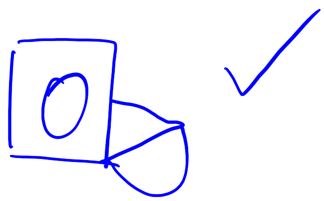
"CENTROID" AND "CENTER OF MASS" BY INTEGRATION

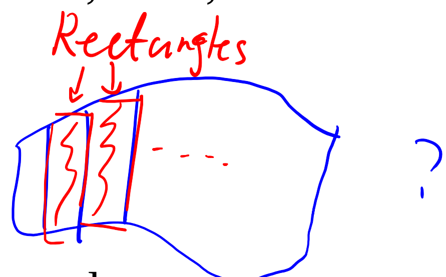
Learning Objectives

1). To determine the *volume, mass, centroid* and *center of mass* using integral calculus.

2). To do an *engineering estimate* of the volume, mass, centroid and center of mass of a body.

Definitions





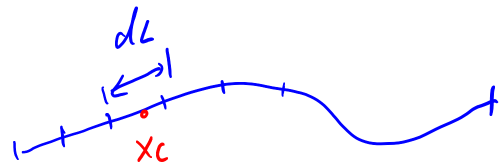
Centroid: **Geometric** *center* of a line, area or volume.

Center of Mass: **Gravitational** *center* of a line, area or volume.

The *centroid* and *center of mass* coincide when the **density** is **uniform** throughout the part.

Centroid by Integration

a). *Line*:



$$L = \int dL \qquad L\,\bar{x} = \int \underset{?}{x_c}\, \underset{?}{dL} \qquad L\,\bar{y} = \int y_c\, dL$$

b). *Area*:

$$A = \int dA \qquad A\,\bar{x} = \int \underset{?}{x_c}\, \underset{?}{dA} \qquad A\,\bar{y} = \int y_c\, dA$$

c). *Volume*:

$$V = \int dV \qquad V\,\bar{x} = \int x_c\, dV$$

$$V\,\bar{y} = \int y_c\, dV \qquad V\,\bar{z} = \int z_c\, dV$$

where: $\bar{x}$, $\bar{y}$, $\bar{z}$ represent the centroid of the line, area or volume.

$(x_c)_i$, $(y_c)_i$, $(z_c)_i$ represent the centroid of the differential element under consideration.

Center of Mass by Integration

$$m = \int dm = \int \rho \; dV$$

$$m \, x_G = \int x_c \, dm = \int x_c \, (\rho \, dV)$$

$$m \, \bar{y} = \int y_c \, dm = \int y_c \, (\rho \, dV)$$

$$m \, \bar{z} = \int z_c \, dm = \int z_c \, (\rho \, dV)$$

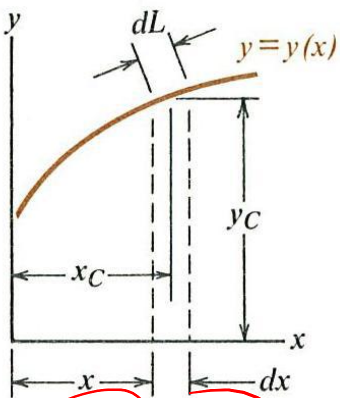
Note:

- For a homogeneous body ρ = constant, thus

$$m = \int \rho \, dV = \rho \int dV = \rho V$$

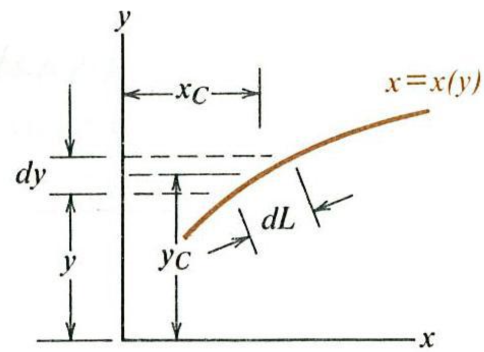
- Tabulated values of the *centroid* and *center of mass* of several standard shapes can be found on the back inside cover of the textbook.

Arch Length



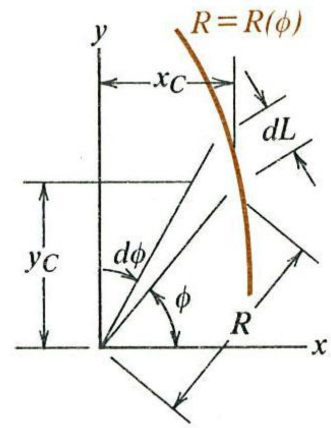
$$dL = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$$

$$x_C = x, \quad y_C = y(x)$$



$$dL = \left[1 + \left(\frac{dx}{dy}\right)^2\right]^{1/2} dy$$

$$x_C = x(y), \quad y_C = y$$

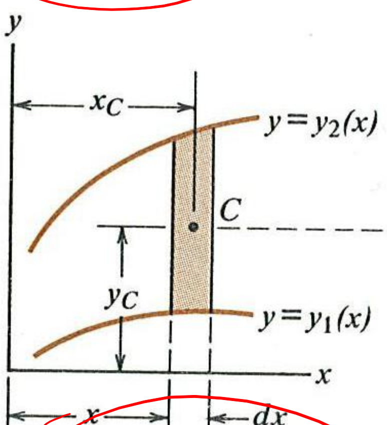


$$dL = \left[\left(\frac{dR}{d\phi}\right)^2 + R^2\right]^{1/2} d\phi$$

$$x_C = R(\phi)\cos\phi$$

$$y_C = R(\phi)\sin\phi$$

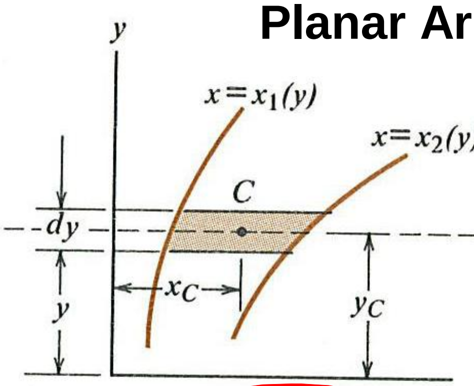
Planar Area



$$d\mathscr{A} = [y_2(x) - y_1(x)]\,dx$$

$$x_C = x$$

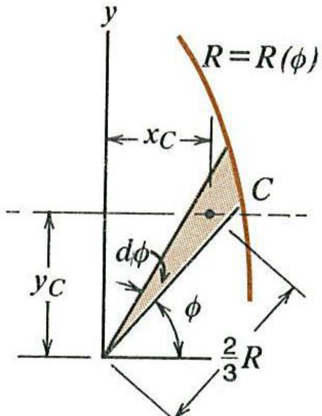
$$y_C = \frac{1}{2}[y_2(x) + y_1(x)]$$



$$d\mathscr{A} = [x_2(y) - x_1(y)]\,dy$$

$$x_C = \frac{1}{2}[x_2(y) + x_1(y)]$$

$$y_C = y$$

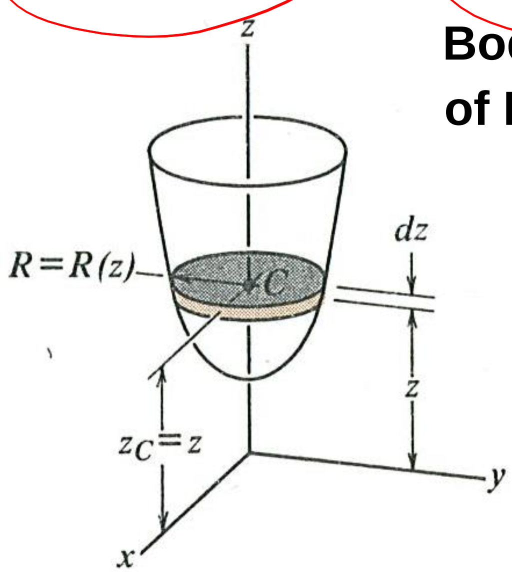


$$d\mathscr{A} = \frac{1}{2}R^2 d\phi$$

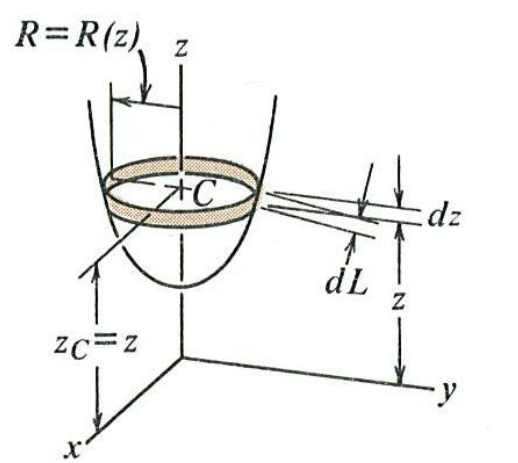
$$x_C = \frac{2}{3}R(\phi)\cos\phi$$

$$y_C = \frac{2}{3}R(\phi)\sin\phi$$

Body or Shell of Revolution



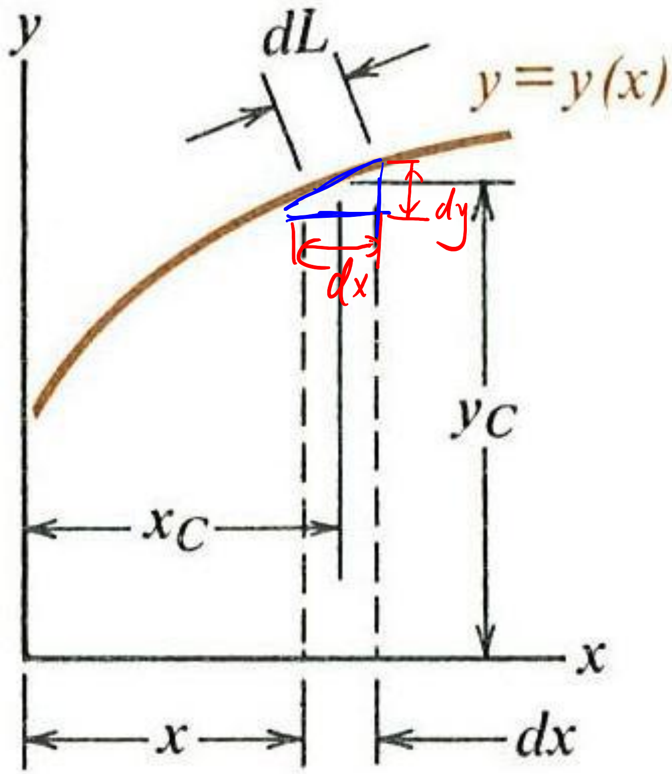
$$d\mathscr{V} = \pi[R(z)]^2 dz$$

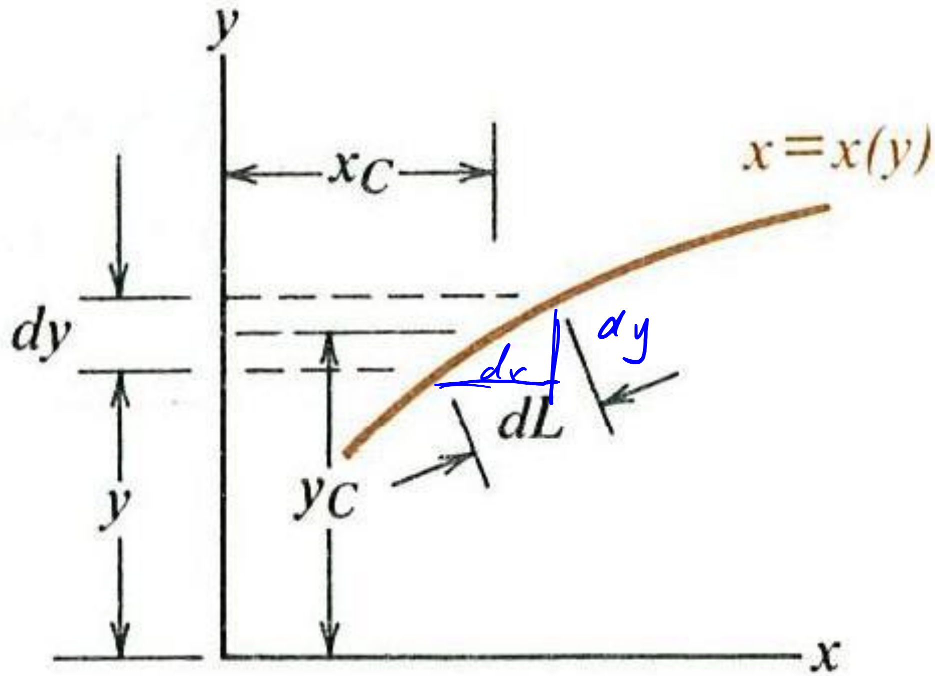


$$d\mathscr{A} = 2\pi R(z)\,dL$$

$$= 2\pi R(z)\left[1 + \left(\frac{dR}{dz}\right)^2\right]^{1/2} dz$$

Arc Length





$$dL = \sqrt{dx^2 + dy^2}$$

$$dL = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$$

$$x_C = x, \; y_C = y(x)$$

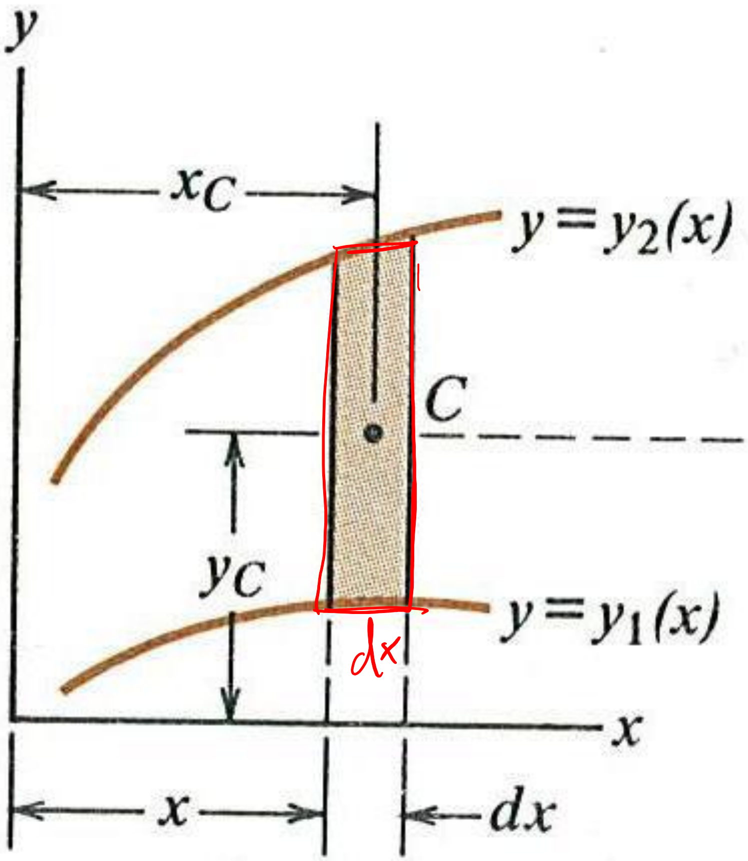
$x_c \rightarrow x$

$y_c \rightarrow y(x)$

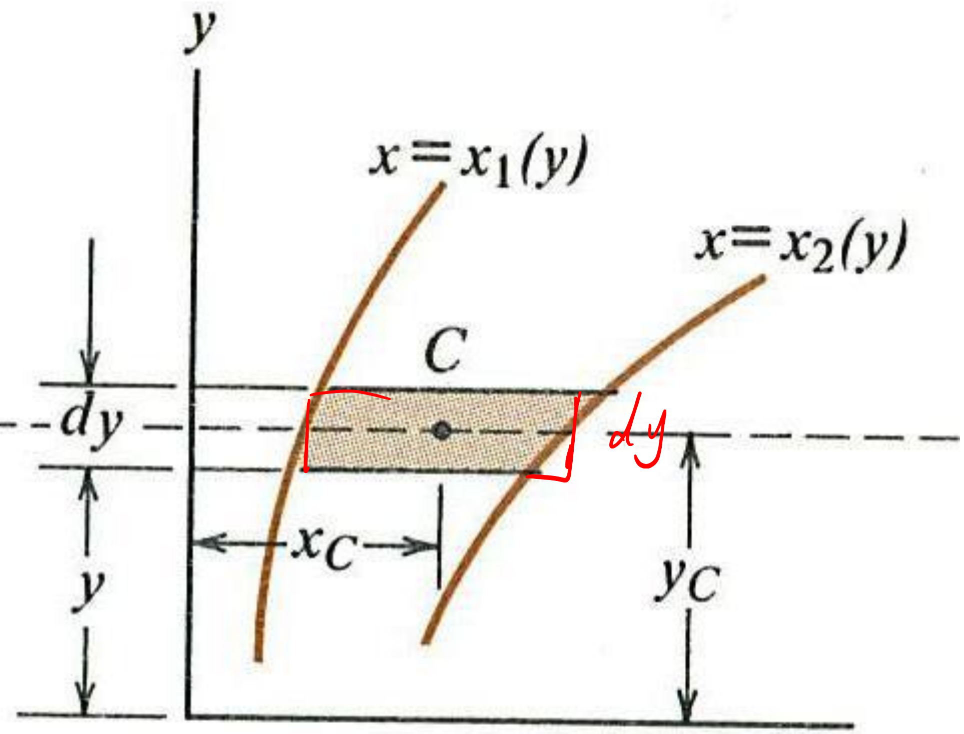
$$dL = \left[1 + \left(\frac{dx}{dy}\right)^2\right]^{1/2} dy$$

$$x_C = x(y), \; y_C = y$$

Planar Area



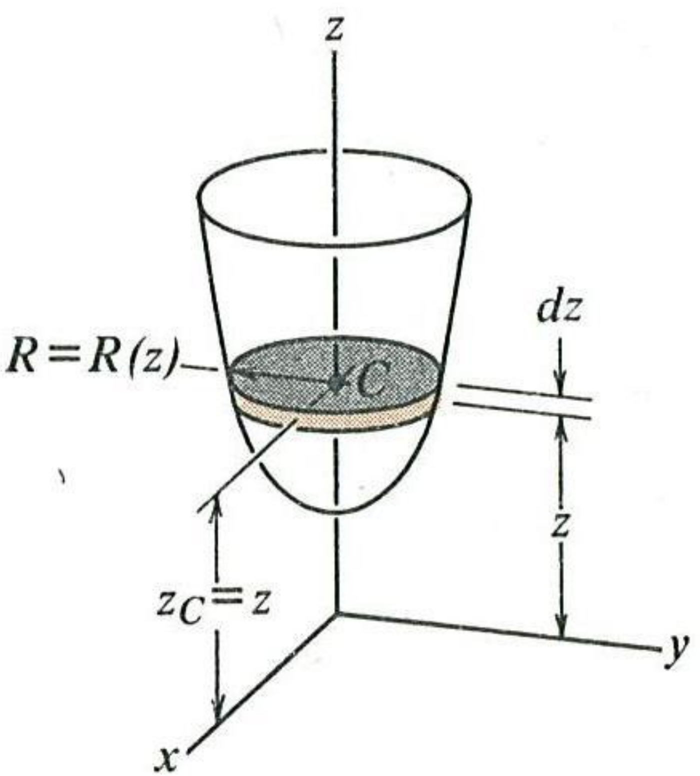
$$d\mathscr{A}=[y_2(x)-y_1(x)]\,dx$$
$$x_C=x$$
$$y_C=\frac{1}{2}[y_2(x)+y_1(x)]$$



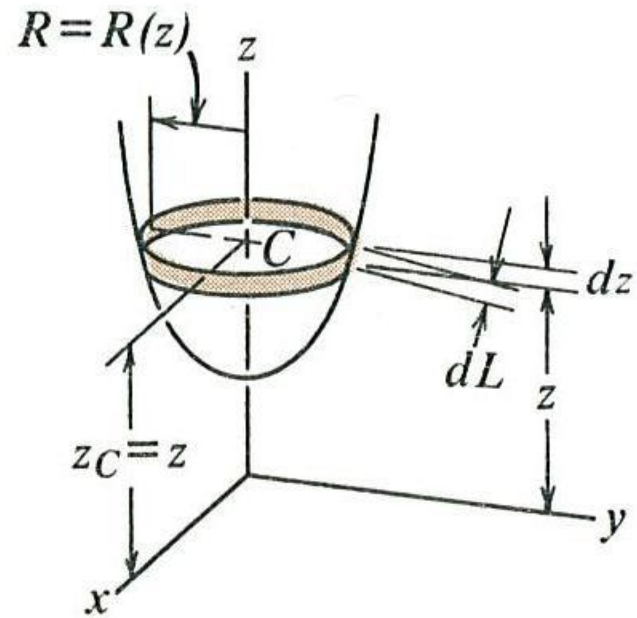
$$d\mathscr{A}=[x_2(y)-x_1(y)]\,dy$$
$$x_C=\frac{1}{2}[x_2(y)+x_1(y)]$$
$$y_C=y$$

We Can also Use double Integral

Body or Shell of Revolution



$$d\mathscr{V} = \pi\,[R(z)]^2 dz$$



$$d\mathscr{A} = 2\pi\,R(z)\,dL$$
$$= 2\pi\,R(z)\left[1 + \left(\frac{dR}{dz}\right)^2\right]^{1/2} dz$$

Centroids and Center of Mass By Integration

Example 1

Given: It is desired to determine the area and centroids of the shaded shape.

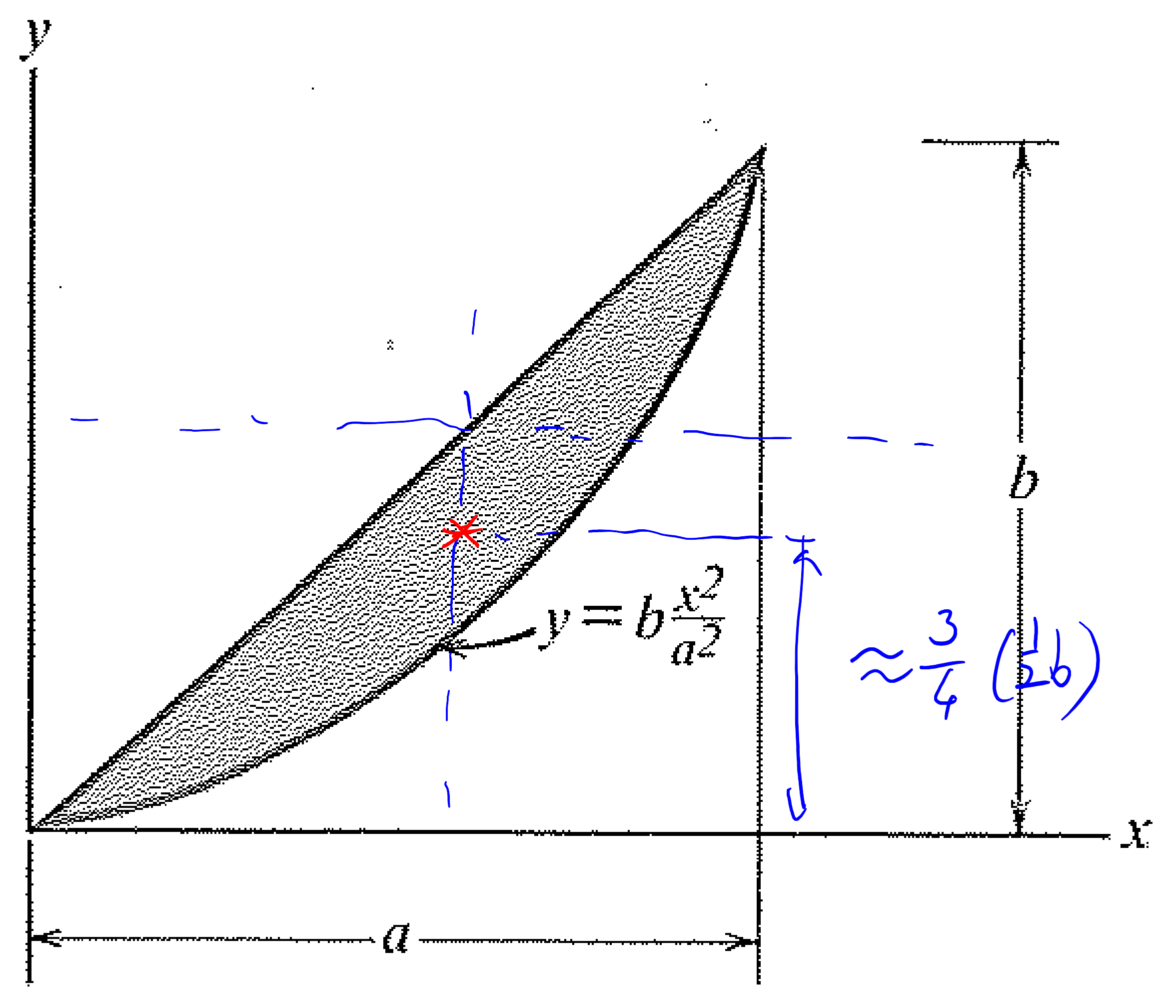
Find: For the shaded shape provided,

a) Estimate the area and the x and y centroids.

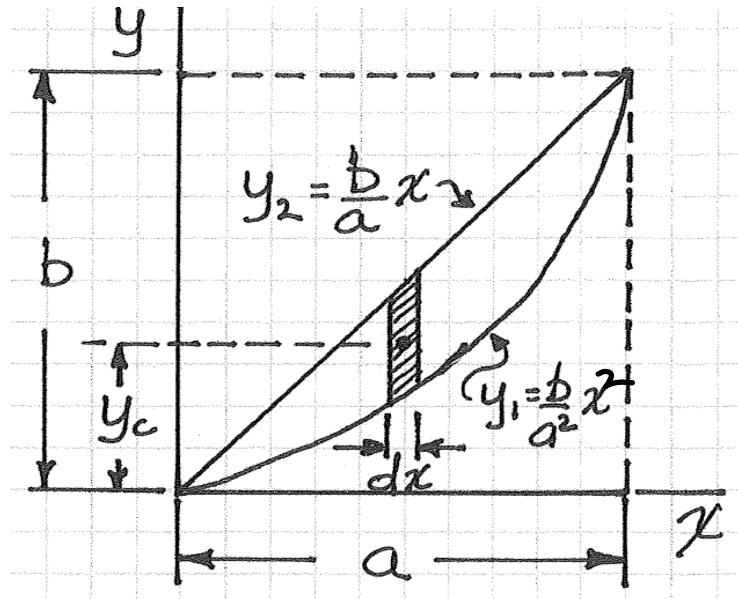
b) Calculate the area of the shape.

c) Calculate the x and y centroids of the shape.

a) Estimate: $A \approx \frac{1}{6} ab$ $\quad x_c \approx \frac{a}{2}$ $\quad y_c \approx \frac{3}{8} b$



b) Total Area



$$A = \int_0^a (y_2 - y_1)\, dx$$

$$= \int_0^a \left(\frac{b}{a}x - \frac{b}{a^2}x^2\right) dx$$

$$= \left(\frac{b}{2a}x^2 - \frac{b}{3a^2}x^3\right)\Big|_0^a$$

$$= \frac{1}{6}ab$$

X- Centroid:

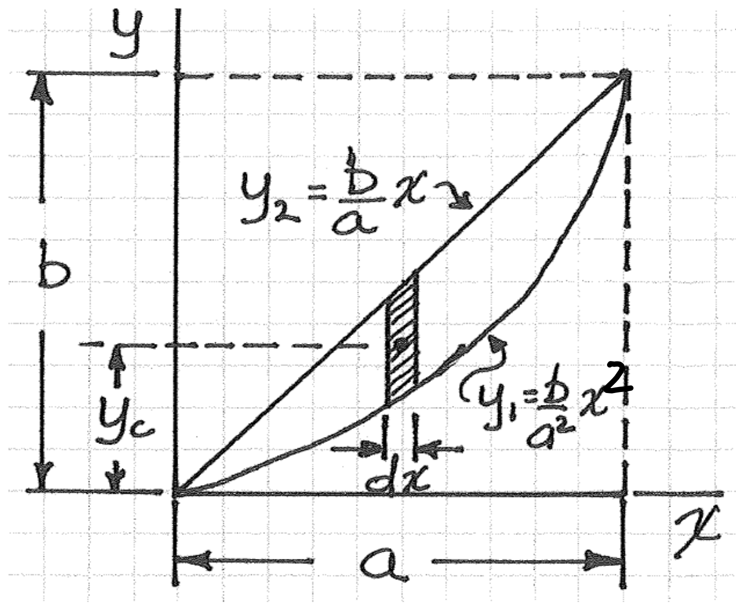
$$A\bar{x} = \int_0^a x_c\, dA = \int_0^a x\,(y_2 - y_1)\, dx$$

$$= \int_0^a \frac{b}{a}x^2 - \frac{b}{a^2}x^3 \quad dx$$

$$= \left(\frac{b}{3a}x^3 - \frac{b}{4a^2}x^4\right)\Big|_0^a$$

$$= \frac{1}{12}a^2b$$

$$\Rightarrow \left(\frac{1}{6}ab\right)\bar{x} = \frac{1}{12}a^2b \Rightarrow \bar{x} = \frac{a}{2}$$



y- Centroid.

$$A\bar{y} = \int_0^a y_c \, dA = \int_0^a \frac{y_2 + y_1}{2} (y_2 - y_1)\, dx$$

$$= \int_0^a \frac{1}{2}\left(\frac{b^2}{a^2}x^2 - \frac{b^2}{a^4}x^4\right)dx$$

$$= \frac{1}{2}\left(\frac{b^2}{3a^2}x^3 - \frac{b^2}{5a^4}x^5\right)\Big|_0^a$$

$$= \frac{ab^2}{15}$$

$$\Rightarrow \frac{1}{6}(ab)\,\bar{y} = \frac{ab^2}{15}$$

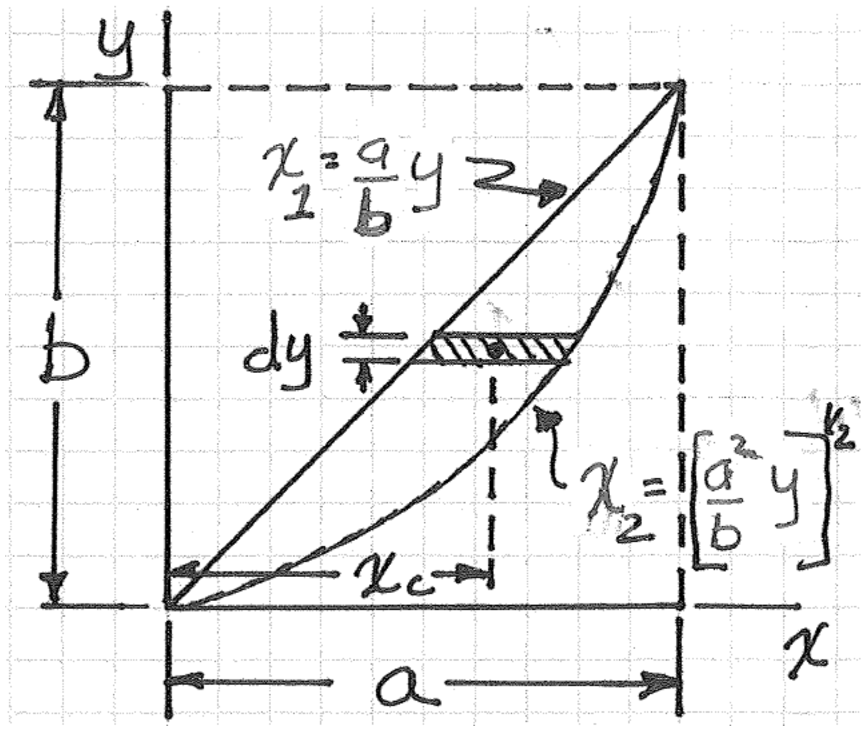
$$\Rightarrow \bar{y} = \frac{2b}{5}$$

$x = x(y)$

$y = \frac{b}{a}x \Rightarrow x_1 = \frac{a}{b}y$

$y = \frac{b}{a^2}x^2 \Rightarrow x_2 = \sqrt{\frac{a^2}{b}y}$

$= \frac{a}{\sqrt{b}}y^{\frac{1}{2}}$



Area:

$$A = \int_0^b (x_2 - x_1)\,dy$$

$$= \int_0^b \frac{a}{\sqrt{b}}y^{\frac{1}{2}} - \frac{a}{b}y \quad dy$$

$$= \left[\frac{2}{3}\frac{a}{\sqrt{b}}y^{\frac{3}{2}} - \frac{a}{2b}y^2\right]\Bigg|_0^b = \frac{1}{6}ab$$

X – Centroid: $A\bar{x} = \int_0^b \frac{x_1+x_2}{2}(x_2 - x_1)\,dy$

$$= \int_0^b \frac{1}{2}\left(\frac{a^2}{b}y - \frac{a^2}{b^2}y^2\right)dy$$

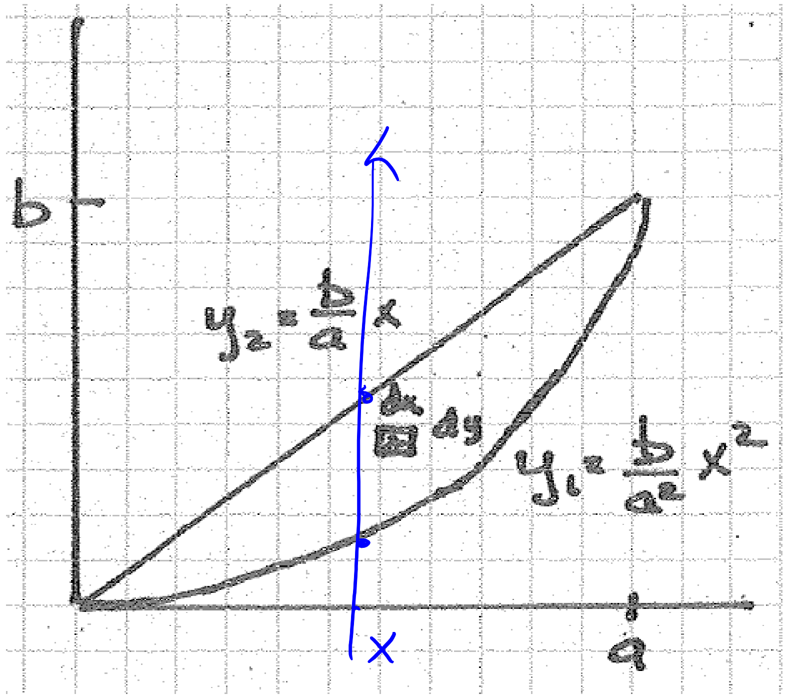
$$= \frac{1}{2}\left(\frac{a^2}{2b}y^2 - \frac{a^2}{3b^2}y^3\right)\Big|_0^b = \frac{1}{12}a^2b$$

y – Centroid: $A\bar{y} = \int_0^b y\,(x_2 - x_1)\,dy$

$$= \int_0^b \frac{a}{\sqrt{b}}y^{\frac{3}{2}} - \frac{a}{b}y^2\,dy$$

$$= \left(\frac{2}{5}\frac{a}{\sqrt{b}}y^{\frac{5}{2}} - \frac{a}{3b}y^3\right)\Big|_0^b = \frac{ab^2}{15}$$

Double Integral:



$$A = \int_0^a \left(\int_{\frac{b}{a^2}x^2}^{\frac{b}{a}x} dy \right) dx$$

$$= \int_0^a \left[(y) \Big|_{\frac{b}{a^2}x^2}^{\frac{b}{a}x} \right] dx$$

$$= \int_0^a \left(\frac{b}{a}x - \frac{b}{a^2}x^2 \right) dx = \frac{1}{6}ab$$

x- Centroid:

$$A\bar{x} = \int_0^a \int_{\frac{b}{a^2}x^2}^{\frac{b}{a}x} x \, dy \, dx = \int_0^a x \left(\frac{b}{a}x - \frac{b}{a^2}x^2 \right) dx$$

$$= \frac{1}{12}a^2 b$$

y- Centroid,

$$A\bar{y} = \int_0^a \int_{\frac{b}{a^2}x^2}^{\frac{b}{a}x} y \, dy \, dx = \int_0^a \left(\frac{y^2}{2} \right) \Big|_{\frac{b}{a^2}x^2}^{\frac{b}{a}x} dx$$

$$= \int_0^a \frac{1}{2} \left(\frac{b^2}{a^2}x^2 - \frac{b^2}{a^4}x^4 \right) dx = \frac{ab^2}{15}$$

Centroids and Center of Mass By Integration

Example 4

Given: The shaded area is bound by two curves.

Find:

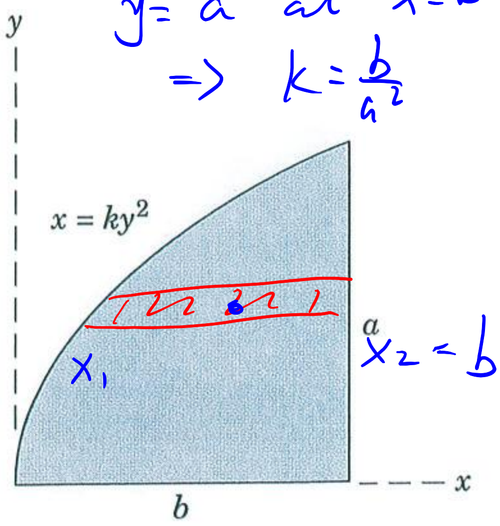
a) ~~Estimate~~ and then calculate the shaded area.

b) ~~Estimate~~ and then calculate the x-centroid of the shaded area.

c) ~~Estimate~~ and then calculate the y-centroid of the shaded area.

$y = a$ at $x = b$

$\Rightarrow k = \frac{b}{a^2}$



$$A = \int_0^a (x_2 - x_1)\, dy$$

$$= \int_0^a (b - ky^2)\, dy$$

$$= \frac{2}{3} ab$$

$$A\bar{x} = \int_0^a \left(\frac{x_2 + x_1}{2}\right)(x_2 - x_1)\,dy$$

$$= \int_0^a \left(b^2 - \frac{b^2}{a^4}y^4\right)dy$$

$$= \frac{4b^2}{10}a$$

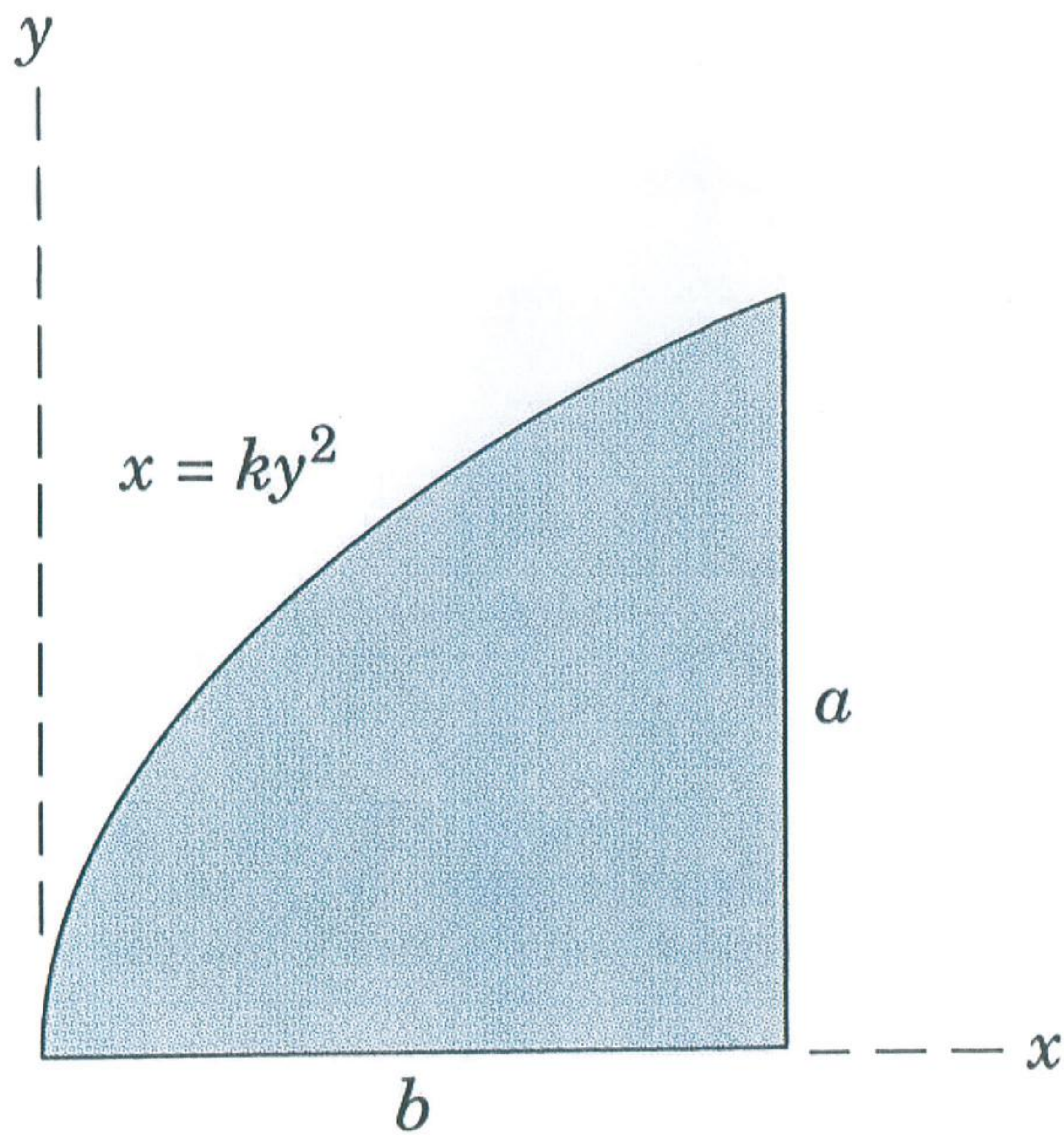
$$\Rightarrow \bar{x} = \frac{3}{5}b$$

$$A\bar{y} = \int_0^a y(x_2 - x_1)\,dy$$

$$= \int_0^a y\left(b - \frac{b}{a^2}y^2\right)dy$$

$$= \frac{ba^2}{4}$$

$$\Rightarrow \bar{y} = \frac{3}{8}a$$


y
x = ky^2
a
x
b

Centers of Mass & Centroids: By Integration
Group Quiz 1

Group #: ____________ Group Members: 1) ______________________________

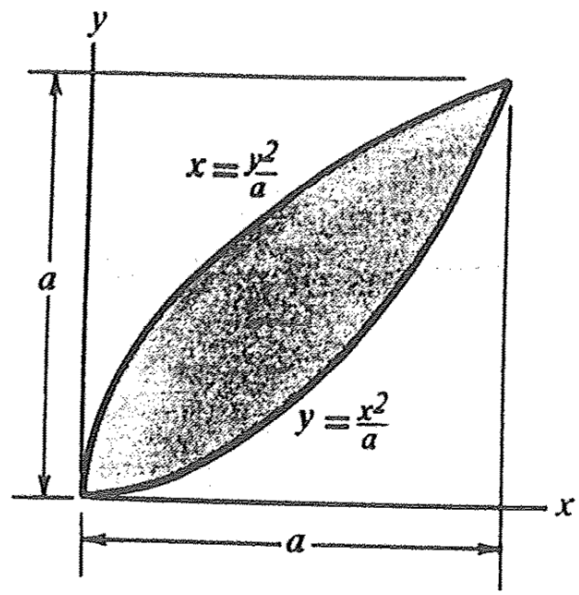
(Present Only)

Date: ______________ Period: _________ 2) ______________________________

3) ______________________________

4) ______________________________

Given: A shaded area is bounded by two lines given by $x = y^2/a$ and $y = x^2/a$.

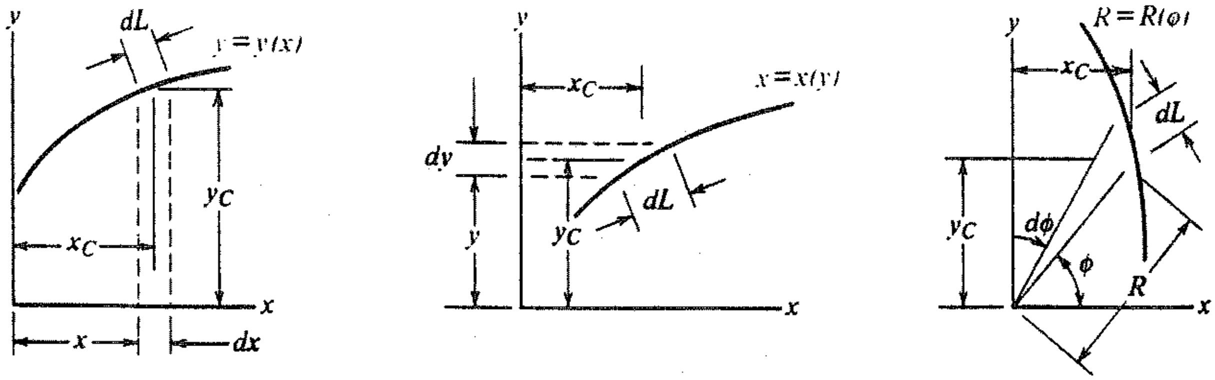


Find:

a) Do an engineering estimate of the shaded area and the centroid of the shaded area $(\bar{x}, \bar{y})$.

b) Determine the location of the centroid $(\bar{x}, \bar{y})$ by the method of integration.

Solution:



$dL = [1 + (\frac{dy}{dx})^2]^{1/2} dx$

$x_C = x,\ y_C = y(x)$

$dL = [1 + (\frac{dx}{dy})^2]^{1/2} dy$

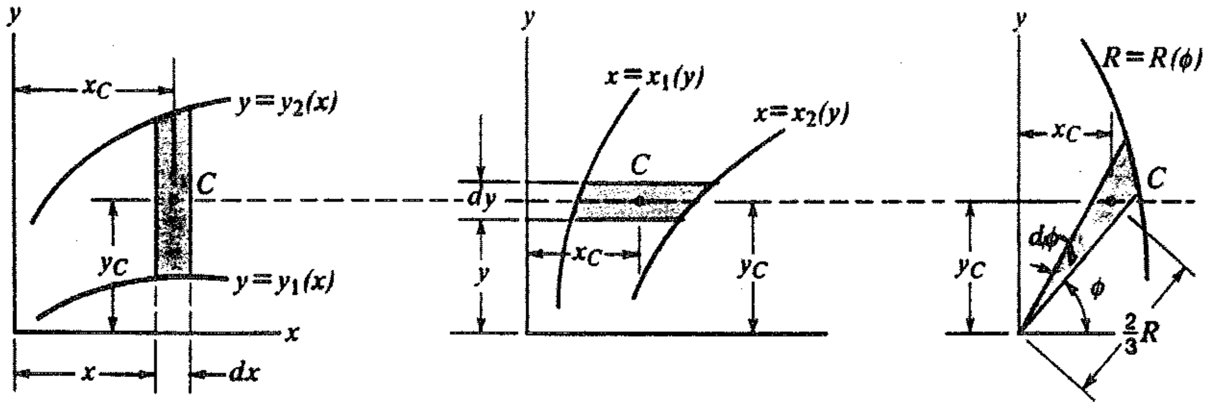
$x_C = x(y),\ y_C = y$

$dL = [(\frac{dR}{d\phi})^2 + R^2]^{1/2} d\phi$

$x_C = R(\phi) \cos \phi$

$y_C = R(\phi) \sin \phi$

FIGURE 8



$d\mathscr{A} = [y_2(x) - y_1(x)] dx$

$x_C = x$

$y_C = \frac{1}{2}[y_2(x) + y_1(x)]$

$d\mathscr{A} = [x_2(y) - x_1(y)] dy$

$x_C = \frac{1}{2}[x_2(y) + x_1(y)]$

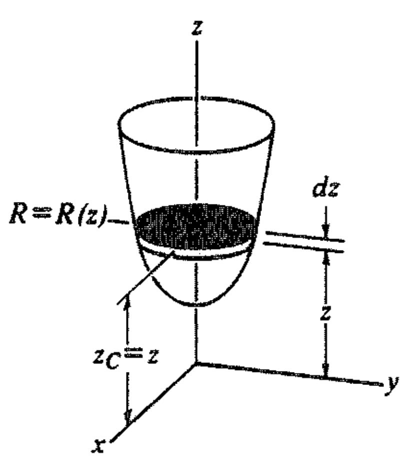
$y_C = y$

$d\mathscr{A} = \frac{1}{2} R^2 d\phi$

$x_C = \frac{2}{3} R(\phi) \cos \phi$

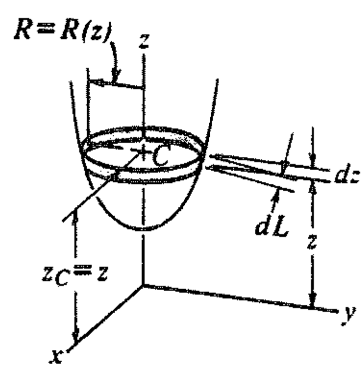
$y_C = \frac{2}{3} R(\phi) \sin \phi$

FIGURE 7



$d\mathscr{V} = \pi [R(z)]^2 dz$

FIGURE 9a



$d\mathscr{A} = 2\pi R(z) dL$

$= 2\pi R(z) [1 + (\frac{dR}{dz})^2]^{1/2} dz$

FIGURE 9b

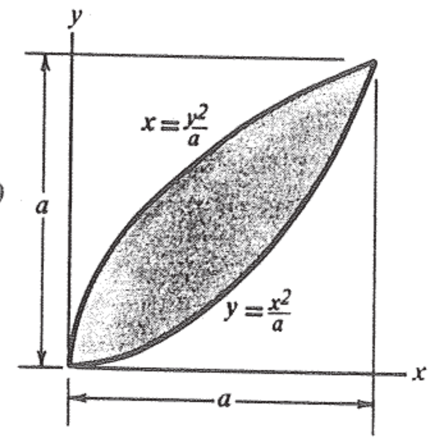
ME 270 - Basic Mechanics I - Group Problems

Your Name: ______________________ Group Members: 1) ____________ 2) ____________

Date: ____________ Period: ____________ 3) ____________ 4) ____________

Group #: ____________

Given: A shaded area is bounded by two lines given by $x = y^2/a$ and $y = x^2/a$.



PROB. 6.41

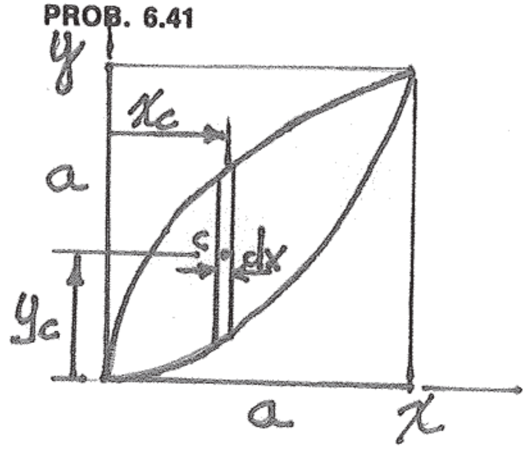
Find: shaded area and the

(a) Do an engineering estimate of the centroid of the shaded area $(\bar{x}, \bar{y})$.

(b) Determine the location of the centroid $(\bar{x}, \bar{y})$ by the method of integration.

Solution: (a) $\bar{x} \approx 0.4a \quad \bar{y} \approx 0.4a$

$A \approx a^2/4$



(b)

$dA = y\,dx, \quad x_c = x,$

$y_c = \left[(ax)^{1/2} + x^2/a\right]/2$

$$A = \int_0^a (y_2 - y_1)\,dx = \int_0^a \left[(ax)^{1/2} - x^2/a\right] dx = \boxed{a^2/3}$$

$$\bar{x}A = \bar{x}(a^2/3) = \int_0^a x_c\,dA = \int_0^a x\left[(ax)^{1/2} - \frac{x^2}{a}\right] dx = \frac{3}{20}a^3$$

$\therefore \boxed{\bar{x} = \frac{9}{20}a}$

$$\bar{y}A = \bar{y}(a^2/3) = \int_0^a y_c\,dA = \int_0^a \left[\frac{(ax)^{1/2} + x^2/a}{2}\right]\left[(ax)^{1/2} - \frac{x^2}{a}\right] dx = \frac{3}{20}a^3$$

$\boxed{\bar{y} = \frac{9}{20}a}$

$$A = \int_0^a \left[(ax)^{1/2} - \frac{x^2}{a} \right] dx$$

$$\left. \frac{a^{1/2} x^{3/2}}{\left(\frac{3}{2}\right)} - \frac{x^3}{3a} \right]_0^a = \frac{2}{3} a^2 - \frac{a^2}{3} = \boxed{\frac{a^2}{3}}$$

$$\bar{x} \left(\frac{a^2}{3} \right) = \int_0^a \left[a^{1/2} x^{3/2} - \frac{x^3}{a} \right] dx = \left. a^{1/2} \frac{x^{5/2}}{5/2} - \frac{x^4}{4a} \right]_0^a$$

$$\bar{x} \left(\frac{a^2}{3} \right) = \frac{2}{5} a^3 - \frac{a^3}{4} = \frac{8a^3}{20} - \frac{5a^3}{20} = \frac{3a^3}{20}$$

$$\therefore \boxed{\bar{x} = \frac{9a}{20}}$$

$$\bar{y} \left(\frac{a^2}{3} \right) = \int_0^a \left[\frac{(ax)^{1/2} + x^2/a}{2} \right] \left[(ax)^{1/2} - \frac{x^2}{a} \right] dx$$

$$= \int_0^a \left[\frac{a x}{2} - \frac{a^{1/2} x^{5/2}}{2a} + \frac{x^{5/2}}{2a^{1/2}} - \frac{x^4}{2a^2} \right] dx$$

$$\left. a \frac{x^2}{4} - \frac{a^{1/2} x^{7/2}}{2a\left(\frac{7}{2}\right)} + \frac{x^{7/2}}{2a^{1/2}\left(\frac{7}{2}\right)} - \frac{x^5}{2(5a^2)} \right]_0^a = \frac{a^3}{4} - \cancel{\frac{a^3}{7}} + \cancel{\frac{a^3}{7}} - \frac{a^3}{10}$$

$$\frac{5a^3}{20} - \frac{2a^3}{20} = \frac{3a^3}{20}$$