Fundamental Graph Algorithms

Part Four

Announcements

- Problem Set One due right now.
  - Due Friday at 2:15PM using one late period.
- Problem Set Two out, due next Friday, July 12 at 2:15PM.
  - Play around with graphs and graph algorithms!

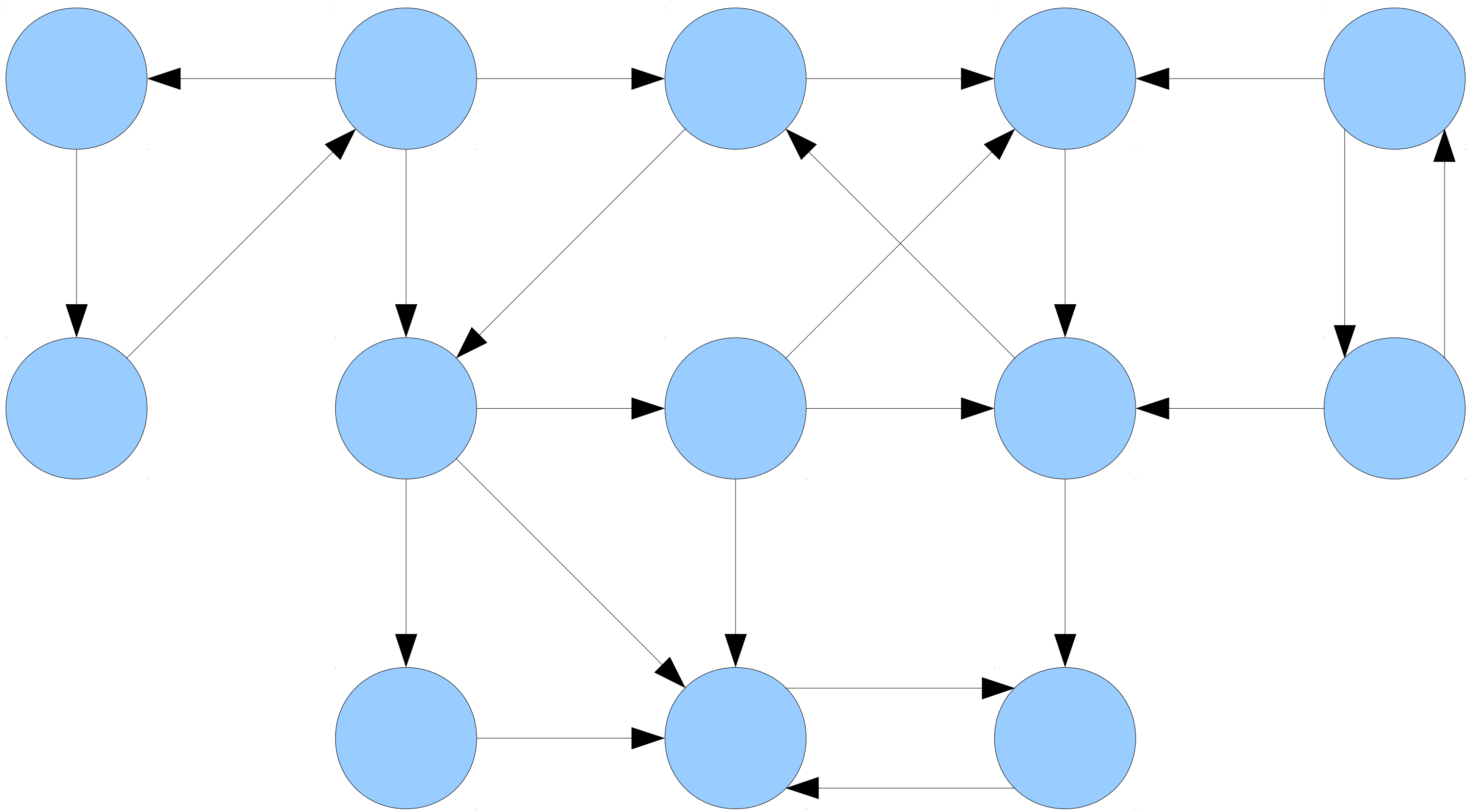
Outline for Today

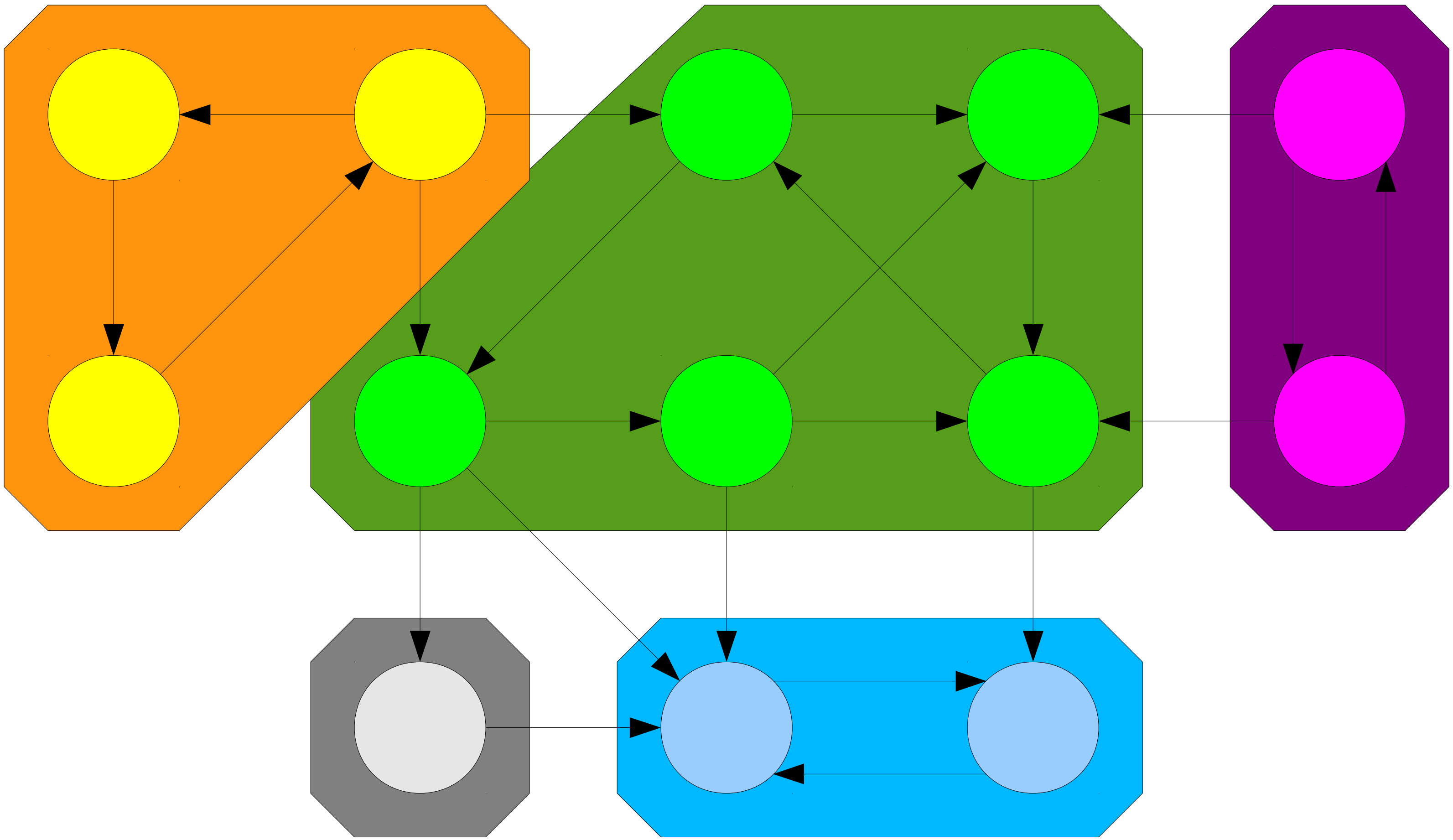
- **Kosaraju's Algorithm, Part II**
  - Completing our algorithm for finding SCCs.
- **Applying Graph Algorithms**
  - How to put these algorithms into practice.

Recap from Last Time

Strongly Connected Components

- Let $G = (V, E)$ be a directed graph.
- Two nodes $u, v \in V$ are called **strongly connected** iff v is reachable from u and u is reachable from v.
- A **strongly connected component** (or **SCC**) of G is a set $C \subseteq V$ such that
  - C is not empty.
  - For any $u, v \in C$: u and v are strongly connected.
  - For any $u \in C$ and $v \in V - C$: u and v are not strongly connected.





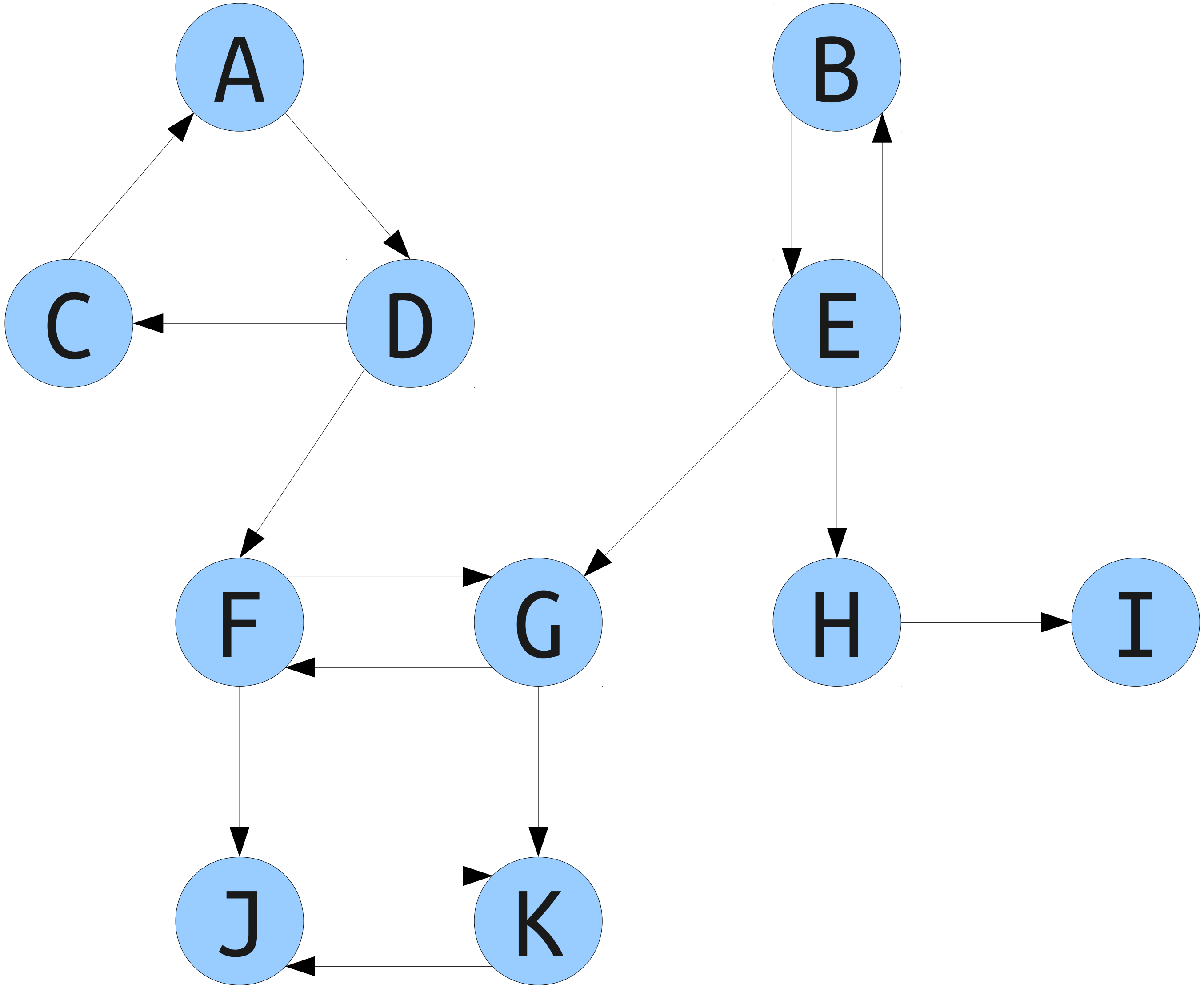
Condensation Graphs

- The **condensation** of a directed graph G is the directed graph G^{SCC} whose nodes are the SCCs of G and whose edges are defined as follows:

$$(C_1, C_2) \text{ is an edge in } G^{SCC} \text{ iff}$$
$$\exists u \in C_1, v \in C_2.\ (u, v) \text{ is an edge in } G.$$

- In other words, if there is an edge in G from *any* node in C_1 to *any* node in C_2, there is an edge in G^{SCC} from C_1 to C_2.
- ***Theorem:*** G^{SCC} is a DAG for any graph G.

How do we find all the SCCs of a graph?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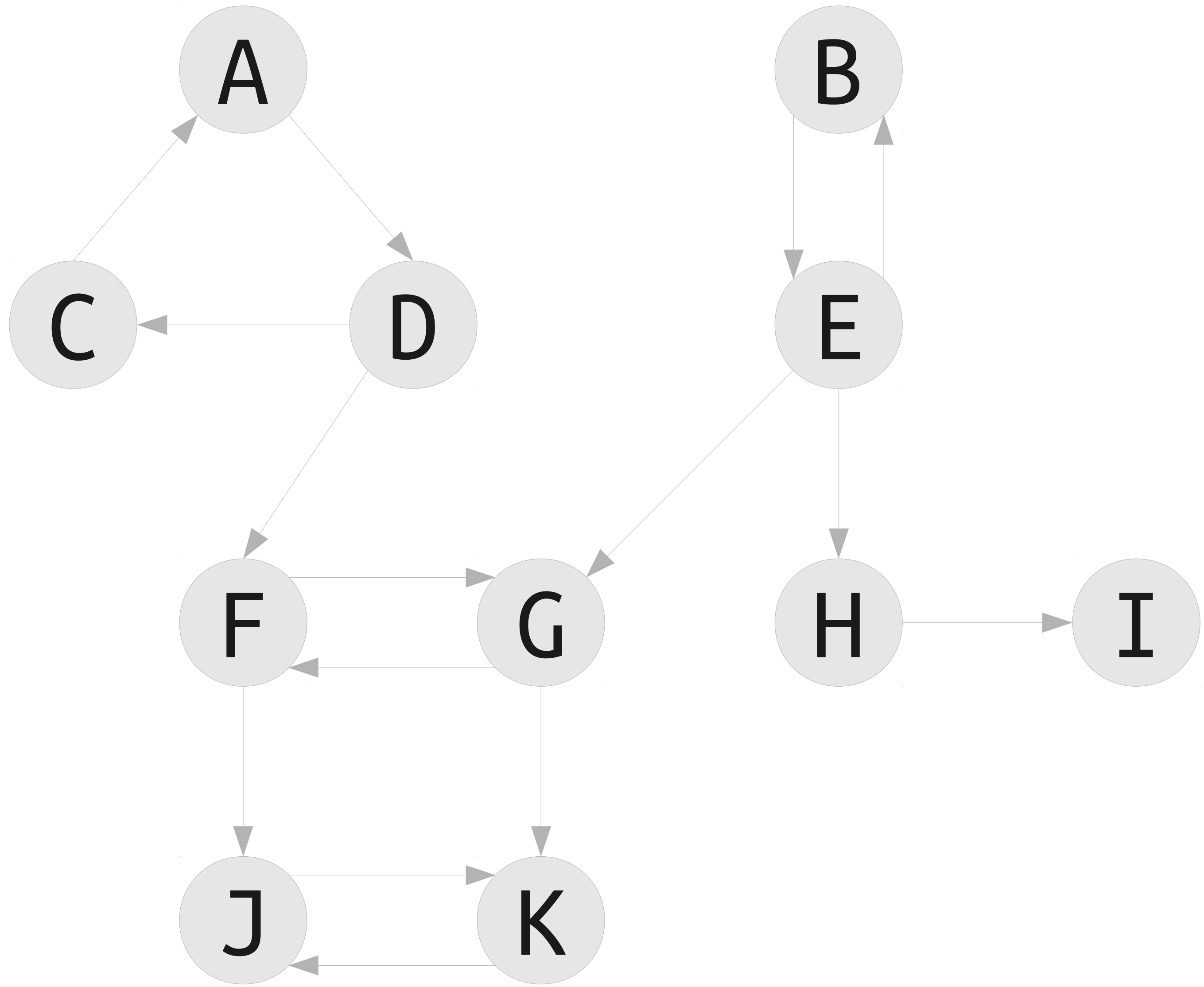
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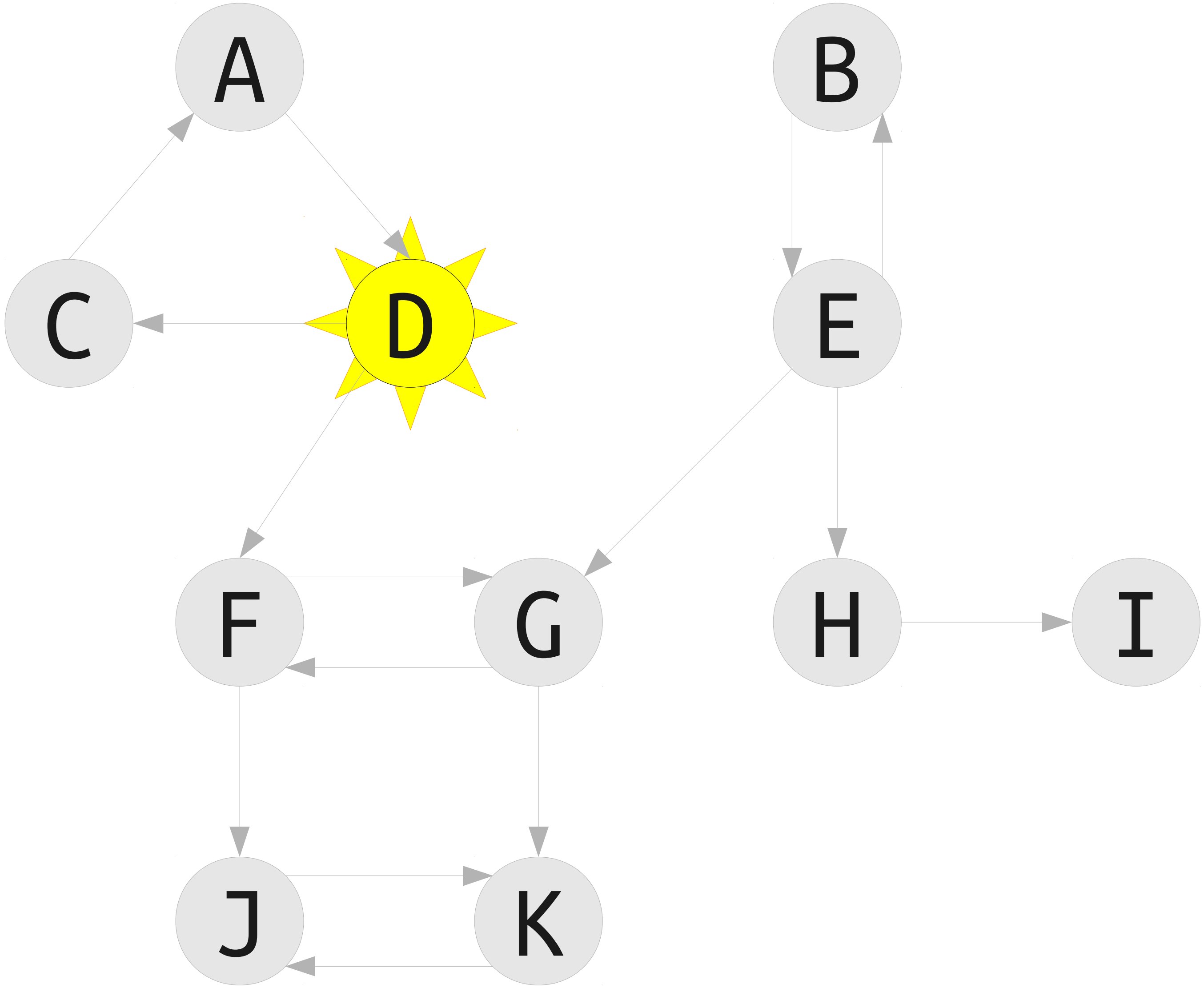
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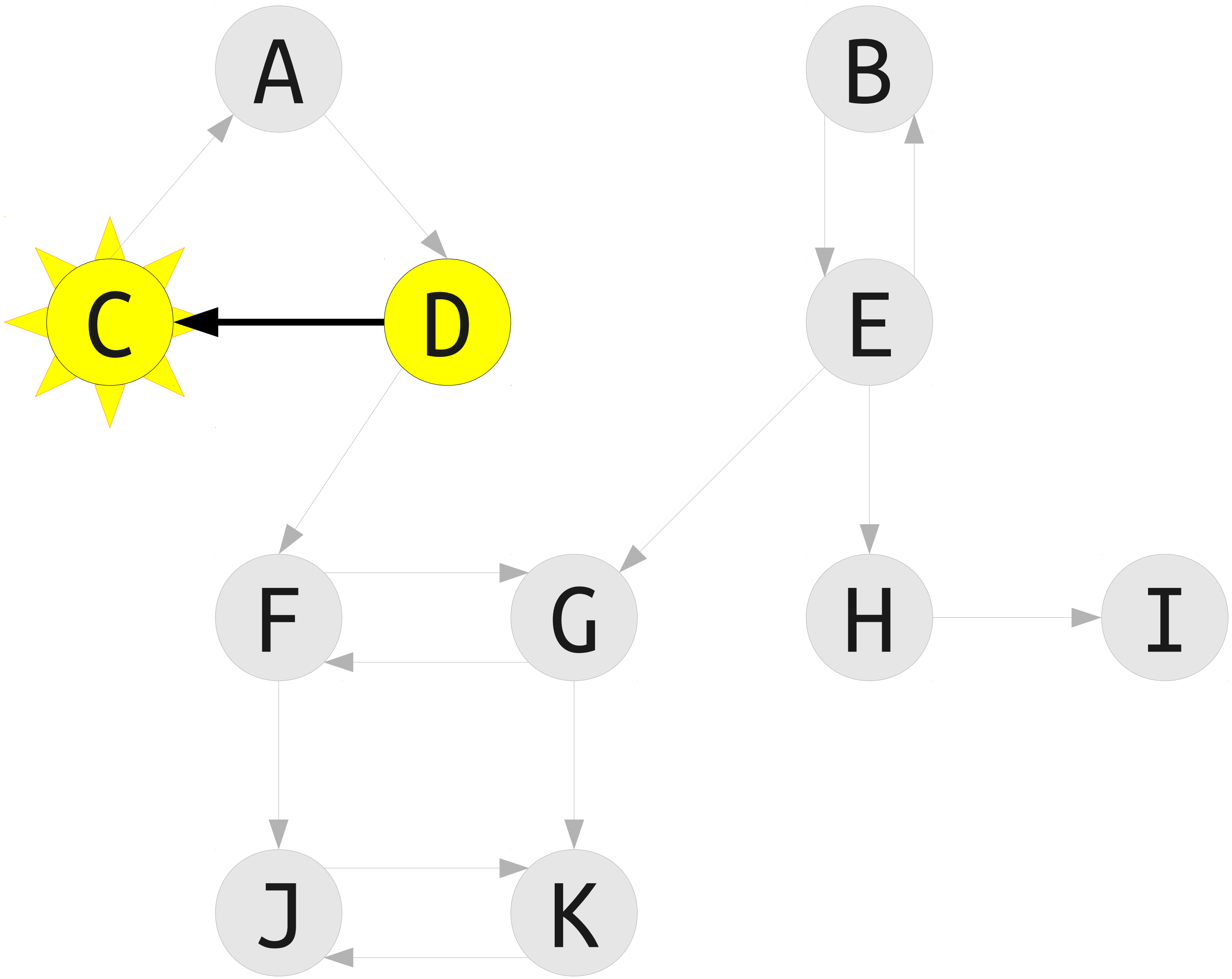
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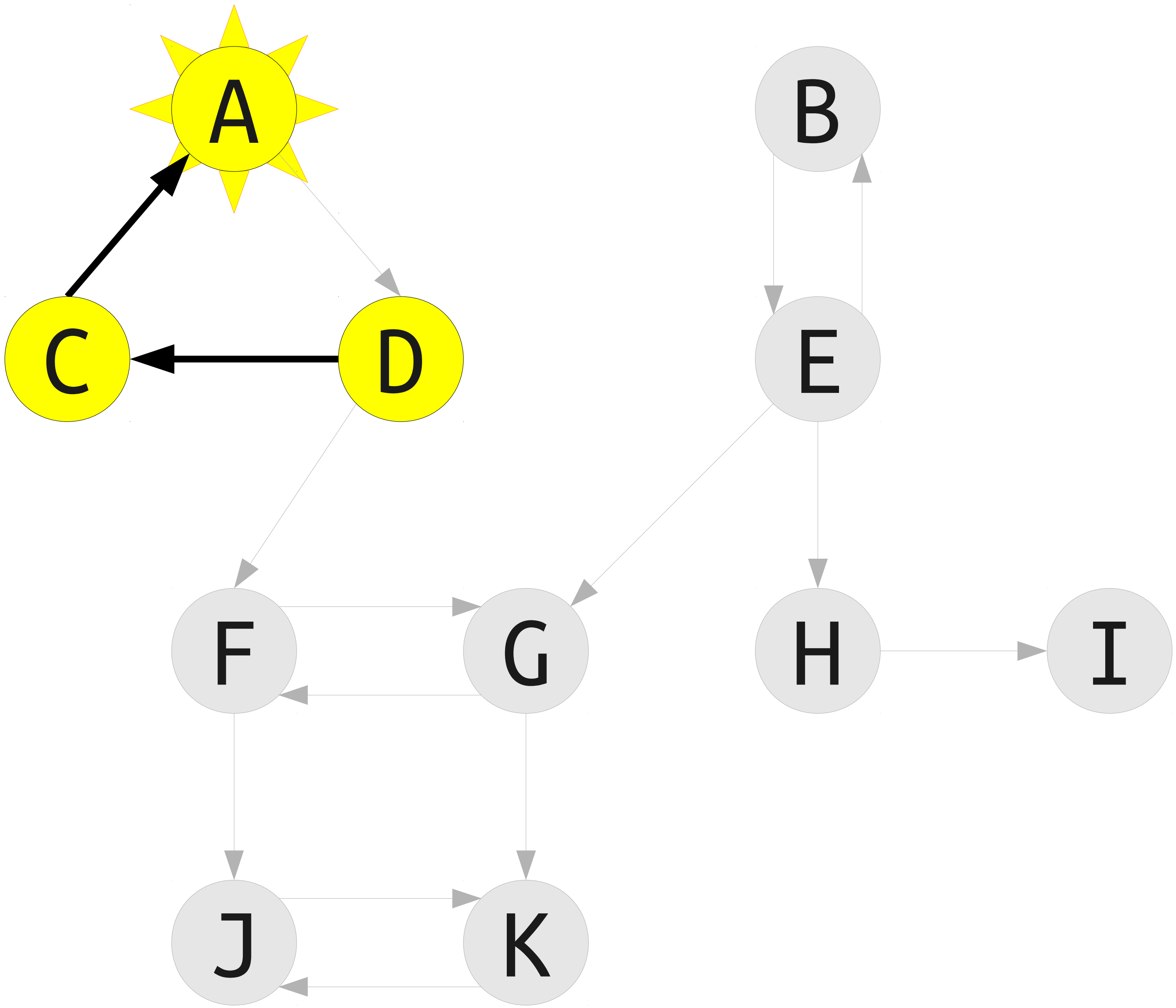
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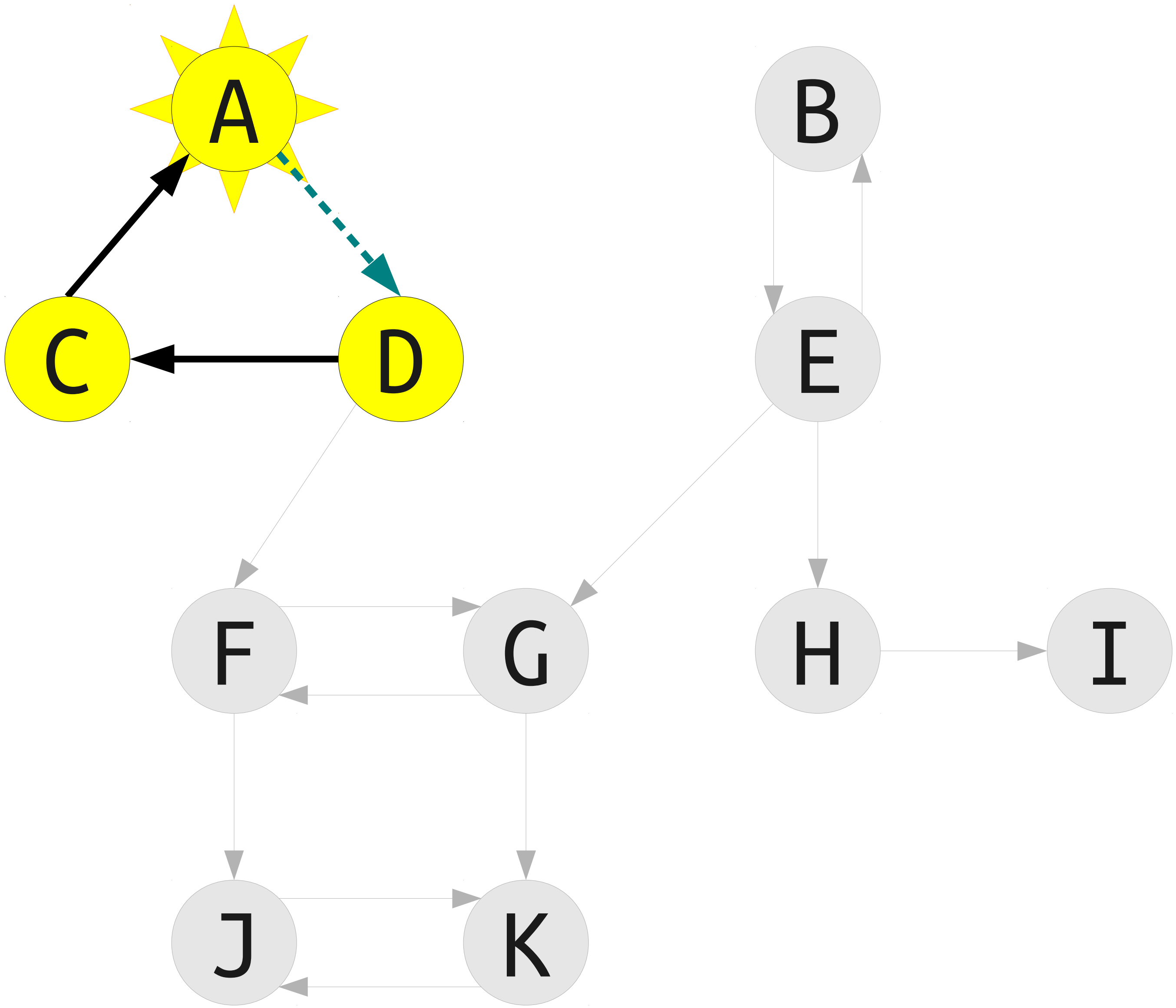
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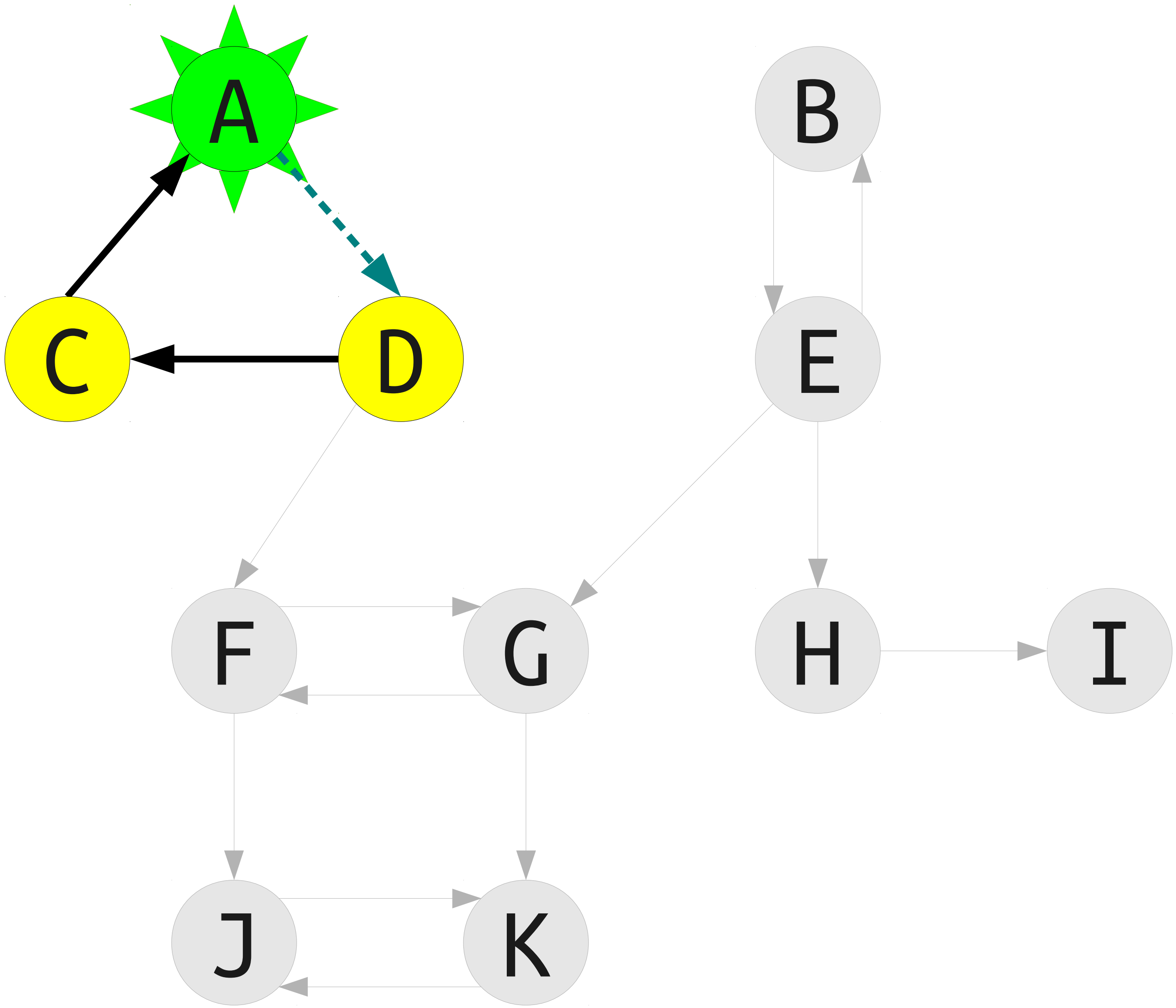
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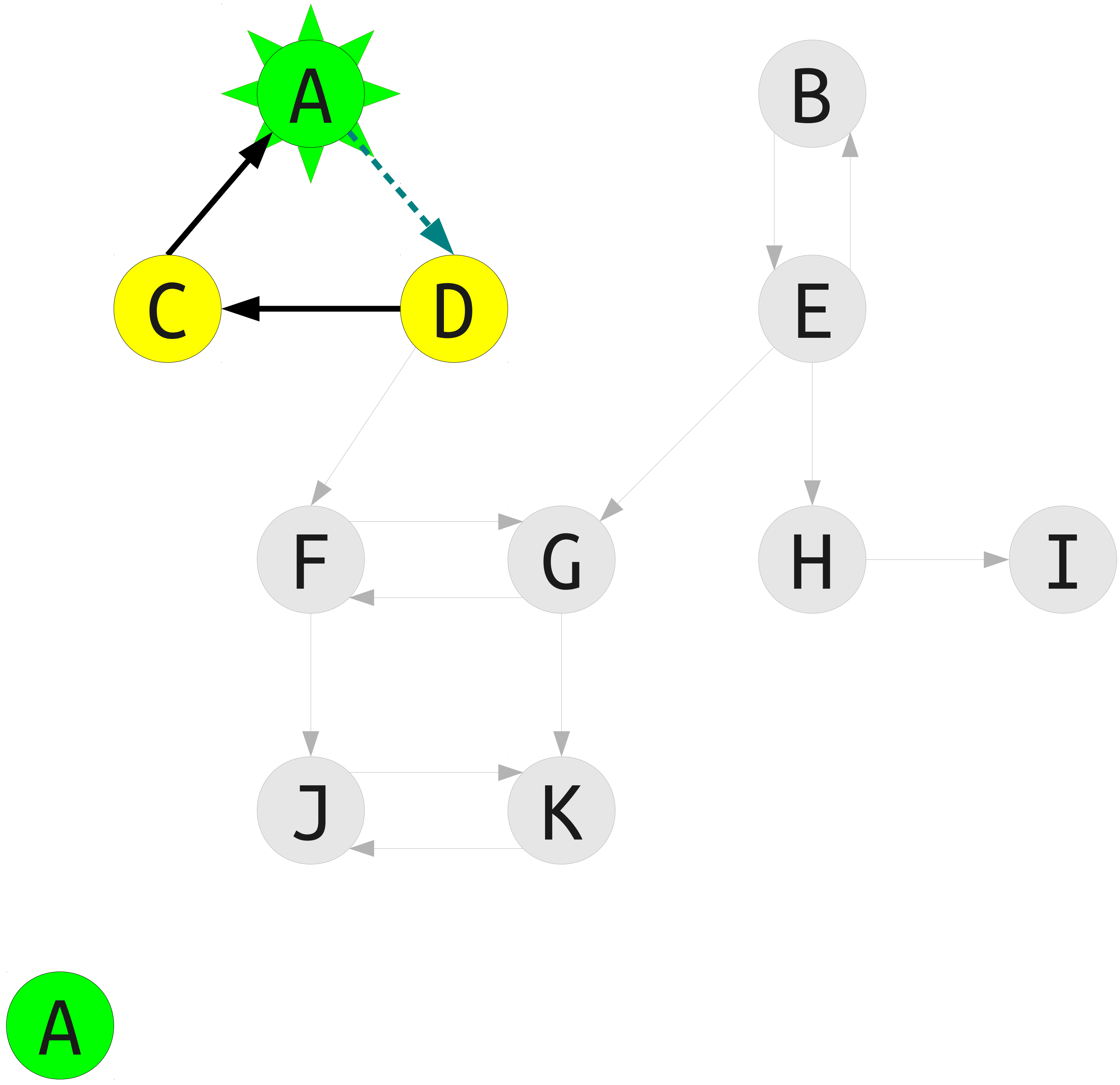
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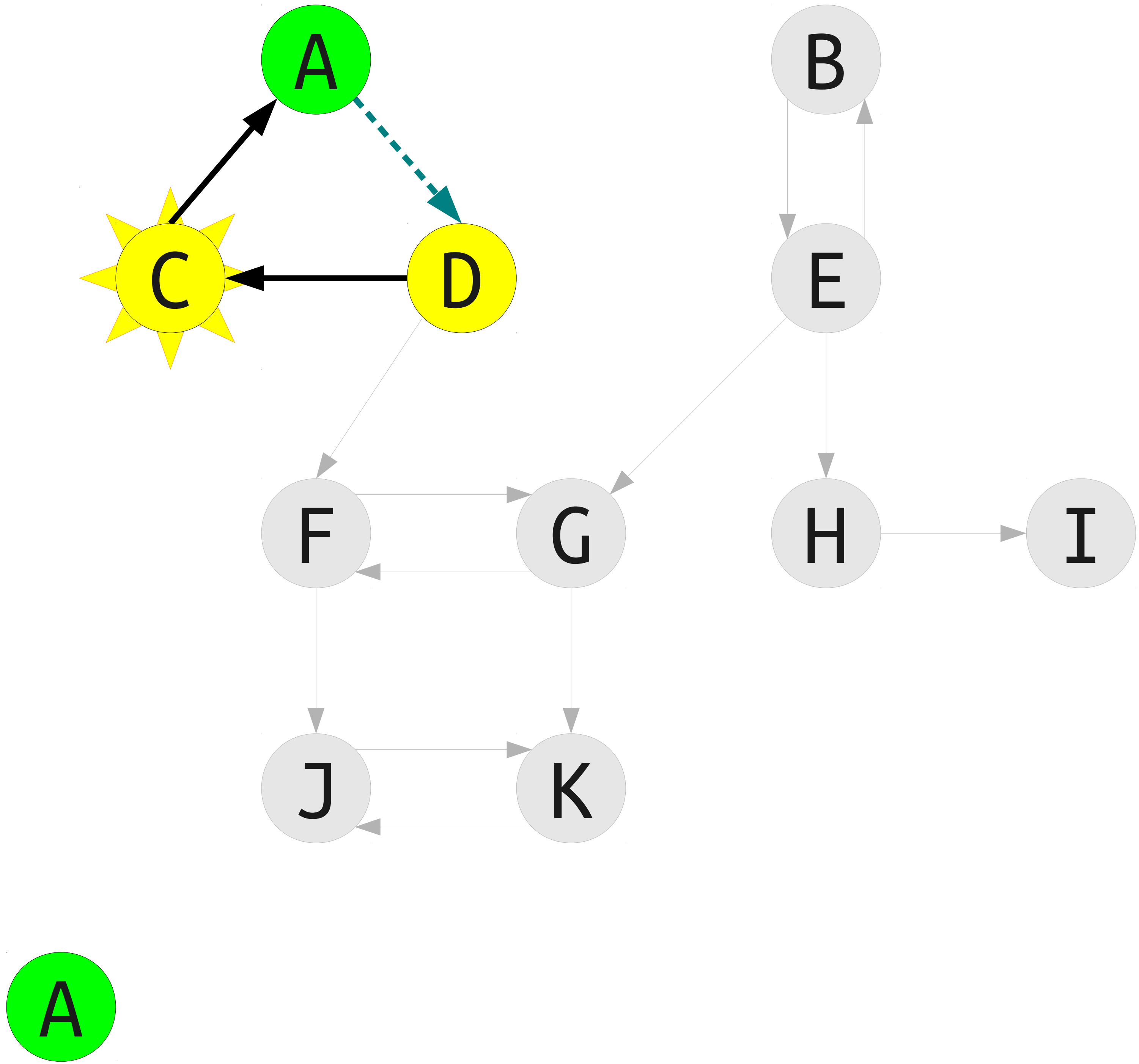

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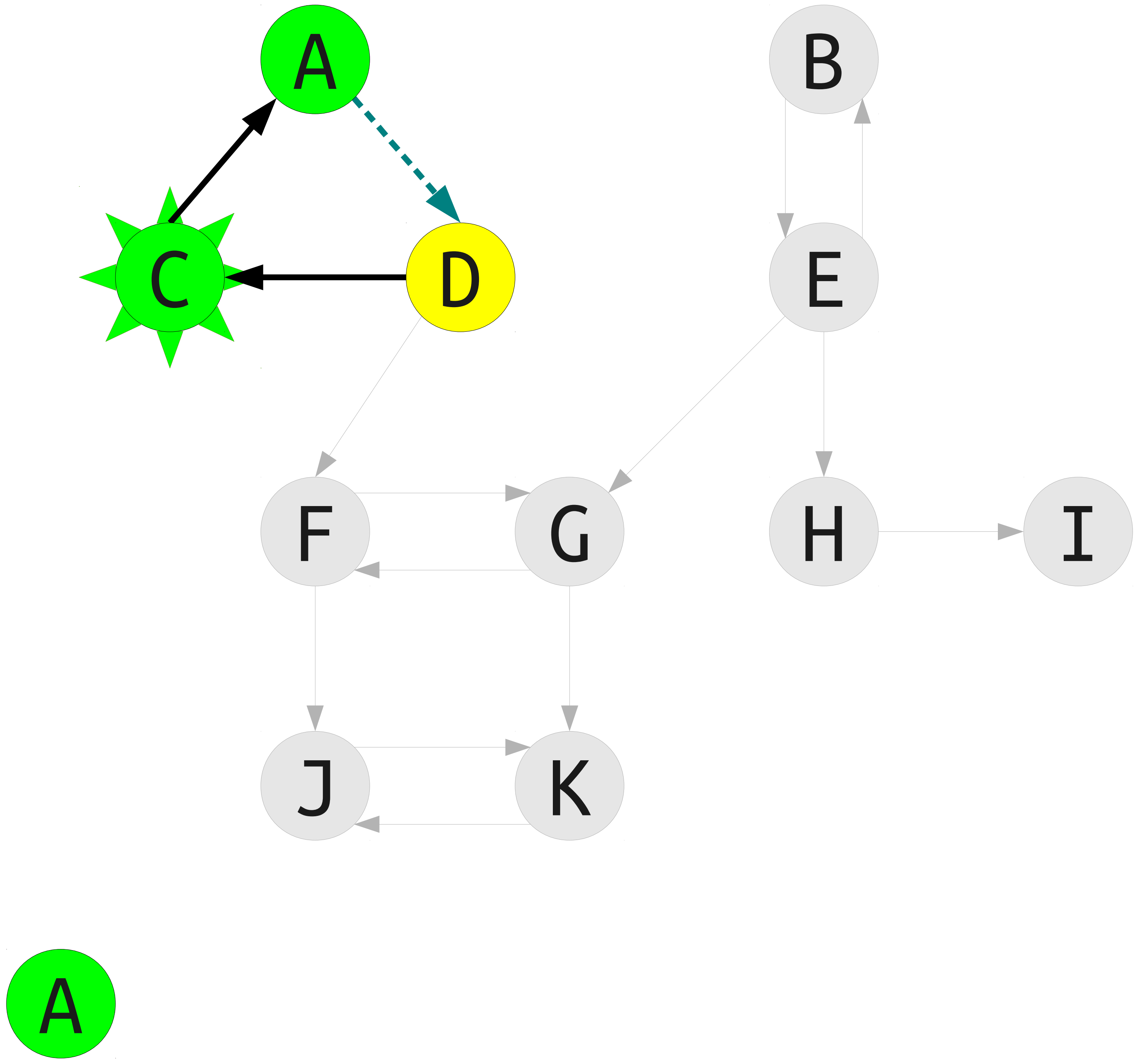
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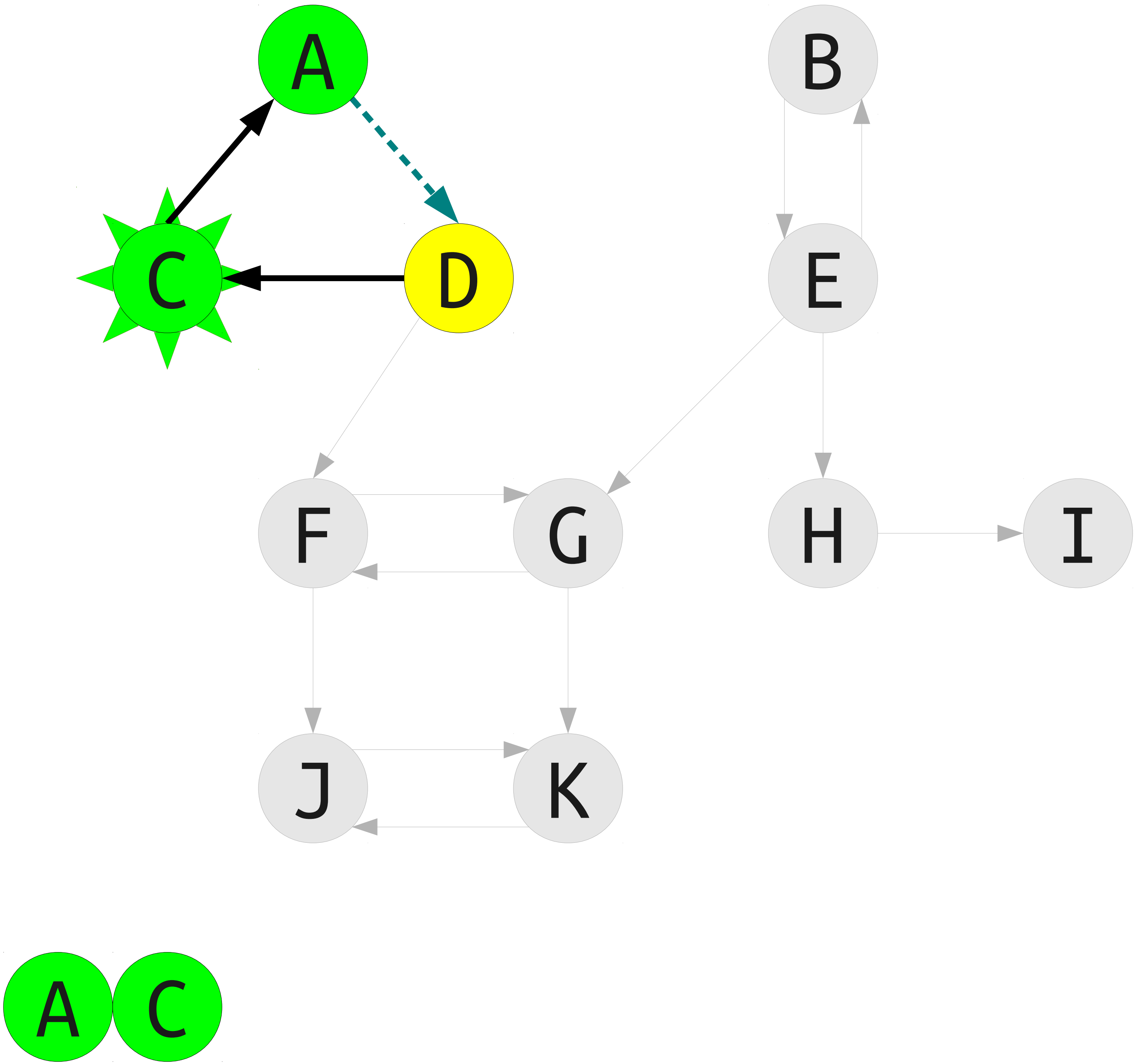
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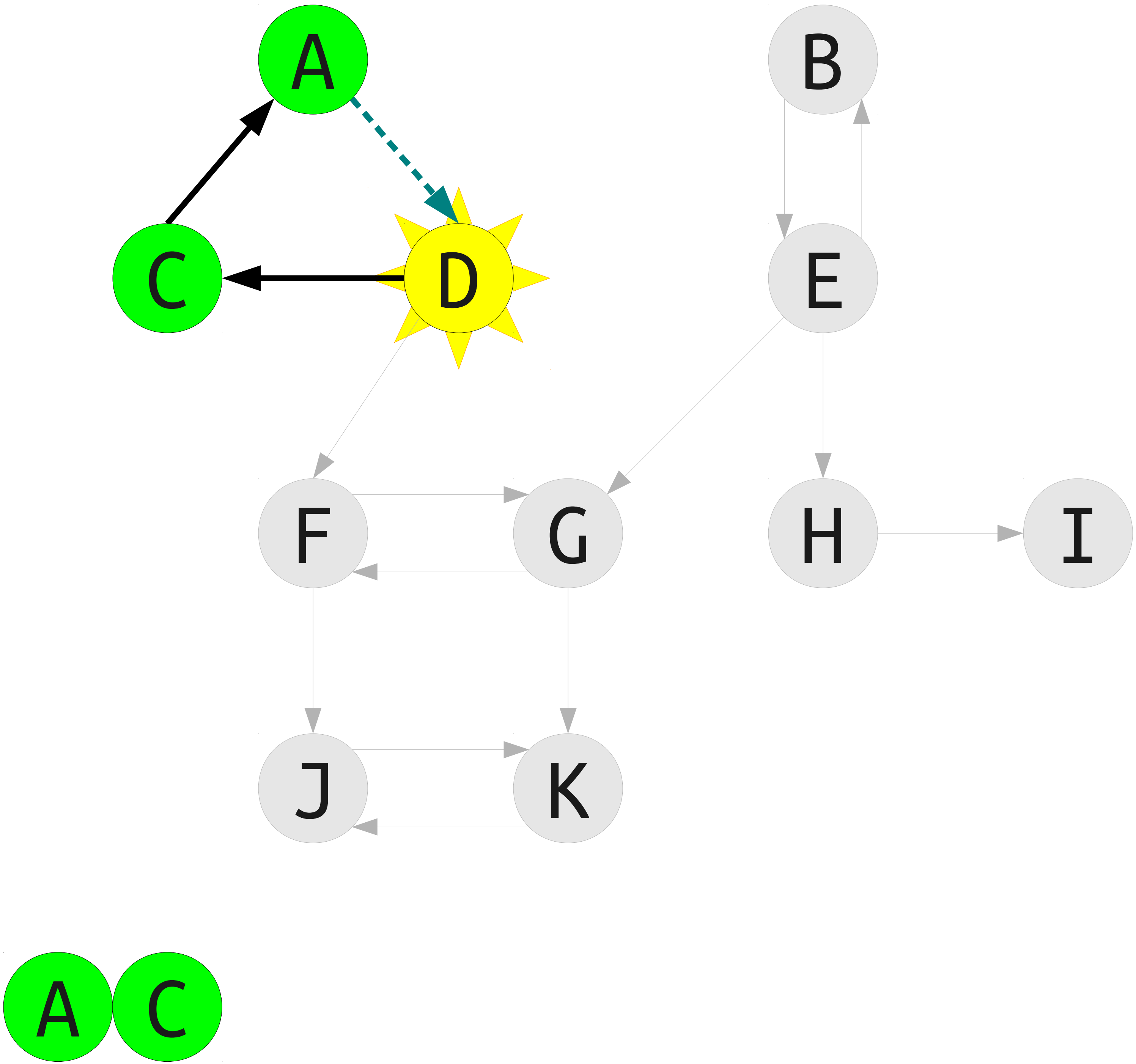
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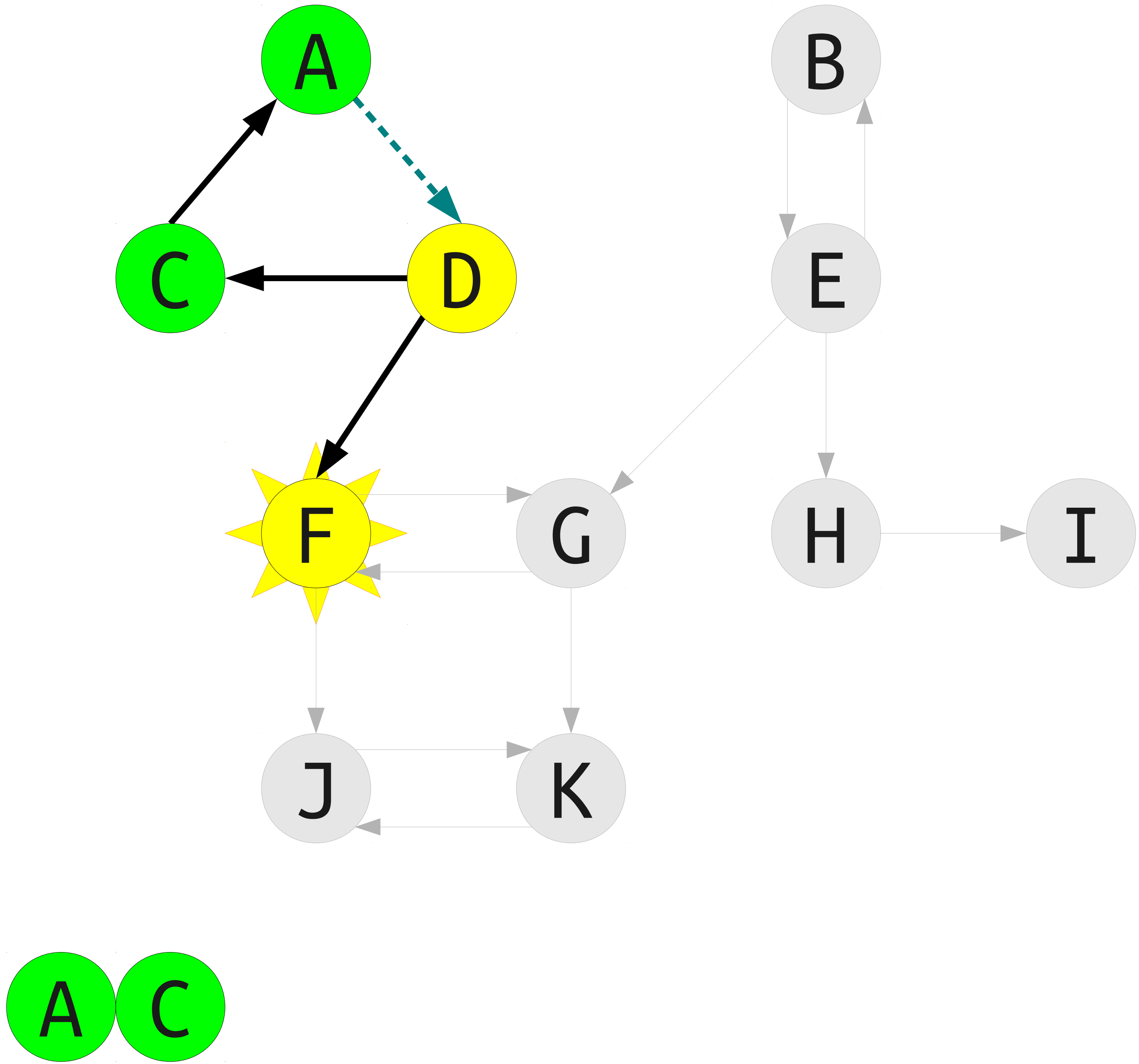
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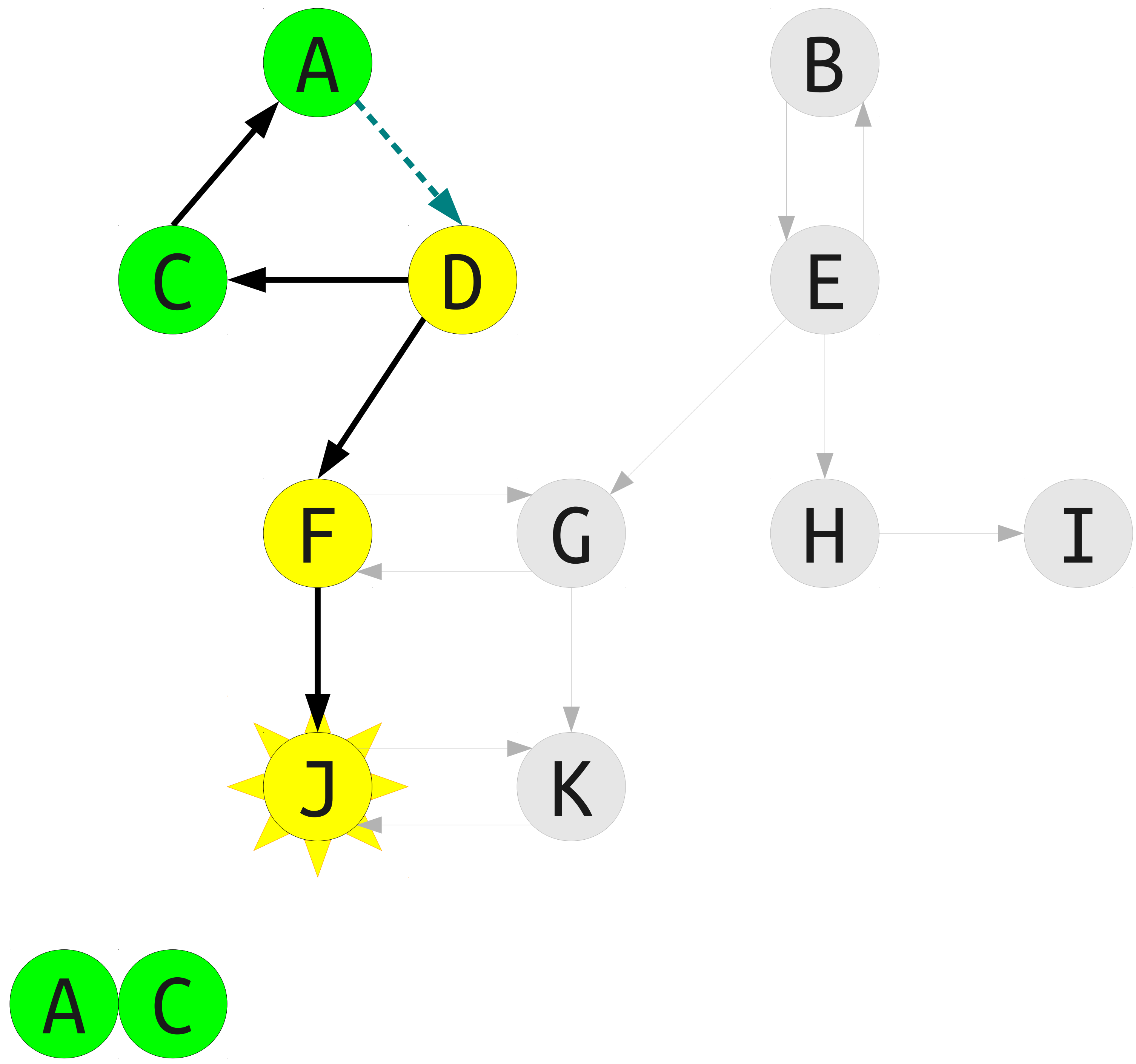

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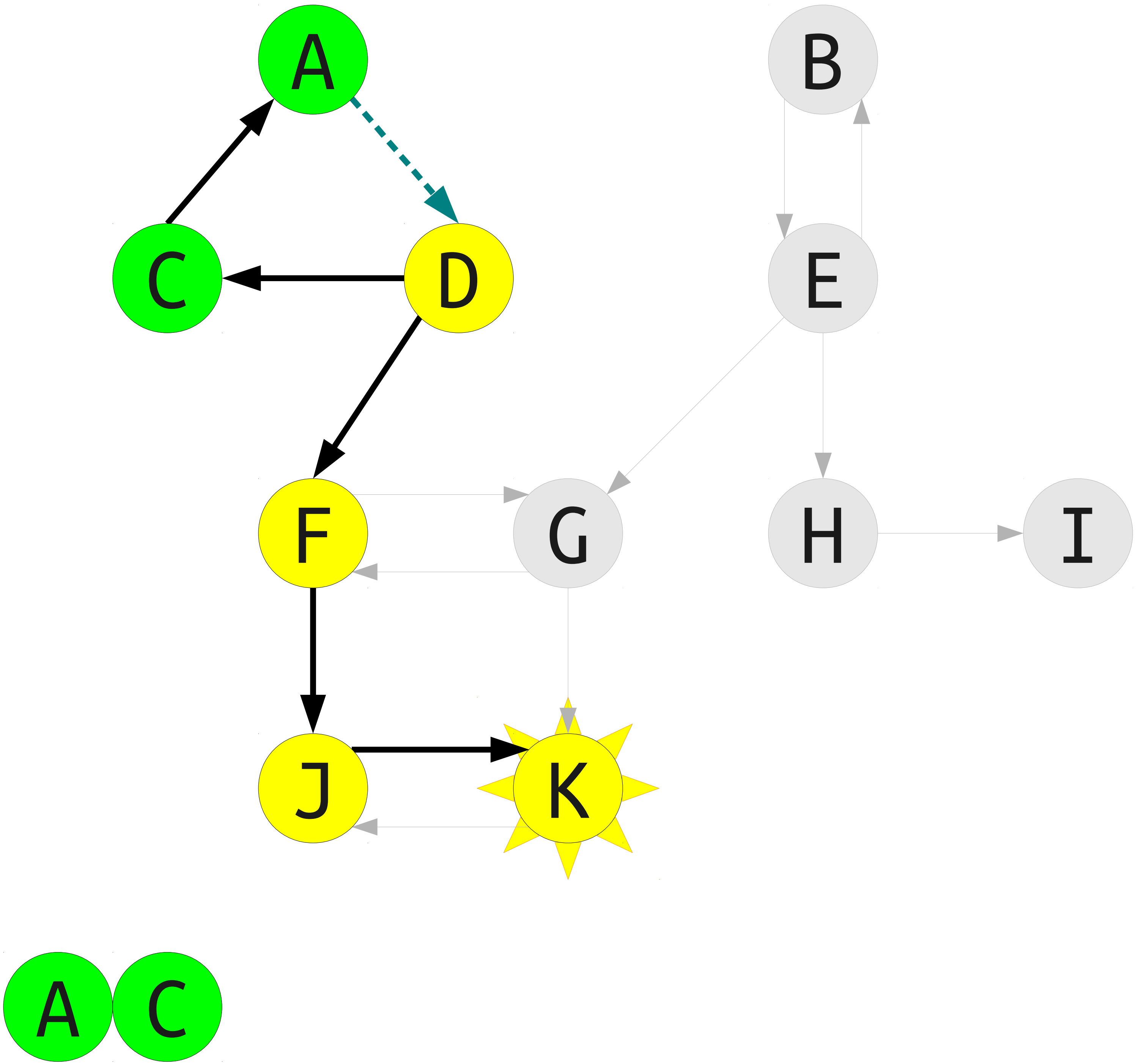
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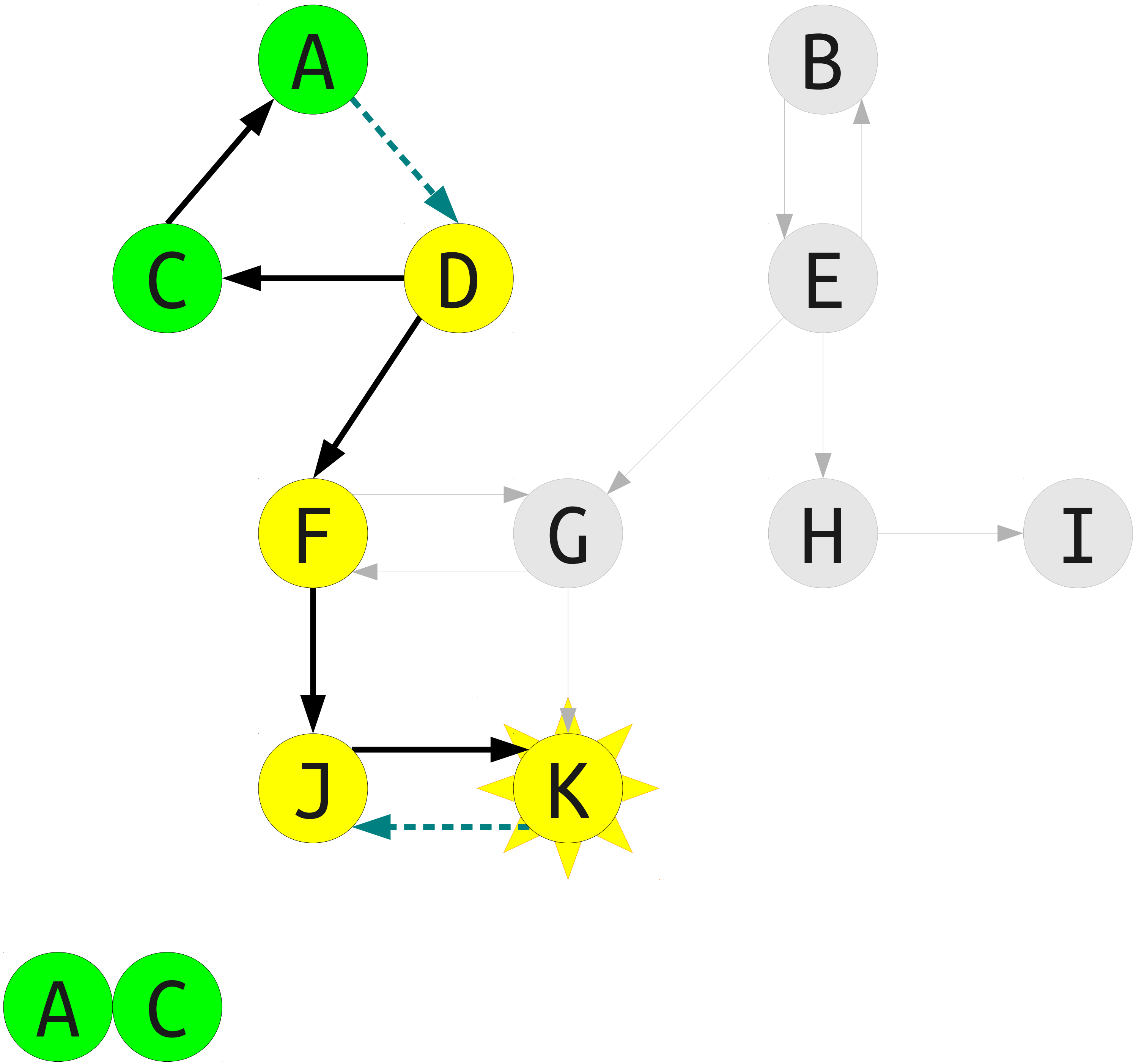
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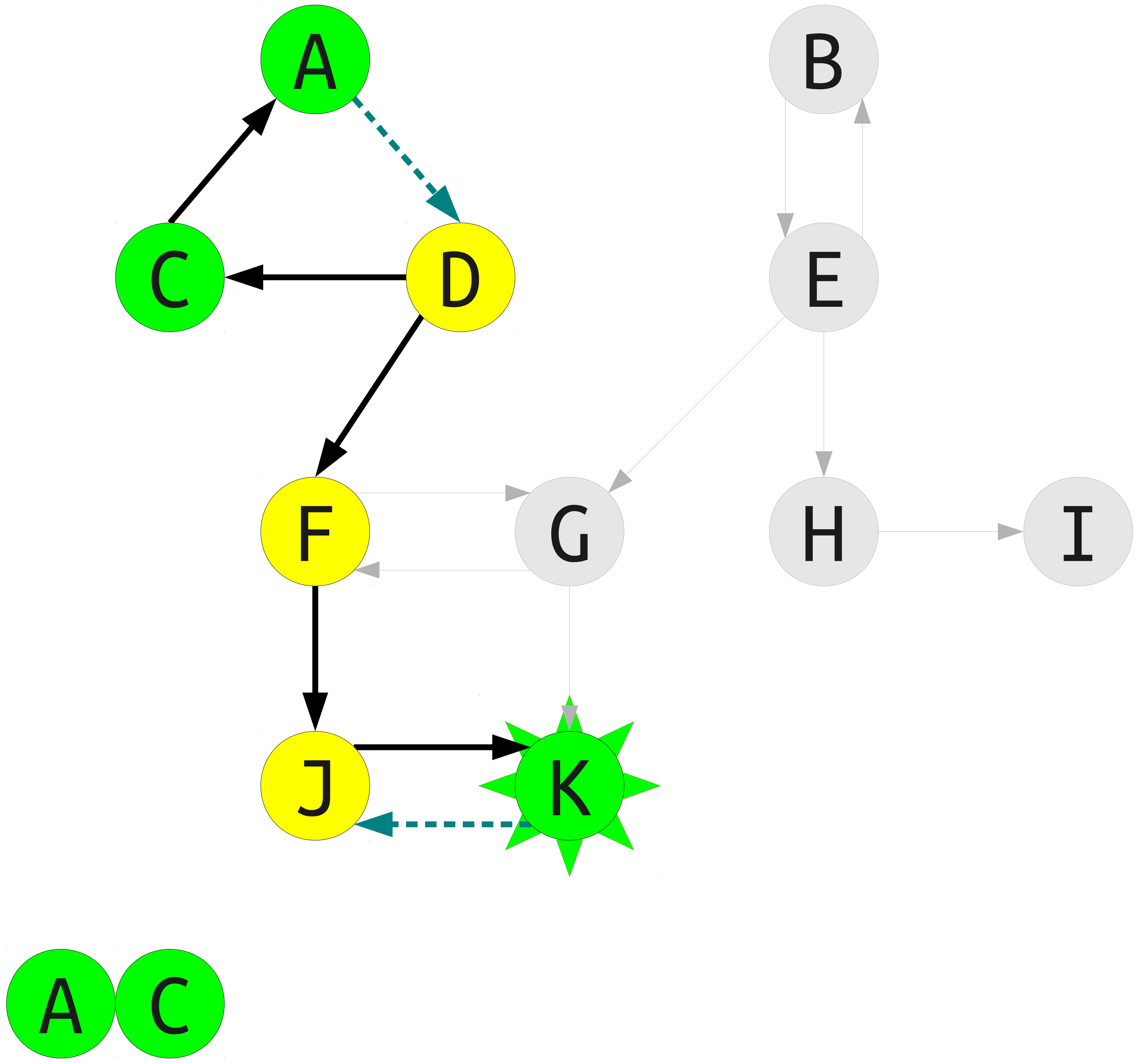
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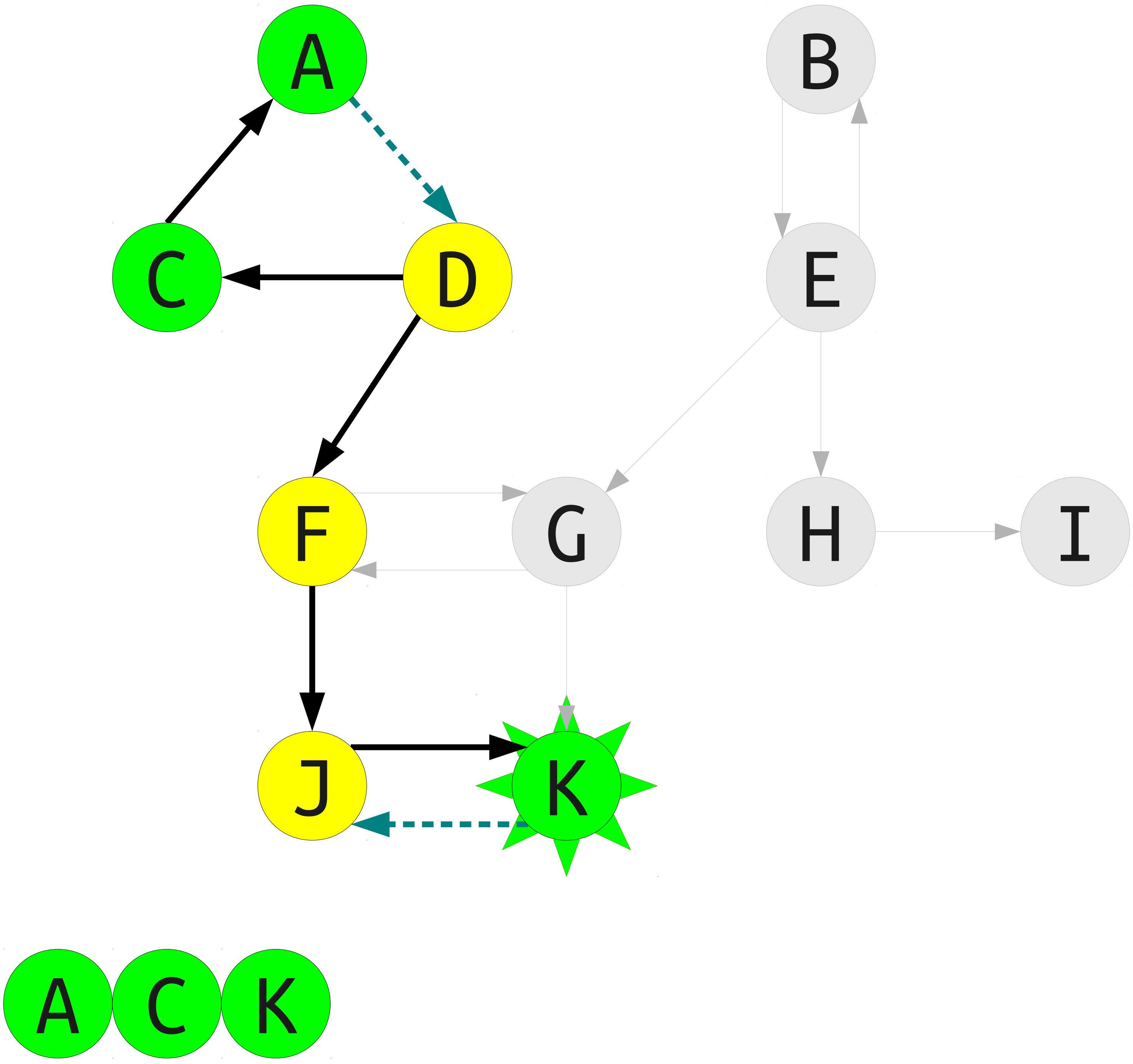
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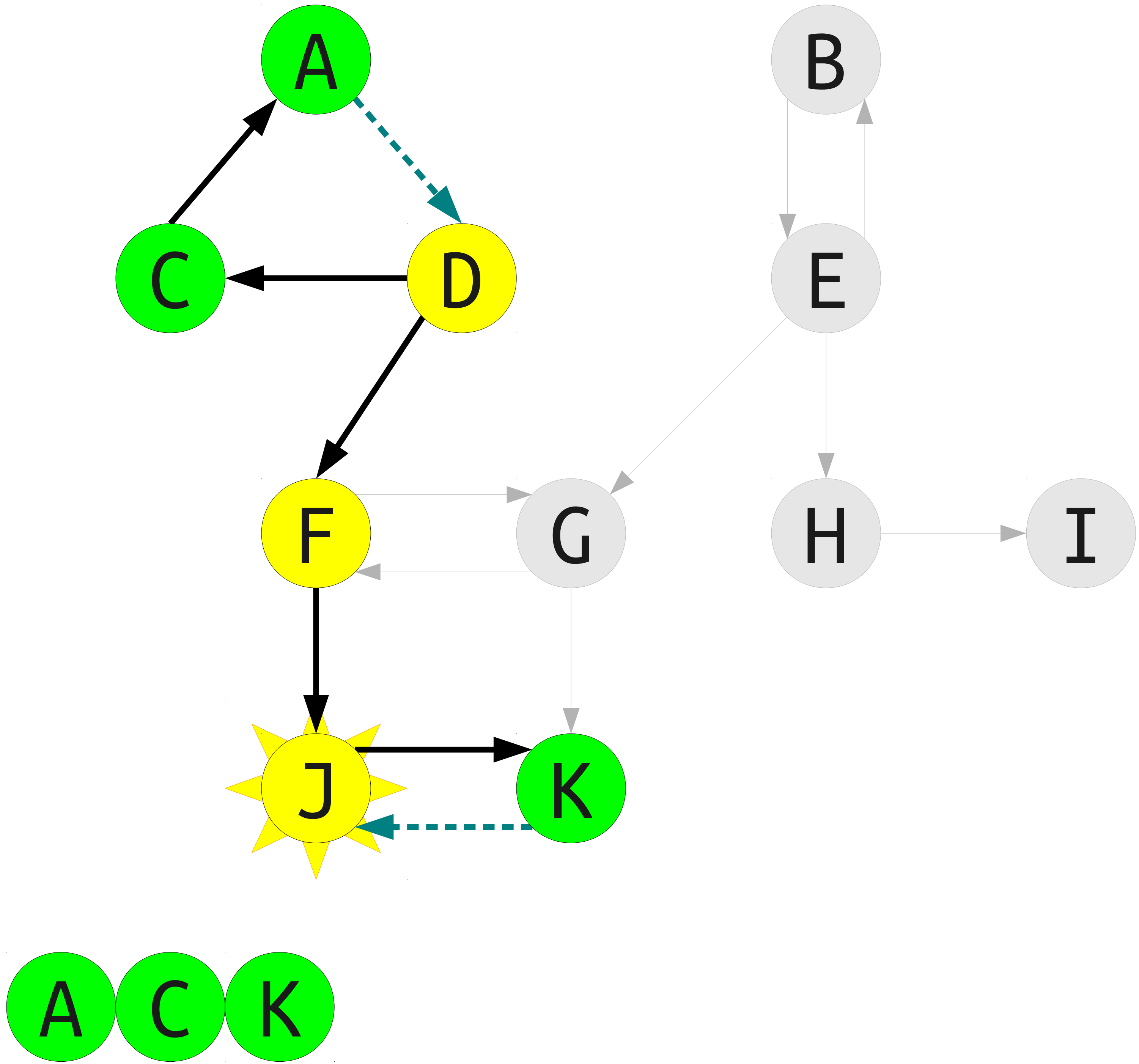
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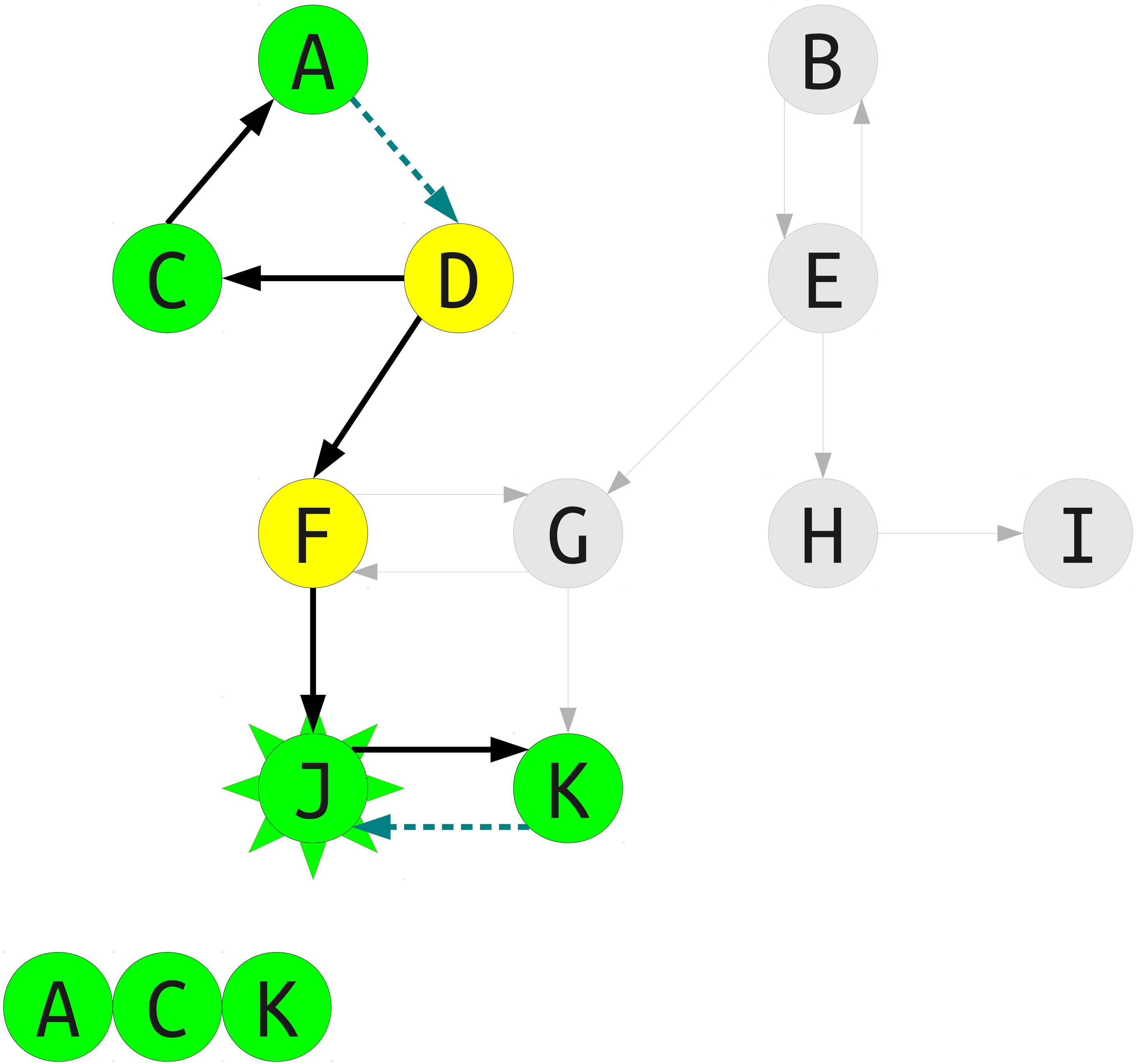
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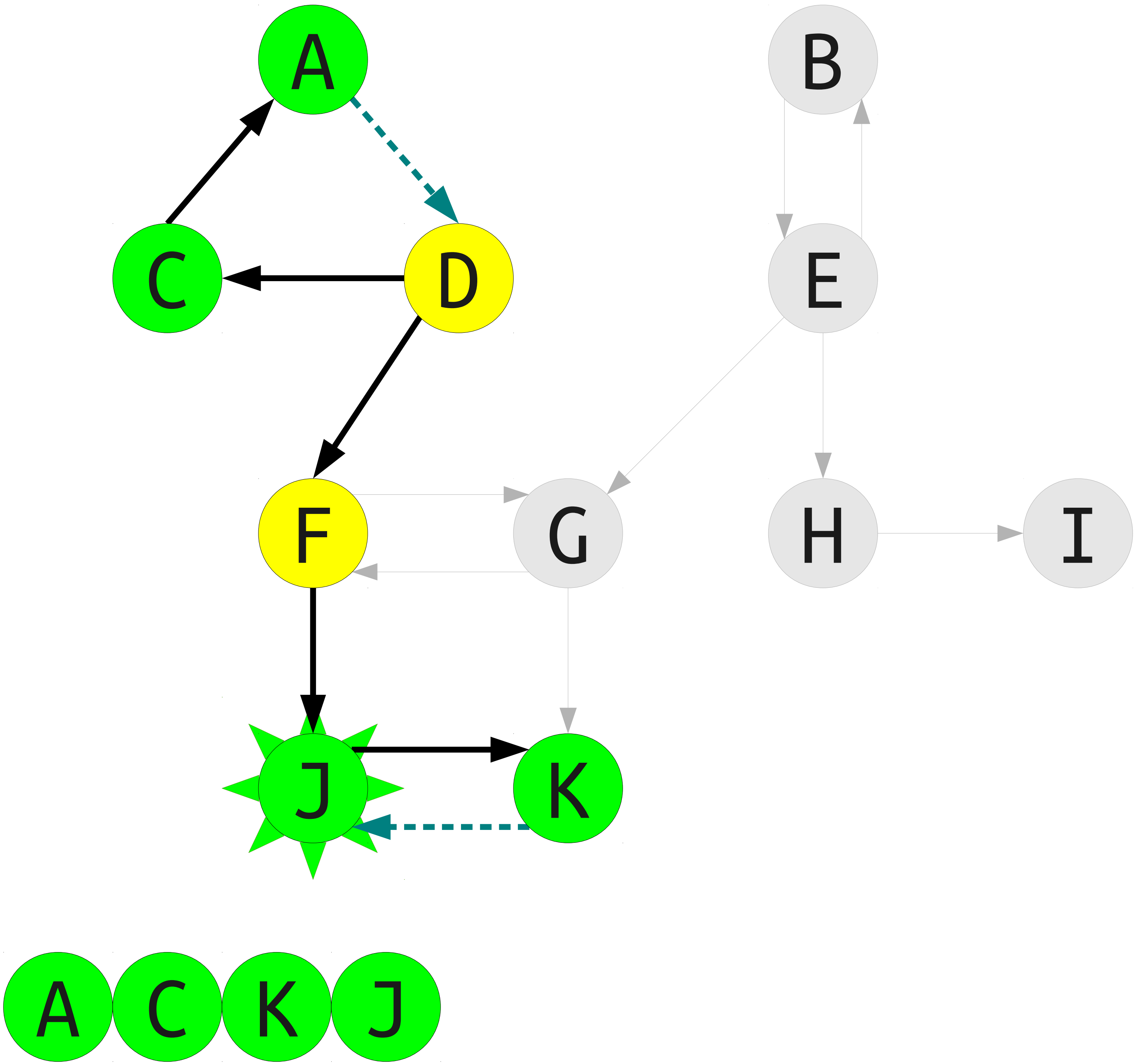

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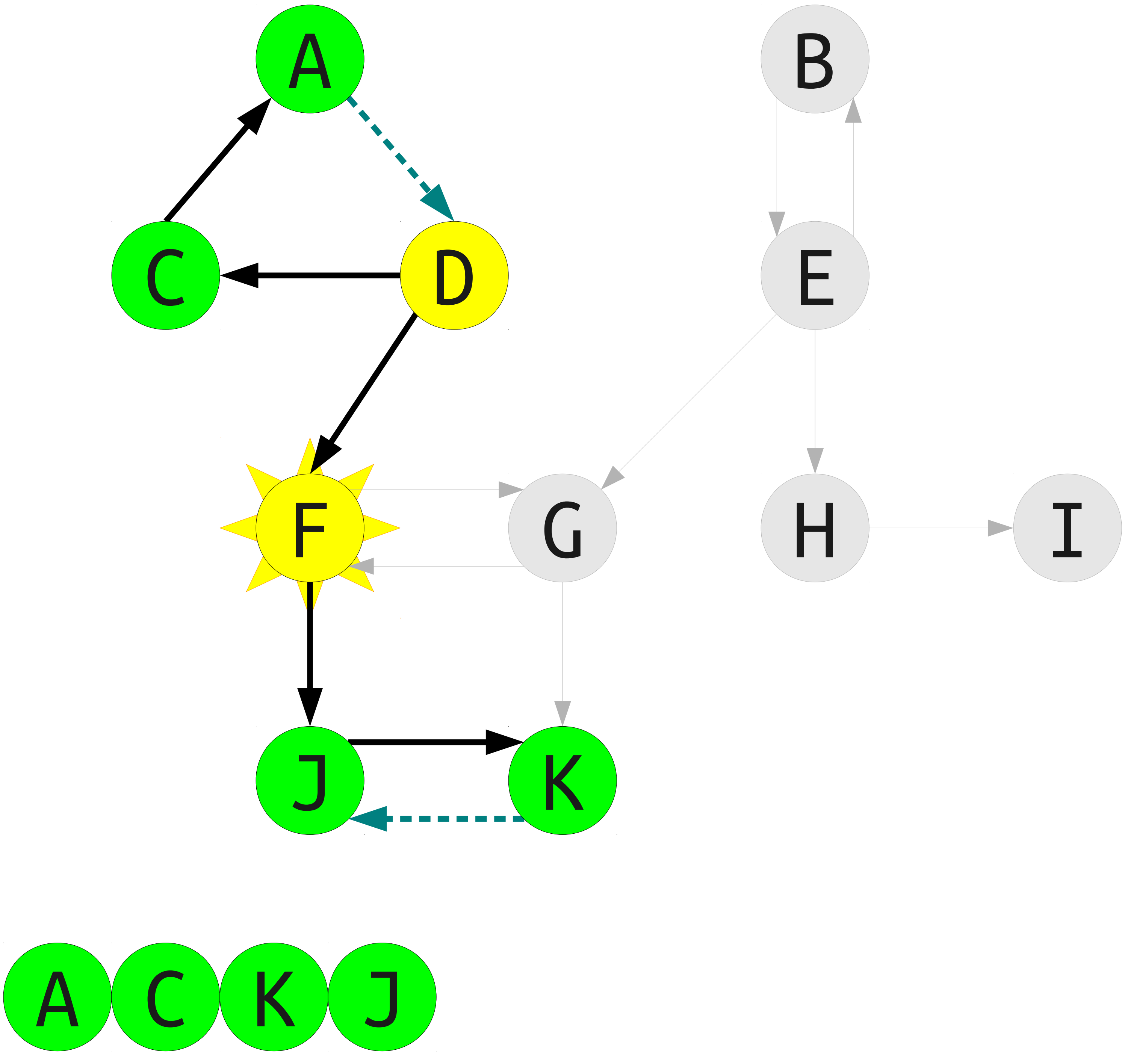
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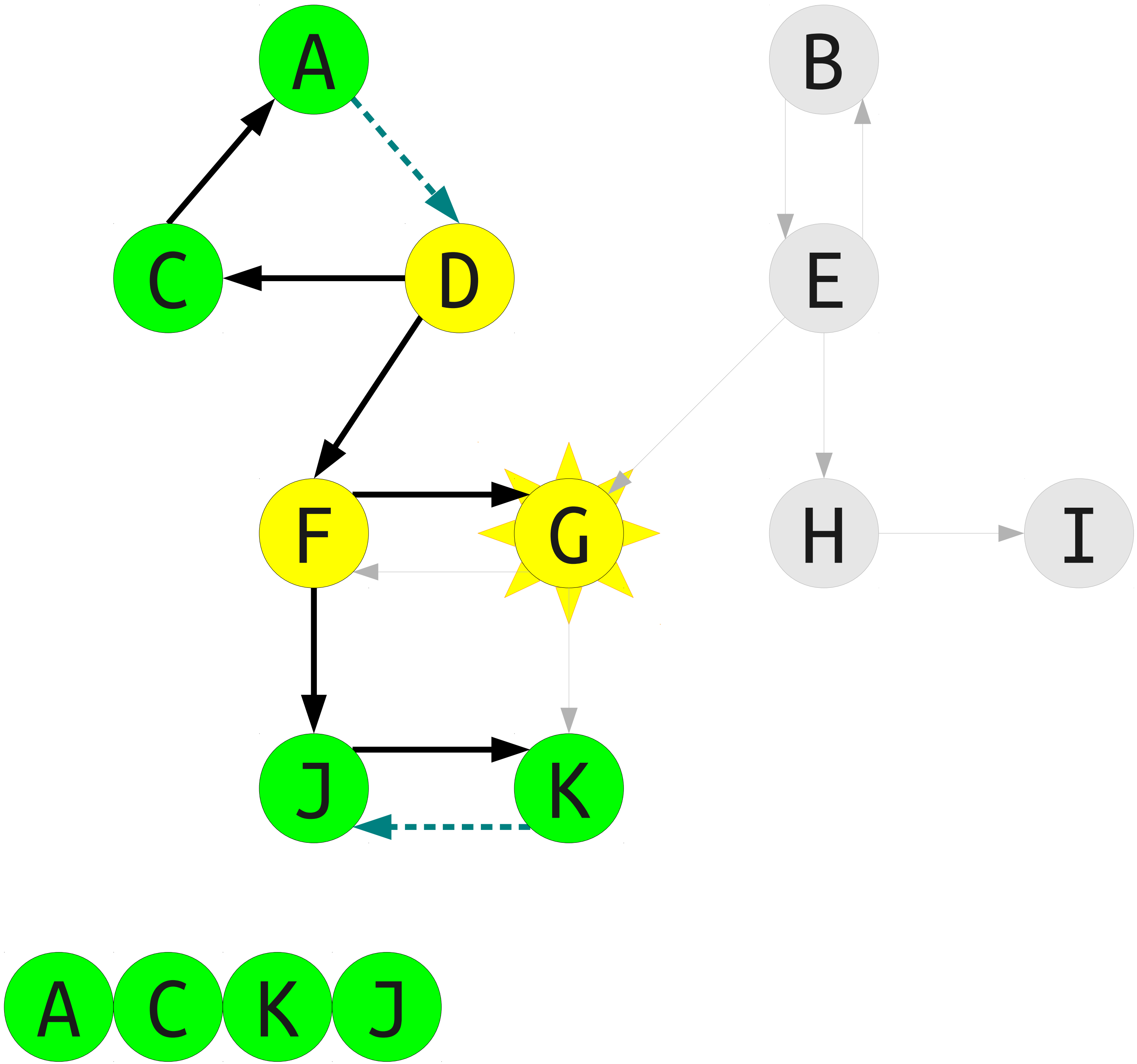
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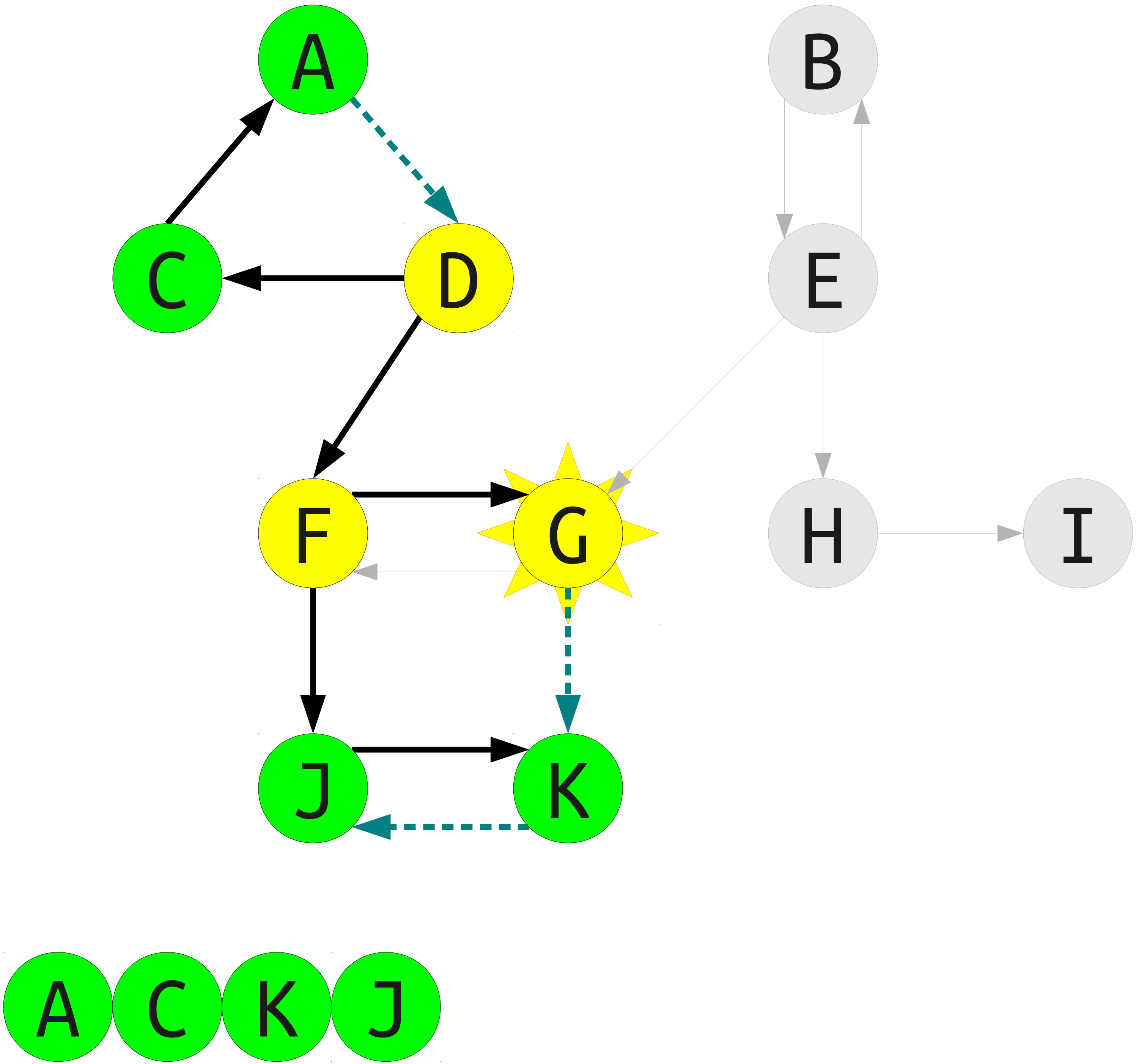
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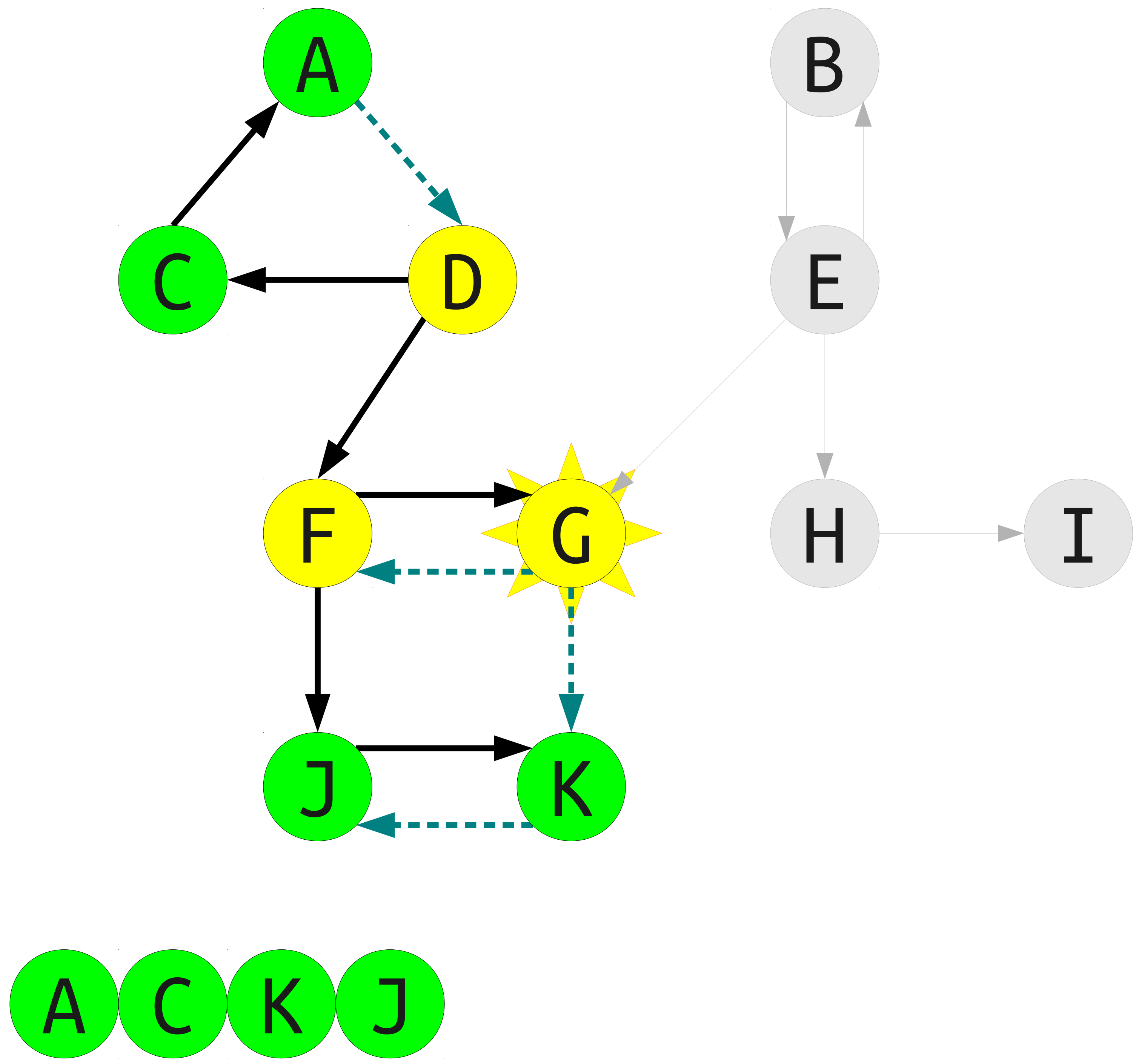
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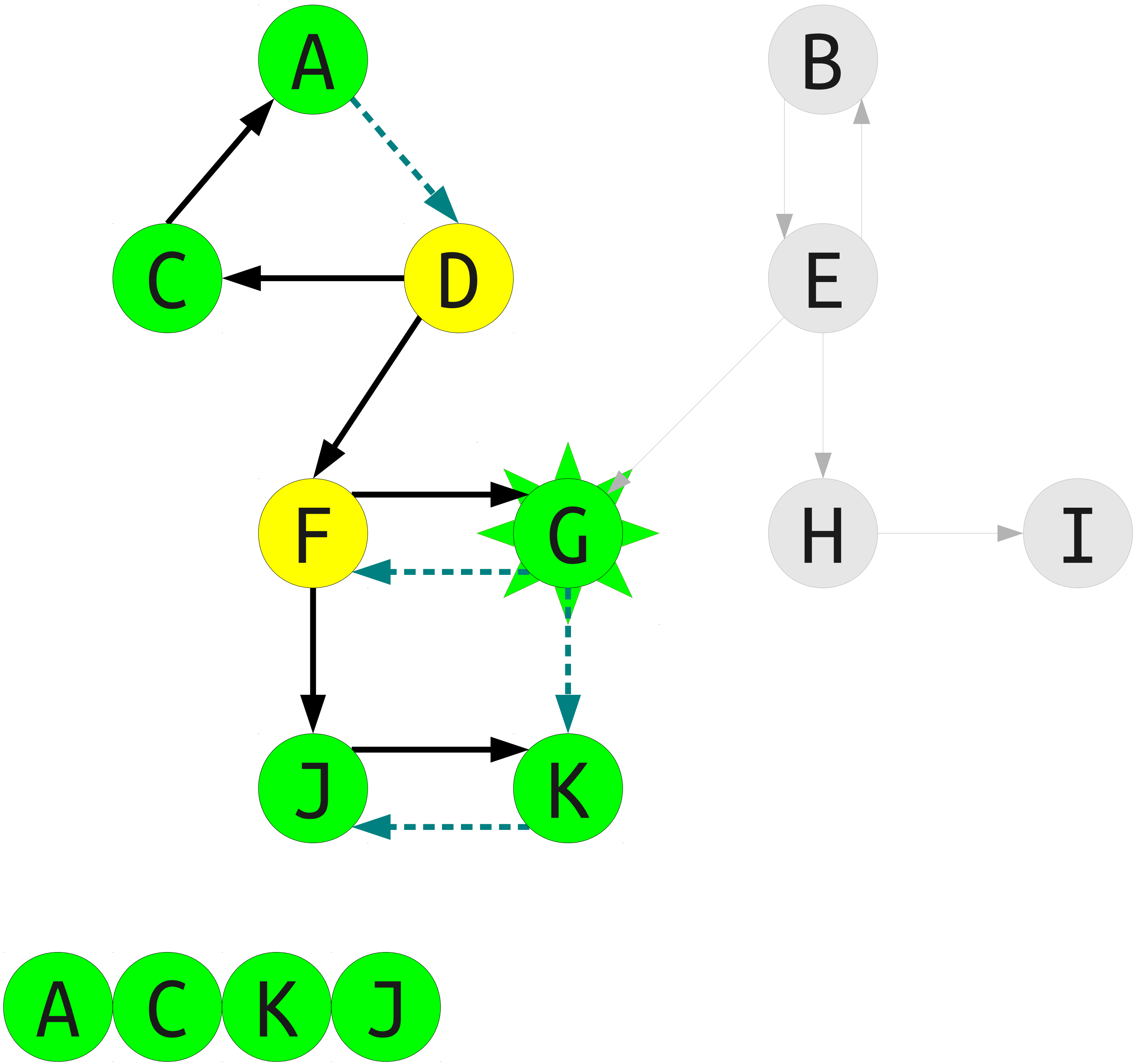
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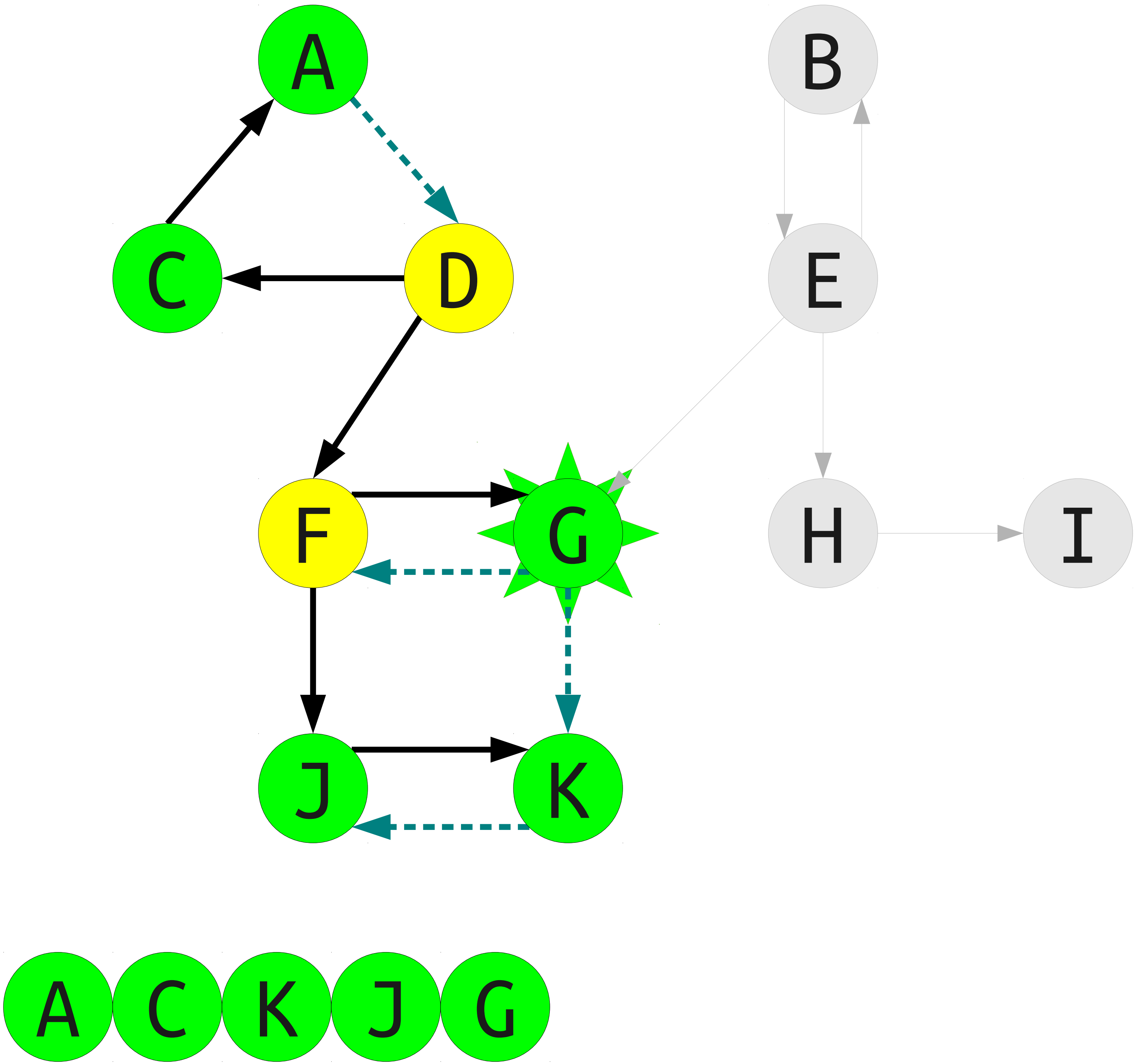
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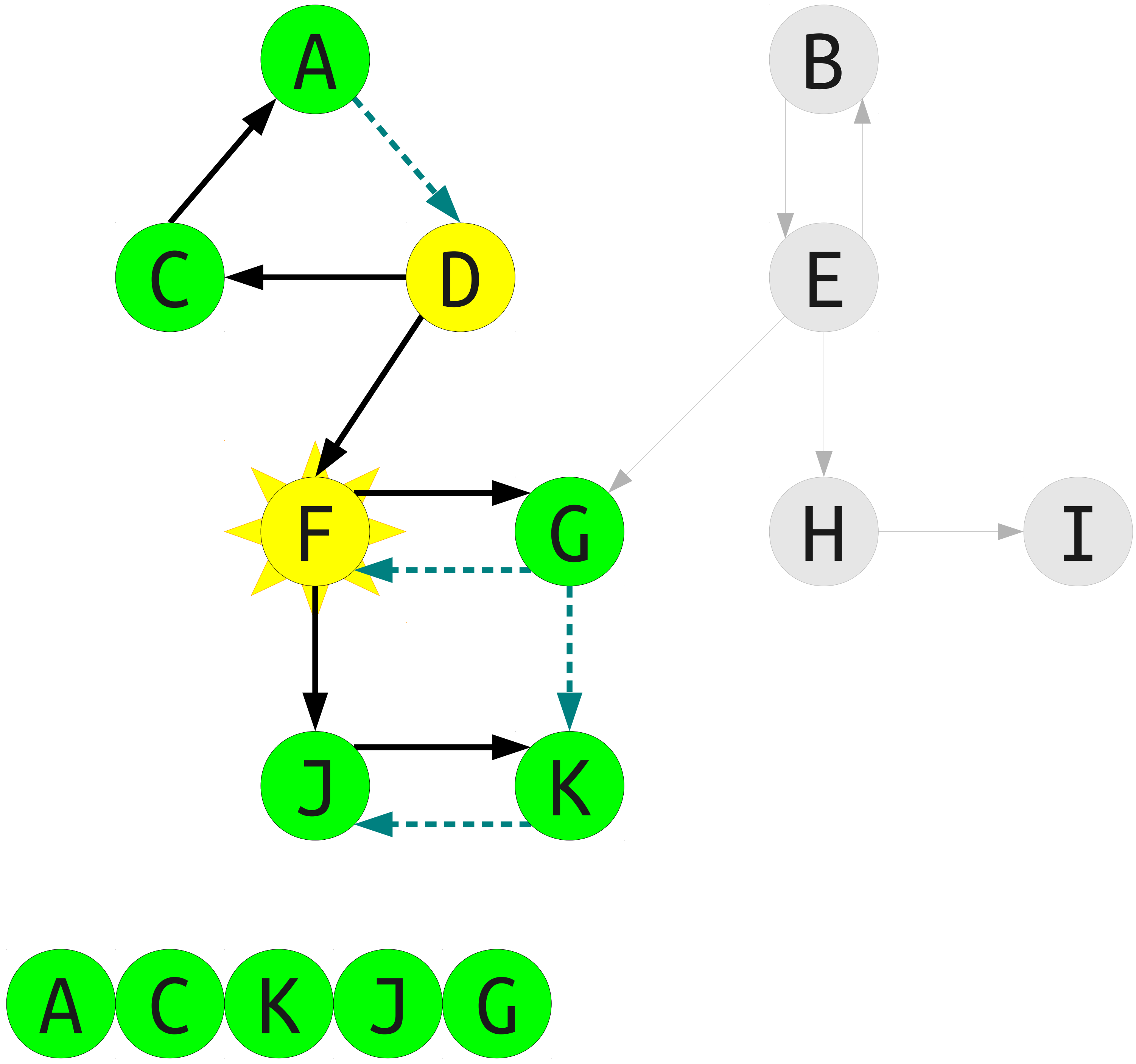
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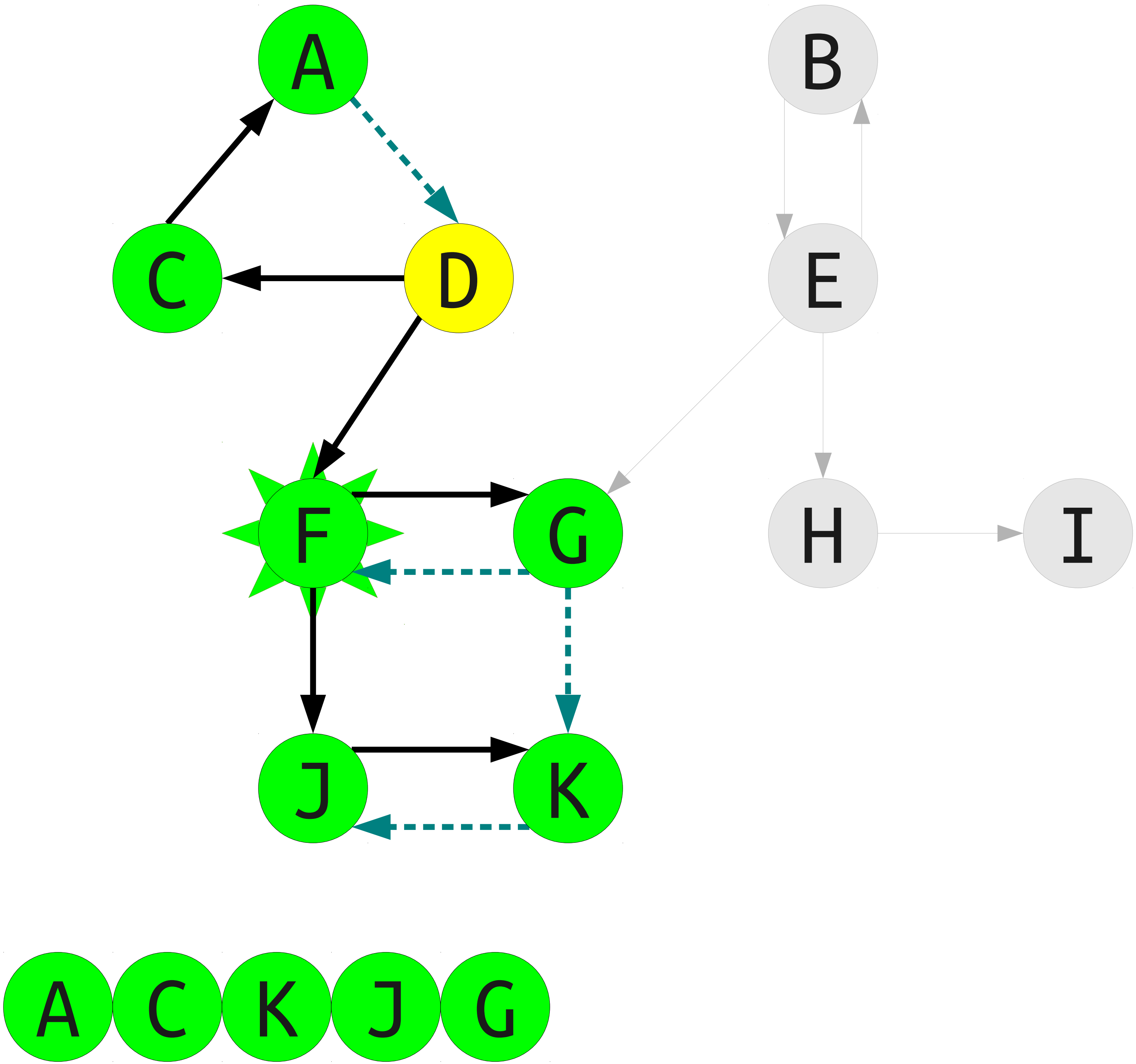
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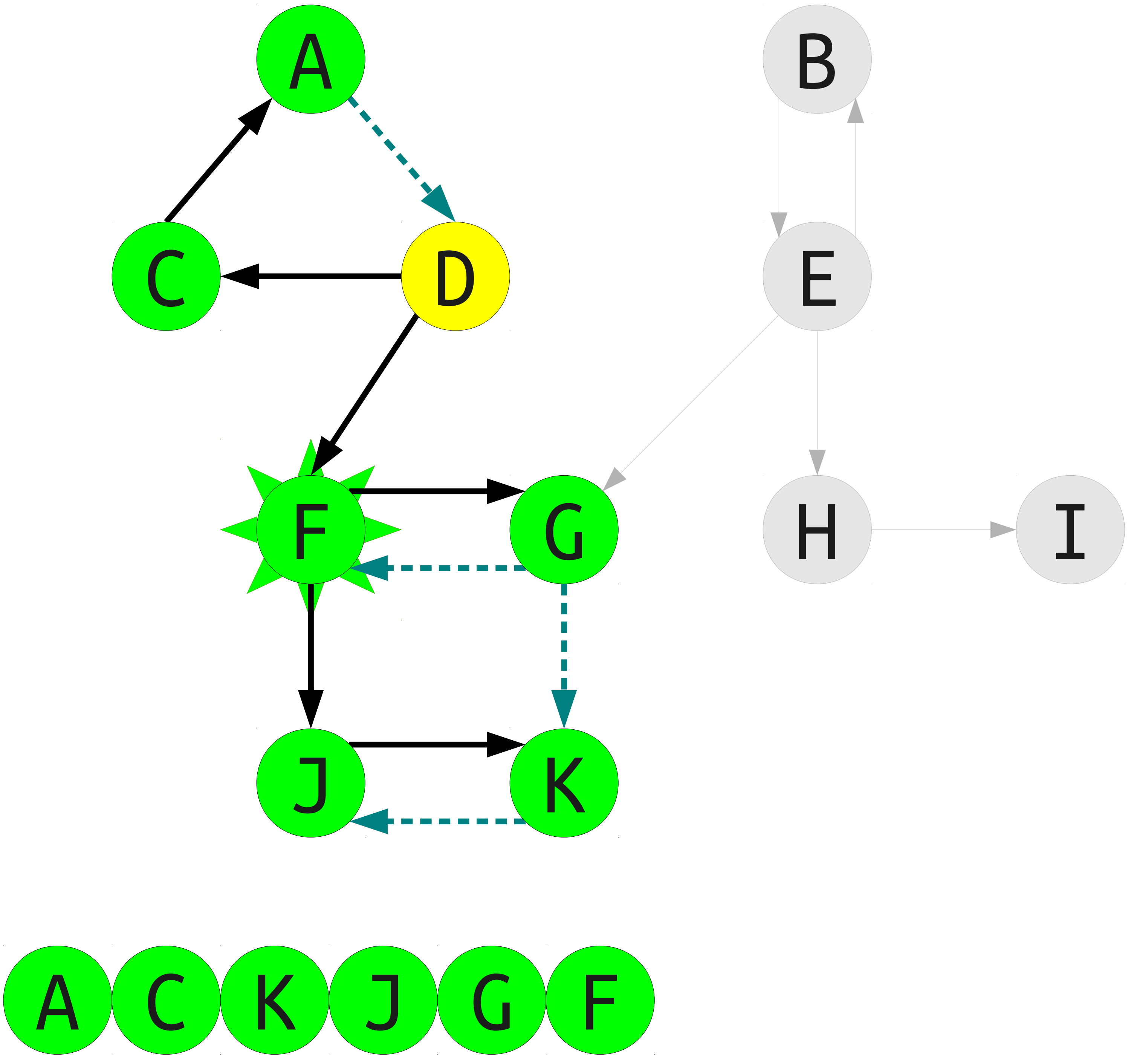
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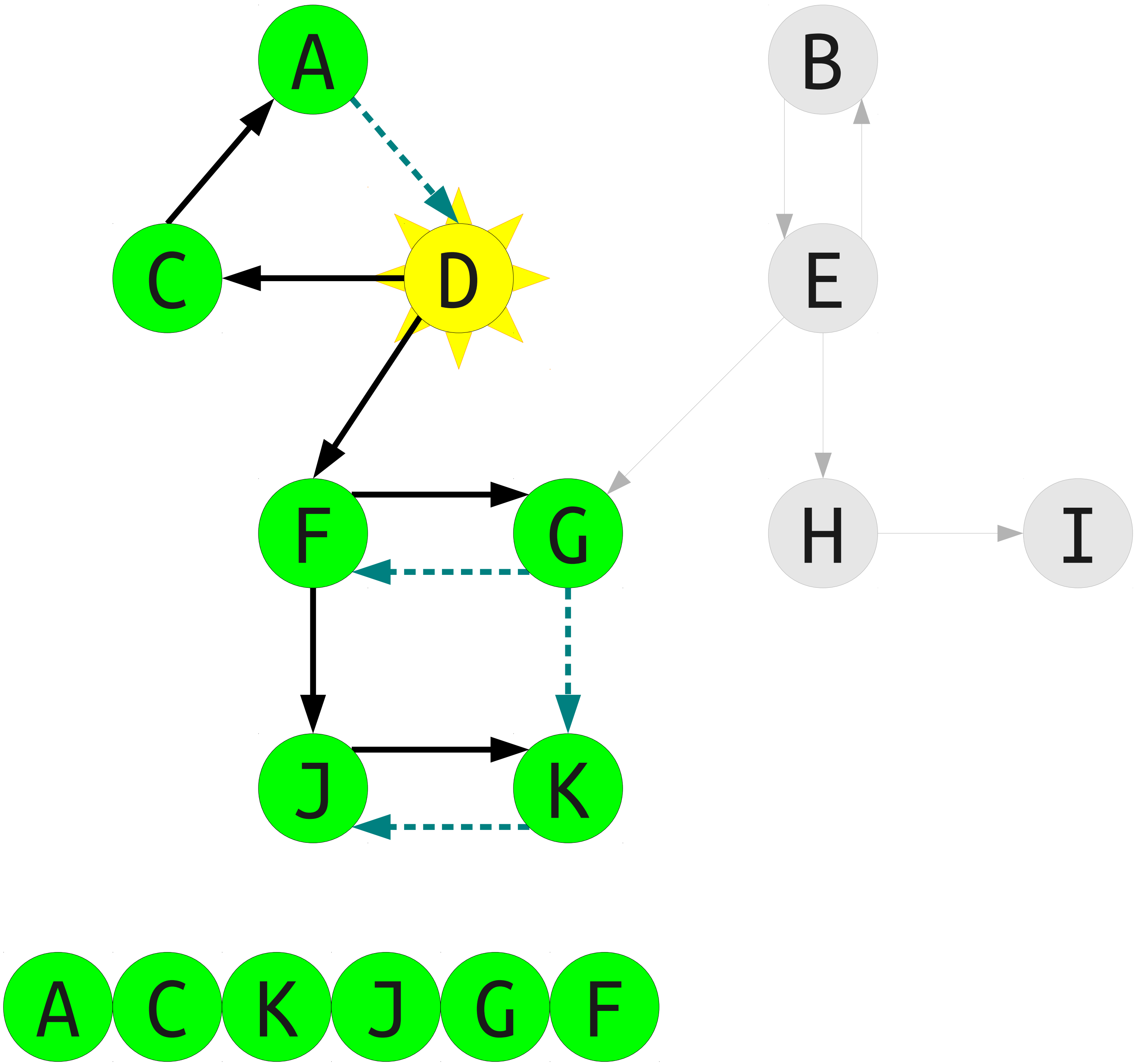
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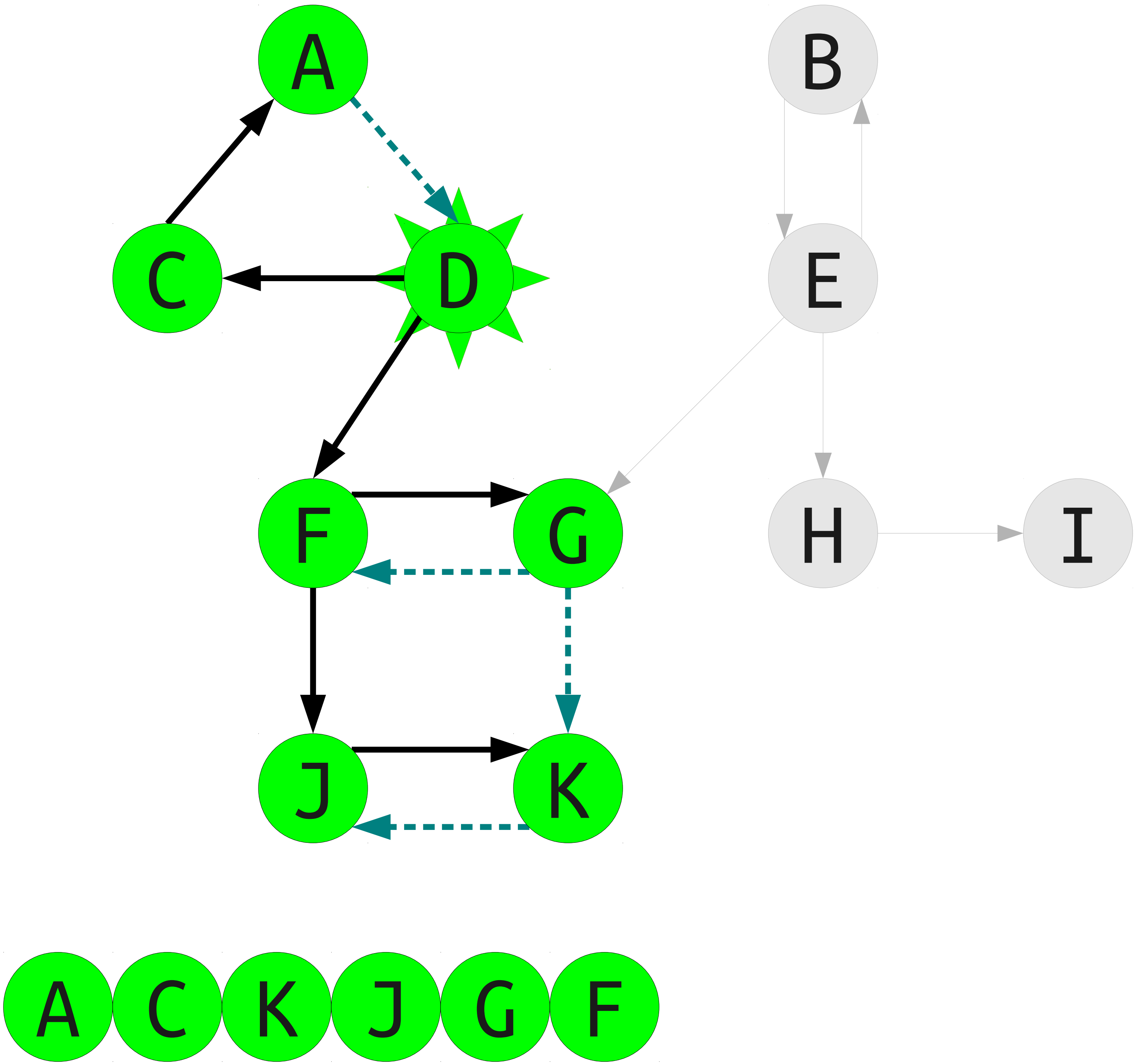

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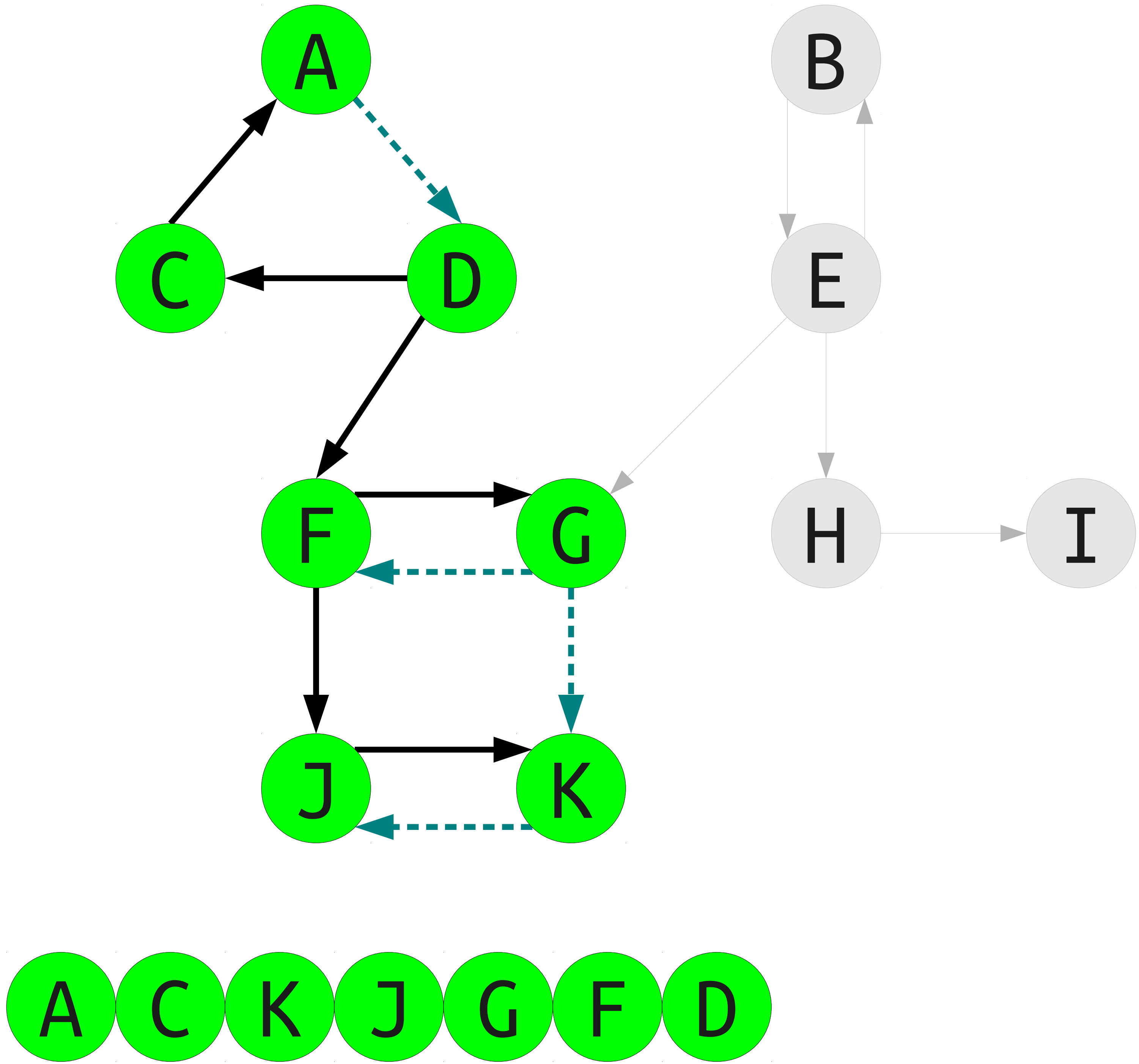
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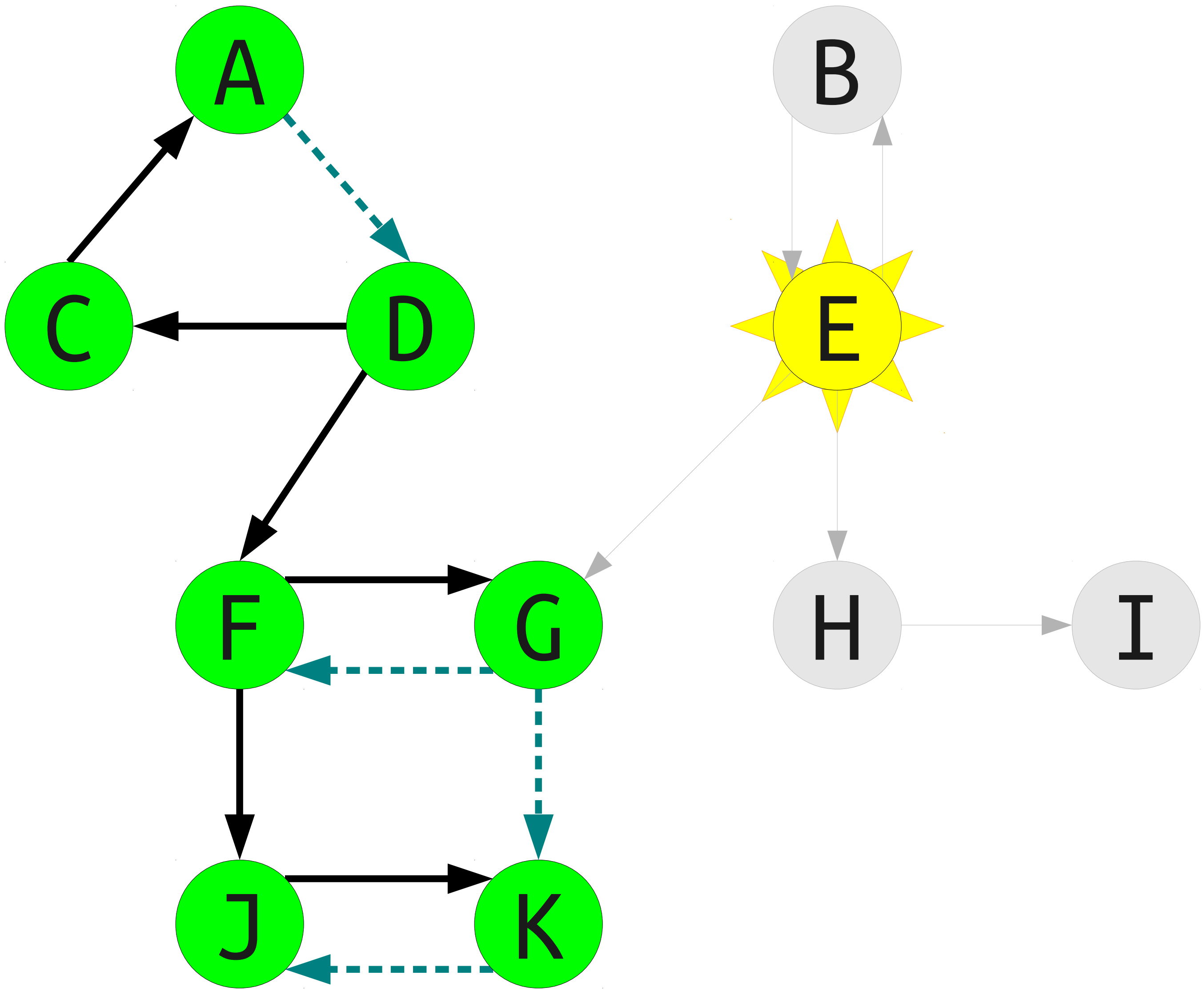
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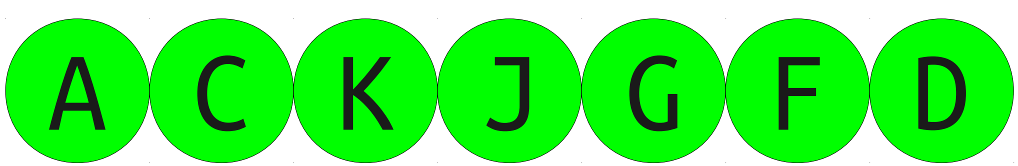
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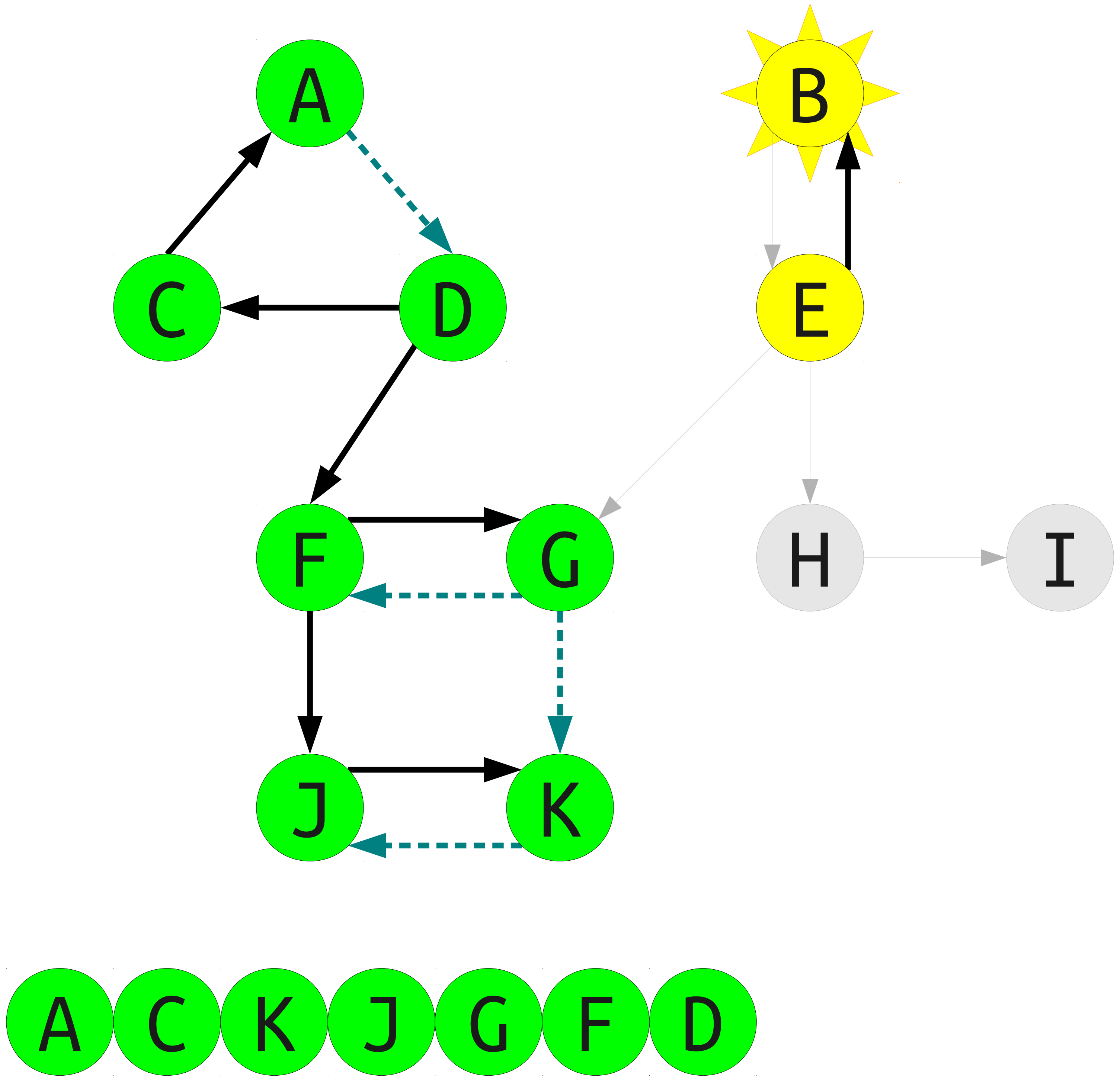
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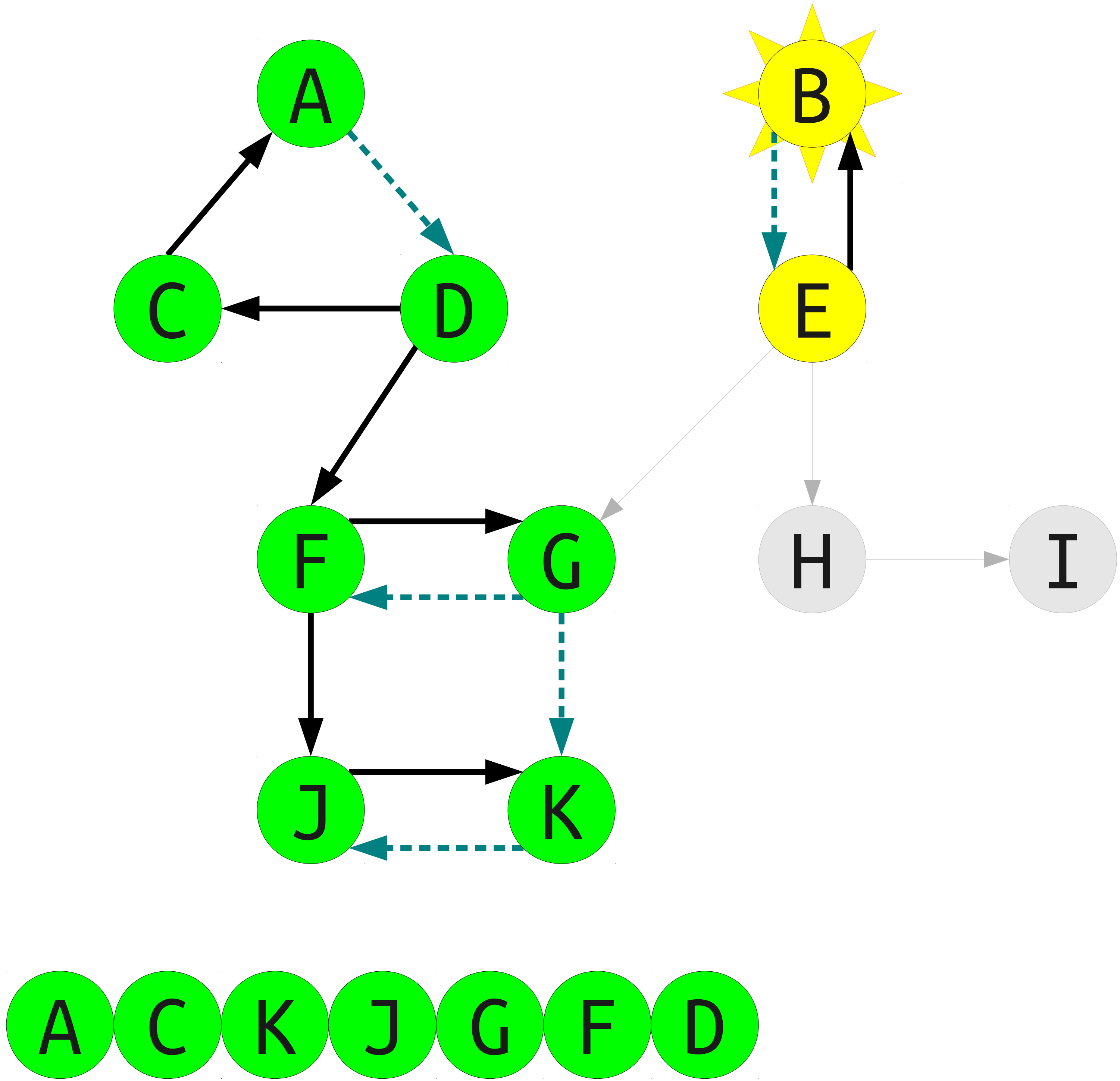
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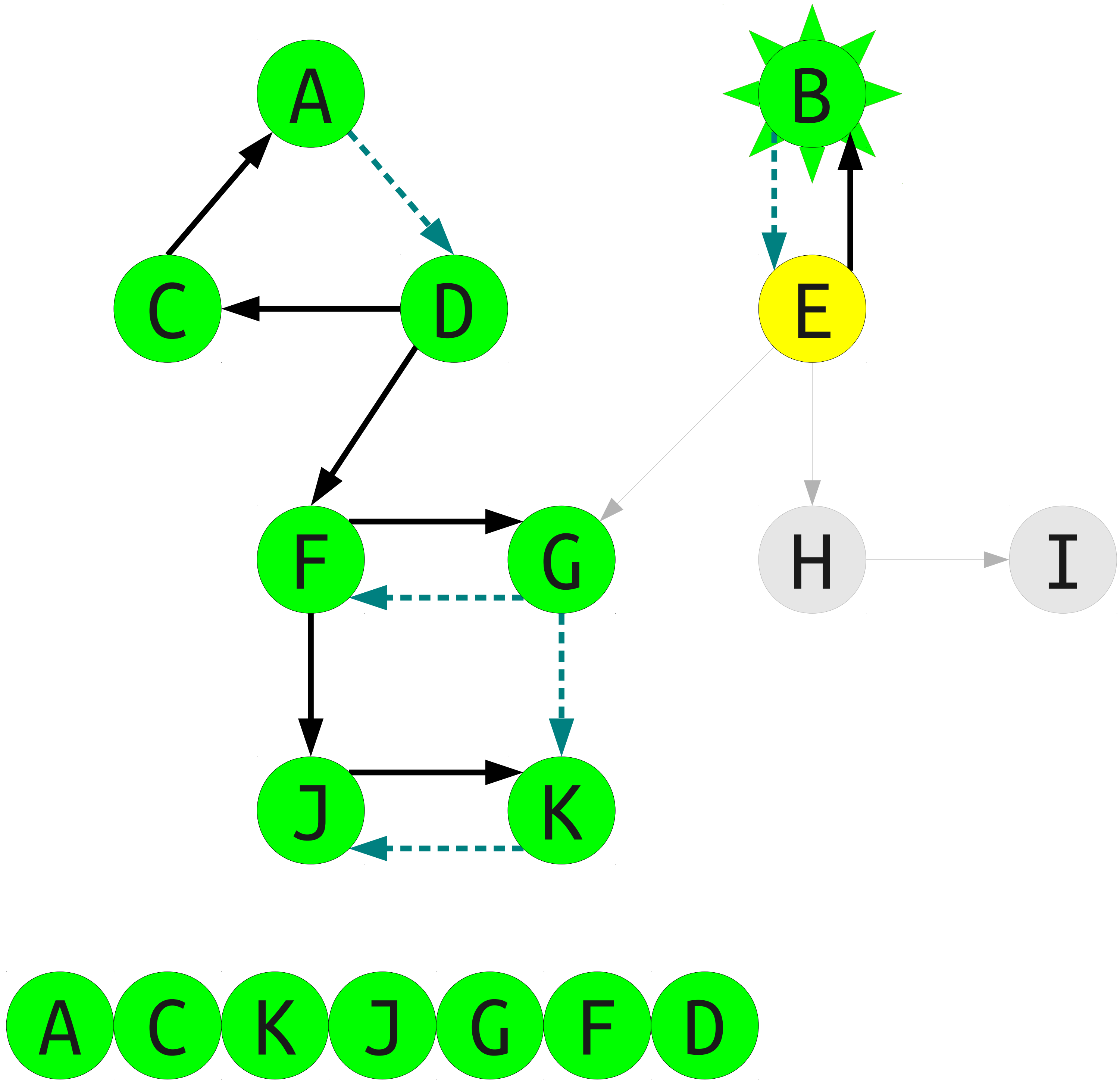
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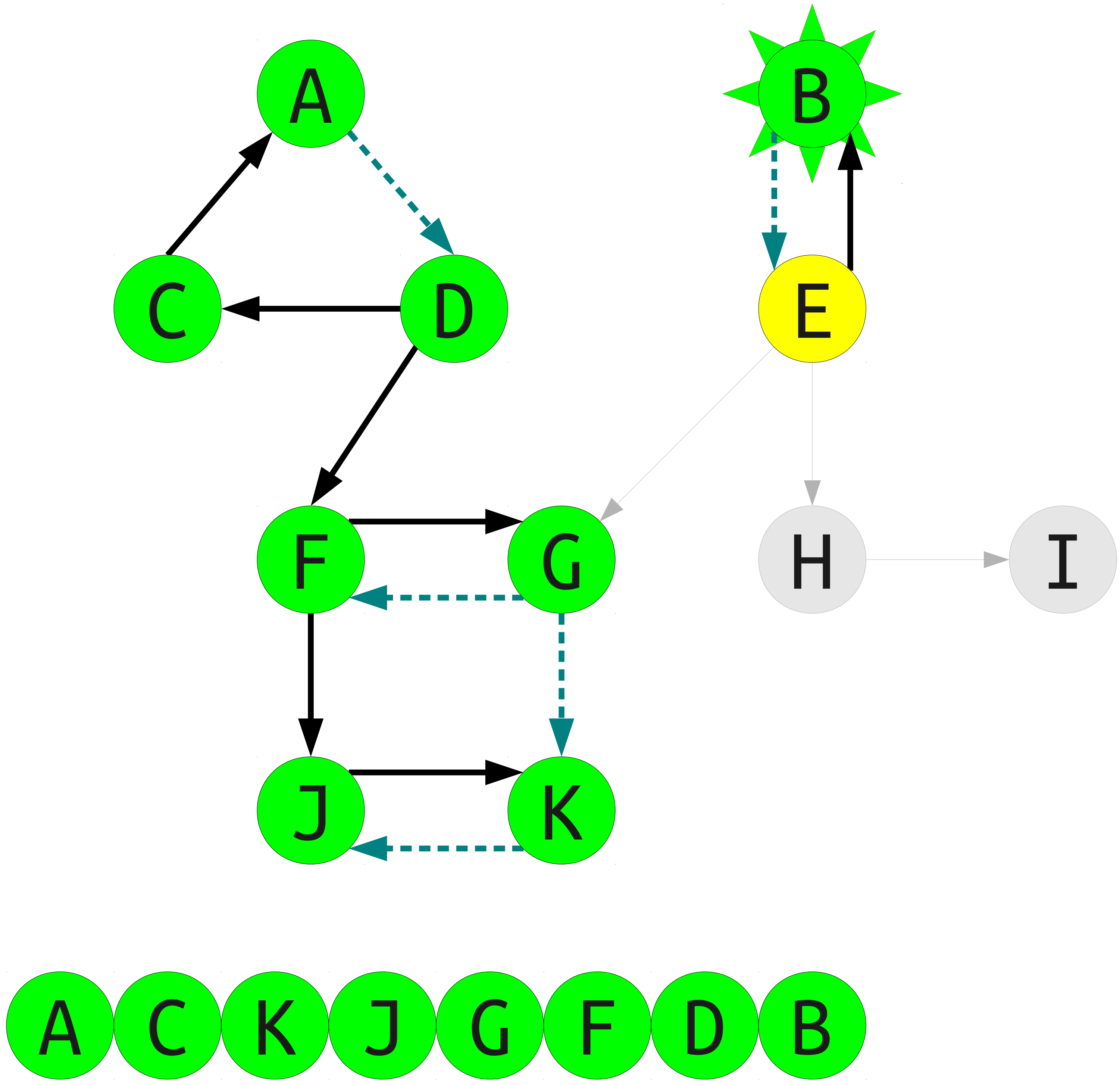
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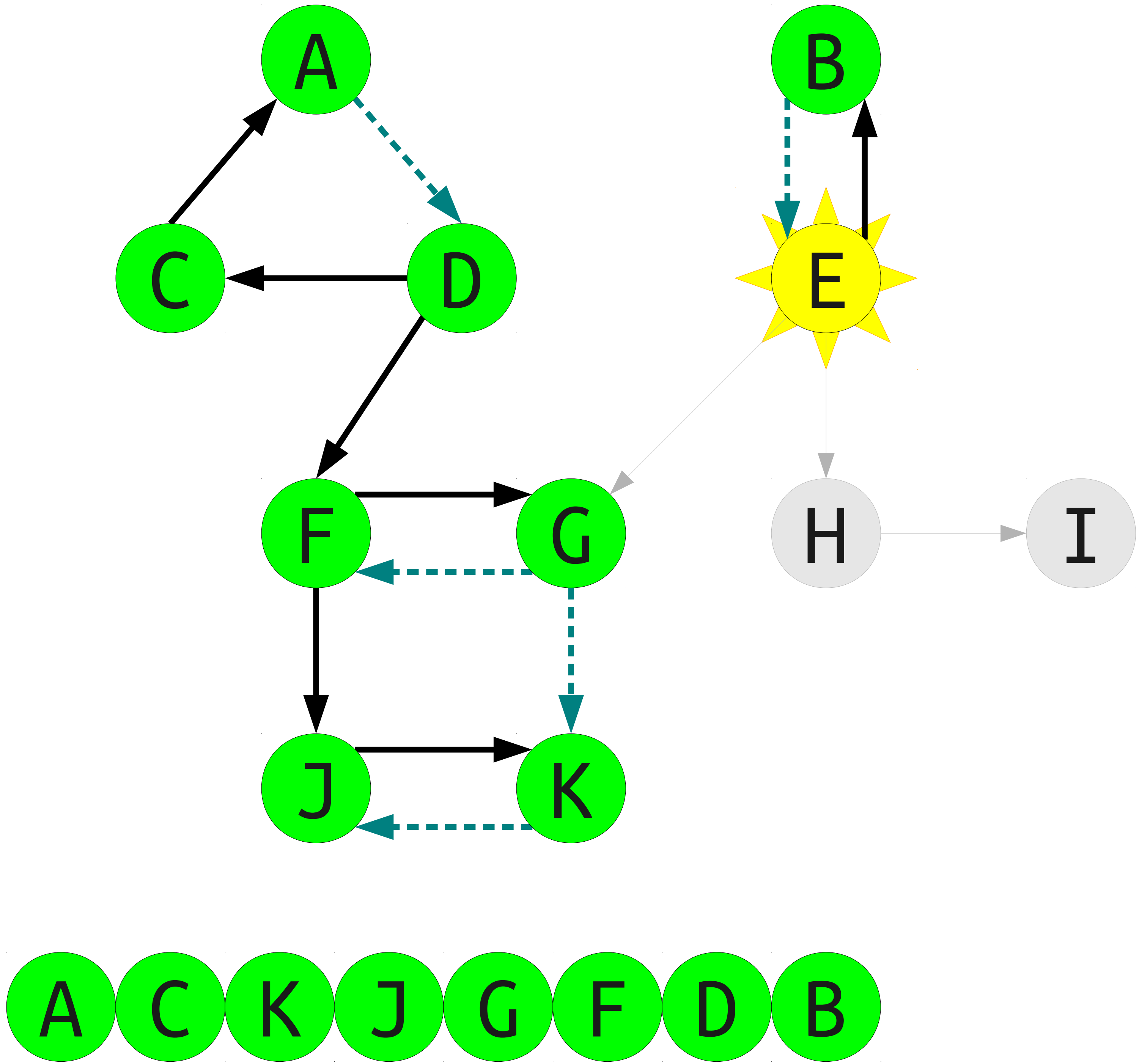
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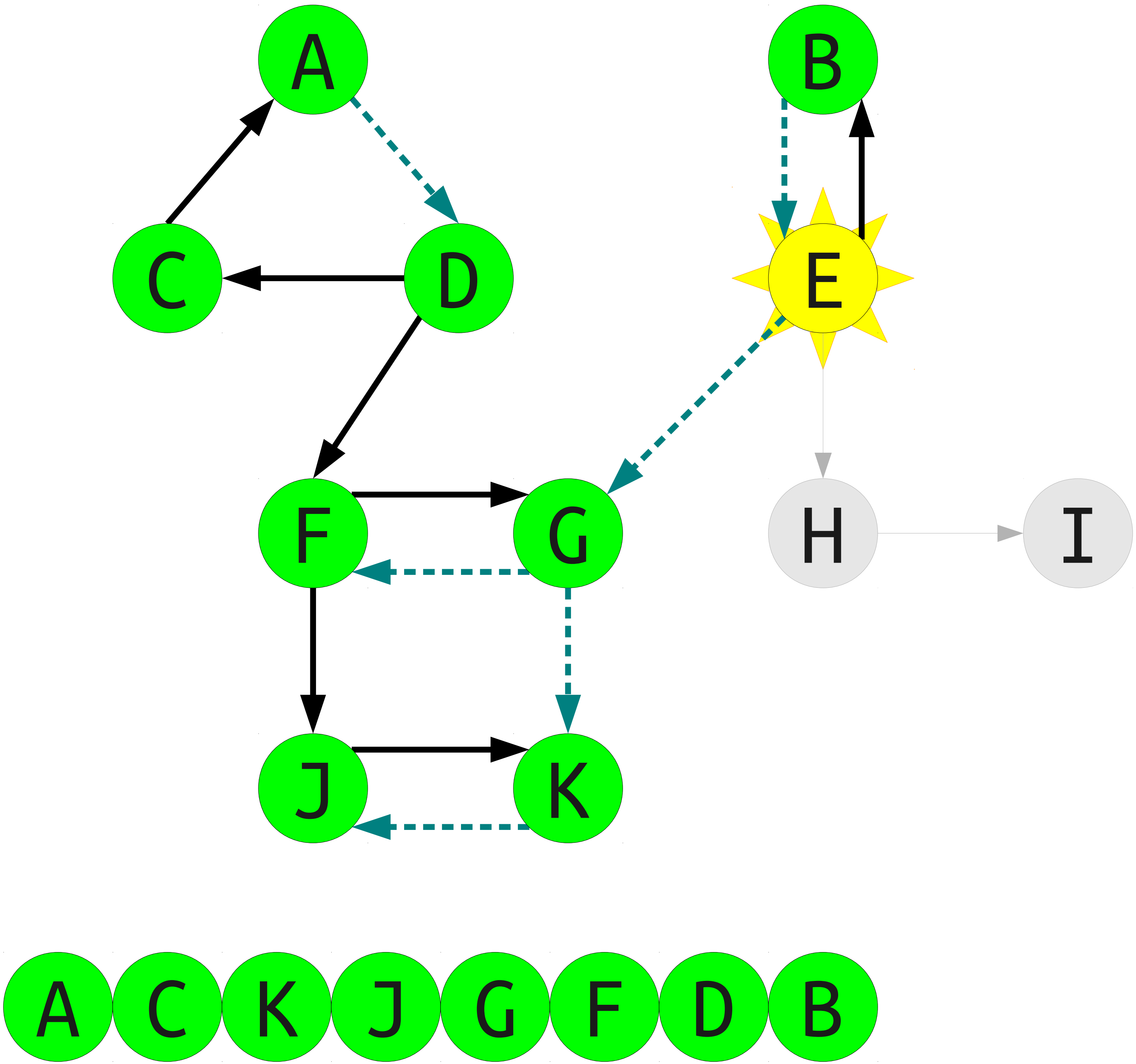
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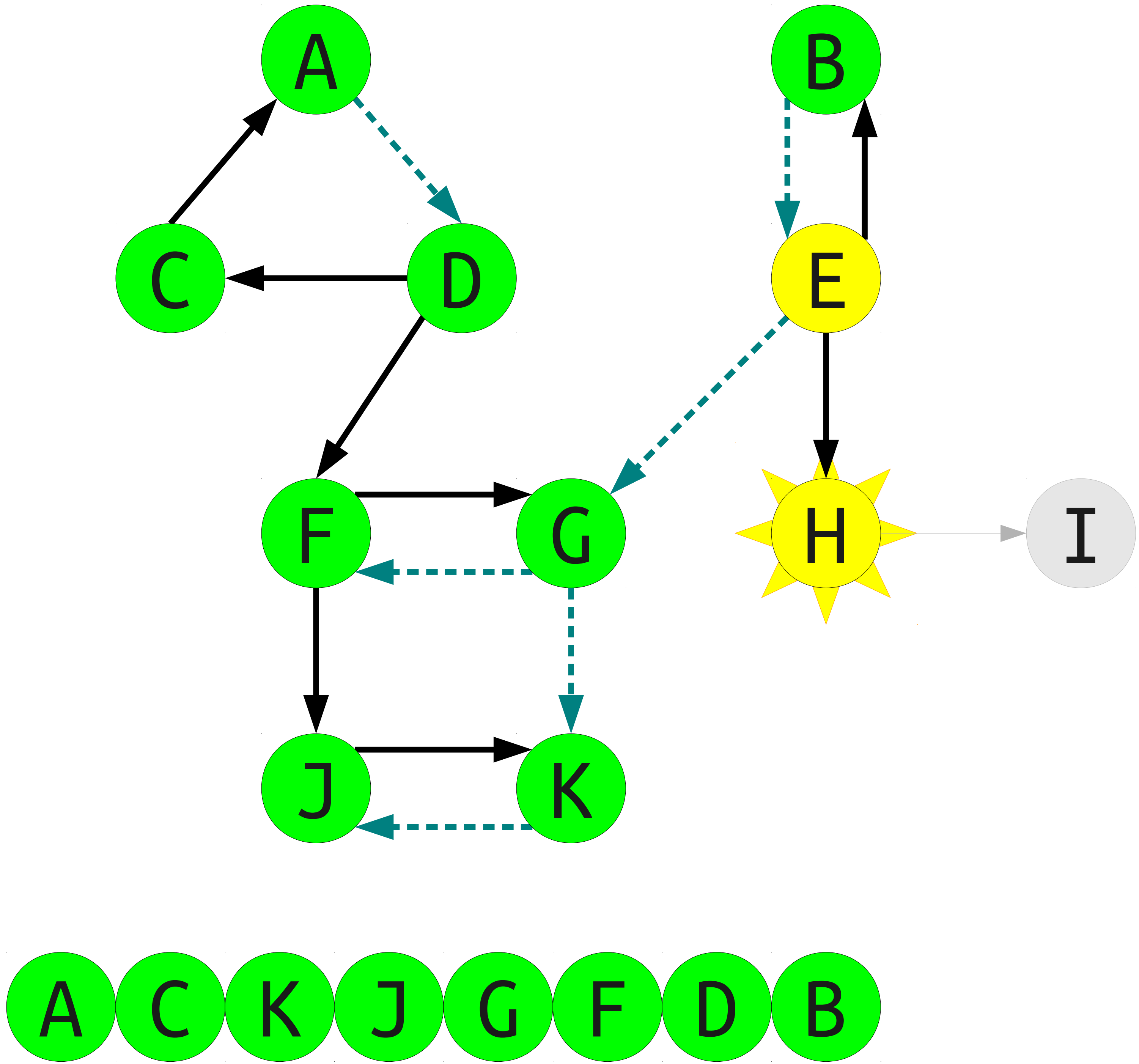
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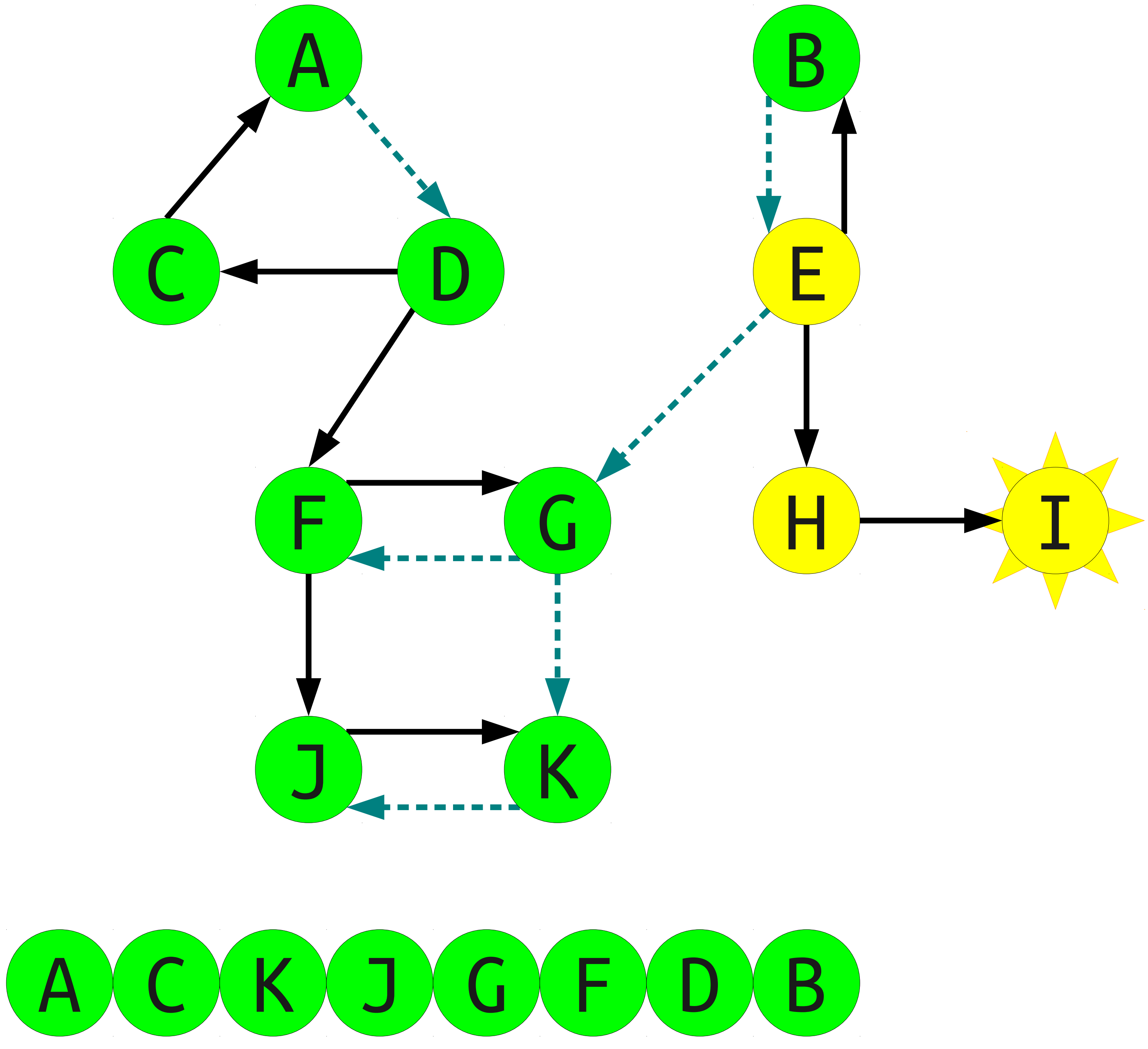
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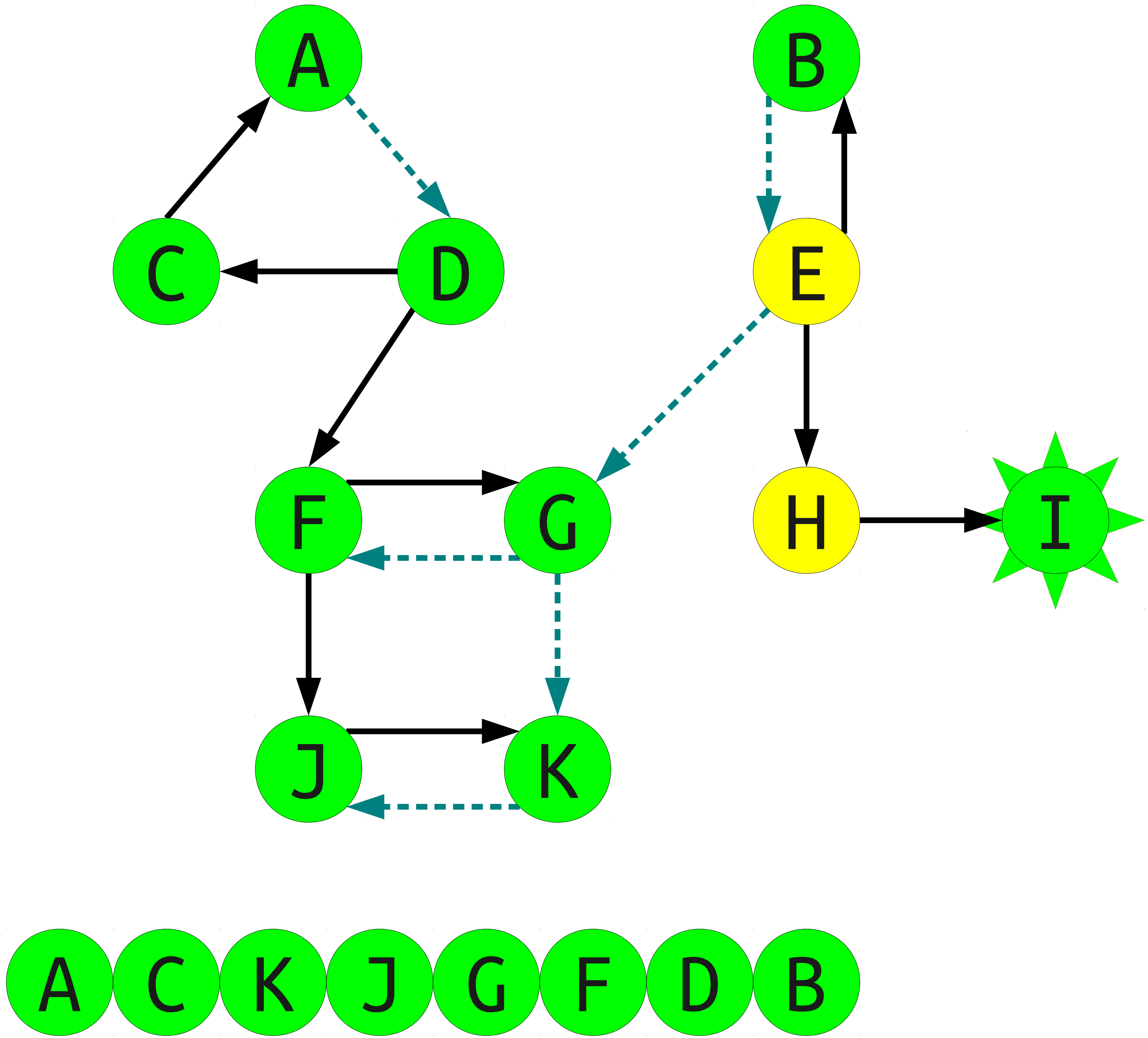
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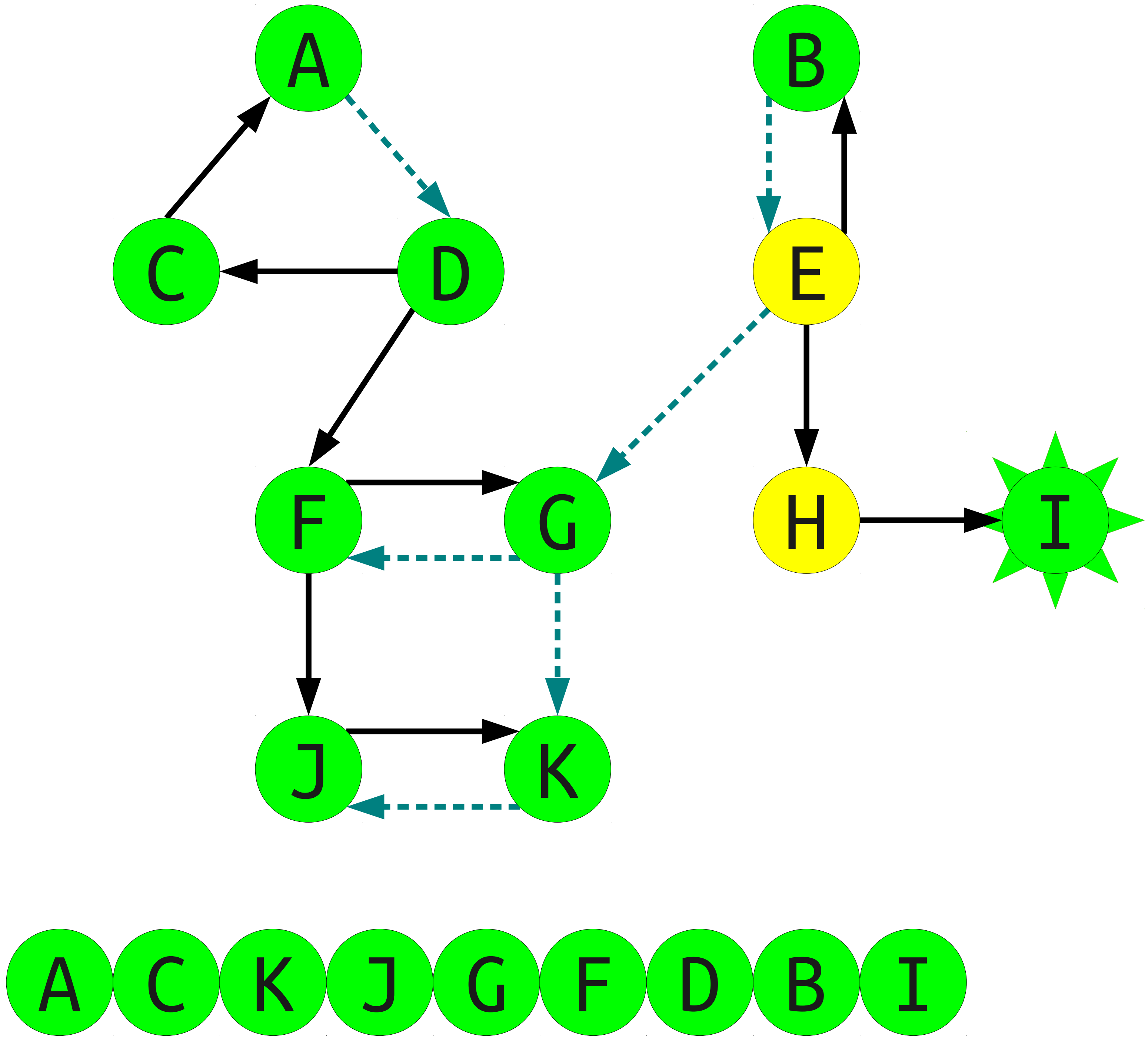
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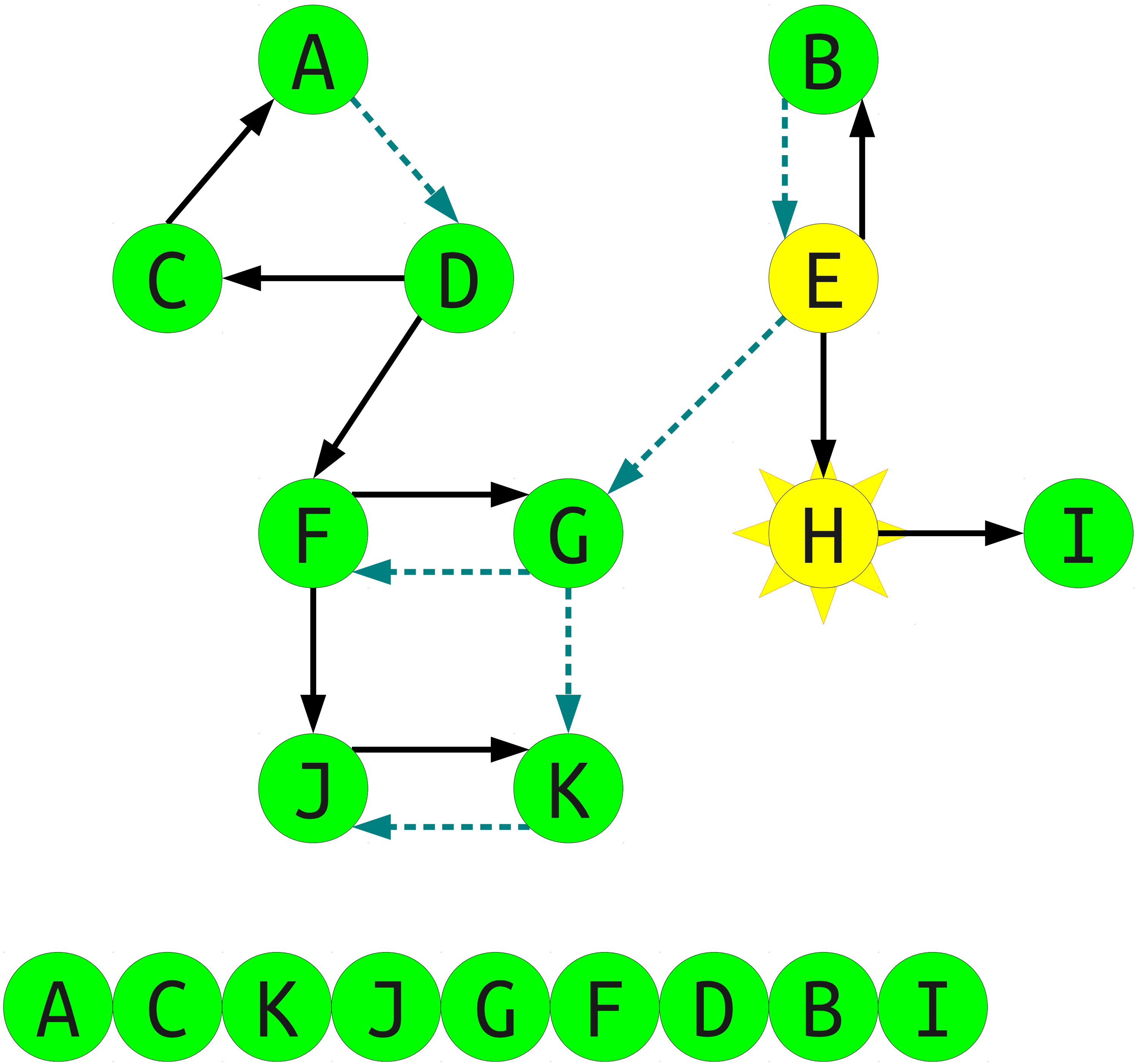
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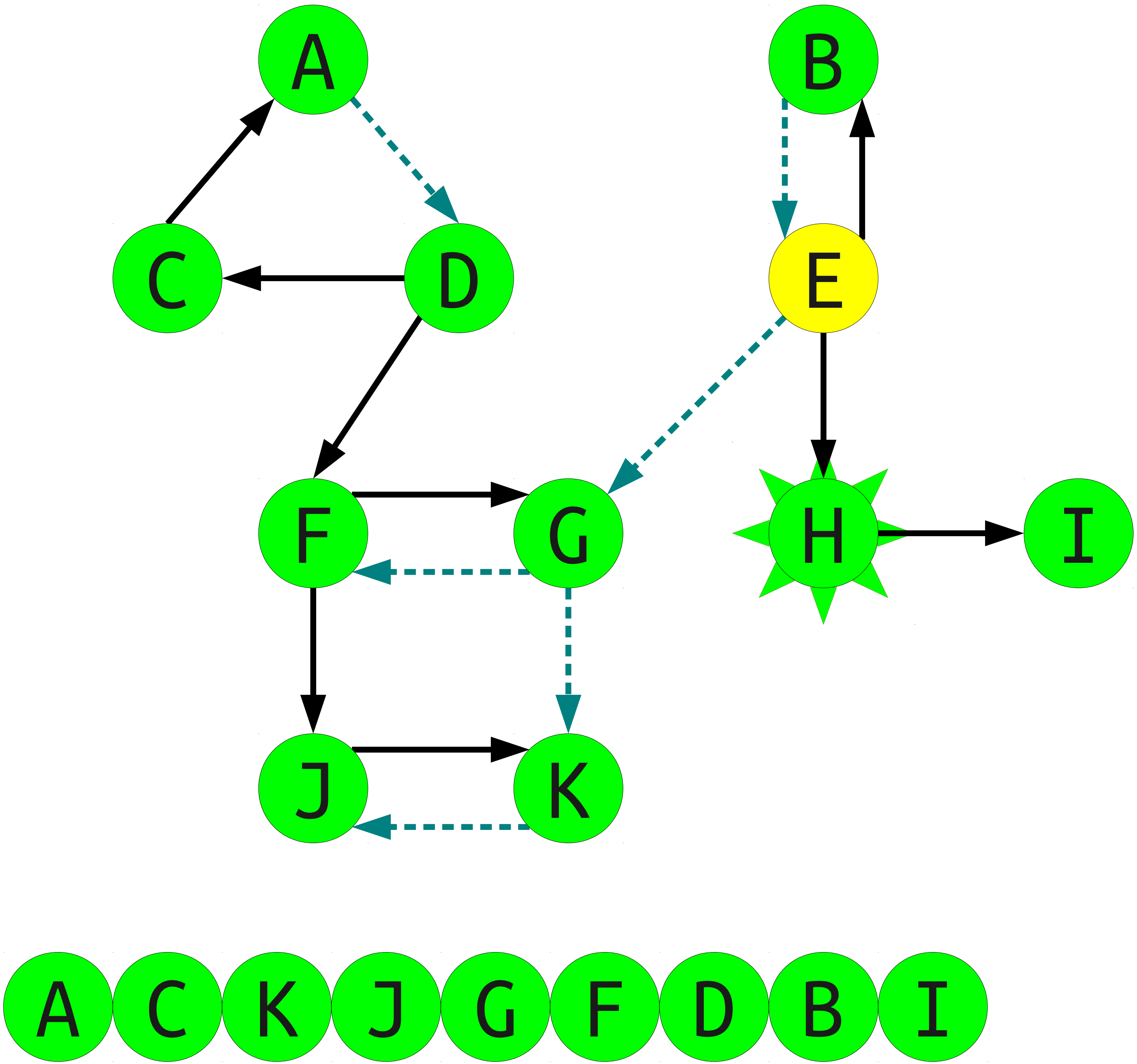
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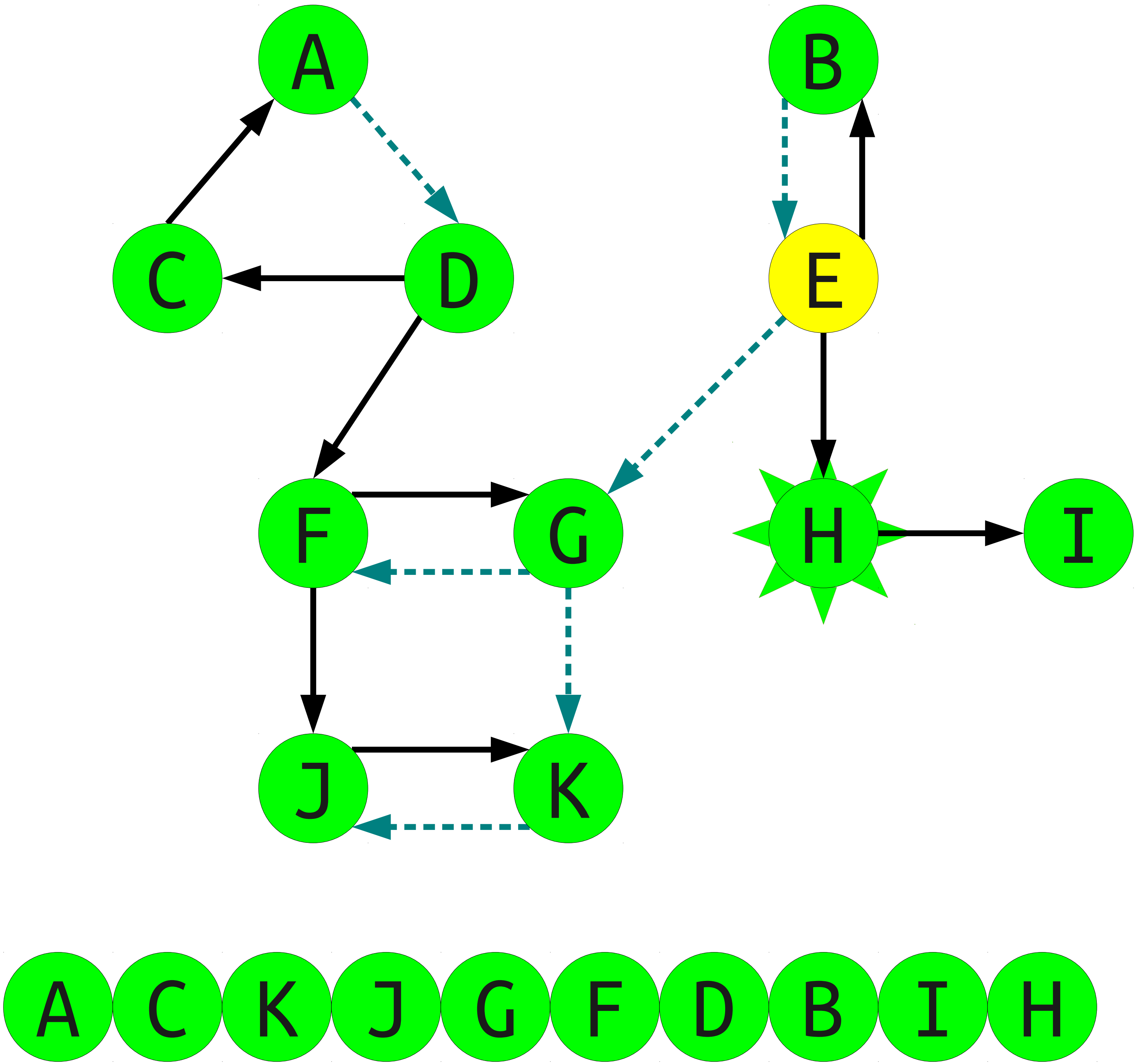
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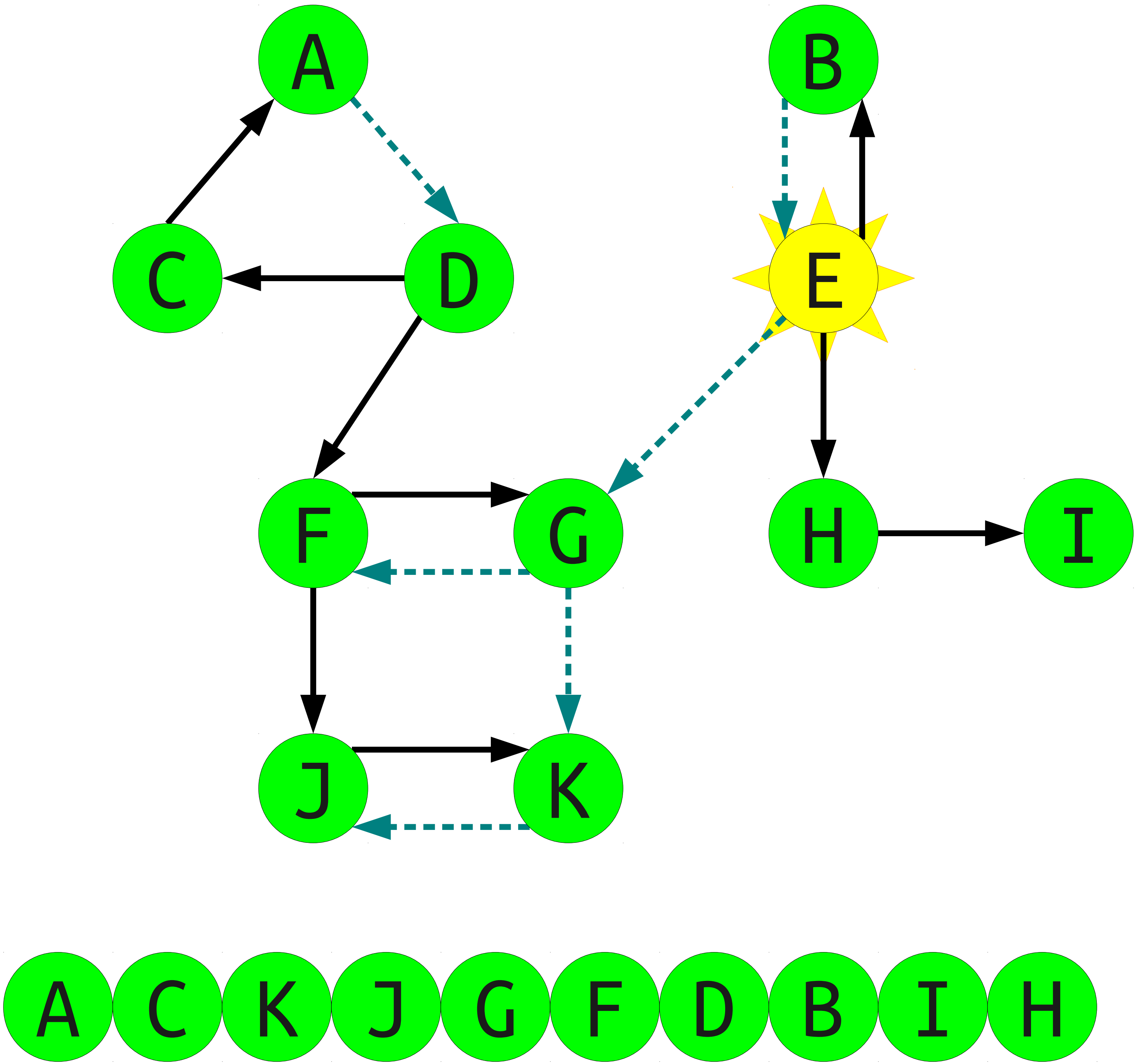
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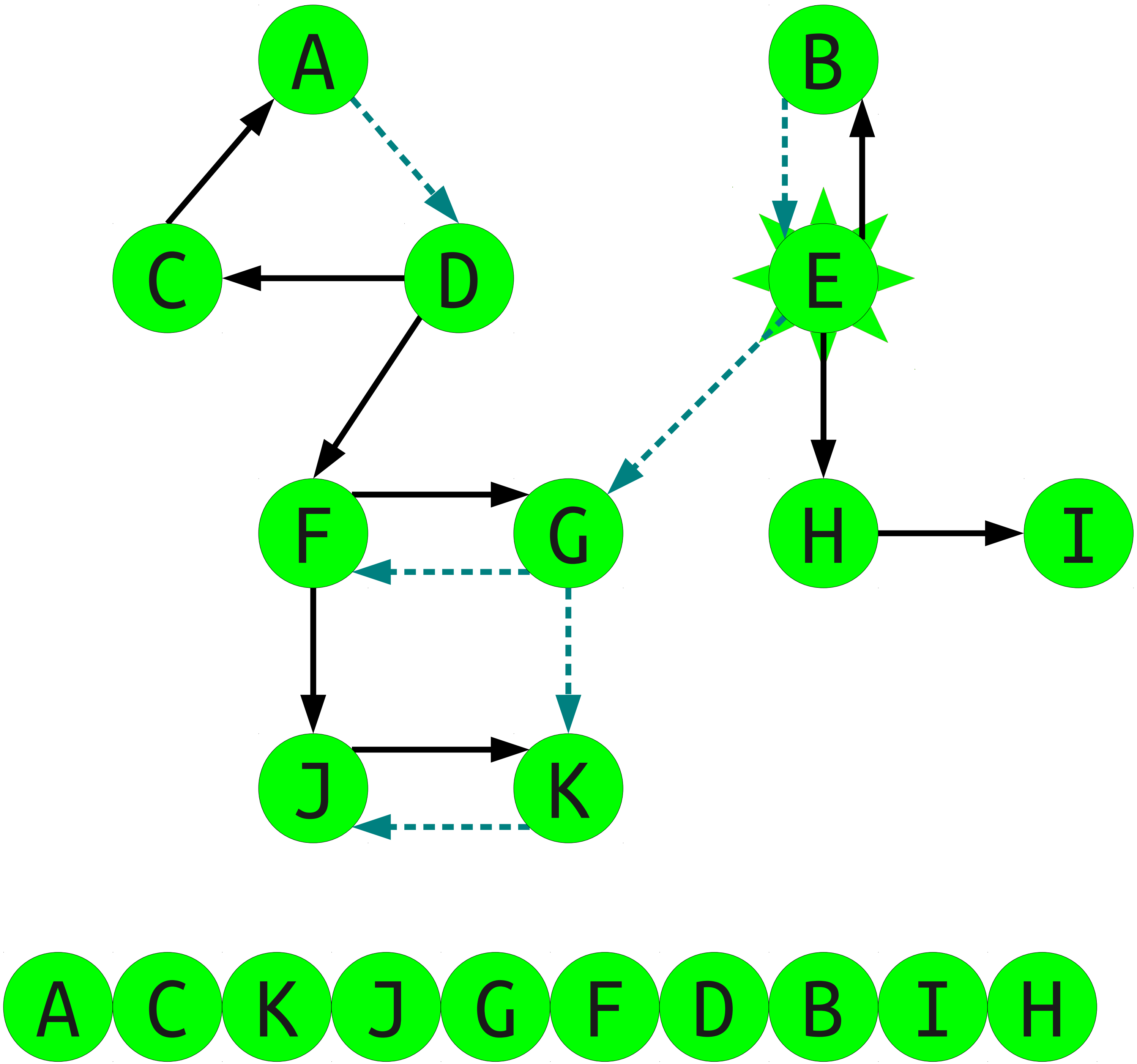
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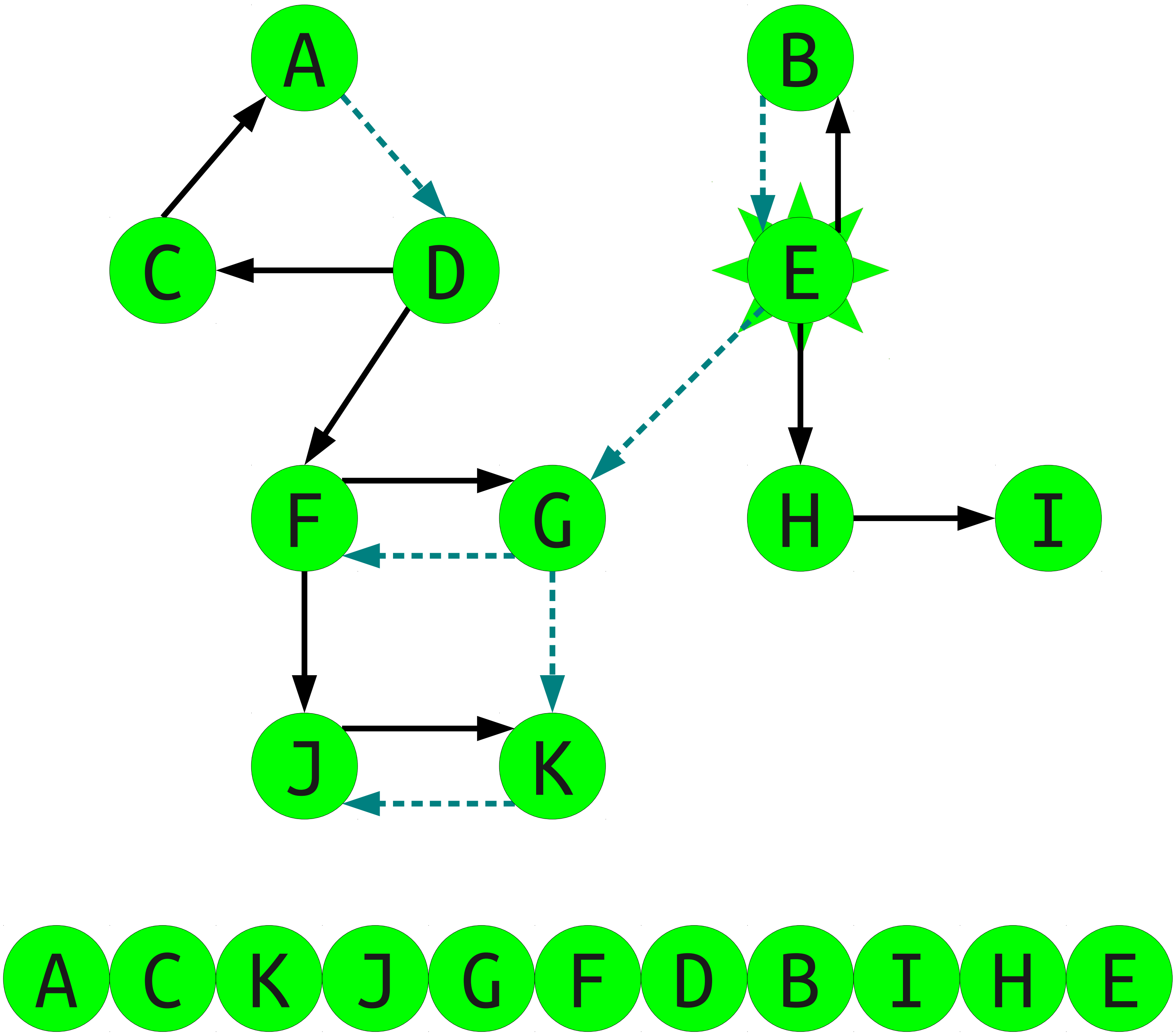
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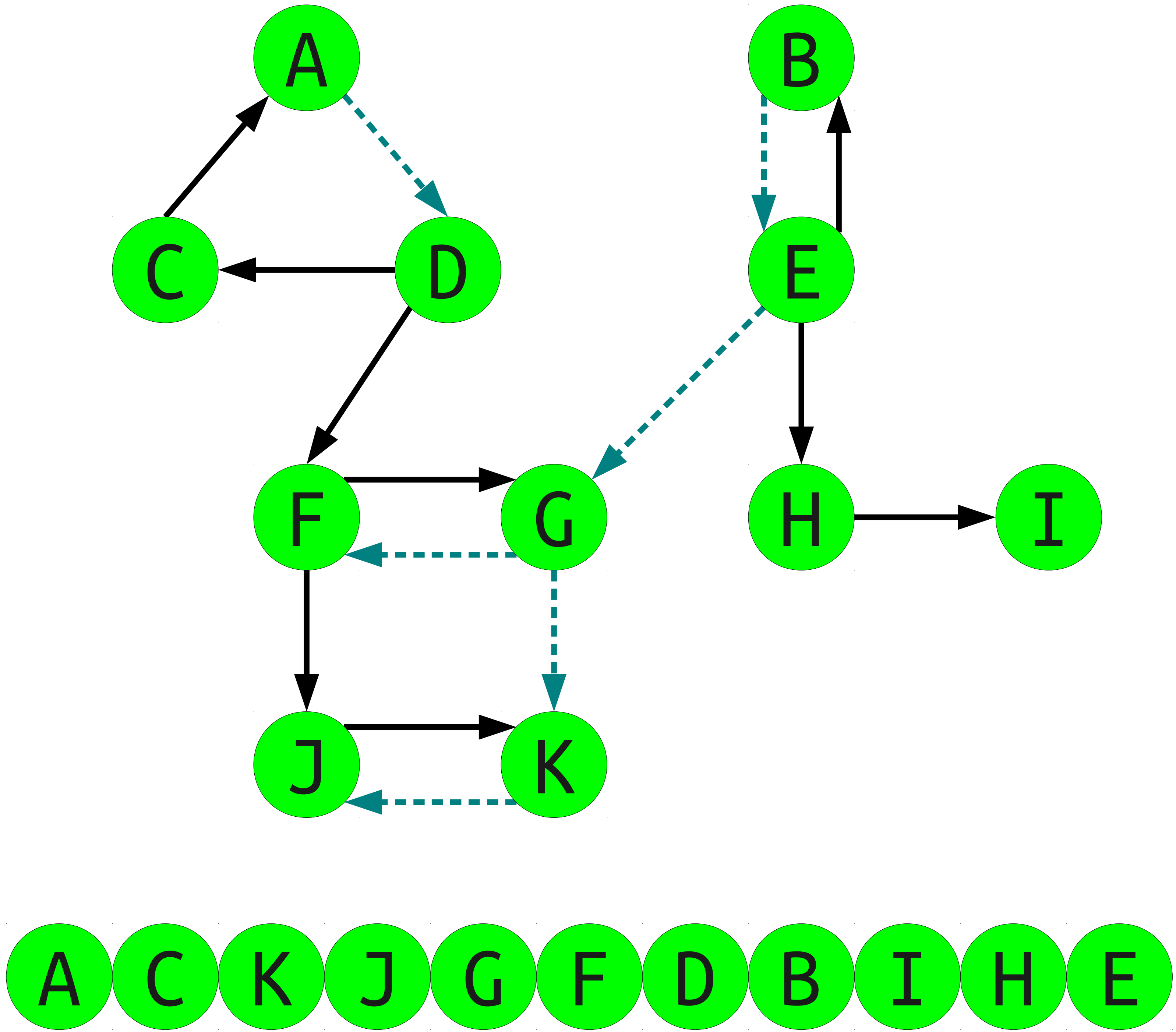
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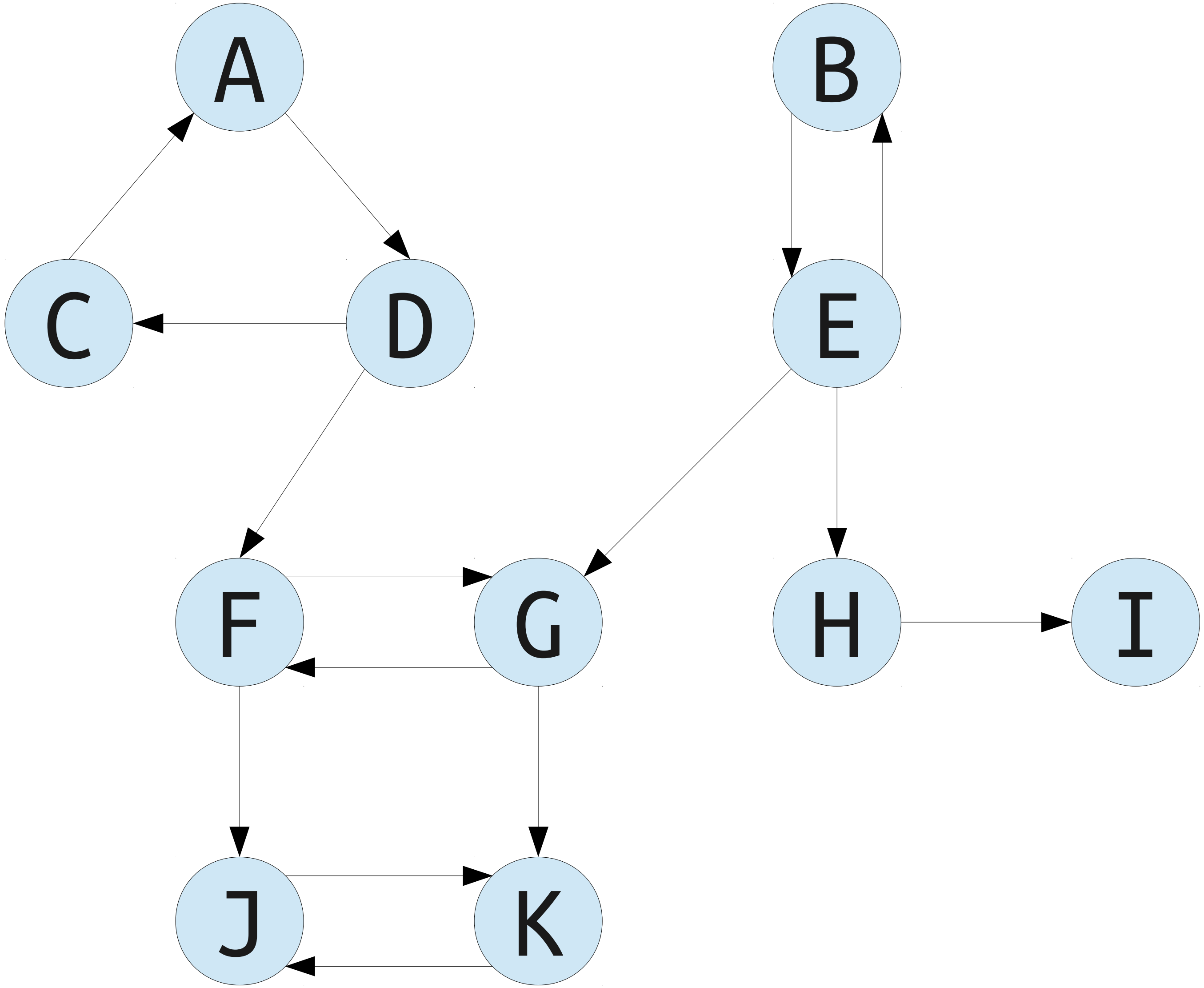
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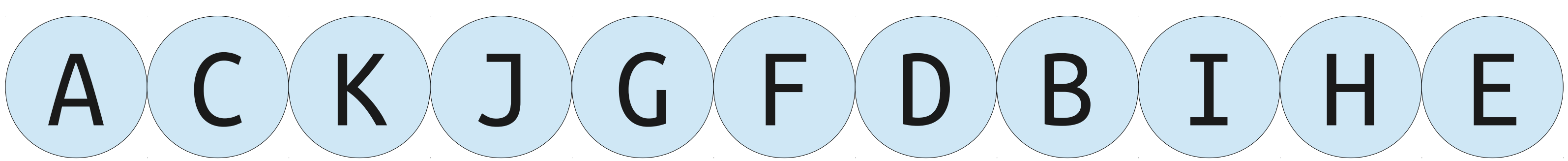
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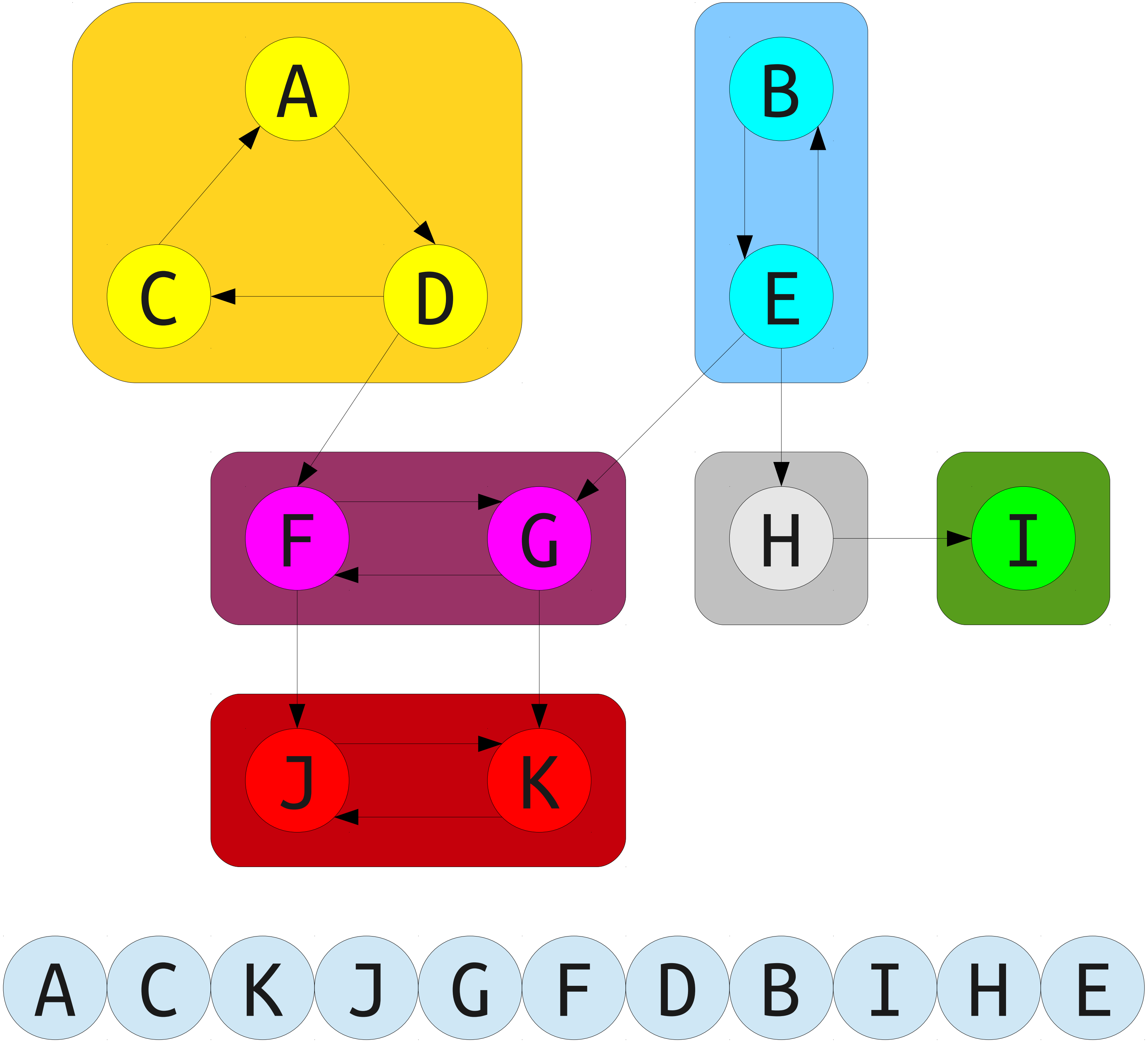
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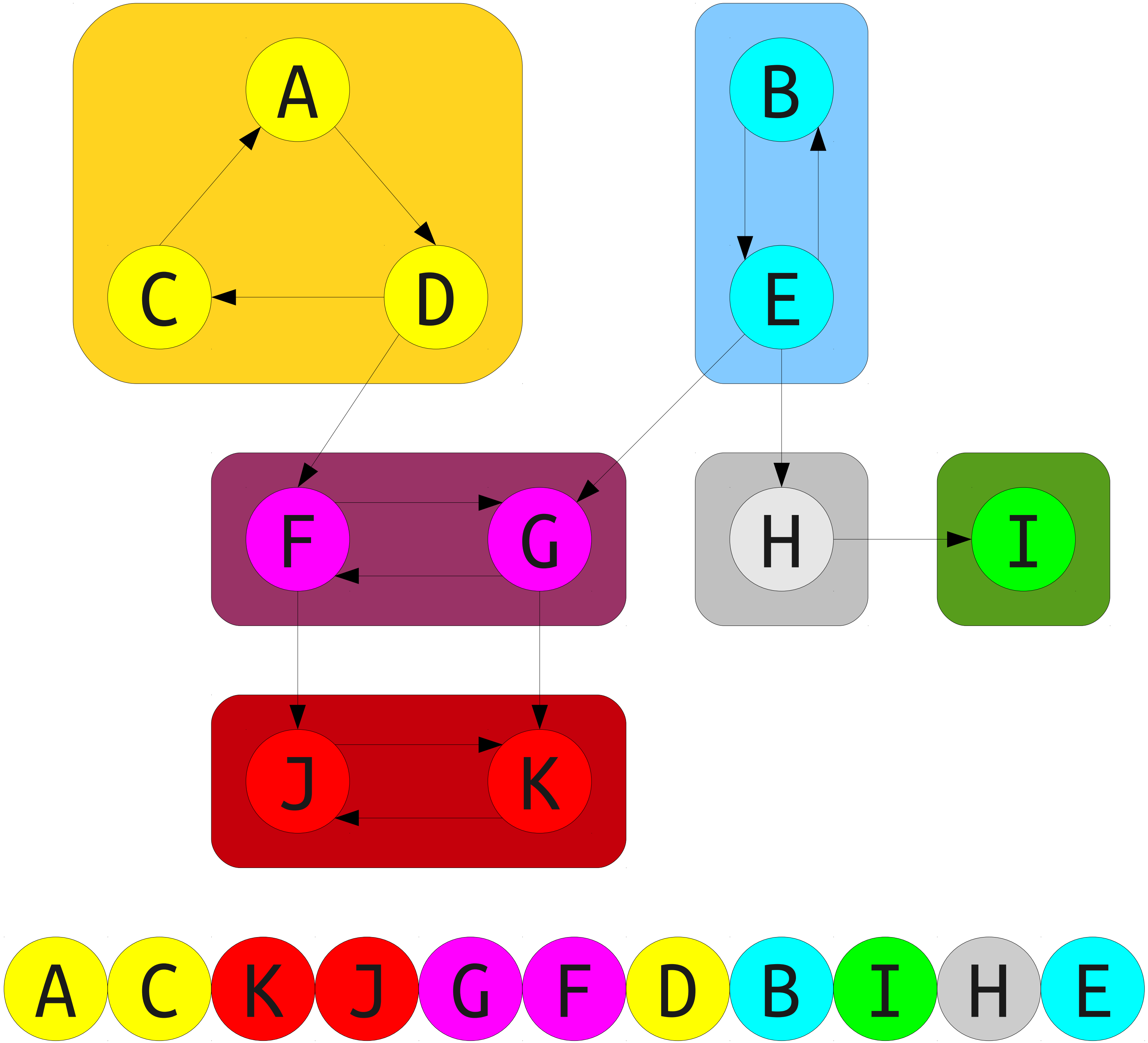
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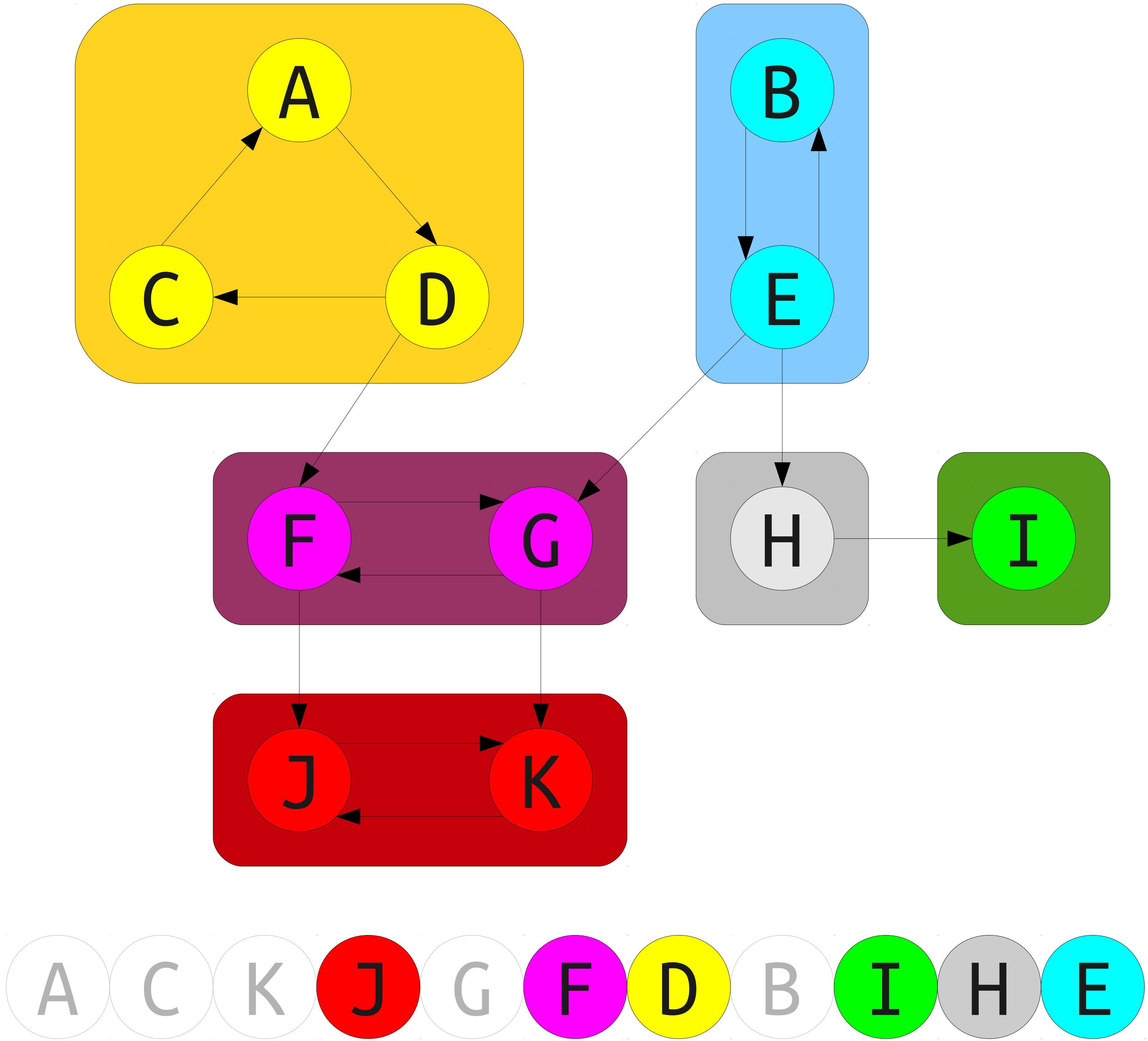

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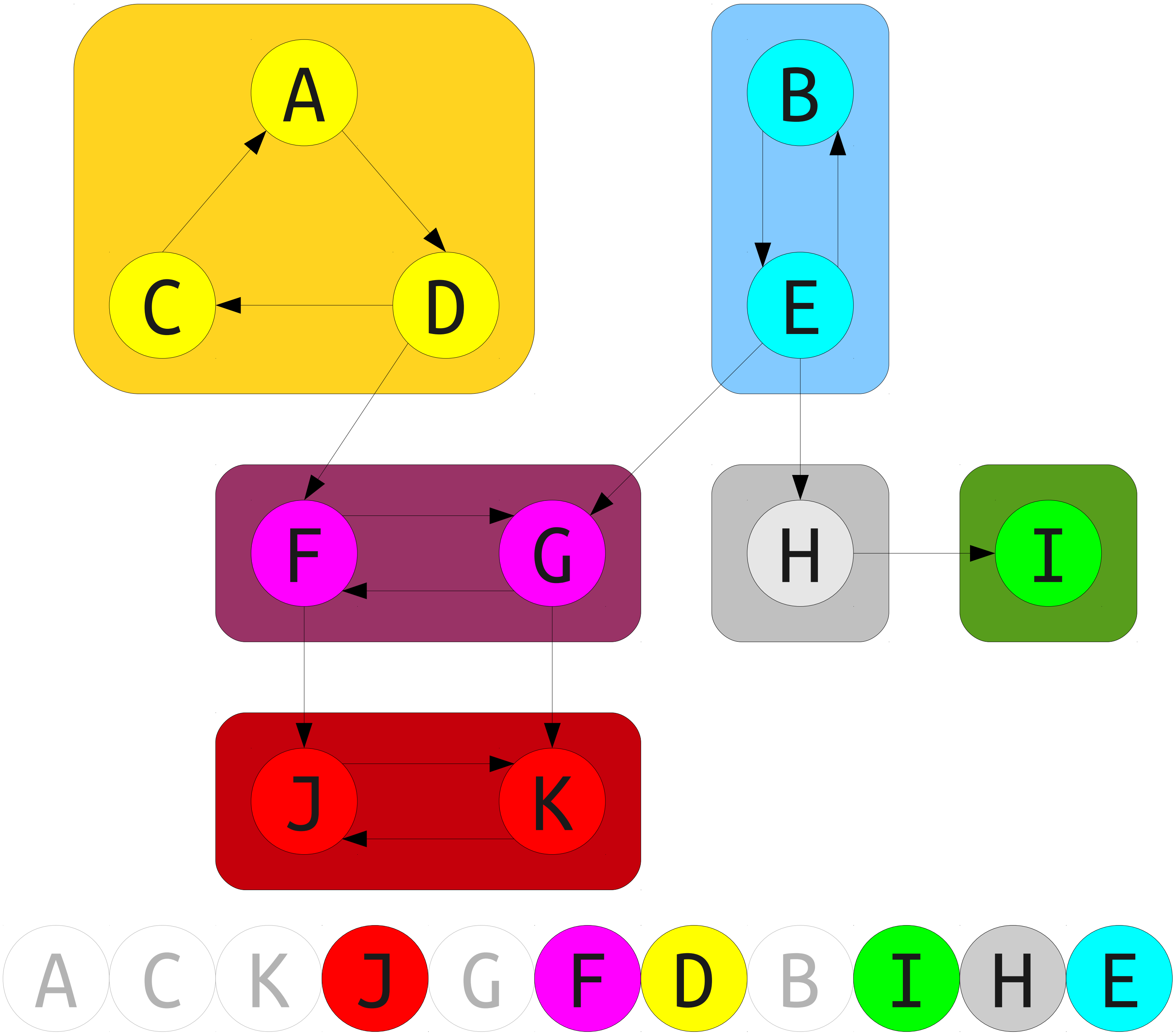
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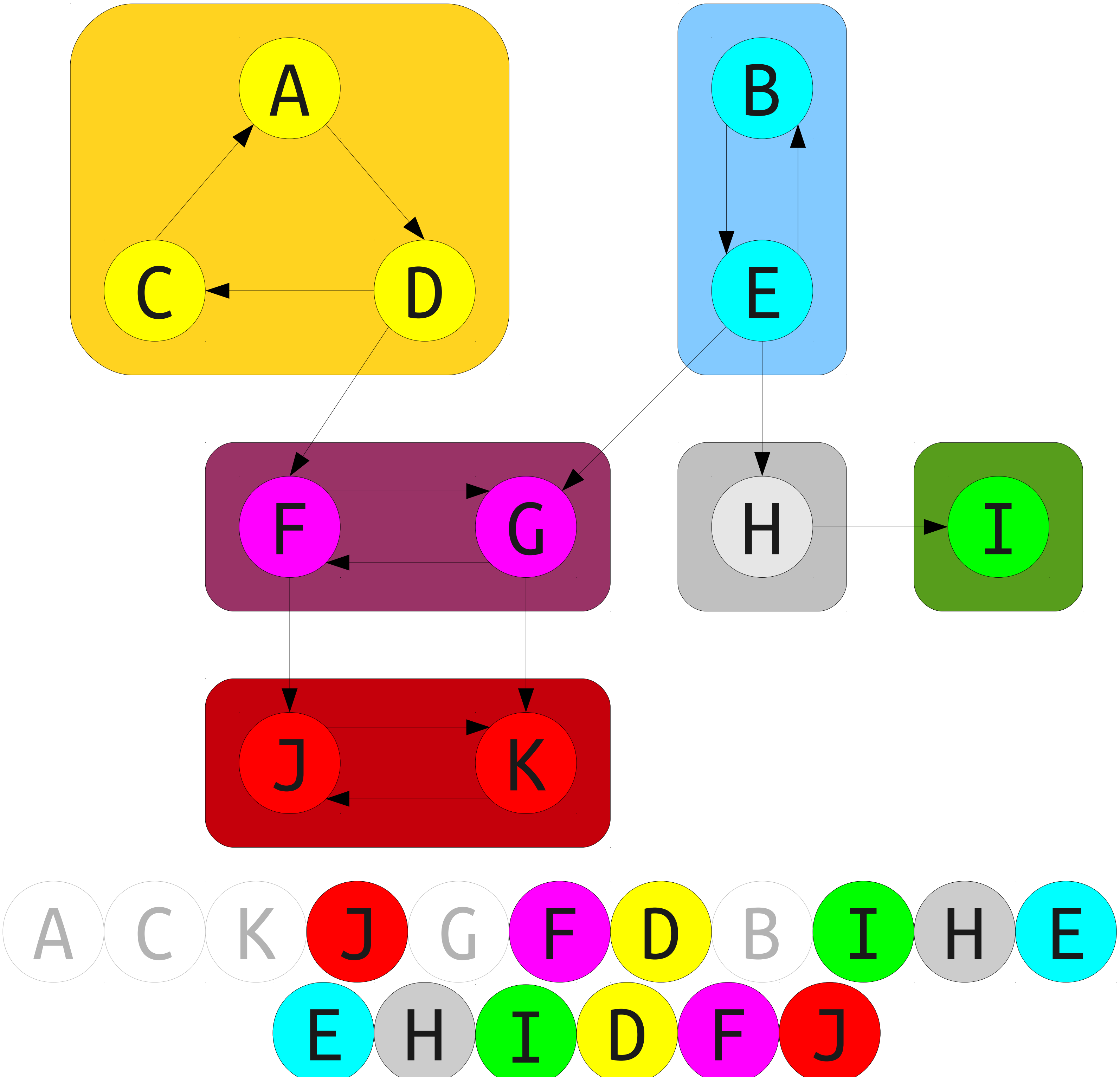

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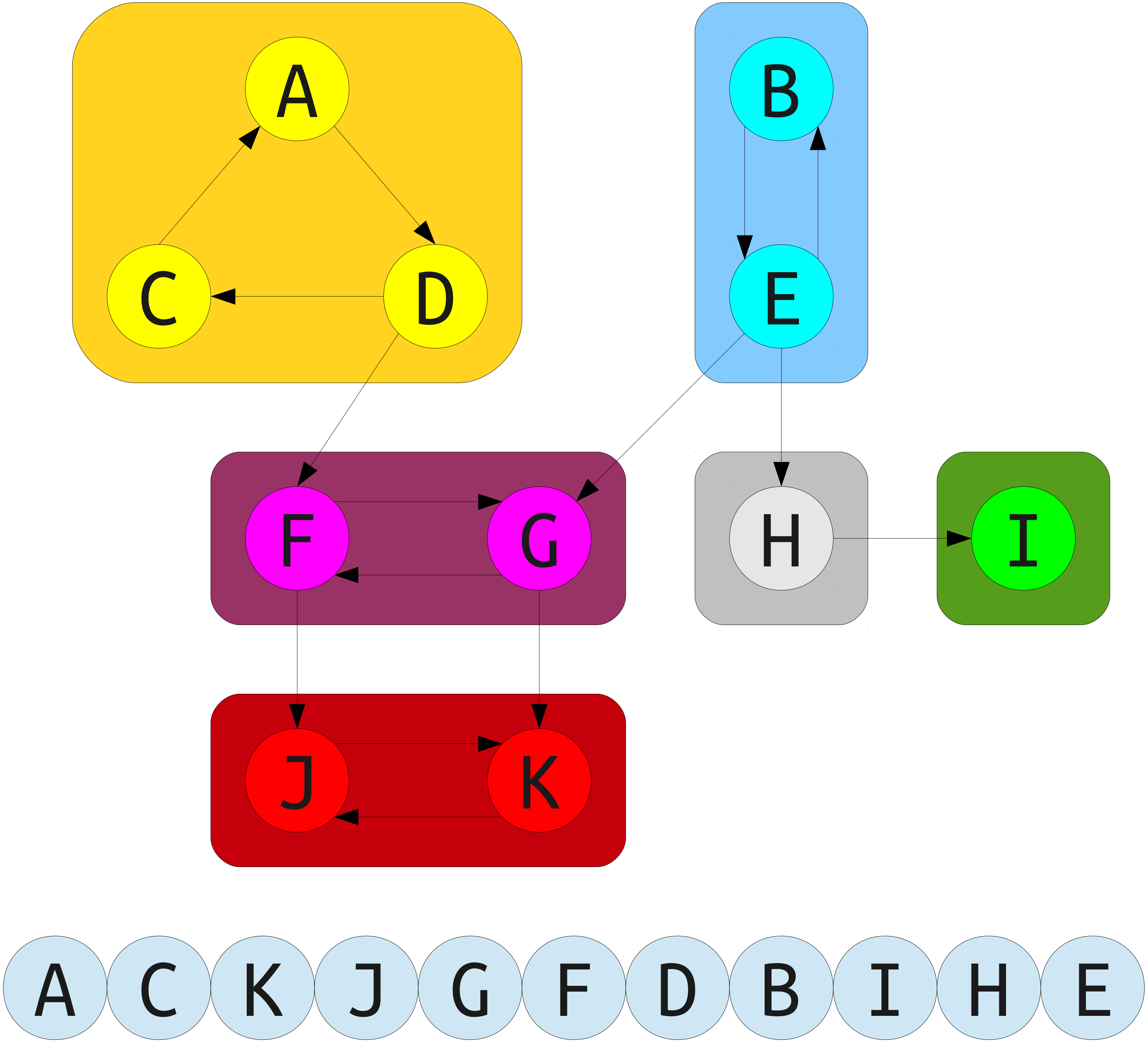
Topological Sort(ish)

- If we look purely at the *last* node from each SCC to turn green, we get a topological sort of G^{SCC} in reverse.
  - Here, each SCC is represented by a single node.
  - We proved this result last time.
- There's still a problem – we still don't have a way of identifying the last node of each SCC!
- We do have one foothold, though…
- **Onward to new content!**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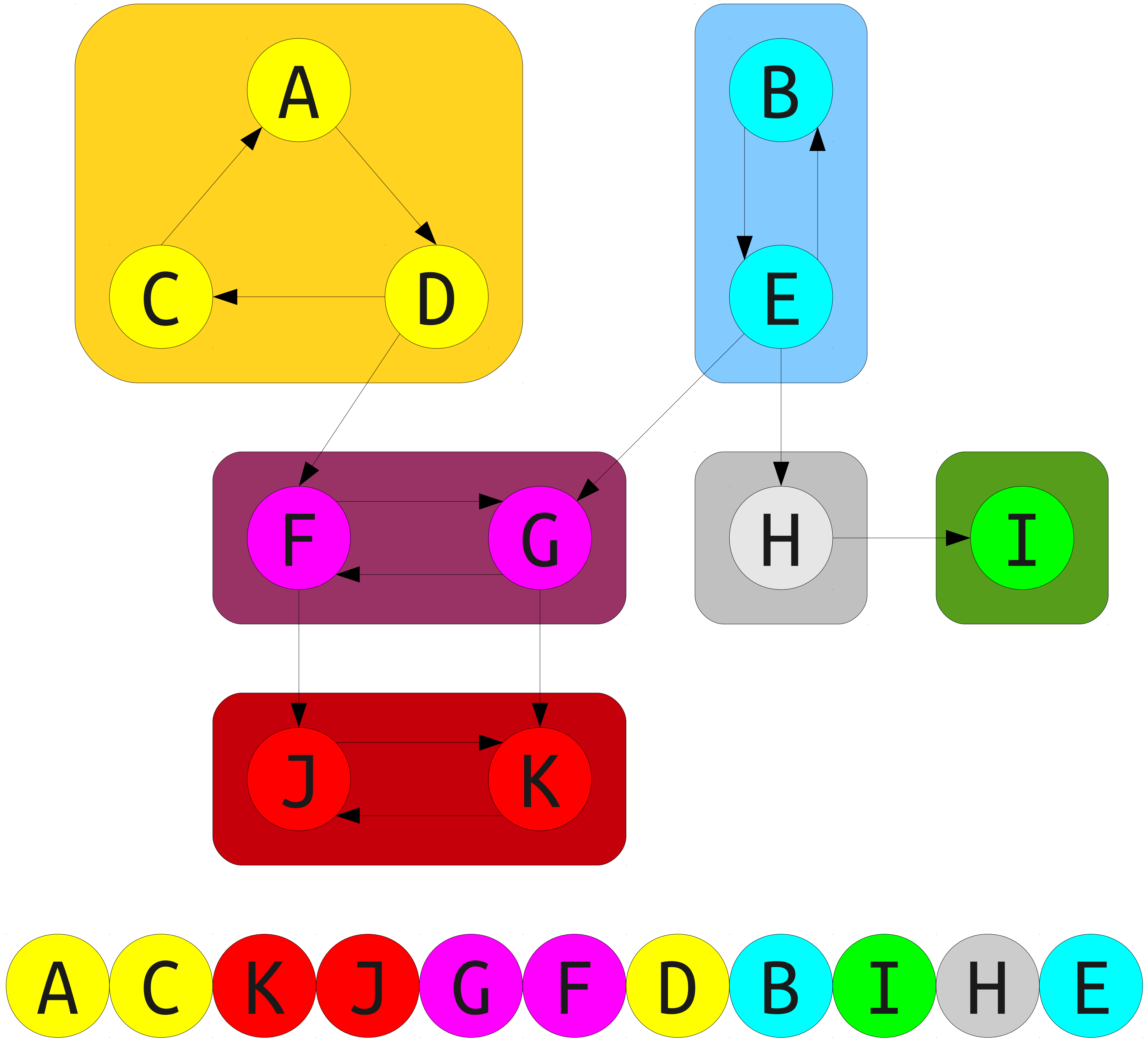
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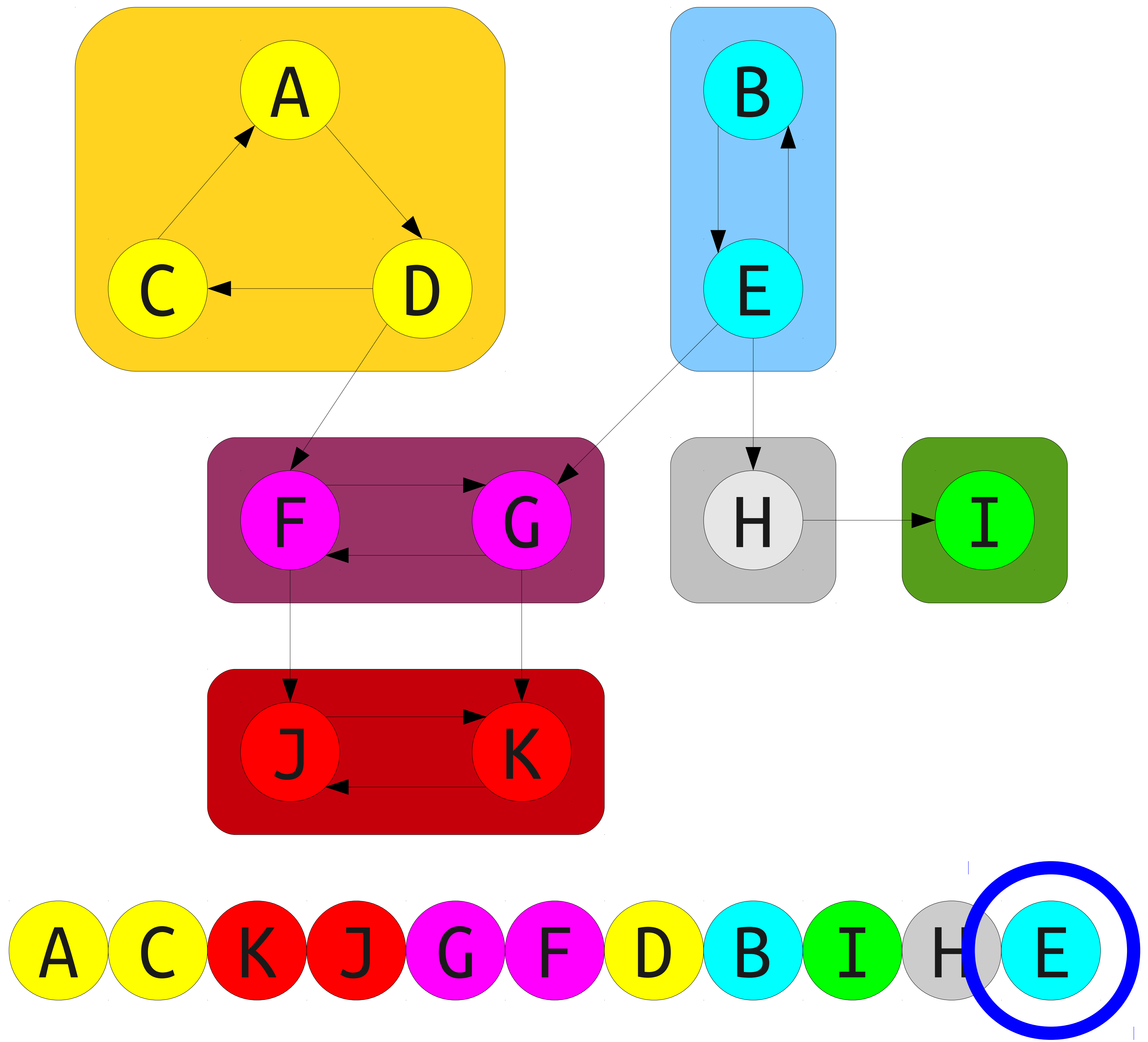
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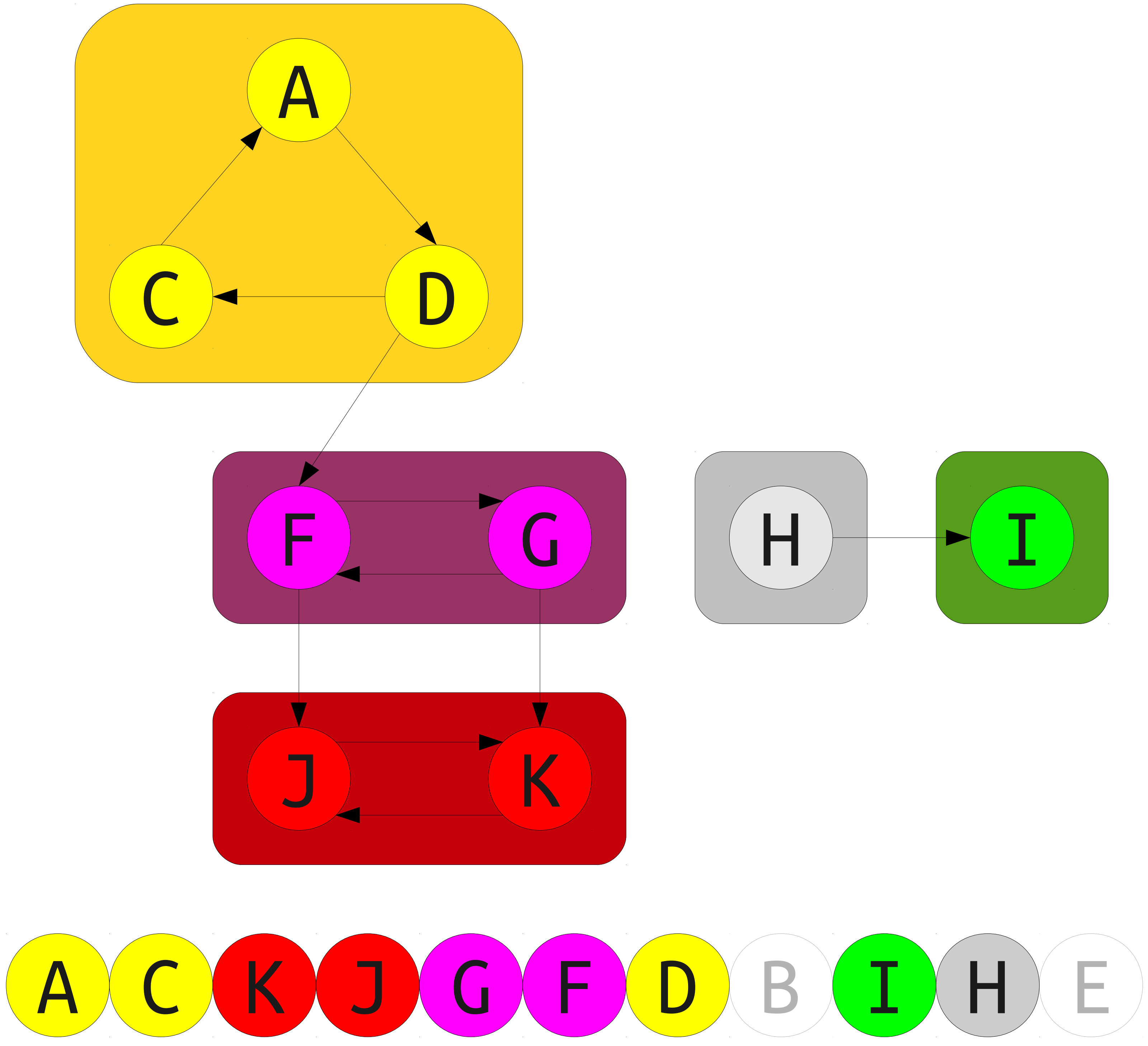
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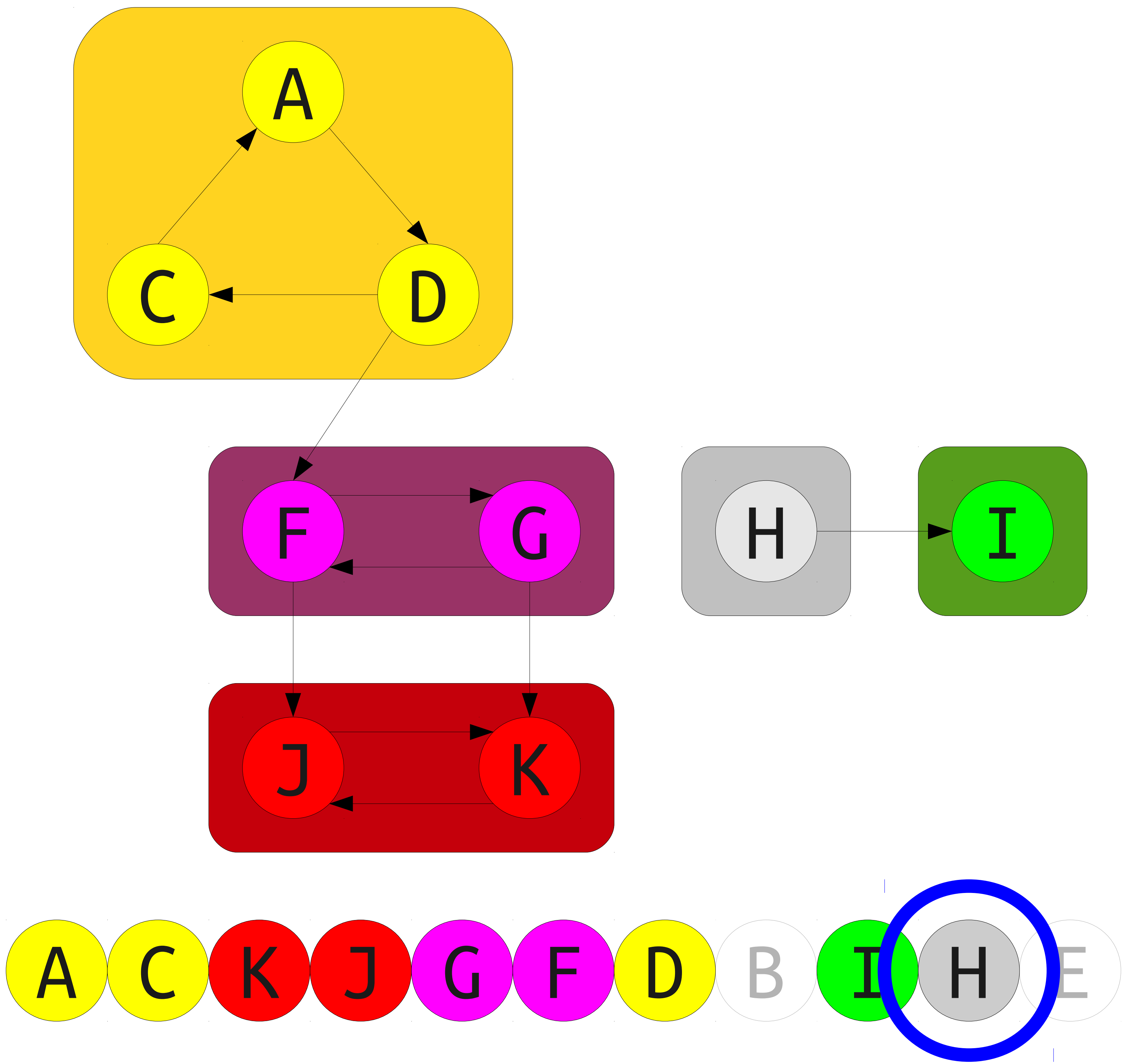
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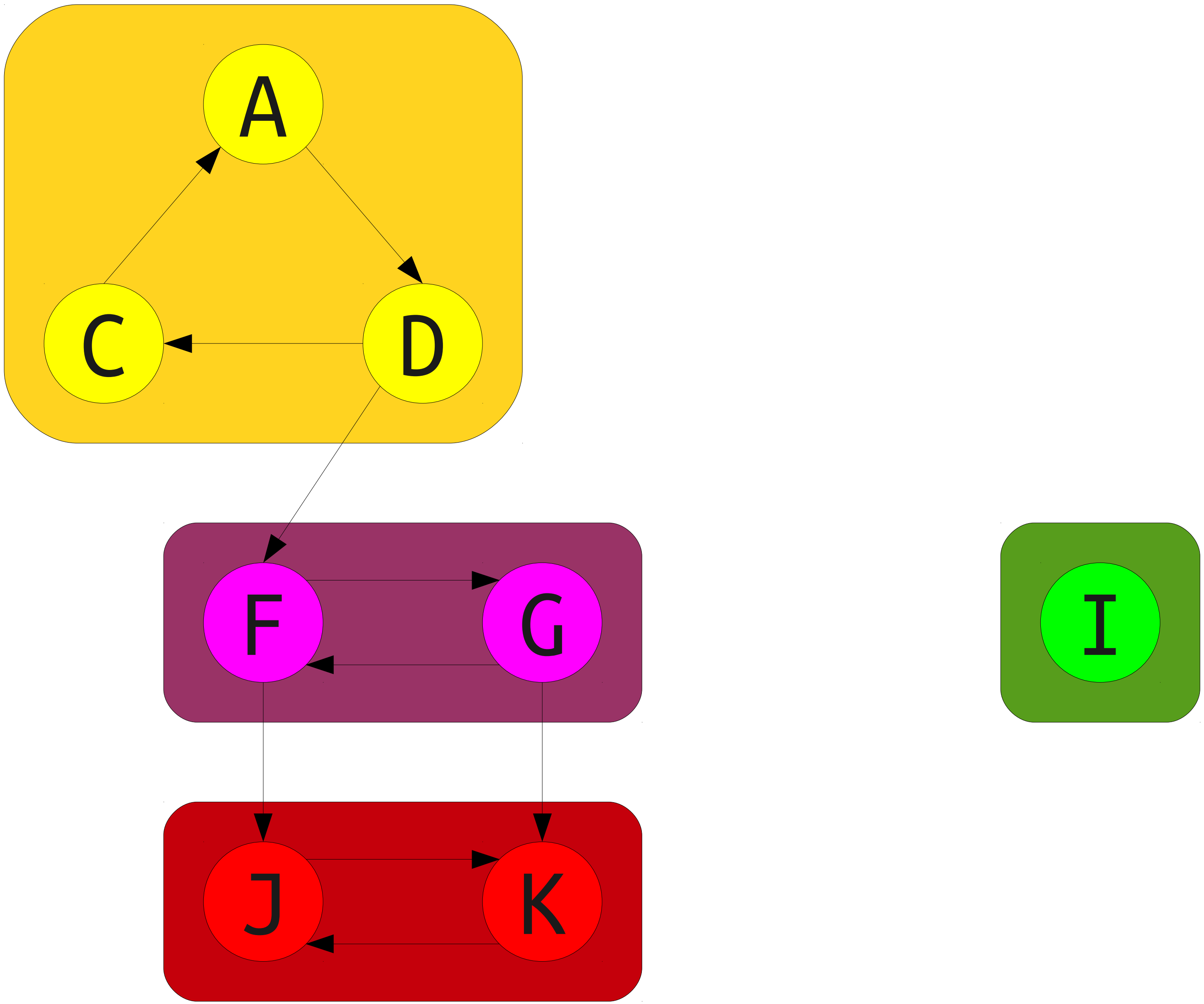
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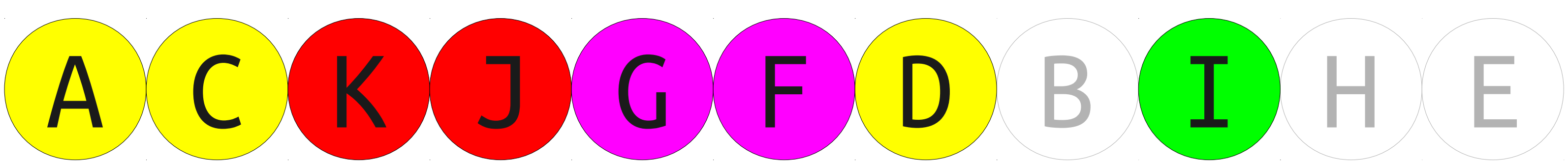
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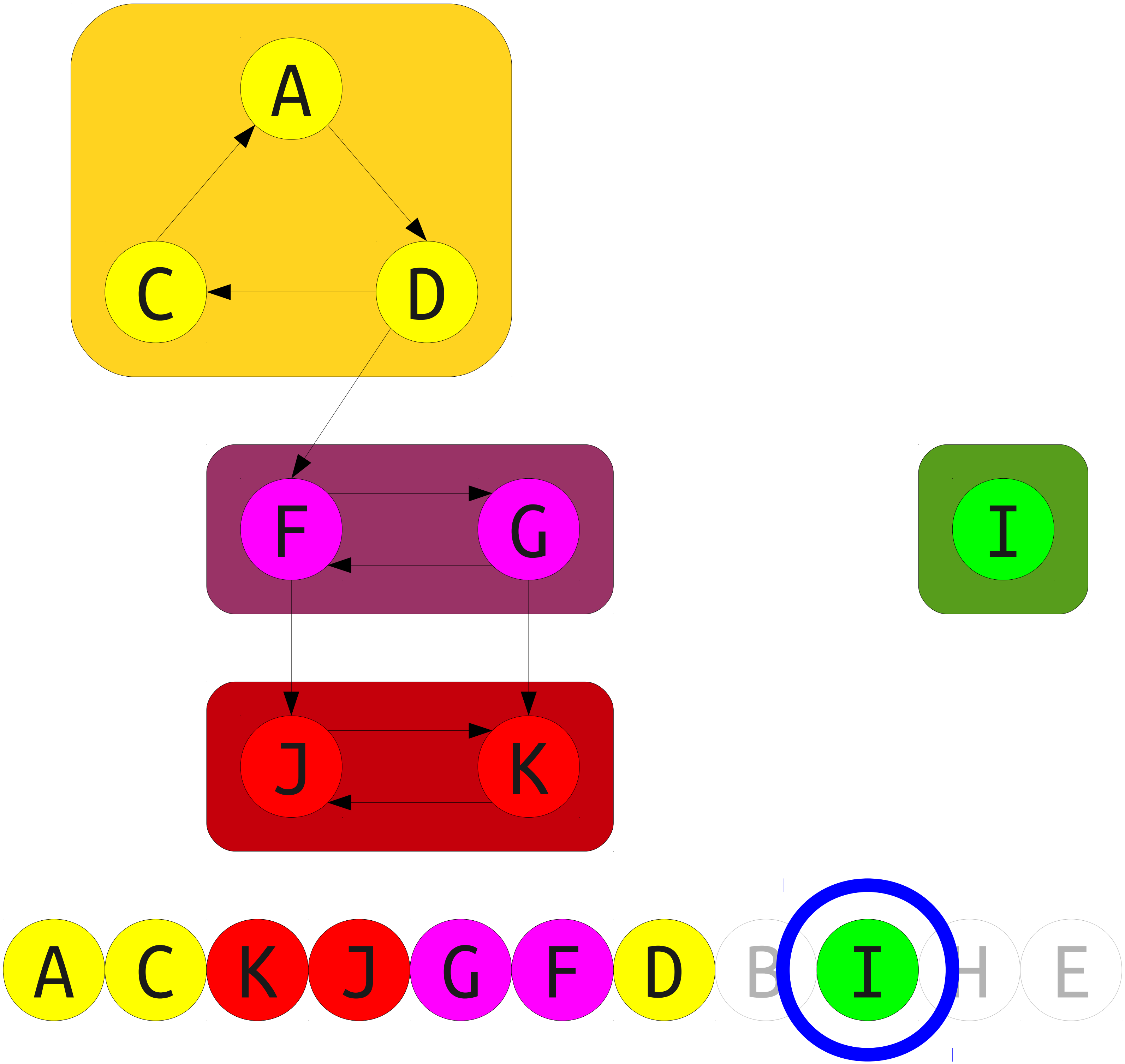

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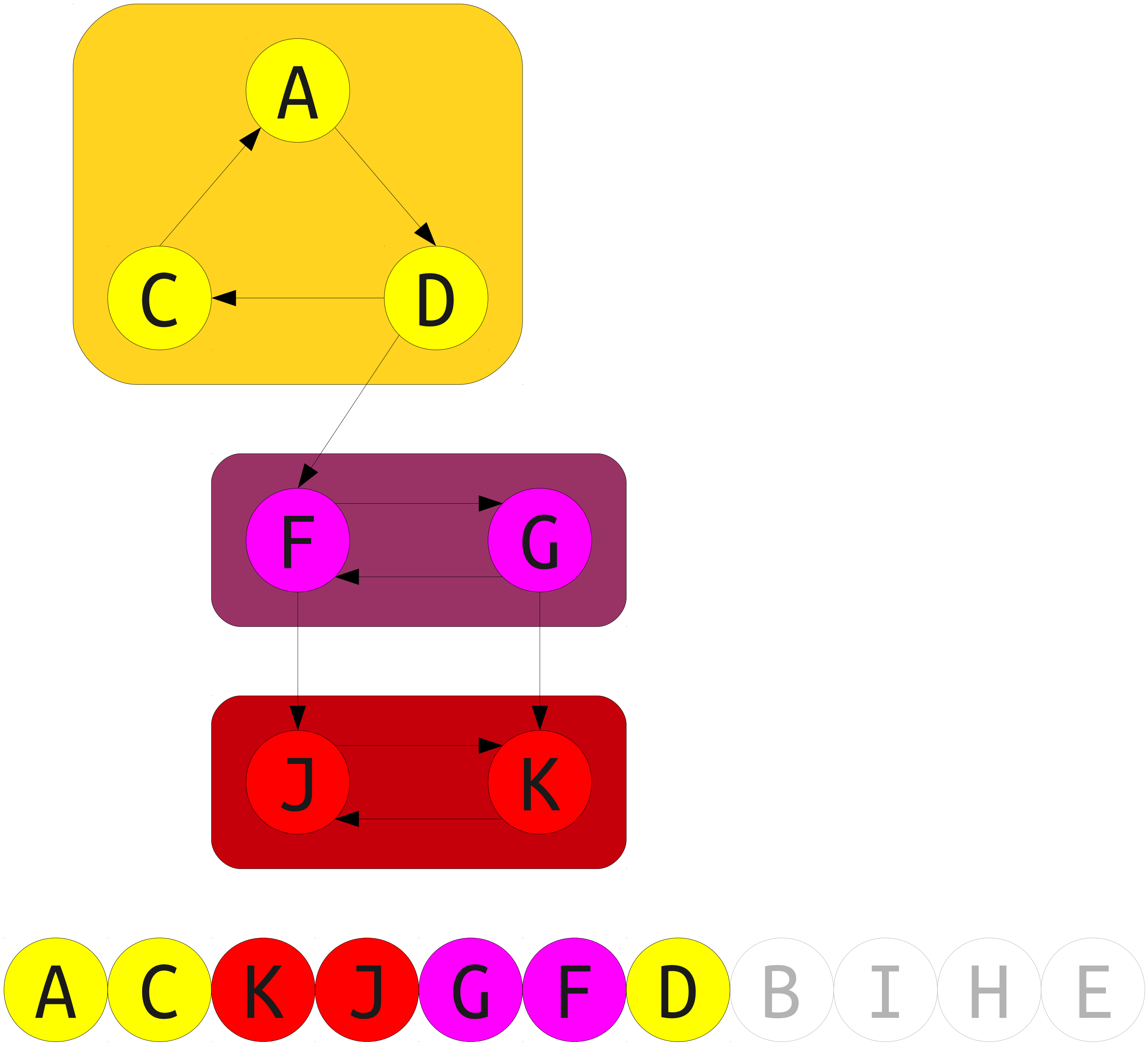
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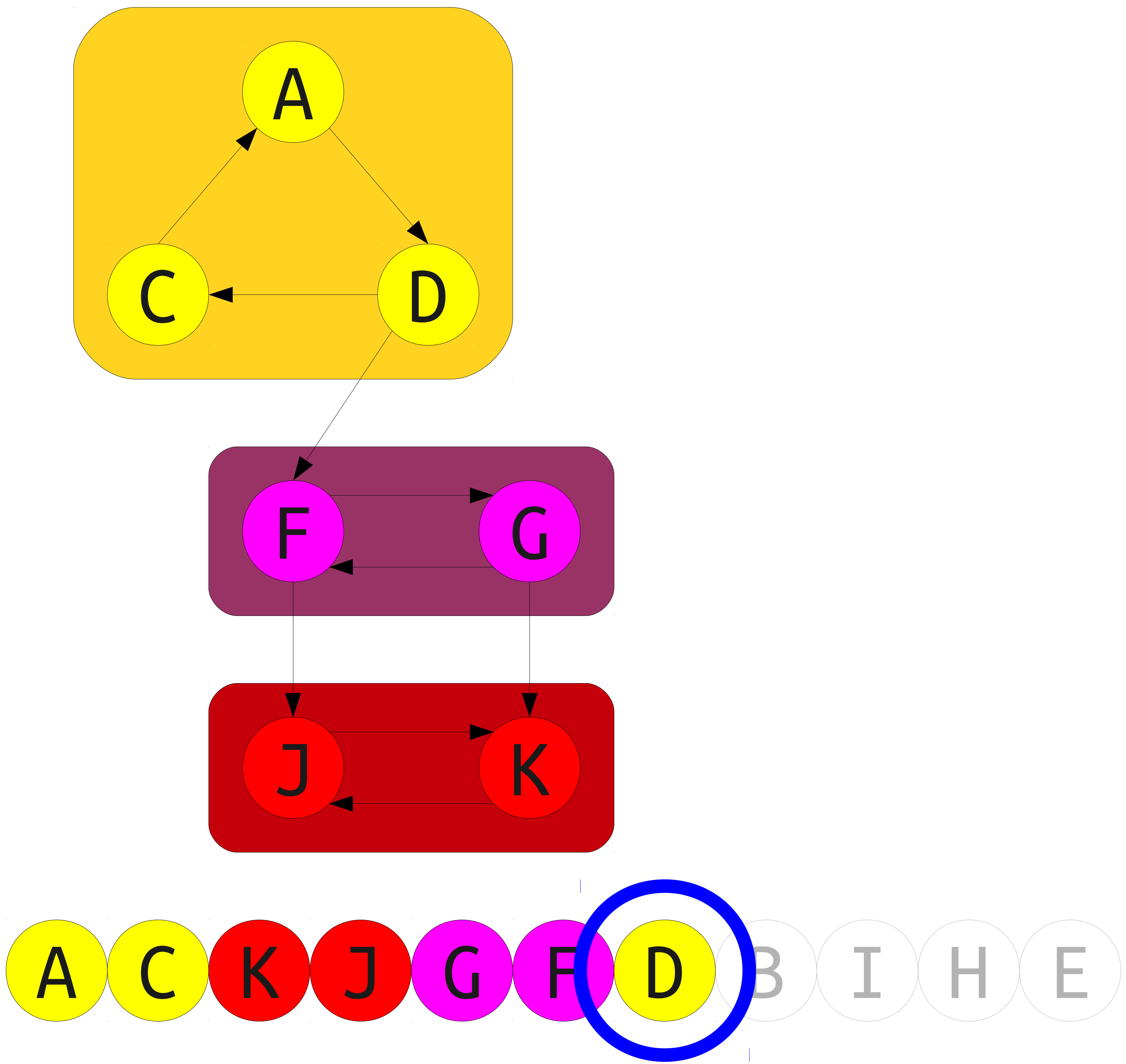
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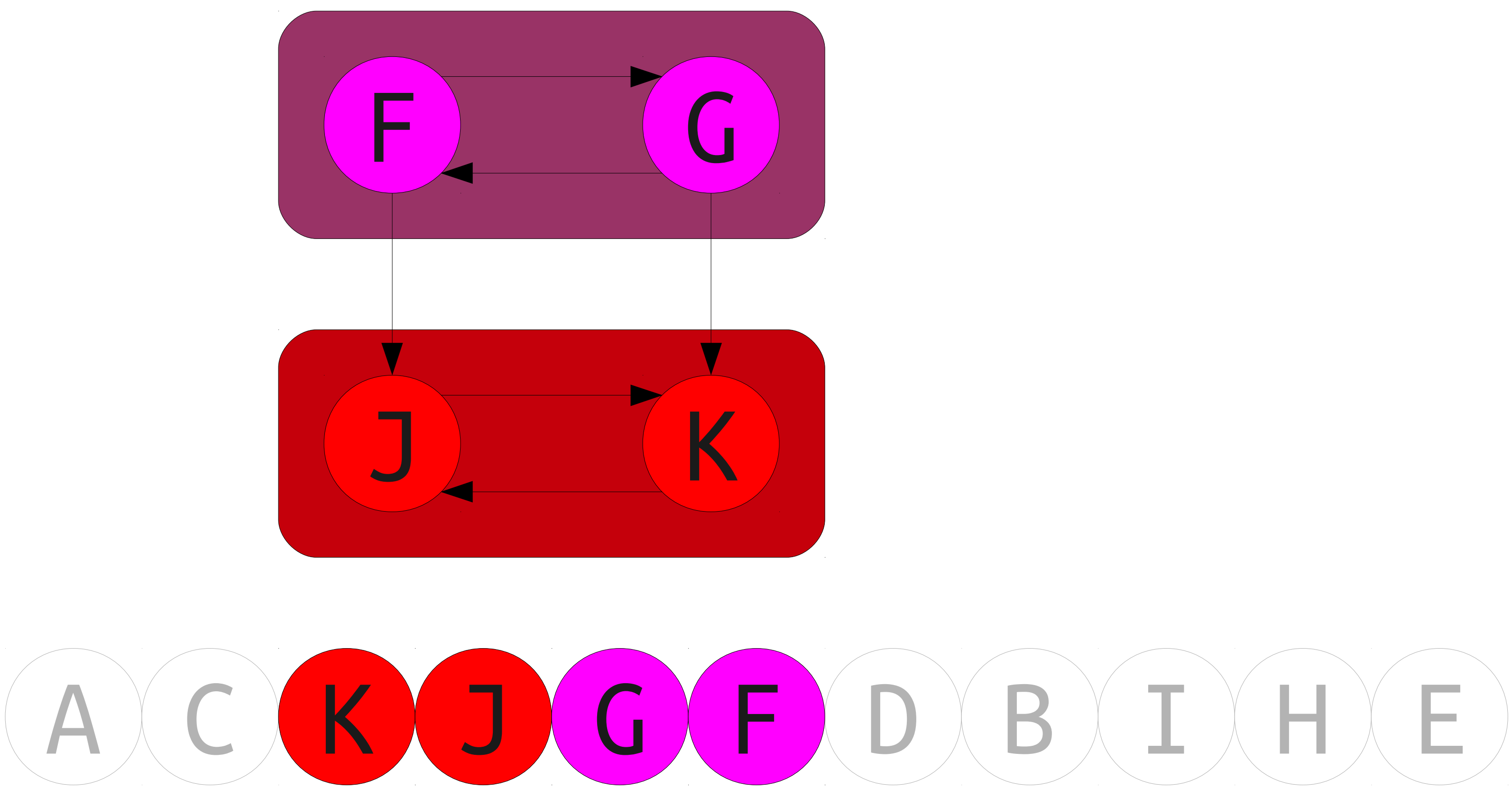
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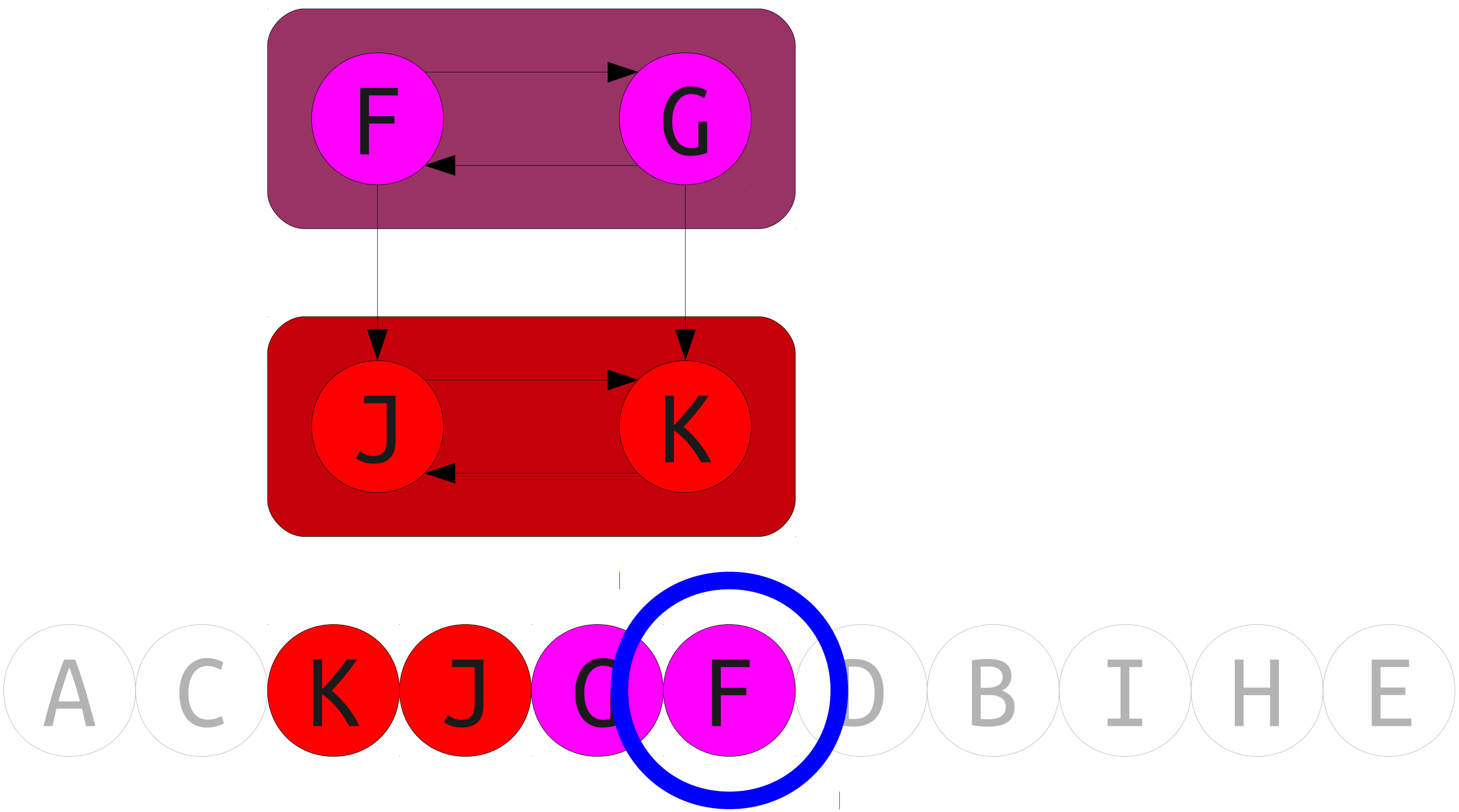
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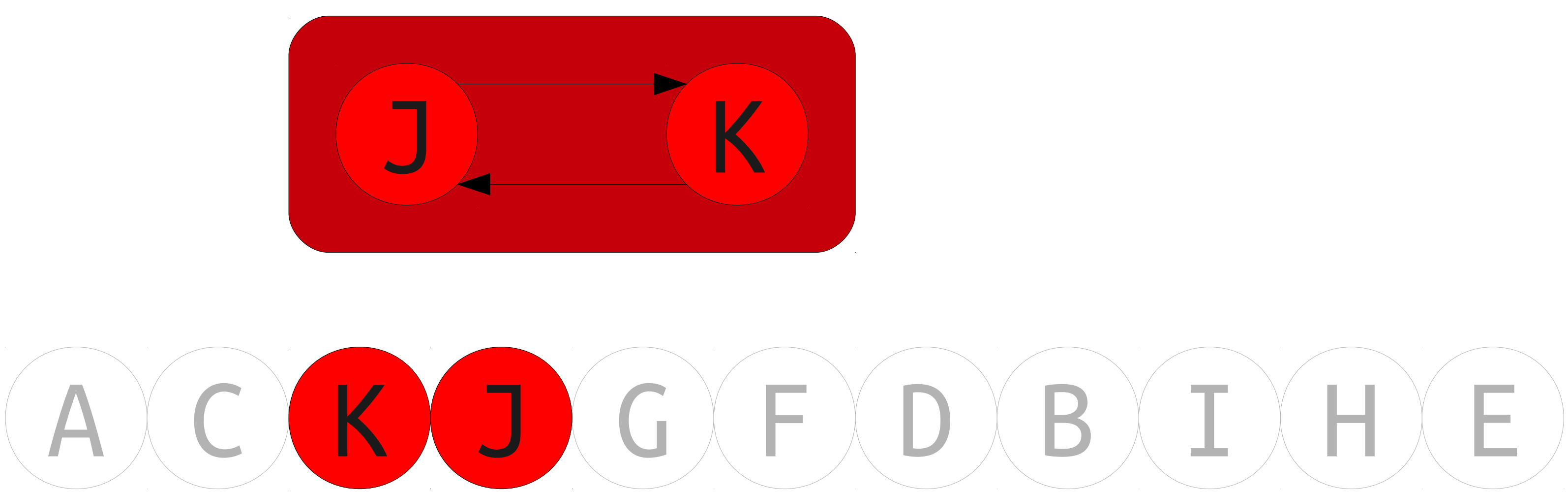
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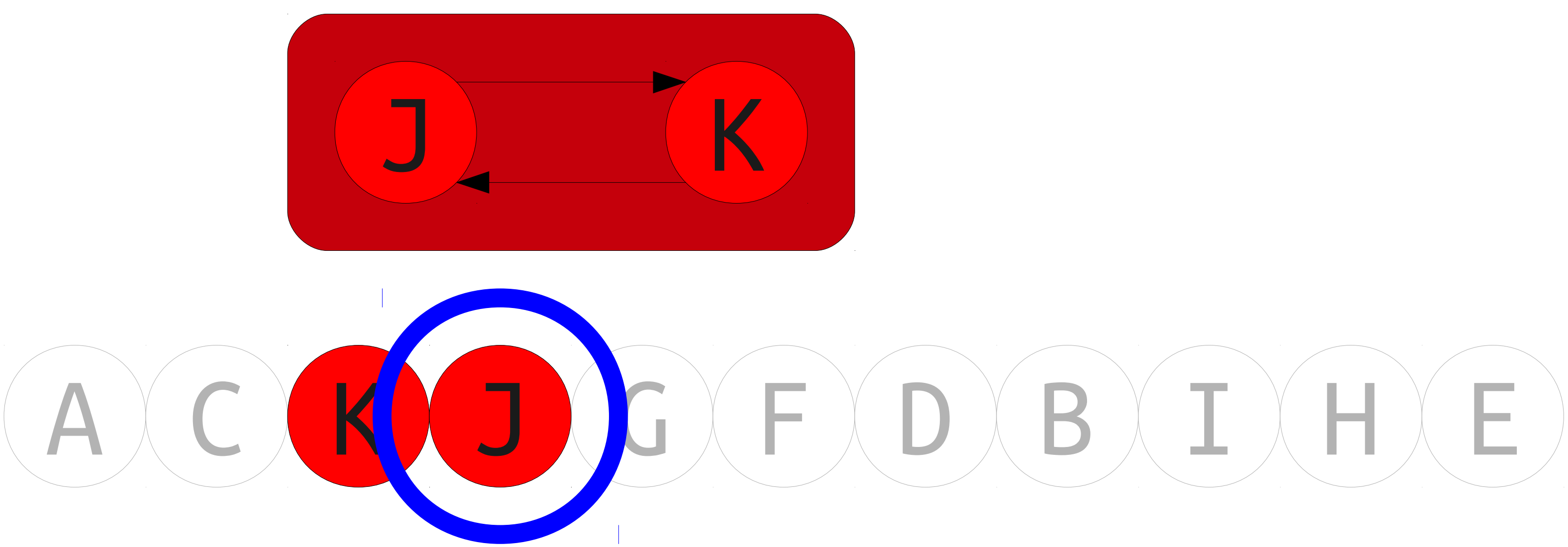

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Making Progress!

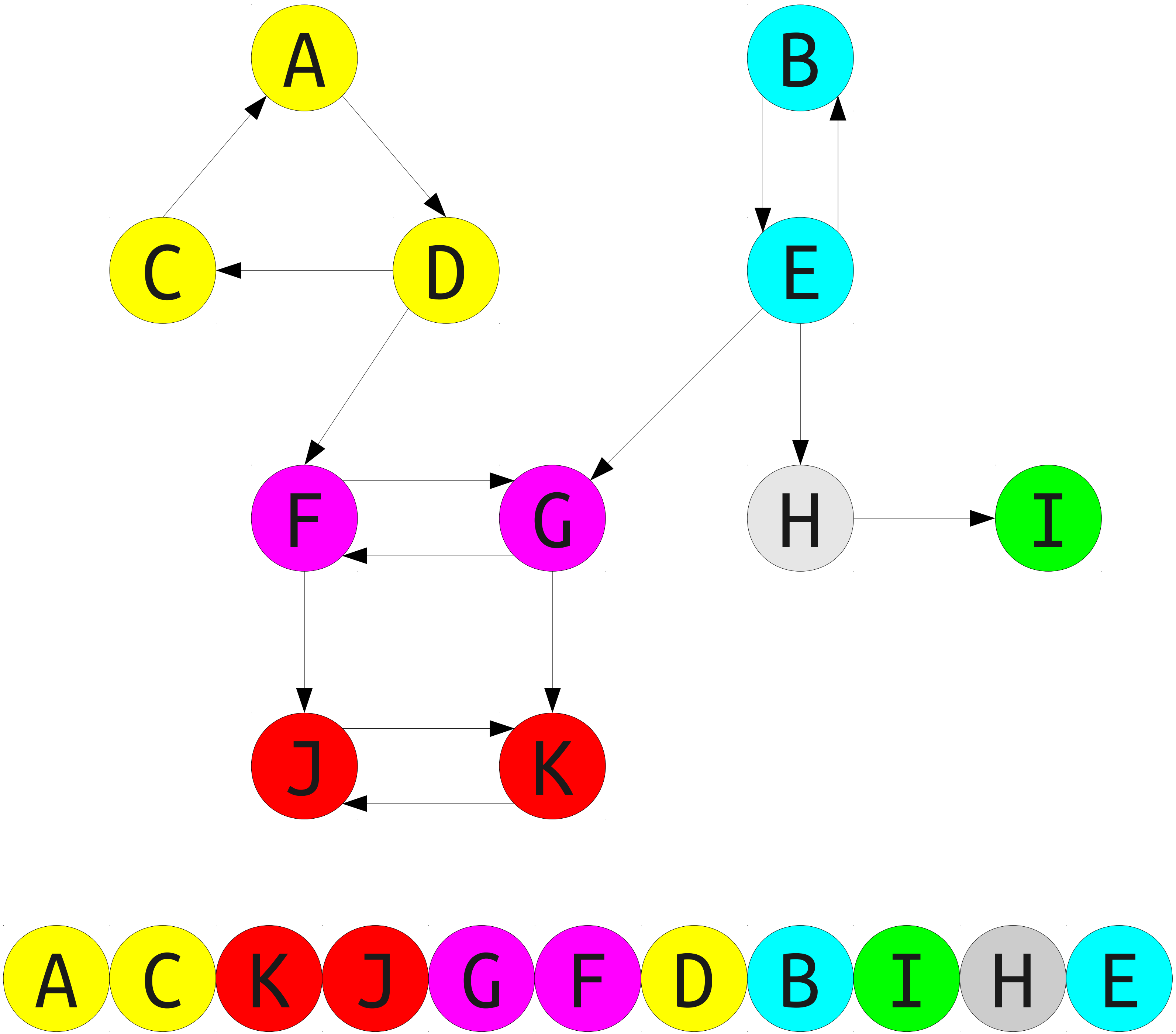
- The last node colored green by DFS must be the last node colored green in some SCC.
- This gives a rough idea for an algorithm:
  - Take the last node in the ordering that hasn't already been put into an SCC.
  - Find all nodes in the same SCC as that node.
  - Repeat.

Making Progress!

The last node colored green by DFS must be the last node colored green in some SCC.

This gives a rough idea for an algorithm:

- Take the last node in the ordering that hasn't already been put into an SCC.
- Find all nodes in the same SCC as that node.
- Repeat.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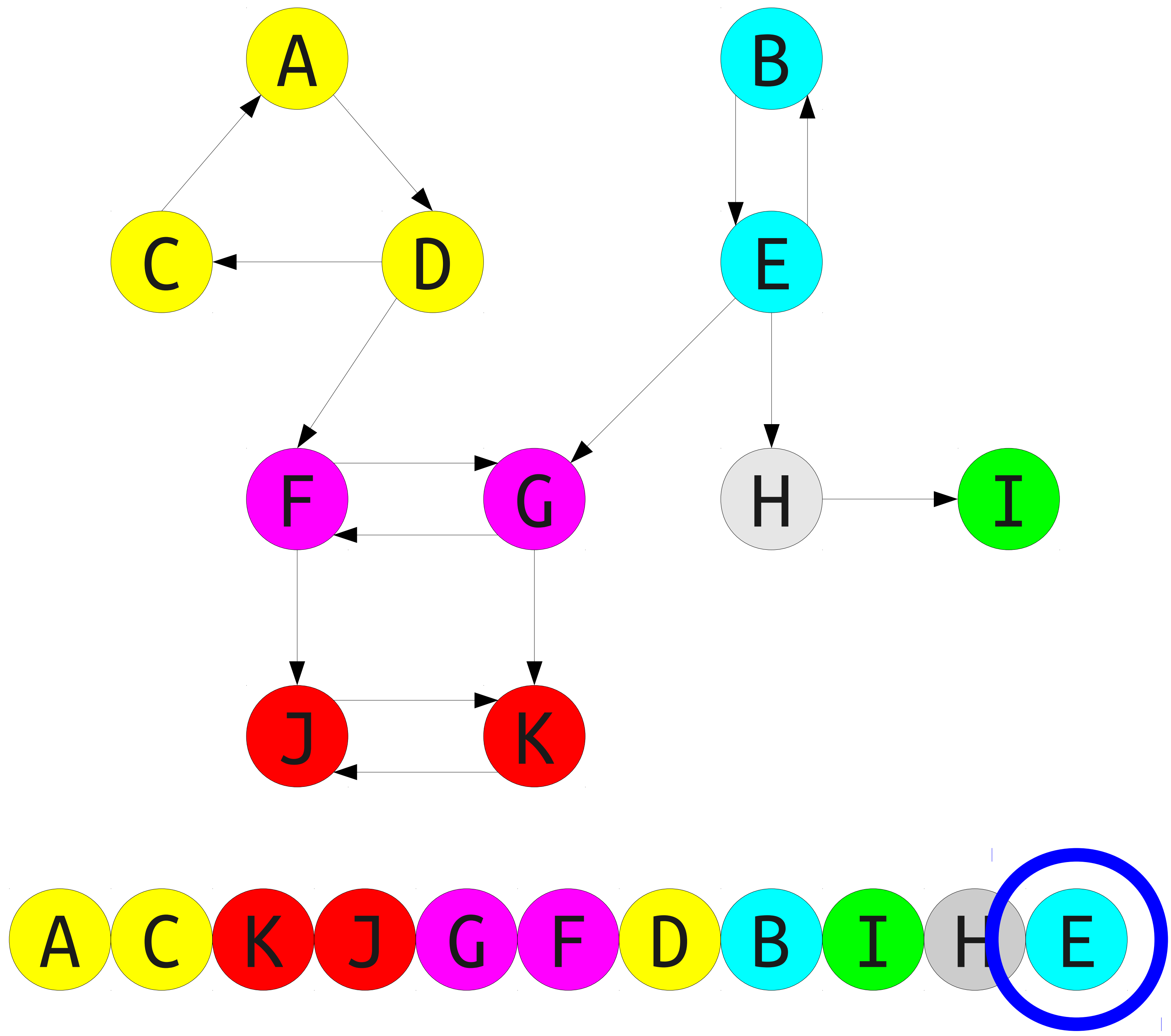
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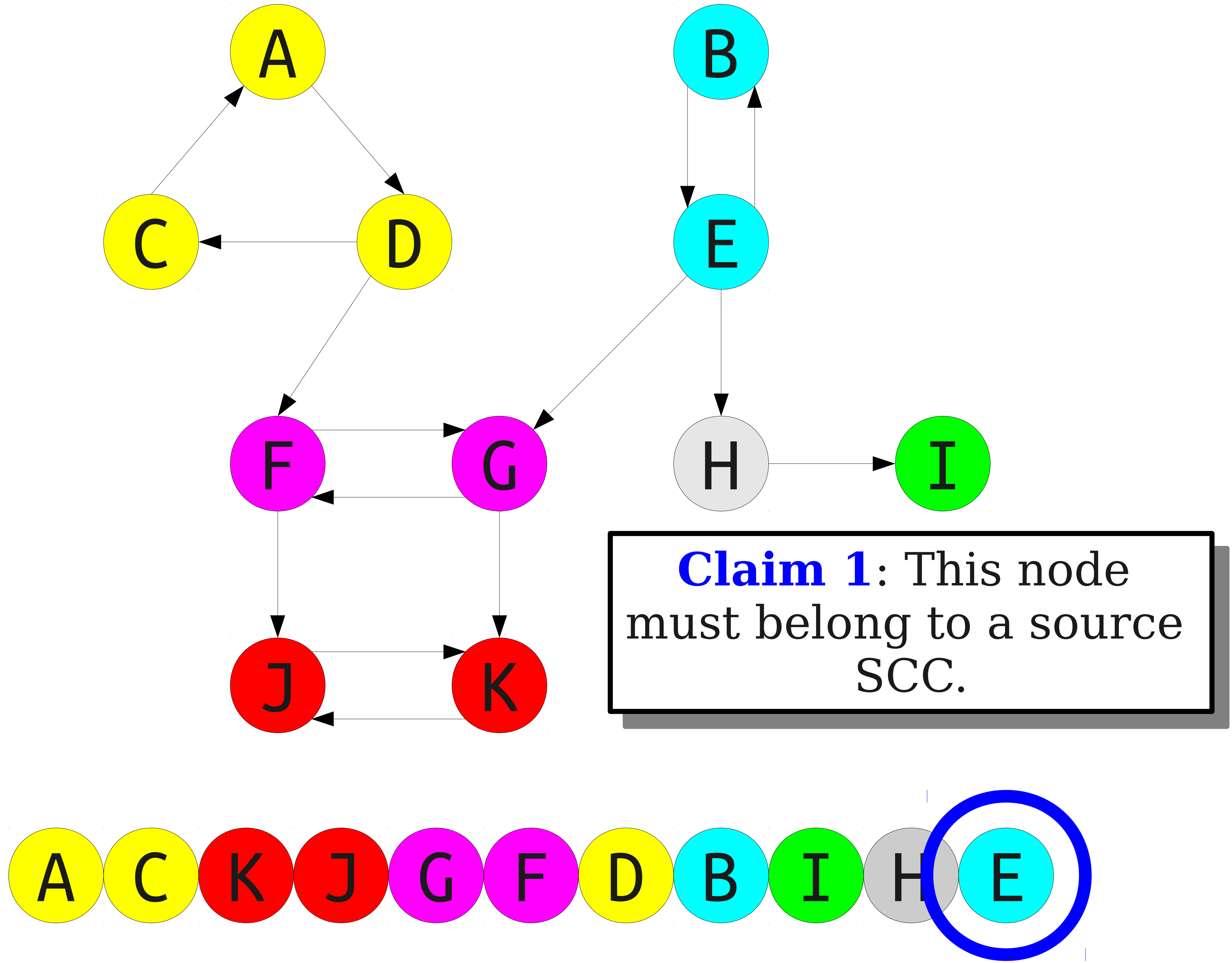
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Claim 1: This node must belong to a source SCC.
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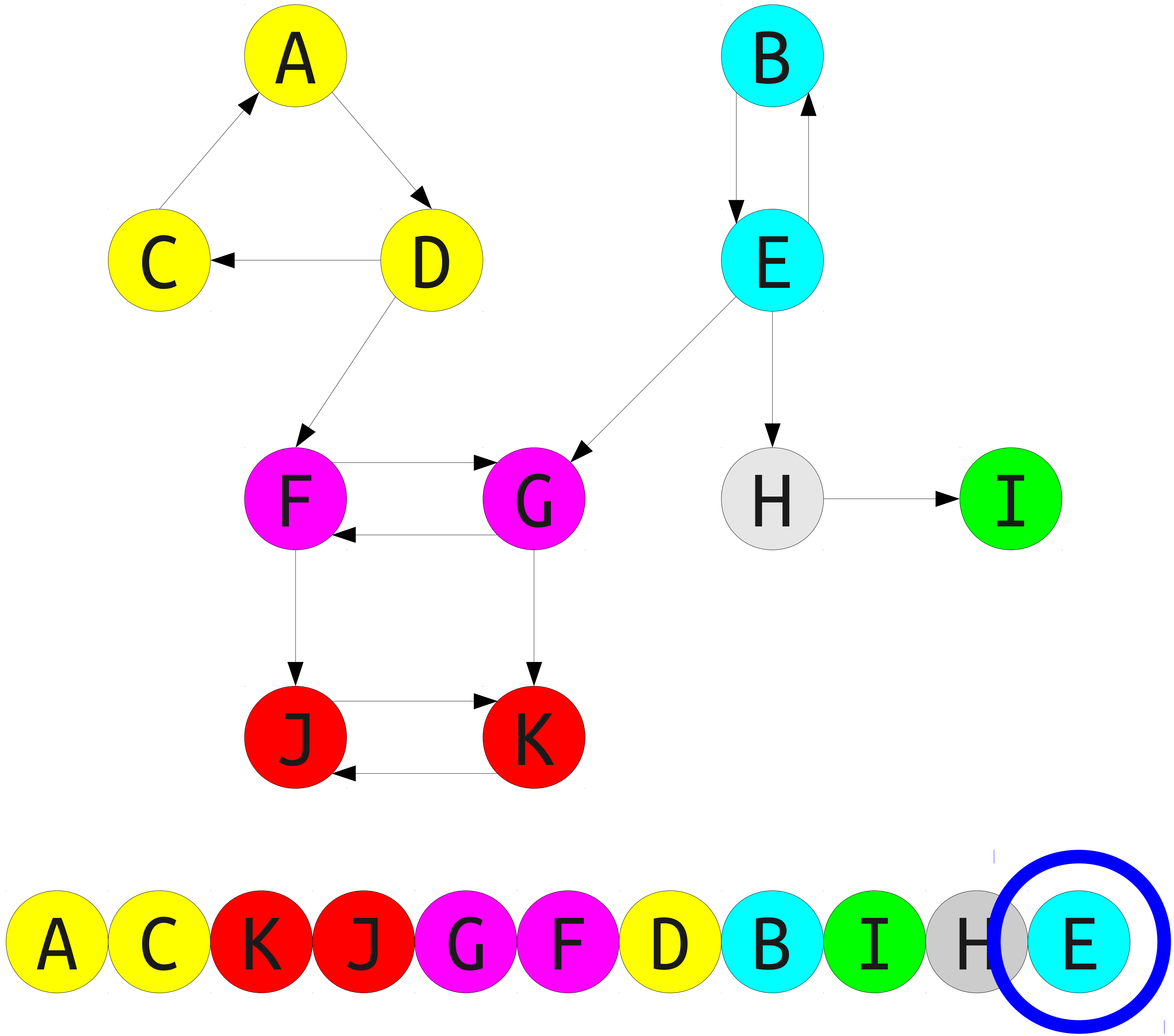
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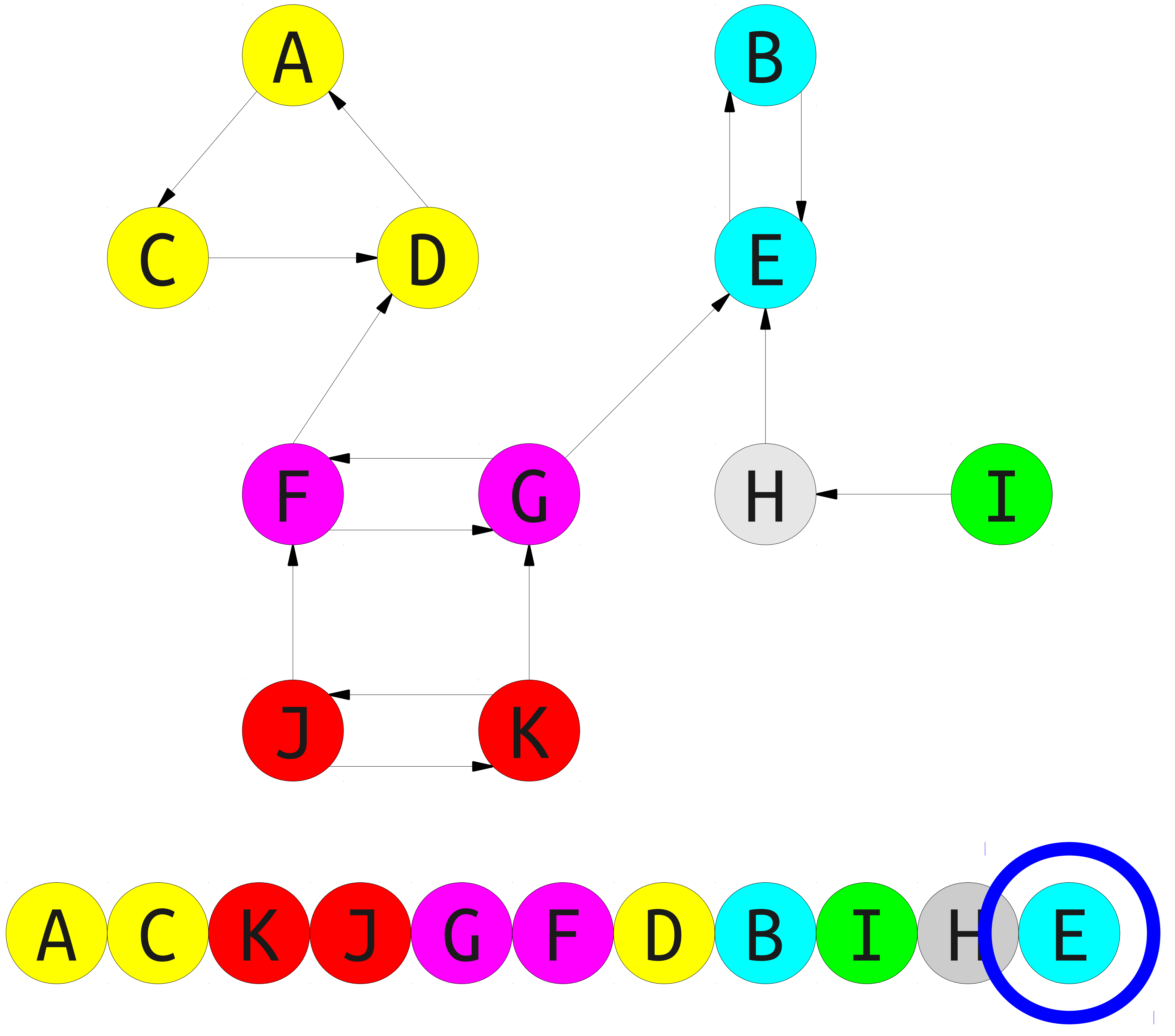
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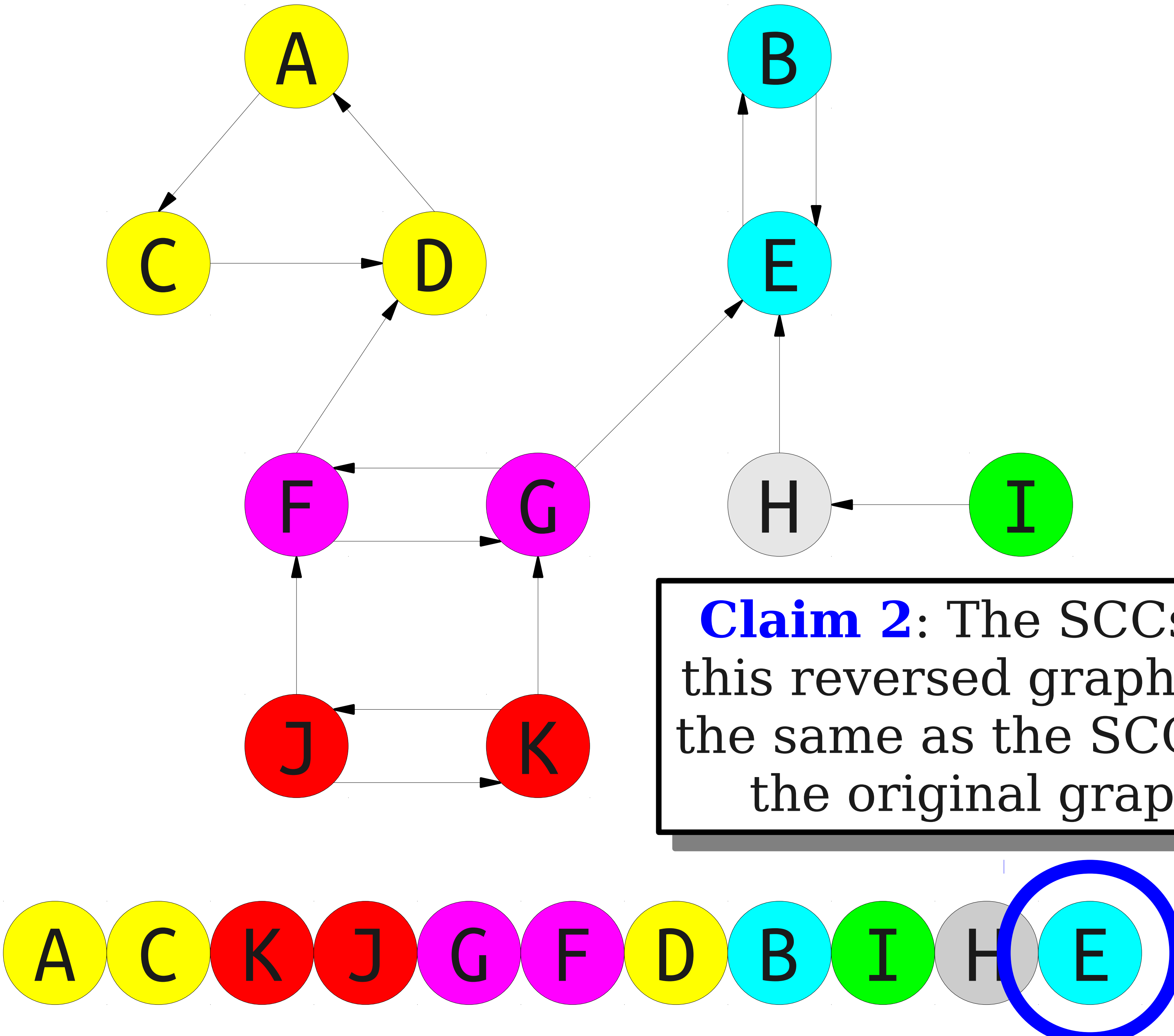
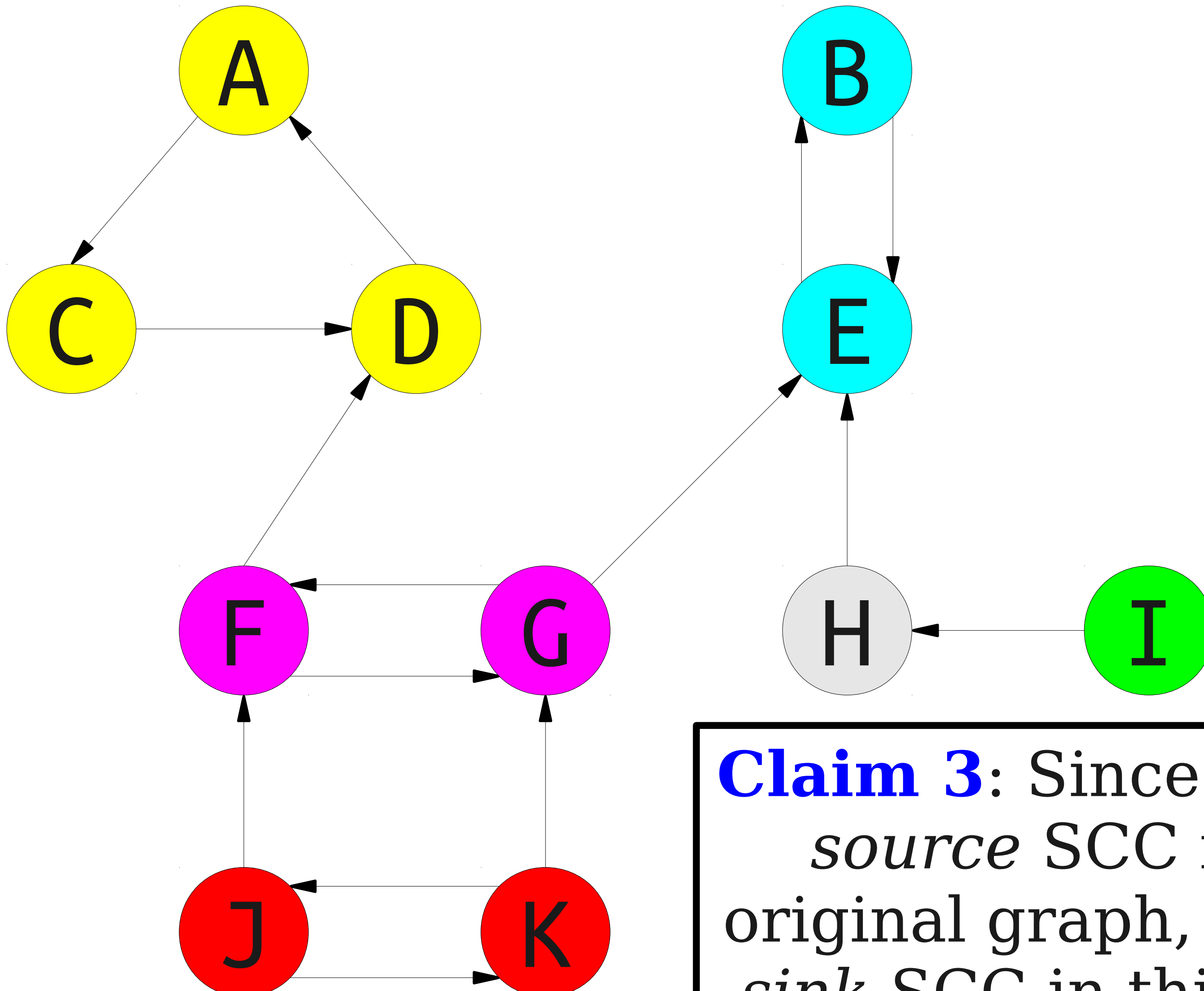
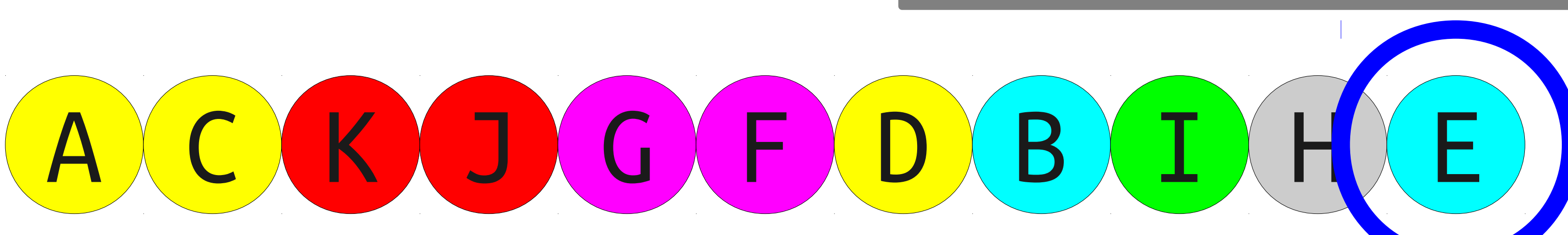
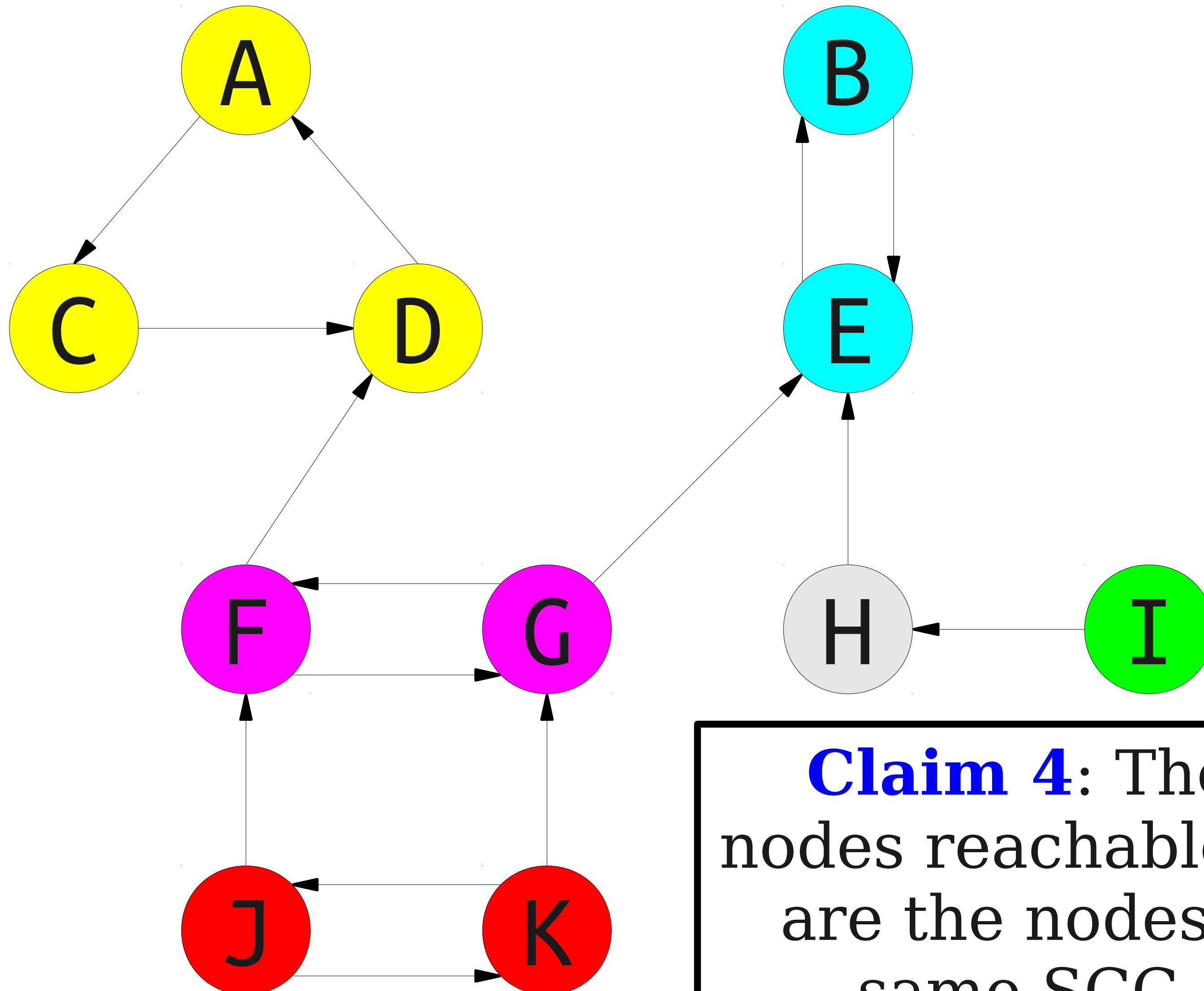


Claim 2: The SCCs of this reversed graph are the same as the SCCs of the original graph.

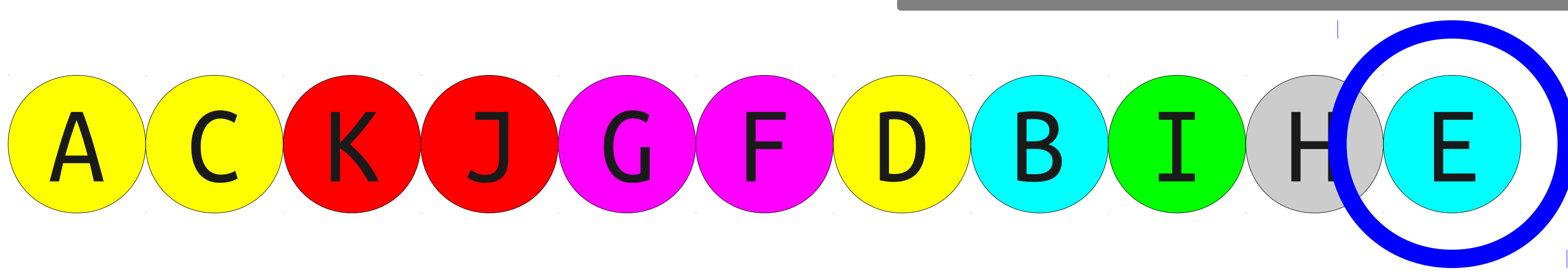


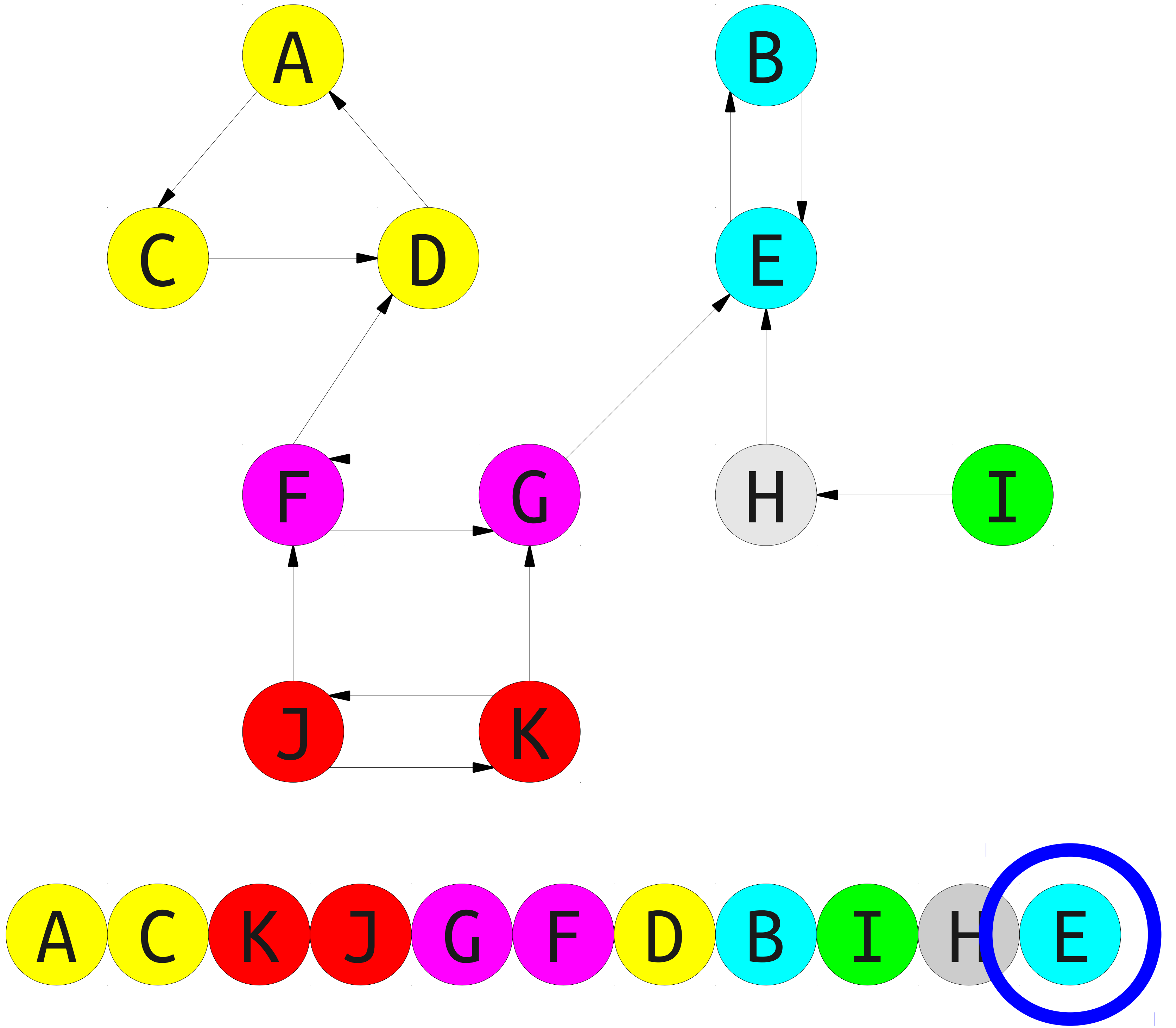
Claim 3: Since **E** is in a *source* SCC in the original graph, **E** is in a *sink* SCC in this graph.





Claim 4: The only nodes reachable from **E** are the nodes in the same SCC as **E**.

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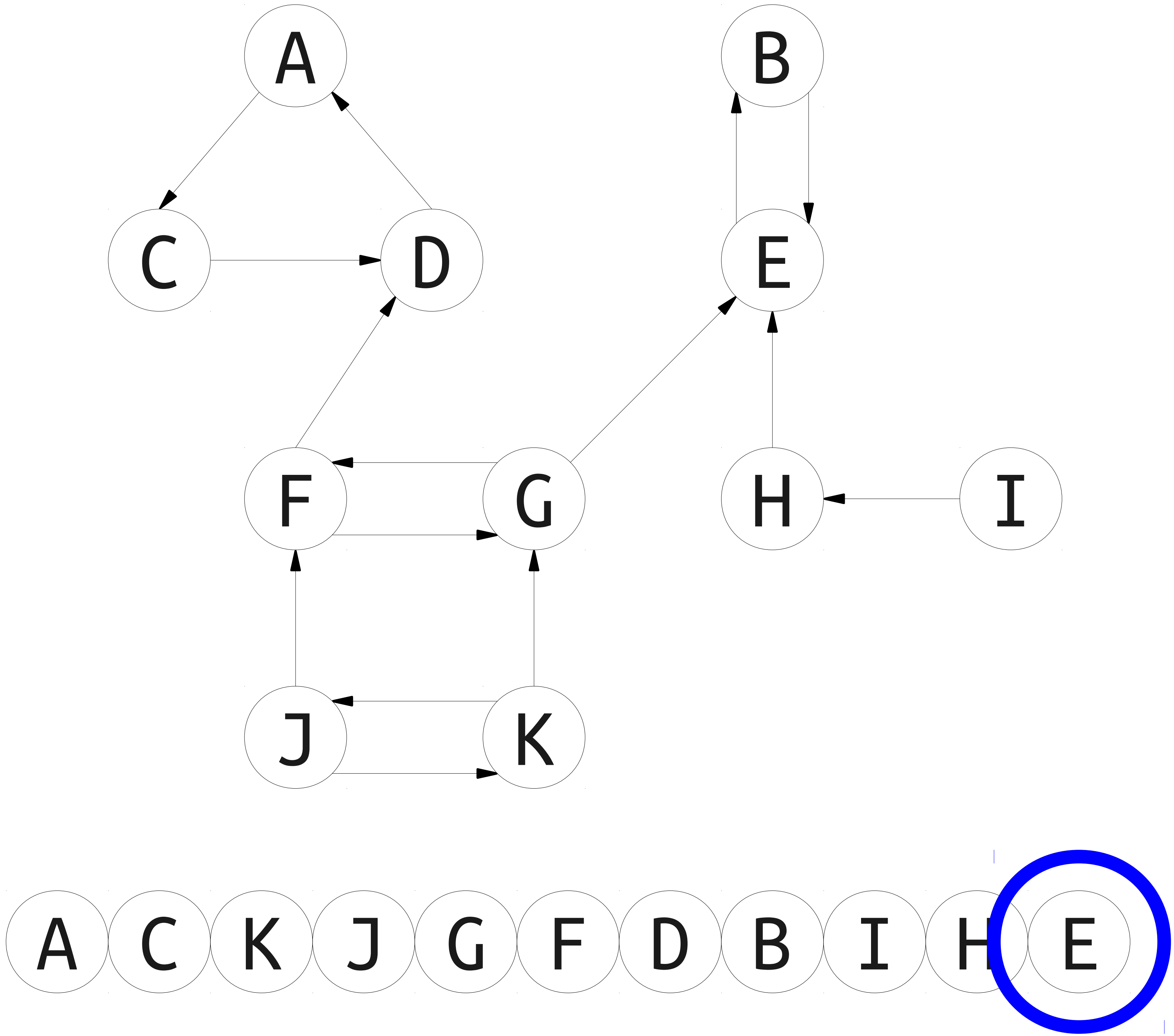
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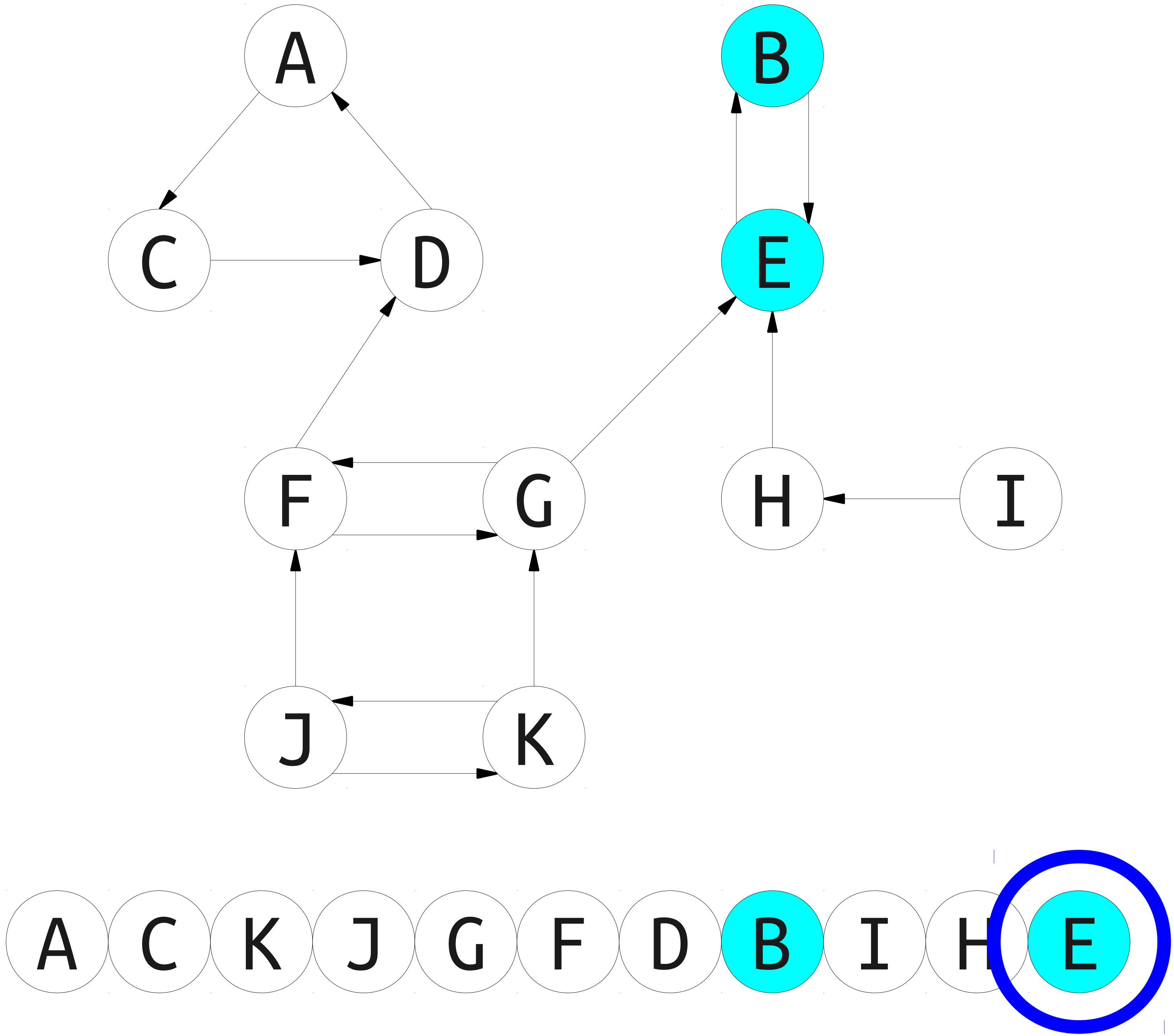
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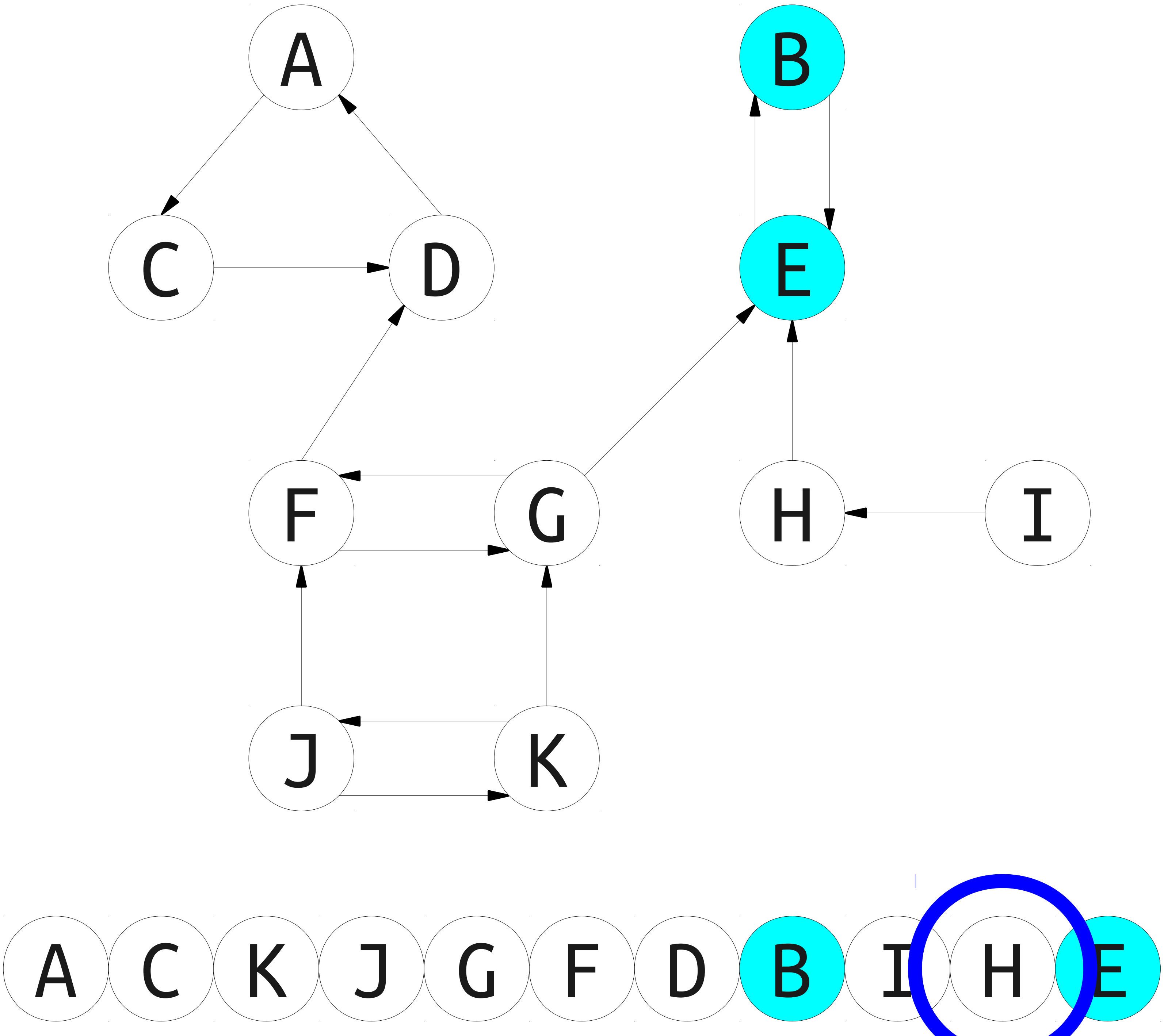
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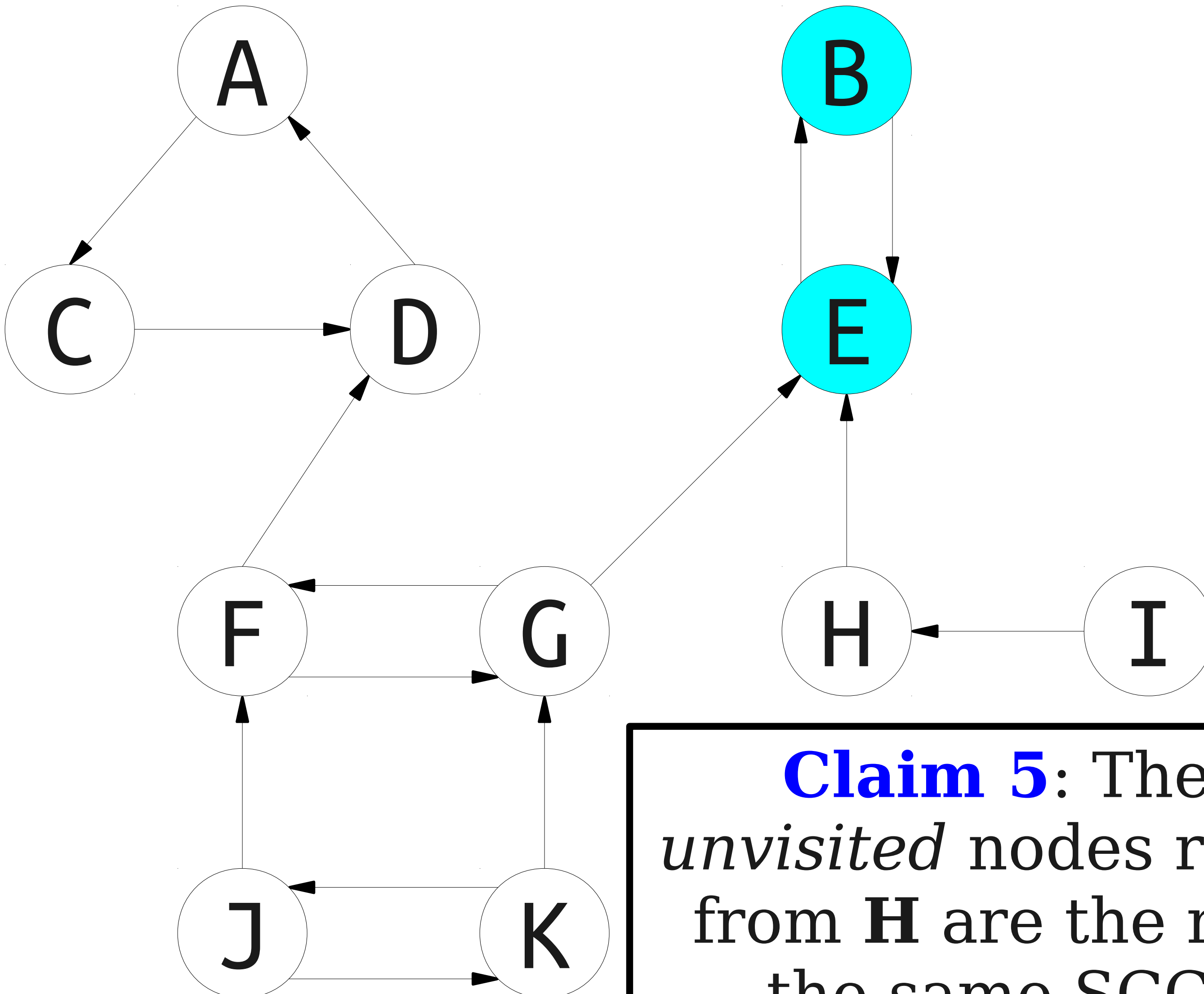
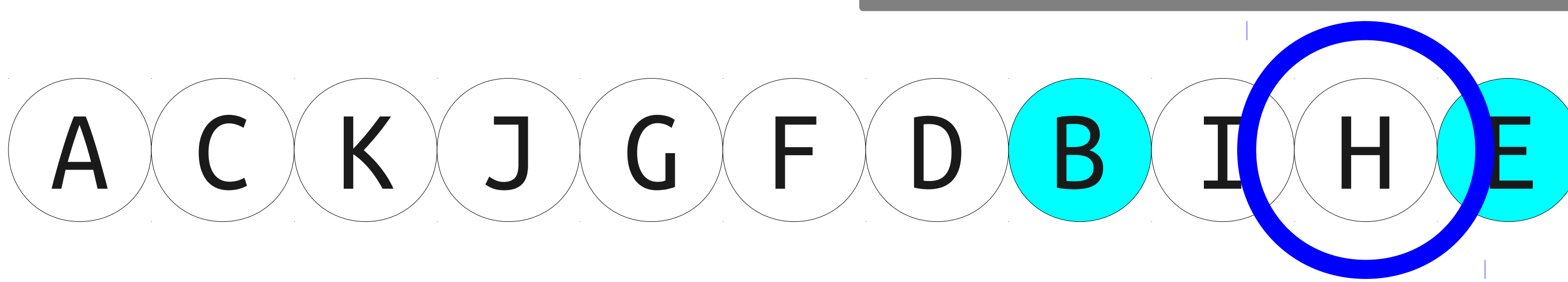
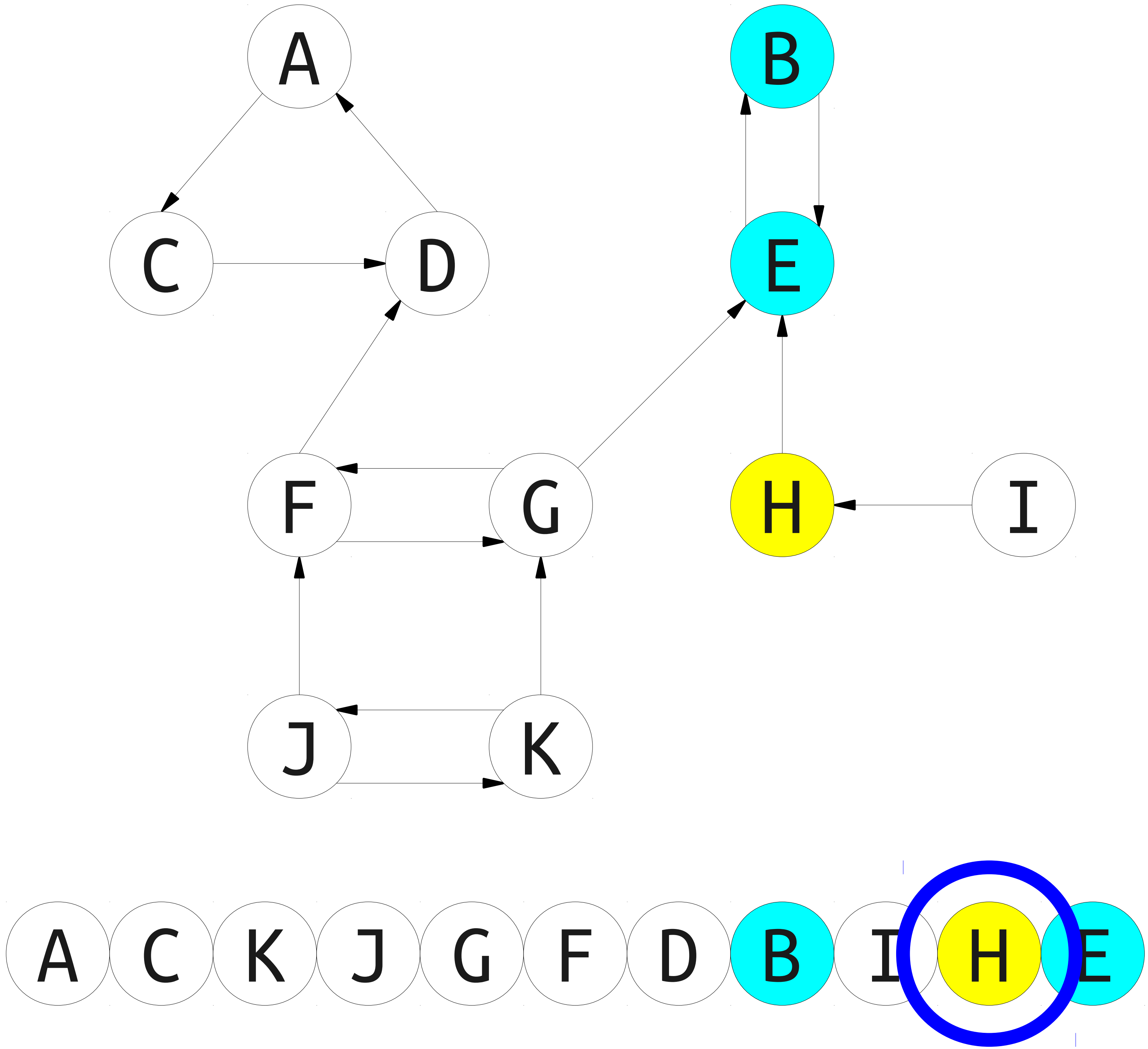


Claim 5: The only *unvisited* nodes reachable from **H** are the nodes in the same SCC as **H**.

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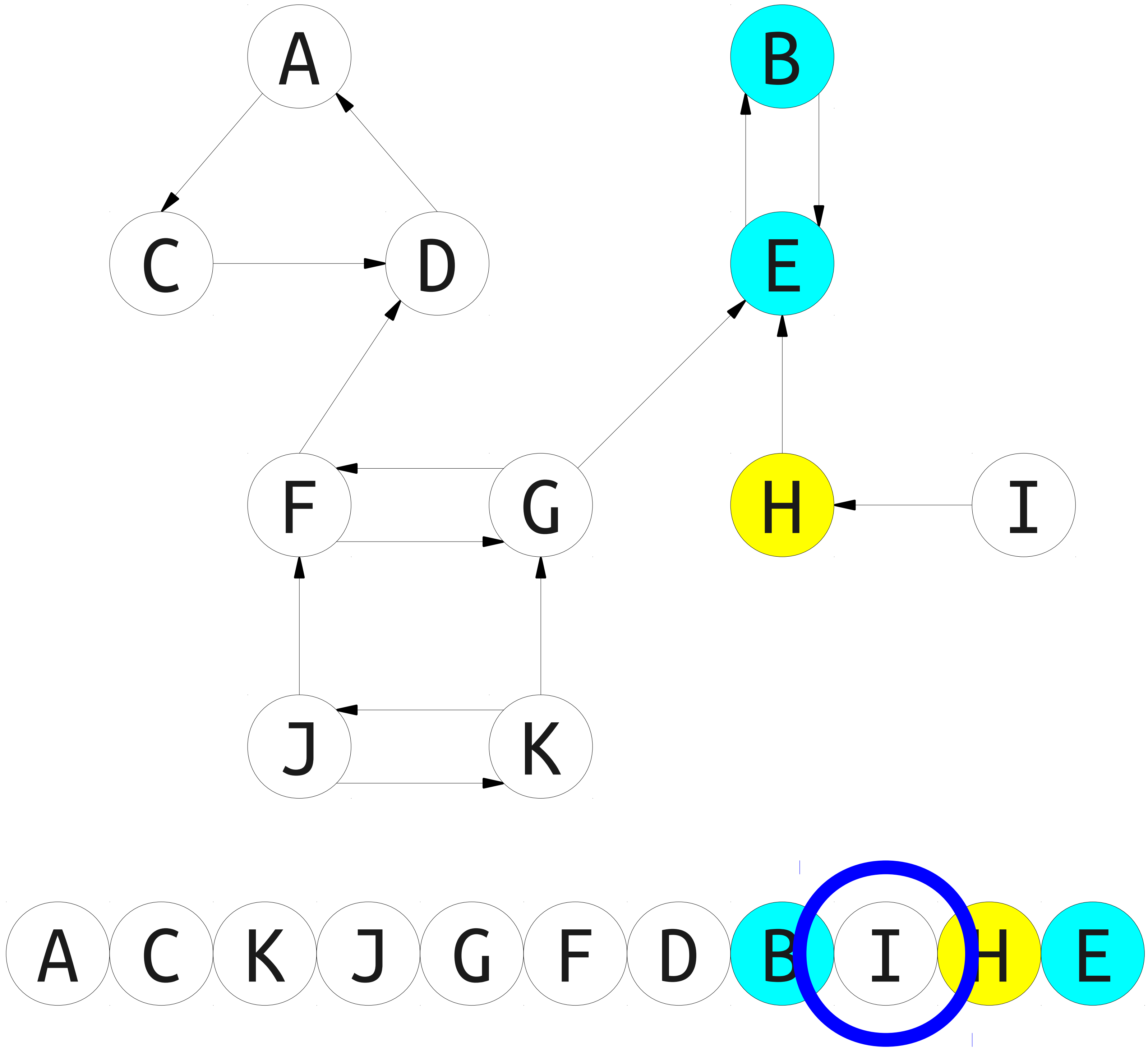
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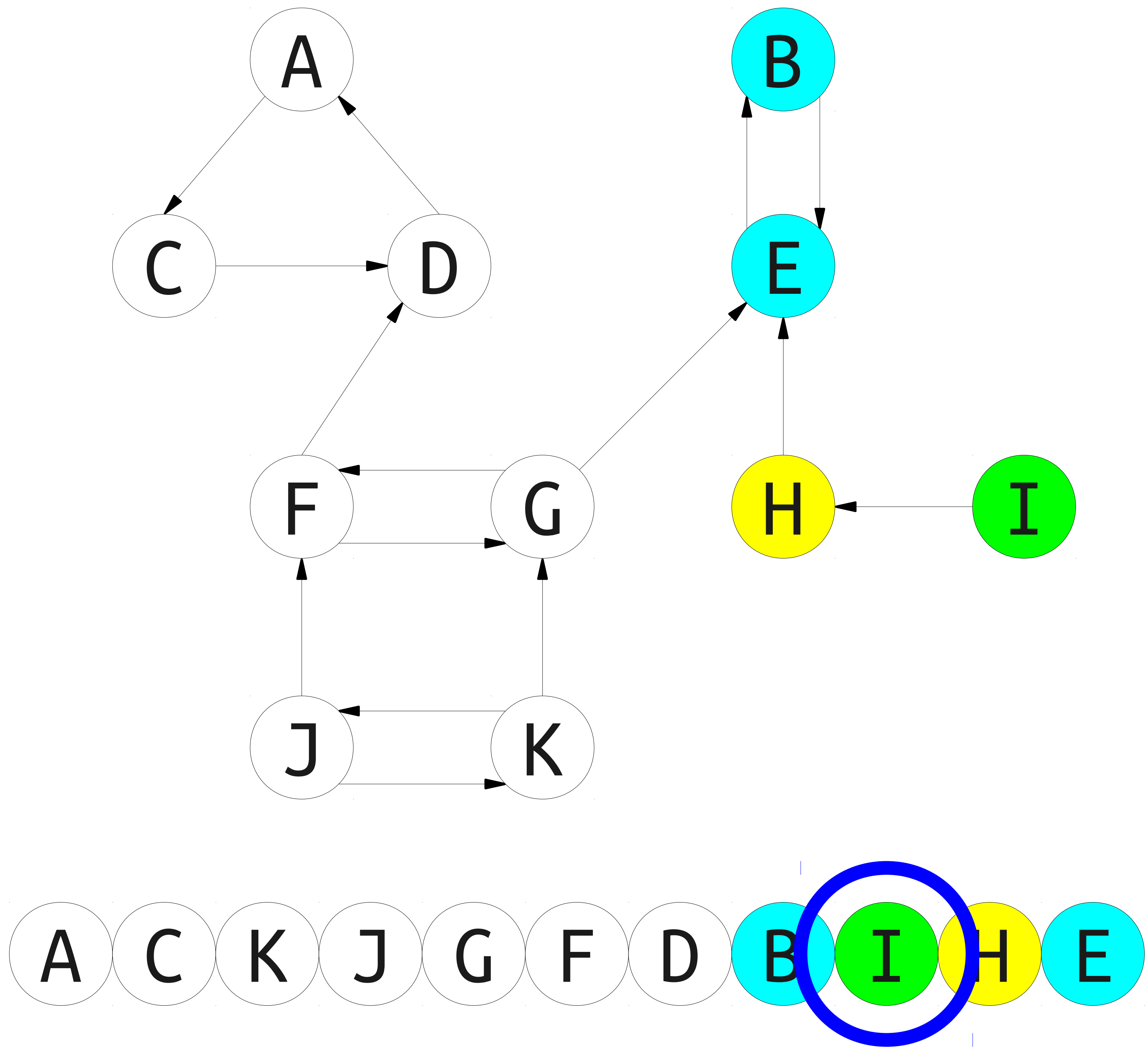
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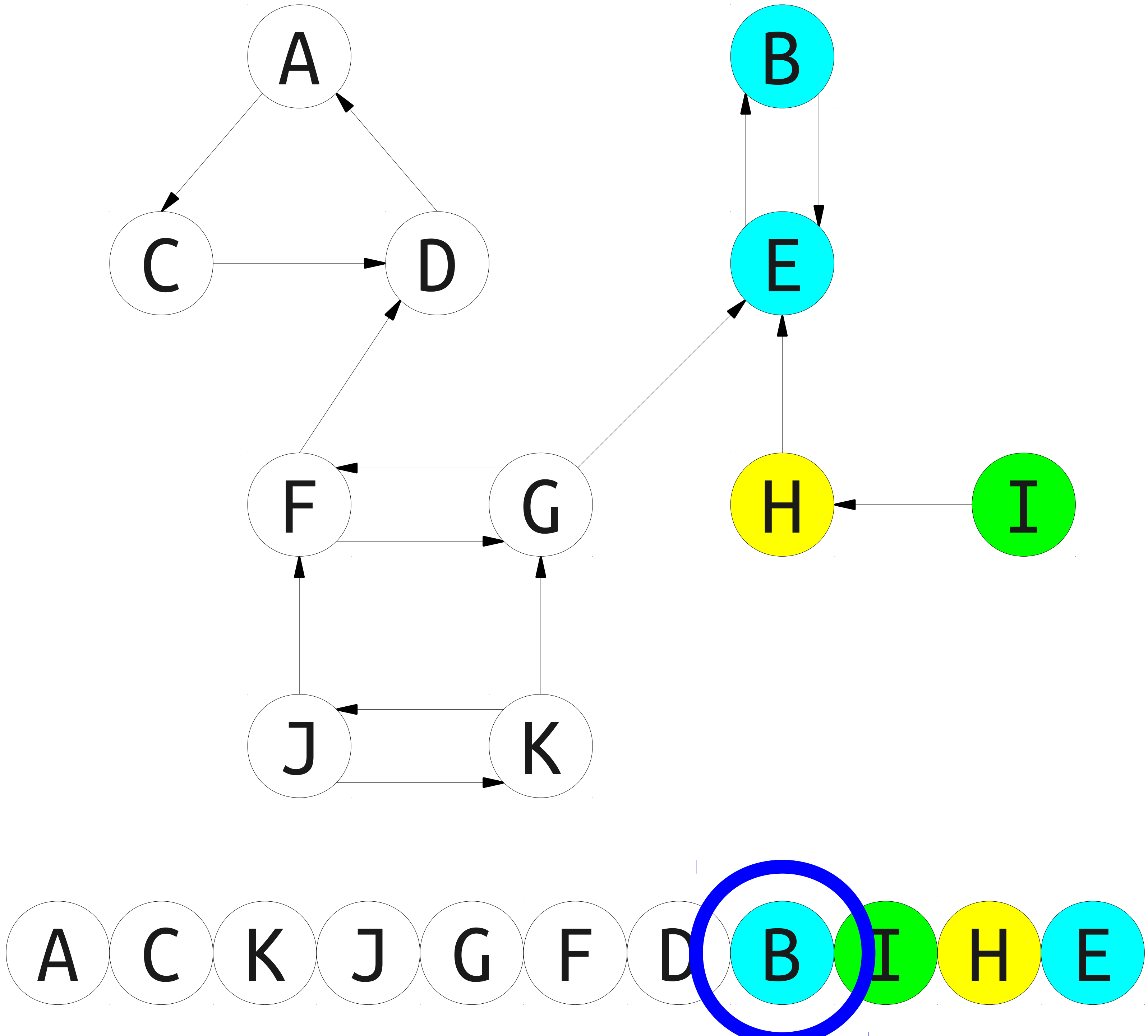
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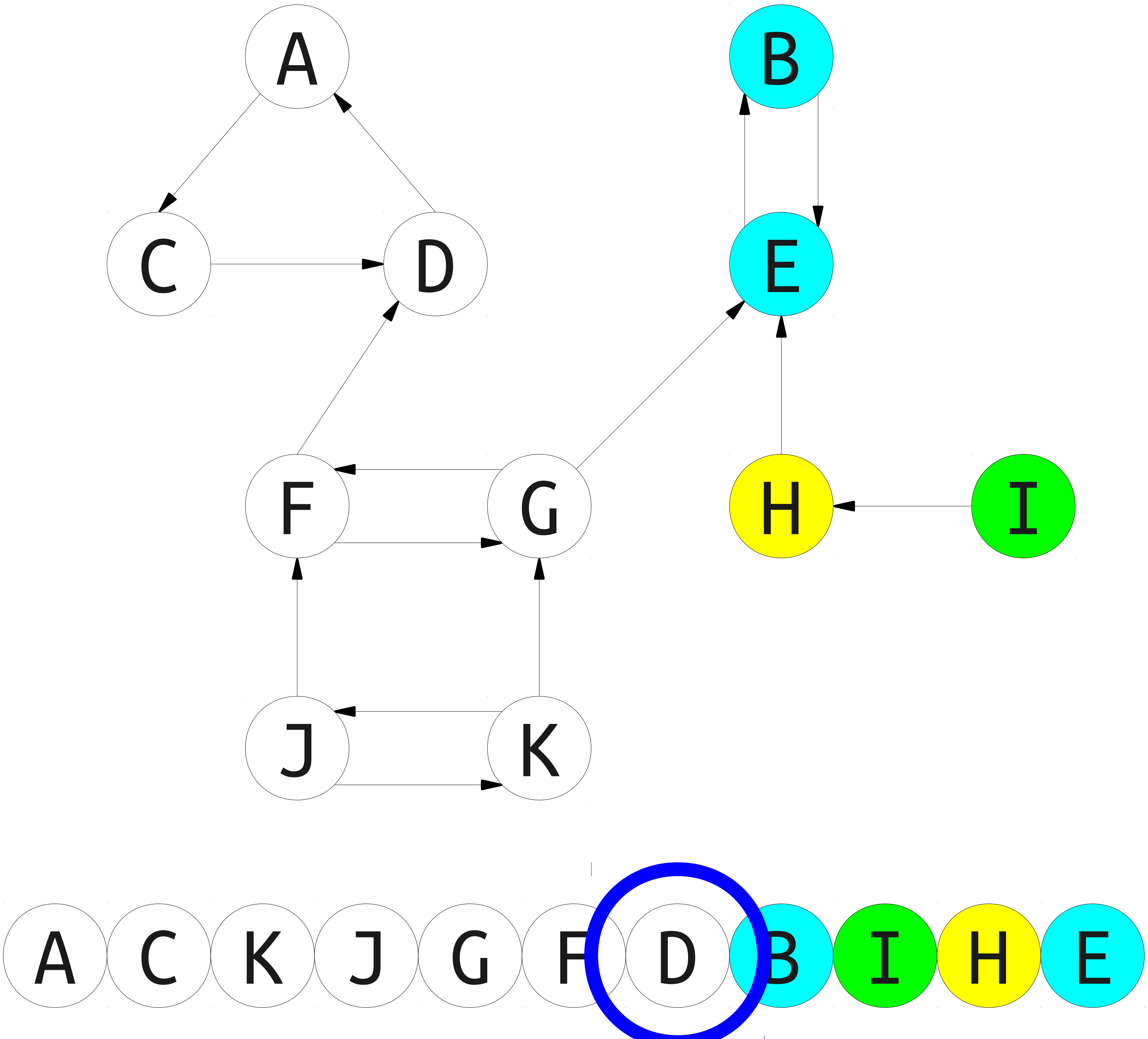
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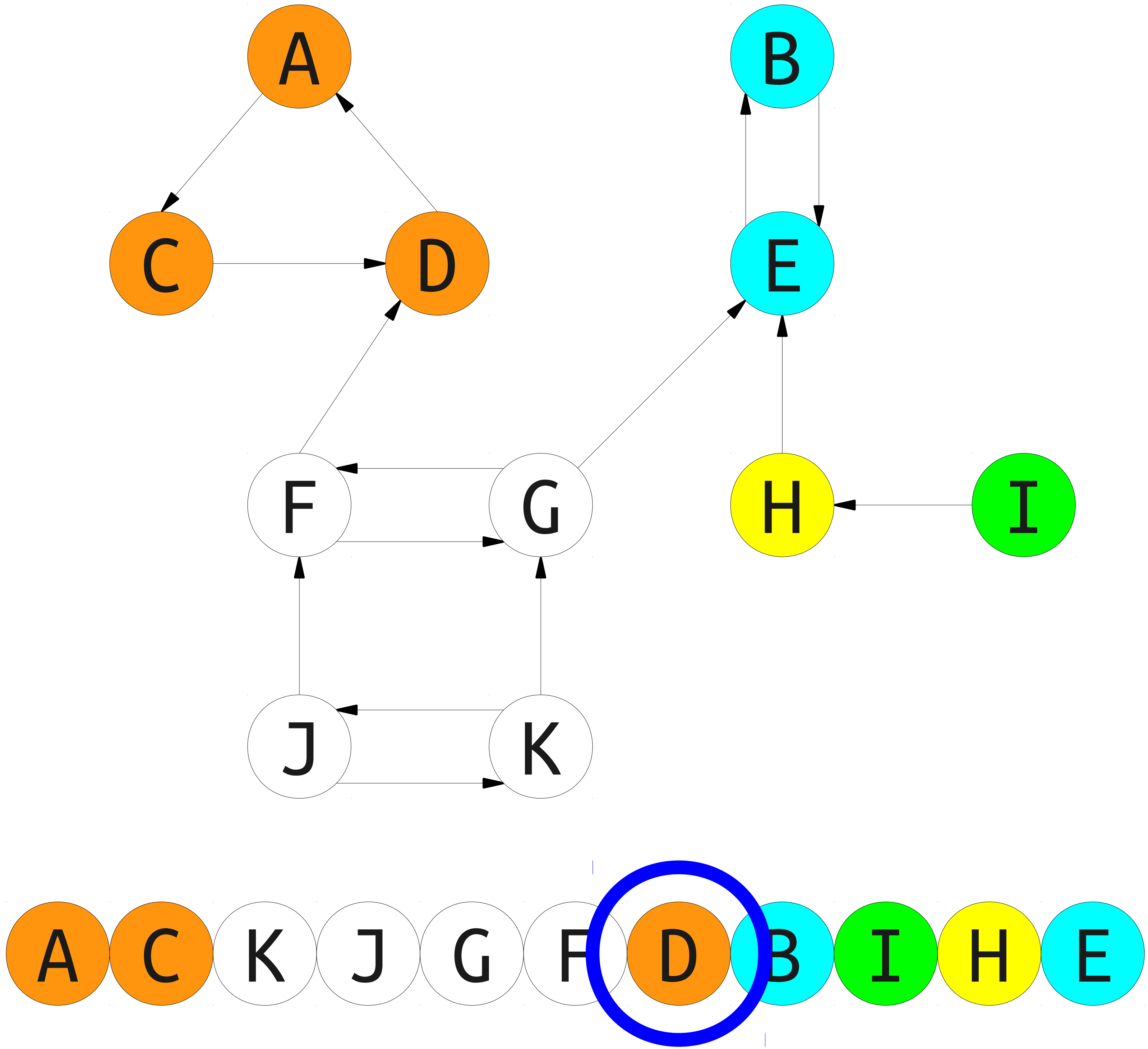
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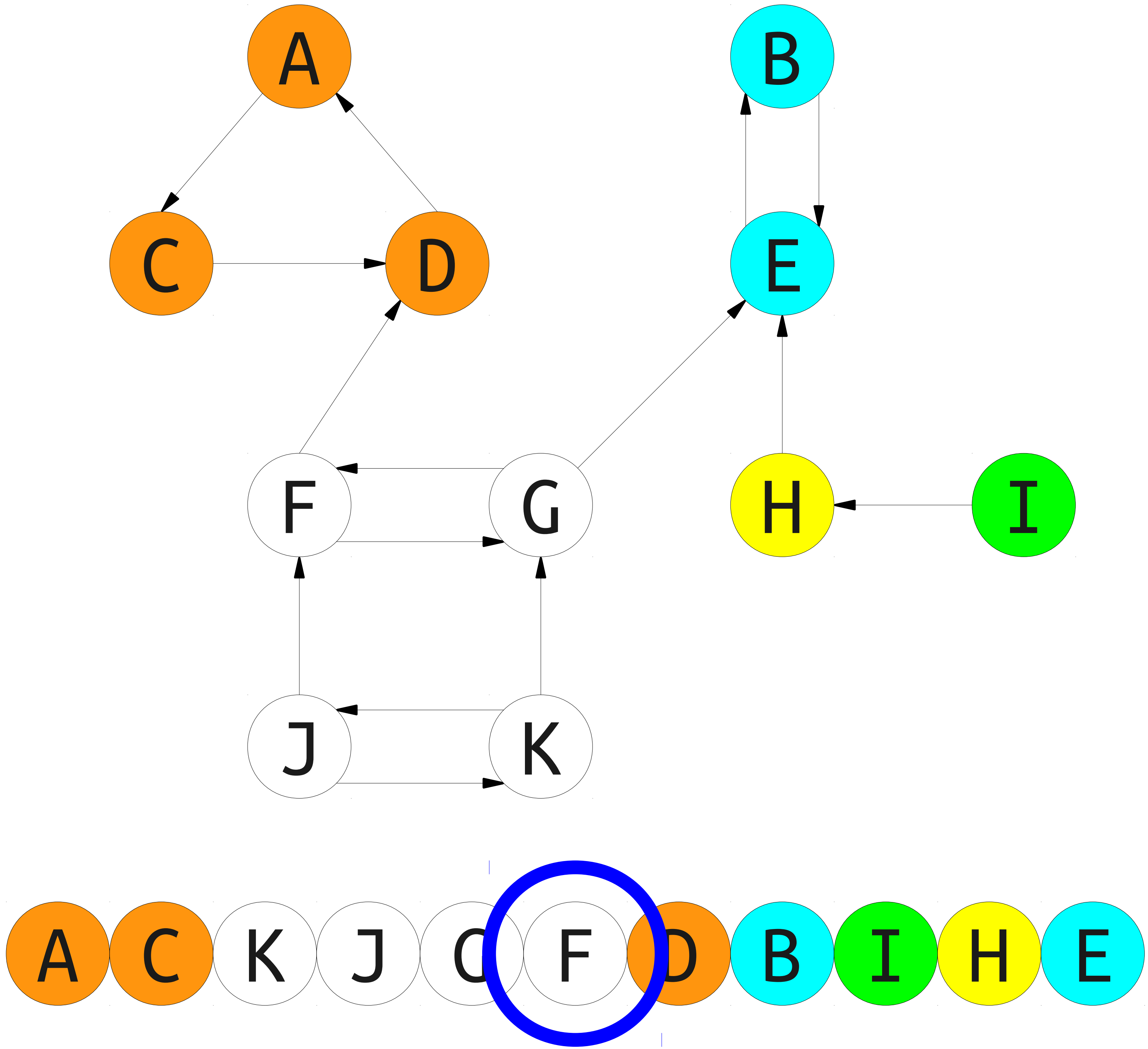
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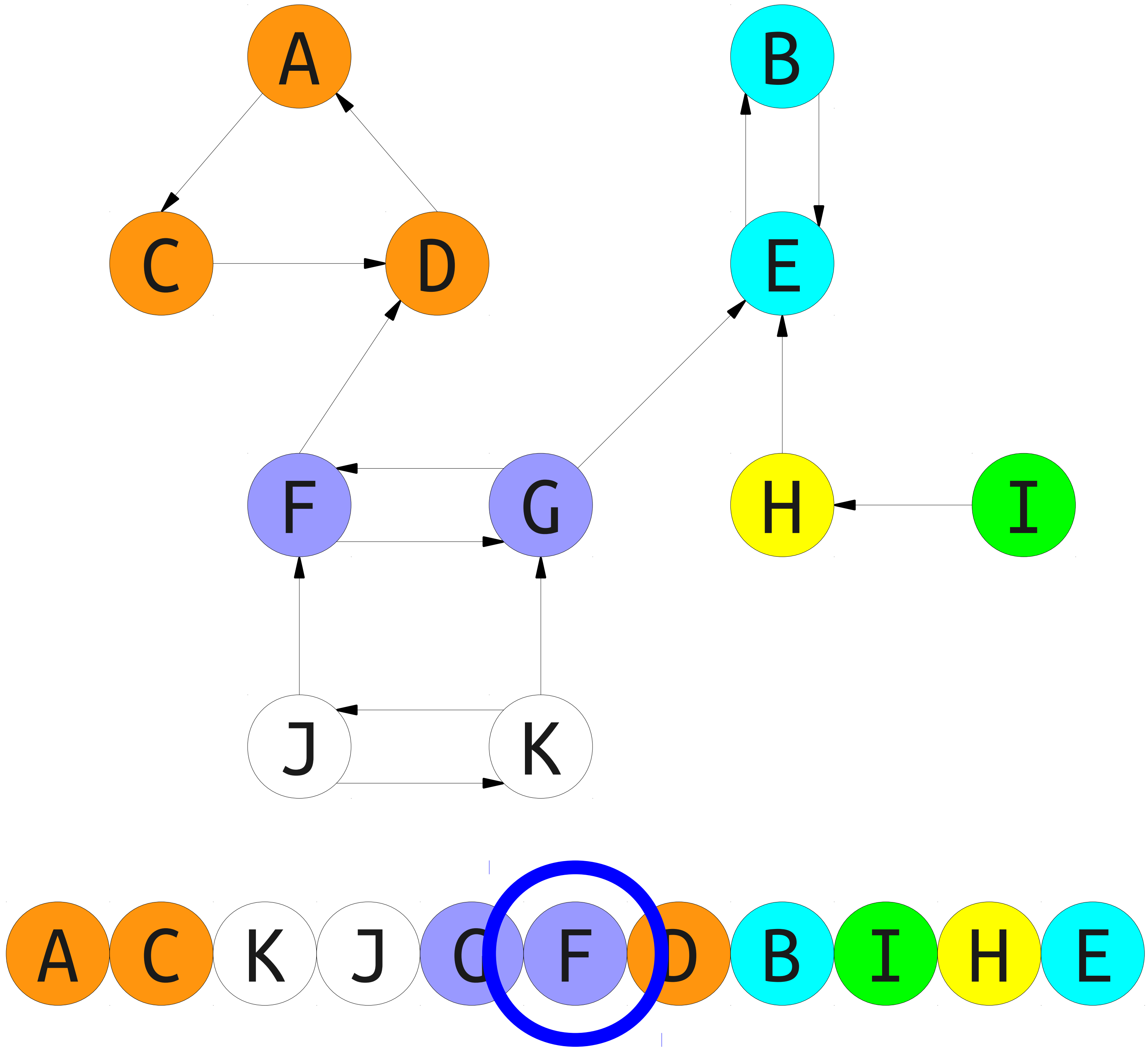
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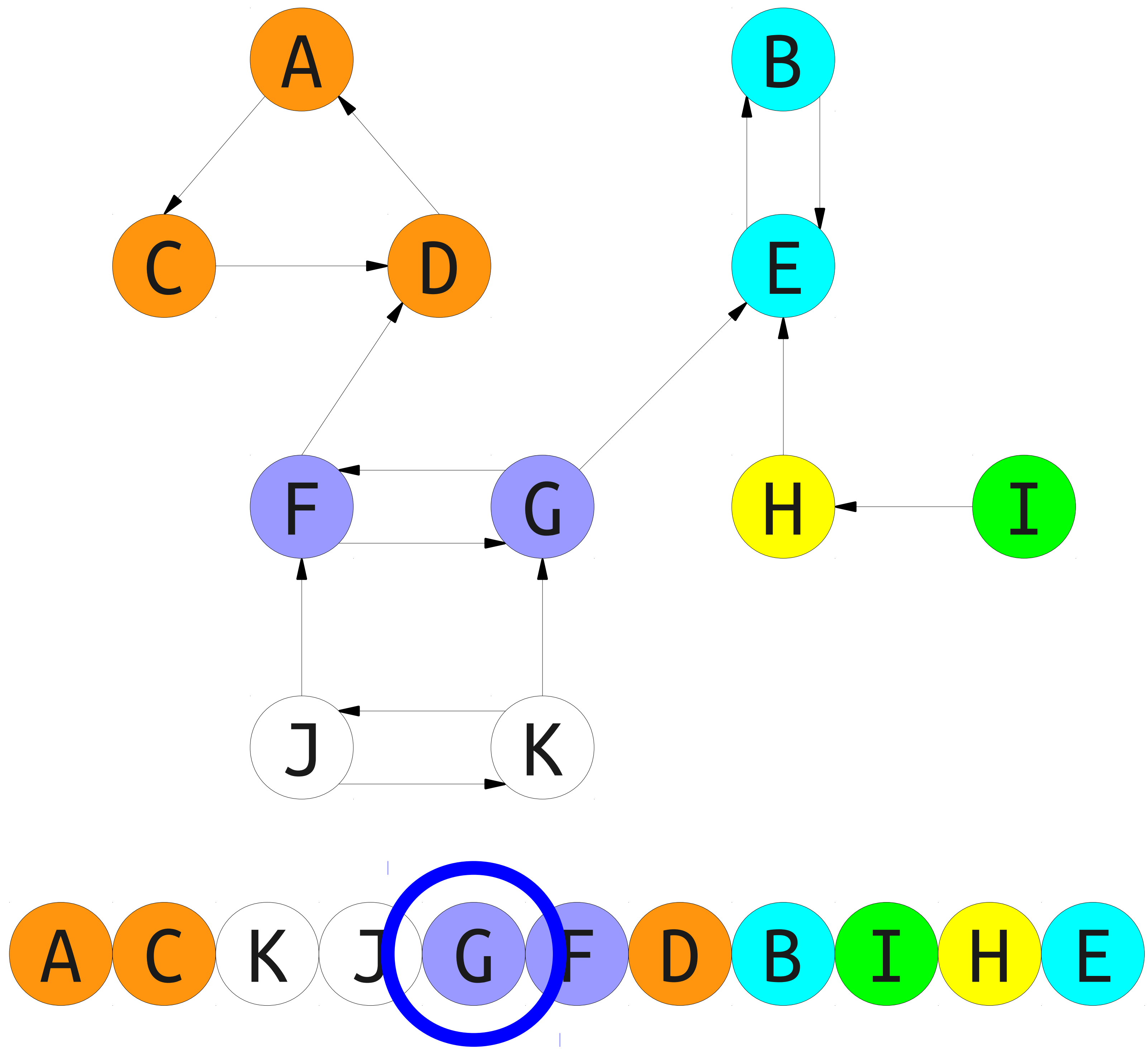
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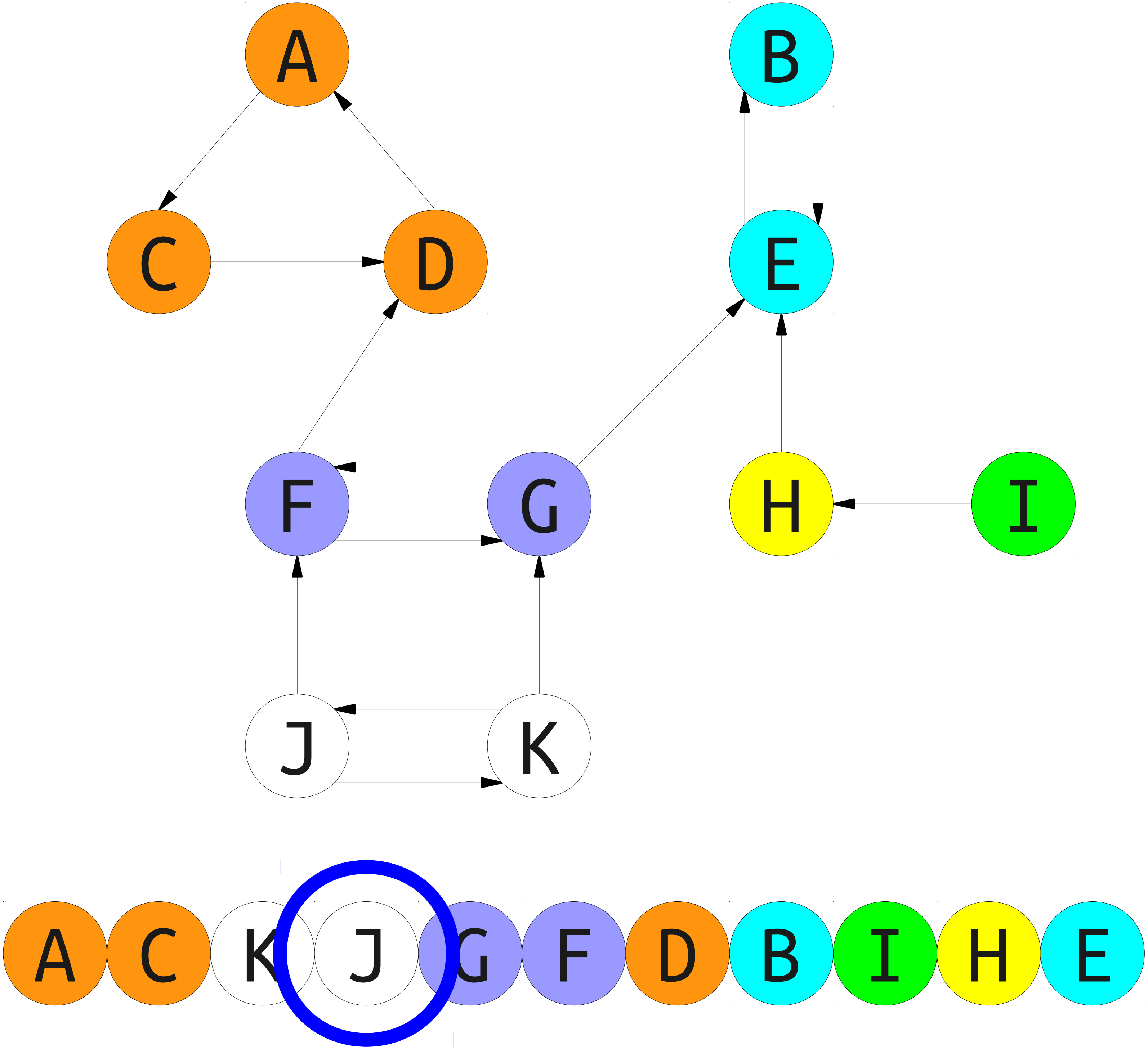
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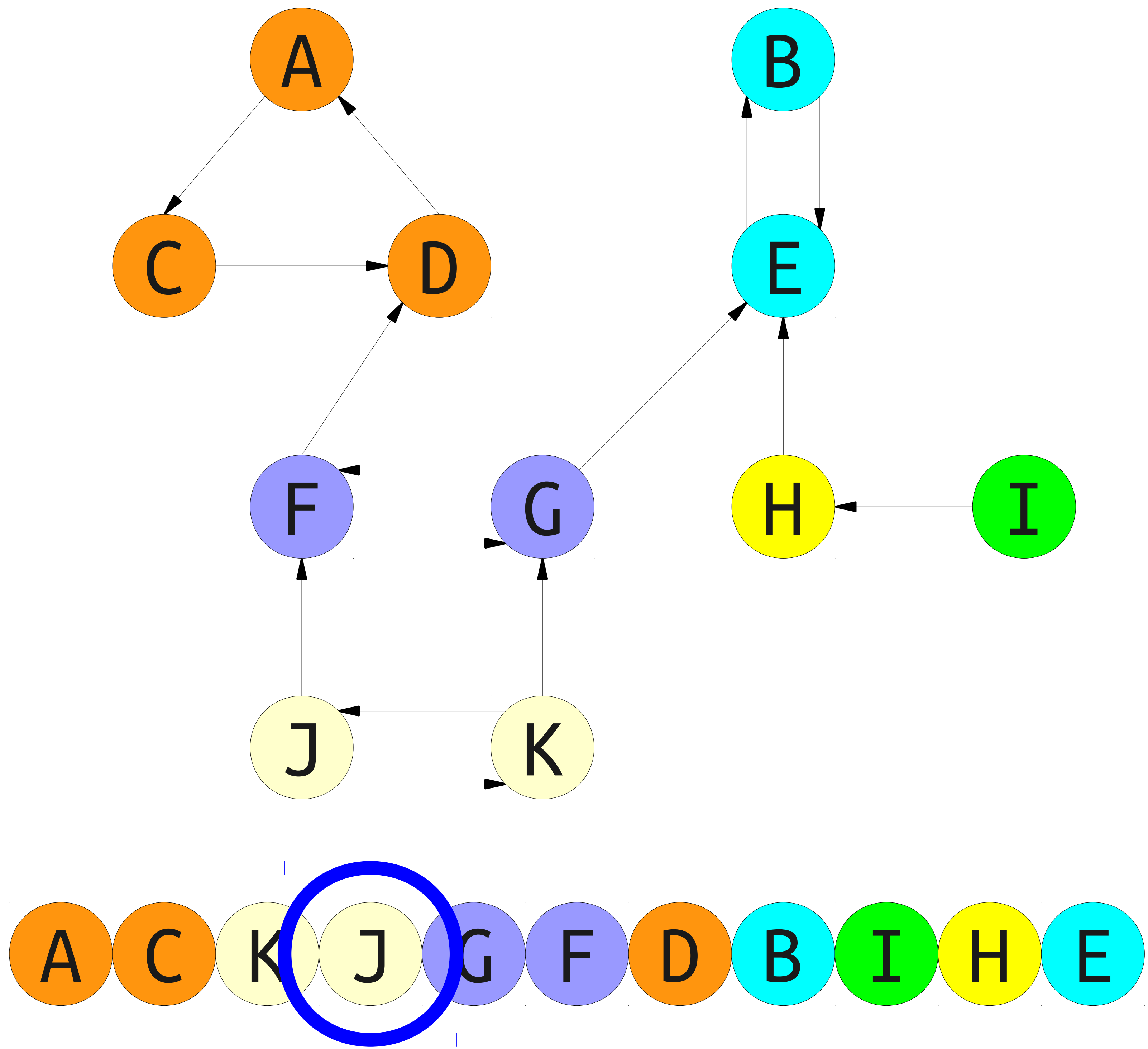

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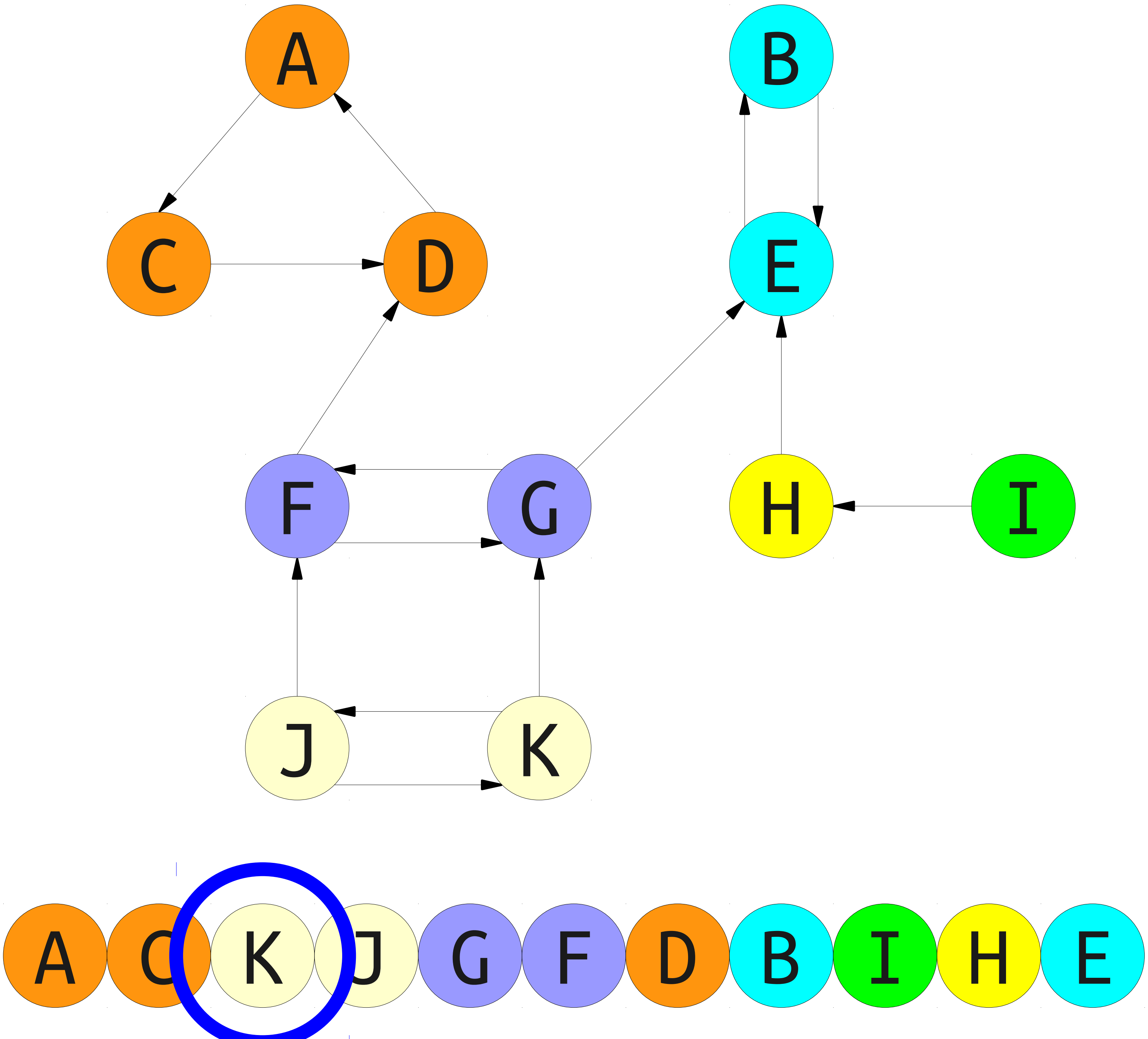
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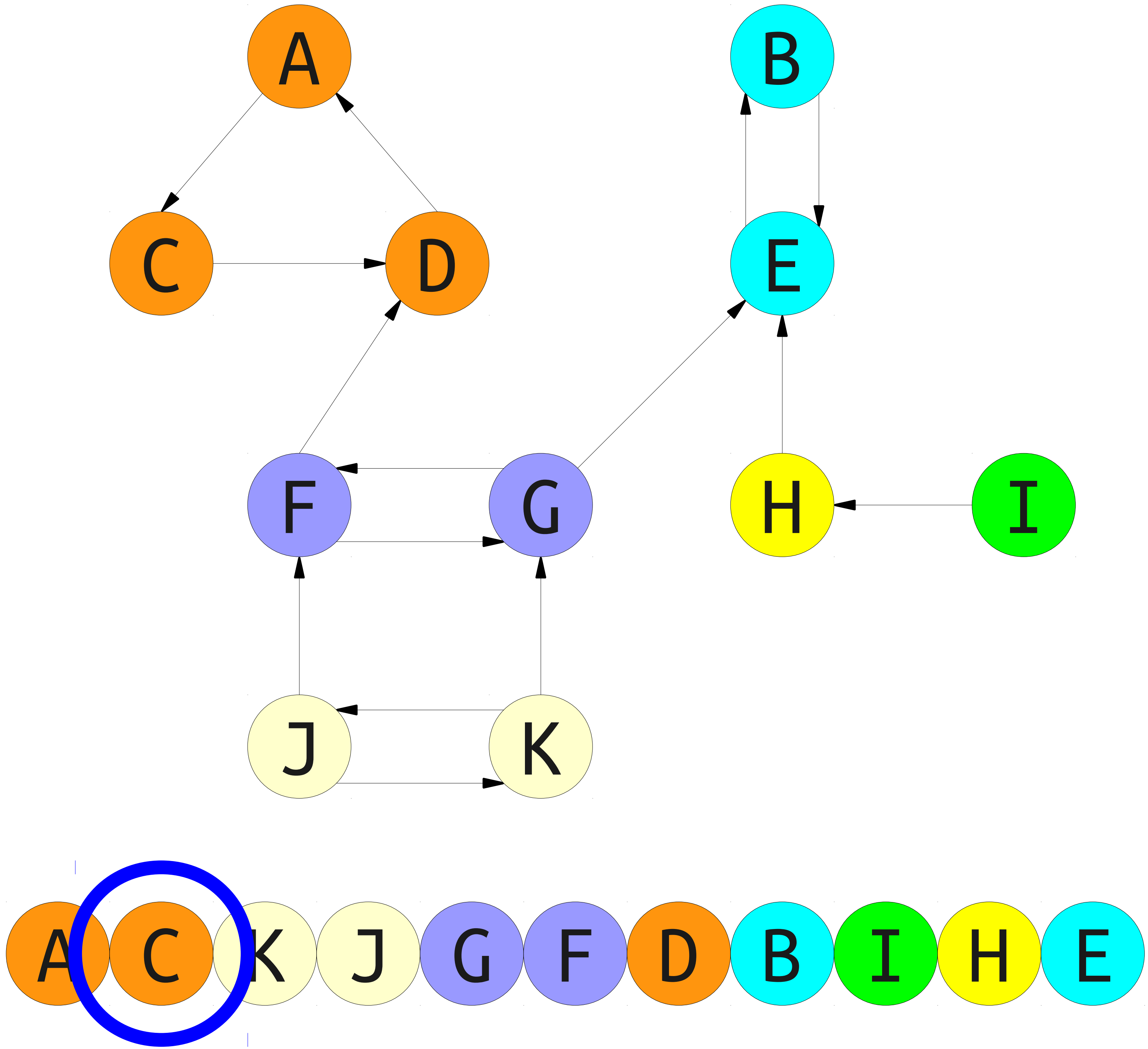
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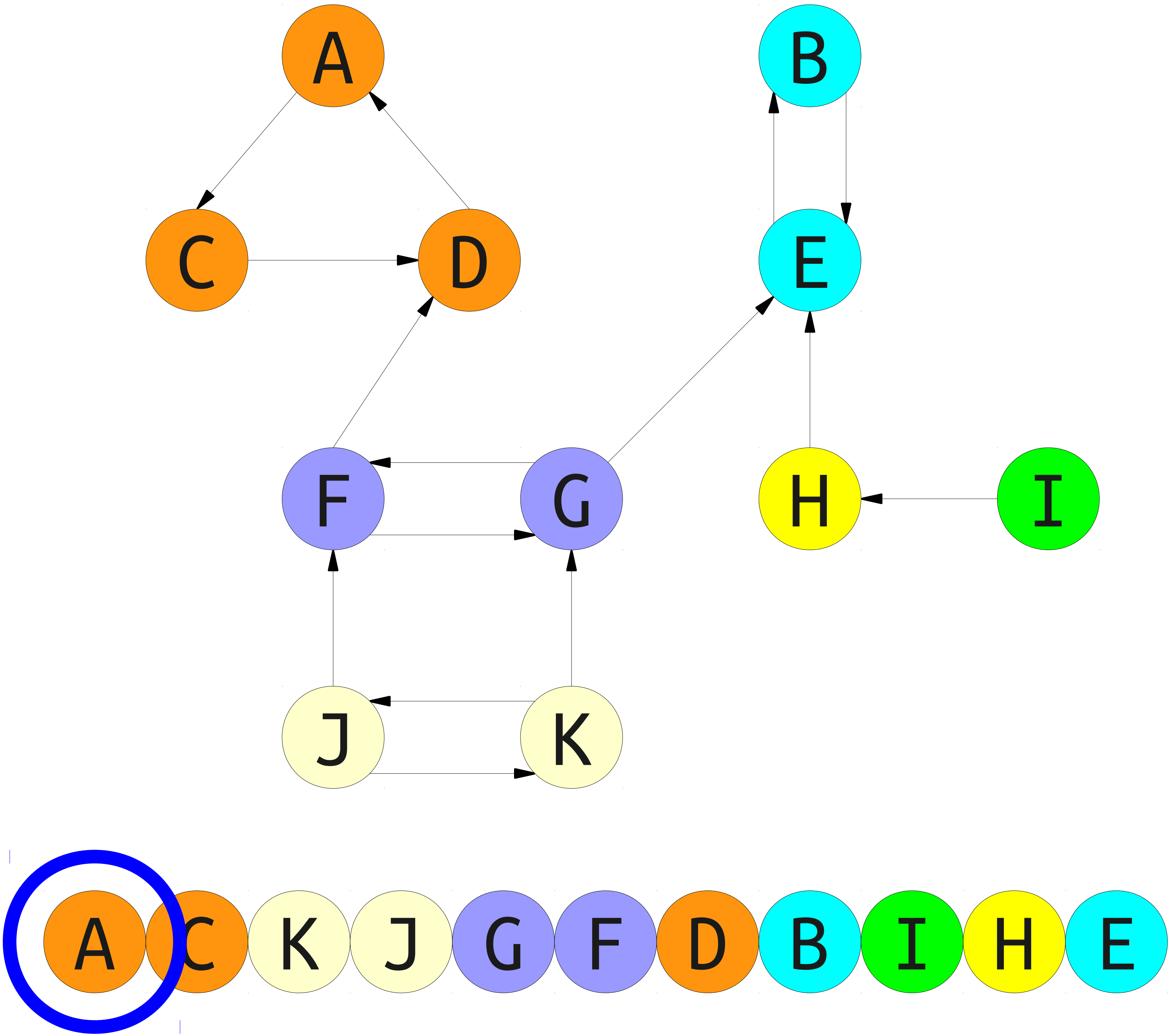
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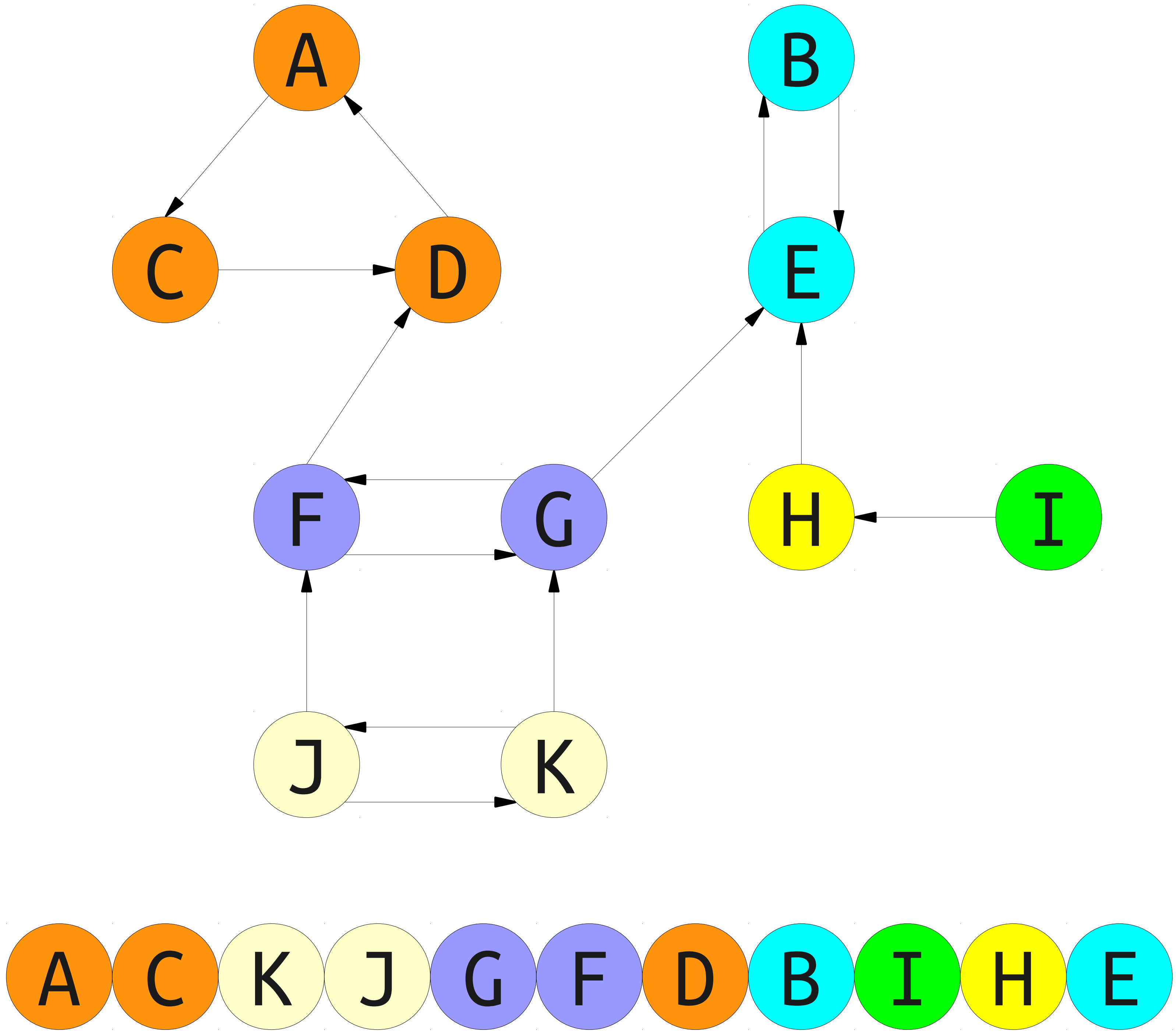
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```
procedure kosarajuSCC(graph G):
   for each node v in G:
      color v gray.

   let L be an empty list.
   for each node v in G:
      if v is gray:
         run DFS starting at v, appending each
         node to list L when it is colored green.

   construct Gᴿ from G.
   for each node v in Gᴿ:
      color v gray.

   let scc be a new array of length n
   let index = 0
   for each node v in L, in reverse order:
      if v is gray:
         run DFS on v in Gᴿ, setting scc[u] = index
         for each node u colored green this way.
      index = index + 1
   return scc
```

Proving Correctness

- Here's a quick sketch of the correctness proof of Kosaraju's algorithm:
  - As proven earlier, the last nodes in each SCC will be returned in reverse topological order.
  - Each time we do a DFS in the *reverse* graph starting from some node, we only reach nodes in the same SCC or in ancestor SCCs.
  - Since we process the SCCs in topological order, at each point the only unvisited nodes reachable are nodes in the same SCC.

Kosaraju's Algorithm Runtime

- What is the runtime of the Kosaraju's algorithm?
  - Runtime for running DFS starting from each node in the graph: $\Theta(m + n)$.
  - Runtime for reversing the graph and coloring all nodes gray: $\Theta(m + n)$.
  - Runtime for running DFS in the reversed graph: $\Theta(m + n)$.
  - Total runtime: $\boldsymbol{\Theta(m + n)}$.
- This is a **linear-time algorithm!**

Why All This Matters

- Depth-first search is an important building block for many other algorithms, including topological sorting, finding connected components, and Kosaraju's algorithm.
- We can find CCs and SCCs in (asymptotically) the same amount of time.
- Further reading: look up **Tarjan's SCC algorithm** for a way to find SCCs with a single DFS!

Applied Graph Algorithms

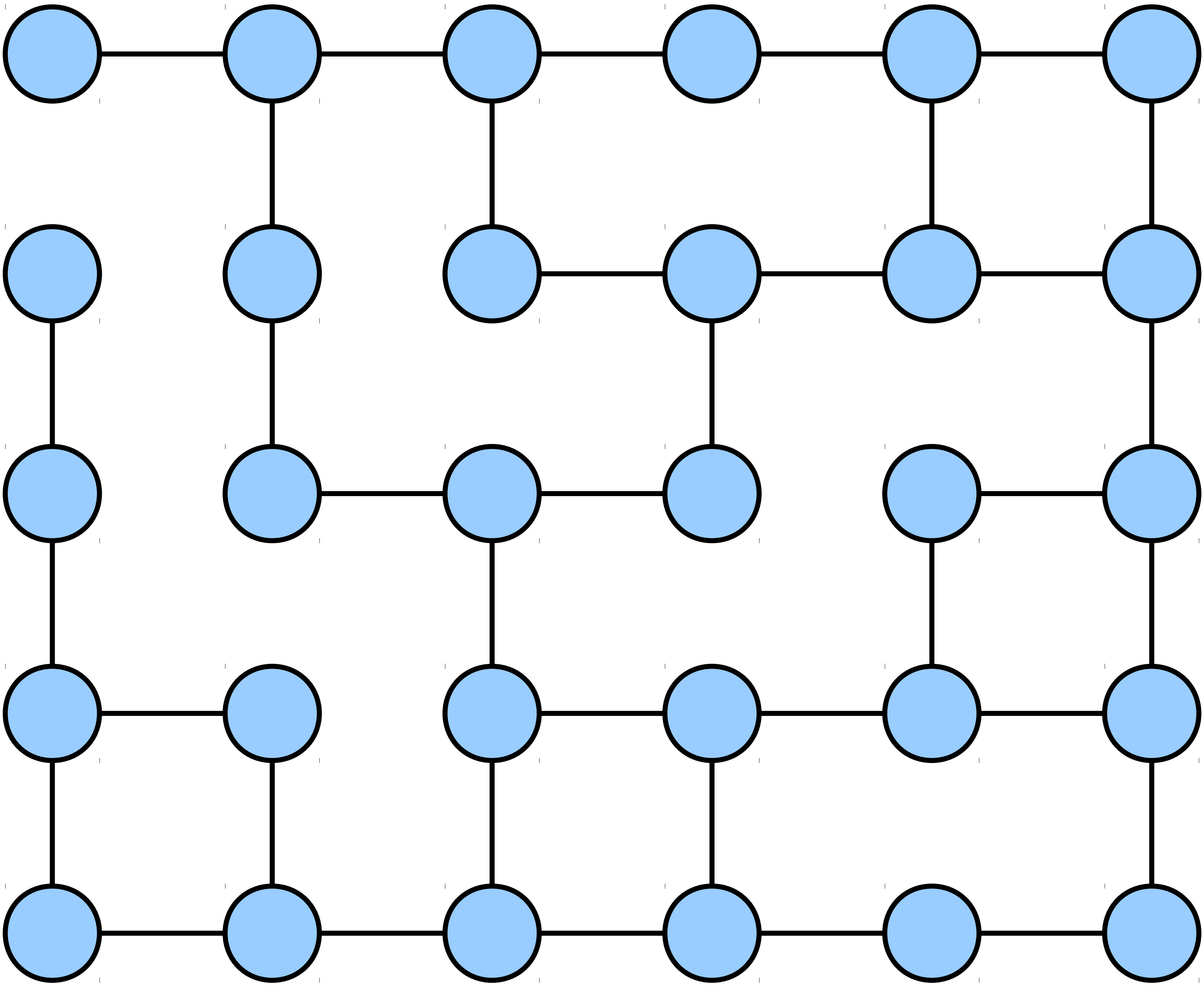
The Story So Far

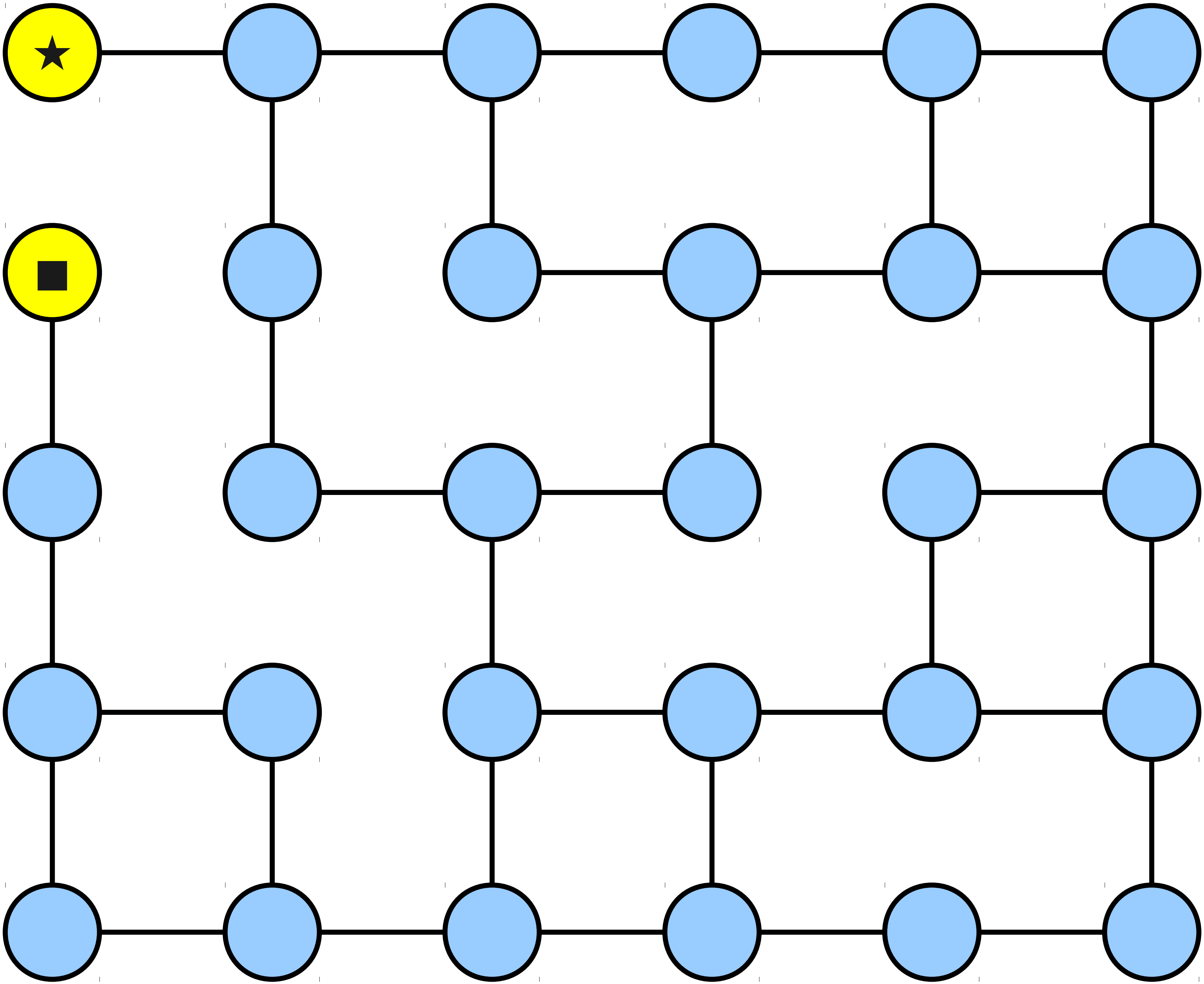
- We have now seen many algorithms that operate on graphs:
  - BFS
  - DFS
  - Dijkstra's algorithm
  - Topological sort (x2)
  - Finding CCs
  - Kosaraju's algorithm
- How do we apply these in practice?

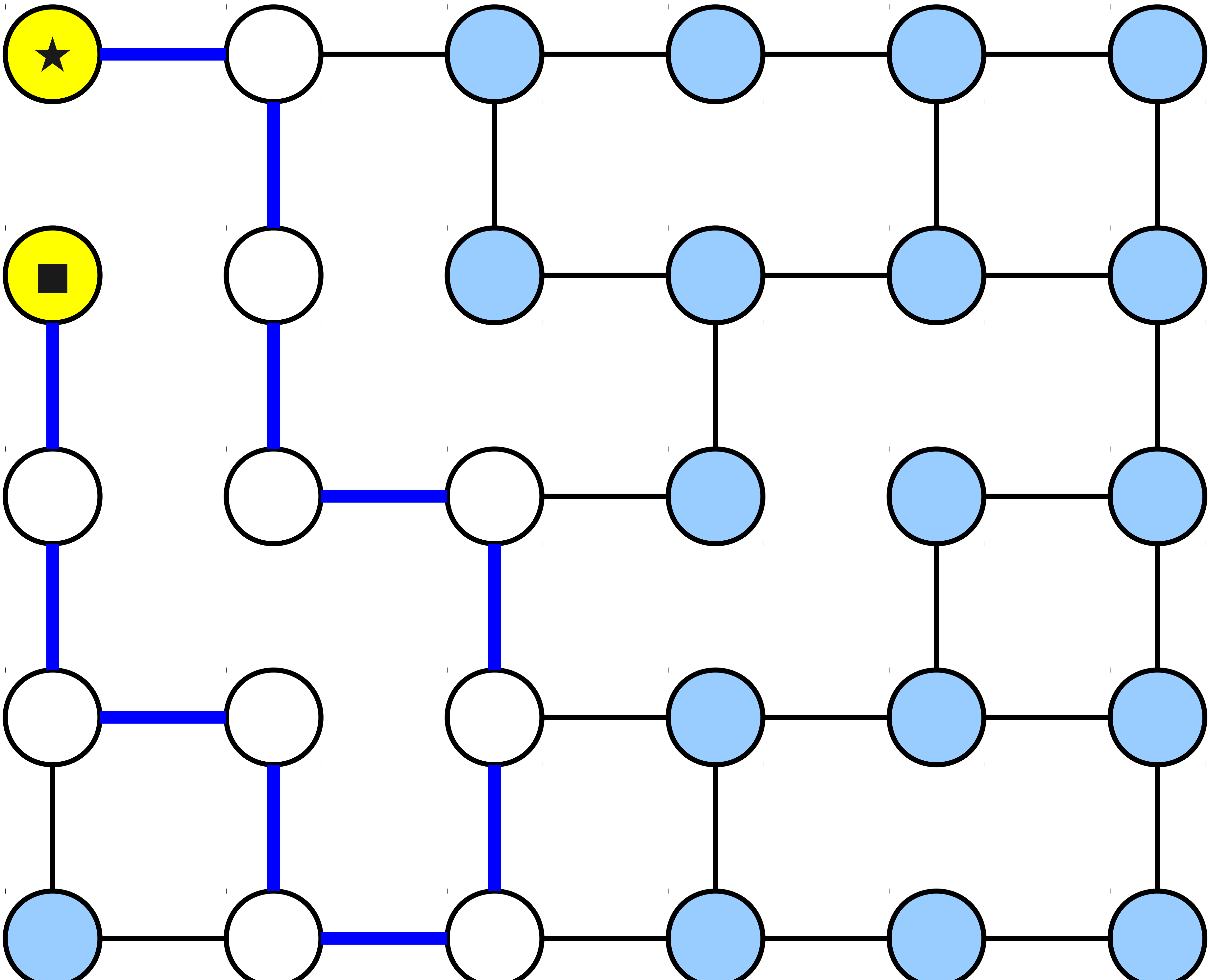
Reusing Algorithms

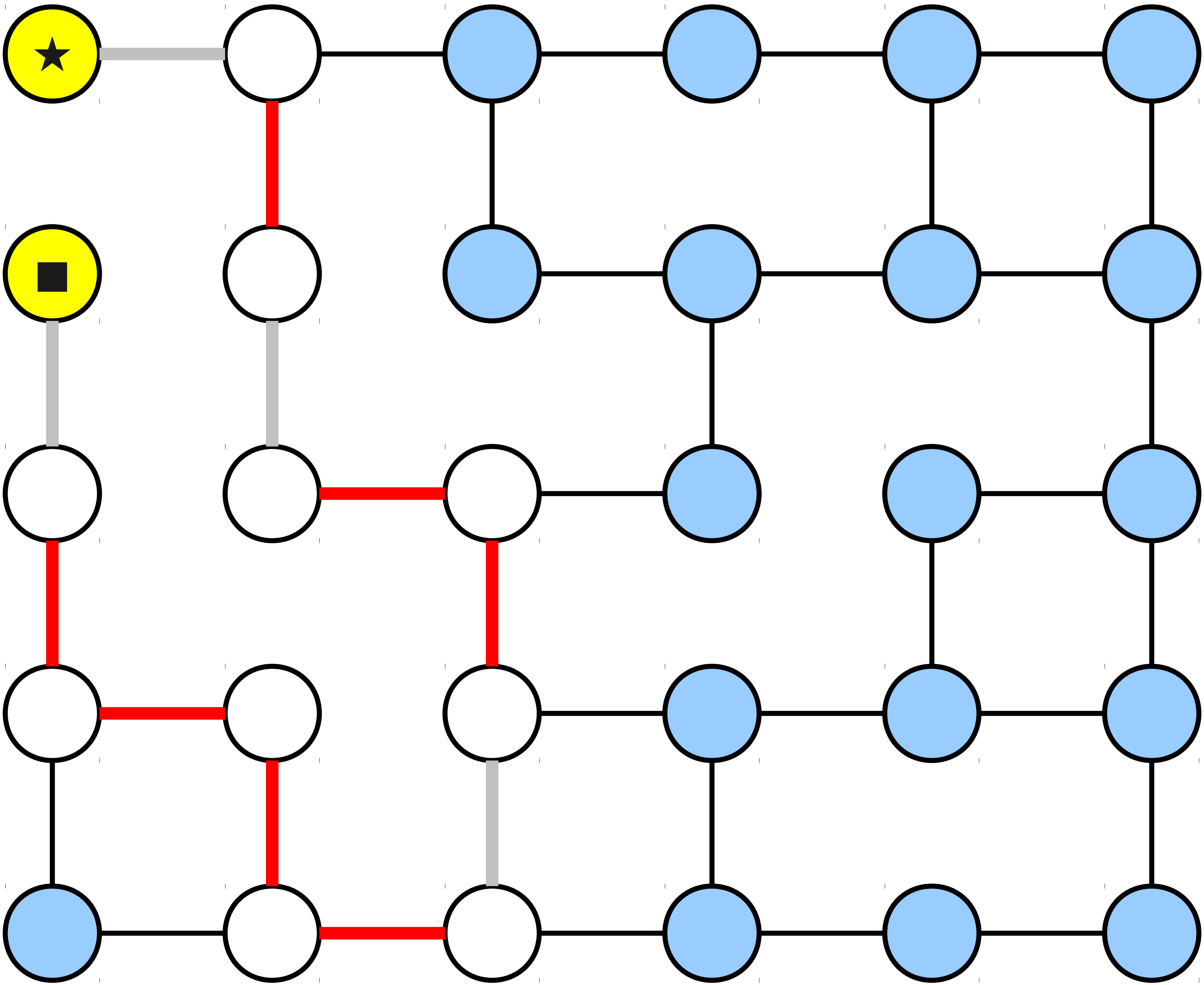
- Developing new graph algorithms is ***hard!***
- Often, it is easier to solve a problem on graphs by reusing existing graph algorithms.
- **Key idea:** Use an existing graph algorithm as a "black box" with known properties and a known runtime.
  - Makes algorithm easier to write: can just use an off-the-shelf implementation.
  - Makes correctness proof easier: can "piggyback" on top of the existing correctness proof.
  - Makes algorithm easier to analyze: runtime of key subroutine is known.

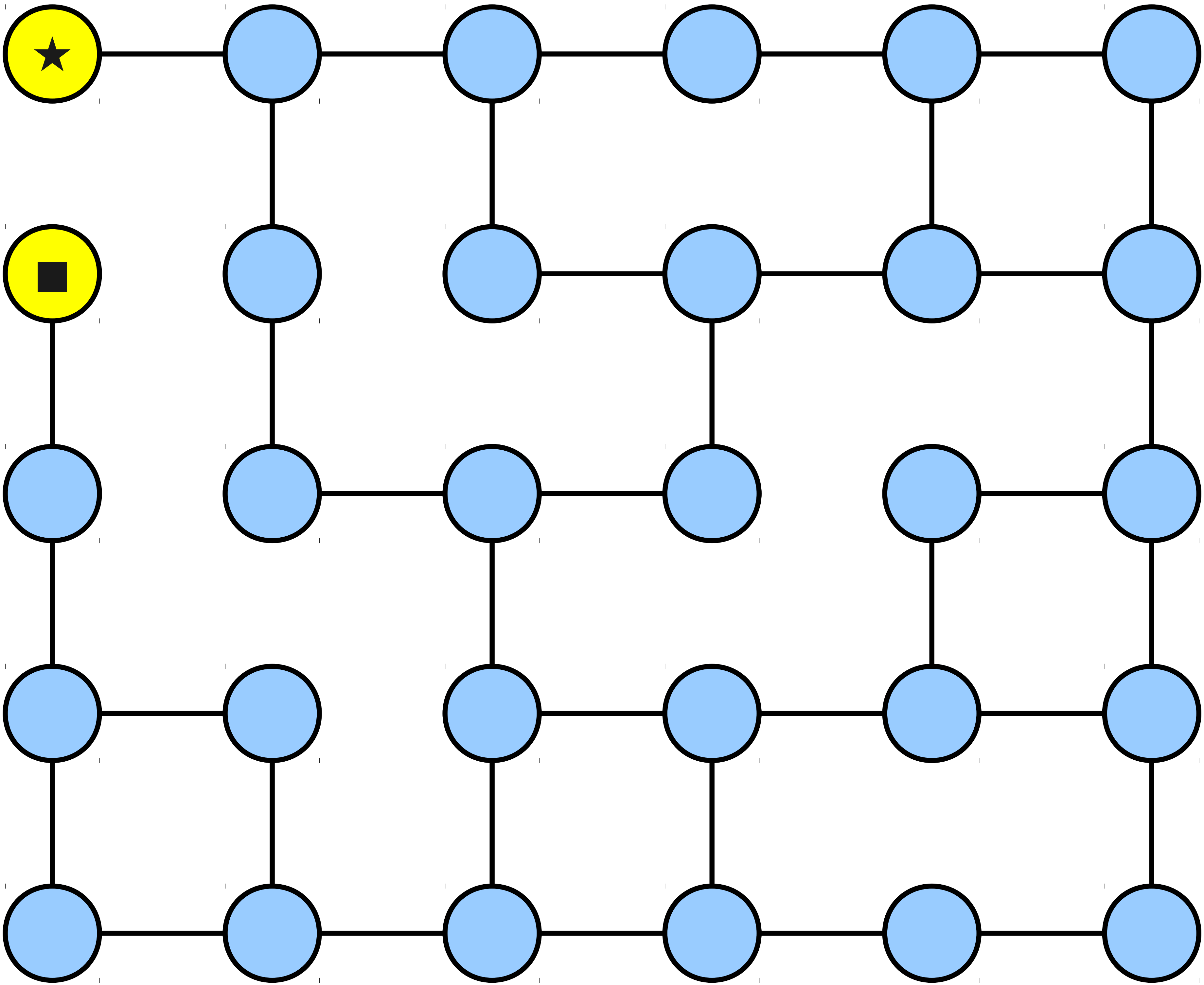
Sample Problem: **Minimizing Turns**

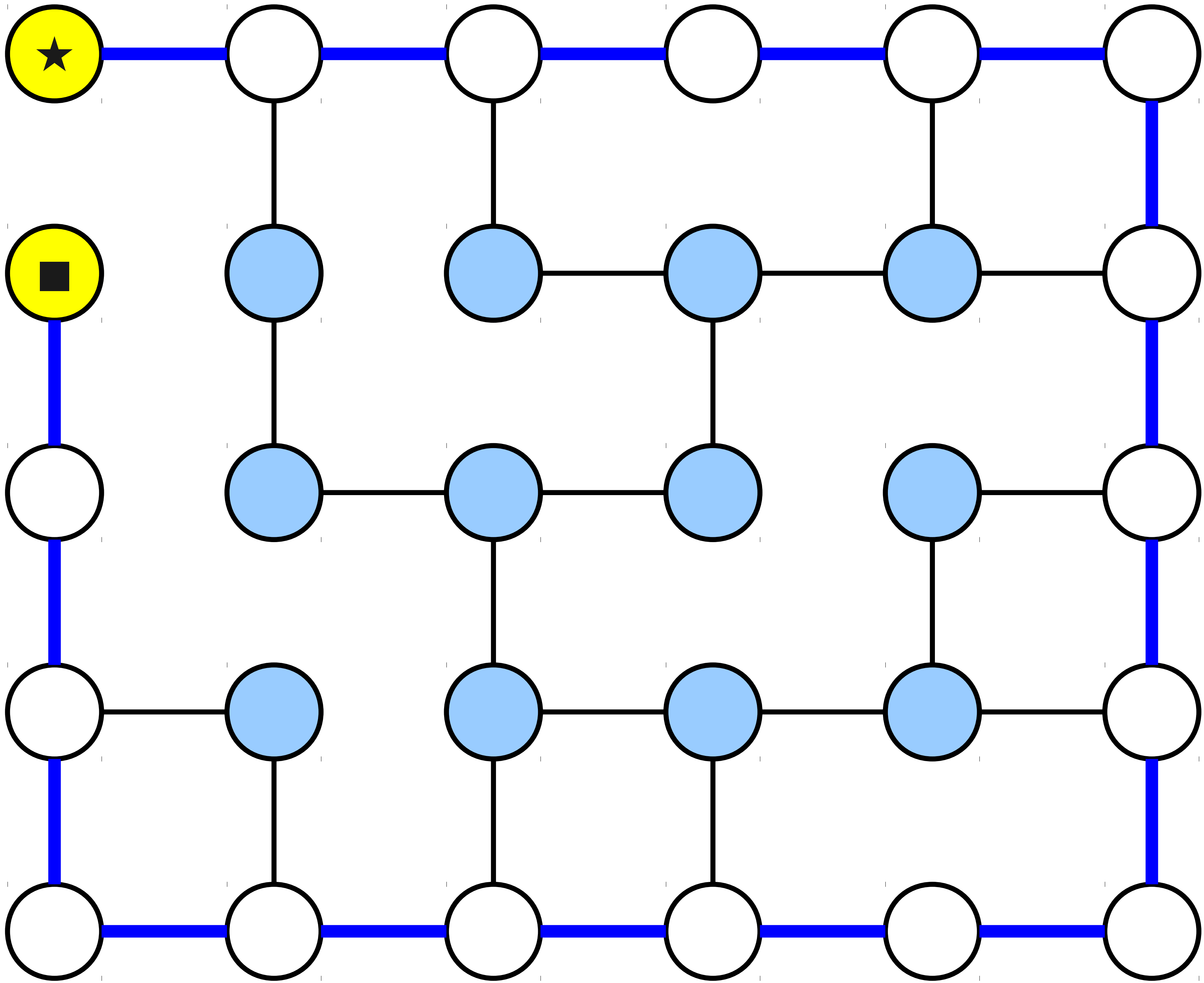


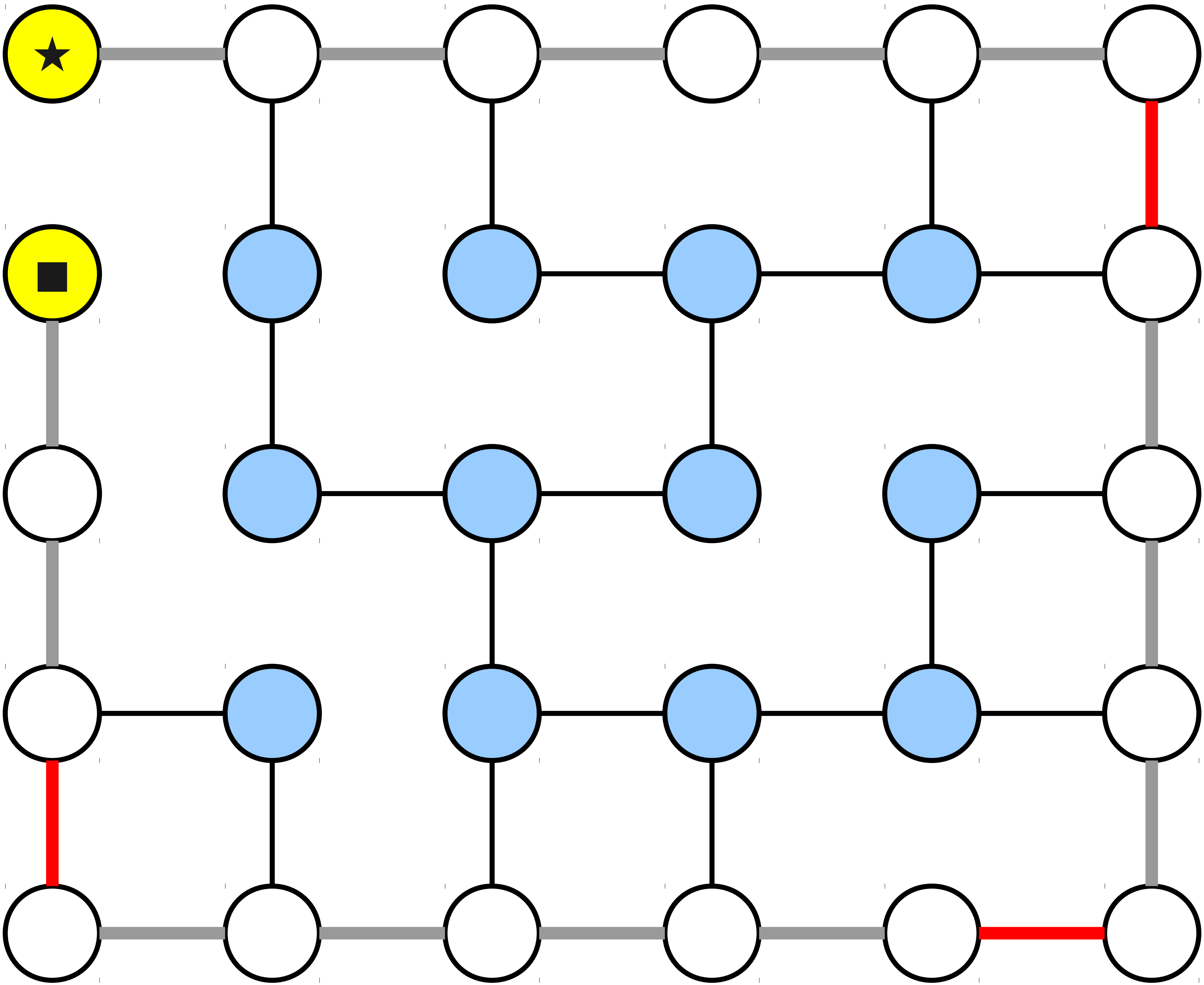












Minimizing Turns

- You are given a (possibly directed) graph $G = (V, E)$ where each edge goes either north, south, east, or west.
- You begin driving in some direction d.
- **Goal**: Find the path from $s \in V$ to $t \in V$ that minimizes the total number of turns made.

What This Looks Like

- This problem doesn't exactly match any of the algorithms we've seen so far.
- Similar to a shortest path problem, but we're charged whenever we make a turn, rather than whenever we follow an edge.
- Could we relate this back to BFS or Dijkstra's algorithm?

Shortest Paths as a Black Box

- Here's what we have now:

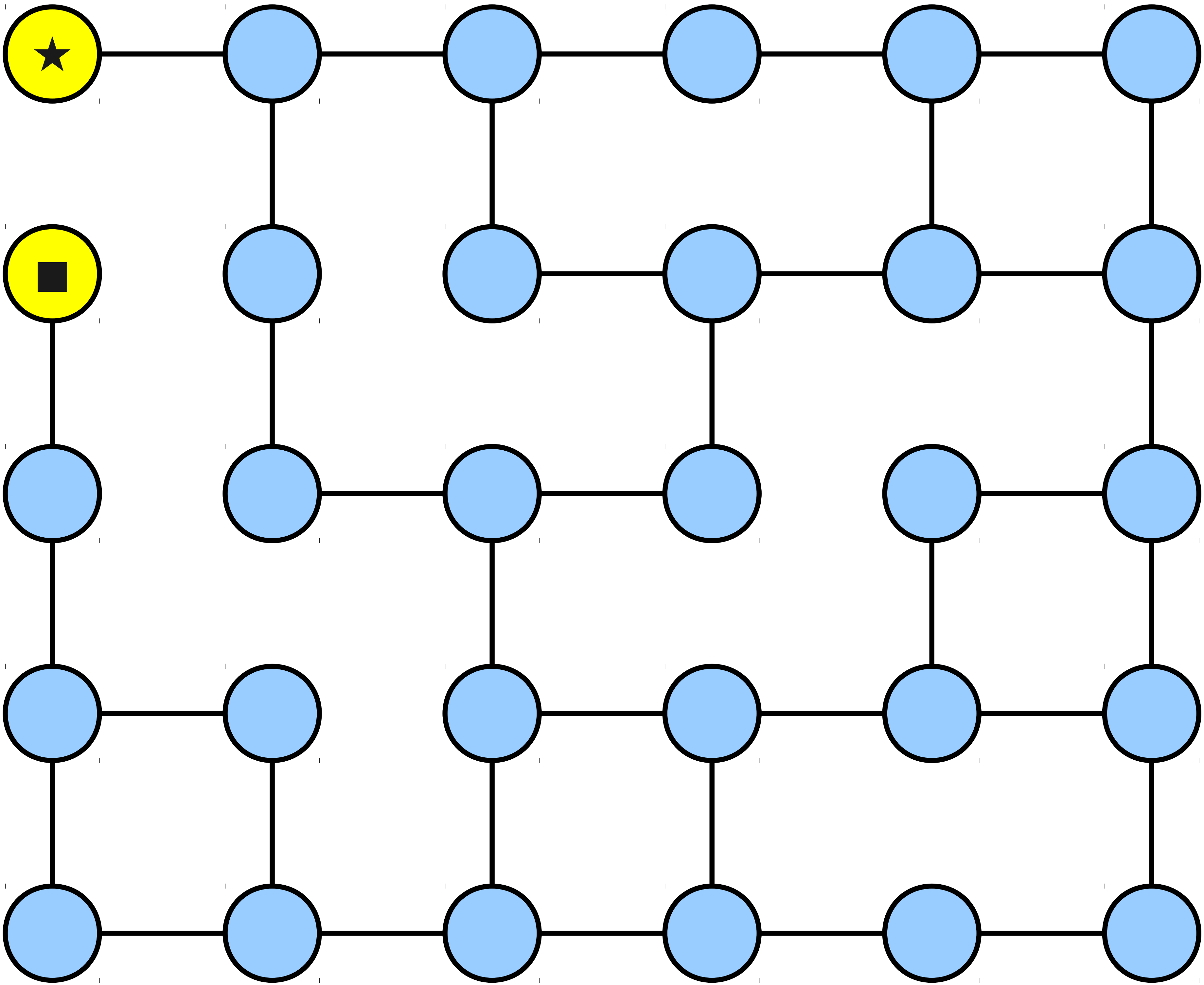
- Here are two options for solving our problem:
  - Open up the black box and try to change how it finds shortest paths. (Harder)
  - Change which input we put into the black box to trick it into solving our problem. (Easier)

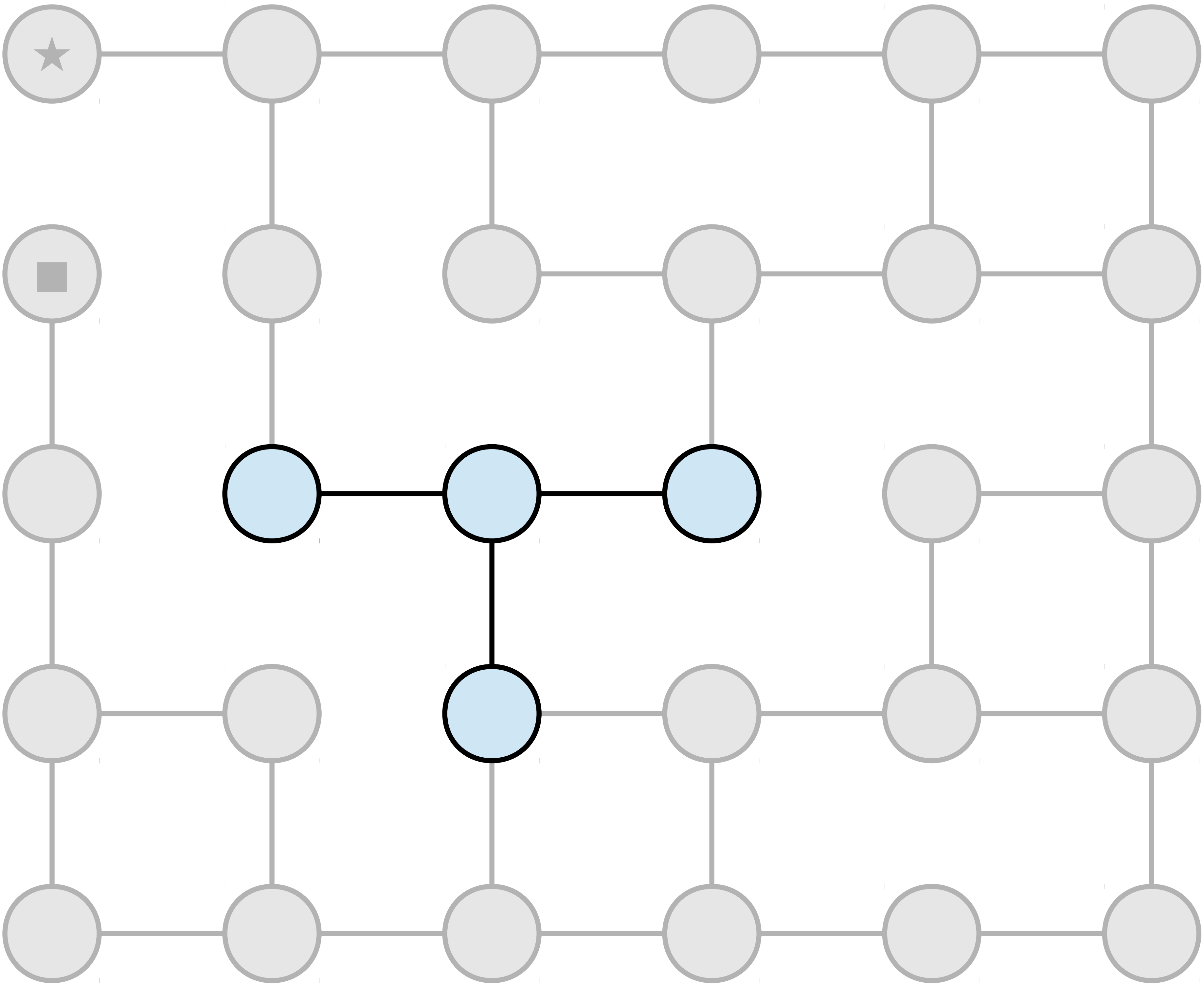
Reductions

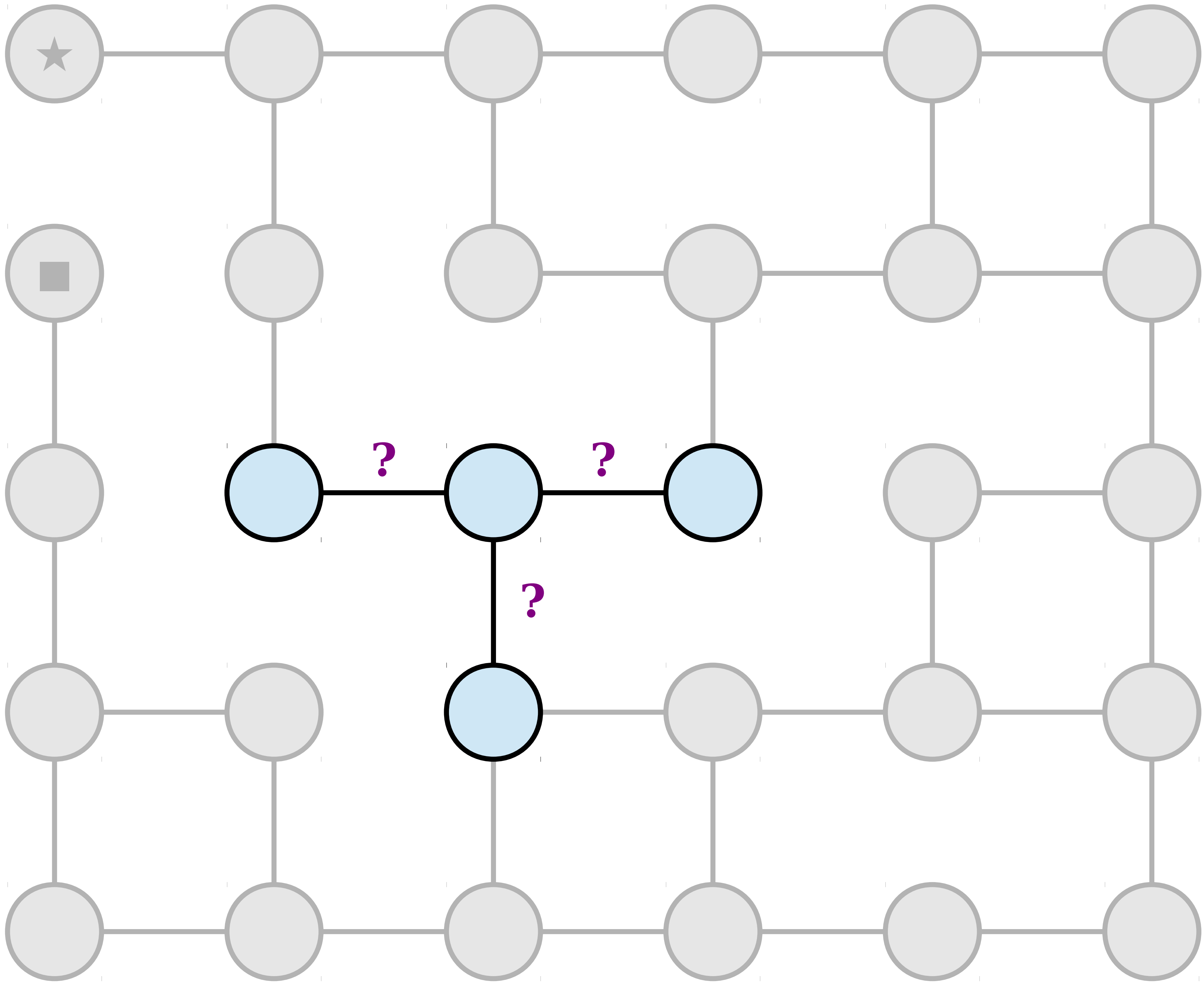
- Goal: Take our given graph $G = (V, E)$, starting node s, and starting direction d, then build a new graph $G' = (V', E')$ such that the following holds:

Shortest paths in G' correspond to minimum-turn paths in G.

- If we can build this graph G', our algorithm will be the following:
  - Build the graph G' out of G, s, and d.
  - Use an existing algorithm for finding shortest paths to find shortest paths in G'.
  - Using the shortest paths found in G', determine the minimum-turn path from s to t.

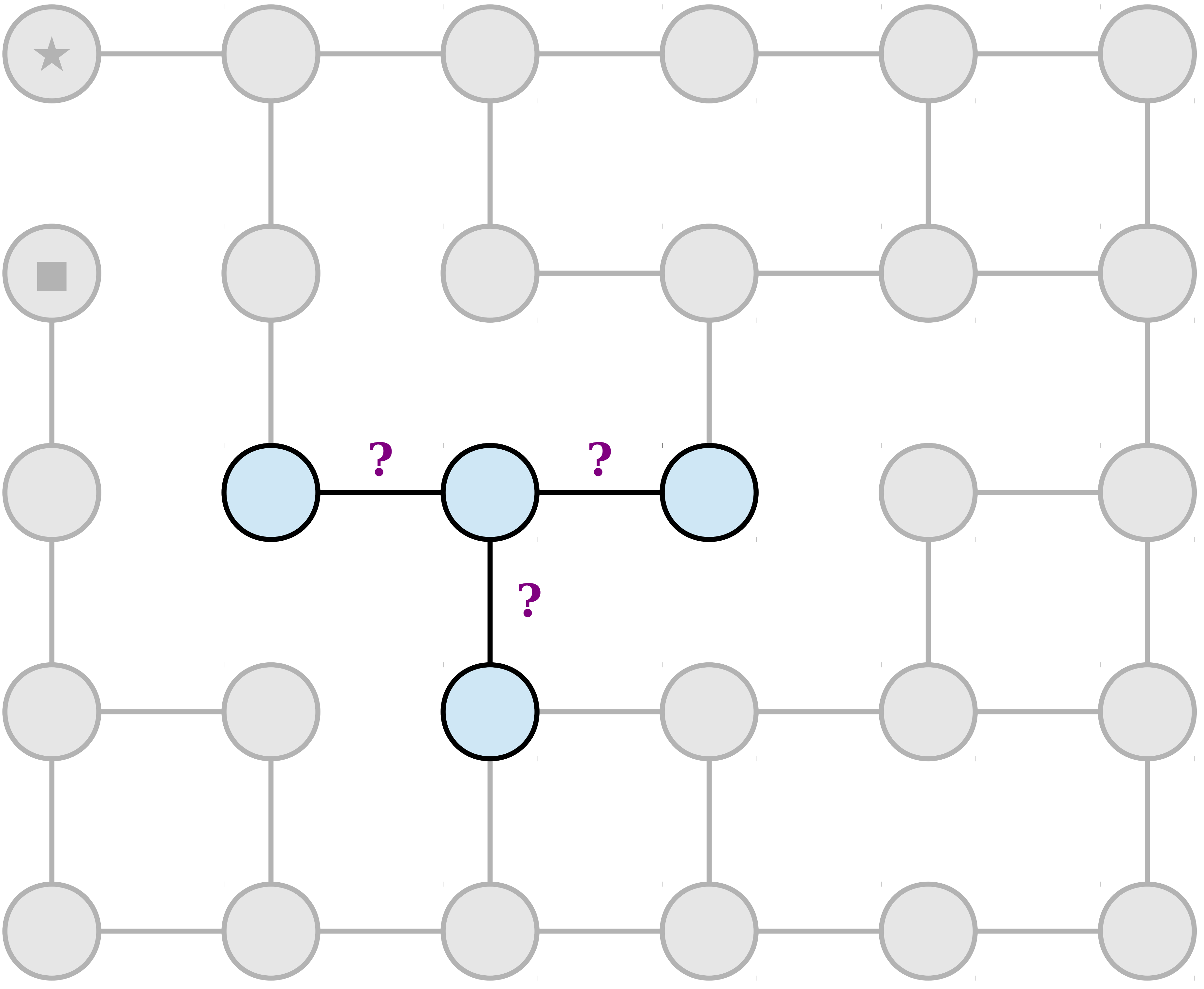


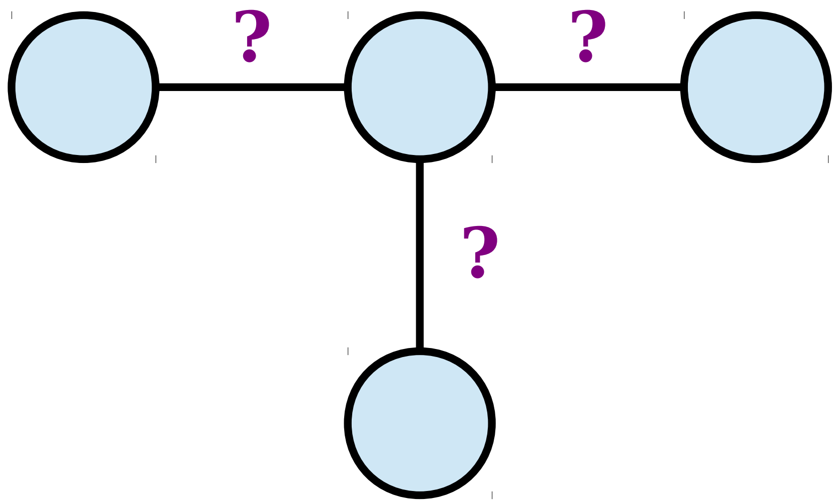


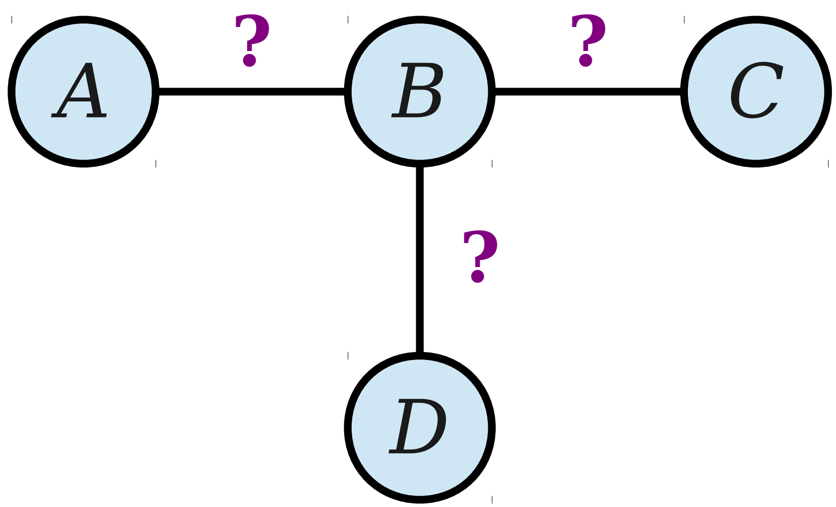


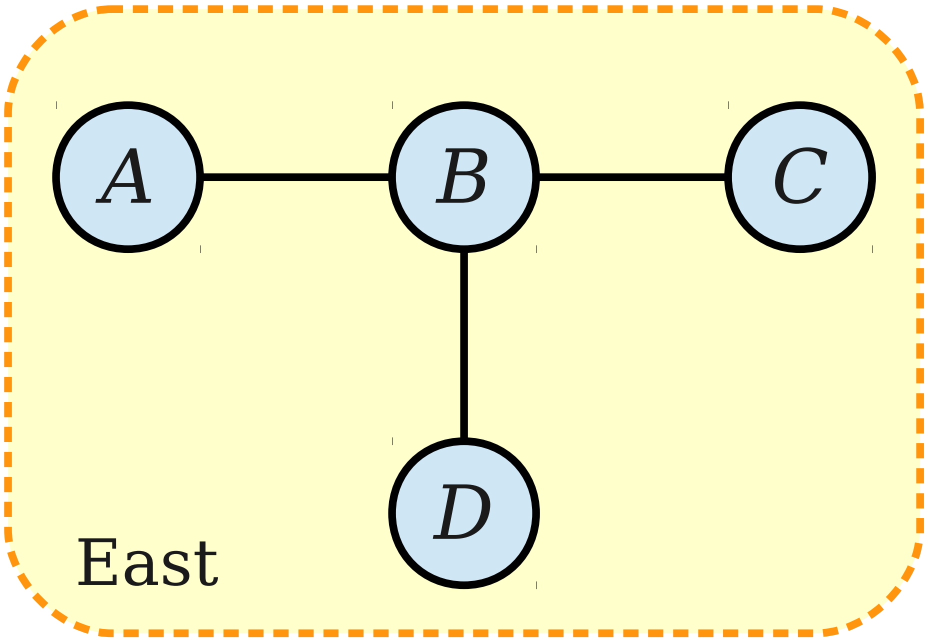
A Major Observation

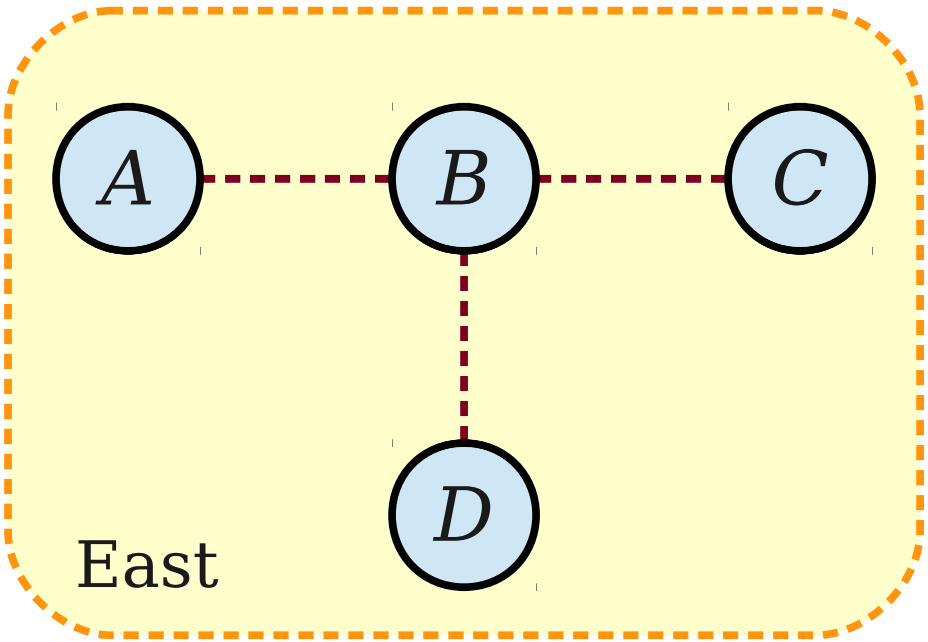
- When computing shortest paths in a graph, each node represents a possible “position” we can be in.
- In our problem, though, “position” also includes the direction you are currently facing.
- **Useful technique:** What if we create one node in the graph for each combination of a position in the original graph and a current direction?



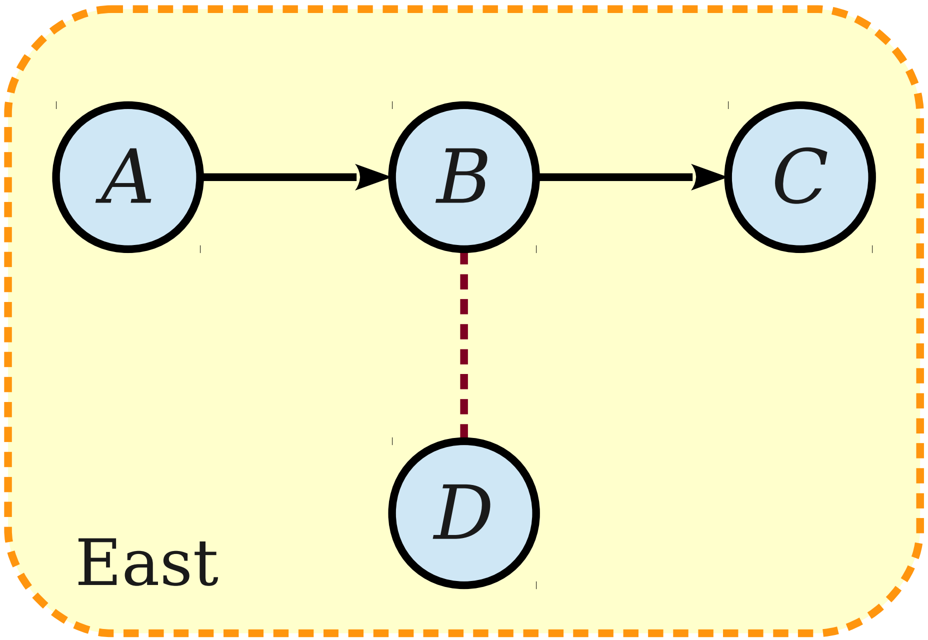



A
B
C
D
?
?
?

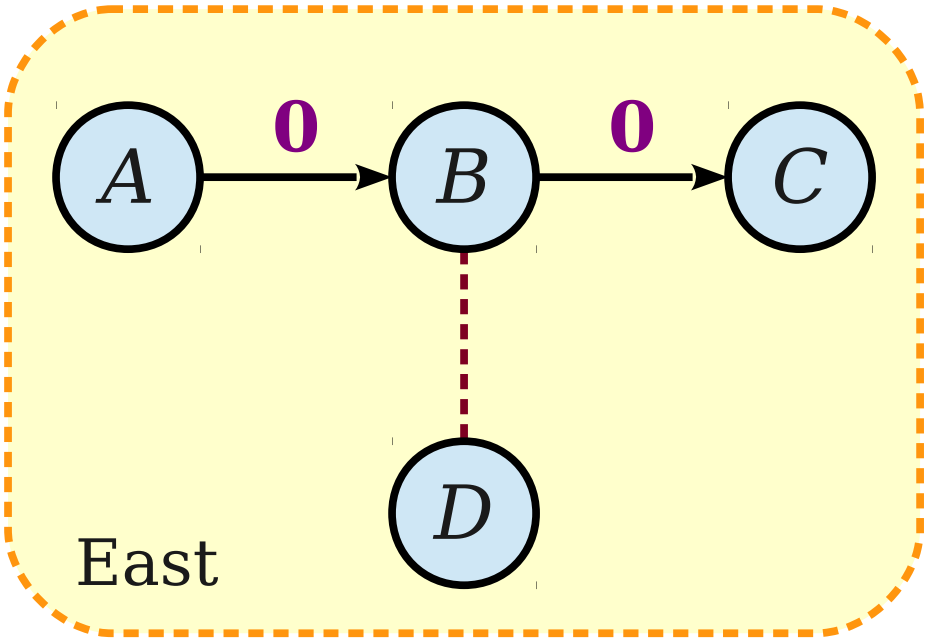

A
B
C
D
East



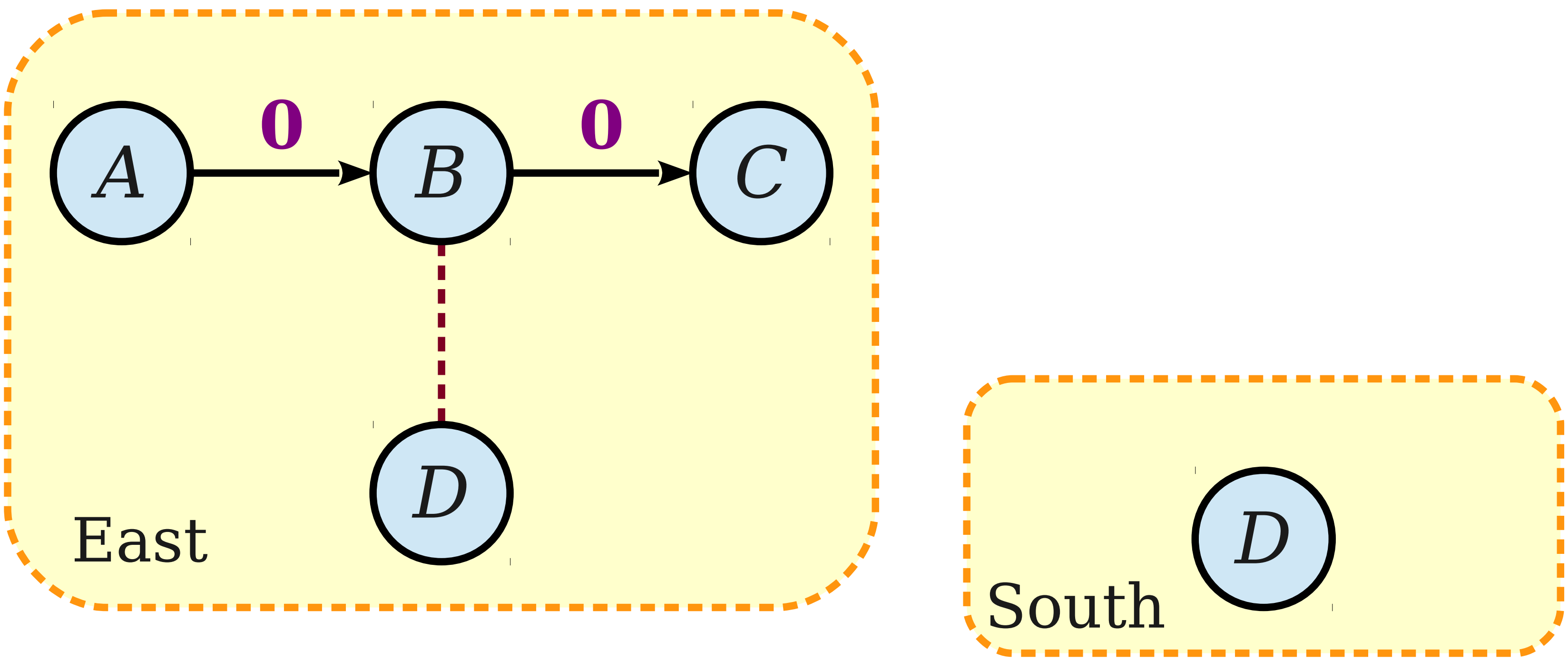
A
B
C
D
East



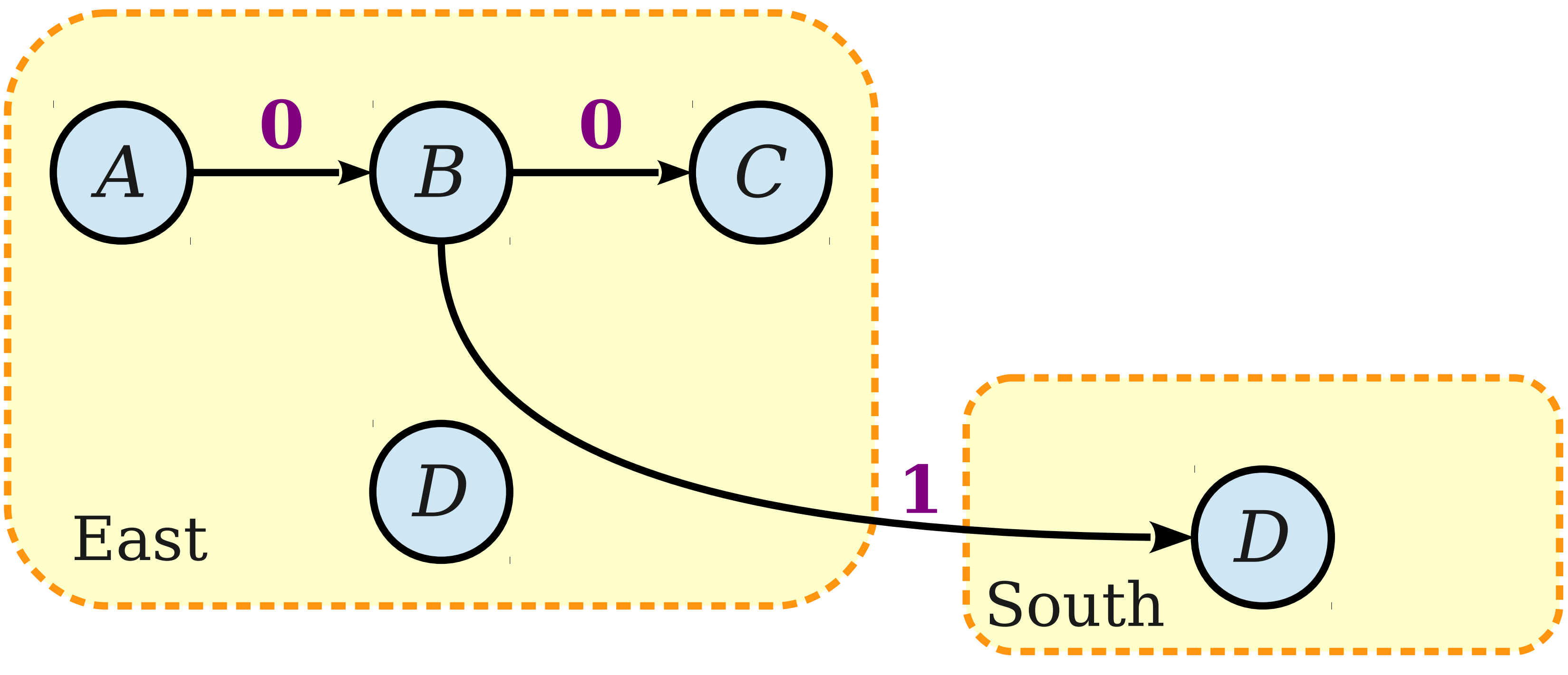
A
B
C
D
East

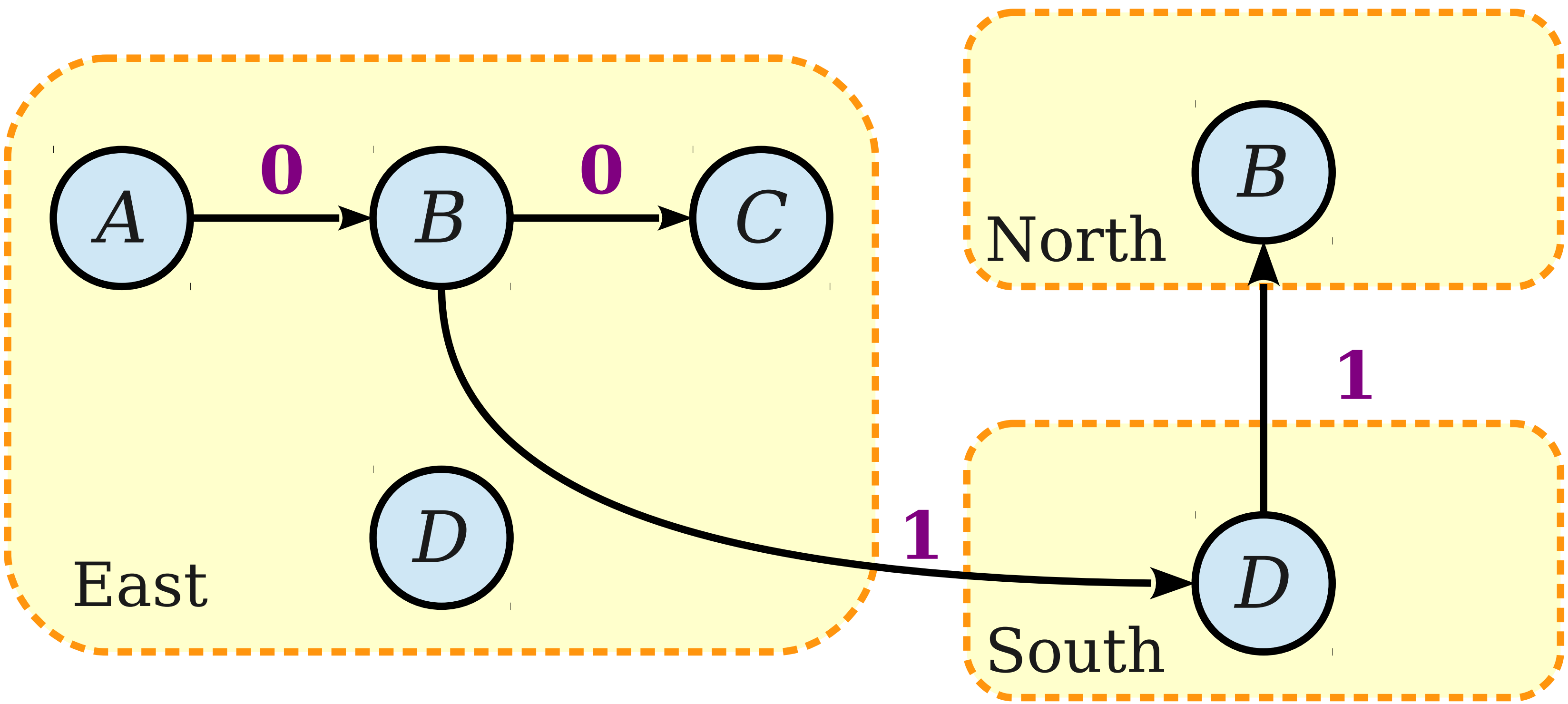


A
0
B
0
C
D
East

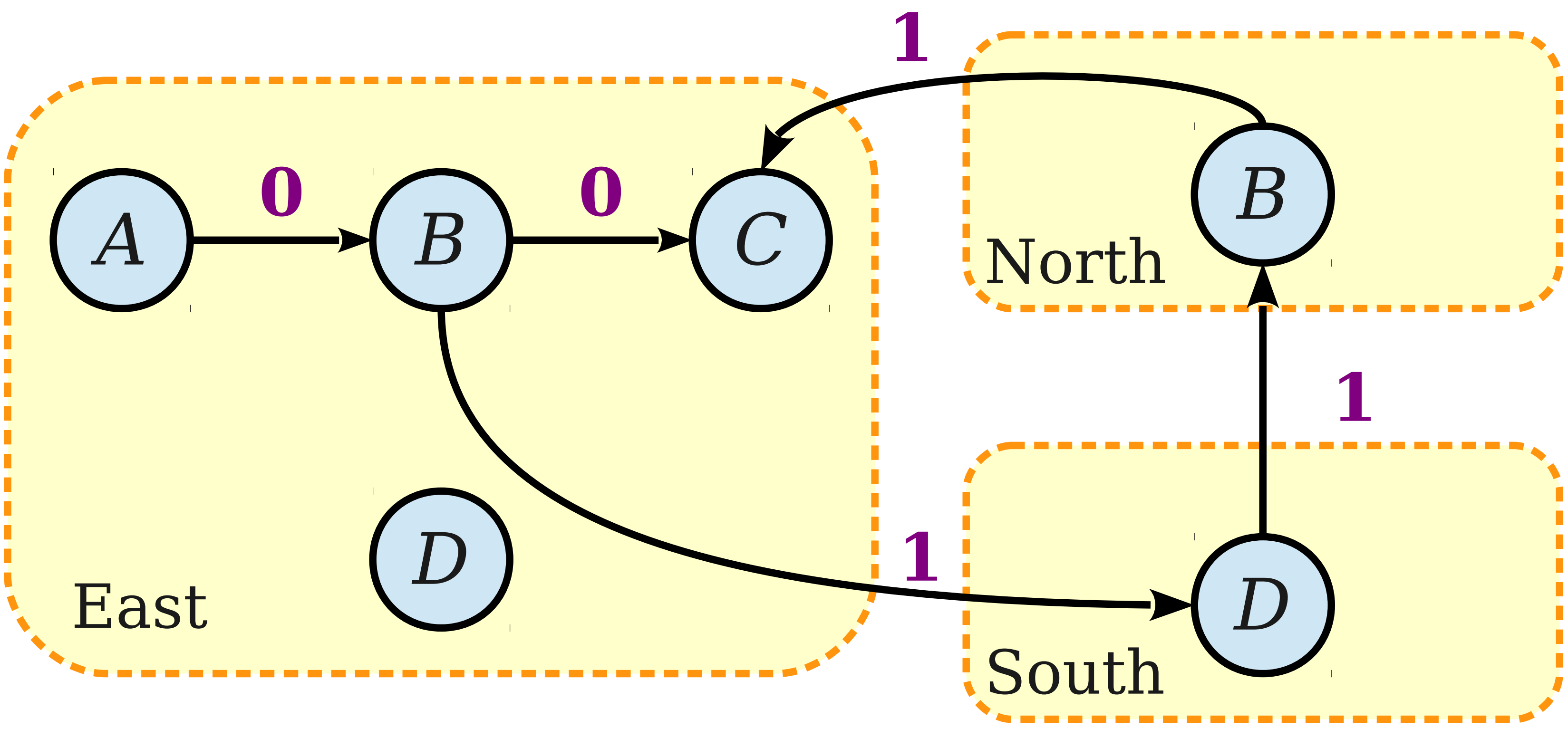


0
0
A
B
C
D
East
D
South


0
0
A
B
C
D
East
1
D
South



A
0
B
0
C
D
East
B
North
1
1
D
South



1
0
0
A
B
C
B
North
1
D
1
D
East
South

The Construction

- For each $v \in V$, construct four nodes:

$$\boldsymbol{v_{\mathrm{N}}, v_{\mathrm{S}}, v_{\mathrm{E}}, v_{\mathrm{W}}}$$

- For each edge $(u, v) \in E$ that goes in direction d, construct four edges:

$$\boldsymbol{(u_{\mathrm{N}}, v_{\mathrm{d}}), (u_{\mathrm{S}}, v_{\mathrm{d}}), (u_{\mathrm{E}}, v_{\mathrm{d}}), (u_{\mathrm{W}}, v_{\mathrm{d}})}$$

- Assign costs as follows:
  - $l(u_{d_1}, v_{d_2}) = 0$ if $d_1 = d_2$
  - $l(u_{d_1}, v_{d_2}) = 1$ if $d_1 \neq d_2$
- New graph has $4n$ nodes and $4m$ edges.

```
procedure minTurnPath(graph G, node s,
                      node t, direction d):
    construct G' from G as described earlier.

    run Dijkstra's algorithm to find shortest
    paths from s_d to each other node in G'.

    return the shortest of the following paths:
        the shortest path from s_d to t_N
        the shortest path from s_d to t_S
        the shortest path from s_d to t_E
        the shortest path from s_d to t_W
```

Correctness Proof Sketch

- Suppose we start at node s facing direction d. Our goal is to get to node t minimizing turns.
- Consider the length, in the new graph, of the shortest path P from s_d to t_x for any direction x.
- $l(P)$ is the sum of all the edge costs in path P. Edges that continue in the same direction cost 0 and edges that change direction cost 1, so $l(P)$ is the number of turns in P.
- Since P is chosen to minimize $l(P)$, P has the fewest number of turns of any path from s_d to t_x.
- The minimum-turn path from s to t is then the cheapest of the paths from s_d to t_N, t_S, t_E, t_W.

Formalizing the Proof

- To be more formal, we should prove the following results:
- ***Lemma 1:*** There is a path in G' from s_{d_1} to t_{d_2} iff there is a path in G from s to t that starts in direction d_1 and ends in direction d_2.
- ***Lemma 2:*** There is a path in G' from s_{d_1} to t_{d_2} of cost k iff there is a path in G from s to t that starts in direction d_1, ends in direction d_2, and makes k turns.
- **We will expect this level of detail in the problem sets.**

Analyzing the Runtime

- Time required to construct the new graph: $\Theta(n + m)$, since there are $4n$ nodes and $4m$ edges and each can be built in $\Theta(1)$ time.
- Time required to find the shortest paths in this graph: $O(n^2)$, or better if we use a faster Dijkstra's implementation.
- Overall runtime: **$O(n^2)$**.

Speeding Things Up

- The algorithm we've described is *correct,* but it can be made more efficient.
- Observation: Every edge in the graph has cost 0 or 1.
- Our algorithm uses Dijkstra's algorithm in this graph.
- Can we speed up Dijkstra's algorithm if all edges cost 0 or 1?

Some Observations

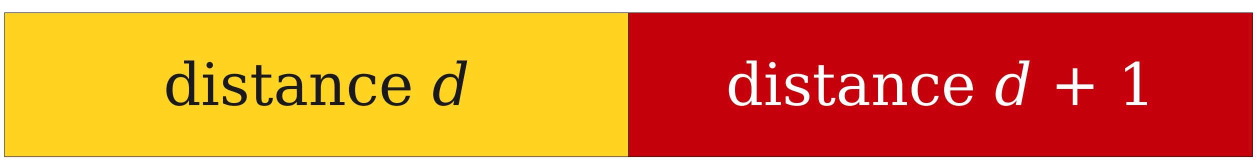
- Dijkstra's algorithm works by
  - Choosing the lowest-cost node in the fringe.
  - Updating costs to all adjacent nodes.
- **Fact 1:** Once Dijkstra's algorithm dequeues a node at distance *d*, all further nodes dequeued will be at distance $\geq d$.
- Can prove this inductively: Initial distance is 0, and all other distances are formed by adding edge costs (which are nonnegative) to the distance of the most recently-dequeued node.

Some Observations

- **Fact 2:** If all edge costs are 0 or 1, every node in the queue will either be at distance d or distance $d + 1$ for some d.
- Can prove this by induction:
  - Initially, all nodes in the queue are at distance 0.
  - If all nodes are at distance d or $d + 1$, we dequeue a node at distance d. All nodes connected to it will then be reinserted at distance either d or $d + 1$.

A Better Queue Structure

- Store the queue as a doubly-linked list. Elements at the front are at distance d and elements at the back are at distance $d + 1$.
  - Enqueue: Compare distance to distance at front. If equal, put at front. If greater, put at back.
  - Dequeue: Remove first element.
  - If a distance decreases from $d + 1$ to d, move that element to the front.
- All operations can be done in O(1) time.



Optimized Dijkstra's Algorithm

***Theorem:* In a graph where all edge costs are 0 or 1, Dijkstra's algorithm runs in time O(m + n).**

Proof Sketch: Use this new queue structure to store the nodes. Dijkstra's algorithm takes time O(m + n) plus the time required for O(m + n) queue operations, which with the new structure run in time O(1) each. Thus the runtime is O(m + n). ■

***Corollary:* The minimum-turns path problem can be solved in linear time.**

Why All This Matters

- Look at the structure of our solution:
  - Show how to solve the new problem (minimizing turns) using a solver for an existing algorithm.
  - Argue correctness using the fact that the existing algorithm is correct.
  - Argue runtime using the runtime of the existing algorithm.
  - ***(Optional)*** Speed up the algorithm by showing how to faithfully simulate the original algorithm in less time.
- Many problems can be solved this way.

Next Time

- Divide-and-Conquer Algorithms
- Mergesort
- Solving Recurrences