $s$ and the vector projection $\mathbf{u}_\mathbf{v}$ of vector $\mathbf{u}$ along vector $\mathbf{v}$

Note that $|s|$ is the length of the line segment along the line of $\mathbf{v}$ obtained by dropping perpendiculars to that line from the tail and head of $\mathbf{u}$. (See Figure 10.21.) Also, $s$ is negative if $\theta>90^\circ$.

It is often necessary to express a vector as a sum of two other vectors parallel and perpendicular to a given direction.

**EXAMPLE 5** Express the vector $3\mathbf{i}+\mathbf{j}$ as a sum of vectors $\mathbf{u}+\mathbf{v}$, where $\mathbf{u}$ is parallel to the vector $\mathbf{i}+\mathbf{j}$ and $\mathbf{v}$ is perpendicular to $\mathbf{u}$.

**Solution**

**METHOD I (Using vector projection)** Note that $\mathbf{u}$ must be the vector projection of $3\mathbf{i}+\mathbf{j}$ in the direction of $\mathbf{i}+\mathbf{j}$. Thus,

$$\begin{aligned}\mathbf{u}&=\frac{(3\mathbf{i}+\mathbf{j})\bullet(\mathbf{i}+\mathbf{j})}{|\mathbf{i}+\mathbf{j}|^2}(\mathbf{i}+\mathbf{j})=\frac{4}{2}(\mathbf{i}+\mathbf{j})=2\mathbf{i}+2\mathbf{j}\\ \mathbf{v}&=3\mathbf{i}+\mathbf{j}-\mathbf{u}=\mathbf{i}-\mathbf{j}.\end{aligned}$$

**METHOD II (From basic principles)** Since $\mathbf{u}$ is parallel to $\mathbf{i}+\mathbf{j}$ and $\mathbf{v}$ is perpendicular to $\mathbf{u}$, we have

$$\mathbf{u} = t(\mathbf{i}+\mathbf{j}) \qquad \text{and} \qquad \mathbf{v} \bullet (\mathbf{i}+\mathbf{j}) = 0,$$

for some scalar $t$. We want $\mathbf{u}+\mathbf{v} = 3\mathbf{i}+\mathbf{j}$. Take the dot product of this equation with $\mathbf{i}+\mathbf{j}$:

$$\begin{aligned} &\mathbf{u} \bullet (\mathbf{i}+\mathbf{j}) + \mathbf{v} \bullet (\mathbf{i}+\mathbf{j}) = (3\mathbf{i}+\mathbf{j}) \bullet (\mathbf{i}+\mathbf{j}) \\ &t(\mathbf{i}+\mathbf{j}) \bullet (\mathbf{i}+\mathbf{j}) + 0 = 4. \end{aligned}$$

Thus $2t = 4$, so $t = 2$. Therefore,

$$\mathbf{u} = 2\mathbf{i}+2\mathbf{j} \qquad \text{and} \qquad \mathbf{v} = 3\mathbf{i}+\mathbf{j}-\mathbf{u} = \mathbf{i}-\mathbf{j}.$$

## Vectors in *n*-Space

All the above ideas make sense for vectors in spaces of any dimension. Vectors in $\mathbb{R}^n$ can be expressed as linear combinations of the $n$ unit vectors

$$\begin{array}{lll} \mathbf{e}_1 & \text{from the origin to the point} & (1,0,0,\ldots,0) \\ \mathbf{e}_2 & \text{from the origin to the point} & (0,1,0,\ldots,0) \\ & \vdots & \\ \mathbf{e}_n & \text{from the origin to the point} & (0,0,0,\ldots,1). \end{array}$$

These vectors constitute a *standard basis* in $\mathbb{R}^n$. The $n$-vector $x$ with components $x_1, x_2, \ldots, x_n$ is expressed in the form

$$x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

The length of $\mathbf{x}$ is $|x| = \sqrt{x_1{}^2 + x_2{}^2 + \cdots + x_n{}^2}$. The angle between two vectors $\mathbf{x}$ and $\mathbf{y}$ is

$$\theta = \cos^{-1} \frac{x \bullet y}{|x||y|},$$

where

$$x \bullet y = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

We will not make much use of $n$-vectors for $n > 3$, but you should be aware that everything said up until now for 2-vectors or 3-vectors extends to $n$-vectors.

## EXERCISES 10.2

**1.** Let $A = (-1,2)$, $B = (2,0)$, $C = (1,-3)$, $D = (0,4)$. Express each of the following vectors as a linear combination of the standard basis vectors $\mathbf{i}$ and $\mathbf{j}$ in $\mathbb{R}^2$.

(a) $\overrightarrow{AB}$, (b) $\overrightarrow{BA}$, (c) $\overrightarrow{AC}$, (d) $\overrightarrow{BD}$, (e) $\overrightarrow{DA}$,

(f) $\overrightarrow{AB} - \overrightarrow{BC}$, (g) $\overrightarrow{AC} - 2\overrightarrow{AB} + 3\overrightarrow{CD}$, and

(h) $\dfrac{\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}}{3}$.

In Exercises 2–3, calculate the following for the given vectors $\mathbf{u}$ and $\mathbf{v}$:

(a) $\mathbf{u}+\mathbf{v}$, $\quad \mathbf{u}-\mathbf{v}$, $\quad 2\mathbf{u}-3\mathbf{v}$,

(b) the lengths $|\mathbf{u}|$ and $|\mathbf{v}|$,

(c) unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ in the directions of $\mathbf{u}$ and $\mathbf{v}$, respectively,

(d) the dot product $\mathbf{u} \bullet \mathbf{v}$,

(e) the angle between $\mathbf{u}$ and $\mathbf{v}$,

(f) the scalar projection of $\mathbf{u}$ in the direction of $\mathbf{v}$,

(g) the vector projection of $\mathbf{v}$ along $\mathbf{u}$.

**2.** $\mathbf{u} = \mathbf{i} - \mathbf{j}$ and $\mathbf{v} = \mathbf{j} + 2\mathbf{k}$

**3.** $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$

**4.** Use vectors to show that the triangle with vertices $(-1, 1)$, $(2, 5)$, and $(10, -1)$ is right-angled.

In Exercises 5–8, prove the stated geometric result using vectors.

**5.** The line segment joining the midpoints of two sides of a triangle is parallel to and half as long as the third side.

**6.** If $P$, $Q$, $R$, and $S$ are midpoints of sides $AB$, $BC$, $CD$, and $DA$, respectively, of quadrilateral $ABCD$, then $PQRS$ is a parallelogram.

**7.** The diagonals of any parallelogram bisect each other.

**8.** The medians of any triangle meet in a common point. (A median is a line joining one vertex to the midpoint of the opposite side. The common point is the *centroid* of the triangle.)

**9.** A weather vane mounted on the top of a car moving due north at 50 km/h indicates that the wind is coming from the west. When the car doubles its speed, the weather vane indicates that the wind is coming from the northwest. From what direction is the wind coming, and what is its speed?

**10.** A straight river 500 m wide flows due east at a constant speed of 3 km/h. If you can row your boat at a speed of 5 km/h in still water, in what direction should you head if you wish to row from point $A$ on the south shore to point $B$ on the north shore directly north of $A$? How long will the trip take?

**11.** In what direction should you head to cross the river in Exercise 10 if you can only row at 2 km/h, and you wish to row from $A$ to point $C$ on the north shore, $k$ km downstream from $B$? For what values of $k$ is the trip not possible?

**12.** A certain aircraft flies with an airspeed of 750 km/h. In what direction should it head in order to make progress in a true easterly direction if the wind is from the northeast at 100 km/h? How long will it take to complete a trip to a city 1,500 km from its starting point?

**13.** For what value of $t$ is the vector $2t\mathbf{i} + 4\mathbf{j} - (10 + t)\mathbf{k}$ perpendicular to the vector $\mathbf{i} + t\mathbf{j} + \mathbf{k}$?

**14.** Find the angle between a diagonal of a cube and one of the edges of the cube.

**15.** Find the angle between a diagonal of a cube and a diagonal of one of the faces of the cube. Give all possible answers.

**16.** **(Direction cosines)** If a vector $\mathbf{u}$ in $\mathbb{R}^3$ makes angles $\alpha$, $\beta$, and $\gamma$ with the coordinate axes, show that

$$\hat{\mathbf{u}} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

is a unit vector in the direction of $\mathbf{u}$, so $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. The numbers $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines* of $\mathbf{u}$.

**17.** Find a unit vector that makes equal angles with the three coordinate axes.

**18.** Find the three angles of the triangle with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$.

**19.** If $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of two points, $P_1$ and $P_2$, and $\lambda$ is a real number, show that

$$\mathbf{r} = (1 - \lambda)\mathbf{r}_1 + \lambda\mathbf{r}_2$$

is the position vector of a point $P$ on the straight line joining $P_1$ and $P_2$. Where is $P$ if $\lambda = 1/2$? if $\lambda = 2/3$? if $\lambda = -1$? if $\lambda = 2$?

**20.** Let $\mathbf{a}$ be a nonzero vector. Describe the set of all points in 3-space whose position vectors $\mathbf{r}$ satisfy $\mathbf{a} \bullet \mathbf{r} = 0$.

**21.** Let $\mathbf{a}$ be a nonzero vector, and let $b$ be any real number. Describe the set of all points in 3-space whose position vectors $\mathbf{r}$ satisfy $\mathbf{a} \bullet \mathbf{r} = b$.

In Exercises 22–24, $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

**22.** Find two unit vectors each of which is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

**23.** Find a vector $\mathbf{x}$ satisfying the system of equations $\mathbf{x} \bullet \mathbf{u} = 9$, $\mathbf{x} \bullet \mathbf{v} = 4$, $\mathbf{x} \bullet \mathbf{w} = 6$.

**24.** Find two unit vectors each of which makes equal angles with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

**25.** Find a unit vector that bisects the angle between any two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$.

**26.** Given two nonparallel vectors $\mathbf{u}$ and $\mathbf{v}$, describe the set of all points whose position vectors $\mathbf{r}$ are of the form $\mathbf{r} = \lambda\mathbf{u} + \mu\mathbf{v}$, where $\lambda$ and $\mu$ are arbitrary real numbers.

**27.** **(The triangle inequality)** Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors.

(a) Show that $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2\mathbf{u} \bullet \mathbf{v} + |\mathbf{v}|^2$.

(b) Show that $\mathbf{u} \bullet \mathbf{v} \le |\mathbf{u}||\mathbf{v}|$.

(c) Deduce from (a) and (b) that $|\mathbf{u} + \mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$.

**28.** (a) Why is the inequality in Exercise 27(c) called a triangle inequality?

(b) What conditions on $\mathbf{u}$ and $\mathbf{v}$ imply that $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$?

**29.** **(Orthonormal bases)** Let $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$, $\mathbf{v} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$, and $\mathbf{w} = \mathbf{k}$.

(a) Show that $|\mathbf{u}| = |\mathbf{v}| = |\mathbf{w}| = 1$ and $\mathbf{u} \bullet \mathbf{v} = \mathbf{u} \bullet \mathbf{w} = \mathbf{v} \bullet \mathbf{w} = 0$. The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular unit vectors and as such are said to constitute an **orthonormal basis** for $\mathbb{R}^3$.

(b) If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, show by direct calculation that

$$\mathbf{r} = (\mathbf{r} \bullet \mathbf{u})\mathbf{u} + (\mathbf{r} \bullet \mathbf{v})\mathbf{v} + (\mathbf{r} \bullet \mathbf{w})\mathbf{w}.$$

**30.** Show that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any three mutually perpendicular unit vectors in $\mathbb{R}^3$ and $\mathbf{r} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$, then $a = \mathbf{r} \bullet \mathbf{u}$, $b = \mathbf{r} \bullet \mathbf{v}$, and $c = \mathbf{r} \bullet \mathbf{w}$.

**31.** **(Resolving a vector in perpendicular directions)** If $\mathbf{a}$ is a nonzero vector and $\mathbf{w}$ is any vector, find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{w} = \mathbf{u} + \mathbf{v}$, $\mathbf{u}$ is parallel to $\mathbf{a}$, and $\mathbf{v}$ is perpendicular to $\mathbf{a}$.

**32.** **(Expressing a vector as a linear combination of two other vectors with which it is coplanar)** Suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{r}$ are position vectors of points $U$, $V$, and $P$, respectively, that $\mathbf{u}$ is not parallel to $\mathbf{v}$, and that $P$ lies in the plane containing the origin, $U$, and $V$. Show that there exist numbers $\lambda$ and $\mu$ such that $\mathbf{r} = \lambda\mathbf{u} + \mu\mathbf{v}$. *Hint:* Resolve both $\mathbf{v}$ and $\mathbf{r}$ as sums of vectors parallel and perpendicular to $\mathbf{u}$ as suggested in Exercise 31.

**33.** Given constants $r$, $s$, and $t$, with $r \neq 0$ and $s \neq 0$, and given a vector $\mathbf{a}$ satisfying $|\mathbf{a}|^2 > 4rst$, solve the system of equations

$$\begin{cases} r\mathbf{x} + s\mathbf{y} = \mathbf{a} \\ \mathbf{x} \bullet \mathbf{y} = t \end{cases}$$

for the unknown vectors $\mathbf{x}$ and $\mathbf{y}$.

**Hanging cables**

**34.** **(A suspension bridge)** If a hanging cable is supporting weight with constant horizontal line density (so that the weight supported by the arc $LP$ in Figure 10.19 is $\delta g x$ rather than $\delta g s$), show that the cable assumes the shape of a parabola rather than a catenary. Such is likely to be the case for the cables of a suspension bridge.

**35.** At a point $P$, 10 m away horizontally from its lowest point $L$, a cable makes an angle $55°$ with the horizontal. Find the length of the cable between $L$ and $P$.

**36.** Calculate the length $s$ of the arc $LP$ of the hanging cable in Figure 10.19 using the equation $y = (1/a)\cosh(ax)$ obtained for the cable. Hence, verify that the magnitude $T = |\mathbf{T}|$ of the tension in the cable at any point $P = (x, y)$ is $T = \delta g y$.

**37.** A cable 100 m long hangs between two towers 90 m apart so that its ends are attached at the same height on the two towers. How far below that height is the lowest point on the cable?

# 10.3 The Cross Product in 3-Space

There is defined, *in 3-space only*, another kind of product of two vectors called a *cross product* or *vector product*, and denoted $\mathbf{u} \times \mathbf{v}$.

**DEFINITION 5**

For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is the unique vector satisfying the following three conditions:

(i) $(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{u} = 0$ and $(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{v} = 0$,

(ii) $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$, where $\theta$ is the angle between $\mathbf{u}$ and $\mathbf{v}$, and

(iii) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ form a right-handed triad.

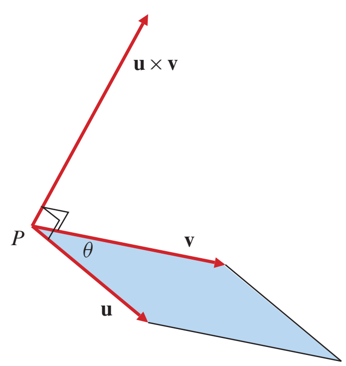



**Figure 10.22** $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ and has length equal to the area of the blue shaded parallelogram

If $\mathbf{u}$ and $\mathbf{v}$ are parallel, condition (ii) says that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the zero vector. Otherwise, through any point in $\mathbb{R}^3$ there is a unique straight line that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. Condition (i) says that $\mathbf{u} \times \mathbf{v}$ is parallel to this line. Condition (iii) determines which of the two directions along this line is the direction of $\mathbf{u} \times \mathbf{v}$; a right-handed screw advances in the direction of $\mathbf{u} \times \mathbf{v}$ if rotated in the direction from $\mathbf{u}$ toward $\mathbf{v}$. (This is equivalent to saying that the thumb, forefinger, and middle finger of the right hand can be made to point in the directions of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$, respectively.)

If $\mathbf{u}$ and $\mathbf{v}$ have their tails at the point $P$, then $\mathbf{u} \times \mathbf{v}$ is normal (i.e., perpendicular) to the plane through $P$ in which $\mathbf{u}$ and $\mathbf{v}$ lie and, by condition (ii), $\mathbf{u} \times \mathbf{v}$ has length equal to the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$. (See Figure 10.22.) These properties make the cross product very useful for the description of tangent planes and normal lines to surfaces in $\mathbb{R}^3$.

The definition of cross product given above does not involve any coordinate system and therefore does not directly show the components of the cross product with respect to the standard basis. These components are provided by the following theorem.

**THEOREM 2**

**Components of the cross product**

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

**PROOF** First, we observe that the vector

$$\mathbf{w} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ since

$$\mathbf{u} \bullet \mathbf{w} = u_1(u_2v_3 - u_3v_2) + u_2(u_3v_1 - u_1v_3) + u_3(u_1v_2 - u_2v_1) = 0,$$

and similarly $\mathbf{v} \bullet \mathbf{w} = 0$. Thus, $\mathbf{u}\times\mathbf{v}$ is parallel to $\mathbf{w}$. Next, we show that $\mathbf{w}$ and $\mathbf{u}\times\mathbf{v}$ have the same length. In fact,

$$\begin{aligned}|\mathbf{w}|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2\\ &= u_2^2v_3^2 + u_3^2v_2^2 - 2u_2v_3u_3v_2 + u_3^2v_1^2 + u_1^2v_3^2\\ &\quad - 2u_3v_1u_1v_3 + u_1^2v_2^2 + u_2^2v_1^2 - 2u_1v_2u_2v_1,\end{aligned}$$

while

$$\begin{aligned}|\mathbf{u}\times\mathbf{v}|^2 &= |\mathbf{u}|^2|\mathbf{v}|^2\sin^2\theta\\ &= |\mathbf{u}|^2|\mathbf{v}|^2(1-\cos^2\theta)\\ &= |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u}\bullet\mathbf{v})^2\\ &= (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2\\ &= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2\\ &\quad - u_1^2v_1^2 - u_2^2v_2^2 - u_3^2v_3^2 - 2u_1v_1u_2v_2 - 2u_1v_1u_3v_3 - 2u_2v_2u_3v_3\\ &= |\mathbf{w}|^2.\end{aligned}$$

Since $\mathbf{w}$ is parallel to, and has the same length as, $\mathbf{u}\times\mathbf{v}$, we must have either $\mathbf{u}\times\mathbf{v} = \mathbf{w}$ or $\mathbf{u}\times\mathbf{v} = -\mathbf{w}$. It remains to be shown that the first of these is the correct choice. To see this, suppose that the triad of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is rigidly rotated in 3-space so that $\mathbf{u}$ points in the direction of the positive $x$-axis and $\mathbf{v}$ lies in the upper half of the $xy$-plane. Then $\mathbf{u} = u_1\mathbf{i}$, and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$, where $u_1 > 0$ and $v_2 > 0$. By the "right-hand rule" $\mathbf{u}\times\mathbf{v}$ must point in the direction of the positive $z$-axis. But $\mathbf{w} = u_1v_2\mathbf{k}$ does point in that direction, so $\mathbf{u}\times\mathbf{v} = \mathbf{w}$, as asserted.

The formula for the cross product in terms of components may seem awkward and asymmetric. As we shall see, however, it can be written more easily in terms of a determinant. We introduce determinants later in this section.

**EXAMPLE 1** **(Calculating cross products)**

(a)
$$\begin{aligned}&\mathbf{i}\times\mathbf{i} = \mathbf{0}, &\quad &\mathbf{i}\times\mathbf{j} = \mathbf{k}, &\quad &\mathbf{j}\times\mathbf{i} = -\mathbf{k},\\ &\mathbf{j}\times\mathbf{j} = \mathbf{0}, & &\mathbf{j}\times\mathbf{k} = \mathbf{i}, & &\mathbf{k}\times\mathbf{j} = -\mathbf{i},\\ &\mathbf{k}\times\mathbf{k} = \mathbf{0}, & &\mathbf{k}\times\mathbf{i} = \mathbf{j}, & &\mathbf{i}\times\mathbf{k} = -\mathbf{j}.\end{aligned}$$

(b)
$$\begin{aligned}&(2\mathbf{i} + \mathbf{j} - 3\mathbf{k})\times(-2\mathbf{j} + 5\mathbf{k})\\ &\quad = \big((1)(5) - (-2)(-3)\big)\mathbf{i} + \big((-3)(0) - (2)(5)\big)\mathbf{j} + \big((2)(-2) - (1)(0)\big)\mathbf{k}\\ &\quad = -\mathbf{i} - 10\mathbf{j} - 4\mathbf{k}.\end{aligned}$$

The cross product has some but not all of the properties we usually ascribe to products. We summarize its algebraic properties as follows:

**Properties of the cross product**

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors in $\mathbb{R}^3$, and $t$ is a real number (a scalar), then

(i) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$,

(ii) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$, (The cross product is **anticommutative**.)

(iii) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$,

(iv) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$,

(v) $(t\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (t\mathbf{v}) = t(\mathbf{u} \times \mathbf{v})$,

(vi) $\mathbf{u} \bullet (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \bullet (\mathbf{u} \times \mathbf{v}) = 0$.

These identities are all easily verified using the components or the definition of the cross product or by using properties of determinants discussed below. They are left as exercises for the reader. Note the absence of an associative law. The cross product is not associative. (See Exercise 21 at the end of this section.) In general,

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}.$$

## Determinants

In order to simplify certain formulas such as the component representation of the cross product, we introduce $2 \times 2$ and $3 \times 3$ **determinants**. General $n \times n$ determinants are normally studied in courses on linear algebra; we will encounter them in Section 10.7 and later chapters. Here we will outline enough of the properties of determinants to enable us to use them as shorthand in some otherwise complicated formulas.

A determinant is an expression that involves the elements of a square array (matrix) of numbers. The determinant of the $2 \times 2$ array of numbers

$$\begin{matrix} a & b \\ c & d \end{matrix}$$

is denoted by enclosing the array between vertical bars, and its value is the number $ad - bc$:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

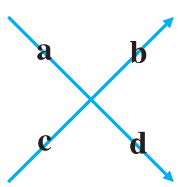



**Figure 10.23** Upward and downward diagonals

This is the product of elements in the *downward diagonal* of the array minus the product of elements in the *upward diagonal*, as shown in Figure 10.23. For example,

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2.$$

Similarly, the determinant of a $3 \times 3$ array of numbers is defined by

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - gec - hfa - idb.$$

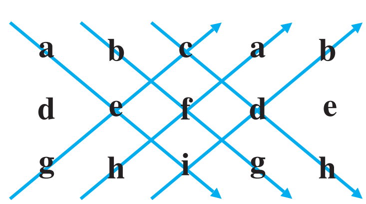



**Figure 10.24** WARNING: This method does not work for 4×4 or higher-order determinants!

Observe that each of the six products in the value of the determinant involves exactly one element from each row and exactly one from each column of the array. As such, each term is the product of elements in a *diagonal* of an *extended* array obtained by repeating the first two columns of the array to the right of the third column, as shown in Figure 10.24. The value of the determinant is the sum of products corresponding to the three complete *downward* diagonals minus the sum corresponding to the three *upward* diagonals. With practice you will be able to form these diagonal products without having to write the extended array.

If we group the terms in the expansion of the determinant to factor out the elements of the first row, we obtain

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$
$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

The $2\times2$ determinants appearing here (called *minors* of the given $3\times3$ determinant) are obtained by deleting the row and column containing the corresponding element from the original $3 \times 3$ determinant. This process is called *expanding* the $3\times3$ determinant *in minors* about the first row.

Such expansions in minors can be carried out about any row or column. Note that if $i + j$ is an *odd* number, a minus sign appears in a term obtained by multiplying the element in the $i$th row and $j$th column and its corresponding minor obtained by deleting that row and column. For example, we can expand the above determinant in minors about the second column as follows:

The pattern of $+$ and $-$ signs used with the terms of an expansion in minors of a $3 \times 3$ determinant is given by

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix}$$
$$= -bdi + bfg + eai - ecg - haf + hcd.$$

(Of course, this is the same value as the one obtained previously.)

**EXAMPLE 2**

$$\begin{vmatrix} 1 & 4 & -2 \\ -3 & 1 & 0 \\ 2 & 2 & -3 \end{vmatrix} = 3\begin{vmatrix} 4 & -2 \\ 2 & -3 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix}$$
$$= 3(-8) + 1 = -23.$$

We expanded about the second row; the third column would also have been a good choice. (Why?)

Any row (or column) of a determinant may be regarded as the components of a vector. Then the determinant is a *linear function* of that vector. For example,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ sx + tl & sy + tm & sz + tn \end{vmatrix} = s\begin{vmatrix} a & b & c \\ d & e & f \\ x & y & z \end{vmatrix} + t\begin{vmatrix} a & b & c \\ d & e & f \\ l & m & n \end{vmatrix}$$

because the determinant is a linear function of its third row. This and other properties of determinants follow directly from the definition. Some other properties are summarized below. These are stated for rows and for $3\times3$ determinants, but similar statements can be made for columns and for determinants of any order.

**Properties of determinants**

(i) If two rows of a determinant are interchanged, then the determinant changes sign:

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

(ii) If two rows of a determinant are equal, the determinant has value 0:

$$\begin{vmatrix} a & b & c \\ a & b & c \\ g & h & i \end{vmatrix} = 0.$$

(iii) If a multiple of one row of a determinant is added to another row, the value of the determinant remains unchanged:

$$\begin{vmatrix} a & b & c \\ d+ta & e+tb & f+tc \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

## The Cross Product as a Determinant

The elements of a determinant are usually numbers because they have to be multiplied to get the value of the determinant. However, it is possible to use vectors as the elements of *one row* (or column) of a determinant. When expanding in minors about that row (or column), the minor for each vector element is a number that determines the scalar multiple of the vector. The formula for the cross product of

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \quad \text{and} \quad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

presented in Theorem 2 can be expressed symbolically as a determinant with the standard basis vectors as the elements of the first row:

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}.$$

The formula for the cross product given in that theorem is just the expansion of this determinant in minors about the first row.

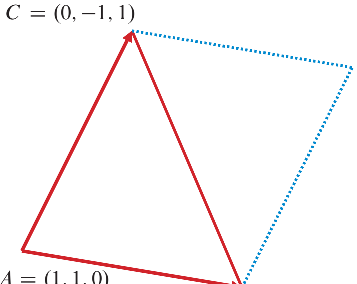



Figure 10.25

**EXAMPLE 3** Find the area of the triangle with vertices at the three points $A = (1, 1, 0)$, $B = (3, 0, 2)$, and $C = (0, -1, 1)$.

***Solution*** Two sides of the triangle (Figure 10.25) are given by the vectors

$$\overrightarrow{AB} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k} \qquad \text{and} \qquad \overrightarrow{AC} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

The area of the triangle is half the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.