1- Consider the scalar system

$$\dot{x} = ax^3$$

(a) Show that Lyapunov's linearization method fails to determine stability of the origin.

(b) Use the Lyapunov function

$$V(x) = x^4$$

to show that the system is globally asymptotically stable for $a < 0$ and unstable for $a > 0$.

(c) What can you say about the system for $a = 0$?

2- Consider the pendulum equation

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2.$$

(a) Assume zero friction, *i.e.* let $k = 0$, and show that the origin is stable. (*Hint*. Show that the energy of the pendulum is constant along all system trajectories.)

(b) Can the pendulum energy be used to show asymptotic stability for the pendulum with non-zero friction, $k > 0$? If not, modify the Lyapunov function to show asymptotic stability of the origin.

3- Consider the system

$$\ddot{x} + d\dot{x}^3 + kx = 0,$$

where $d > 0,\ k > 0$. Show that

$$V(x) = \frac{1}{2}(kx^2 + \dot{x}^2)$$

is a Lyapunov function. Is the system locally stable, locally asymptotically stable, and globally asymptotically stable?

4- Consider the linear system

$$\dot{x} = Ax = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} x$$

(a) Compute the eigenvalues of A and verify that the system is asymptotically stable

(b) From the lectures, we know that an equivalent characterization of stability can be obtained by considering the Lyapunov equation

$$A^T P + PA = -Q$$

where $Q = Q^T$ is any positive definite matrix. The system is asymptotically stable if and only if the solution P to the Lyapunov equation is positive definite.

(i) Let

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

Verify by completing squares that $V(x) = x^T P x$ is a positive definite function if and only if

$$p_{11} > 0, \qquad p_{11}p_{22} - p_{12}^2 > 0$$

(ii) Solve the Lyapunov function with Q as the identity matrix. Is the solution P a positive definite matrix?

(c) Solve the Lyapunov equation in Matlab.

5- As you know, the system

$$\dot{x}(t) = Ax(t), \qquad t \geq 0,$$

is asymptotically stable if all eigenvalues of A have negative real parts. It might be tempted to conjecture that the time-varying system

$$\dot{x}(t) = A(t)x(t), \qquad t \geq 0,$$

is asymptotically stable if the eigenvalues of $A(t)$ have negative real parts for all $t \geq 0$. This is *not* true.

(a) Show this by explicitly deriving the solution of

$$\dot{x} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} x, \qquad t \geq 0.$$

(b) The system is however stable if the eigenvalues of $A(t) + A^T(t)$ have negative real parts for all $t \geq 0$. Prove this by showing that $V = x^T x$ is a Lyapunov function.

6- Consider the system

$$\dot{x}_1 = 4x_1^2 x_2 - f_1(x_1)(x_1^2 + 2x_2^2 - 4)$$
$$\dot{x}_2 = -2x_1^3 - f_2(x_2)(x_1^2 + 2x_2^2 - 4),$$

where the continuous functions f_1 and f_2 have the same sign as their arguments, i.e. $x_i f_i(x_i) \geq 0$ (note that $f_i(0) = 0$). Show that almost all trajectories of the system tend towards the invariant set $x_1^2 + 2x_2^2 = 4$ independently of the explicit expressions of f_1 and f_2. However, this set will NOT be a limit cycle as the system will have singular points belonging to this set. What equilibrium points does the system have? Show that $x_1^2 + 2x_2^2 = 4$ is an attractive invariant set. How will the trajectories of the system behave?

7- Consider the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_1 - 2x_2 - 4x_1^3.$$

Use the function

$$V(x) = 4x_1^2 + 2x_2^2 + 4x_1^4$$

to show that

(a) the system is globally stable around the origin.

(b) the origin is globally asymptotically stable.

8- Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= -x_1 - x_2 + g(x)\end{aligned}$$

(a) Show that $V(x) = 0.5x^T x$ is a Lyapunov function for the system when $g(x) = 0$.

(b) Use this Lyapunov function to show that the system is globally asymptotically stable for all $g(x)$ that satisfy

$$g(x) = g(x_2)$$

and $\text{sign}(g(x_2)) = -\text{sign}(x_2)$.

(c) Let $g(x) = x_2^3$. This term does not satisfy the conditions in (a). However, we can apply Lyapunov's linearzation method to show that the origin is still locally asymptotically stable.

For large initial values, on the other hand, simulations reveal that the system is unstable. It would therefore be interesting to find the set of "safe" initial values, such that all trajectories that start in this set tend to the origin. This set is called the *region of attraction* of the origin. We will now illustrate how quadratic Lyapunov functions can be used to estimate the region of attraction.

(i) Show that $\dot{V}(x) < 0$ for $|x_2| < 1$. This means that $V(x)$ decreases for all solutions that are confined in the strip $|x_2(t)| \leq 1$ for all t.

(ii) Recall that level sets for the Lyapunov function are invariant. Thus, solutions that start inside a proper level set remain there for all future times. Conclude that the region of attraction can be estimated as the largest level set

$$\Omega = \{x : V(x) \leq \gamma\}$$

for which $|x_2| < 1$. Compute the maxiumum value of γ such that Ω is a region of attraction.

9- Use Krasovskii's method to justify Lyapunov's linearization m
ethod.