Impact Mechanics

1D Elastic-Plastic Waves

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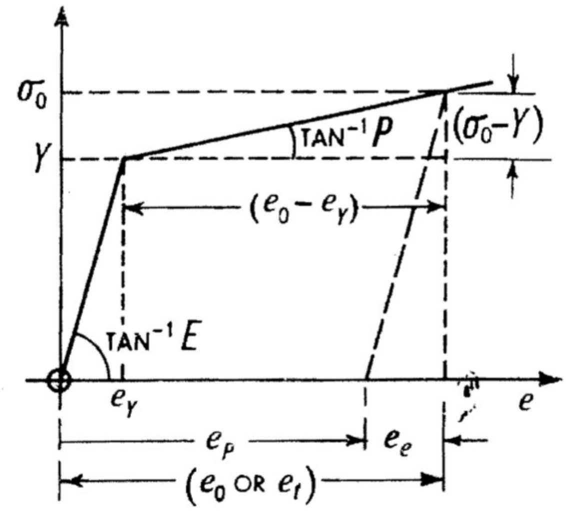
Elastic-Plastic Waves in a Long Uniform Bar

Elastic-plastic Waves

- We consider the dynamic loading in tension of a long thin bar, the loaded end of which moves in some prescribed way so as to initiate elastic and plastic longitudinal waves.
- We assume throughout that the tensile stress-tensile strain relation for monotonic loading is at all times independent of the rate of strain.
- Donnell apparently first investigated longitudinal plastic wave propagation in a paper published in 1930.
- If the nominal stress suddenly reached at the end of the bar is $\sigma_0 < Y$, it will be propagated through the unstrained bar at a speed of $\sqrt{E/\rho}$.

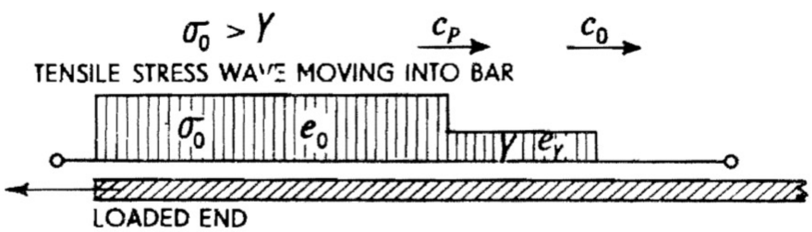
Elastic-plastic Waves

- Further, if a stress σ_0 is applied where $\sigma_0 > Y$ so that the nominal stress increment $(\sigma_0 - Y)$ and strain increment $(e_0 - e_y)$ are related through the modulus P, then this excess stress should be propagated with a speed of $\sqrt{P/\rho_0}$ through the bar in its unstrained configuration.



Elastic-plastic Waves

- If the speed at which the free end of the bar is moved at an instant results in a tensile stress $\sigma_0 > Y$, this stress may be expected to be transmitted by two waves, which start at the same instant from the loaded end of the bar but move at the different speeds $c_0 = \sqrt{E/\rho}$ and $c_p = \sqrt{P/\rho}$.



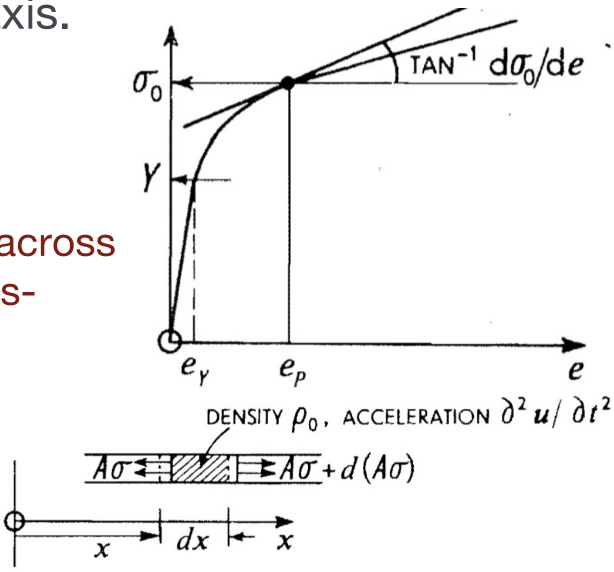
- As t increases the distance between the head of each of the two waves increases. Thus elastic stress Y and corresponding elastic strain e_y are propagated at speed c_0 whilst the plastic wave following up at speed c_p increases Y to σ_0.

Elastic-plastic Waves

- A slight generalization may now be arrived at for the speed of propagation of longitudinal stress wave through a bar of material which has a continuously turning $\sigma_0 - e$ curve, concave to the strain axis.
- Equation of motion:

$$d(A\sigma) = \rho_0 A_0 dx.\,\partial^2 u/\partial t^2$$

  - σ is the true longitudinal stress across the element whose current cross-sectional area is A.
  - However, as $A_0\sigma_0 = A\sigma$ where A_0 is the initial cross-sectional area of the bar,



$$d(A_0\sigma_0) = \rho_0 A_0 dx.\,\partial^2 u/\partial t^2$$

Elastic-plastic Waves

- Hence,

$$\frac{d\sigma_0}{de} = \rho_0 \frac{dx}{de} \cdot \frac{\partial^2 u}{\partial t^2}$$

- Since $e = \partial u/\partial x$, $de/dx = \partial^2 u/\partial x^2$,

$$\frac{\partial^2 u}{\partial t^2} = \frac{d\sigma_0/de}{\rho_0} \cdot \frac{\partial^2 u}{\partial x^2}$$

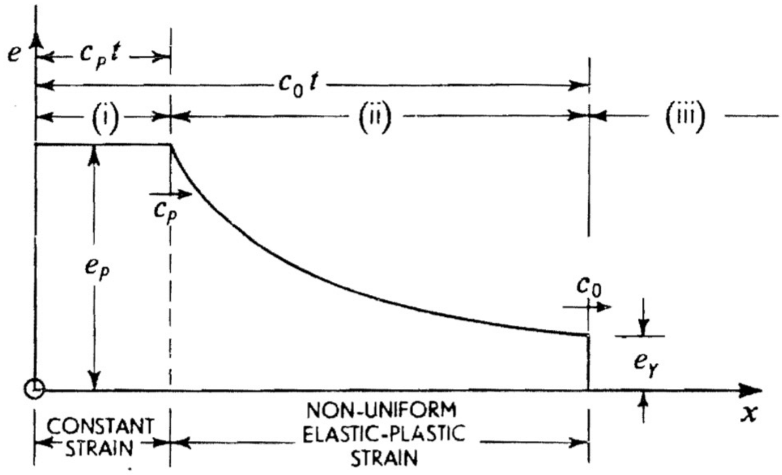
- Thus the speed of wave propagation along the x-axis is,

$$c_p = \sqrt{(d\sigma_0/de)/\rho_0}$$

  - Note that for elastic wave $d\sigma_0/de = E$, and for a bilinear plasticity, $d\sigma_0/de = P$.

Elastic-plastic Waves

- If the long bar is loaded to a nominal stress level of σ_0 instantaneously, with strain e_p, then over the elastic range of stress, the wave speed is constant at $c_0 = \sqrt{E/\rho}$, whilst for every stress level $\sigma_0 > Y$, the wave speed is less, at $\sqrt{(d\sigma_0/de)/\rho_0}$, being the smaller, the greater is σ_0.



Elastic-plastic Waves

- Three distinct regions at given time t may be identified by reference to the position of the unstressed bar,
  - (i) Between $x = 0$ and $x = c_p t$, the strain is constant at e_p; $c_p = \sqrt{(d\sigma_0/de)/\rho_0}$, where σ_0 is the greatest nominal stress imposed.
  - (ii) Between $x = c_p t$ and $x = c_0 t$, there is a variable distribution of strain between e_p and e_Y.
  - (iii) For $x > c_p t$ i.e. ahead of the elastic wave, the bar is physically unstressed.

Particle Speed

- A simple and direct derivation of the speed of movement at the end of a bar in order to produce a strain of e_p, is obtained after considering a small element of the bar in its unstretched state of length dx, and noting that the time, dt, taken for it to propagate a force increment $d(A_0\sigma_0)$ at stress level σ_0 is dx/c_p.
  - We emphasize that the speed here has reference to the unstretched length of the element or the space occupied by it.

$$dt = \frac{dx}{\sqrt{(d\sigma_0/de)/\rho_0}}$$

- However, applying the momentum equation,

$$(\rho_0 A_0 dx) \cdot dv = d(A_0\sigma_0)dt$$

Particle Speed

- Hence, eliminating dt,

$$dv = \frac{d\sigma_0}{\rho_0\sqrt{\frac{d\sigma_0/de}{\rho_0}}}$$

- Thus the total speed acquired by the element,

$$v = \int_0^{e_P} \sqrt{\frac{d\sigma_0/de}{\rho_0}} \cdot de = \int_0^{e_P} c_0\sqrt{\frac{d\sigma_0/de}{E}} \cdot de$$

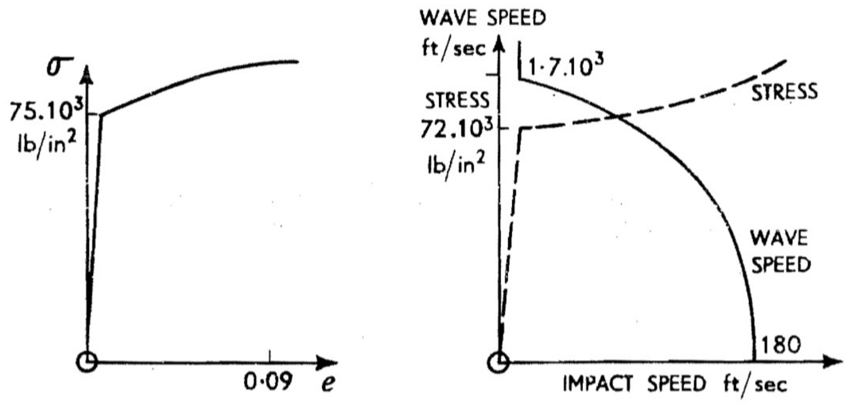
- For the bilinear nominal stress-engineering curve,

$$v = e_Y\sqrt{E/\rho_0} + (e_P - e_Y)\sqrt{P/\rho_0}$$

$$v = e_Y c_0 + (e_P - e_Y)c_1$$

Critical Impact Speed

- The critical impact speed for a bar, v_c, is the speed of the loaded end which will cause fracture in the bar.
- We have, $v_c = \int_0^{e_u} \sqrt{\frac{d\sigma_0/de}{\rho_0}} \cdot de$



- From the stress-strain curve given, the computed value of v_c was 150 ft/sec and rupture of the specimen actually occurred at an impact speed of 171 ft/sec.

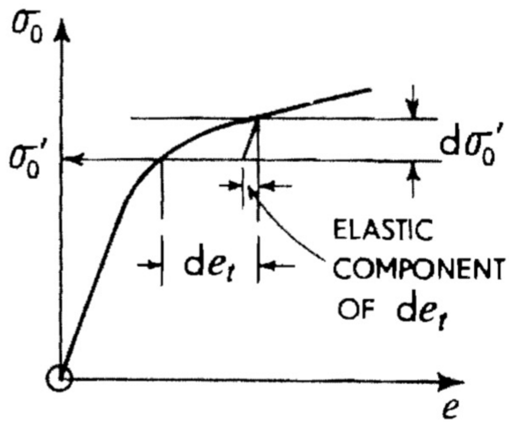
Shock Wave

- In all the cases so far considered the stress-strain curve has been concave towards the strain axis.
- In a few cases (e.g. nickel-chrome steel and poly-crystalline material), the slope increases with strain, the speed of propagation of a stress wave in a long rod increases with increase in stress intensity.
- The largest stress or strain imposed is propagated at a faster rate than the early or lower stresses and strains so that, if the bar is sufficiently long, following waves will overtake early waves and eventually all should coalesce to give one strong wave front, a shock wave.

Shock Wave

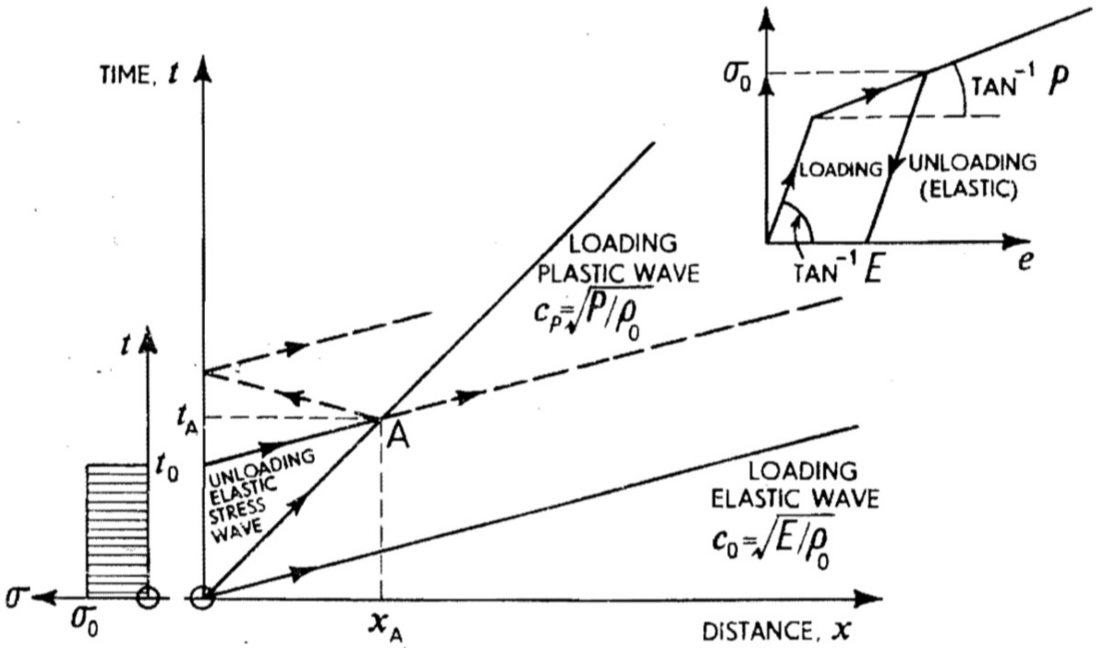
Elastic Precursor

- For a bar preloaded to plastic stress σ_0', the stress increment $d\sigma_0'$ would propagate at a speed of $\sqrt{(d\sigma'_0/de_e)/\rho_0}$.
- Experiment shows however that both elastic and plastic waves are propagated by the extra load.
- The elastic wave is commonly referred to as the precursor.
  - In fact, imposing stress increment $d\sigma_0'$ causes a total strain increment de_t which is made up of elastic de_e, and plastic de_p, strain increments; the elastic strain increment must be propagated at the elastic wave speed.



Unloading Waves

- If the load is completely and instantaneously removed, this will take place as an unloading elastic wave, effectively as an elastic compressive wave.



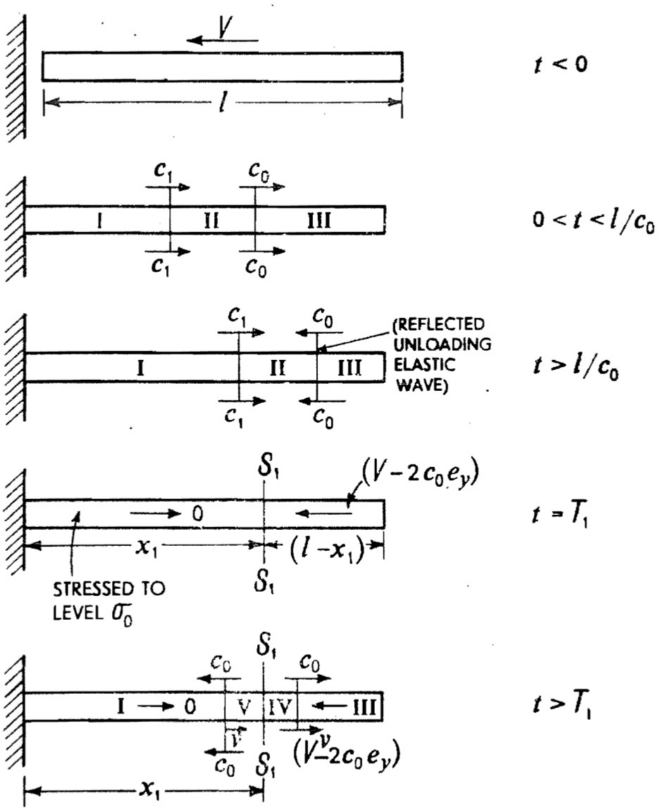
Unloading Waves

- It will be propagated into the bar and at time t_A will have overtaken the slower moving loading plastic wave at distance x_A from the original loaded end.
- When the two waves meet, depending upon the original intensity of the applied stress σ_0, the plastic wave may propagate further into the bar or it may be arrested and an elastic wave only, continue.
- Also, an elastic wave is reflected from a section distance x_A into the bar, back towards the now free end, there to be reflected yet again up the bar.

Impact of a Uniform Bar of Linear Hardening Material with a Rigid Flat Anvil

Elastic-Plastic Impact

- Let a uniform bar of elastic linear strain-hardening material, impinge normally on a rigid flat anvil and the nominal stress rise to $\sigma_0 > Y$, the yield stress.
  - For $0 < t < l/c_0$, three distinct regions in the bar; region I which is that traversed by both the elastic and the plastic waves, region II which is traversed thus far only by elastic waves, and region III which is undisturbed.



Elastic-Plastic Impact

- The minimum velocity necessary to initiate plastic strains is just $Y/\rho c_0 = c_0 e_Y$.
- The speed of particles in region II is $v = V - c_0 e_Y$ and since this speed is reduced to zero when the plastic 'shock' front passes, the compressive stress jump is,

$$\sigma_0 - Y = \rho_0 c_1 (V - c_0 e_Y)$$

- So that, the compressive stress in region I is,

$$\sigma_0 = Y + \rho_0 c_1 (V - c_0 e_Y)$$

- Also the total compressive strain is,

$$e_t = e_Y + \frac{\sigma_0 - Y}{P} = e_Y + \frac{V - c_0 e_Y}{c_1}$$

Elastic-Plastic Impact

- If there is no further plastic strain in region I, then the residual plastic strain e_P, is given by,

$$e_{\text{t}} = e_e + e_P$$

- Hence,

$$\begin{aligned} e_P &= \left(e_Y + \frac{V - c_0 e_Y}{c_1}\right) - \frac{\sigma_0}{E} \\ &= e_Y + \frac{V - c_0 e_Y}{c_1} - \left(\frac{Y + \rho_0 c_1 (V - c_0 e_Y)}{E}\right) \\ &= (V - c_0 e_Y)\left(\frac{1}{c_1} - \frac{c_1}{c_0^2}\right) \\ &= \frac{c_0^2 - c_1^2}{c_0^2 c_1} \cdot (V - c_0 e_Y) \end{aligned}$$

Elastic-Plastic Impact

- Just after time $t = l/c_0$, the elastic wave will be reflected from the free end of the bar and the slower moving plastic wave front will have advanced further to the right.
  - The effect of reflecting the elastic stress wave will be to progressively and completely unload the right hand end of the bar or much of region III; the velocity of region III will be decreased to $(V - 2c_0 e_Y)$.
  - The approaching reflected elastic wave and the advancing plastic stress wave eventually meet at time T_1 after it has advanced x_1 from the end of the bar at which impact first took place,

$$T_1 = \frac{x_1}{c_1} = \frac{2l - x_1}{c_0}$$

  - So,

$$\frac{x_1}{l} = \frac{2c_1/c_0}{1 + c_1/c_0}, \quad \rightarrow \quad T_1 = \frac{2l}{c_0 + c_1}$$

Elastic-Plastic Impact

- Since c_1/c_0 is often about 0.1 for many materials, $x_1/l \cong 0.18$.
- When the reflected unloading wave and the outward going plastic wave meet, at some section S_1S_1 at time T_1, it will be as if the length of bar $(l - x_1)$ having a speed $(V - 2c_0e_Y)$ or $(2c_0e_Y - V)$, suddenly impinges on a stationary bar of length x_1, which is already subjected to a compressive stress σ_0.
- The result of this impact is that just after time T_1, waves will be reflected back into each part of the bar.
- If it is assumed that both the reflected waves are elastic, then immediately after impact the particle speed must be the same on both sides of S_1S_1.

Elastic-Plastic Impact

- The *change* in speed of that part of region I embraced by the reflected elastic wave from S_1, i.e. region V is then just v; since this particle speed is oppositely directed to that of the wave, it implies the propagation of a tensile stress wave and the imposition of tensile strain.
- And because region V is already loaded in compression and has zero speed, therefore the tensile wave elastically unloads region V by amount $\rho_0 c_0 v$ to $[P(e_t - e_Y) + Ee_Y - \rho_0 c_0 v]$.
- At the same time, the stress in that part of region III which is traversed by the rightward moving reflected elastic stress wave, i.e. region IV, becomes $[\rho_0 c_0 (V - 2c_0 e_Y + v)]$.

Elastic-Plastic Impact

- However, the forces at the interface $S_1 S_1$ must be the same for both region V and IV, so that,

$$P(e_t - e_Y) + Ee_Y - \rho_0 c_0 v = \rho_0 c_0 (V - 2c_0 e_Y + v)$$

- Substituting for e_t,

$$P\left[\frac{V - c_0 e_Y}{c_1} + e_Y - e_Y\right] + Ee_Y = 2\rho_0 c_0 v + \rho_0 c_0 (V - 2c_0 e_Y)$$

- This equation may be reduced to,

$$v = \frac{(c_1 - 3c_0)(V - c_0 e_Y)}{2c_0} + V$$

Elastic-Plastic Impact

- The elastic strain e_1 engendered in region IV is,

$$e_1 = \frac{\rho_0 c_0}{E}\left[\frac{(c_1 - 3c_0)(V - c_0 e_Y)}{2c_0} + 2V - 2c_0 e_Y\right]$$
$$= \frac{(c_1 + c_0)(V - c_0 e_Y)}{2c_0^2}$$

- Now the greatest value which e_1 can take and region IV still remain elastic, is e_Y, so that if $e_1 = e_Y$, then

$$V = c_0 e_Y\left(1 + \frac{2c_0}{c_0 + c_1}\right)$$

Elastic-Plastic Impact

- The compressive strain e' remaining in regions I and V, after traversal by the elastic wave reflected from S_1S_1 results from a change in compressive strain $e_Y + (V - c_0 e_Y)/c_1$, by an amount of tensile strain $\rho c_0 v/E$,

$$e' = \left(e_Y + \frac{V - c_0 e_Y}{c_1}\right) - \frac{\rho_0 c_0 v}{E}$$
$$= \left(e_Y + \frac{V - c_0 e_Y}{c_1}\right) - \frac{\rho_0 c_0}{E}\left[\frac{(c_1 - 3c_0)(V - c_0 e_Y)}{2c_0} + V\right]$$
$$= (V - c_0 e_Y)\left[\frac{-c_1^2 + c_1 c_0 + 2c_0^2}{2c_0^2 c_1}\right]$$

Elastic-Plastic Impact

- Provided that only elastic waves leave section S_1S_1,elastic waves will travel up and down the bar after contact with the anvil ceases and the total strain at any section will vary; but the plastic strains—particularly in region I—will remain constant.
- Thus provided that,

$$c_0 e_Y < V < c_0 e_Y \left(1 + \frac{2c_0}{c_0 + c_1}\right)$$

$$1 < \frac{V}{c_0 e_Y} < 3 - \frac{2c_1/c_0}{1 + c_1/c_0}$$

- There will only be one region embracing an original length of bar of extent x_1 in which plastic deformation occurs.

Elastic-Plastic Impact

- S_1S_1 is known as a stationary front of second order discontinuity in strain.
- When V is just sufficient to initiate further plastic deformation, i.e. beyond S_1S_1, we find,

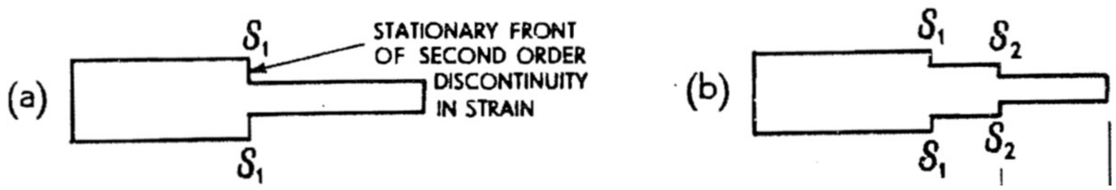
$$e_t = e_Y \left[1 + \frac{2(c_0/c_1)^2}{1 + c_0/c_1}\right]$$

$$e' = e_Y \left[2\frac{c_0}{c_1} - 1\right]$$

$$e_P = e_Y \left[\frac{c_0}{c_1} - 1\right]$$

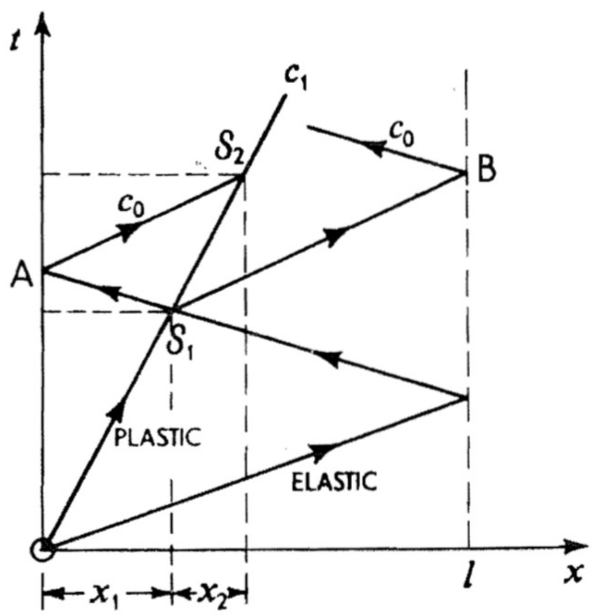
Elastic-Plastic Impact

- If $V > c_0 e_Y (1 + 2c_0/(c_0 + c_1))$, then a further plastic wave propagates to the right from S_1S_1, as well as elastic waves in both directions.
- Either of these latter elastic waves after reflection from the ends of the bar, may later intercept the plastic wave and cause a second stationary front of second order strain discontinuity S_2S_2.
- The bar, if only one stationary front arises, will appear, as in (a) and if two are created as in (b).



Elastic-Plastic Impact

- The sequence of events and the interaction of the wave motions may be well represented in the characteristic (x, t) plane as shown.
  - Which of the two elastic waves from S_1S_1 first meets the plastic wave depends on the ratio c_0/c_1.
  - The critical case arises when both elastic waves meet the plastic wave simultaneously.
  - Let this happen after further time T_2 at a further distance along the unstrained bar from S_1S_1 of x_2 , i.e. at section S_2S_2.



Elastic-Plastic Impact

- For the elastic wave which travels to the left of the plastic wave,

$$T_2 = \frac{x_1}{c_0} + \frac{x_1 + x_2}{c_0}$$

- For the other elastic wave,

$$T_2 = \frac{2(l - x_1) - x_2}{c_0}$$

- and for the plastic wave,

$$T_2 = \frac{\dot{x}_2}{c_1}$$

- So,

$$\frac{x_2}{l} = \frac{2c_1(c_0 - c_1)}{(c_0 + c_1)^2} = \frac{2(1 - c_1/c_0)(c_1/c_0)}{(1 + c_1/c_0)^2}$$

Elastic-Plastic Impact

$$\frac{x_1 + x_2}{l} = \frac{4c_0c_1}{(c_0 + c_1)^2} = 4 \cdot \frac{c_1/c_0}{[1 + c_1/c_0]^2}$$

$$\frac{l}{c_1} \cdot \frac{2(1 - c_1/c_0)(c_1/c_0)}{[1 + c_1/c_0]^2} = \frac{l}{c_0} \cdot \frac{2(c_1/c_0)}{1 + c_1/c_0} + \frac{l}{c_0} \cdot \frac{4(c_1/c_0)}{\left[1 + (c_1/c_0)^2\right]}$$

- Which reduces to, $c_0^2 - 4c_0c_1 - c_1^2 = 0$

- So, $c_0/c_1 = 2 + \sqrt{5} \simeq 4.24$
  - Thus if $c_0/c_1 > 4.24$, then the leftward moving elastic wave from section S_1 first intercepts the plastic wave.

Analysis of the Dynamic Compression of a Short Cylinder Between Rigid Dies

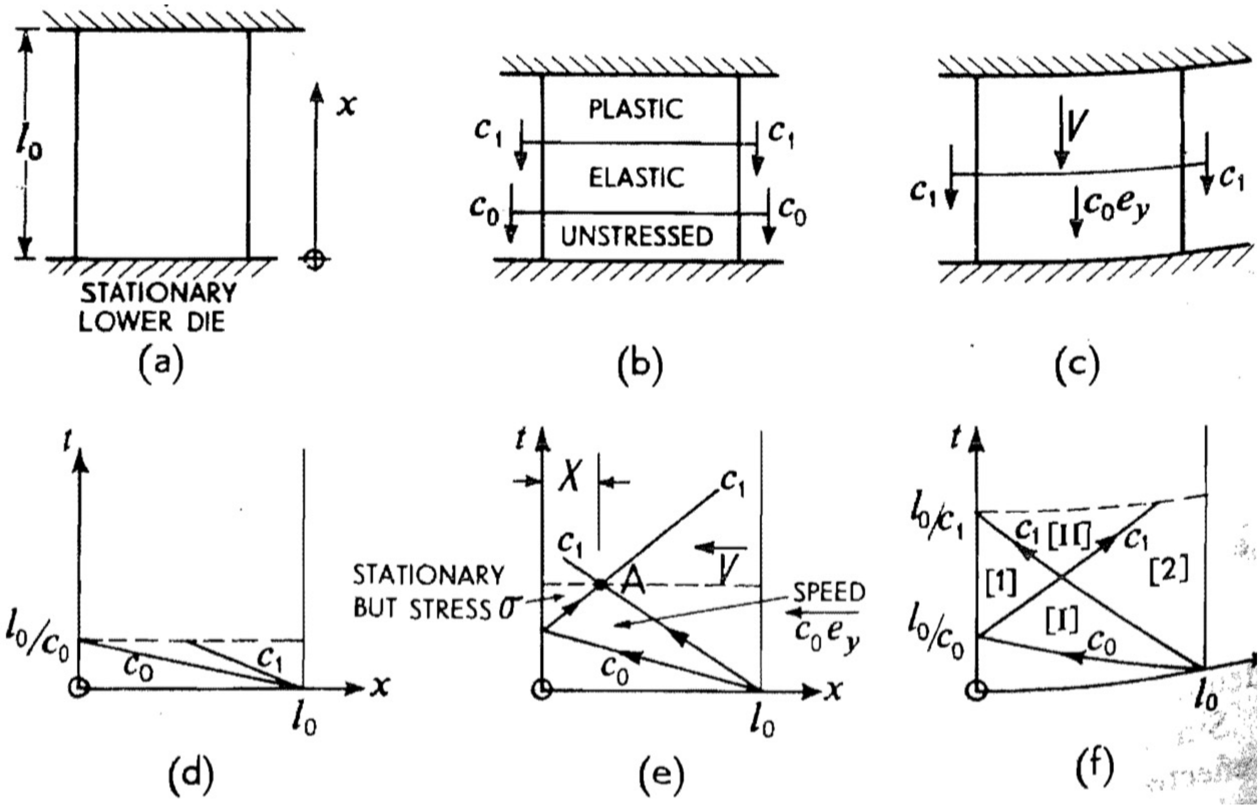
Dynamic Compression

- We consider a short cylindrical block of elastic-linear strain-hardening material, situated on a frictionless flat rigid bottom die, which is compressed by an identical upper die moving with a speed V which remains constant for a period of time $6l_0/c_1$, where $c_1 = \sqrt{P/\rho_0}$ is the plastic wave speed and l_0 is the original height of the cylinder.
- All material engulfed by the plastic wave will be moving at speed V, i.e. the die speed, and that through which the elastic wave only has passed will have a speed u where,

$$Y = \rho_0 c_0 u \quad \text{or} \quad u = \frac{Y}{\rho_0 c_0} = \frac{Y}{E} . \frac{E}{\rho_0} . \frac{1}{c_0}$$

$$u = c_0 e_Y$$

Dynamic Compression



Dynamic Compression

- Since the particles in contact with the lower die are at rest, then the wave reflected from it must be such as to change the incident elastic wave particle speed from $c_0 e_Y$ to zero.
- Further, since the material is already stressed to the compressive yield stress, the reflected wave must be a plastic wave.

$$\sigma - Y = \rho_0 c_1 \cdot \Delta V$$

- where ΔV is the change in particle speed that the plastic wave brings about. Now $\Delta V = c_0 e_Y$ and thus,

$$\sigma = Y + \rho_0 c_1 \cdot c_0 e_Y = Y\left(1 + c_1 c_0 \frac{\rho_0}{E}\right)$$

$$= Y\left(1 + \frac{c_1}{c_0}\right)$$

Dynamic Compression

- The plastic wave from the lower die and the initiated plastic wave meet at A, distant X, from the bottom die.
- From this meeting of the two plastic waves, only two identical (or continuing) plastic waves can be produced.
  - no elastic effects intervene since the cylinder is still being compressively loaded by the top die moving with speed V.
- At time $t = l_0/c_1$, the stress level in zone [2] is,

$$Y + (V - c_0 e_Y)\rho_0 c_1 = Y\left(1 - \frac{c_1}{c_0}\right) + \rho_0 c_1 V$$

- Let the particle speed in zone [II] be w, then from a consideration of zones [1] and [II], the stress in zone [II] is,

$$Y\left(1 + \frac{c_1}{c_0}\right) + \rho_0 c_1 w$$

Dynamic Compression

- From a consideration of zones [2] and [II], the stress in zone [II] is,

$$Y\left(1 - \frac{c_1}{c_0}\right) + \rho_0 c_1 V + \rho_0 c_1 (V - w)$$

- However,

$$Y\left(1 + \frac{c_1}{c_0}\right) + \rho_0 c_1 w = Y\left(1 - \frac{c_1}{c_0}\right) + \rho_0 c_1 V + \rho_0 c_1 (V - w)$$

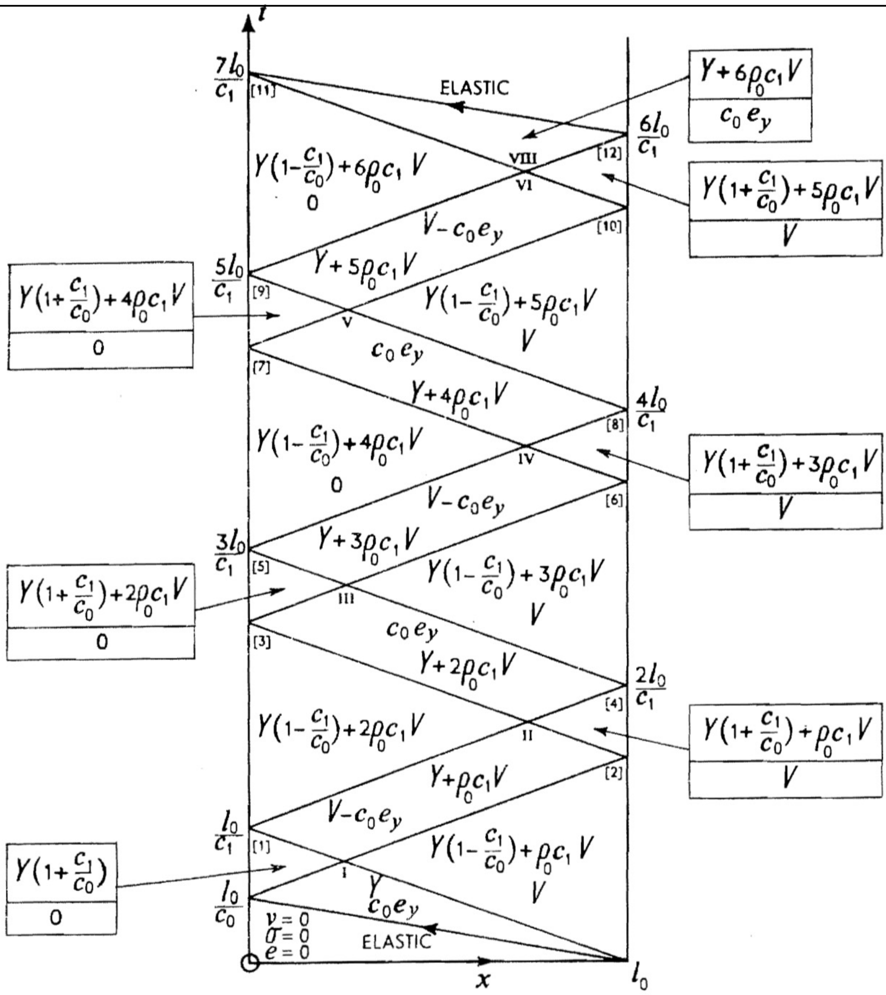
- So, $w = V - c_0 e_Y$

Dynamic Compression

- Thus the stress in zone [II] is,

$$\sigma = Y\left(1 + \frac{c_1}{c_0}\right) + \rho_0 c_1 (V - c_0 e_Y)$$
$$= Y + \rho_0 c_1 V$$

- It is straightforward to arrive at the strain and stress levels in each region from observations about the particle speed.
- Note that when the upper die is arrested at $t = 6l_0/c_1$, an unloading wave of intensity $(Y + 6\rho_0 c_1 V)$ is propagated into the cylinder and results in it springing away from the bottom die to some extent in due course.



Dynamic Compression

- Some typical figures for steel
  $(Y = 172\ MPa, c_1/c_0 = 0.1,\ V = 18\ m/s)$,

$$c_0 = (E/\rho_0)^{1/2} \simeq \left(5 \cdot 10^3 \text{ m/sec}\right)$$

$$\rho_0 c_1 V = \frac{E}{c_0^2} \cdot c_1 V = \left(75 \cdot 10^6 \text{ N/m}^2\right).$$

$$Y\left(1 + \frac{c_1}{c_0}\right) = 25\left(1 + \frac{1}{10}\right) \cdot 10^3 = \left(190 \cdot 10^6 \text{ N/m}^2\right)$$

$$Y\left(1 - \frac{c_1}{c_0}\right) = 25\left(1 - \frac{1}{10}\right) \cdot 10^3 = \left(155 \cdot 10^6 \text{ N/m}^2\right)$$

$$c_0 e_Y = 2 \cdot 10^5 \cdot \frac{25 \cdot 10^3}{30 \cdot 10^6} = (4.3 \text{ m/sec})$$

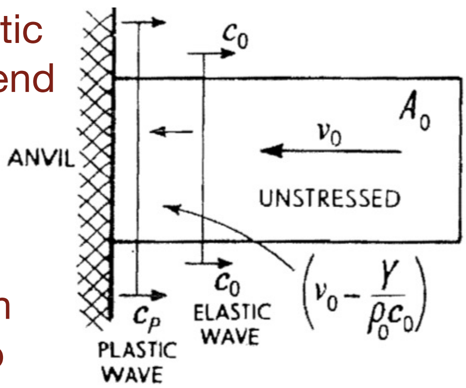
High Speed Impact of Perfectly Plastic Solid Cylindrical Blocks with a Flat Rigid Anvil

Bar Impact with a Rigid Anvil

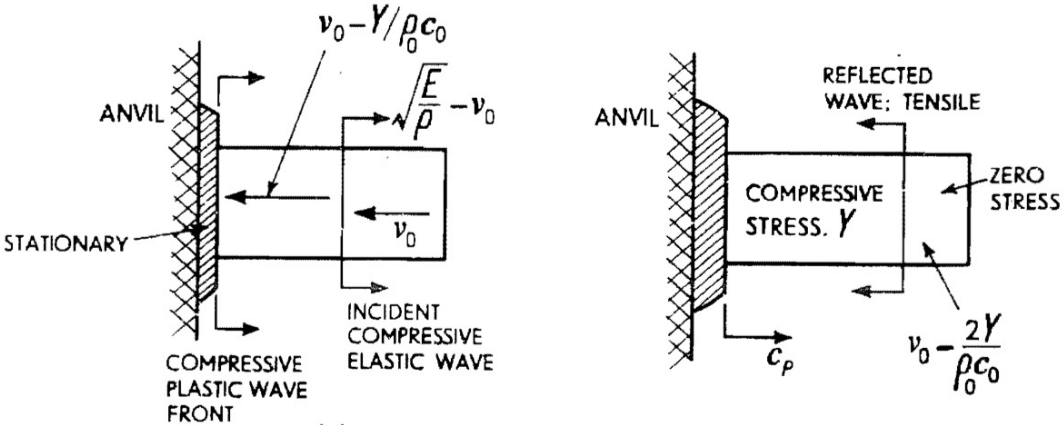
- In this section, we consider the normal, high speed impact with consequent plastic deformation of a short, solid cylindrical bar with a rigid anvil.
- The end at which impact takes place 'mushrooms', and this shape is characteristic of this type of process.
- The analysis which follows is originally due to Taylor.
- The approach is applicable in cases where $\rho_0 v_0^2 \cong Y$ where v_0 is the initial bar or projectile speed.
- The aim is to account for the 'mushrooming' of bullets or projectiles during the impact process.
- The analysis here is for rigid-perfectly plastic materials.

Taylor's Momentum Approach

- At impact with the rigid anvil, two waves are initiated and move out from the anvil;
  - One travels with the elastic wave speed c_0 and the other—a plastic wave—at a much slower speed which is to be determined.
  - The stress in the bar immediately rises to the elastic limit and particularly the elastic compressive stress, travels to the free end of the bar, giving rise to a change in particle speed of $Y/\rho_0 c_0$ to the right.
  - If the initial speed of the bar was v_0 the speed of the particles in the bar through which this wave has travelled relative to the fixed anvil, is reduced to $v_0 - Y/\rho_0 c_0$.



Taylor's Momentum Approach



- At the free end of the bar, the compressive elastic wave is reflected as a tension wave, so progressively unloading the bar.
- There are three regions in the bar; one which is plastically strained, a second which is not strained at all and a third- between the plastic wavefront and the elastic wavefront in which elastic strains of magnitude Y/E are imposed.

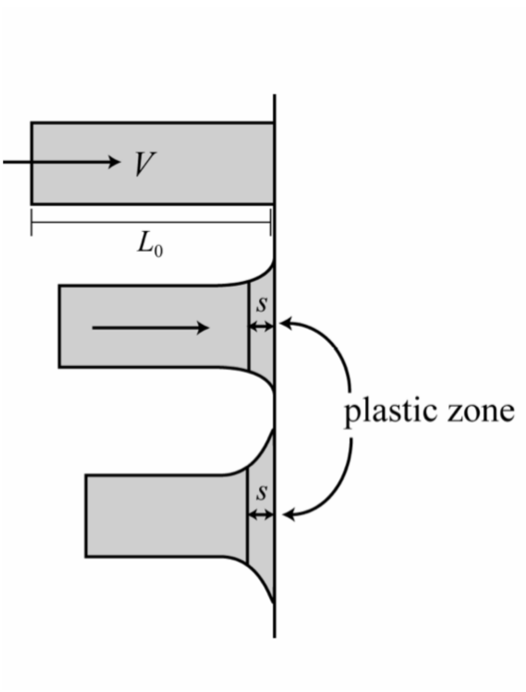
Taylor's Momentum Approach

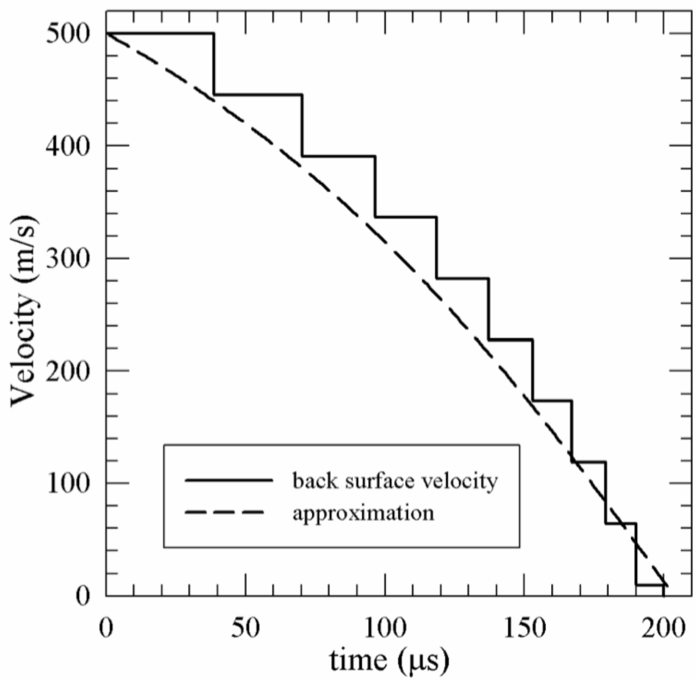
- The reflected wave of tension moving to the left, of itself causes a change in particle speed of magnitude $Y/\rho_0 c_0$ to the right, so that the speed of the unloaded rear end of the bar is as a whole $(v_0 - 2Y/\rho_0 c_0)$.
- Thus, when the rightward-moving compressive plastic wave and the leftward-moving, unloading, tensile elastic wave meet, the whole of the bar to the right of the plastic wave front has the speed $(v_0 - 2Y/\rho_0 c_0)$.
- The continuous passage of the elastic wave up and down the rear portion of the bar, which is reflected from the slowly advancing plastic wavefront and the free end of the bar, feeds energy forward for its subsequent dissipation plastically and slowly, after many traversals of the rear part of the bar, brings it to rest.

Taylor's Momentum Approach

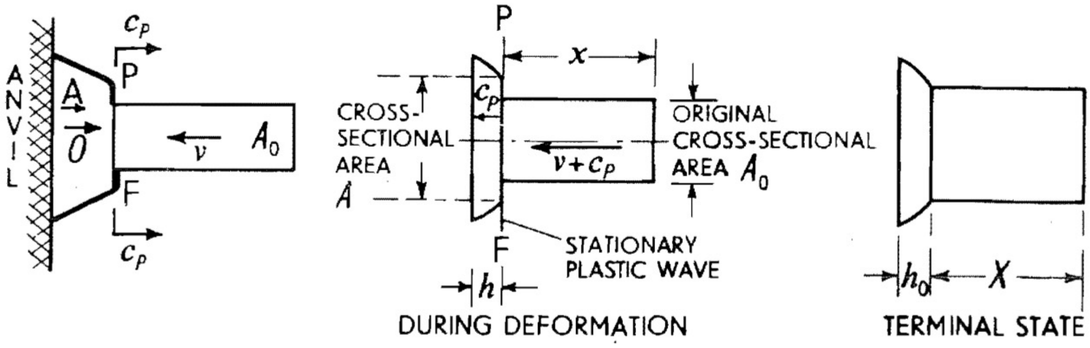
- This rear portion, i.e. that which is not plastically deformed at a given time, may thus reasonably be treated as a continuously retarded rigid body whose motion is determined by events at the plastic wavefront.
- The momentum flux at the plastic wavefront decreases with time and hence the plastic strain developed also decreases with time; thus a mushroom shape is expected.
- A simple theoretical model may now be conceived in which the portion of the bar moving through the plastic wavefront (or shock) is brought to rest and, in doing so, the material spreads out laterally undergoing compressive plastic deformation.

Aluminum Cylinder





Basic Equations



- In the figure, bring the plastic wave front, PF, which has an absolute speed of c_p to relative rest, so that rigid material from the right moves across it with speed $(v + c_p)$; v is the instantaneous absolute speed of the end of the bar of initial cross-sectional area A_0.
- Thus, the equation for no change in volume gives,

$$A_0(v + c_p) = Ac_p$$

Basic Equations

- Also at PF, the net force is $Y(A - A_0)$; the pressure in the 'shock' is everywhere the same at magnitude Y.
  - This is equal to the rate of change of momentum across the shock plane, i.e. the mass arriving per unit time is $\rho_0 A_0(v + c_p)$ and its change in velocity is from $(v + c_p)$ to c_p.
- Hence, $$\rho_0 A_0(v + c_p) \cdot \{(v + c_p) - c_p\} = Y(A - A_0)$$

$$\rho_0 A_0(v + c_p)v = Y(A - A_0)$$

  - If the longitudinal compressive engineering strain in any element of original length dl_0 which has been plastically compressed to length dl is e, then

$$e = \frac{dl_0 - dl}{dl_0} = \frac{\frac{V}{A_0} - \frac{V}{A}}{\frac{V}{A_0}} = \frac{A - A_0}{A} = 1 - \frac{A_0}{A}$$

Basic Equations

- From $A_0(v + c_p) = Ac_p$,

$$c_p = \frac{v}{\frac{A}{A_0} - 1}$$

- Substituting,

$$\rho_0 A_0 \left\{ v + \frac{v}{\frac{A}{A_0} - 1} \right\} v = Y\left(\frac{A}{A_0} - 1\right) A_0$$

- Which on simplifying reduces to,

$$\frac{\rho_0 v^2}{Y} = \frac{\left(\frac{A}{A_0} - 1\right)^2}{\frac{A}{A_0}} = \frac{\left(\frac{1}{1-e} - 1\right)^2}{\frac{1}{1-e}} = \frac{e^2}{1 - e}$$

Deformed Length

- For the two principal portions of the bar which are of current increasing length h and decreasing length x, we have,

$$\frac{dh}{dt} = c_p \qquad \frac{dx}{dt} = -(v + c_p)$$

- Further, for the undeformed portion of the rod, applying Newton's second law,

$$YA_0 = -\rho_0 A_0 x \cdot \frac{d}{dt}(v + c_p) = -\rho_0 A_0 x \frac{dv}{dt}$$

$$\frac{dv}{dt} = -\frac{Y}{\rho_0 x}$$

- Eliminating dt,

$$\frac{dx}{dv} = \frac{v + c_p}{Y/\rho_0 x}$$

Deformed Length

- But from $A_0(v + c_p) = Ac_p$,

$$c_p = \frac{v}{\frac{A}{A_0} - 1} = \frac{v}{\frac{1}{1-e} - 1} = \frac{1-e}{e} \cdot v$$

- Substituting,

$$\frac{dx}{dv} = \frac{v\rho_0 x}{eY} \quad \text{or} \quad \frac{dx}{x} = \frac{\rho_0 v \cdot dv}{eY}$$

- The term $v.dv$ can be eliminated using $\rho_0 v^2/Y = e^2/(1-e)$,

$$\frac{2\rho_0 v \cdot dv}{Y} = \frac{2e - e^2}{(1-e)^2} \cdot de$$

- So,

$$2\frac{dx}{x} = \frac{2-e}{(1-e)^2} \cdot de$$

Deformed Length

- We may now integrate, noting that when $x = L$, i.e. at the beginning of the plastic deformation, $e = e_0$. Hence,

$$\left[\ln x^2\right]_L^x = \left[-\ln(1-e) + \frac{1}{(1-e)}\right]_{e_0}^{e}$$

$$\ln\left(\frac{x}{L}\right)^2 = \ln\left(\frac{1-e_0}{1-e}\right) + \frac{e-e_0}{(1-e)(1-e_0)}$$

- Also at the end of plastic deformation $e = 0$, and we may denote the remaining undeformed length of bar by X so that from equation,

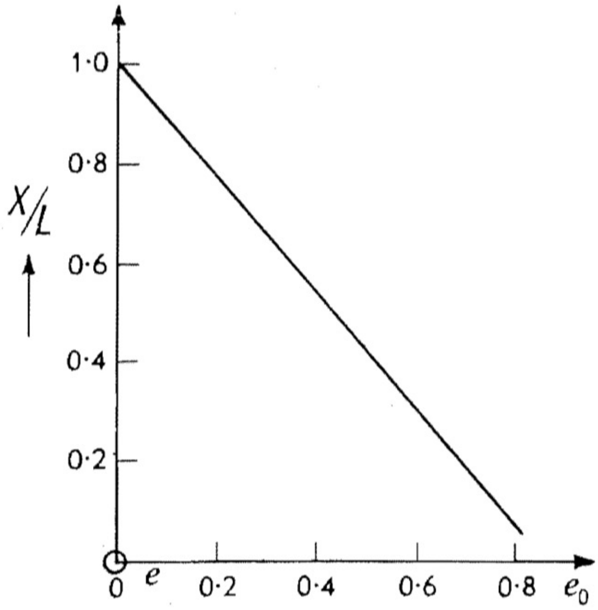
$$\ln\left(\frac{X}{L}\right)^2 = \ln(1-e_0) - \frac{e_0}{(1-e_0)}$$

Deformed Length

- Or,

$$\ln \frac{L}{X} = \frac{1}{2}\left[\frac{e_0}{(1-e_0)} + \ln\left(\frac{1}{1-e_0}\right)\right]$$

- Figure shows how X/L varies with e_0



Example

- Consider $e_0 = 0.5$, then,

$$\frac{\rho_0 v_0^2}{Y} = \frac{e_0^2}{1-e_0} = \frac{0.5^2}{1-0.5} = 0.5$$

$$\ln\left(\frac{X}{L}\right)^2 = \ln(1-0.5) - 0.5/(1-0.5) = -1.69$$

- and $X/L = 0.43$.

e	0	0·1	0·2	0·3	0·4	0·5
x/L	0·43	0·48	0·54	0·635	0·7	1·0
h/L see (5.50) below	0·38	0·34	0·28	0·21	0·12	0·0
d/d_0	1·00	1·05	1·12	1·20	1·29	1·41
v_0t/L	0·34	0·29	0·24	0·18	0·10	0·0

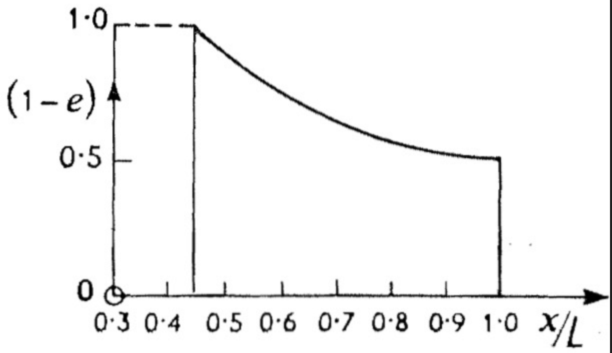
Example

- To facilitate depicting the sequence of states of deformation, the corresponding values of h are required.

$$\frac{dh}{dx} = -\frac{c_p}{c_p + v} = -1 + e$$

$$\frac{h}{L} = \int_{X/L}^{1} (1 - e) d(x/L)$$

- The integral can be graphically interpreted after plotting x/L versus $(1 - e)$,



Profile Shape

- The profile of the 'mushroom' for each value of e or at each section x/L, is given by the ratio of plastically deformed diameter as it becomes stationary on crossing the plastic wave front d, to the original bar diameter d_0,

$$\frac{d}{d_0} = \sqrt{\frac{A}{A_0}} = \frac{1}{\sqrt{1 - e}}$$

- Values of d/d_0 corresponding to values of e appear in Table 5.1.

e	0	0·1	0·2	0·3	0·4	0·5
x/L	0·43	0·48	0·54	0·635	0·7	1·0
h/L see (5.50) below	0·38	0·34	0·28	0·21	0·12	0·0
d/d_0	1·00	1·05	1·12	1·20	1·29	1·41
v_0t/L	0·34	0·29	0·24	0·18	0·10	0·0

Time-wave Position

- Finally, the time t which elapses since impact first occurred, for the plastic wave front to reach distance h from the anvil, is,

$$\int dt = \int -\frac{\rho_0 x dv}{Y}$$

- But,

$$v^2 = \frac{Ye^2}{\rho_0(1-e)} \quad \text{so that} \quad dv = \frac{1-e/2}{(1-e)^{3/2}} \cdot \sqrt{\frac{Y}{\rho_0}} \cdot de$$

- So,

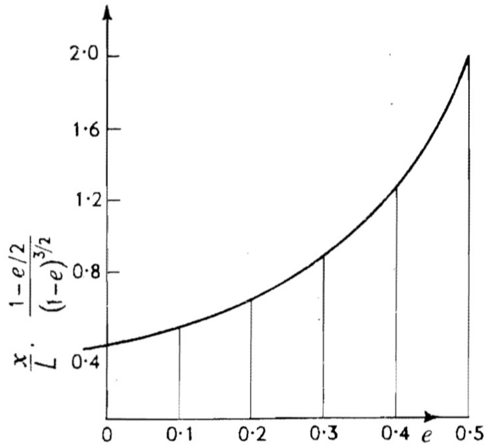
$$\int_0^t dt = -L\sqrt{\frac{\rho_0}{Y}} \int_{e_0}^{e} \left(\frac{x}{L}\right) \cdot \frac{(1-e/2)}{(1-e)^{3/2}} \cdot de$$

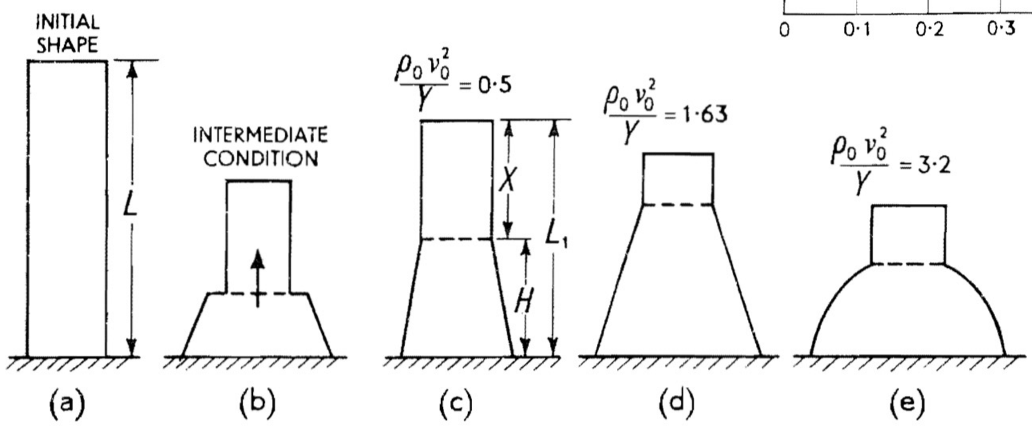
$$\frac{v_0 t}{L} = \frac{e_0}{\sqrt{(1-e_0)}} \int_{e}^{e_0} \left(\frac{x}{L}\right) \cdot \frac{1-e/2}{(1-e)^{3/2}} \cdot de$$

  - with the help of $\rho_0 v_0^2 Y = e_0^2/(1-e_0)$

Time-wave Position

- The function $(x/L)(1-e/2)/(1 - e)^{3/2}$ versus e, is plotted for the case considered, i.e. $\rho_0 v_0^2/Y$=0.5
- Below, three terminal profiles for $\rho_0 v_0^2/Y$ =0.5, 1.63 and 3.2 are shown as calculated by Taylor.





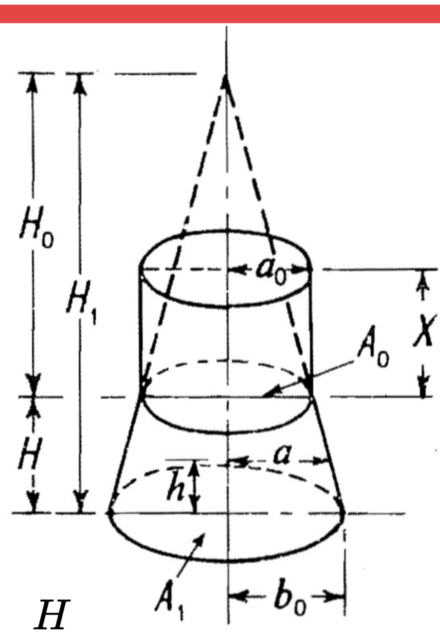
Simplified Calculation

- For $\rho_0 v_0^2/Y$ =0.5, and 1.63, the 'mushroomed' end appears as an almost straight-sided conical frustrum.
- For each particular value of $\rho_0 v_0^2/Y$, the final depth of the mushroomed head, H, requires to be numerically calculated as above.
- However, the amount of labor involved in calculating H/L may be avoided without too great an error.

Simplified Calculation

- The volume of the deformed head, is expressed as,

$$(L - X)A_0 = \frac{1}{3}(A_1 H_1 - A_0 H_0)$$



- Thus,

$$1 - \frac{X}{L} = \frac{1}{3}\left(\frac{1}{1-e_0} \cdot \frac{d_1}{d_1 - d_0} - \frac{d_0}{d_1 - d_0}\right) \cdot \frac{H}{L}$$

$$\frac{H}{L} = \frac{3(1 - X/L)(1 - e_0)\left(1 - \sqrt{1 - e_0}\right)}{1 - (1 - e_0)^{3/2}}$$

Simplified Calculation

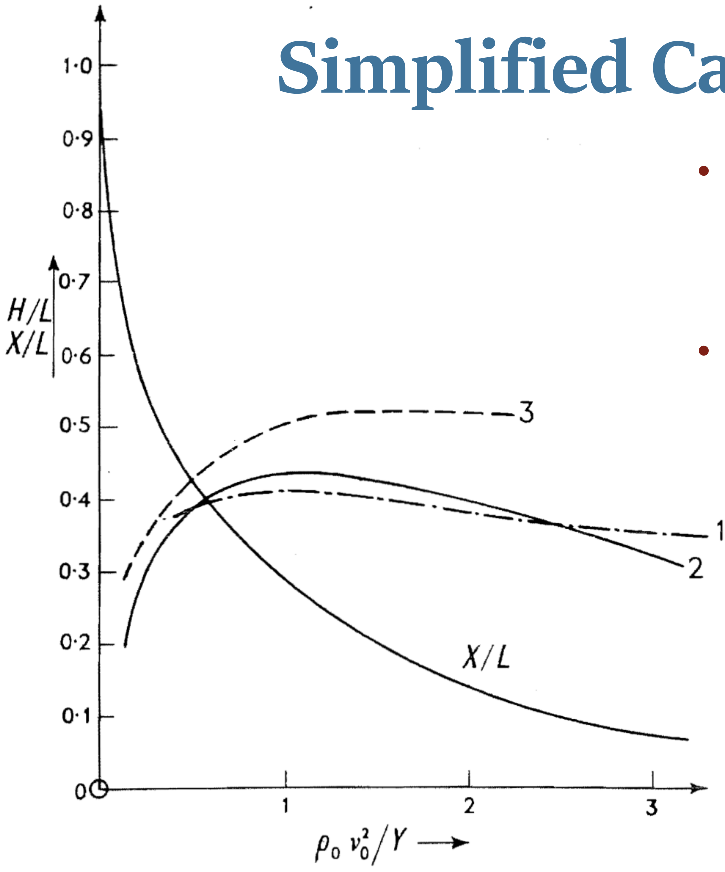


Fig. 5.26 Showing how H/L varies with $\rho_0 V_0^2/Y$ as (1) calculated by Taylor[5.3], (2) following equation (5.54) and (3) following equation (5.65).

- It is interesting to note that H/L has a maximum of about 0.43 when $\rho_0 v_0^2/Y \cong 1$.
- The values of H/L appear to be the same as those given by Taylor and significant curvature of the sides of the mushroom is evidently only taken on when $\rho_0 v_0^2/Y$>1.63, a feature of no great importance since the solution is obviously in error at this level of $\rho_0 v_0^2/Y$.

Plastic Wave Speed

- Plots of h/L against $v_0 t/L$ show that the plastic wave speed is nearly constant.
- For the case considered, i.e. $\rho_0 v_0^2/Y$=0.5, if this is denoted by c_p, then h/v_0 is the slope of the line made by plotting h/L by $v_0 t/L$ from Table 5.1; in this case $c_p \cong 1.12 v_0$.
- By treating c_p as constant,

$$\frac{1}{\rho_0} \cdot \frac{dx}{x} = \frac{(v + c_p)}{Y} \cdot dv$$

- Integrating,

$$\frac{Y}{\rho_0} \ln x = \frac{1}{2} v^2 + c_P \cdot v + c'$$

Plastic Wave Speed

- When $v = v_0,\ x = L$ and thus,

$$\frac{Y}{c_0} = -\frac{1}{2}v_0^2 - c_p v_0 + (Y \ln L)\rho_0$$

- Hence, $\ln\left(\frac{x}{L}\right) + \frac{1}{2}(v_0^2 - v^2) + c_P(v_0 - v) = 0$

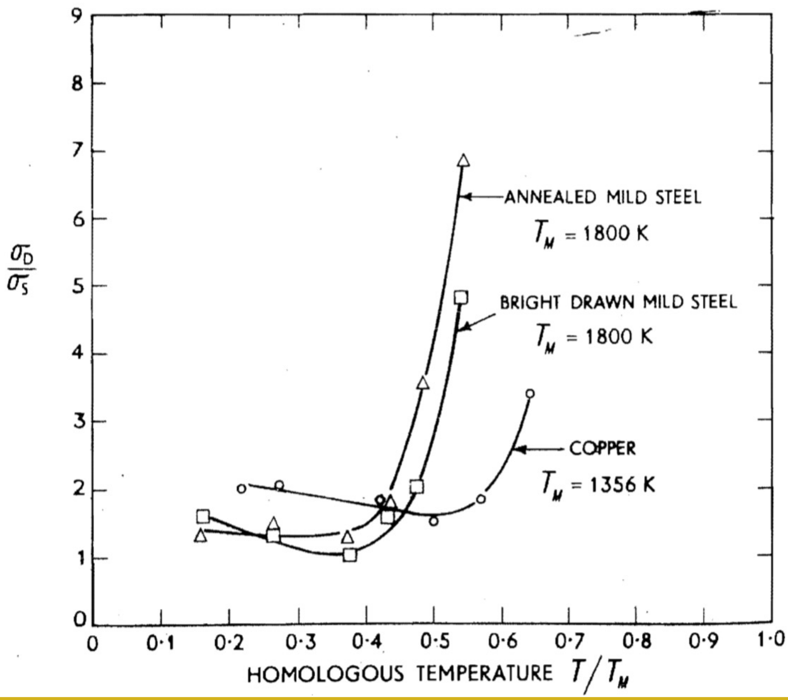
- Since $v = 0$, when $x = X$,

$$\frac{Y}{\rho_0}\ln\left(\frac{X}{L}\right) + \frac{1}{2}v_0^2 + c_p \cdot v_0 = 0$$

$$\frac{c_p}{v_0} = \frac{\ln(L/X)}{\rho_0 v_0^2/Y} - 0.5$$

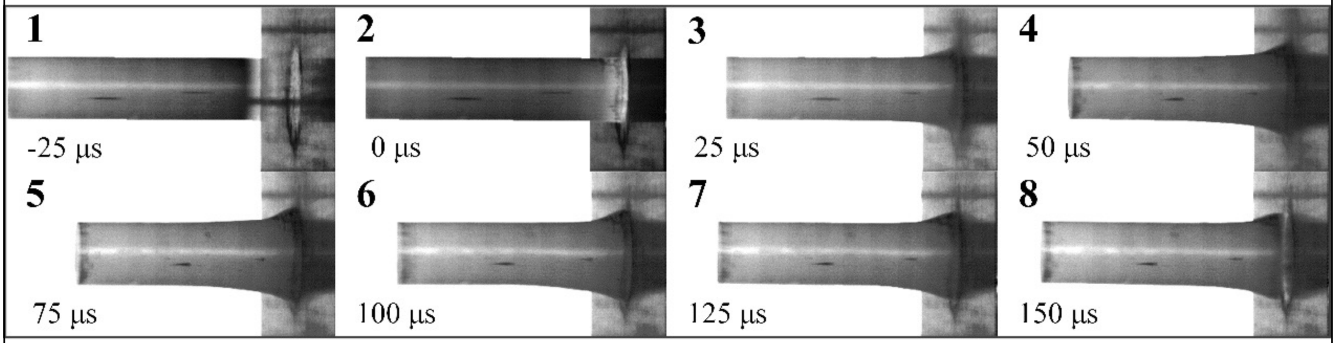
Strain Rate Effects

- Strain rate effects are the more pronounced the higher the homologous temperature at which a test is conducted.

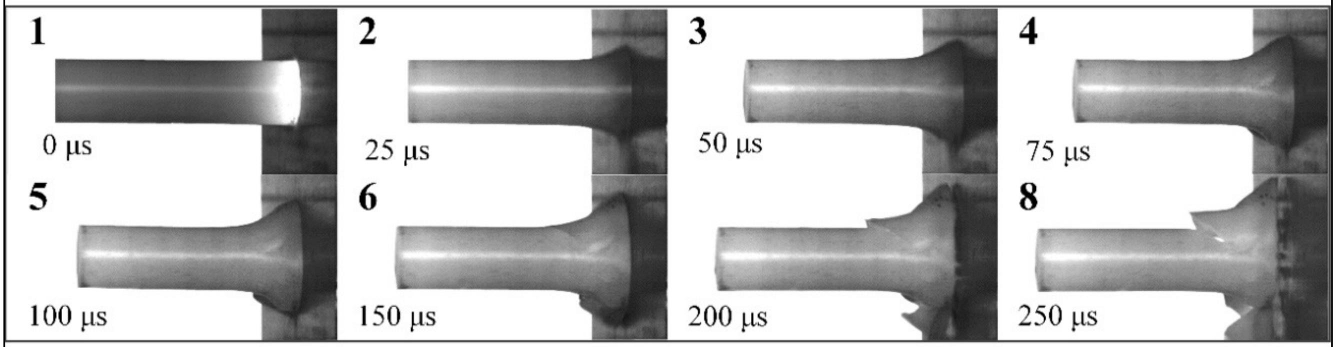


Xu, et al., Semi-Crystalline Polymers Applied to Taylor Impact Test: Constitutive, Experimental and FEM Analysis, Polymers 2020, 12, 1615

- Taylor test of PA66 rod at 128 m/s:

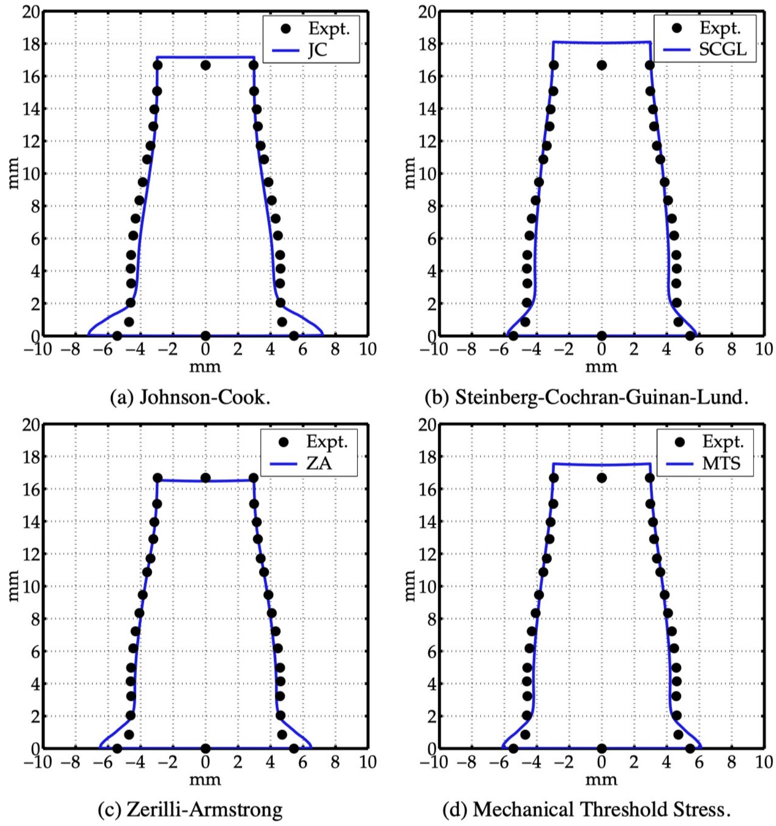


- at 168 m/s:



Banerjee, Taylor impact tests: detailed report, C-SAFE Internal Report No. C-SAFE-CD-IR-05-001, 2005.

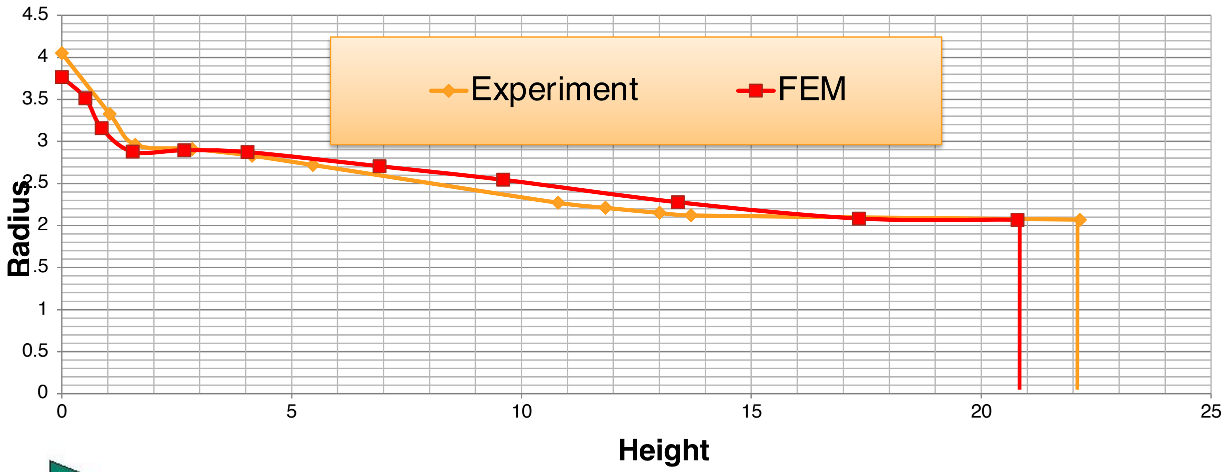
- Computed versus experimental profiles for Taylor test of copper.
- $L_0 = 30\ mm$
  $D_0 = 6.0\ mm$
  $V_0 = 188\ m/s$
  $T_0 = 718\ K$

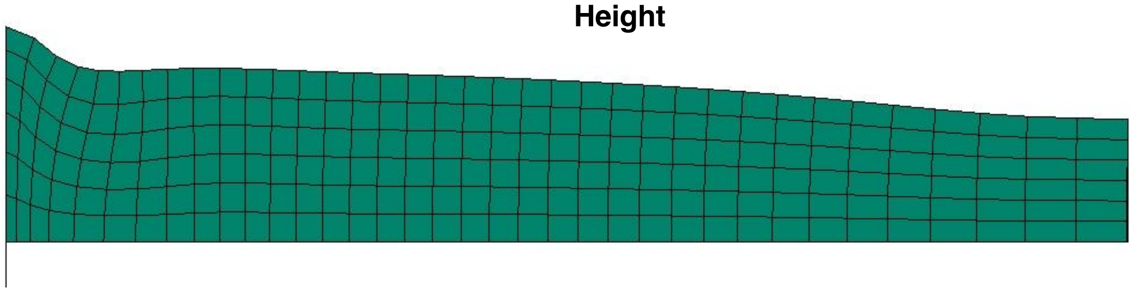


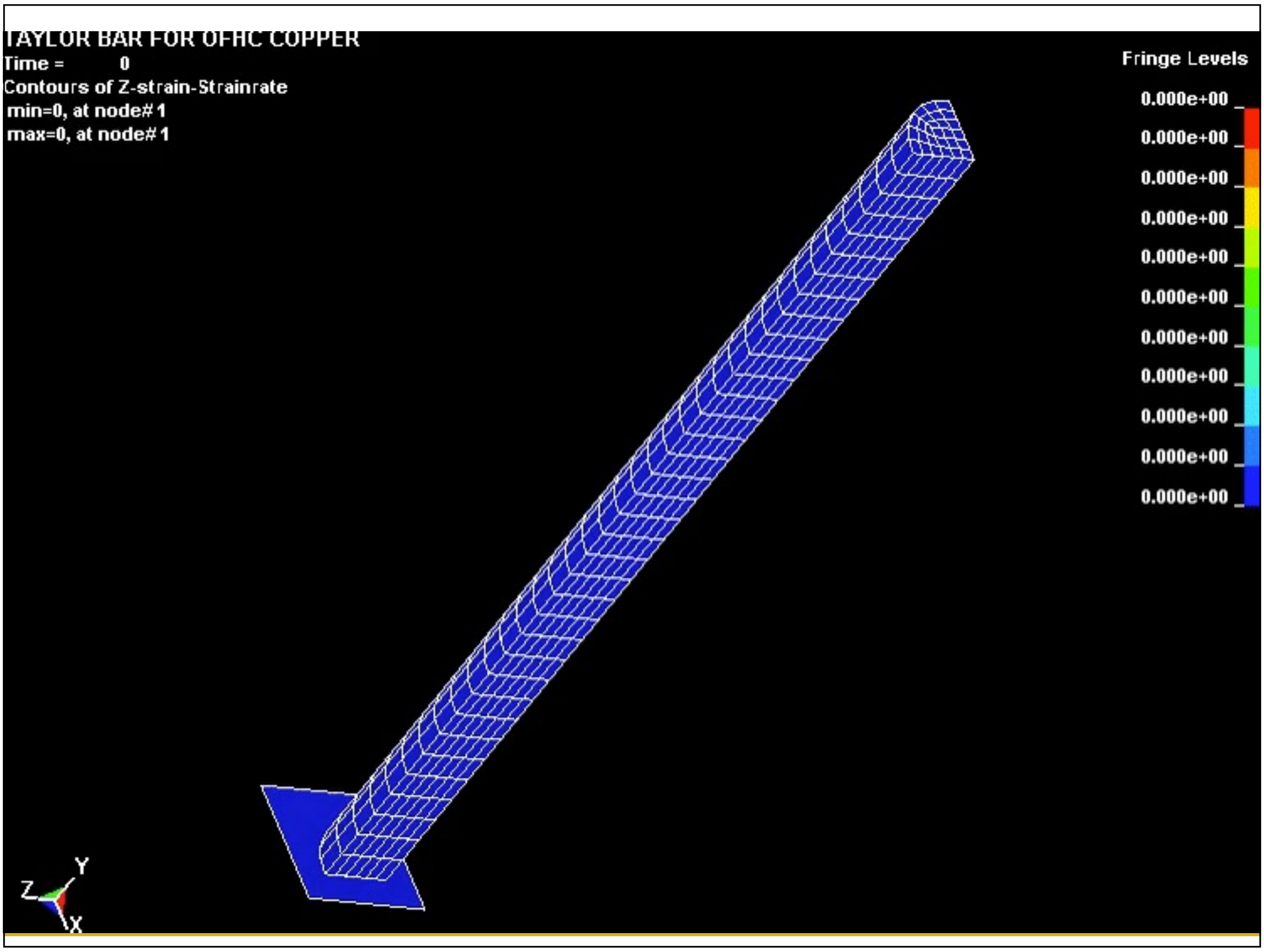
(a) Johnson-Cook.
(b) Steinberg-Cochran-Guinan-Lund.
(c) Zerilli-Armstrong
(d) Mechanical Threshold Stress.

Corelation

- OFHC copper:







Energy Method Based on Deformation into a Frustum

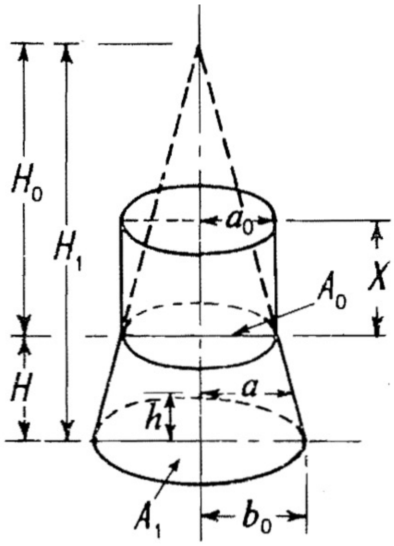
Non-hardening Material

- If the final 'mushroomed' head of the projectile is assumed to be frustum-shaped, the compressive strain distribution may be presumed to be implicitly specified as proportional to distance from the anvil and thus the plastic work done in arriving at this state may be found.

- From the figure,

$$\frac{a - a_0}{b_0 - a_0} = \frac{H - h}{H}$$

$$\frac{da}{b_0 - a_0} = -\frac{dh}{H}$$



Non-hardening Material

- For a transverse element of height dh at h from the anvil, the plastic work done in expanding its radius from a_0 to a is

$$dW = Y \cdot \ln\left(\frac{dh_0}{dh}\right) \cdot \text{volume of element}$$

- For the whole frustum,

$$\frac{W}{Y} = \int_0^H \ln\left(\frac{dh_0}{dh}\right) \cdot \pi a^2 \cdot dh$$

- Now, $A_0 dh_0 = A dh$, where A and A_0 denote initial and final cross-sectional areas and hence,

$$\frac{dh_0}{dh} = \frac{A}{A_0} = \frac{a^2}{a_0^2}$$

Non-hardening Material

- Thus,

$$\frac{W}{Y} = 2\pi \int_0^H \ln\left(\frac{a}{a_0}\right) \cdot a^2 dh$$

$$\frac{W}{Y} = 2\pi \int_{b_0}^{a} -\frac{Ha^2}{b_0 - a_0} \cdot \ln\left(\frac{a}{a_0}\right) da$$

$$= \frac{2\pi H}{b_0 - a_0} \int_{a_0}^{b_0} a^2 \ln\left(\frac{a}{a_0}\right) \cdot da$$

$$= \frac{2\pi H a_0^3}{9(b_0 - a_0)} \cdot \left[3\left(\frac{b_0}{a_0}\right)^3 \ln\left(\frac{b_0}{a_0}\right) - \left(\frac{b_0}{a_0}\right)^3 + 1\right]$$

- Or,

$$\frac{W}{Y} = \frac{2\pi H a_0^3}{9(b_0 - a_0)} \cdot \phi\left(\frac{b_0}{a_0}\right)$$

Non-hardening Material

- Now the energy for doing the plastic work is derived from the kinetic energy of the projectile and thus,

$$\frac{W}{Y} = \frac{1}{2} \cdot \frac{(\rho_0 \pi a_0^2 L)}{Y} \cdot v_0^2 = \frac{\pi a_0^2 L}{2} \cdot \frac{\rho_0 v_0^2}{Y}$$

- Simplifying,

$$\frac{H}{L} = \frac{9(b_0 - a_0)}{4a_0} \cdot \frac{\rho_0 v_0^2 / Y}{\left[3\left(\frac{b_0}{a_0}\right)^3 \ln\left(\frac{b_0}{a_0}\right) - \left(\frac{b_0}{a_0}\right)^3 + 1\right]}$$

Non-hardening Material

- However, from previous work,

$$\frac{\rho_0 v_0^2}{Y} = \frac{e_0^2}{1-e_0} = \frac{\left(\frac{x^2-1}{x^2}\right)^2}{1-\left(\frac{x^2-1}{x^2}\right)} = \frac{\left(x^2-1\right)^2}{x^2}$$

- where $x = b_0/a_0$. Substituting,

$$\frac{H}{L} = \frac{9}{4} \cdot \frac{(x-1)\left(x^2-1\right)^2}{x^2\left[3x^2 \ln x - x^3 + 1\right]}$$

x	1·2	1·414	1·6	1·7	2·0	2·5
e_0	0·305	0·5	0·61	0·65	0·75	0·84
$\rho_0 v_0^2/Y$	0·135	0·5	0·975	1·24	2·25	4·42
H/L	0·29	0·42	0·505	0·51	0·515	0·52

Linear Hardening Material

- By following a similar approach to that just presented, it may be shown for a material which is rigid/linear strain-hardening, i.e. $\sigma = Y + P\varepsilon$, where P is the plastic modulus, that the plastic work required to be done to attain the frustum shape, determined on the same assumptions as before, is:

$$\frac{W}{Y} = \frac{2\pi H a_0^3}{9(b_0-a_0)}\left[3\left(\frac{b_0}{a_0}\right)^3 \ln\left(\frac{b_0}{a_0}\right)^3 - \left(\frac{b_0}{a_0}\right)^3 + 1 + \frac{P}{Y}\cdot\left(\frac{b_0}{a_0}\right)^3\left\{\left(3\ln\left(\frac{b_0}{a_0}\right) - 2\right)\ln\frac{b_0}{a_0} + \frac{2}{3}\left(1-\left(\frac{a_0}{b_0}\right)^3\right)\right\}\right]$$

Mean Strain After Impact

- The mean strain ε_m imparted, for the non-hardening material is,

$$\varepsilon_m = \frac{W}{Y.\text{volume of frustum}} = \frac{1}{3}\left[\frac{3\left(\frac{b_0}{a_0}\right)^3 \ln\left(\frac{b_0}{a_0}\right) - \left(\frac{b_0}{a_0}\right)^3 + 1}{\left(\frac{b_0}{a_0}\right)^3 - 1}\right]$$

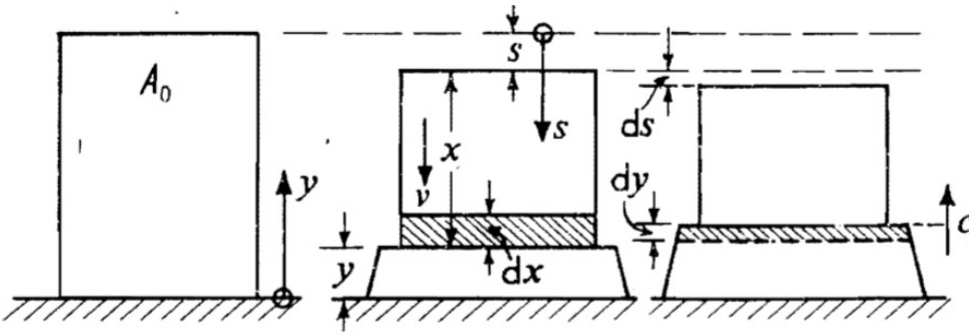
$$= \frac{2\cdot\left(\frac{b_0}{a_0}\right)^3 \ln\left(\frac{b_0}{a_0}\right)}{\left(\frac{b_0}{a_0}\right)^3 - 1} - \frac{2}{3}$$

- For b_0/a_0=1.5, 2, and 3, the corresponding values of ε_m are 0.48, 0.92, and 1.63, respectively.

Hawkyard's Energy Method

Energy Balance Equation

- In this section, with all the same assumptions as before, the consequences of establishing an energy balance across the discontinuity at the plastic wavefront are examined—an investigation originally made by Hawkyard.
- An elemental cylindrical length, dx, passes through the plastic wavefront and is so deformed as to acquire an area A, and height dy; the plastic wave speed is thus $c_p = dy/dt$.



Energy Balance Equation

- In the same time, the rear portion of the cylinder moves forward a distance ds so that $v = ds/dt$.
- The rate at which plastic work is dissipated in crossing the wavefront is,

$$\frac{dW}{dt} = \frac{d}{dt}\left(A_0 . dx . Y . \ln \frac{A}{A_0}\right)$$

$$\frac{dW}{dt} = A_0(v + c_p) \cdot Y . \ln \frac{A}{A_0}$$

- Since, $dx = ds + dy$, i.e.

$$dx/dt = ds/dt + dy/dt = v + c_p$$

Energy Balance Equation

- The loss of kinetic energy in the undeformed rear portion in decreasing in speed from v to $(v - dv)$ is equal to the arresting force times the distance it moves, i.e. YA_0ds and the deformed element loses its entire kinetic energy, i.e. $A_0\rho_0\, dx\, v^2/2$.
- Thus the rate of loss of energy of the projectile is

$$\frac{dE}{dt} = \frac{d}{dt}\left[\frac{1}{2}A_0\rho_0 dx v^2 + YA_0 ds\right] = A_0 v\left[\frac{1}{2}\rho_0 v(v + c_p) + Y\right]$$

- Since $dE/dt = dW/dt$,

$$A_0(v + c_p)Y \ln\frac{A}{A_0} = A_0 v[\frac{1}{2}\rho_0 v(v + c_p) + Y]$$

- Combining,

$$\frac{1}{2}\rho_0 v^2 = Y\left[\ln\frac{A}{A_0} - \left(1 - \frac{A_0}{A}\right)\right]$$

Velocity-Strain Equation

- Re-writing previous equation,

$$\frac{1}{2}\rho_0 v^2 = Y\left[\ln\left(\frac{1}{1-e}\right) - e\right]$$

- With the initial condition $v = v_0$, the initial strain e_0 is given by,

$$\frac{1}{2}\rho_0 v_0^2 = Y\left[\ln\left(\frac{1}{1-e_0}\right) - e_0\right]$$

- The equation of motion for the rear undeformed portion of the cylinder is,

$$YA_0 = -\rho_0 A_0 x\left(v \cdot \frac{dv}{ds}\right)$$

Undeformed Length and Strain

- But by differentiating with respect to s,

$$\rho_0 v \frac{dv}{ds} = Y\left(\frac{e}{1-e}\right)\frac{de}{ds}$$

- Hence, $$-\frac{ds}{x} = \frac{e}{1-e} \cdot de$$

- But $ds = -edx$, so $$\frac{dx}{x} = \frac{de}{1-e}$$
- Integrating,

$$\ln\frac{L}{x} = \ln\left(\frac{1-e}{1-e_0}\right) \quad \rightarrow \quad x = L\left(\frac{1-e_0}{1-e}\right)$$

$$\rightarrow \quad X = L(1-e_0)$$

Deformed Length and Strain

- We have for the element which crosses the plastic wavefront, $A_0 . dx = A . dy$

$$dy = \frac{A_0}{A}dx = -(1-e)dx$$

$$dy = -(1-e)x \cdot \frac{de}{(1-e)} = -x \cdot de$$

- Substituting for x

$$dy = -L(1-e_0)\frac{de}{1-e}$$

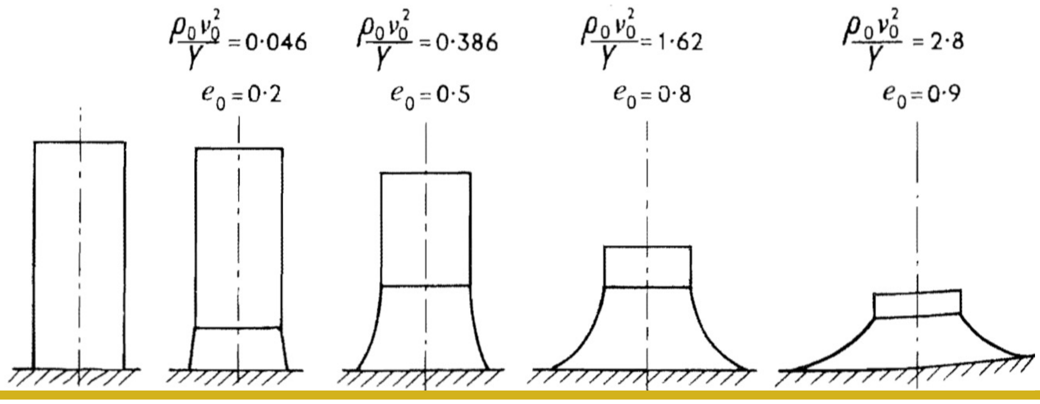
- Integrating, $y = L(1-e_0)\ln\left[(1-e)/(1-e_0)\right]$
- The final deformed length is H and

$$H = L(1-e_0)\ln\left[\frac{1}{(1-e_0)}\right]$$

  - and the final length $L_1 = X + H$.

Terminal Profile

- The final profile of the deformed end of the cylinder can be obtained by using previous equation.
  - Terminal profile shapes for various values of $\rho_0 v_0^2/Y$ as calculated by Hawkyard are shown.
  - All profiles are of concave form and have the shape given by experimental results; the profiles differ significantly from those given by Taylor particularly for the larger values of $\rho_0 v_0^2/Y$



Plastic Wave Speed

- From equations (5.37) and (5.36) (slides 49, 50),

$$c_p = \frac{1-e}{e} \cdot v$$

- and using (5.71) (slide 76) for v,

$$c_p = \left(\frac{Y}{\rho_0}\right)^{1/2} \left[2\left(\ln\left(\frac{1}{1-e}\right) - e\right)\right]^{1/2} \cdot \left(\frac{1-e}{e}\right)$$

- The momentum balance theory gives,

$$c_p = \left(\frac{1-e}{e}\right)\left[\frac{Ye^2}{\rho_0(1-e)}\right]^{1/2} = \left(\frac{Y}{\rho_0}\right)^{1/2} \cdot (1-e)^{1/2}$$

  - For a given value of e, the cp of (5.87) is greater than that of (5.86) and the difference increases as e increases.