Chapter 9: Center of Mass and Linear Momentum

- ✓ **Center of Mass**
- ✓ **Newton's Second Law for a System of Particles**
- ✓ **Linear Momentum**
- ✓ **Impulse**
- ✓ **Collision**

Session 19:

- ✓ Impulse
- ✓ Collision
- ✓ Examples

Impulse

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}}$$

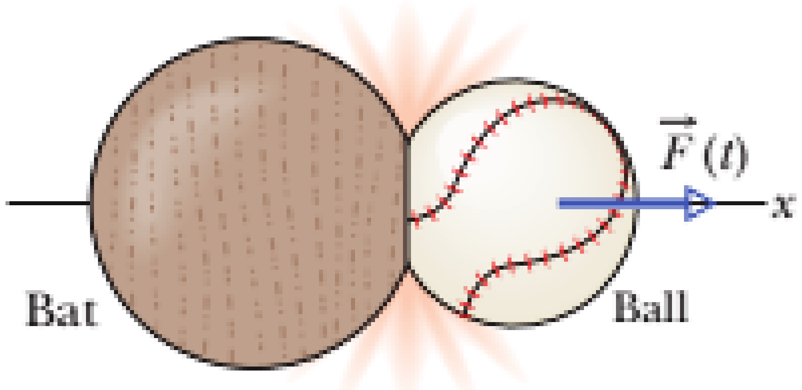
$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt}$$

$$\vec{\mathbf{F}}(t) = \frac{d\vec{\mathbf{p}}}{dt} \;\Rightarrow\; d\vec{\mathbf{p}} = \vec{\mathbf{F}}(t)\,dt \;\Rightarrow\; \int_{\vec{\mathbf{p}}_i}^{\vec{\mathbf{p}}_f} d\vec{\mathbf{p}} = \int_{t_i}^{t_f} \vec{\mathbf{F}}(t)\,dt$$

$$\Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i = \int_{t_i}^{t_f} \vec{\mathbf{F}}(t)\,dt = \vec{\mathbf{J}}$$

$$\vec{\mathbf{J}} = \Delta\vec{\mathbf{p}}$$

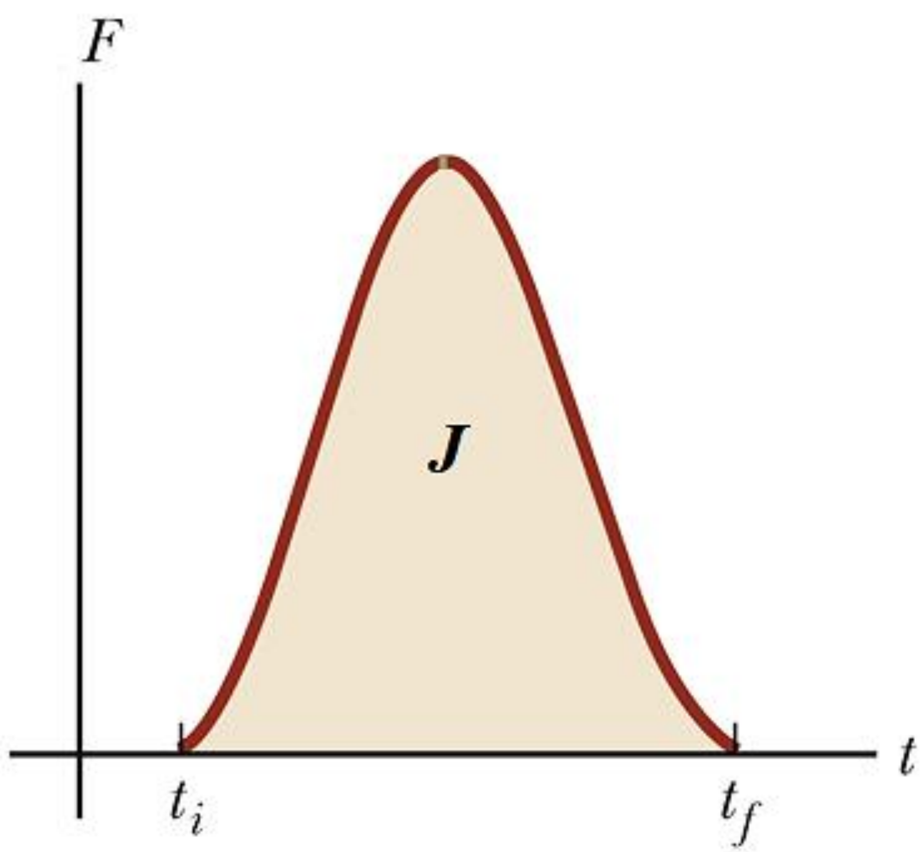
Impulse-momentum theorem

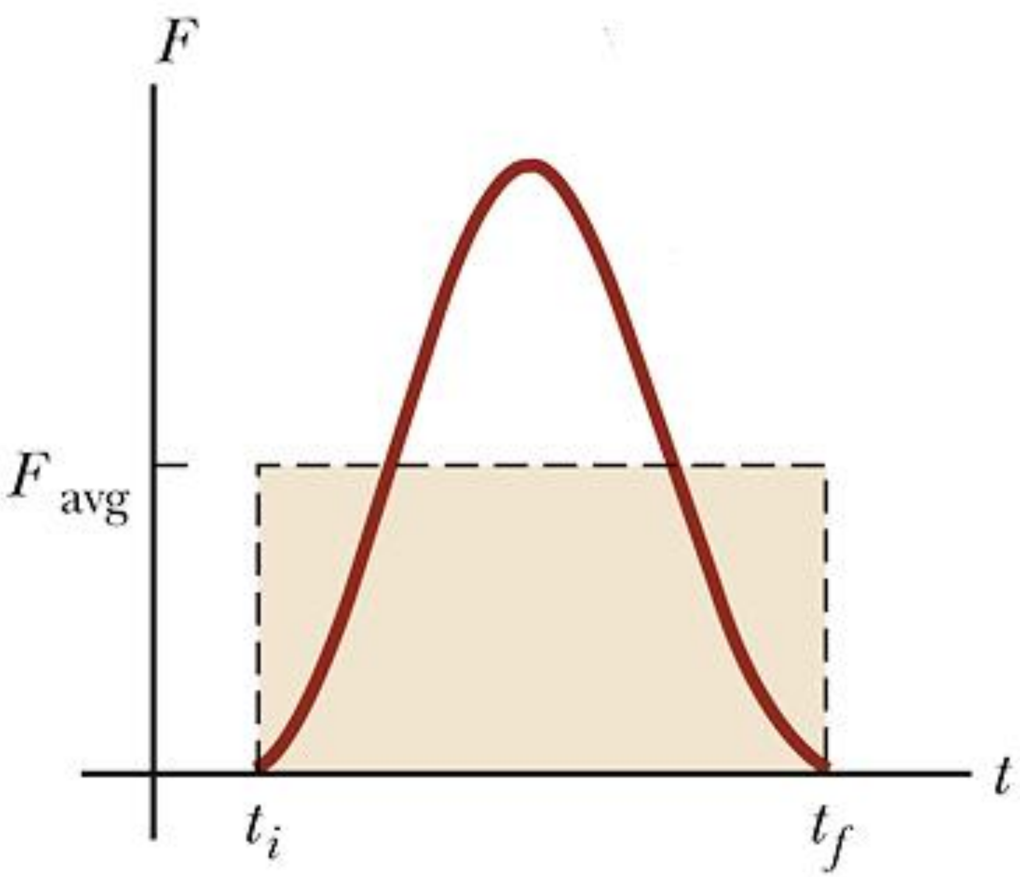


Impulse

$$\vec{\mathbf{J}} = \Delta\vec{\mathbf{p}} = \int_{t_i}^{t_f} \vec{\mathbf{F}}(t)\,dt$$

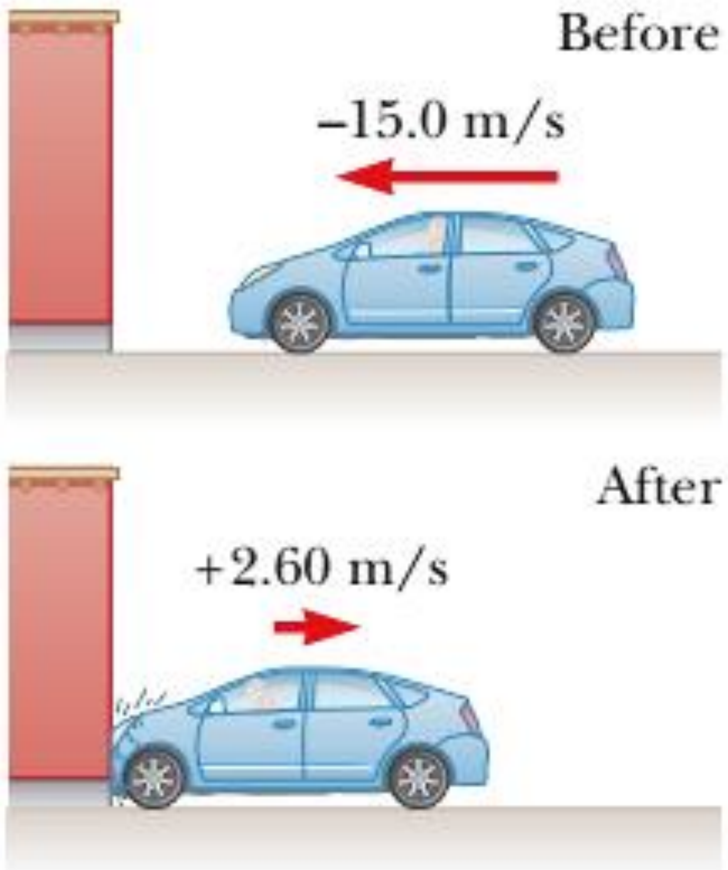
$$\begin{cases} J_x = \Delta \mathbf{p}_x = \int_{t_i}^{t_f} \mathbf{F}_x(t)\,dt \\ J_y = \Delta \mathbf{p}_y = \int_{t_i}^{t_f} \mathbf{F}_y(t)\,dt \\ J_z = \Delta \mathbf{p}_z = \int_{t_i}^{t_f} \mathbf{F}_z(t)\,dt \end{cases}$$





$$\mathbf{J} = \mathbf{F}_{avg}(t_f - t_i) = \mathbf{F}_{avg}\,\Delta t$$

Ex 11: In a particular crash test, a car of mass **1500 kg** collides with a wall as shown in Figure. The initial and final velocities of the car are $\vec{\mathbf{v}}_i = -15\,\hat{\mathbf{i}}$ m/s and $\vec{\mathbf{v}}_f = 2.6\,\hat{\mathbf{i}}$ m/s, respectively. If the collision lasts **0.150 s**, find the **impulse** caused by the collision and the **average net force** exerted on the car.



$$\vec{\mathbf{J}} = \Delta\vec{\mathbf{p}} = m\vec{\mathbf{v}}_f - m\vec{\mathbf{v}}_i$$

$$\vec{\mathbf{J}} = m(\vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i) = 1500\left[2.6\hat{\mathbf{i}} - (-15\hat{\mathbf{i}})\right] \Rightarrow \vec{\mathbf{J}} = 2.64\times10^4\,\hat{\mathbf{i}} \quad (kg.m/s)$$

$$\vec{\mathbf{J}} = \vec{\mathbf{F}}_{avg}\,\Delta t \Rightarrow \vec{\mathbf{F}}_{avg} = \frac{\vec{\mathbf{J}}}{\Delta t} = \frac{2.64\times10^4\,\hat{\mathbf{i}}}{0.150} = 1.76\times10^5\,\hat{\mathbf{i}}\ N$$

combination of the **normal force on the car from the wall** and any **friction force between the tires and the ground** as the front of the car crumples

Ex 12: (Problem 9.25 Halliday)

A **1.2 kg** ball drops vertically onto a floor, hitting with a speed of **25 m/s**. It rebounds with an initial speed of **10 m/s**. (a) What impulse acts on the ball during the contact? (b) If the ball is in contact with the floor for **0.020 s**, what is the magnitude of the average force on the floor from the ball?

$$\vec{\mathbf{J}} = \Delta\vec{\mathbf{p}} = m\vec{\mathbf{v}}_f - m\vec{\mathbf{v}}_i$$

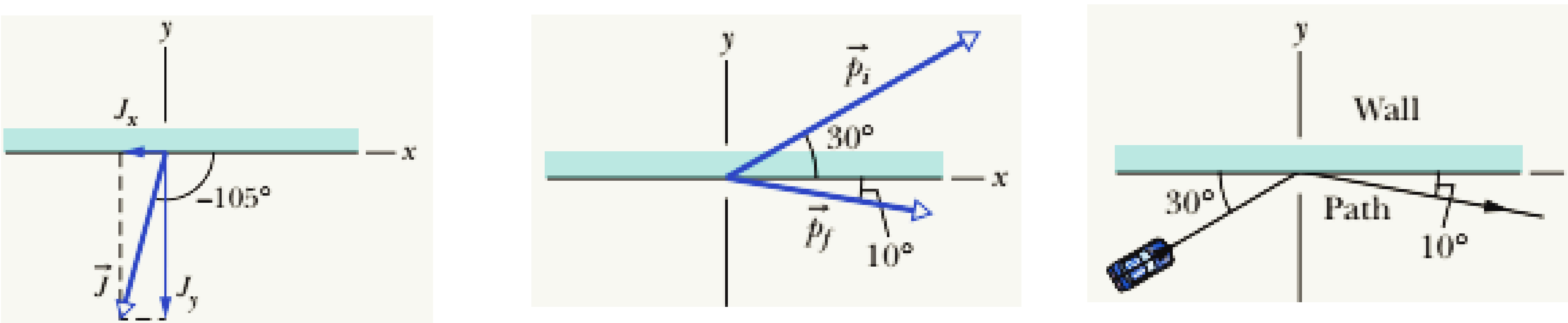
$$\vec{\mathbf{J}} = m(\vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i) = 1.2\left[10\hat{\mathbf{j}} - (-25\hat{\mathbf{j}})\right] = 42\ \hat{\mathbf{j}} \quad kg.m/s$$

$$\left|\vec{\mathbf{F}}_{avg}\right| = \frac{\left|\vec{\mathbf{J}}\right|}{\Delta t} = \frac{42}{0.020} = 2.1\times 10^3\ \ N$$

$$\left|\vec{\mathbf{F}}_{avg}\right| \gg mg \approx 10\ \ N$$

Impulse Approximation: one force acting on a particle acts for a short time (impulsive force), but is much greater than any other force present.

Ex 13: Figure below is an overhead view of the path taken by a race car driver as his car collides with the racetrack wall. Just before the collision, he is traveling at speed $\mathbf{v_i = 70\ m/s}$ along a straight line at **30°** from the wall. Just after the collision, he is traveling at speed $\mathbf{v_f = 50\ m/s}$ along a straight line at **10°** from the wall. His mass is **80 kg**. What is the impulse on the driver due to the collision?



$$\vec{\mathbf{J}} = \Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i$$

$$\begin{cases} J_x = \Delta \mathbf{p}_x = m(\mathbf{v}_{fx} - \mathbf{v}_{ix}) \\ J_y = \Delta \mathbf{p}_y = m(\mathbf{v}_{fy} - \mathbf{v}_{iy}) \end{cases}$$

$$\begin{cases} J_x = 80\left[50\cos(-10) - 70\cos 30\right] = -910\ \ kg.m/s \\ J_y = 80\left[50\sin(-10) - 70\sin 30\right] = -3495\ \ kg.m/s \end{cases}$$

$$\vec{\mathbf{J}} = -910\hat{\mathbf{i}} - 3495\hat{\mathbf{j}}$$

$$\left|\vec{\mathbf{J}}\right| = \sqrt{(910)^2 + (3495)^2} = 3616\ kg.m/s$$

$$\theta = \tan^{-1}\frac{J_y}{J_x} = \tan^{-1}\left(\frac{-3495}{-910}\right) = 75^\circ\ (-105^\circ)$$

Ex 14: **(Problem 9.37 Halliday)**

A soccer player kicks a soccer ball of mass **0.45 kg** that is initially at **rest**. The foot of the player is in contact with the ball for **3 × 10^{-3} s**, and the force of the kick is given by

$$F(t) = (6\times10^6)t - (2\times10^9)t^2 \quad N; \quad 0 \le t \le 3\times10^{-3}\, s$$

where **t** is in seconds. Find the magnitudes of (a) the **impulse** on the ball due to the kick, (b) the **average force** on the ball from the player's foot during the period of contact, (c) the **maximum force** on the ball from the player's foot during the period of contact, and (d) the **ball's velocity** immediately after it loses contact with the player's foot.

$$\mathbf{J} = \Delta\mathbf{p} = \int_{t_i}^{t_f} \mathbf{F}(t)\,dt$$

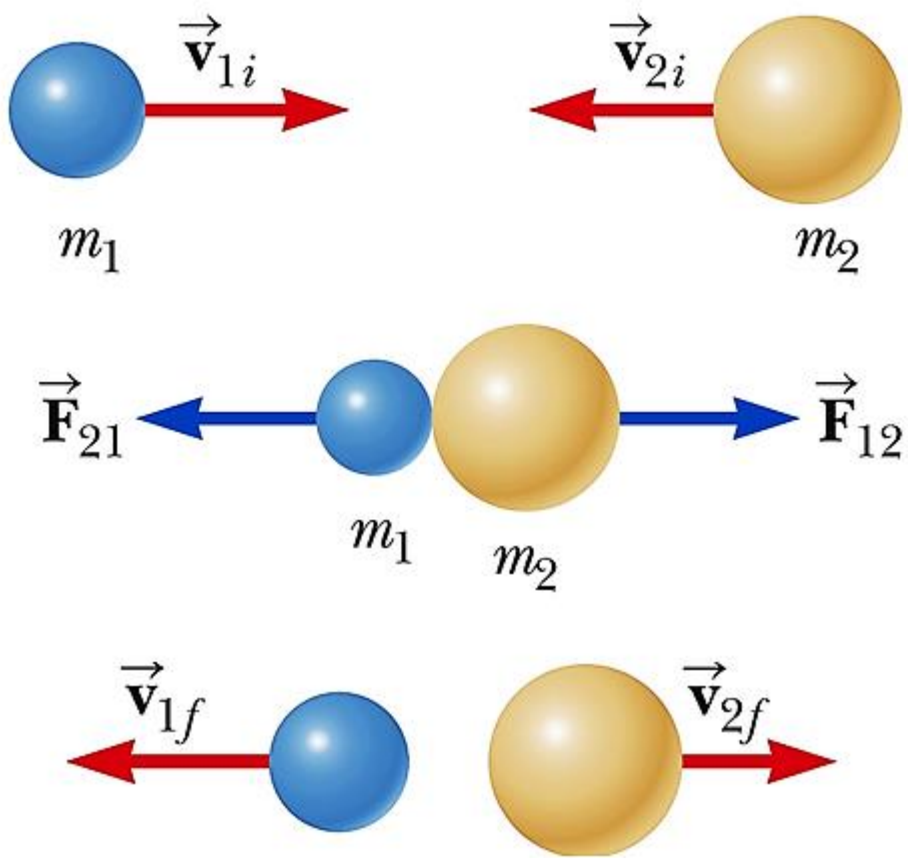
$$\mathbf{J} = \Delta\mathbf{p} = \int_0^{3\times10^{-3}} \left[(6\times10^6)t - (2\times10^9)t^2\right]dt = (6\times10^6)\frac{t^2}{2} - (2\times10^9)\frac{t^3}{3}\Bigg|_0^{3\times10^{-3}} = 9\ N.s$$

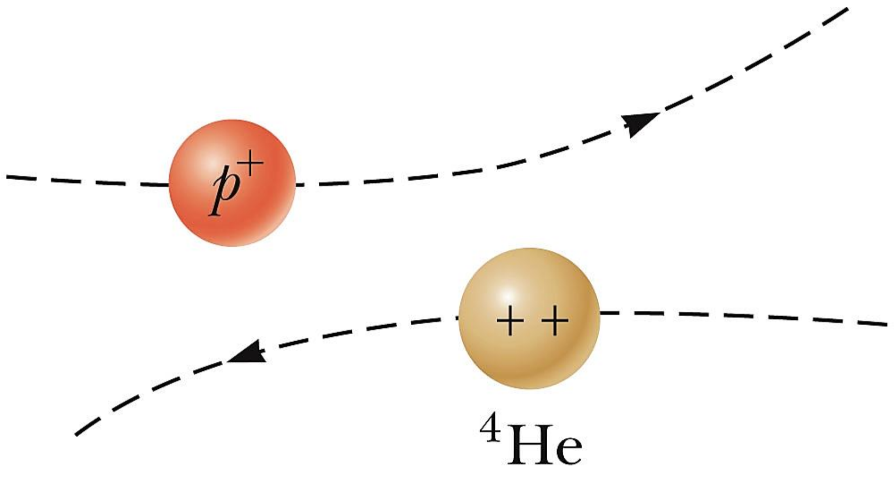
$$\left|\vec{\mathbf{F}}_{avg}\right| = \frac{\left|\vec{\mathbf{J}}\right|}{\Delta t} = \frac{9}{3\times10^{-3}} = 3\times10^3\ N$$

$$\frac{dF(t)}{dt} = 0 \Rightarrow (6\times10^6) - 2(2\times10^9)t = 0 \;\rightarrow\; t = 1.5\times10^{-3}\ s \;\rightarrow\; F_{\max} = 4.5\times10^3\ N$$

$$\mathbf{J} = \Delta\mathbf{p} = mv_2 - m\underbrace{v_1}_{0} = 9\ N.s \;\rightarrow\; v_2 = 20\ m/s$$

Collision





$$\vec{F}_{net} = \frac{d\vec{p}_{com}}{dt}$$

$$\vec{p}_{com} = \vec{p}_1 + \vec{p}_2 = \vec{p}_{tot}$$

$$\vec{F}_{net} = 0 \Rightarrow \frac{d\vec{p}_{com}}{dt} = 0 \Rightarrow \vec{p}_{com} = \text{constant}$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

Conservation of linear momentum

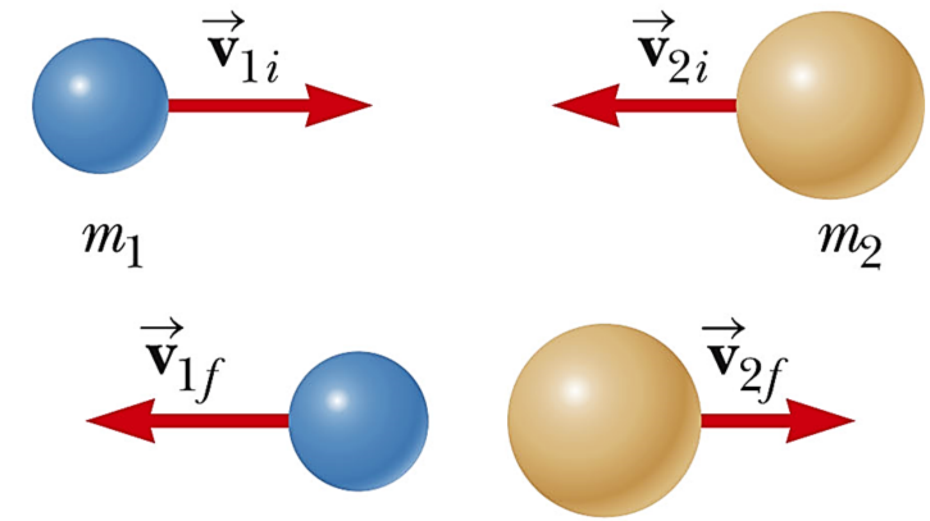
$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}$$

Kinetic Energy in Collisions

- **Collision**
  - **Elastic** — Kinetic energy is conserved
  - **Inelastic** — Kinetic energy is not conserved

Elastic:

$$\begin{cases} m_1\vec{\mathbf{v}}_{1i} + m_2\vec{\mathbf{v}}_{2i} = m_1\vec{\mathbf{v}}_{1f} + m_2\vec{\mathbf{v}}_{2f} \\ \frac{1}{2}m_1\mathbf{v}_{1i}^2 + \frac{1}{2}m_2\mathbf{v}_{2i}^2 = \frac{1}{2}m_1\mathbf{v}_{1f}^2 + \frac{1}{2}m_2\mathbf{v}_{2f}^2 \end{cases}$$

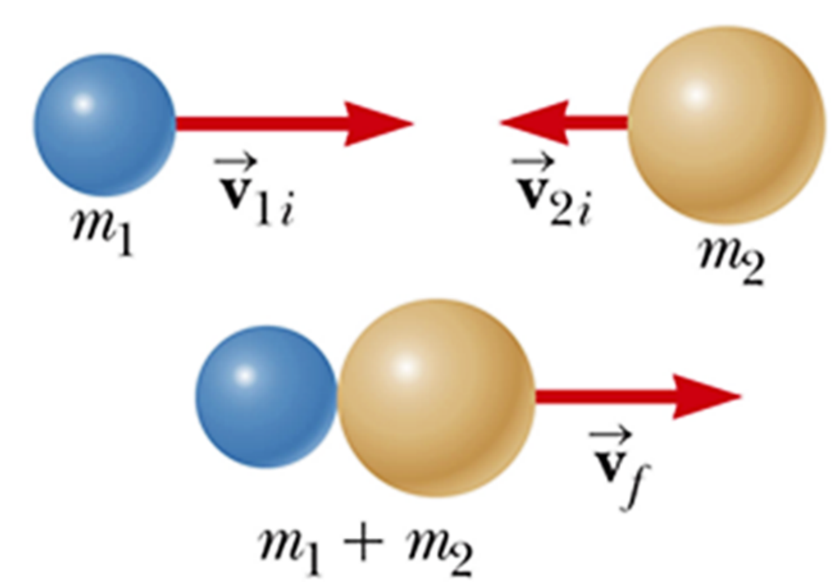


Inelastic:

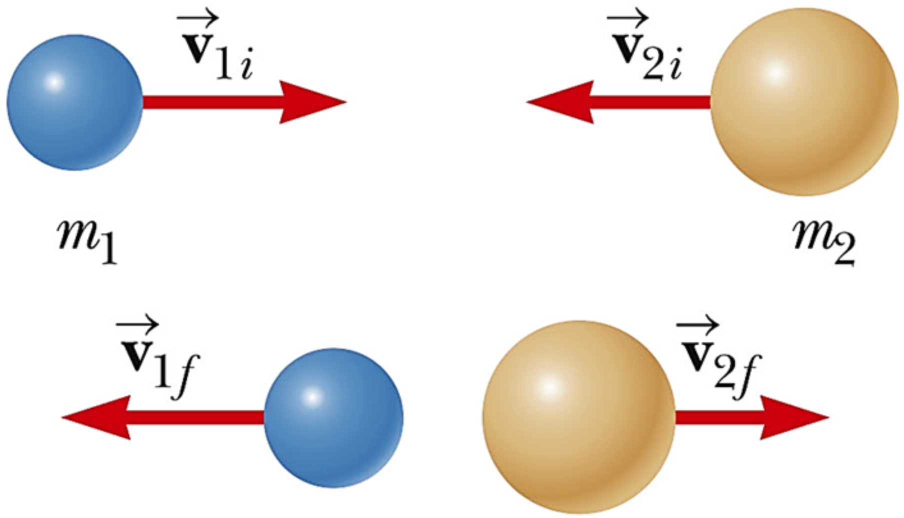
$$\begin{cases} m_1\vec{\mathbf{v}}_{1i} + m_2\vec{\mathbf{v}}_{2i} = m_1\vec{\mathbf{v}}_{1f} + m_2\vec{\mathbf{v}}_{2f} \\ \frac{1}{2}m_1\mathbf{v}_{1i}^2 + \frac{1}{2}m_2\mathbf{v}_{2i}^2 \neq \frac{1}{2}m_1\mathbf{v}_{1f}^2 + \frac{1}{2}m_2\mathbf{v}_{2f}^2 \end{cases}$$

Perfectly Inelastic:

$$m_1\vec{\mathbf{v}}_{1i} + m_2\vec{\mathbf{v}}_{2i} = (m_1 + m_2)\vec{\mathbf{v}}_f$$



Elastic Collision in One Dimension



$$m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i} = m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f}$$

$$\frac{1}{2}m_1\mathbf{v}_{1i}^2 + \frac{1}{2}m_2\mathbf{v}_{2i}^2 = \frac{1}{2}m_1\mathbf{v}_{1f}^2 + \frac{1}{2}m_2\mathbf{v}_{2f}^2$$

$$\mathbf{v}_{1f} = \frac{m_1 - m_2}{m_1 + m_2}\mathbf{v}_{1i} + \frac{2m_2}{m_1 + m_2}\mathbf{v}_{2i}$$

$$\mathbf{v}_{2f} = \frac{2m_1}{m_1 + m_2}\mathbf{v}_{1i} + \frac{m_2 - m_1}{m_1 + m_2}\mathbf{v}_{2i}$$

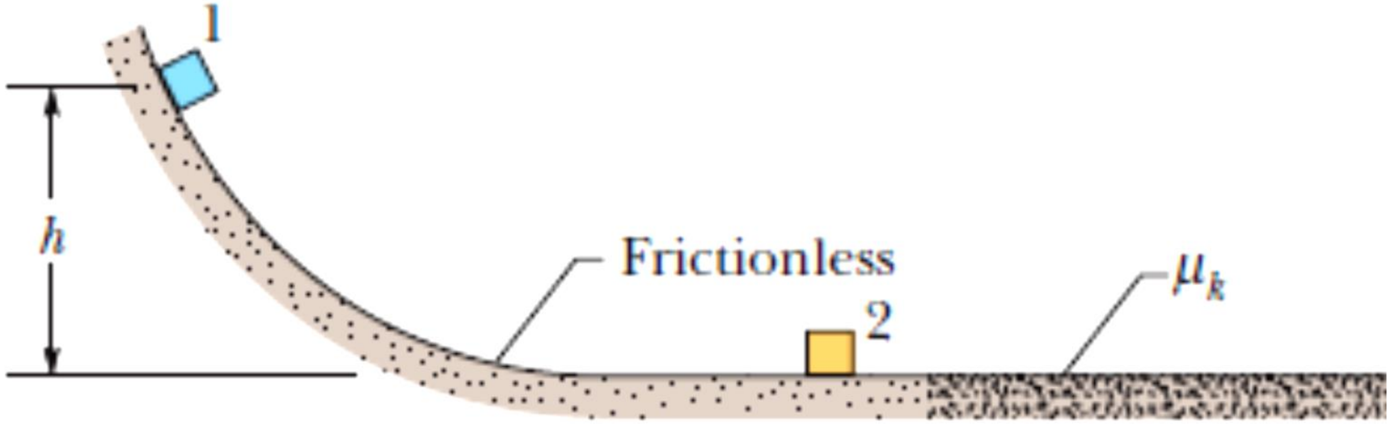
$$(\mathbf{v}_{1i} - \mathbf{v}_{2i}) = -(\mathbf{v}_{1f} - \mathbf{v}_{2f})$$

$$if\ \ m_1 = m_2 = m \Rightarrow \begin{cases} \mathbf{v}_{1f} = \mathbf{v}_{2i} \\ \mathbf{v}_{2f} = \mathbf{v}_{1i} \end{cases}$$

$$if\ \ \mathbf{v}_{2i} = 0 \Rightarrow \begin{cases} \mathbf{v}_{1f} = \dfrac{m_1 - m_2}{m_1 + m_2}\mathbf{v}_{1i} \\ \mathbf{v}_{2f} = \dfrac{2m_1}{m_1 + m_2}\mathbf{v}_{1i} \end{cases}$$

Ex 15: **(Problem 9.37 Halliday)**

Block 1 of mass $\mathbf{m_1}$ slides from **rest** along a frictionless ramp from height **h = 2.5 m** and then collides with **stationary block 2**, which has mass $\mathbf{m_2 = 2\,m_1}$. After the collision, block 2 slides into a region where the coefficient of kinetic friction μ_k is **0.500** and comes to a **stop** in distance **d** within that region. What is the value of distance **d** if the collision is (a) elastic and (b) completely inelastic?



$$K_i + U_i = K_f + U_f \qquad 0 + m_1 g h = \frac{1}{2} m_1 v_1^2 + 0$$

$$v_1 = \sqrt{2gh} = \sqrt{2(9.8)(2.5)} = 7 \;\; m/s$$

Elastic: $\mathbf{v}_{2i} = 0$ →

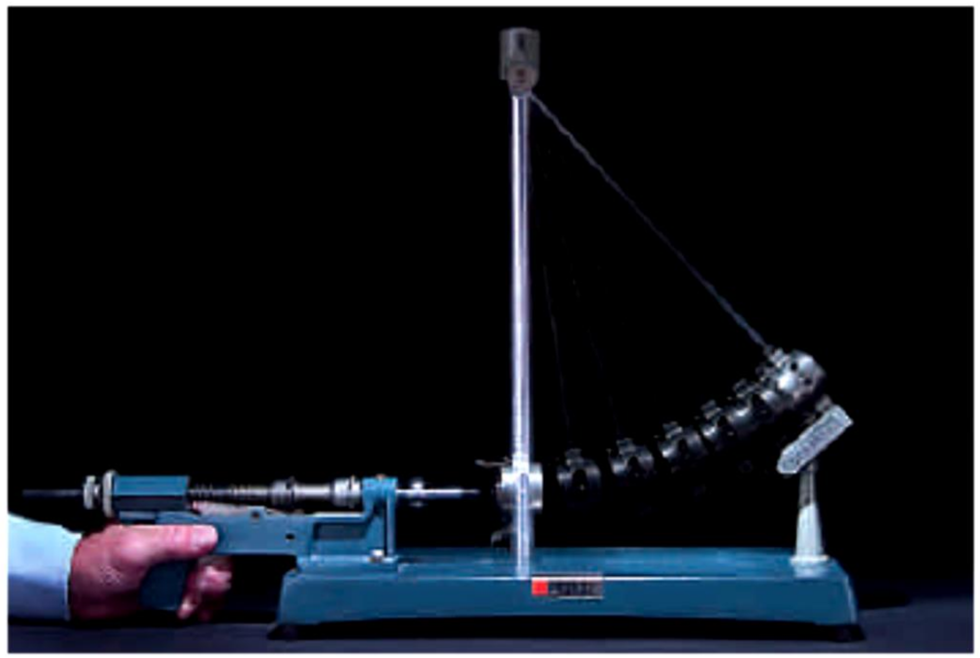
$$\begin{cases} \mathbf{v}_{1f} = \dfrac{m_1 - m_2}{m_1 + m_2}\mathbf{v}_{1i} \\ \mathbf{v}_{2f} = \dfrac{2m_1}{m_1 + m_2}\mathbf{v}_{1i} \end{cases} \rightarrow \begin{cases} \mathbf{v}_{1f} = -\dfrac{1}{3}\mathbf{v}_{1i} = -\dfrac{7}{3}\, m/s \\ \mathbf{v}_{2f} = \dfrac{2}{3}\mathbf{v}_{1i} = \dfrac{14}{3}\, m/s \end{cases}$$

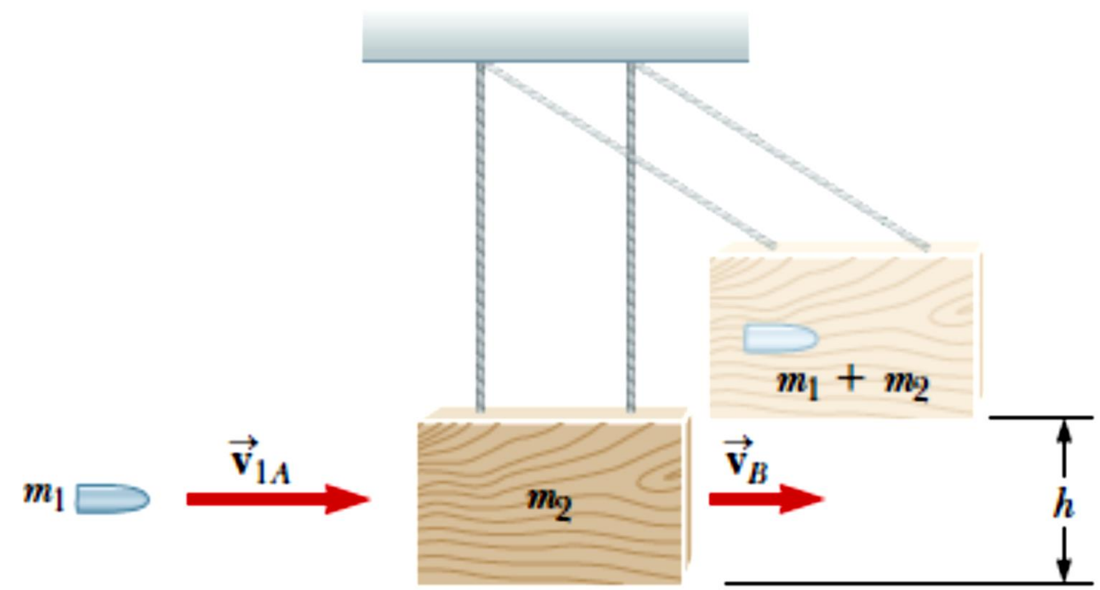
$$(0 - \frac{1}{2} m_2 v_{2f}^2) = -\mu_k m_2 g\, d \rightarrow d = \frac{v_{2f}^2}{2\mu_k g} = \frac{(4.66)^2}{2(0.5)(9.8)} = 2.22\ m$$

Perfectly Inelastic: $m_1\mathbf{v}_{1i} + m_2 \underbrace{\mathbf{v}_{2i}}_{0} = (m_1 + m_2)\mathbf{v}_f$ → $\mathbf{v}_f = \dfrac{m_1}{m_1 + m_2}\mathbf{v}_{1i} = \dfrac{\mathbf{v}_{1i}}{3} = \dfrac{7}{3}\, m/s$

$$d = \frac{v_f^2}{2\mu_k g} = \frac{(2.33)^2}{2(0.5)(9.8)} = 0.55\ m$$

Ex 16: The **ballistic pendulum** is an apparatus used to measure the speed of a fast-moving projectile such as a bullet. A projectile of mass $\mathbf{m_1}$ is fired into a large block of wood of mass $\mathbf{m_2}$ suspended from some light wires. The projectile embeds in the block, and the entire system swings through a height **h**. How can we determine the speed of the projectile from a measurement of **h**?



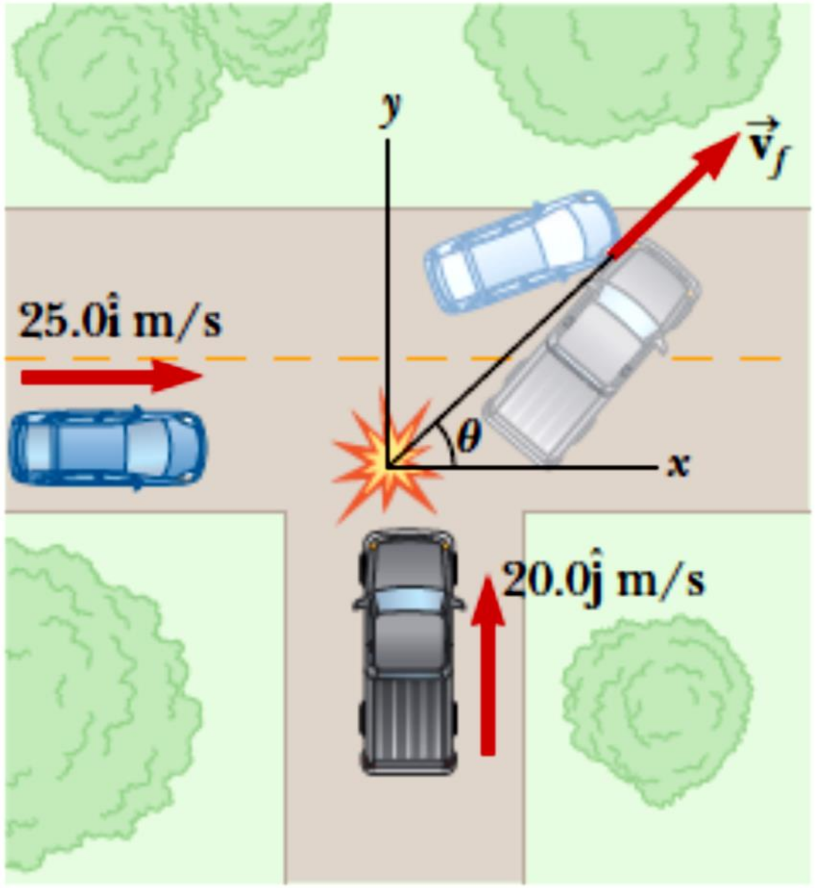


Perfectly Inelastic: $m_1\mathbf{v}_{1A} + m_2 \underbrace{\mathbf{v}_{2i}}_{0} = (m_1 + m_2)\mathbf{v}_B$ ➡ $\mathbf{v}_B = \dfrac{m_1}{m_1 + m_2}\mathbf{v}_{1A}$

$K_i + U_i = K_f + U_f$ ➡ $\dfrac{1}{2}(m_1 + m_2)\mathbf{v}_B^{\ 2} + 0 = 0 + (m_1 + m_2)gh$

$\dfrac{1}{2}(m_1 + m_2)(\dfrac{m_1}{m_1 + m_2}\mathbf{v}_{1A})^2 = (m_1 + m_2)gh$ ➡ $\mathbf{v}_{1A} = (\dfrac{m_1 + m_2}{m_1})\sqrt{2gh}$

Ex 17: A **1500-kg car** traveling east with a speed of **25 m/s** collides at an intersection with a **2500-kg truck** traveling north at a speed of **20 m/s** as shown in Figure. Find the **direction and magnitude of the velocity** of the wreckage after the collision, assuming the **vehicles stick together** after the collision.



$$\vec{p}_i = \vec{p}_f \Rightarrow \begin{cases} p_{xi} = p_{xf} \\ p_{yi} = p_{yf} \end{cases}$$

$$\begin{cases} m_1 v_{1i} = (m_1 + m_2) v_f \cos\theta \\ m_2 v_{2i} = (m_1 + m_2) v_f \sin\theta \end{cases}$$

$$\frac{m_2 v_{2i}}{m_1 v_{1i}} = \frac{\sin\theta}{\cos\theta} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{2500 \times 20}{1500 \times 25}\right) = 53.1^\circ$$

$$v_f = \frac{m_1 v_{1i}}{(m_1 + m_2)\cos\theta} = \frac{(1500)(25)}{(4000)\cos(53.1)} = 15.6 \ m/s$$