Nonlinear Programming
2nd Edition

Solutions Manual

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NOTE

This solutions manual is continuously updated and improved. Portions of the manual, involving primarily theoretical exercises, have been posted on the internet at the book's www page

http://www.athenasc.com/nonlinbook.html

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Solutions Chapter 1

SECTION 1.1

1.1.9 (www)

For any $x, y \in \mathcal{R}^n$, from the second order expansion (see Appendix A, Proposition A.23) we have

$$f(y) - f(x) = (y - x)'\nabla f(x) + \frac{1}{2}(y - x)'\nabla^2 f(z)(y - x), \tag{1}$$

where z is some point of the line segment joining x and y. Setting $x = 0$ in (1) and using the given property of f, it can be seen that f is coercive. Therefore, there exists $x^* \in \mathcal{R}^n$ such that $f(x^*) = \inf_{x \in \mathcal{R}^n} f(x)$ (see Proposition A.8 in Appendix A). The condition

$$m||y||^2 \le y'\nabla^2 f(x)y, \qquad \forall\ x, y \in \mathcal{R}^n,$$

is equivalent to strong convexity of f. Strong convexity guarantees that there is a unique global minimum x^*. By using the given property of f and the expansion (1), we obtain

$$(y - x)'\nabla f(x) + \frac{m}{2}||y - x||^2 \le f(y) - f(x) \le (y - x)'\nabla f(x) + \frac{M}{2}||y - x||^2.$$

Taking the minimum over $y \in \mathcal{R}^n$ in the expression above gives

$$\min_{y \in \mathcal{R}^n} \left((y - x)'\nabla f(x) + \frac{m}{2}||y - x||^2 \right) \le f(x^*) - f(x) \le \min_{y \in \mathcal{R}^n} \left((y - x)'\nabla f(x) + \frac{M}{2}||y - x||^2 \right).$$

Note that for any $a > 0$

$$\min_{y \in \mathcal{R}^n} \left((y - x)'\nabla f(x) + \frac{a}{2}||y - x||^2 \right) = -\frac{1}{2a}||\nabla f(x)||^2,$$

and the minimum is attained for $y = x - \frac{\nabla f(x)}{a}$. Using this relation for $a = m$ and $a = M$, we obtain

$$-\frac{1}{2m}||\nabla f(x)||^2 \le f(x^*) - f(x) \le -\frac{1}{2M}||\nabla f(x)||^2.$$

The first chain of inequalities follows from here. To show the second relation, use the expansion (1) at the point $x = x^*$, and note that $\nabla f(x^*) = 0$, so that

$$f(y) - f(x^*) = \frac{1}{2}(y - x^*)'\nabla^2 f(z)(y - x^*).$$

The rest follows immediately from here and the given property of the function f.

1.1.11 (www)

Since x^* is a nonsingular strict local minimum, we have that $\nabla^2 f(x^*) > 0$. The function f is twice continuously differentiable over $\Re^n$, so that there exists a scalar $\delta > 0$ such that

$$\nabla^2 f(x) > 0, \qquad \forall\ x, \ \text{ with } \ ||x - x^*|| \leq \delta.$$

This means that the function f is strictly convex over the open sphere $B(x^*, \delta)$ centered at x^* with radius δ. Then according to Proposition 1.1.2, x^* is the only stationary point of f in the sphere $B(x^*, \delta)$.

If f is not twice continuously differentiable, then x^* need not be an isolated stationary point. The example function f does not have the second derivative at $x = 0$. Note that $f(x) > 0$ for $x \neq 0$, and by definition $f(0) = 0$. Hence, $x^* = 0$ is the unique (singular) global minimum. The first derivative of $f(x)$ for $x \neq 0$ can be calculated as follows:

$$\begin{aligned}
f'(x) &= 2x\left(\sqrt{2} - \sin\left(\frac{5\pi}{6} - \sqrt{3}\ln(x^2)\right) + \sqrt{3}\cos\left(\frac{5\pi}{6} - \sqrt{3}\ln(x^2)\right)\right) \\
&= 2x\left(\sqrt{2} - 2\cos\frac{\pi}{3}\sin\left(\frac{5\pi}{6} - \sqrt{3}\ln(x^2)\right) + 2\sin\frac{\pi}{3}\cos\left(\frac{5\pi}{6} - \sqrt{3}\ln(x^2)\right)\right) \\
&= 2x\left(\sqrt{2} + 2\sin\left(\frac{\pi}{3} - \frac{5\pi}{6} + \sqrt{3}\ln(x^2)\right)\right) \\
&= 2x\left(\sqrt{2} - 2\cos(2\sqrt{3}\ln x)\right).
\end{aligned}$$

Solving $f'(x) = 0$, gives $x^k = e^{\frac{(1-8k)\pi}{8\sqrt{3}}}$ and $y^k = e^{\frac{-(1+8k)\pi}{8\sqrt{3}}}$ for k integer. The second derivative of $f(x)$, for $x \neq 0$, is given by

$$f''(x) = 2\left(\sqrt{2} - 2\cos(2\sqrt{3}\ln x) + 4\sqrt{3}\sin(2\sqrt{3}\ln x)\right).$$

Thus:

$$\begin{aligned}
f''(x^k) &= 2\left(\sqrt{2} - 2\cos\frac{\pi}{4} + 4\sqrt{3}\sin\frac{\pi}{4}\right) \\
&= 2\left(\sqrt{2} - 2\frac{\sqrt{2}}{2} + 4\sqrt{3}\frac{\sqrt{2}}{2}\right) \\
&= 4\sqrt{6}.
\end{aligned}$$

Similarly

$$\begin{aligned}
f''(y^k) &= = 2\left(\sqrt{2} - 2\cos\left(\frac{-\pi}{4}\right) + 4\sqrt{3}\sin\left(\frac{-\pi}{4}\right)\right) \\
&= 2\left(\sqrt{2} - 2\frac{\sqrt{2}}{2} - 4\sqrt{3}\frac{\sqrt{2}}{2}\right) \\
&= -4\sqrt{6}.
\end{aligned}$$

Hence, $\{x^k \mid k \geq 0\}$ is a sequence of nonsingular local minima, which evidently converges to x^*, while $\{y^k \mid k \geq 0\}$ is a sequence of nonsingular local maxima converging to x^*.

1.1.12 (www)

(a) Let x^* be a strict local minimum of f. Then there is δ such that $f(x^*) < f(x)$ for all x in the closed sphere centered at x^* with radius δ. Take any local sequence $\{x^k\}$ that minimizes f, i.e. $||x^k - x^*|| \leq \delta$ and $\lim_{k\to\infty} f(x^k) = f(x^*)$. Then there is a subsequence $\{x^{k_i}\}$ and the point $\overline{x}$ such that $x^{k_i} \to \overline{x}$ and $||\overline{x} - x^*|| \leq \delta$. By continuity of f, we have

$$f(\overline{x}) = \lim_{i\to\infty} f(x^{k_i}) = f(x^*).$$

Since x^* is a strict local minimum, it follows that $\overline{x} = x^*$. This is true for any convergent subsequence of $\{x^k\}$, therefore $\{x^k\}$ converges to x^*, which means that x^* is locally stable. Next we will show that for a continuous function f every locally stable local minimum must be strict. Assume that this is not true, i.e., there is a local minimum x^* which is locally stable but is not strict. Then for any $\theta > 0$ there is a point $x^\theta \neq x^*$ such that

$$0 < ||x^\theta - x^*|| < \theta \quad \text{and} \quad f(x^\theta) = f(x^*). \tag{1}$$

Since x^* is a stable local minimum, there is a $\delta > 0$ such that $x^k \to x^*$ for all $\{x^k\}$ with

$$\lim_{k\to\infty} f(x^k) = f(x^*) \quad \text{and} \quad ||x^k - x^*|| < \delta. \tag{2}$$

For $\theta = \delta$ in (1), we can find a point $x^0 \neq x^*$ for which $0 < ||x^0 - x^*|| < \delta$ and $f(x^0) = f(x^*)$. Then, for $\theta = \frac{1}{2}||x^0 - x^*||$ in (1), we can find a point x^1 such that $0 < ||x^1 - x^*|| < \frac{1}{2}||x^0 - x^*||$ and $f(x^1) = f(x^*)$. Then, again, for $\theta = \frac{1}{2}||x^1 - x^*||$ in (1), we can find a point x^2 such that $0 < ||x^2 - x^*|| < \frac{1}{2}||x^1 - x^*||$ and $f(x^2) = f(x^*)$, and so on. In this way, we have constructed a sequence $\{x^k\}$ of distinct points such that $0 < ||x^k - x^*|| < \delta$, $f(x^k) = f(x^*)$ for all k, and $\lim_{k\to\infty} x^k = x^*$. Now, consider the sequence $\{y^k\}$ defined by

$$y^{2m} = x^m, \quad y^{2m+1} = x^0, \quad \forall\ m \geq 0.$$

Evidently, the sequence $\{y^k\}$ is contained in the sphere centered at x^* with the radius δ. Also we have that $f(y^k) = f(x^*)$, but $\{y^k\}$ does not converge to x^*. This contradicts the assumption that x^* is locally stable. Hence, x^* must be strict local minimum.

(b) Since x^* is a strict local minimum, we can find $\delta > 0$, such that $f(x) > f(x^*)$ for all $x \neq x^*$ with $||x - x^*|| \leq \delta$. Then $\min_{||x-x^*||=\delta} f(x) = f^\delta > f(x^*)$. Let $G^\delta = \max_{||x-x^*||\leq\delta} |g(x)|$. Now, we have

$$f(x) - \epsilon G^\delta \leq f(x) + \epsilon g(x) \leq f(x) + \epsilon G^\delta, \qquad \forall\ \epsilon > 0, \qquad \forall\ x\ \ ||x - x^*|| < \delta.$$

Choose ϵ^δ such that

$$f^\delta - \epsilon^\delta G^\delta > f(x^*) + \epsilon^\delta G^\delta,$$

and notice that for all $0 \le \epsilon \le \epsilon^\delta$ we have

$$f^\delta - \epsilon G^\delta > f(x^*) + \epsilon G^\delta.$$

Consider the level sets

$$L(\epsilon) = \{x \mid f(x) + \epsilon g(x) \le f(x^*) + \epsilon G^\delta, \;\; ||x - x^*|| \le \delta\}, \qquad 0 \le \epsilon \le \epsilon^\delta.$$

Note that

$$L(\epsilon^1) \subset L(\epsilon^2) \subset B(x^*, \delta), \qquad \forall\, 0 \le \epsilon^1 < \epsilon^2 \le \epsilon^\delta, \tag{3}$$

where $B(x^*, \delta)$ is the open sphere centered at x^* with radius δ. The relation (3) means that the sequence $\{L(\epsilon)\}$ decreases as ϵ decreases. Observe that for any $\epsilon \ge 0$, the level set $L(\epsilon)$ is compact. Since x^* is strictly better than any other point $x \in B(x^*, \delta)$, and $x^* \in L(\epsilon)$ for all $0 \le \epsilon \le \epsilon^\delta$, we have

$$\cap_{0 \le \epsilon \le \epsilon^\delta} L(\epsilon) = \{x^*\}. \tag{4}$$

According to Weierstrass' theorem, the continuous function $f(x) + \epsilon g(x)$ attains its minimum on the compact set $L(\epsilon)$ at some point $x_\epsilon \in L(\epsilon)$. From (3) it follows that $x_\epsilon \in B(x^*, \delta)$ for any ϵ in the range $[0, \epsilon^\delta]$. Finally, since $x_\epsilon \in L(\epsilon)$, from (4) we see that $\lim_{\epsilon \to \infty} x_\epsilon = x^*$.

1.1.13 www

In the solution to the Exercise 1.1.12 we found the numbers $\delta > 0$ and $\epsilon^\delta > 0$ such that for all $\epsilon \in [0, \epsilon^\delta)$ the function $f(x) + \epsilon g(x)$ has a local minimum x_ϵ within the sphere $B(x^*, \delta) = \{x \mid ||x - x^*|| < \delta\}$. The Implicit Function Theorem can be applied to the continuously differentiable function $G(\epsilon, x) = \nabla f(x) + \epsilon \nabla g(x)$ for which $G(0, x^*) = 0$. Thus, there are an interval $[0, \epsilon_0)$, a number δ_0 and a continuously differentiable function $\phi : [0, \epsilon_0) \mapsto B(x^*, \delta_0)$ such that $\phi(\epsilon) = x'_\epsilon$ and

$$\nabla \phi(\epsilon) = -\nabla_\epsilon G\left(\epsilon, \phi(\epsilon)\right) \left(\nabla_x G\left(\epsilon, \phi(\epsilon)\right)\right)^{-1}, \qquad \forall\, \epsilon \in [0, \epsilon_0).$$

We may assume that ϵ_0 is small enough so that the first order expansion for $\phi(\epsilon)$ at $\epsilon = 0$ holds, namely

$$\phi(\epsilon) = \phi(0) + \epsilon \nabla \phi(0) + o(\epsilon), \qquad \forall\, \epsilon \in [0, \epsilon_0). \tag{1}$$

It can be seen that $\nabla_x G\left(0, \phi(0)\right) = \nabla_x G(0, x^*) = \nabla^2 f(x^*)$, and $\nabla_\epsilon G\left(0, \phi(0)\right) = \nabla g(x^*)'$, which combined with $\phi(\epsilon) = x'_\epsilon, \;\; \phi(0) = (x^*)'$ and (1) gives the desired relation.

SECTION 1.2

1.2.5 (www)

(a) Given a bounded set A, let $r = \sup\{\|x\| \mid x \in A\}$ and $B = \{x \mid \|x\| \le r\}$. Let $L = \max\{\|\nabla^2 f(x)\| \mid x \in B\}$, which is finite because a continuous function on a compact set is bounded. For any $x, y \in A$ we have

$$\nabla f(x) - \nabla f(y) = \int_0^1 \nabla^2 f\big(tx + (1-t)y\big)(x-y)dt.$$

Notice that $tx + (1-t)y \in B$, for all $t \in [0,1]$. It follows that

$$\|\nabla f(x) - f(y)\| \le L\|x - y\|,$$

as desired.

(b) The key idea is to show that x^k stays in the bounded set

$$A = \big\{x \mid f(x) \le f(x^0)\big\}$$

and to use a stepsize α^k that depends on the constant L corresponding to this bounded set. Let

$$R = \max\{\|x\| \mid x \in A\},$$

$$G = \max\{\|\nabla f(x)\| \mid x \in A\},$$

and

$$B = \{x \mid \|x\| \le R + 2G\}.$$

Using condition (i) in the exercise, there exists some constant L such that $\|\nabla f(x) - \nabla f(y)\| \le L\|x - y\|$, for all $x, y \in B$. Suppose the stepsize α^k satisfies

$$0 < \epsilon \le \alpha^k \le (2-\epsilon)\gamma^k \min\{1, 1/L\},$$

where

$$\gamma^k = \frac{|\nabla f(x^k)' d^k|}{\|d^k\|^2}.$$

Let $\beta^k = \alpha^k(\gamma^k - L\alpha^k/2)$, which can be seen to satisfy $\beta^k \ge \epsilon^2\gamma^k/2$ by our choice of α^k. We will, show by induction on k that with such a choice of stepsize, we have $x^k \in A$ and

$$f\big(x^{k+1}\big) \le f(x^k) - \beta^k\|d^k\|^2, \qquad (*)$$

for all $k \geq 0$.

To start the induction, we note that $x^0 \in A$, by the definition of A. Suppose that $x^k \in A$. By the definition of γ^k, we have

$$\gamma^k \|d^k\|^2 = \big|\nabla f(x^k)' d^k\big| \leq \big\|\nabla f(x^k)\big\| \cdot \|d^k\|.$$

Thus, $\|d^k\| \leq \big\|\nabla f(x^k)\big\| / \gamma^k \leq G/\gamma^k$. Hence,

$$\|x^k + \alpha^k d^k\| \leq \|x^k\| + \alpha^k G/\gamma^k \leq R + 2G,$$

which shows that $x^k + \alpha^k d^k \in B$. In order to prove Eq. (*), we now proceed as in the proof of Prop. 1.2.3. A difficulty arises because Prop. A.24 assumes that the inequality $\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$ holds for all x, y, whereas in this exercise this inequality holds only for $x, y \in B$. We thus essentially repeat the proof of Prop. A.24, to obtain

$$\begin{aligned} f(x^{k+1}) &= f(x^k + \alpha^k d^k) \\ &= \int_0^1 \alpha^k \nabla f(x^k + \tau \alpha^k d^k)' d^k \, d\tau \\ &\leq \alpha^k \nabla f(x^k)' d^k + \left| \int_0^1 \alpha^k \Big(\nabla f\big(x^k + \alpha^k \tau d^k\big) - \nabla f(x^k) \Big)' d^k \, d\tau \right| \\ &\leq \alpha^k \nabla f(x^k)' d^k + (\alpha^k)^2 \|d^k\|^2 \int_0^1 L\tau \, d\tau \\ &= \alpha^k \nabla f(x^k)' d^k + \frac{L(\alpha^k)^2}{2} \|d^k\|^2. \end{aligned} \tag{**}$$

We have used here the inequality

$$\big\|\nabla f\big(x^k + \alpha^k \tau d^k\big) - \nabla f(x^k)\big\| \leq \alpha^k L \tau \|d^k\|,$$

which holds because of our definition of L and because $x^k \in A \subset B$, $x^k + \alpha^k d^k \in B$ and (because of the convexity of B) $x^k + \alpha^k \tau d^k \in B$, for $\tau \in [0, 1]$.

Inequality (*) now follows from Eq. (**) as in the proof of Prop. 1.2.3. In particular, we have $f(x^{k+1}) \leq f(x^k) \leq f(x^0)$ and $x^{k+1} \in A$. This completes the induction. The remainder of the proof is the same as in Prop. 1.2.3.

1.2.10 (www)

We have

$$\nabla f(x) - \nabla f(x^*) = \int_0^1 \nabla^2 f\big(x^* + t(x - x^*)\big)(x - x^*) dt$$

and since

$$\nabla f(x^*) = 0,$$

we obtain

$$(x - x^*)'\nabla f(x) = \int_0^1 (x - x^*)'\nabla^2 f(x^* + t(x - x^*))(x - x^*)dt \geq m \int_0^1 \|x - x^*\|^2 dt.$$

Using the Cauchy-Schwartz inequality $(x - x^*)'\nabla f(x) \leq \|x - x^*\|\|\nabla f(x)\|$, we have

$$m \int_0^1 \|x - x^*\|^2 dt \leq \|x - x^*\|\|\nabla f(x)\|,$$

and

$$\|x - x^*\| \leq \frac{\|\nabla f(x)\|}{m}.$$

Now define for all scalars t,

$$F(t) = f(x^* + t(x - x^*))$$

We have

$$F'(t) = (x - x^*)'\nabla f(x^* + t(x - x^*))$$

and

$$F''(t) = (x - x^*)'\nabla^2 f(x^* + t(x - x^*))(x - x^*) \geq m\|x - x^*\|^2 \geq 0.$$

Thus F' is an increasing function, and $F'(1) \geq F'(t)$ for all $t \in [0, 1]$. Hence

$$\begin{aligned} f(x) - f(x^*) = F(1) - F(0) &= \int_0^1 F'(t)dt \\ &\leq F'(1) = (x - x^*)'\nabla f(x) \\ &\leq \|x - x^*\|\|\nabla f(x)\| \leq \frac{\|\nabla f(x)\|^2}{m}, \end{aligned}$$

where in the last step we used the result shown earlier.

1.2.11 www

Assume condition (i). The same reasoning as in proof of Prop. 1.2.1, can be used here to show that

$$0 \leq \nabla f(\bar{x})'\bar{p}, \qquad (1)$$

where $\bar{x}$ is a limit point of $\{x^k\}$, namely $\{x^k\}_{k \in \bar{\mathcal{K}}} \longrightarrow \bar{x}$, and

$$p^k = \frac{d^k}{||d^k||}, \qquad \{p^k\}_{k \in \bar{\mathcal{K}}} \to \bar{p}. \qquad (2)$$

Since ∇f is continuous, we can write

$$\begin{aligned} \nabla f(\bar{x})'\bar{p} &= \lim_{k \to \infty,\ k \in \bar{\mathcal{K}}} \nabla f(x^k)'p^k \\ &= \liminf_{k \to \infty,\ k \in \bar{\mathcal{K}}} \nabla f(x^k)'p^k \\ &\leq \frac{\liminf_{k \to \infty,\ k \in \bar{\mathcal{K}}} \nabla f(x^k)'d^k}{\limsup_{k \to \infty,\ k \in \bar{\mathcal{K}}} ||d^k||} < 0, \end{aligned}$$

which contradicts (1). The proof for the other choices of stepsize is the same as in Prop.1.2.1.

Assume condition (ii). Suppose that $\nabla f(x^k) \neq 0$ for all k. For the minimization rule we have

$$f(x^{k+1}) = \min_{\alpha \geq 0} f(x^k + \alpha d^k) = \min_{\theta \geq 0} f(x^k + \theta p^k), \tag{3}$$

for all k, where $p^k = \frac{d^k}{||d^k||}$. Note that

$$\nabla f(x^k)' p^k \leq -c||\nabla f(x^k)||, \qquad \forall\ k. \tag{4}$$

Let $\hat{x}^{k+1} = x^k + \hat{\alpha}_k p^k$ be the iterate generated from x^k via the Armijo rule, with the corresponding stepsize $\hat{\alpha}_k$ and the descent direction p^k. Then from (3) and (4), it follows that

$$f(x^{k+1}) - f(x^k) \leq f(\hat{x}^{k+1}) - f(x^k) \leq \sigma \hat{\alpha}_k \nabla f(x^k)' p^k \leq -\sigma c \hat{\alpha}_k ||\nabla f(x^k)||^2. \tag{5}$$

Hence, either $\{f(x^k)\}$ diverges to $-\infty$ or else it converges to some finite value. Suppose that $\{x^k\}_{k \in \mathcal{K}} \to \bar{x}$ and $\nabla f(\bar{x}) \neq 0$. Then, $\lim_{k \to \infty, k \in \mathcal{K}} f(x^k) = f(\bar{x})$, which combined with (5) implies that

$$\lim_{k \to \infty, k \in \mathcal{K}} \hat{\alpha}_k ||\nabla f(x^k)||^2 = 0.$$

Since $\lim_{k \to \infty, k \in \mathcal{K}} \nabla f(x^k) = \nabla f(\bar{x}) \neq 0$, we must have $\lim_{k \to \infty, k \in \mathcal{K}} \hat{\alpha}_k = 0$. Without loss of generality, we may assume that $\lim_{k \to \infty, k \in \mathcal{K}} p^k = \bar{p}$. Now, we can use the same line of arguments as in the proof of the Prop. 1.2.1 to show that (1) holds. On the other hand, from (4) we have that

$$\lim_{k \to \infty, k \in \mathcal{K}} \nabla f(x^k)' p^k = \nabla f(\bar{x})' \bar{p} \leq -c||\nabla f(\bar{x})|| < 0.$$

This contradicts (1), so that $\nabla f(\bar{x}) = 0$.

1.2.13 (www)

Consider the stepsize rule (i). From the Descent Lemma (cf. the proof of Prop. 1.2.3), we have for all k

$$f(x^{k+1}) \leq f(x^k) - \alpha^k \left(1 - \frac{\alpha^k L}{2}\right) \|\nabla f(x^k)\|^2.$$

From this relation, we obtain for any minimum x^* of f,

$$f(x^*) \leq f(x^0) - \frac{\epsilon}{2} \sum_{k=0}^{\infty} \|\nabla f(x^k)\|^2.$$

It follows that $\nabla f(x^k) \to 0$, that $\{f(x^k)\}$ converges, and that $\sum_{k=0}^{\infty} \|\nabla f(x^k)\|^2 < \infty$, from which

$$\sum_{k=0}^{\infty} \|x^{k+1} - x^k\|^2 < \infty,$$

since $\nabla f(x^k) = (x^k - x^{k+1})/\alpha^k$.

Using the convexity of f, we have for any minimum x^* of f,

$$\begin{aligned}\|x^{k+1} - x^*\|^2 - \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2 &\leq -2(x^* - x^k)'(x^{k+1} - x^k)\\ &= 2\alpha^k (x^* - x^k)'\nabla f(x^k)\\ &\leq 2\alpha^k \big(f(x^*) - f(x^k)\big)\\ &\leq 0,\end{aligned}$$

so that

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 + \|x^{k+1} - x^k\|^2.$$

Hence, for any m,

$$\|x^m - x^*\|^2 \leq \|x^0 - x^*\|^2 + \sum_{k=0}^{m-1} \|x^{k+1} - x^k\|^2.$$

It follows that $\{x^k\}$ is bounded. Let $\overline{x}$ be a limit point of $\{x^k\}$, and for any $\epsilon > 0$, let $\overline{k}$ be such that

$$\|x^{\overline{k}} - \overline{x}\|^2 \leq \epsilon, \qquad \sum_{i=\overline{k}}^{\infty} \|x^{i+1} - x^i\|^2 \leq \epsilon.$$

Since $\overline{x}$ is a minimum of f, using the preceding relations, for any $k > \overline{k}$, we have

$$\|x^k - \overline{x}\|^2 \leq \|x^{\overline{k}} - \overline{x}\|^2 + \sum_{i=\overline{k}}^{k-1} \|x^{i+1} - x^i\|^2 \leq 2\epsilon.$$

Since ϵ is arbitrarily small, it follows that the entire sequence $\{x^k\}$ converges to $\overline{x}$.

The proof for the case of the stepsize rule (ii) is similar. Using the assumptions $\alpha^k \to 0$ and $\sum_{k=0}^{\infty} \alpha^k = \infty$, and the Descent Lemma, we show that $\nabla f(x^k) \to 0$, that $\{f(x^k)\}$ converges, and that

$$\sum_{k=0}^{\infty} \|x^{k+1} - x^k\|^2 < \infty.$$

From this point, the preceding proof applies.

1.2.14 www

(a) We have

$$\begin{aligned}\|x^{k+1} - y\|^2 &= \|x^k - y - \alpha^k \nabla f(x^k)\|^2\\ &= \left(x^k - y - \alpha^k \nabla f(x^k)\right)' \left(x^k - y - \alpha^k \nabla f(x^k)\right)\\ &= \|x^k - y\|^2 - 2\alpha^k (x^k - y)'\nabla f(x^k) + \left(\alpha^k \|\nabla f(x^k)\|\right)^2\\ &= \|x^k - y\|^2 + 2\alpha^k (y - x^k)'\nabla f(x^k) + \left(\alpha^k \|\nabla f(x^k)\|\right)^2\\ &\leq \|x^k - y\|^2 + 2\alpha^k \left(f(y) - f(x^k)\right) + \left(\alpha^k \|\nabla f(x^k)\|\right)^2\\ &= \|x^k - y\|^2 - 2\alpha^k \left(f(x^k) - f(y)\right) + \left(\alpha^k \|\nabla f(x^k)\|\right)^2,\end{aligned}$$

where the inequality follows from Prop. B.3, which states that f is convex if and only if

$$f(y) - f(x) \geq (y - x)'\nabla f(x), \quad \forall\, x, y.$$

(b) Assume the contrary; that is, $\liminf_{k\to\infty} f(x^k) \neq \inf_{x\in\Re^n} f(x)$. Then, for some $\delta > 0$, there exists y such that $f(y) < f(x^k) - \delta$ for all $k \geq \bar{k}$, where $\bar{k}$ is sufficiently large. From part (a), we have

$$\|x^{k+1} - y\|^2 \leq \|x^k - y\|^2 - 2\alpha^k \left(f(x^k) - f(y)\right) + \left(\alpha^k \|\nabla f(x^k)\|\right)^2.$$

Summing over all k sufficiently large, we have

$$\sum_{k=\bar{k}}^{\infty} \|x^{k+1} - y\|^2 \leq \sum_{k=\bar{k}}^{\infty} \left\{ \|x^k - y\|^2 - 2\alpha^k \left(f(x^k) - f(y)\right) + \left(\alpha^k \|\nabla f(x^k)\|\right)^2 \right\},$$

or

$$0 \leq \|x^{\bar{k}} - y\|^2 - \sum_{k=\bar{k}}^{\infty} 2\alpha^k \delta + \sum_{k=\bar{k}}^{\infty} \left(\alpha^k \|\nabla f(x^k)\|\right)^2 = \|x^{\bar{k}} - y\|^2 - \sum_{k=\bar{k}}^{\infty} \alpha^k \left(2\delta - \alpha^k \|\nabla f(x^k)\|^2\right).$$

By taking $\bar{k}$ large enough, we may assume (using $\alpha^k \|\nabla f(x^k)\|^2 \to 0$) that $\alpha^k \|\nabla f(x^k)\|^2 \leq \delta$ for $k \geq \bar{k}$. So we obtain

$$0 \leq \|x^{\bar{k}} - y\|^2 - \delta \sum_{k=\bar{k}}^{\infty} \alpha^k.$$

Since $\sum \alpha^k = \infty$, the term on the right is equal to $-\infty$, yielding a contradiction. Therefore we must have $\liminf_{k\to\infty} f(x^k) = \inf_{x\in\Re^n} f(x)$.

(c) Let y be some x^* such that $f(x^*) \leq f(x^k)$ for all k. (If no such x^* exists, the desired result follows trivially). Then

$$\begin{aligned}
\|x^{k+1} - y\|^2 &\leq \|x^k - y\|^2 - 2\alpha^k \left(f(x^k) - f(y)\right) + \left(\alpha^k \|\nabla f(x^k)\|\right)^2 \\
&\leq \|x^k - y\|^2 + \left(\alpha^k \|\nabla f(x^k)\|\right)^2 \\
&= \|x^k - y\|^2 + \left(\frac{s^k}{\|\nabla f(x^k)\|} \|\nabla f(x^k)\|\right)^2 \\
&= \|x^k - y\|^2 + (s^k)^2 \\
&\leq \|x^{k-1} - y\|^2 + (s^{k-1})^2 + (s^k)^2 \\
&\leq \cdots \leq \|x^0 - y\|^2 + \sum_{i=0}^{k} (s^i)^2 < \infty.
\end{aligned}$$

Thus $\{x^k\}$ is bounded. Since f is continuously differentiable, we then have that $\{\nabla f(x^k)\}$ is bounded. Let M be an upper bound for $\|\nabla f(x^k)\|$. Then

$$\sum \alpha^k = \sum \frac{s^k}{\|\nabla f(x^k)\|} \geq \frac{1}{M} \sum s^k = \infty.$$

Furthermore,

$$\alpha^k \|\nabla f(x^k)\|^2 = s^k \|\nabla f(x^k)\| \le s^k M.$$

Since $\sum (s^k)^2 < \infty, s^k \to 0$. Then $\alpha^k \|\nabla f(x^k)\|^2 \to 0$. We can thus apply the results of part (b) to show that $\liminf_{k\to\infty} f(x^k) = \inf_{x\in\Re^n} f(x)$.

Now, since $\liminf_{k\to\infty} f(x^k) = \inf_{x\in\Re^n} f(x)$, there must be a subsequence $\{x^k\}_K$ such that $\{x^k\}_K \to \bar{x}$, for some $\bar{x}$ where $f(\bar{x}) = \inf_{x\in\Re^n} f(x)$ so that $\bar{x}$ is a global minimum. We have

$$\|x^{k+1} - \bar{x}\|^2 \le \|x^k - \bar{x}\|^2 + (s^k)^2,$$

so that

$$\|x^{k+N} - \bar{x}\|^2 \le \|x^k - \bar{x}\|^2 + \sum_{m=k}^{N} (s^m)^2, \quad \forall\, k, N \ge 1.$$

For any $\epsilon > 0$, we can choose $\bar{k} \in K$ to be sufficiently large so that for all $k \in K$ with $k \ge \bar{k}$ we have

$$\|x^k - \bar{x}\|^2 \le \epsilon \quad \text{and} \sum_{m=k}^{\infty} (s^m)^2 \le \epsilon.$$

Then

$$\|x^{k+N} - \bar{x}\|^2 \le 2\epsilon, \quad \forall\, N \ge 1.$$

Since $\epsilon > 0$ is arbitrary, we see that $\{x^k\}$ converges to $\bar{x}$.

1.2.17 www

By using the descent lemma (Proposition A.24 of Appendix A), we obtain

$$\begin{aligned} f(x^{k+1}) - f(x^k) &\le -\alpha^k \nabla f(x^k)'(\nabla f(x^k) + e^k) + \frac{L}{2}(\alpha^k)^2 ||\nabla f(x^k) + e^k||^2 \\ &= -\alpha^k \left(1 - \frac{L}{2}\alpha^k\right) ||\nabla f(x^k)||^2 + \frac{L}{2}(\alpha^k)^2 ||e^k||^2 - \alpha^k(1 - L\alpha^k)\nabla f(x^k)'e^k. \end{aligned}$$

Assume that $\alpha^k < \frac{1}{L}$ for all k, so that $1 - L\alpha^k > 0$ for every k. Then, using the estimates

$$1 - \frac{L}{2}\alpha^k \ge 1 - L\alpha^k,$$

$$\nabla f(x^k)'e^k \ge -\frac{1}{2}\left(||\nabla f(x^k)||^2 + ||e^k||^2\right),$$

and the assumption $||e^k|| \le \delta$ for all k, in the inequality above, we obtain

$$f(x^{k+1}) - f(x^k) \le -\frac{\alpha^k}{2}(1 - L\alpha^k)\left(||\nabla f(x^k)||^2 - \delta^2\right) + (\alpha^k)^2 \frac{L\delta^2}{2}. \tag{1}$$

Let δ' be an arbitrary number satisfying $\delta' > \delta$. Consider the set $\mathcal{K} = \{k \mid ||\nabla f(x^k)|| < \delta'\}$. If the set $\mathcal{K}$ is infinite, then we are done. Suppose that the set $\mathcal{K}$ is finite. Then, there is some index k_0 such that $||\nabla f(x^k)|| \geq \delta'$ for all $k \geq k_0$. By substituting this in (1), we can easily find that

$$f(x^{k+1}) - f(x^k) \leq -\frac{\alpha^k}{2}\left((1 - L\alpha^k)\left({\delta'}^2 - \delta^2\right) - \alpha^k L\delta^2\right), \qquad \forall\ k \geq k_0.$$

By choosing $\underline{\alpha}$ and $\overline{\alpha}$ such that $0 < \underline{\alpha} < \bar{\alpha} < \min\{\frac{{\delta'}^2 - \delta^2}{{\delta'}^2 L}, \frac{1}{L}\}$, and $\alpha^k \in [\underline{\alpha}, \bar{\alpha}]$ for all $k \geq k_0$, we have that

$$f(x^{k+1}) - f(x^k) \leq -\frac{1}{2}\underline{\alpha}\left({\delta'}^2 - \delta^2 - \bar{\alpha}L{\delta'}^2\right), \qquad \forall\ k \geq k_0. \tag{2}$$

Since ${\delta'}^2 - \delta^2 - \bar{\alpha}L{\delta'}^2 > 0$ for $k \geq k_0$, the sequence $\{f(x^k) \mid k \geq k_0\}$ is strictly decreasing. Summing the inequalities in (2) over k for $k_0 \leq k \leq N$, we get

$$f(x^{N+1}) - f(x^{k_0}) \leq -\frac{(N - k_0)}{2}\underline{\alpha}\left({\delta'}^2 - \delta^2 - \bar{\alpha}L{\delta'}^2\right), \qquad \forall\ N > k_0.$$

Taking the limit as $N \longrightarrow \infty$, we obtain $\lim_{N\to\infty} f(x^N) = -\infty$.

1.2.19 www

(a) Note that

$$\nabla f(x) = \nabla_x F(x, g(x)) + \nabla g(x)\nabla_y F(x, g(x)).$$

We can write the given method as

$$x^{k+1} = x^k + \alpha^k d^k = x^k - \alpha^k \nabla_x F(x^k, g(x^k)) = x^k + \alpha^k\left(-\nabla f(x^k) + \nabla g(x^k)\nabla_y F(x^k, g(x^k))\right),$$

so that this method is essentially steepest descent with error

$$e^k = -\nabla g(x^k)\nabla_y F(x^k, g(x^k)).$$

Claim: The directions d^k are gradient related.

Proof: We first show that d^k is a descent direction. We have

$$\begin{aligned}
\nabla f(x^k)'d^k &= \left(\nabla_x F(x^k, g(x^k)) + \nabla g(x)\nabla_y F(x^k, g(x^k))\right)'\left(-\nabla_x F(x^k, g(x^k))\right) \\
&= -\left\|\nabla_x F(x^k, g(x^k))\right\|^2 - \left(\nabla g(x)\nabla_y F(x^k, g(x^k))\right)'\left(\nabla_x F(x^k, g(x^k))\right) \\
&\leq -\left\|\nabla_x F(x^k, g(x^k))\right\|^2 + \left\|\nabla g(x)\nabla_y F(x^k, g(x^k))\right\|\left\|\nabla_x F(x^k, g(x^k))\right\| \\
&\leq -\left\|\nabla_x F(x^k, g(x^k))\right\|^2 + \gamma\left\|\nabla_x F(x^k, g(x^k))\right\|^2 \\
&= (-1 + \gamma)\left\|\nabla_x F(x^k, g(x^k))\right\|^2 \\
&< 0 \quad \text{for} \quad \|\nabla_x F(x^k, g(x^k))\| \neq 0.
\end{aligned}$$

It is straightforward to show that $\|\nabla_x F(x^k, g(x^k))\| = 0$ if and only if $\|\nabla f(x^k)\| = 0$, so that we have $\nabla f(x^k)' d^k < 0$ for $\|\nabla f(x^k)\| \neq 0$. Hence d^k is a descent direction if x^k is nonstationary. Furthermore, for every subsequence $\{x^k\}_{k \in K}$ that converges to a nonstationary point $\bar{x}$, we have

$$\begin{aligned}
\|d^k\| &= \frac{1}{1-\gamma} \left[\|\nabla_x F(x^k, g(x^k))\| - \gamma \|\nabla_x F(x^k, g(x^k))\| \right] \\
&\leq \frac{1}{1-\gamma} \left[\|\nabla_x F(x^k, g(x^k))\| - \|\nabla g(x) \nabla_y F(x^k, g(x^k))\| \right] \\
&\leq \frac{1}{1-\gamma} \|\nabla_x F(x^k, g(x^k)) + \nabla g(x) \nabla_y F(x^k, g(x^k))\| \\
&= \frac{1}{1-\gamma} \|\nabla f(x^k)\|,
\end{aligned}$$

and so $\{d^k\}$ is bounded. We have from Eq. (1), $\nabla f(x^k)' d^k \leq -(1-\gamma) \|\nabla_x F(x^k, g(x^k))\|^2$. Hence if $\lim_{k \to \infty} \inf_{k \in K} \nabla f(x^k)' d^k = 0$, then $\lim_{k \to \infty, k \in K} \|\nabla F(x^k, g(x^k))\| = 0$, from which $\|\nabla F(\bar{x}, g(\bar{x}))\| = 0$. So $\nabla f(\bar{x}) = 0$, which contradicts the nonstationarity of $\bar{x}$. Hence,

$$\lim_{k \to \infty} \inf_{k \in K} \nabla f(x^k)' d^k < 0,$$

and it follows that the directions d^k are gradient related.

From Prop. 1.2.1, we then have the desired result.

(b) Let's assume that in addition to being continuously differentiable, h has a continuous and nonsingular gradient matrix $\nabla_y h(x, y)$. Then from the Implicit Function Theorem (Prop. A.33), there exists a continuously differentiable function $\phi : \Re^n \to \Re^m$ such that $h(x, \phi(x)) = 0$, for all $x \in \Re^n$. If, furthermore, there exists a $\gamma \in (0, 1)$ such that

$$\|\nabla \phi(x) \nabla_y f(x, \phi(x))\| \leq \gamma \|\nabla_x f(x, \phi(x))\|, \quad \forall\, x \in \Re^n,$$

then from part (a), the method described is convergent.

1.2.20 (www)

(a) Consider a function $g(\alpha) = f(x^k + \alpha d^k)$ for $0 < \alpha < \alpha^k$, which is convex over I^k. Suppose that $\overline{x}^k = x^k + \overline{\alpha} d^k \in I^k$ minimizes $f(x)$ over I^k. Then $g'(\overline{\alpha}) = 0$ and from convexity it follows that $g'(\alpha^k) = \nabla f(x^{k+1})' d^k > 0$ (since $g'(0) = \nabla f(x^k)' d^k < 0$). Therefore the stepsize will be reduced after this iteration. Now, assume that $\overline{x}^k \notin I^k$. This means that the derivative $g'(\alpha)$ does not change the sign for $0 < \alpha < \alpha^k$, i.e. for all α in the interval $(0, \alpha^k)$ we have $g'(\alpha) < 0$. Hence, $g'(\alpha^k) = \nabla f(x^{k+1})' d^k \leq 0$ and we can use the same stepsize α^k in the next iteration.

(b) Here we will use conditions on $\nabla f(x)$ and d^k which imply

$$\begin{aligned}\nabla f(x^{k+1})'d^k &\le \nabla f(x^k)'d^k + ||\nabla f(x^{k+1}) - \nabla f(x^k)|| \cdot ||d^k|| \\ &\le \nabla f(x^k)'d^k + \alpha^k L||d^k||^2 \\ &\le -(c_1 - c_2\alpha^k L)\|\nabla f(x^k)\|^2.\end{aligned}$$

When the stepsize becomes small enough so that $c_1 - c_2\alpha^{\hat{k}}L \ge 0$ for some $\hat{k}$, then $\nabla f(x^{k+1})'d^k \le 0$ for all $k \ge \hat{k}$ and no further reduction will ever be needed.

(c) The result follows in the same way as in the proof of Prop.1.2.4. Every limit point of $\{x^k\}$ is a stationary point of f. Since f is convex, every limit point of $\{x^k\}$ must be a global minimum of f.

1.2.21 (www)

By using the descent lemma (Prop. A.24 of Appendix A), we obtain

$$f(x^{k+1}) - f(x^k) \le \alpha^k \nabla f(x^k)'(d^k + e^k) + (\alpha^k)^2 \frac{L}{2}||d^k + e^k||^2. \tag{1}$$

Taking into account the given properties of d^k, e^k, the Schwartz inequality, and the inequality $||y|| \cdot ||z|| \le ||y||^2 + ||z||^2$, we obtain

$$\begin{aligned}\nabla f(x^k)'(d^k + e^k) &\le -(c_1 - p\alpha_k)||\nabla f(x^k)||^2 + q\alpha^k||\nabla f(x^k)|| \\ &\le -\left(c_1 - (p+1)\alpha_k\right)||\nabla f(x^k)||^2 + \alpha^k q^2.\end{aligned}$$

To estimate the last term in the right hand-side of (1), we again use the properties of d^k, e^k, and the inequality $\frac{1}{2}||y+z||^2 \le ||y||^2 + ||z||^2$, which gives

$$\begin{aligned}\frac{1}{2}||d^k + e^k||^2 &\le ||d^k||^2 + ||e^k||^2 \\ &\le 2\left(c_2^2 + (p\alpha^k)^2\right)||\nabla f(x^k)||^2 + 2\left(c_2^2 + (q\alpha^k)^2\right) \\ &\le 2(c_2^2 + p^2)||\nabla f(x^k)||^2 + 2(c_2^2 + q^2), \qquad \forall\ k \ge k_0,\end{aligned}$$

where k_0 is such that $\alpha_k \le 1$ for all $k \ge k_0$.

By substituting these estimates in (1), we get

$$f(x^{k+1}) - f(x^k) \le -\alpha^k(c_1 - C)||\nabla f(x^k)||^2 + (\alpha^k)^2 b_2, \qquad \forall\ k \ge k_0,$$

where $C = 1 + p + 2L(c_2^2 + p^2)$ and $b_2 = q^2 + 2L(c_2^2 + q^2)$. By choosing k_0 large enough, we can have

$$f(x^{k+1}) - f(x^k) \le -\alpha^k b_1||\nabla f(x^k)||^2 + (\alpha^k)^2 b_2, \qquad \forall\ k \ge k_0.$$

Summing up these inequalities over k for $k_0 \le K \le k \le N$ gives

$$f(x^{N+1}) + b_1 \sum_{k=K}^{N} \alpha^k ||\nabla f(x^k)||^2 \le f(x^K) + b_2 \sum_{k=K}^{N} (\alpha^k)^2, \qquad \forall\ k_0 \le K \le k \le N. \tag{2}$$

Therefore

$$\limsup_{N\to\infty} f(x^{N+1}) \le f(x^K) + b_2 \sum_{k=K}^{\infty} (\alpha^k)^2, \qquad \forall\ K \ge k_0.$$

Since $\sum_{k=0}^{\infty} (\alpha^k)^2 < \infty$, the last inequality implies

$$\limsup_{N\to\infty} f(x^{N+1}) \le \liminf_{K\to\infty} f(x^K),$$

i.e. $\lim_{k\to\infty} f(x^k)$ exists (possibly infinite). In particular, the relation (2) implies

$$\sum_{k=0}^{\infty} \alpha^k ||\nabla f(x^k)||^2 < \infty.$$

Thus we have $\liminf_{k\to\infty} ||\nabla f(x^k)|| = 0$ (see the proof of Prop. 1.2.4). To prove that $\lim_{k\to\infty} ||\nabla f(x^k)|| = 0$, assume the contrary, i.e.

$$\limsup_{k\to\infty} ||\nabla f(x^k)|| \ge \epsilon > 0. \tag{3}$$

Let $\{m_j\}$ and $\{n_j\}$ be sequences such that

$$m_j < n_j < m_{j+1},$$

$$\frac{\epsilon}{3} < ||\nabla f(x^k)|| \qquad \text{for} \quad m_j \le k < n_j,$$

$$||\nabla f(x^k)|| \le \frac{\epsilon}{3} \qquad \text{for} \quad n_j \le k < m_{j+1}. \tag{4}$$

Let $\bar{j}$ be large enough so that

$$\alpha_k \le 1, \qquad \forall\ k \ge \bar{j},$$

$$\sum_{k=m_{\bar{j}}}^{\infty} \alpha^k ||\nabla f(x^k)||^2 \le \frac{\epsilon^3}{27L(2c_2 + q + p)}.$$

For any $j \geq \bar{j}$ and any m with $m_j \leq m \leq n_j - 1$, we have

$$\begin{aligned}
||\nabla f(x^{n_j}) - \nabla f(x^m)|| &\leq \sum_{k=m}^{n_j-1} ||\nabla f(x^{k+1}) - \nabla f(x^k)|| \\
&\leq L \sum_{k=m}^{n_j-1} ||x^{k+1} - x^k|| \\
&\leq L \sum_{k=m}^{n_j-1} \alpha_k \left(||d^k|| + ||e^k||\right) \\
&\leq L(c_2+q)\left(\sum_{k=m}^{n_j-1} \alpha_k\right) + L(c_2+p)\sum_{k=m}^{n_j-1} \alpha_k ||\nabla f(x^k)|| \\
&\leq \left(L(c_2+q)\frac{9}{\epsilon^2} + L(c_2+p)\frac{3}{\epsilon}\right)\sum_{k=m}^{n_j-1} \alpha_k ||\nabla f(x^k)||^2 \\
&\leq \frac{9L(2c_2+p+q)}{\epsilon^2}\sum_{k=m}^{n_j-1} \alpha_k ||\nabla f(x^k)||^2 \\
&\leq \frac{9L(2c_2+p+q)}{\epsilon^2}\frac{\epsilon^3}{27L(2c_2+q+p)} \\
&= \frac{\epsilon}{3}.
\end{aligned}$$

Therefore

$$||\nabla f(x^m)|| \leq ||\nabla f(x^{n_j})|| + \frac{\epsilon}{3} \leq \frac{2\epsilon}{3}, \qquad \forall\, j \geq \bar{j},\ m_j \leq m \leq n_j - 1.$$

From here and (4), we have

$$||\nabla f(x^m)|| \leq \frac{2\epsilon}{3}, \qquad \forall\, m \geq m_j$$

which contradicts Eq. (3). Hence $\lim_{k\to\infty} \nabla f(x^k) = 0$. If $\bar{x}$ is a limit point of $\{x^k\}$, then $\lim_{k\to\infty} f(x^k) = f(\bar{x})$. Thus, we have $\lim_{k\to\infty} \nabla f(x^k) = 0$, implying that $\nabla f(\bar{x}) = 0$.

SECTION 1.3

1.3.2 www

Let β be any scalar with $0 < \beta < 1$ and $B(x^*, \bar{\epsilon}) = \{x \mid ||x - x^*|| \leq \bar{\epsilon}\}$ be a closed sphere centered at x^* with the radius $\bar{\epsilon} > 0$ such that for all $x, y \in B(x^*, \bar{\epsilon})$ the following hold

$$\nabla^2 f(x) > 0, \qquad ||\nabla^2 f(x)^{-1}|| \leq M_1, \tag{1}$$

$$||\nabla f(x) - \nabla f(y)|| \leq M_2 ||x - y||, \qquad M_2 = \sup_{x \in B(x^*, \epsilon)} ||\nabla^2 f(x)||, \tag{2}$$

$$||\nabla^2 f(x) - \nabla^2 f(y)|| \leq \frac{\beta}{2M_1} \tag{3}$$

$$||d(x) + \nabla^2 f(x)^{-1} \nabla f(x)|| \leq \frac{\beta}{2M_2} ||\nabla f(x)||. \tag{4}$$

Then, by using these relations and $\nabla f(x^*) = 0$, for any $x \in B(x^*, \overline{\epsilon})$ one can obtain

$$\begin{aligned}
||x + d(x) - x^*|| &\leq ||x - x^* - \nabla^2 f(x)^{-1} \nabla f(x)|| + ||d(x) + \nabla^2 f(x)^{-1} \nabla f(x)|| \\
&\leq ||\nabla^2 f(x)^{-1} \left(\nabla^2 f(x)(x - x^*) - \nabla f(x)\right)|| + \frac{\beta}{2M_2} ||\nabla f(x)|| \\
&\leq M_1 ||\nabla^2 f(x)(x - x^*) - \nabla f(x) + \nabla f(x^*)|| + \frac{\beta}{2M_2} ||\nabla f(x) - \nabla f(x^*)|| \\
&\leq M_1 ||\nabla^2 f(x)(x - x^*) - \int_0^1 \nabla^2 f\left((x^* + t(x - x^*)\right)' (x - x^*) dt|| + \frac{\beta}{2} ||x - x^*|| \\
&\leq M_1 \left(\int_0^1 ||\nabla^2 f(x) - \nabla^2 f\left((x^* + t(x - x^*)\right)|| dt \right) ||x - x^*|| + \frac{\beta}{2} ||x - x^*|| \\
&\leq \beta ||x - x^*||.
\end{aligned}$$

This means that if $x^0 \in B(x^*, \overline{\epsilon})$ and $\alpha^k = 1$ for all k, then we will have

$$||x^k - x^*|| \leq \beta^k ||x^0 - x^*||, \qquad \forall \;\; k \geq 0. \tag{5}$$

Now, we have to prove that for $\overline{\epsilon}$ small enough the unity initial stepsize will pass the test of Armijo rule. By the mean value theorem, we have

$$f(x + d(x)) - f(x) = \nabla f(x)' d(x) + \frac{1}{2} d(x)' \nabla^2 f(\overline{x}) d(x),$$

where $\overline{x}$ is a point on the line segment joining x and $x + d(x)$. We would like to have

$$\nabla f(x)' d(x) + \frac{1}{2} d(x)' \nabla^2 f(\overline{x}) d(x) \leq \sigma \nabla f(x)' d(x), \tag{6}$$

for all x in some neighborhood of x^*. Therefore, we must find how small $\overline{\epsilon}$ should be that this holds in addition to the conditions given in (1)–(4). By defining

$$p(x) = \frac{\nabla f(x)}{||\nabla f(x)||}, \qquad q(x) = \frac{d(x)}{||\nabla f(x)||},$$

the condition (6) takes the form

$$(1 - \sigma) p(x)' q(x) + \frac{1}{2} q(x)' \nabla^2 f(\overline{x}) q(x) \leq 0. \tag{7}$$

The condition on $d(x)$ is equivalent to

$$q(x) = -\left(\nabla^2 f(x^*)\right)^{-1} p(x) + \nu(x),$$

where $\nu(x)$ denotes a vector function with $\nu(x) \to 0$ as $x \to x^*$. By using the above relation and the fact $\nabla^2 f(\overline{x}) \to \nabla^2 f(x^*)$ as $x \to x^*$, we may write Eq.(7) as

$$(1-\sigma)p(x)'\left(\nabla^2 f(x^*)\right)^{-1} p(x) - \frac{1}{2}p(x)'\left(\nabla^2 f(x^*)\right)^{-1} p(x) \geq \gamma(x),$$

where $\{\gamma(x)\}$ is some scalar sequence with $\lim_{x\to x^*} \gamma(x) = 0$. Thus Eq.(7) is equivalent to

$$\left(\frac{1}{2} - \sigma\right) p(x)'\left(\nabla^2 f(x^*)\right)^{-1} p(x) \geq \gamma(x). \tag{8}$$

Since $1/2 > \sigma$, $||p(x)|| = 1$, and $\nabla^2 f(x^*) > 0$, the above relation holds in some neighborhood of point x^*. Namely, there is some $\epsilon \in (0, \overline{\epsilon})$ such that (1)–(4) and (8) hold. Then for any initial point $x^0 \in B(X^*, \epsilon)$ the unity initial stepsize passes the test of Armijo rule, and (5) holds for all k. This completes the proof.

1.3.8 www

In this case, the gradient method has the form $x^{k+1} = x^k - \alpha\nabla f(x^k)$. From the descent lemma (Prop. A.24 of Appendix A), we have

$$f(x^{k+1}) - f(x^k) \leq -\alpha c||\nabla f(x^k)||^2, \tag{1}$$

where $\alpha < \frac{2}{L}$, and $c = 1 - \alpha L/2$. By using the same arguments as in the proof of Prop. 1.3.3, we can show that

$$\lim_{k\to\infty} d(x^k, X^*) = 0. \tag{2}$$

We assume that $d(x^k, X^*) \neq 0$, otherwise the method will terminate in a finite number of iterations. Convexity of the function f implies that

$$f(x^k) - f(x^*) \leq \nabla f(x^k)'(x^k - x^*) \leq ||\nabla f(x^k)|| \cdot ||x^k - x^*||, \quad \forall\ x^* \in X^*,$$

from which, by minimizing over $x^* \in X^*$, we have

$$f(x^k) - f^* \leq ||\nabla f(x^k)||d(x^k, X^*). \tag{3}$$

Let $e^k = f(x^k) - f^*$. Then, inequalities (1) and (3) imply that

$$e^{k+1} \leq e^k - \alpha c\frac{(e^k)^2}{d^2(x^k, X^*)}, \qquad \forall\ k.$$

The rest of the proof is exactly the same as the proof of Prop. 1.3.3, starting from the relation

$$f(x^{k+1}) \leq f(x^k) - \frac{c^2||\nabla f(x^k)||^2}{2L}.$$

1.3.9 (www)

Without loss of generality we assume that $c = 0$ (otherwise we make the change of variables $x = y - Q^{-1}c$). The iteration becomes

$$\begin{pmatrix} x_{k+1} \\ x_k \end{pmatrix} = \begin{pmatrix} (1+\beta)I - \alpha Q & -\beta I \\ I & 0 \end{pmatrix} \begin{pmatrix} x_k \\ x_{k-1} \end{pmatrix}$$

Define

$$A = \begin{pmatrix} (1+\beta)I - \alpha Q & -\beta I \\ I & 0 \end{pmatrix}.$$

If μ is an eigenvalue of A, then for some vectors u and w, which are not both 0, we have

$$A \begin{pmatrix} u \\ w \end{pmatrix} = \mu \begin{pmatrix} u \\ w \end{pmatrix},$$

or equivalently,

$$u = \mu w \qquad \text{and} \qquad \big((1+\beta)I - \alpha Q\big)u - \beta w = \mu u.$$

If we had $\mu = 0$, then it is seen from the above equations that $u = 0$ and also $w = 0$, which is not possible. Therefore, $\mu \neq 0$ and A is invertible. We also have from the above equations that

$$u = \mu w \qquad \text{and} \qquad \big((1+\beta)I - \alpha Q\big)u = \left(\mu + \frac{\beta}{\mu}\right) u,$$

so that $\mu + \beta/\mu$ is an eigenvalue of $(1+\beta)I - \alpha Q$. Hence, if μ and λ satisfy the equation $\mu + \beta/\mu = 1 + \beta - \alpha\lambda$, then μ is an eigenvalue of A if and only if λ is an eigenvalue of Q.

Now, if

$$0 < \alpha < 2\left(\frac{1+\beta}{M}\right),$$

where M is the maximum eigenvalue of Q, then we have

$$|1 + \beta - \alpha\lambda| < 1 + \beta$$

for every eigenvalue λ of Q, and therefore also

$$\left|\mu + \frac{\beta}{\mu}\right| < 1 + \beta$$

for every eigenvalue μ of A. Let the complex number μ have the representation $\mu = |\mu| e^{j\theta}$. Then, since $\mu + \beta/\mu$ is a real number, its imaginary part is 0, or

$$|\mu| \sin\theta - \beta(1/|\mu|)\sin\theta = 0.$$

If $\sin\theta \neq 0$, we have $|\mu|^2 = \beta < 1$, while if $\sin\theta = 0$, μ is a real number and the relation $|\mu + \beta/\mu| < 1 + \beta$ is written as $\mu^2 + \beta < (1+\beta)|\mu|$ or $(|\mu| - 1)(|\mu| - \beta) < 0$. Therefore,

$\beta < |\mu| < 1$. Thus, for all values of θ, we have $\beta \leq |\mu| < 1$. Thus, all the eigenvalues of A are strictly within the unit circle, implying that $x_k \to 0$; that is, the method converges to the unique optimal solution.

Assume for the moment that α and β are fixed. From the preceding analysis we have that μ is an eigenvalue of A if and only if $\mu^2 + \beta = 1 + \beta - \alpha\lambda$, where λ is an eigenvalue of Q. Thus, the set of eigenvalues of A is

$$\left\{ \frac{1 + \beta - \alpha\lambda \pm \sqrt{(1 + \beta - \alpha\lambda)^2 - 4\beta}}{2} \;\Big|\; \lambda \text{ is an eigenvalue of } Q \right\},$$

so that the spectral radius of A is

$$\rho(A) = \max \left\{ \left| \frac{|1 + \beta - \alpha\lambda| + \sqrt{(1 + \beta - \alpha\lambda)^2 - 4\beta}}{2} \right| \;\Big|\; \lambda \text{ is an eigenvalue of } Q \right\}.$$

For any scalar $c \geq 0$, consider the function $g : R^+ \mapsto R^+$ given by

$$g(r) = |r + \sqrt{r^2 - c}|.$$

We claim that

$$g(r) \geq \max\{\sqrt{c}, 2r - \sqrt{c}\}.$$

Indeed, let us show this relation in each of two cases: *Case 1*: $r \geq \sqrt{c}$. Then it is seen that $\sqrt{r^2 - c} \geq r - \sqrt{c}$, so that $g(r) \geq 2r - \sqrt{c} \geq \sqrt{c}$. *Case 2*: $r < \sqrt{c}$. Then $g(r) = \sqrt{r^2 + (c - r^2)} = \sqrt{c} \geq 2r - \sqrt{c}$.

We now apply the relation $g(r) \geq \max\{\sqrt{c}, 2r - \sqrt{c}\}$ to Eq. (3), with $c = 4\beta$ and with $r = |1 + \beta - \alpha\lambda|$, where λ is an eigenvalue of Q. We have

$$\rho^2(A) \geq \frac{1}{4} \max\{4\beta, \max\{2(1 + \beta - \alpha\lambda)^2 - 4\beta \mid \lambda \text{ is an eigenvalue of } Q\}\}.$$

Therefore,

$$\rho^2(A) \geq \frac{1}{4} \max\{4\beta, 2(1 + \beta - \alpha m)^2 - 4\beta, 2(1 + \beta - \alpha M)^2 - 4\beta\}$$

or

$$\rho^2(A) \geq \max \left\{ \beta, \frac{1}{2}(1 + \beta - \alpha m)^2 - \beta,\ \frac{1}{2}(1 + \beta - \alpha M)^2 - \beta \right\}.$$

It is easy to verify that for every β,

$$\max \left\{ \frac{1}{2}(1 + \beta - \alpha m)^2 - \beta,\ \frac{1}{2}(1 + \beta - \alpha M)^2 - \beta \right\} \geq \frac{1}{2}(1 + \beta - \alpha' m)^2 - \beta,$$

where α' corresponds to the intersection point of the graphs of the functions of α inside the braces, satisfying

$$\frac{1}{2}(1 + \beta - \alpha' m)^2 - \beta = \frac{1}{2}(1 + \beta - \alpha' M)^2 - \beta$$

or

$$\alpha' = \frac{2(1+\beta)}{m+M}.$$

From Eqs. (4), (5), and the above formula for α', we obtain

$$\rho^2(A) \geq \max\left\{\beta, \frac{1}{2}\left((1+\beta)\frac{M-m}{m+M}\right)^2 - \beta\right\}$$

Again, consider the point β' that corresponds to the intersection point of the graphs of the functions of β inside the braces, satisfying

$$\beta' = \frac{1}{2}\left((1+\beta')\frac{M-m}{m+M}\right)^2 - \beta'.$$

We have

$$\beta' = \left(\frac{\sqrt{M}-\sqrt{m}}{\sqrt{M}+\sqrt{m}}\right)^2,$$

and

$$\max\left\{\beta, \frac{1}{2}\left((1+\beta)\frac{M-m}{m+M}\right)^2 - \beta\right\} \geq \beta'.$$

Therefore,

$$\rho(A) \geq \sqrt{\beta'} = \frac{\sqrt{M}-\sqrt{m}}{\sqrt{M}+\sqrt{m}}.$$

Note that equality in Eq. (6) is achievable for the (optimal) values

$$\beta' = \left(\frac{\sqrt{M}-\sqrt{m}}{\sqrt{M}+\sqrt{m}}\right)^2$$

and

$$\alpha' = \frac{2(1+\beta)}{m+M}.$$

In conclusion, we have

$$\min_{\alpha,\beta} \rho(A) = \frac{\sqrt{M}-\sqrt{m}}{\sqrt{M}+\sqrt{m}}$$

and the minimum is attained by some values $\alpha' > 0$ and $\beta' \in [0,1)$. Therefore, the convergence rate of the heavy ball method (2) with optimal choices of stepsize α and parameter β is governed by

$$\frac{\|x^{k+1}\|}{\|x^k\|} \leq \frac{\sqrt{M}-\sqrt{m}}{\sqrt{M}+\sqrt{m}}.$$

It can be seen that

$$\frac{\sqrt{M}-\sqrt{m}}{\sqrt{M}+\sqrt{m}} \leq \frac{M-m}{M+m},$$

so the convergence rate of the heavy ball iteration (2) is faster than the one of the steepest descent iteration (cf. Section 1.3.2).

1.3.10 (www)

By using the given property of the sequence $\{e^k\}$, we can obtain

$$||e^{k+1} - e^k|| \leq \beta^{k+1-\bar{k}} ||e^{\bar{k}} - e^{\bar{k}-1}||, \qquad \forall\ k \geq \bar{k}.$$

Thus, we have

$$\begin{aligned} ||e^m - e^k|| &\leq ||e^m - e^{m-1}|| + ||e^{m-1} - e^{m-2}|| + \ldots + ||e^{k+1} - e^k|| \\ &\leq \left(\beta^{m-\bar{k}+1} + \beta^{m-\bar{k}} + \ldots + \beta^{k-\bar{k}+1}\right) ||e^{\bar{k}} - e^{\bar{k}-1}|| \\ &\leq \beta^{1-\bar{k}}\ ||e^{\bar{k}} - e^{\bar{k}-1}||\ \sum_{j=k}^{m} \beta^j. \end{aligned}$$

By choosing $k_0 \geq \bar{k}$ large enough, we can make $\sum_{j=k}^{m} \beta^j$ arbitrarily small for all $m, k \geq k_0$. Therefore, $\{e^k\}$ is a Cauchy sequence. Let $\lim_{m\to\infty} e^m = e^*$, and let $m \to \infty$ in the inequality above, which results in

$$||e^k - e^*|| \leq \beta^{1-\bar{k}}\ ||e^{\bar{k}} - e^{\bar{k}-1}||\ \sum_{j=k}^{\infty} \beta^j = \beta^{1-\bar{k}}\ ||e^{\bar{k}} - e^{\bar{k}-1}||\ \frac{\beta^k}{1-\beta} = q^{\bar{k}}\beta^k, \tag{1}$$

for all $k \geq \bar{k}$, where $q^{\bar{k}} = \frac{\beta^{1-\bar{k}}}{1-\beta} ||e^{\bar{k}} - e^{\bar{k}-1}||$. Define the sequence $\{q^k \mid 0 \leq k < \bar{k}\}$ as follows

$$q^k = \frac{||e^k - e^*||}{\beta^k}, \qquad \forall\ k,\ \ 0 \leq k < \bar{k}. \tag{2}$$

Combining (1) and (2), it can be seen that

$$||e^k - e^*|| \leq q\beta^k, \qquad \forall\ k,$$

where $q = \max_{0 \leq k \leq \bar{k}} q^k$.

1.3.11 (www)

Since α^k is determined by Armijo rule, we know that $\alpha^k = \beta^{m_k} s$, where m_k is the first index m for which

$$f\left(x^k - \beta^m s \nabla f(x^k)\right) - f(x^k) \leq -\sigma\beta^m s ||\nabla f(x^k)||^2. \tag{1}$$

The second order expansion of f yields

$$f\left(x^k - \beta^i s \nabla f(x^k)\right) - f(x^k) = -\beta^i s ||\nabla f(x^k)||^2 + \frac{(\beta^i s)^2}{2} \nabla f(x^k)' \nabla^2 f(\bar{x}) \nabla f(x^k),$$

for some $\bar{x}$ that lies in the segment joining the points $x^k - \beta^i s \nabla f(x^k)$ and x^k. From the given property of f, it follows that

$$f\left(x^k - \beta^i s \nabla f(x^k)\right) - f(x^k) \leq -\beta^i s \left(1 - \frac{\beta^i s M}{2}\right) ||\nabla f(x^k)||^2. \tag{2}$$

Now, let i_k be the first index i for which $1 - \frac{M}{2}\beta^i s \geq \sigma$, i.e.

$$1 - \frac{M}{2}\beta^i s < \sigma \qquad \forall\ i,\ \ 0 \leq i \leq i_k, \qquad \text{and} \qquad 1 - \frac{M}{2}\beta^{i_k} s \geq \sigma. \tag{3}$$

Then, from (1)-(3), we can conclude that $m_k \leq i_k$. Therefore $\alpha^k \geq \hat{\alpha}^k$, where $\hat{\alpha}^k = \beta^{i_k} s$. Thus, we have

$$f\left(x^k - \alpha^k \nabla f(x^k)\right) - f(x^k) \leq -\sigma \hat{\alpha}^k ||\nabla f(x^k)||^2. \tag{4}$$

Note that (3) implies

$$\sigma > 1 - \frac{M}{2}\beta^{i_k - 1} s = 1 - \frac{M}{2\beta}\hat{\alpha}^k.$$

Hence, $\hat{\alpha}^k \geq 2\beta(1-\sigma)/M$. By substituting this in (4), we obtain

$$f(x^{k+1}) - f(x^*) \leq f(x^k) - f(x^*) - \frac{2\beta\sigma(1-\sigma)}{M}||\nabla f(x^k)||^2. \tag{5}$$

The given property of f implies that (see Exercise 1.1.9)

$$f(x) - f(x^*) \leq \frac{1}{2m}||\nabla f(x)||^2, \qquad \forall\ x \in \mathcal{R}^n, \tag{6}$$

$$\frac{m}{2}||x - x^*||^2 \leq f(x) - f(x^*), \qquad \forall\ x \in \mathcal{R}^n. \tag{7}$$

By combining (5) and (6), we obtain

$$f(x^{k+1}) - f(x^*) \leq r\left(f(x^k) - f(x^*)\right),$$

with $r = 1 - \frac{4m\beta\sigma(1-\sigma)}{M}$. Therefore, we have

$$f(x^k) - f(x^*) \leq r^k\left(f(x^0) - f(x^*)\right), \qquad \forall\ k,$$

which combined with (7) yields

$$||x^k - x^*||^2 \leq q r^k, \qquad \forall\ k,$$

with $q = \frac{2}{m}\left(f(x^0) - f(x^*)\right)$.

SECTION 1.4

1.4.2 (www)

From the proof of Prop. 1.4.1, we have

$$\|x^{k+1} - x^*\| \le M \left(\int_0^1 \|\nabla g(x^*) - \nabla g(x^* + t(x^k - x^*))\| dt \right) \|x^k - x^*\|.$$

By continuity of ∇g, we can take δ sufficiently small to ensure that the term under the integral sign is arbitrarily small. Let δ_1 be such that the term under the integral sign is less than r/M. Then

$$\|x^{k+1} - x^*\| \le r\|x^k - x^*\|.$$

Now, let

$$M(x) = \int_0^1 \nabla g\left(x^* + t(x - x^*)\right)' dt.$$

We then have $g(x) = M(x)(x - x^*)$. Note that $M(x^*) = \nabla g(x^*)$. We have that $M(x^*)$ is invertible. By continuity of ∇g, we can take δ to be such that the region S_δ around x^* is sufficiently small so the $M(x)'M(x)$ is invertible. Let δ_2 be such that $M(x)'M(x)$ is invertible. Then the eigenvalues of $M(x)'M(x)$ are all positive. Let γ and Γ be such that

$$0 < \gamma \le \min_{\|x - x^*\| \le \delta_2} eig\left(M(x)'M(x)\right) \le \max_{\|x - x^*\| \le \delta_2} eig\left(M(x)'M(x)\right) \le \Gamma.$$

Then, since $\|g(x)\|^2 = (x - x^*)'M'(x)M(x)(x - x^*)$, we have

$$\gamma\|x - x^*\|^2 \le \|g(x)\|^* \le \Gamma\|x - x^*\|^2,$$

or

$$\frac{1}{\sqrt{\Gamma}}\|g(x^{k+1})\| \le \|x^{k+1} - x^*\| \text{ and } r\|x^k - x^*\| \le \frac{r}{\sqrt{\gamma}}\|g(x^k)\|.$$

Since we've already shown that $\|x^{k+1} - x^*\| \le r\|x^k - x^*\|$, we have

$$\|g(x^{k+1})\| \le \frac{r\sqrt{\Gamma}}{\sqrt{\gamma}}\|g(x^k)\|.$$

Let $\hat{r} = \frac{r\sqrt{\Gamma}}{\sqrt{\gamma}}$. By letting $\hat{\delta}$ be sufficiently small, we can have $\hat{r} < r$. Letting $\delta = \min\{\hat{\delta}, \delta_2\}$ we have for any r, both desired results.

1.4.5 (www)

Since $\{x^k\}$ converges to nonsingular local minimum x^* of twice continuously differentiable function f and

$$\lim_{k \to \infty} ||H^k - \nabla^2 f(x^k)|| = 0,$$

we have that

$$\lim_{k\to\infty} ||H^k - \nabla^2 f(x^*)|| = 0. \tag{1}$$

Let m^k and m denote the smallest eigenvalues of H^k and $\nabla^2 f(x^*)$, respectively. The positive definiteness of $\nabla^2 f(x^*)$ and the Eq. (1) imply that for any $\epsilon > 0$ with $m - \epsilon > 0$ and k_0 large enough, we have

$$0 < m - \epsilon \le m^k \le m + \epsilon, \qquad \forall\ k \ge k_0. \tag{2}$$

For the truncated Newton method, the direction d^k is such that

$$\frac{1}{2} d^{k\prime} H^k d^k + \nabla f(x^k)' d^k < 0, \qquad \forall\ k \ge 0. \tag{3}$$

Define $q^k = \frac{d^k}{||\nabla f(x^k)||}$ and $p^k = \frac{\nabla f(x^k)}{||\nabla f(x^k)||}$. Then Eq. (3) can be written as

$$\frac{1}{2} q^{k\prime} H^k q^k + p^{k\prime} q^k < 0, \qquad \forall\ k \ge 0.$$

By the positive definiteness of H^k, we have

$$\frac{m^k}{2} ||q^k||^2 < ||q^k||, \qquad \forall\ k \ge 0,$$

where we have used the fact that $||p^k|| = 1$. Combining this and Eq. (2) we obtain that the sequence $\{q^k\}$ is bounded. Thus, we have

$$\begin{aligned}
\lim_{k\to\infty} \frac{||d^k + (\nabla^2 f(x^*))^{-1} \nabla f(x^k)||}{||\nabla f(x^k)||} &\le M \lim_{k\to\infty} \frac{||\nabla^2 f(x^*) d^k + \nabla f(x^k)||}{||\nabla f(x^k)||} \\
&= M \lim_{k\to\infty} ||\nabla^2 f(x^*) q^k + p^k|| \\
&\le M \lim_{k\to\infty} ||\nabla^2 f(x^*) - H^k|| \cdot ||q^k|| + M \lim_{k\to\infty} ||H^k q^k + p^k|| \\
&= 0,
\end{aligned}$$

where $M = ||(\nabla^2 f(x^*))^{-1}||$. Now we have that all the conditions of Prop. 1.3.2 are satisfied, so $\{||x^k - x^*||\}$ converges superlinearly.

1.4.6 (www)

For the function $f(x) = \|x\|^3$, we have

$$\nabla f(x) = 3\|x\|x, \quad \nabla^2 f(x) = 3\|x\| + \frac{3}{\|x\|} xx' = \frac{3}{\|x\|}(\|x\|^2 I + xx').$$

Using the formula $(A + CBC')^{-1} = A^{-1} - A^{-1}C(B^{-1} + C'A^{-1}C)^{-1}C'A^{-1}$ [Eq. (A.7) from Appendix A], we have

$$(\|x\|^2 I + xx')^{-1} = \frac{1}{\|x\|^2}\left(I - \frac{1}{2\|x\|^2} xx'\right),$$

and so

$$(\nabla^2 f(x))^{-1} = \frac{1}{3\|x\|}\left(I - \frac{1}{2\|x\|^2}xx'\right).$$

Newton's method is then

$$\begin{aligned} x^{k+1} &= x^k - \alpha\left(\nabla^2 f(x^k)\right)^{-1}\nabla f(x^k) \\ &= x^k - \alpha\frac{1}{3\|x^k\|}\left(I - \frac{1}{2\|x^k\|^2}x^k(x^k)'\right)3\|x^k\|x^k \\ &= x^k - \alpha\left(x^k - \frac{1}{2\|x^k\|^2}x^k\|x^k\|^2\right) \\ &= x^k - \alpha\left(x^k - \frac{1}{2}x^k\right) \\ &= \left(1 - \frac{\alpha}{2}\right)x^k. \end{aligned}$$

Thus for $0 < \alpha < 2$, Newton's method converges linearly to $x^* = 0$. For $\alpha^0 = 2$ method converges in one step. Note that the method also converges linearly for $2 < \alpha < 4$. Proposition 1.4.1 does not apply since $\nabla^2 f(0)$ is not invertible. Otherwise, we would have superlinear convergence.

Alternatively, instead of inverting $\nabla^2 f(x)$, we can calculate the Newton direction at a vector x by guessing (based on symmetry) that it has the form γx for some scalar γ, and by determining the value of γ through the equation $\nabla^2 f(x)(\gamma x) = -\nabla f(x)$. In this way, we can verify that $\gamma = -1/2$.

SECTION 1.6

1.6.3 (www)

We have that

$$f(x^{k+1}) \leq \max_i \left(1 + \lambda_i P^k(\lambda_i)\right)^2 f(x^0), \tag{1}$$

for any polynomial P^k of degree k and any k, where $\{\lambda_i\}$ is the set of the eigenvalues of Q. Chose P^k such that

$$1 + \lambda P^k(\lambda) = \frac{(z_1 - \lambda)}{z_1} \cdot \frac{(z_2 - \lambda)}{z_2} \cdots \frac{(z_k - \lambda)}{z_k}.$$

Define $I_j = [z_j - \delta_j, z_j + \delta_j]$ for $j = 1, \ldots, k$. Since $\lambda_i \in I_j$ for some j, we have

$$\left(1 + \lambda_i P^k(\lambda_i)\right)^2 \leq \max_{\lambda \in I_j}\left(1 + \lambda P^k(\lambda)\right)^2.$$

Hence

$$\max_i \left(1 + \lambda_i P^k(\lambda_i)\right)^2 \leq \max_{1 \leq j \leq k} \max_{\lambda \in I_j} \left(1 + \lambda P^k(\lambda)\right)^2. \tag{2}$$

For any j and $\lambda \in I_j$ we have

$$\begin{aligned}
\left(1 + \lambda P^k(\lambda)\right)^2 &= \frac{(z_1 - \lambda)^2}{z_1^2} \cdot \frac{(z_2 - \lambda)^2}{z_2^2} \cdots \frac{(z_k - \lambda)^2}{z_k^2} \\
&\leq \frac{(z_j + \delta_j - z_1)^2 (z_j + \delta_j - z_2)^2 \cdots (z_j + \delta_j - z_{j-1})^2 \delta_j^2}{z_1^2 \cdots z_j^2}.
\end{aligned}$$

Here we used the fact that $\lambda \in I_j$ implies $\lambda < z_l$ for $l = j+1, \ldots, k$, and therefore $\frac{(z_l - \lambda)^2}{z_l^2} \leq 1$ for all $l = j+1, \ldots, k$. Thus, from (2) we obtain

$$\max_i \left(1 + \lambda_i P^k(\lambda_i)\right)^2 \leq R, \tag{3}$$

where

$$R = \left\{ \frac{\delta_1^2}{z_1^2}, \frac{\delta_2^2 (z_2 + \delta_2 - z_1)^2}{z_1^2 z_2^2}, \ldots, \frac{\delta_k^2 (z_k + \delta_k - z_1)^2 \cdots (z_k + \delta_k - z_{k-1})^2}{z_1^2 z_2^1 \cdots z_k^2} \right\}.$$

The desired estimate follows from (1) and (3).

1.6.4 (www)

It suffices to show that the subspace spanned by $g^0, g^1, \ldots, g^{k-1}$ is the same as the subspace spanned by $g^0, Qg^0, \ldots, Q^{k-1}g^0$, for $k = 1, \ldots, n$. We will prove this by induction. Clearly, for $k = 1$ the statement is true. Assume it is true for $k - 1 < n - 1$, i.e.

$$\text{span}\{g^0, g^1, \ldots, g^{k-1}\} = \text{span}\{g^0, Qg^0, \ldots, Q^{k-1}g^0\},$$

where $\text{span}\{v^0, \ldots, v^l\}$ denotes the subspace spanned by the vectors $v^0, \ldots, v^l$. Assume that $g^k \neq 0$ (i.e. $x^k \neq x^*$). Since $g^k = \nabla f(x^k)$ and x^k minimizes f over the manifold $x^0 + \text{span}\{g^0, g^1, \ldots, g^{k-1}\}$, from our assumption we have that

$$g^k = Qx^k - b = Q\left(x^0 + \sum_{i=0}^{k-1} \xi_i Q^i g^0\right) - b = Qx^0 - b + \sum_{i=0}^{k-1} \xi_i Q^{i+1} g^0.$$

The fact that $g^0 = Qx^0 - b$ yields

$$g^k = g^0 + \xi_0 Q g^0 + \xi_1 Q^2 g^0 + \ldots + \xi_{k-2} Q^{k-1} g^0 + \xi_{k-1} Q^k g^0. \tag{1}$$

If $\xi_{k-1} = 0$, then from (1) and the inductive hypothesis it follows that

$$g^k \in \text{span}\{g^0, g^1, \ldots, g^{k-1}\}. \tag{2}$$

We know that g^k is orthogonal to $g^0, \ldots, g^{k-1}$. Therefore (2) is possible only if $g^k = 0$ which contradicts our assumption. Hence, $\xi_{k-1} \neq 0$. If $Q^k g^0 \in \text{span}\{g^0, Qg^0, \ldots, Q^{k-1}g^0\}$, then (1) and our inductive hypothesis again imply (2) which is not possible. Thus the vectors $g^0, Qg^0, \ldots, Q^{k-1}g^0, Q^k g^0$ are linearly independent. This combined with (1) and linear independence of the vectors $g^0, \ldots, g^{k-1}, g^k$ implies that

$$\text{span}\{g^0, g^1, \ldots, g^{k-1}, g^k\} = \text{span}\{g^0, Qg^0, \ldots, Q^{k-1}g^0, Q^k g^0\},$$

which completes the proof.

1.6.5 (www)

Let x^k be the sequence generated by the conjugate gradient method, and let d^k be the sequence of the corresponding Q-conjugate directions. We know that x^{k+1} minimizes f over

$$x^0 + \text{span}\,\{d^0, d^1, \ldots, d^k\}.$$

Let $\tilde{x}^k$ be the sequence generated by the method described in the exercise. In particular, $\tilde{x}^1$ is generated from x^0 by steepest descent and line minimization, and for $k \geq 1$, $\tilde{x}^{k+1}$ minimizes f over the two-dimensional linear manifold

$$\tilde{x}^k + \text{span}\,\{\tilde{g}^k \text{ and } \tilde{x}^k - \tilde{x}^{k-1}\},$$

where $\tilde{g}^k = \nabla f(\tilde{x}^k)$. We will show by induction that $x^k = \tilde{x}^k$ for all $k \geq 1$.

Indeed, we have by construction $x^1 = \tilde{x}^1$. Suppose that $x^i = \tilde{x}^i$ for $i = 1, \ldots, k$. We will show that $x^{k+1} = \tilde{x}^{k+1}$. We have that $\tilde{g}^k$ is equal to $g^k = \beta^k d^{k-1} - d^k$ so it belongs to the subspace spanned by d^{k-1} and d^k. Also $\tilde{x}^k - \tilde{x}^{k-1}$ is equal to $x^k - x^{k-1} = \alpha^{k-1} d^{k-1}$. Thus

$$\text{span}\,\{\tilde{g}^k \text{ and } \tilde{x}^k - \tilde{x}^{k-1}\} = \text{span}\,\{d^{k-1} \text{ and } d^k\}.$$

Observe that x^k belongs to

$$x^0 + \text{span}\,\{d^0, d^1, \ldots, d^{k-1}\},$$

so

$$x^0 + \text{span}\,\{d^0, d^1, \ldots, d^{k-1}\} \supset x^k + \text{span}\,\{d^{k-1} \text{ and } d^k\} \supset x^k + \text{span}\,\{d^k\}.$$

The vector x^{k+1} minimizes f over the linear manifold on the left-hand side above, and also over the linear manifold on the right-hand side above (by the definition of a conjugate direction method). Moreover, $\tilde{x}^{k+1}$ minimizes f over the linear manifold in the middle above. Hence $x^{k+1} = \tilde{x}^{k+1}$.

1.6.6 (PARTAN)

Suppose that $x^1, \ldots, x^k$ have been generated by the method of Exercise 1.6.5, which by the result of that exercise, is equivalent to the conjugate gradient method. Let y^k and x^{k+1} be generated by the two line searches given in the exercise.

By the definition of the congugate gradient method, x^k minimizes f over

$$x^0 + \text{span}\,\{g^0, g^1, \ldots, g^{k-1}\},$$

so that

$$g^k \perp \text{span}\,\{g^0, g^1, \ldots, g^{k-1}\},$$

and in particular

$$g^k \perp g^{k-1}. \tag{1}$$

Also, since y^k is the vector that minimizes f over the line $y_\alpha = x^k - \alpha g^k$, $\alpha \geq 0$, we have

$$g^k \perp \nabla f(y^k). \tag{2}$$

Any vector on the line passing through x^{k-1} and y^k has the form

$$y = \alpha x^{k-1} + (1-\alpha) y^k, \qquad \alpha \in \Re,$$

and the gradient of f at such a vector has the form

$$\begin{aligned} \nabla f\big(\alpha x^{k-1} + (1-\alpha)y^k\big) &= Q\big(\alpha x^{k-1} + (1-\alpha)y^k\big) - b \\ &= \alpha(Qx^{k-1} - b) + (1-\alpha)(Qy^k - b) \\ &= \alpha g^{k-1} + (1-\alpha)\nabla f(y^k). \end{aligned} \tag{3}$$

From Eqs. (1)-(3), it follows that g^k is orthogonal to the gradient $\nabla f(y)$ of any vector y on the line passing through x^{k-1} and y^k.

In particular, for the vector x^{k+1} that minimizes f over this line, we have that $\nabla f(x^{k+1})$ is orthogonal to g^k. Furthermore, because x^{k+1} minimizes f over the line passing through x^{k-1} and y^k, $\nabla f(x^{k+1})$ is orthogonal to $y^k - x^{k-1}$. Thus, $\nabla f(x^{k+1})$ is orthogonal to

$$\text{span}\,\{g^k, y^k - x^{k-1}\},$$

and hence also to

$$\text{span}\,\{g^k, x^k - x^{k-1}\},$$

since x^{k-1}, x^k, and y^k form a triangle whose side connecting x^k and y^k is proportional to g^k. Thus x^{k+1} minimizes f over

$$x^k + \text{span}\,\{g^k, x^k - x^{k-1}\},$$

and it is equal to the one generated by the algorithm of Exercise 1.6.5.

1.6.7 (www)

The objective is to minimize over $\Re^n$, the positive semidefinite quadratic function

$$f(x) = \frac{1}{2}x'Qx + b'x.$$

The value of x^k following the kth iteration is

$$x^k = \arg\min\left\{f(x)|x = x^0 + \sum_{i=1}^{k-1}\gamma^i d^i, \gamma^i \in \Re\right\} = \arg\min\left\{f(x)|x = x^0 + \sum_{i=1}^{k-1}\delta^i g^i, \delta^i \in \Re\right\},$$

where d^i are the conjugate directions, and g^i are the gradient vectors. At the beginning of the $(k+1)$st iteration, there are two possibilities:

(1) $g^k = 0$: In this case, x^k is the global minimum since $f(x)$ is a convex function.

(2) $g^k \neq 0$: In this case, a new conjugate direction d^k is generated. Here, we also have two possibilities:

   (a) A minimum is attained along the direction d^k and defines x^{k+1}.

   (b) A minimum along the direction d^k does not exist. This occurs if there exists a direction d in the manifold spanned by $d^0, \ldots, d^k$ such that $d'Qd = 0$ and $b'd \neq 0$. The problem in this case has no solution.

If the problem has no solution (which occurs if there is some vector d such that $d'Qd = 0$ but $b'd \neq 0$), the algorithm will terminate because the line minimization problem along such a direction d is unbounded from below.

If the problem has infinitely many solutions (which will happen if there is some vector d such that $d'Qd = 0$ and $b'd = 0$), then the algorithm will proceed as if the matrix Q were positive definite, i.e. it will find one of the solutions (case 1 occurs).

However, in both situations the algorithm will terminate in at most m steps, where m is the rank of the matrix Q, because the manifold

$$\{x \in \Re^n | x = x^0 + \sum_{i=0}^{k-1}\gamma^i d^i, \gamma^i \in \Re\}$$

will not expand for $k > m$.

1.6.8 (www)

Let S_1 and S_2 be the subspaces with $S_1 \cap S_2$ being a proper subspace of $\Re^n$ (i.e. a subspace of $\Re^n$ other than $\{0\}$ and $\Re^n$ itself). Suppose that the subspace $S_1 \cap S_2$ is spanned by linearly independent vectors $v_k,\ k \in K \subseteq \{1, 2, \ldots, n\}$. Assume that x^1 and x^2 minimize the given quadratic function f over the manifolds M_1 and M_2 that are parallel to subspaces S_1 and S_2, respectively, i.e.

$$x^1 = \arg \min_{x \in M_1} f(x) \quad \text{and} \quad x^2 = \arg \min_{x \in M_2} f(x)$$

where $M_1 = y^1 + S_1,\ M_2 = y^2 + S_2$, with some vectors $y^1, y^2 \in \Re^n$. Assume also that $x^1 \neq x^2$. Without loss of generality we may assume that $f(x^2) > f(x^1)$. Since $x^2 \notin M_1$, the vectors $x^2 - x^1$ and $\{v_k \mid k \in K\}$ are linearly independent. From the definition of x^1 and x^2 we have that

$$\frac{d}{dt} f(x^1 + tv^k)\bigg|_{t=0} = 0 \qquad \text{and} \qquad \frac{d}{dt} f(x^2 + tv^k)\bigg|_{t=0} = 0,$$

for any v^k. When this is written out, we get

$$x^{1\prime} Q v^k - b' v^k = 0 \qquad \text{and} \qquad x^{2\prime} Q v^k - b' v^k = 0.$$

Subtraction of the above two equalities yields

$$(x^1 - x^2)' Q v^k = 0, \qquad \forall\ k \in K.$$

Hence, $x^1 - x^2$ is Q-conjugate to all vectors in the intersection $S_1 \cap S_2$. We can use this property to construct a conjugate direction method that does not evaluate gradients and uses only line minimizations in the following way.

Initialization: Choose any direction d^1 and points y^1 and z^1 such that $M_1^1 = y^1 + \text{span}\{d^1\}$, $M_2^1 = z^1 + \text{span}\{d^1\},\ M_1^1 \neq M_2^1$. Let $d^2 = x_1^1 - x_1^2$, where $x_1^i = \arg\min_{x \in M_i^1} f(x)$ for $i = 1, 2$.

Generating new conjugate direction: Suppose that Q-conjugate directions $d^1, d^2, \ldots, d^k$, $k < n$ have been generated. Let $M_1^k = y^k + \text{span}\{d^1, \ldots d^k\}$ and $x_k^1 = \arg\min_{x \in M_1^k} f(x)$. If x_k^1 is not optimal there is a point z^k such that $f(z^k) < f(x_k^1)$. Starting from point z^k we again search in the directions $d^1, d^2, \ldots, d^k$ obtaining a point x_k^2 which minimizes f over the manifold M_2^k generated by z^k and $d^1, d^2, \ldots, d^k$. Since $f(x_k^2) \leq f(z^k)$, we have

$$f(x_k^2) < f(x_k^1).$$

As both x_k^1 and x_k^2 minimize f over the manifolds that are parallel to $\text{span}\{d^1, \ldots, d^k\}$, setting $d^{k+1} = x_k^2 - x_k^1$ we have that $d^1, \ldots, d^k, d^{k+1}$ are Q-conjugate directions (here we have used the established property).

In this procedure it is important to have a step which given a nonoptimal point x generates a point y for which $f(y) < f(x)$. If x is an optimal solution then the step must indicate this fact. Simply, the step must first determine whether x is optimal, and if x is not optimal, it must find a better point. A typical example of such a step is one iteration of the cyclic coordinate descent method, which avoids calculation of derivatives.

SECTION 1.7

1.7.1 (www)

The proof is by induction. Suppose the relation $D^k q^i = p^i$ holds for all k and $i \leq k-1$. The relation $D^{k+1} q^i = p^i$ also holds for $i = k$ because of the following calculation

$$D^{k+1} q^k = D^k q^k + \frac{y^k y^{k\prime} q^k}{q^{k\prime} y^k} = D^k q^k + y^k = D^k q^k + (p^k - D^k q^k) = p^k.$$

For $i < k$, we have, using the induction hypothesis $D^k q^i = p^i$,

$$D^{k+1} q^i = D^k q^i + \frac{y^k (p^k - D^k q^k)' q^i}{q^{k\prime} y^k} = p^i + \frac{y^k (p^{k\prime} q^i - q^{k\prime} p^i)}{q^{k\prime} y^k}.$$

Since $p^{k\prime} q^i = p^{k\prime} Q p^i = q^{k\prime} p^i$, the second term in the right-hand side vanishes and we have $D^{k+1} q^i = p^i$. This completes the proof.

To show that $(D^n)^{-1} = Q$, note that from the equation $D^{k+1} q^i = p^i$, we have

$$D^n = \left[p^0 \ \cdots \ p^{n-1} \right] \left[q^0 \ \cdots \ q^{n-1} \right]^{-1}, \qquad (*)$$

while from the equation $Qp^i = Q(x^{i+1} - x^i) = (Qx^{i+1} - b) - (Qx^i - b) = \nabla f(x^{i+1}) - \nabla f(x^i) = q^i$, we have

$$Q \left[p^0 \ \cdots \ p^{n-1} \right] = \left[q^0 \ \cdots \ q^{n-1} \right],$$

or equivalently

$$Q = \left[q^0 \ \cdots \ q^{n-1} \right] \left[p^0 \ \cdots \ p^{n-1} \right]^{-1}. \qquad (**)$$

(Note here that the matrix $\left[p^0 \ \cdots \ p^{n-1} \right]$ is invertible, since both Q and $\left[q^0 \ \cdots \ q^{n-1} \right]$ are invertible by assumption.) By comparing Eqs. (*) and (**), it follows that $(D^n)^{-1} = Q$.

1.7.2 (www)

For simplicity, we drop superscripts. The BFGS update is given by

$$\begin{aligned}
\bar{D} &= D + \frac{pp'}{p'q} - \frac{Dqq'D}{q'Dq} + q'Dq \left(\frac{p}{p'q} - \frac{Dq}{q'Dq} \right) \left(\frac{p}{p'q} - \frac{Dq}{q'Dq} \right)' \\
&= D + \frac{pp'}{p'q} - \frac{Dqq'D}{q'Dq} + q'Dq \left(\frac{pp'}{(p'q)^2} - \frac{Dqp' + pq'D}{(p'q)(q'Dq)} + \frac{Dqq'D}{(q'Dq)^2} \right). \\
&= D + \left(1 + \frac{q'Dq}{p'q} \right) \frac{pp'}{p'q} - \frac{Dqp' + pq'D}{p'q}
\end{aligned}$$

1.7.3 (www)

(a) For simplicity, we drop superscripts. Let $V = I - \rho q p'$, where $\rho = 1/(q'p)$. We have

$$\begin{aligned} V'DV + \rho pp' &= (I - \rho qp')' D (I - \rho qp') + \rho pp' \\ &= D - \rho(Dqp' + pq'D) + \rho^2 pq'Dqp' + \rho pp' \\ &= D - \frac{Dqp' + pq'D}{q'p} + \frac{(q'Dq)(pp')}{(q'p)^2} + \frac{pp'}{q'p} \\ &= D + \left(1 + \frac{q'Dq}{p'q}\right)\frac{pp'}{p'q} - \frac{Dqp' + pq'D}{p'q} \end{aligned}$$

and the result now follows using the alternative BFGS update formula of Exercise 1.7.2.

(b) We have, by using repeatedly the update formula for D of part (a),

$$\begin{aligned} D^k &= V^{k-1'} D^{k-1} V^{k-1} + \rho^{k-1} p^{k-1} p^{k-1'} \\ &= V^{k-1'} V^{k-2'} D^{k-2} V^{k-2} V^{k-1} + \rho^{k-2} V^{k-1'} p^{k-2} p^{k-2'} V^{k-1} + \rho^{k-1} p^{k-1} p^{k-1'}, \end{aligned}$$

and proceeding similarly,

$$\begin{aligned} D^k = {} & V^{k-1'} V^{k-2'} \cdots V^{0'} D^0 V^0 \cdots V^{k-2} V^{k-1} \\ & + \rho^0 V^{k-1'} \cdots V^{1'} p^0 p^{0'} V^1 \cdots V^{k-1} \\ & + \rho^1 V^{k-1'} \cdots V^{2'} p^1 p^{1'} V^2 \cdots V^{k-1} \\ & + \cdots \\ & + \rho^{k-2} V^{k-1'} p^{k-2} p^{k-2'} V^{k-1} \\ & + \rho^{k-1} p^{k-1} p^{k-1'} \end{aligned} .$$

Thus to calculate the direction $-D^k \nabla f(x^k)$, we need only to store D^0 and the past vectors p^i, q^i, $i = 0, 1, \ldots, k-1$, and to perform the matrix-vector multiplications needed using the above formula for D^k. Note that multiplication of a matrix V^i or $V^{i'}$ with any vector is relatively simple. It requires only two vector operations: one inner product, and one vector addition.

1.7.4 (www)

Suppose that D is updated by the DFP formula and H is updated by the BFGS formula. Thus the update formulas are

$$\bar{D} = D + \frac{pp'}{p'q} - \frac{Dqq'D}{q'Dq},$$

$$\bar{H} = H + \left(1 + \frac{p'Hp}{q'p}\right)\frac{qq'}{q'p} - \frac{Hpq' + qp'H}{q'p}.$$

If we assume that HD is equal to the identity I, and form the product $\bar{H}\bar{D}$ using the above formulas, we can verify with a straightforward calculation that $\bar{H}\bar{D}$ is equal to I. Thus if the initial H and D are inverses of each other, the above updating formulas will generate (at each step) matrices that are inverses of each other.

1.7.5 (www)

(a) By pre- and postmultiplying the DFP update formula

$$\bar{D} = D + \frac{pp'}{p'q} - \frac{Dqq'D}{q'Dq},$$

with $Q^{1/2}$, we obtain

$$Q^{1/2}\bar{D}Q^{1/2} = Q^{1/2}DQ^{1/2} + \frac{Q^{1/2}pp'Q^{1/2}}{p'q} - \frac{Q^{1/2}Dqq'DQ^{1/2}}{q'Dq}.$$

Let

$$\bar{R} = Q^{1/2}\bar{D}Q^{1/2}, \qquad R = Q^{1/2}DQ^{1/2},$$

$$r = Q^{1/2}p, \qquad q = Qp = Q^{1/2}r.$$

Then the DFP formula is written as

$$\bar{R} = R + \frac{rr'}{r'r} - \frac{Rrr'R}{r'Rr}.$$

Consider the matrix

$$P = R - \frac{Rrr'R}{r'Rr}.$$

From the interlocking eigenvalues lemma, the eigenvalues $\mu_1, \ldots, \mu_n$ satisfy

$$\mu_1 \le \lambda_1 \le \mu_2 \le \cdots \le \mu_n \le \lambda_n,$$

where $\lambda_1, \ldots \lambda_n$ are the eigenvalues of R. We have $Pr = 0$, so 0 is an eigenvalue of P and r is a corresponding eigenvector. Hence, since $\lambda_1 > 0$, we have $\mu_1 = 0$. Consider the matrix

$$\bar{R} = P + \frac{rr'}{r'r}.$$

We have $\bar{R}r = r$, so 1 is an eigenvalue of $\bar{R}$. The other eigenvalues are the eigenvalues $\mu_2, \ldots, \mu_n$ of P, since their corresponding eigenvectors $e_2, \ldots, e_n$ are orthogonal to r, so that

$$\bar{R}e_i = Pe_i = \mu_i e_i, \qquad i = 2, \ldots, n.$$

(b) We have

$$\lambda_1 \le \frac{r'Rr}{r'r} \le \lambda_n,$$

so if we multiply the matrix R with $r'r/r'Rr$, its eigenvalue range shifts so that it contains 1. Since

$$\frac{r'r}{r'Rr} = \frac{p'Qp}{p'Q^{1/2}RQ^{1/2}p} = \frac{p'q}{q'Q^{-1/2}RQ^{-1/2}q} = \frac{p'q}{q'Dq},$$

multiplication of R by $r'r/r'Rr$ is equivalent to multiplication of D by $p'q/q'Dq$.

(c) In the case of the BFGS update

$$\bar{D} = D + \left(1 + \frac{q'Dq}{p'q}\right)\frac{pp'}{p'q} - \frac{Dqp' + pq'D}{p'q},$$

(cf. Exercise 1.7.2) we again pre- and postmultiply with $Q^{1/2}$. We obtain

$$\bar{R} = R + \left(1 + \frac{r'Rr}{r'r}\right)\frac{rr'}{r'r} - \frac{Rrr' + rr'R}{r'r},$$

and an analysis similar to the ones in parts (a) and (b) goes through.

1.7.6 (www)

(a) We use induction. Assume that the method coincides with the conjugate gradient method up to iteration k. For simplicity, denote for all k,

$$g^k = \nabla f(x^k).$$

We have, using the facts $p^{k'} g^{k+1} = 0$ and $p^k = \alpha^k d^k$,

$$\begin{aligned} d^{k+1} &= -D^{k+1} g^{k+1} \\ &= -\left(I + \left(1 + \frac{q^{k'}q^k}{p^{k'}q^k}\right)\frac{p^k p^{k'}}{p^{k'}q^k} - \frac{q^k p^{k'} + p^k q^{k'}}{p^{k'}q^k}\right) g^{k+1} \\ &= -g^{k+1} + \frac{p^k q^{k'} g^{k+1}}{p^{k'}q^k} \\ &= -g^{k+1} + \frac{(g^{k+1} - g^k)' g^{k+1}}{d^{k'}q^k} d^k. \end{aligned}$$

The argument given at the end of the proof of Prop. 1.6.1 shows that this formula is the same as the conjugate gradient formula.

(b) Use a scaling argument, whereby we work in the transformed coordinate system $y = D^{-1/2}x$, where the matrix D becomes the identity.