-۱

a) $f(z) = z|z|^2$

$z = x + iy \Rightarrow f(z) = x(x^2+y^2) + iy(x^2+y^2)$

$u = x(x^2+y^2) \quad , \quad V = y(x^2+y^2)$

$\frac{\partial u}{\partial x} = \frac{\partial V}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial V}{\partial x}$

① $x^2+y^2+2x^2 = x^2+y^2+2y^2 \Rightarrow x = y$

② $2xy = -2xy \Rightarrow x = y = 0$ در نقطه صفر مشتق دارد.

b) $f(z) = \operatorname{Re} z + \operatorname{Im} z \qquad z = x + iy$

$f(z) = x + y \qquad u = x + y \qquad V = 0$

$\frac{\partial u}{\partial x} = \frac{\partial V}{\partial y} \Rightarrow 1 = 0$

$\frac{\partial u}{\partial y} = -\frac{\partial V}{\partial x} \Rightarrow 1 = 0$

-۲

a) $f(z) = z^3 \Rightarrow f(z) = (x+iy)^3 = x^3 + i3x^2y + 3x(iy)^2 + (iy)^3 =$

$x^3 - 3xy^2 + i(3x^2y - y^3)$

$u = x^3 - 3xy^2 \quad , \quad v = 3x^2y - y^3$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 3x^2 - 3y^2 = 3x^2 - 3y^2$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow -6xy = -6xy$$

در تمام نقاط مشتق پذیر و تحلیلی است.

d) $f(z) = (1+i)(x-y)^2 = \underbrace{(x-y)^2}_{u} + i\underbrace{(x-y)^2}_{v}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2(x-y) = -2(x-y) \Rightarrow \boxed{x = y}$$ تحلیلی است

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow -2(x-y) = -2(x-y)$$

-۳

-۴

a: $u = e^{ax} \cos by$

$$u_{xx} = a^2 e^{ax} \cos by \qquad u_{yy} = -b^2 e^{ax} \cos by$$

$$u_{xx} + u_{yy} = 0 \Rightarrow a^2 e^{ax} \cos by - b^2 e^{ax} \cos by = 0$$

$$\Rightarrow e^{ax} \cos by \, (a^2 - b^2) = 0 \Rightarrow a^2 = b^2 \Rightarrow a = \pm b$$

به شرط مساوی بودن a و b ، خواهیم داشت.

$$u = e^{ax} \cos ay \Rightarrow \frac{\partial u}{\partial x} = a e^{ax} \cos ay = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -a e^{ax} \sin ay$$

$$v(y) = e^{ax} \sin ay + \phi(x) \Rightarrow \frac{\partial v}{\partial y} = -a e^{ax} \cos ay - \phi'(x)$$

$$\phi(x) = -x \qquad -\phi'(x) = 1 \Rightarrow \phi'(x) = -1$$

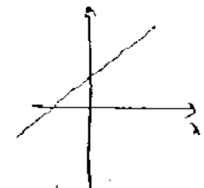
$$\Rightarrow v(y) = e^{ax} \sin ay - x$$

---

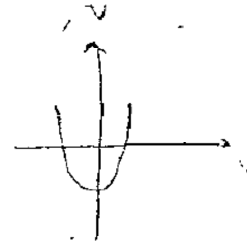
c) $y = 1+x$

$$\begin{cases} u = x^2 - (1+x)^2 \\ v = 2x(1+x) \end{cases} \Rightarrow \begin{cases} u = 2x+1 \\ v = 2x^2+2x \end{cases}$$

$$\Rightarrow x = \frac{u-1}{2} \Rightarrow v = \frac{u^2-1}{2}$$



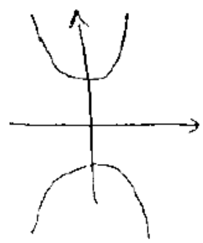
منحنی z



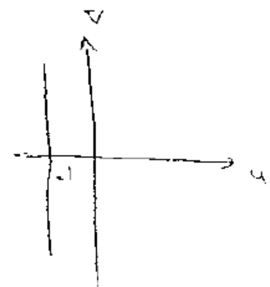
منحنی w

d) $y^2 = x^2+1$

$$\begin{cases} u = x^2 - x^2 - 1 \Rightarrow u = -1 \\ v = 2x\sqrt{x^2+1} \end{cases}$$



منحنی z



منحنی w

-۲

-۳

b) $-\frac{\pi}{4} < \arg z < \frac{\pi}{4} \implies -\frac{\pi}{4} < \arg w < \frac{\pi}{4}$

c) $0 < x < 1 \qquad 0 < y < 1$

$$w = \frac{1}{z} \to z = \frac{1}{w} = \frac{u - iv}{u^2+v^2} \quad \left| \begin{array}{l} x = \dfrac{u}{u^2+v^2} \\[2ex] y = -\dfrac{v}{u^2+v^2} \end{array} \right.$$

$x = 0 \to u = 0$

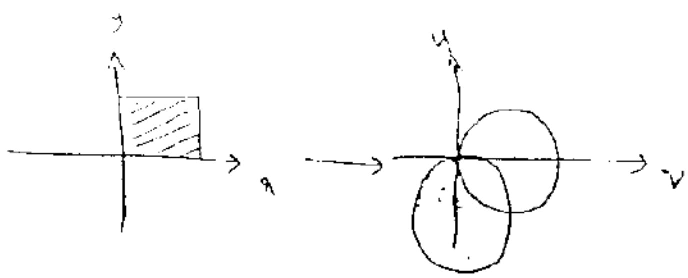
$x = 1 \Rightarrow u^2+v^2 = u \to (u-\frac{1}{2})^2 + v^2 = \frac{1}{4} \to |u - \frac{1}{2} + iv| = \frac{1}{2}$

$|w - \frac{1}{2}| = \frac{1}{2}$

$y = 0 \to v = 0$

$y = 1 \to 1 = -\frac{v}{u^2+v^2} \to u^2+v^2+v = 0 \to u^2 + (v+\frac{1}{2})^2 = \frac{1}{4}$

$|u + iv + \frac{1}{2}i| = \frac{1}{2} \to |w + \frac{i}{2}| = \frac{1}{2}$



-۴

-۵

d) $w = -iz^2 \to w = r^2 e^{2i\theta} e^{-\frac{\pi}{2}i} \to -\frac{\pi}{2} < \arg w < 0$

-۶

۵- سه نقطه‌ی ۰ و ۲ و ۱ را به ترتیب روی ۱ و $\frac{1}{2}$ و $\frac{1}{3}$ بنگارد (z_1, z_2, z_3 ؛ w_1, w_2, w_3)

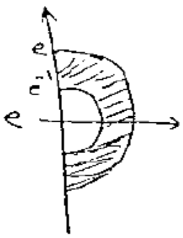
$$\frac{w-1}{w-\frac{1}{3}} \times \frac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{2}-1} = \frac{z-0}{z-2} \times \frac{1-2}{1}$$

-۱۲

a) $-\frac{\pi}{2} < y < \frac{\pi}{2}$ و $-1 < x < 1$

$$w = e^{x} e^{iy} \qquad \rho = e^{x} \quad , \quad -1 < x < 1 \Rightarrow e^{-1} < \rho < e$$

و از $\varphi = y \qquad -\frac{\pi}{2} < \varphi < \frac{\pi}{2}$



-۲۳

۲۳- [illegible]

[illegible]:

$\sin z = \sin x \cosh y + i \sinh y \cos x = \cosh 4 + i0$

$\cos x \sinh y = 0 \Rightarrow \begin{cases} y = 0 \\ \text{or} \\ x = k\pi + \frac{\pi}{2} = (2k+1)\frac{\pi}{2} \end{cases}$

$\sin x \cdot \cosh y = \cosh 4$

if $y = 0 \Rightarrow \sin x = \cosh 4$ ✗ $\Rightarrow y \neq 0$

if $x = (2k+1)\frac{\pi}{2}$, فرد k $\Rightarrow -\cosh y = \cosh 4$ ✗

if $x = (2k+1)\frac{\pi}{2}$, زوج k $\Rightarrow \cosh y = \cosh 4$

باتوجه به زوج بودن $\cosh y$ داریم:

$y = -4, 4 \Rightarrow \begin{cases} x = (2k+1)\frac{\pi}{2} , \text{ زوج } k \\ y = -4, 4 \end{cases}$

-۱

b) $f(z)=|z|^2$ ، C دایره واحد.

*حل ،

$f(z)=x^2+y^2 \qquad z=e^{i\theta} \qquad dz=ie^{i\theta}\,d\theta$

$$\int_C f(z)\,dz=\int_0^{2\pi} 1\times ie^{i\theta}\,d\theta=i\left[\frac{e^{i\theta}}{i}\right]_0^{2\pi}=\left[\frac{\overset{1}{e^{2\pi i}}-\overset{1}{e^{0i}}}{i}\right]i=0$$

تابع تحلیلی است پس انتگرال مستقل از مسیر می باشد.

$$\int_{1-i}^{1+i}\cos z\,dz=\left[\sin z\right]_{1-i}^{1+i}=\sin(1+i)-\sin(1-i)$$

$$=\frac{e^{1+i}-e^{-1-i}-e^{1-i}+e^{i-1}}{2i}=\frac{e^{i}(e^{1}+e^{-1})}{2i}-\frac{e^{-i}(e^{1}+e^{-1})}{2i}$$

$$=\left(e^{-1}+e^{1}\right)\left(\frac{e^{i}-e^{-i}}{2i}\right)=2\sinh(1)\sin(1)$$

-۱۶

$$b)\ z\cos\frac{1}{z}=z\left(1-\frac{1}{2!\,z^2}+\frac{1}{4!\,z^4}-\frac{1}{6!\,z^6}+\dots\right)$$

$$=z-\frac{1}{2!\,z}+\frac{1}{4!\,z^3}-\frac{1}{6!\,z^5}+\dots ,$$

-۱۷

۱۷ـ [illegible]

a) $\dfrac{e^z}{(z-1)^۲}$; $z_0=1$

ابتدا سری تیلور e^z را حول $z_0=1$ می‌نویسیم.

$$e^z = 1+(z-1)+\frac{(z-1)^۲}{۲!}+\frac{(z-1)^۳}{۳!}+\cdots$$

$$f(z) = (z-1)^{-۲}e^z = (z-1)^{-۲}+(z-1)^{-1}+\frac{1}{۲!}(z-1)^0+\frac{(z-1)}{۳!}+\cdots$$

-۲۲

ب- برای تعیین نقاط تکین [illegible]

b) $\dfrac{z}{z^۴-1} = \dfrac{z}{(z-i)(z+i)(z-1)(z+1)}$ $\qquad z=\pm i\ ,\pm 1$

$$\operatorname*{Res}_{z=1}\ \cancel{(z-1)}\frac{z}{\cancel{(z-1)}(z+1)(z-i)(z+i)} = \lim_{z\to 1}\frac{z}{(z+1)(z-i)(z+i)} = \frac{1}{۲\times۲} = \frac{1}{٤}$$

برای بقیه نقاط هم به همین ترتیب عمل کنید.

۴ ـ ٤z

-۲۳

c) $$\oint \frac{\sin \pi z}{z^۴}\,dz = ۲\pi i \operatorname*{Res}_{z=0} f(z) = ۲\pi i\, C_{-1}$$

$$\sin \pi z = z-\frac{z^۳}{۳!}+\frac{z^۵}{۵!}-\cdots \qquad \frac{\sin \pi z}{z^۴} = z^{-۳}-\frac{z^{-1}}{۳!}+\frac{z}{۵!}-\cdots \Rightarrow C_{-1}=-\frac{1}{۶}$$

$$\Rightarrow \oint_{|z|=1} f(z) = ۲\pi i\cdot\frac{-1}{۶} = \frac{-\pi i}{۳}$$

e) $$\oint_{|z|=1} \frac{dz}{z^4+5z^3+z^2} = 2\pi i \operatorname*{Res}_{z=0} f(z)$$

$$z^4+6z^1+5z^3 = z^2(z^2+5z+6) = z^2(z+2)(z+3) = 0$$

$z=0$ درون ناحیه است.

$$c_{-1} = \lim \frac{d}{dz}\left(z^2 \frac{1}{z^2(z+2)(z+3)}\right) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{1}{z^2+5z+6}\right)$$

$$= \lim_{z\to 0} \frac{-2z+5}{(z^2+5z+6)^2} = \frac{5}{36}$$

$$\oint f(z)\,dz = 2\pi i \,\frac{5}{36} = \frac{5\pi i}{18}$$

$$a)\ \int_{۰}^{۲\pi} \frac{d\theta}{۲+\cos\theta} \Rightarrow \qquad \cos\theta = \frac{۱}{۲}\left(z+\frac{۱}{z}\right) \qquad d\theta = \frac{dz}{iz}$$

$$I = \oint_{|z|=۱} \frac{dz}{iz\left(۲+\frac{z}{۲}+\frac{۱}{۲z}\right)} = \oint \frac{۲\,dz}{۴iz+iz^{۲}+i}$$

$$z_۱ = \frac{-۴i+i\sqrt{۱۲}}{۲i} \qquad z_۲ = \frac{-۴i-i\sqrt{۱۲}}{۲i}$$

$$z_۱ = \frac{\sqrt{۱۲}-۴}{۲} = \sqrt{۳}-۲ \qquad z_۲ = -۲-\sqrt{۳}$$

در دایره قرار دارد   در دایره قرار ندارد

$$I = ۲\pi i\, \mathrm{Res}\left[\frac{۱}{z+۲+\sqrt{۳}}\right] = ۲\pi i \lim_{z\to\sqrt{۳}-۲} \frac{۱}{z+۲+\sqrt{۳}} = ۲\pi i \times \frac{۱}{۲\sqrt{۳}}$$

$$b) = \int_{۰}^{\pi} \frac{d\theta}{۱+\frac{1}{3}\cos\theta} = \oint_{|z|=۱} \frac{\frac{dz}{iz}}{۱+\frac{z^{۲}+۱}{۶z}} = -۶i \oint_{|z|=۱} \frac{dz}{z^{۲}+۶z+۱}$$

$$= (-۶i)(۲\pi i)\, \mathrm{Res}_{z=-۳+۲\sqrt{۲}}\, f(z) = ۱۲\pi \lim_{z\to -۳+۲\sqrt{۲}} \frac{۱}{۲z+۶} = ۱۲\pi\alpha$$

b) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2-2x+2)^2}\,dx = 2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res} f(z)$

$f(z) = \dfrac{z}{(z^2-2z+2)^2}$ $\qquad z^2-2z+2=0 \Rightarrow z = 1\pm i$

$(z^2-2z+2)^2 = (z-1-i)^2(z-1+i)^2$ نقطه $z=1+i$ در $\operatorname{Im}(z)>0$ وجود دارد.

$\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \operatorname{Res}_{z=1+i} f(z)$ $\qquad \displaystyle\operatorname{Res}_{z=1+i} f(z) = \lim \frac{d}{dz}(z-1-i)^2 \cdot \frac{z}{(z-1-i)^2(z-1-}$

$\displaystyle = \lim_{z\to 1+i} \frac{d}{dz} \frac{z}{(z-1+i)^2} = \lim \frac{(z-1+i)-2z}{(z-1+i)^3} = \alpha$ $\qquad \displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i\alpha$

d) $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)(x^2+9)} = 2\pi i \sum_{\operatorname{Im}(z)>0} \operatorname{Res} f(z)$ $\qquad z^2+1=0 \quad z=\pm i$

$z^2+9=0 \quad z=\pm 3i$

$\Rightarrow z_0 = i,\ z_1 = 3i$ $\qquad \displaystyle\operatorname{Res}_{z=i} f(z) = \lim_{z\to i}(z-i)f(z) = \lim \frac{1}{(z+i)(z^2+9)} = \alpha$

$\displaystyle\operatorname{Res}_{z=3i} f(z) = \lim_{z\to 3i} \frac{1}{(z^2+1)(z+3i)} = \beta$ $\qquad I = 2\pi i(\alpha+\beta)$