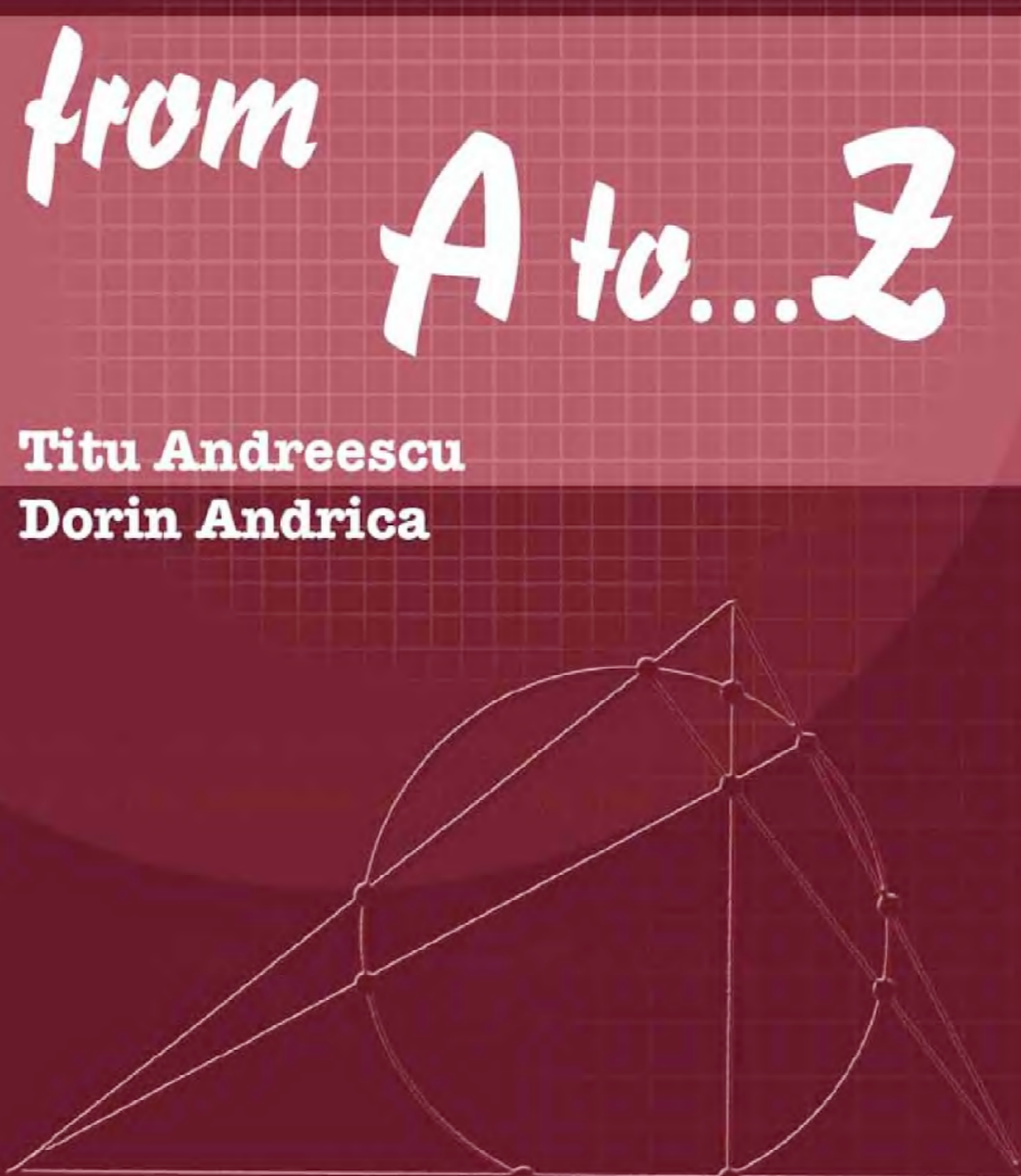
# Complex Numbers from A to...Z

Titu Andreescu
Dorin Andrica



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# About the Authors

**Titu Andreescu** received his BA, MS, and PhD from the West University of Timisoara, Romania. The topic of his doctoral dissertation was "Research on Diophantine Analysis and Applications." Professor Andreescu currently teaches at the University of Texas at Dallas. Titu is past chairman of the USA Mathematical Olympiad, served as director of the MAA American Mathematics Competitions (1998–2003), coach of the USA International Mathematical Olympiad Team (IMO) for 10 years (1993–2002), Director of the Mathematical Olympiad Summer Program (1995–2002) and leader of the USA IMO Team (1995–2002). In 2002 Titu was elected member of the IMO Advisory Board, the governing body of the world's most prestigious mathematics competition. Titu received the Edyth May Sliffe Award for Distinguished High School Mathematics Teaching from the MAA in 1994 and a "Certificate of Appreciation" from the president of the MAA in 1995 for his outstanding service as coach of the Mathematical Olympiad Summer Program in preparing the US team for its perfect performance in Hong Kong at the 1994 IMO. Titu's contributions to numerous textbooks and problem books are recognized worldwide.

**Dorin Andrica** received his PhD in 1992 from "Babeş-Bolyai" University in Cluj-Napoca, Romania, with a thesis on critical points and applications to the geometry of differentiable submanifolds. Professor Andrica has been chairman of the Department of Geometry at "Babeş-Bolyai" since 1995. Dorin has written and contributed to numerous mathematics textbooks, problem books, articles and scientific papers at various levels. Dorin is an invited lecturer at university conferences around the world—Austria, Bulgaria, Czech Republic, Egypt, France, Germany, Greece, the Netherlands, Serbia, Turkey, and USA. He is a member of the Romanian Committee for the Mathematics Olympiad and member of editorial boards of several international journals. Dorin has been a regular faculty member at the Canada–USA Mathcamps since 2001.

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*The shortest path between two truths in the real domain passes through the complex domain.*

Jacques Hadamard

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# Contents

# Preface

Solving algebraic equations has been historically one of the favorite topics of mathematicians. While linear equations are always solvable in real numbers, not all quadratic equations have this property. The simplest such equation is $x^2 + 1 = 0$. Until the 18th century, mathematicians avoided quadratic equations that were not solvable over $\mathbb{R}$. Leonhard Euler broke the ice introducing the "number" $\sqrt{-1}$ in his famous book *Elements of Algebra* as " ... *neither nothing, nor greater than nothing, nor less than nothing* ... " and observed " ... *notwithstanding this, these numbers present themselves to the mind; they exist in our imagination and we still have a sufficient idea of them;* ... *nothing prevents us from making use of these imaginary numbers, and employing them in calculation*". Euler denoted the number $\sqrt{-1}$ by $i$ and called it the imaginary unit. This became one of the most useful symbols in mathematics. Using this symbol one defines complex numbers as $z = a + bi$, where $a$ and $b$ are real numbers. The study of complex numbers continues and has been enhanced in the last two and a half centuries; in fact, it is impossible to imagine modern mathematics without complex numbers. All mathematical domains make use of them in some way. This is true of other disciplines as well: for example, mechanics, theoretical physics, hydrodynamics, and chemistry.

Our main goal is to introduce the reader to this fascinating subject. The book runs smoothly between key concepts and elementary results concerning complex numbers. The reader has the opportunity to learn how complex numbers can be employed in solving algebraic equations, and to understand the geometric interpretation of complex numbers and the operations involving them. The theoretical part of the book is augmented by rich exercises and problems of various levels of difficulty. In Chapters 3 and 4 we cover important applications in Euclidean geometry. Many geometry problems may be solved efficiently and elegantly using complex numbers. The wealth of examples we provide, the presentation of many topics in a personal manner, the presence of numerous original problems, and the attention to detail in the solutions to selected exercises and problems are only some of the key features of this book.

Among the techniques presented, for example, are those for the real and the complex product of complex numbers. In complex number language, these are the analogues of the scalar and cross products, respectively. Employing these two products turns out to be efficient in solving numerous problems involving complex numbers. After covering this part, the reader will appreciate the use of these techniques.

A special feature of the book is Chapter 5, an outstanding selection of genuine Olympiad and other important mathematical contest problems solved using the methods already presented.

This work does not cover all aspects pertaining to complex numbers. It is not a complex analysis book, but rather a stepping stone in its study, which is why we have not used the standard notation $e^{it}$ for $z = \cos t + i \sin t$, or the usual power series expansions.

The book reflects the unique experience of the authors. It distills a vast mathematical literature, most of which is unknown to the western public, capturing the essence of an abundant problem-solving culture.

Our work is partly based on a Romanian version, *Numere complexe de la A la . . . Z*, authored by D. Andrica and N. Bişboacă and published by Millennium in 2001 (see our reference [10]). We are preserving the title of the Romanian edition and about 35% of the text. Even this 35% has been significantly improved and enhanced with up-to-date material.

The targeted audience includes high school students and their teachers, undergraduates, mathematics contestants such as those training for Olympiads or the W. L. Putnam Mathematical Competition, their coaches, and any person interested in essential mathematics.

This book might spawn courses such as Complex Numbers and Euclidean Geometry for prospective high school teachers, giving future educators ideas about things they could do with their brighter students or with a math club. This would be quite a welcome development.

Special thanks are given to Daniel Văcăreţu, Nicolae Bişboacă, Gabriel Dospinescu, and Ioan Şerdean for the careful proofreading of the final version of the manuscript. We

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Titu Andreescu
Dorin Andrica
October 2004

# Notation

| | |
|---|---|
| $\mathbb{Z}$ | the set of integers |
| $\mathbb{N}$ | the set of positive integers |
| $\mathbb{Q}$ | the set of rational numbers |
| $\mathbb{R}$ | the set of real numbers |
| $\mathbb{R}^*$ | the set of nonzero real numbers |
| $\mathbb{R}^2$ | the set of pairs of real numbers |
| $\mathbb{C}$ | the set of complex numbers |
| $\mathbb{C}^*$ | the set of nonzero complex numbers |
| $[a, b]$ | the set of real numbers $x$ such that $a \leq x \leq b$ |
| $(a, b)$ | the set of real numbers $x$ such that $a < x < b$ |
| $\overline{z}$ | the conjugate of the complex number $z$ |
| $\lvert z \rvert$ | the modulus or absolute value of complex number $z$ |
| $\overrightarrow{AB}$ | the vector $AB$ |
| $(AB)$ | the open segment determined by $A$ and $B$ |
| $[AB]$ | the closed segment determined by $A$ and $B$ |
| $(AB$ | the open ray of origin $A$ that contains $B$ |
| area$[F]$ | the area of figure $F$ |
| $U_n$ | the set of $n^{\text{th}}$ roots of unity |
| $\mathcal{C}(P; n)$ | the circle centered at point $P$ with radius $n$ |

# 1

# Complex Numbers in Algebraic Form

## 1.1 Algebraic Representation of Complex Numbers

### 1.1.1 Definition of complex numbers

In what follows we assume that the definition and basic properties of the set of real numbers $\mathbb{R}$ are known.

Let us consider the set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) | \, x, y \in \mathbb{R}\}$. Two elements $(x_1, y_1)$ and $(x_2, y_2)$ of $\mathbb{R}^2$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$. The operations of addition and multiplication are defined on the set $\mathbb{R}^2$ as follows:

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$$

and

$$z_1 \cdot z_2 = (x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) \in \mathbb{R}^2,$$

for all $z_1 = (x_1, y_1) \in \mathbb{R}^2$ and $z_2 = (x_2, y_2) \in \mathbb{R}^2$.

The element $z_1 + z_2 \in \mathbb{R}^2$ is called the *sum* of $z_1, z_2$ and the element $z_1 \cdot z_2 \in \mathbb{R}^2$ is called the *product* of $z_1, z_2$.

**Remarks.** 1) If $z_1 = (x_1, 0) \in \mathbb{R}^2$ and $z_2 = (x_2, 0) \in \mathbb{R}^2$, then $z_1 \cdot z_2 = (x_1x_2, 0)$.
(2) If $z_1 = (0, y_1) \in \mathbb{R}^2$ and $z_2 = (0, y_2) \in \mathbb{R}^2$, then $z_1 \cdot z_2 = (-y_1y_2, 0)$.

**Examples.** 1) Let $z_1 = (-5, 6)$ and $z_2 = (1, -2)$. Then

$$z_1 + z_2 = (-5, 6) + (1, -2) = (-4, 4)$$

and

$$z_1 z_2 = (-5, 6) \cdot (1, -2) = (-5 + 12, 10 + 6) = (7, 16).$$

(2) Let $z_1 = \left(-\frac{1}{2}, 1\right)$ and $z_2 = \left(-\frac{1}{3}, \frac{1}{2}\right)$. Then

$$z_1 + z_2 = \left(-\frac{1}{2} - \frac{1}{3}, 1 + \frac{1}{2}\right) = \left(-\frac{5}{6}, \frac{3}{2}\right)$$

and

$$z_1 z_2 = \left(\frac{1}{6} - \frac{1}{2}, -\frac{1}{4} - \frac{1}{3}\right) = \left(-\frac{1}{3}, -\frac{7}{12}\right).$$

**Definition.** The set $\mathbb{R}^2$, together with the addition and multiplication operations, is called the *set of complex numbers*, denoted by $\mathbb{C}$. Any element $z = (x, y) \in \mathbb{C}$ is called a *complex number*.

The notation $\mathbb{C}^*$ is used to indicate the set $\mathbb{C} \setminus \{(0, 0)\}$.

### 1.1.2 Properties concerning addition

The addition of complex numbers satisfies the following properties:

(a) **Commutative law**

$$z_1 + z_2 = z_2 + z_1 \text{ for all } z_1, z_2 \in \mathbb{C}.$$

(b) **Associative law**

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \text{ for all } z_1, z_2, z_3 \in \mathbb{C}.$$

Indeed, if $z_1 = (x_1, y_1) \in \mathbb{C}$, $z_2 = (x_2, y_2) \in \mathbb{C}$, $z_3 = (x_3, y_3) \in \mathbb{C}$, then

$$(z_1 + z_2) + z_3 = [(x_1, y_1) + (x_2, y_2)] + (x_3, y_3)$$

$$= (x_1 + x_2, y_1 + y_2) + (x_3, y_3) = ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3),$$

and

$$z_1 + (z_2 + z_3) = (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)]$$

$$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3) = (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3)).$$

The claim holds due to the associativity of the addition of real numbers.

(c) **Additive identity** There is a unique complex number $0 = (0, 0)$ such that

$$z + 0 = 0 + z = z \text{ for all } z = (x, y) \in \mathbb{C}.$$

(d) **Additive inverse** For any complex number $z = (x, y)$ there is a unique $-z = (-x, -y) \in \mathbb{C}$ such that

$$z + (-z) = (-z) + z = 0.$$

The reader can easily prove the claims (a), (c) and (d).

The number $z_1 - z_2 = z_1 + (-z_2)$ is called the *difference* of the numbers $z_1$ and $z_2$. The operation that assigns to the numbers $z_1$ and $z_2$ the number $z_1 - z_2$ is called *subtraction* and is defined by

$$z_1 - z_2 = (x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2) \in \mathbb{C}.$$

## 1.1.3 Properties concerning multiplication

The multiplication of complex numbers satisfies the following properties:

(a) **Commutative law**

$$z_1 \cdot z_2 = z_2 \cdot z_1 \text{ for all } z_1, z_2 \in \mathbb{C}.$$

(b) **Associative law**

$$(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3) \text{ for all } z_1, z_2, z_3 \in \mathbb{C}.$$

(c) **Multiplicative identity** There is a unique complex number $1 = (1, 0) \in \mathbb{C}$ such that

$$z \cdot 1 = 1 \cdot z = z \text{ for all } z \in \mathbb{C}.$$

A simple algebraic manipulation is all that is needed to verify these equalities:

$$z \cdot 1 = (x, y) \cdot (1, 0) = (x \cdot 1 - y \cdot 0, x \cdot 0 + y \cdot 1) = (x, y) = z$$

and

$$1 \cdot z = (1, 0) \cdot (x, y) = (1 \cdot x - 0 \cdot y, 1 \cdot y + 0 \cdot x) = (x, y) = z.$$

(d) **Multiplicative inverse** For any complex number $z = (x, y) \in \mathbb{C}^*$ there is a unique number $z^{-1} = (x', y') \in \mathbb{C}$ such that

$$z \cdot z^{-1} = z^{-1} \cdot z = 1.$$

To find $z^{-1} = (x', y')$, observe that $(x, y) \neq (0, 0)$ implies $x \neq 0$ or $y \neq 0$ and consequently $x^2 + y^2 \neq 0$.

The relation $z \cdot z^{-1} = 1$ gives $(x, y) \cdot (x', y') = (1, 0)$, or equivalently

$$\begin{cases} xx' - yy' = 1 \\ yx' + xy' = 0. \end{cases}$$

Solving this system with respect to $x'$ and $y'$, one obtains

$$x' = \frac{x}{x^2 + y^2} \text{ and } y' = -\frac{y}{x^2 + y^2},$$

hence the multiplicative inverse of the complex number $z = (x, y) \in \mathbb{C}^*$ is

$$z^{-1} = \frac{1}{z} = \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right) \in \mathbb{C}^*.$$

By the commutative law we also have $z^{-1} \cdot z = 1$.

Two complex numbers $z_1 = (z_1, y_1) \in \mathbb{C}$ and $z = (x, y) \in \mathbb{C}^*$ uniquely determine a third number called their *quotient*, denoted by $\frac{z_1}{z}$ and defined by

$$\frac{z_1}{z} = z_1 \cdot z^{-1} = (x_1, y_1) \cdot \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right)$$
$$= \left(\frac{x_1 x + y_1 y}{x^2 + y^2}, \frac{-x_1 y + y_1 x}{x^2 + y^2}\right) \in \mathbb{C}.$$

**Examples.** 1) If $z = (1, 2)$, then

$$z^{-1} = \left(\frac{1}{1^2 + 2^2}, \frac{-2}{1^2 + 2^2}\right) = \left(\frac{1}{5}, \frac{-2}{5}\right).$$

2) If $z_1 = (1, 2)$ and $z_2 = (3, 4)$, then

$$\frac{z_1}{z_2} = \left(\frac{3 + 8}{9 + 16}, \frac{-4 + 6}{9 + 16}\right) = \left(\frac{11}{25}, \frac{2}{25}\right).$$

An integer power of a complex number $z \in \mathbb{C}^*$ is defined by

$$z^0 = 1; \quad z^1 = z; \quad z^2 = z \cdot z;$$

$$z^n = \underbrace{z \cdot z \cdots z}_{n \text{ times}} \text{ for all integers } n > 0$$

and $z^n = (z^{-1})^{-n}$ for all integers $n < 0$.

The following properties hold for all complex numbers $z, z_1, z_2 \in \mathbb{C}^*$ and for all integers $m, n$:

1) $z^m \cdot z^n = z^{m+n}$;
2) $\frac{z^m}{z^n} = z^{m-n}$;
3) $(z^m)^n = z^{mn}$;
4) $(z_1 \cdot z_2)^n = z_1^n \cdot z_2^n$;
5) $\left(\frac{z_1}{z_2}\right)^n = \frac{z_1^n}{z_2^n}$.

When $z = 0$, we define $0^n = 0$ for all integers $n > 0$.

e) **Distributive law**

$$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3 \text{ for all } z_1, z_2, z_3 \in \mathbb{C}.$$

The above properties of addition and multiplication show that the set $\mathbb{C}$ of all complex numbers, together with these operations, forms a field.

### 1.1.4 Complex numbers in algebraic form

For algebraic manipulation it is not convenient to represent a complex number as an ordered pair. For this reason another form of writing is preferred.

To introduce this new algebraic representation, consider the set $\mathbb{R} \times \{0\}$, together with the addition and multiplication operations defined on $\mathbb{R}^2$. The function

$$f : \mathbb{R} \to \mathbb{R} \times \{0\}, \quad f(x) = (x, 0)$$

is bijective and moreover,

$$(x, 0) + (y, 0) = (x + y, 0) \text{ and } (x, 0) \cdot (y, 0) = (xy, 0).$$

The reader will not fail to notice that the algebraic operations on $\mathbb{R} \times \{0\}$ are similar to the operations on $\mathbb{R}$; therefore we can identify the ordered pair $(x, 0)$ with the number $x$ for all $x \in \mathbb{R}$. Hence we can use, by the above bijection $f$, the notation $(x, 0) = x$.

Setting $i = (0, 1)$ we obtain

$$z = (x, y) = (x, 0) + (0, y) = (x, 0) + (y, 0) \cdot (0, 1)$$

$$= x + yi = (x, 0) + (0, 1) \cdot (y, 0) = x + iy.$$

In this way we obtain

**Proposition.** *Any complex number $z = (x, y)$ can be uniquely represented in the form*

$$z = x + yi,$$

*where $x, y$ are real numbers. The relation $i^2 = -1$ holds.*

The formula $i^2 = -1$ follows directly from the definition of multiplication: $i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (-1, 0) = -1$.

The expression $x + yi$ is called the *algebraic representation* (*form*) of the complex number $z = (x, y)$, so we can write $\mathbb{C} = \{x + yi \mid x \in \mathbb{R},\ y \in \mathbb{R},\ i^2 = -1\}$. From now on we will denote the complex number $z = (x, y)$ by $x + iy$. The real number $x = \operatorname{Re}(z)$ is called the *real part* of the complex number $z$ and similarly, $y = \operatorname{Im}(z)$ is called the *imaginary part* of $z$. Complex numbers of the form $iy$, $y \in \mathbb{R}$ — in other words, complex numbers whose real part is 0 — are called *imaginary*. On the other hand, complex numbers of the form $iy$, $y \in \mathbb{R}^*$ are called *purely imaginary* and the complex number $i$ is called the *imaginary unit*.

The following relations are easy to verify:

a) $z_1 = z_2$ if and only if $\text{Re}(z)_1 = \text{Re}(z)_2$ and $\text{Im}(z)_1 = \text{Im}(z)_2$.

b) $z \in \mathbb{R}$ if and only if $\text{Im}(z) = 0$.

c) $z \in \mathbb{C} \setminus \mathbb{R}$ if and only if $\text{Im}(z) \neq 0$.

Using the algebraic representation, the usual operations with complex numbers can be performed as follows:

**1. Addition**

$$z_1 + z_2 = (x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i \in \mathbb{C}.$$

It is easy to observe that the sum of two complex numbers is a complex number whose real (imaginary) part is the sum of the real (imaginary) parts of the given numbers:

$$\text{Re}(z_1 + z_2) = \text{Re}(z)_1 + \text{Re}(z)_2;$$
$$\text{Im}(z_1 + z_2) = \text{Im}(z)_1 + \text{Im}(z)_2.$$

**2. Multiplication**

$$z_1 \cdot z_2 = (x_1 + y_1 i)(x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i \in \mathbb{C}.$$

In other words,

$$\text{Re}(z_1 z_2) = \text{Re}(z)_1 \cdot \text{Re}(z)_2 - \text{Im}(z)_1 \cdot \text{Im}(z)_2$$

and

$$\text{Im}(z_1 z_2) = \text{Im}(z)_1 \cdot \text{Re}(z)_2 + \text{Im}(z)_2 \cdot \text{Re}(z)_1.$$

For a real number $\lambda$ and a complex number $z = x + yi$,

$$\lambda \cdot z = \lambda(x + yi) = \lambda x + \lambda y i \in \mathbb{C}$$

is the product of a real number with a complex number. The following properties are obvious:

1) $\lambda(z_1 + z_2) = \lambda z_1 + \lambda z_2$;

2) $\lambda_1(\lambda_2 z) = (\lambda_1 \lambda_2)z$;

3) $(\lambda_1 + \lambda_2)z = \lambda_1 z + \lambda_2 z$ for all $z, z_1, z_2 \in \mathbb{C}$ and $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$.

Actually, relations 1) and 3) are special cases of the distributive law and relation 2) comes from the associative law of multiplication for complex numbers.

**3. Subtraction**

$$z_1 - z_2 = (x_1 + y_1 i) - (x_2 + y_2 i) = (x_1 - x_2) + (y_1 - y_2)i \in \mathbb{C}.$$

That is,

$$\operatorname{Re}(z_1 - z_2) = \operatorname{Re}(z)_1 - \operatorname{Re}(z)_2;$$
$$\operatorname{Im}(z_1 - z_2) = \operatorname{Im}(z)_1 - \operatorname{Im}(z)_2.$$

### 1.1.5 Powers of the number $i$

The formulas for the powers of a complex number with integer exponents are preserved for the algebraic form $z = x + iy$. Setting $z = i$, we obtain

$$i^0 = 1; \quad i^1 = i; \quad i^2 = -1; \quad i^3 = i^2 \cdot i = -i;$$
$$i^4 = i^3 \cdot i = 1; \quad i^5 = i^4 \cdot i = i; \quad i_6 = i^5 \cdot i = -1; \quad i^7 = i^6 \cdot i = -i.$$

One can prove by induction that for any positive integer $n$,

$$i^{4n} = 1; \quad i^{4n+1} = i; \quad i^{4n+2} = -1; \quad i^{4n+3} = -i.$$

Hence $i^n \in \{-1, 1, -i, i\}$ for all integers $n \geq 0$. If $n$ is a negative integer, we have

$$i^n = (i^{-1})^{-n} = \left(\frac{1}{i}\right)^{-n} = (-i)^{-n}.$$

**Examples.** 1) We have

$$i^{105} + i^{23} + i^{20} - i^{34} = i^{4\cdot 26+1} + i^{4\cdot 5+3} + i^{4\cdot 5} - i^{4\cdot 8+2} = i - i + 1 + 1 = 2.$$

2) Let us solve the equation $z^3 = 18 + 26i$, where $z = x + yi$ and $x, y$ are integers. We can write

$$(x + yi)^3 = (x + yi)^2(x + yi) = (x^2 - y^2 + 2xyi)(x + yi)$$
$$= (x^3 - 3xy^2) + (3x^2y - y^3)i = 18 + 26i.$$

Using the definition of equality of complex numbers, we obtain

$$\begin{cases} x^3 - 3xy^2 = 18 \\ 3x^2y - y^3 = 26. \end{cases}$$

Setting $y = tx$ in the equality $18(3x^2y - y^3) = 26(x^3 - 3xy^2)$, let us observe that $x \neq 0$ and $y \neq 0$ implies $18(3t - t^3) = 26(1 - 3t^2)$. The last relation is equivalent to $(3t - 1)(3t^2 - 12t - 13) = 0$.

The only rational solution of this equation is $t = \dfrac{1}{3}$; hence,

$$x = 3, \ y = 1 \text{ and } z = 3 + i.$$

### 1.1.6 Conjugate of a complex number

For a complex number $z = x + yi$ the number $\overline{z} = x - yi$ is called the *complex conjugate* or the *conjugate complex* of $z$.

**Proposition.** *1) The relation $z = \overline{z}$ holds if and only if $z \in \mathbb{R}$.*

*2) For any complex number $z$ the relation $z = \overline{\overline{z}}$ holds.*

*3) For any complex number $z$ the number $z \cdot \overline{z} \in \mathbb{R}$ is a nonnegative real number.*

*4) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ (the conjugate of a sum is the sum of the conjugates).*

*5) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$ (the conjugate of a product is the product of the conjugates).*

*6) For any nonzero complex number $z$ the relation $\overline{z^{-1}} = (\overline{z})^{-1}$ holds.*

*7) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$ (the conjugate of a quotient is the quotient of the conjugates).*

*8) The formulas*

$$\operatorname{Re}(z) = \frac{z + \overline{z}}{2} \quad \textit{and} \quad \operatorname{Im}(z) = \frac{z - \overline{z}}{2i}$$

*are valid for all $z \in \mathbb{C}$.*

*Proof.* 1) If $z = x + yi$, then the relation $z = \overline{z}$ is equivalent to $x + yi = x - yi$. Hence $2yi = 0$, so $y = 0$ and finally $z = x \in \mathbb{R}$.

2) We have $\overline{z} = x - yi$ and $\overline{\overline{z}} = x - (-y)i = x + yi = z$.

3) Observe that $z \cdot \overline{z} = (x + yi)(x - yi) = x^2 + y^2 \geq 0$.

4) Note that

$$\overline{z_1 + z_2} = \overline{(x_1 + x_2) + (y_1 + y_2)i} = (x_1 + x_2) - (y_1 + y_2)i$$

$$= (x_1 - y_1 i) + (x_2 - y_2 i) = \overline{z_1} + \overline{z_2}.$$

5) We can write

$$\overline{z_1 \cdot z_2} = \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)}$$

$$= (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1) = (x_1 - iy_1)(x_2 - iy_2) = \overline{z_1} \cdot \overline{z_2}.$$

6) Because $z \cdot \dfrac{1}{z} = 1$, we have $\overline{\left(z \cdot \dfrac{1}{z}\right)} = \overline{1}$, and consequently $\overline{z} \cdot \overline{\left(\dfrac{1}{z}\right)} = 1$, yielding $(\overline{z^{-1}}) = (\overline{z})^{-1}$.

7) Observe that $\overline{\left(\dfrac{z_1}{z_2}\right)} = \overline{\left(z_1 \cdot \dfrac{1}{z_2}\right)} = \overline{z_1} \cdot \overline{\left(\dfrac{1}{z_2}\right)} = \overline{z_1} \cdot \dfrac{1}{\overline{z_2}} = \dfrac{\overline{z_1}}{\overline{z_2}}$.

8) From the relations

$$z + \overline{z} = (x + yi) + (x - yi) = 2x,$$

$$z - \overline{z} = (x + yi) - (x - yi) = 2yi$$

it follows that

$$\text{Re}(z) = \frac{z + \overline{z}}{2} \quad \text{and} \quad \text{Im}(z) = \frac{z - \overline{z}}{2i}$$

as desired. □

The properties 4) and 5) can be easily extended to give

4′) $\overline{\left(\sum_{k=1}^{n} z_k\right)} = \sum_{k=1}^{n} \overline{z_k}$;

5′) $\overline{\left(\prod_{k=1}^{n} z_k\right)} = \prod_{k=1}^{n} \overline{z_k}$ for all $z_k \in \mathbb{C}$, $k = 1, 2, \ldots, n$.

As a consequence of 5′) and 6) we have

5″) $(\overline{z^n}) = (\overline{z})^n$ for any integers $n$ and for any $z \in \mathbb{C}$.

**Comments.** a) To obtain the multiplication inverse of a complex number $z \in \mathbb{C}^*$ one can use the following approach:

$$\frac{1}{z} = \frac{\overline{z}}{z \cdot \overline{z}} = \frac{x - yi}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} i.$$

b) The complex conjugate allows us to obtain the quotient of two complex numbers as follows:

$$\frac{z_1}{z_2} = \frac{z_1 \cdot \overline{z_2}}{z_2 \cdot \overline{z_2}} = \frac{(x_1 + y_1 i)(x_2 - y_2 i)}{x_2^2 + y_2^2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{-x_1 y_2 + x_2 y_1}{x_2^2 + y_2^2} i.$$

**Examples.** (1) Compute $z = \dfrac{5 + 5i}{3 - 4i} + \dfrac{20}{4 + 3i}$.

**Solution.** We can write

$$z = \frac{(5 + 5i)(3 + 4i)}{9 - 16i^2} + \frac{20(4 - 3i)}{16 - 9i^2} = \frac{-5 + 35i}{25} + \frac{80 - 60i}{25}$$

$$= \frac{75 - 25i}{25} = 3 - i.$$

(2) Let $z_1, z_2 \in \mathbb{C}$. Prove that the number $E = z_1 \cdot \overline{z_2} + \overline{z_1} \cdot z_2$ is a real number.

**Solution.** We have

$$\overline{E} = \overline{z_1 \cdot \overline{z_2} + \overline{z_1} \cdot z_2} = \overline{z_1} \cdot z_2 + z_1 \cdot \overline{z_2} = E, \text{ so } E \in \mathbb{R}.$$

### 1.1.7 Modulus of a complex number

The number $|z| = \sqrt{x^2 + y^2}$ is called the *modulus* or the *absolute value* of the complex number $z = x + yi$. For example, the complex numbers

$$z_1 = 4 + 3i, \quad z_2 = -3i, \quad z_3 = 2$$

have the moduli

$$|z_1| = \sqrt{4^2 + 3^2} = 5, \quad |z_2| = \sqrt{0^2 + (-3)^2} = 3, \quad |z_3| = \sqrt{2^2} = 2.$$

**Proposition.** *The following properties are satisfied:*

*(1)* $-|z| \leq \mathrm{Re}(z) \leq |z|$ *and* $-|z| \leq \mathrm{Im}(z) \leq |z|$.

*(2)* $|z| \geq 0$ *for all* $z \in \mathbb{C}$. *Moreover, we have* $|z| = 0$ *if and only if* $z = 0$.

*(3)* $|z| = |-z| = |\overline{z}|$.

*(4)* $z \cdot \overline{z} = |z|^2$.

*(5)* $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ *(the modulus of a product is the product of the moduli).*

*(6)* $|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|$.

*(7)* $|z^{-1}| = |z|^{-1}$, $z \neq 0$.

*(8)* $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $z_2 \neq 0$ *(the modulus of a quotient is the quotient of the moduli).*

*9)* $|z_1| - |z_2| \leq |z_1 - z_2| \leq |z_1| + |z_2|$.

*Proof.* One can easily check that (1)–(4) hold.

(5) We have $|z_1 \cdot z_2|^2 = (z_1 \cdot z_2)(\overline{z_1 \cdot z_2}) = (z_1 \cdot \overline{z}_1)(z_2 \cdot \overline{z}_2) = |z_1|^2 \cdot |z_2|^2$ and consequently $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$, since $|z| \geq 0$ for all $z \in \mathbb{C}$.

(6) Observe that

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) = |z_1|^2 + z_1 \cdot \overline{z}_2 + \overline{z}_1 \cdot z_2 + |z_2|^2.$$

Because $\overline{z_1 \cdot \overline{z}_2} = \overline{z}_1 \cdot \overline{\overline{z}}_2 = \overline{z}_1 \cdot z_2$ it follows that

$$z_1\overline{z}_2 + \overline{z}_1 \cdot z_2 = 2\ \mathrm{Re}(z_1 \cdot \overline{z}_2) \leq 2|z_1 \cdot \overline{z}_2| = 2|z_1| \cdot |z_2|,$$

hence

$$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2,$$

and consequently, $|z_1 + z_2| \leq |z_1| + |z_2|$, as desired.

In order to obtain inequality on the left-hand side note that

$$|z_1| = |z_1 + z_2 + (-z_2)| \leq |z_1 + z_2| + |-z_2| = |z_1 + z_2| + |z_2|,$$

hence

$$|z_1| - |z_2| \leq |z_1 + z_2|.$$

(7) Note that the relation $z \cdot \dfrac{1}{z} = 1$ implies $|z| \cdot \left|\dfrac{1}{z}\right| = 1$, or $\left|\dfrac{1}{z}\right| = \dfrac{1}{|z|}$. Hence $|z^{-1}| = |z|^{-1}$.

(8) We have

$$\left|\frac{z_1}{z_2}\right| = \left|z_1 \cdot \frac{1}{z_2}\right| = |z_1 \cdot z_2^{-1}| = |z_1| \cdot |z_2^{-1}| = |z_1| \cdot |z_2|^{-1} = \frac{|z_1|}{|z_2|}.$$

(9) We can write $|z_1| = |z_1 - z_2 + z_2| \le |z_1 - z_2| + |z_2|$, so $|z_1 - z_2| \ge |z_1| - |z_2|$. On the other hand,

$$|z_1 - z_2| = |z_1 + (-z_2)| \le |z_1| + |-z_2| = |z_1| + |z_2|. \qquad \square$$

**Remarks.** (1) The inequality $|z_1 + z_2| \le |z_1| + |z_2|$ becomes an equality if and only if $\operatorname{Re}(z_1\overline{z}_2) = |z_1||z_2|$. This is equivalent to $z_1 = tz_2$, where $t$ is a nonnegative real number.

(2) The properties 5) and 6) can be easily extended to give

(5′) $\left|\prod_{k=1}^{n} z_k\right| = \prod_{k=1}^{n} |z_k|$;

(6′) $\left|\sum_{k=1}^{n} z_k\right| \le \sum_{k=1}^{n} |z_k|$ for all $z_k \in \mathbb{C}$, $k = \overline{1, n}$.

As a consequence of(5′) and (7) we have

(5″) $|z^n| = |z|^n$ for any integer $n$ and any complex number $z$.

**Problem 1.** *Prove the identity*

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

*for all complex numbers* $z_1, z_2$.

**Solution.** Using property 4 in the proposition above, we obtain

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) + (z_1 - z_2)(\overline{z}_1 - \overline{z}_2)$$

$$= |z_1|^2 + z_1 \cdot \overline{z}_2 + z_2 \cdot \overline{z}_1 + |z_2|^2 + |z_1|^2 - z_1 \cdot \overline{z}_2 - z_2 \cdot \overline{z}_1 + |z_2|^2$$
$$= 2(|z_1|^2 + |z_2|^2).$$

**Problem 2.** *Prove that if* $|z_1| = |z_2| = 1$ *and* $z_1z_2 \ne -1$, *then* $\dfrac{z_1 + z_2}{1 + z_1z_2}$ *is a real number.*

**Solution.** Using again property 4 in the above proposition, we have

$$z_1 \cdot \overline{z}_1 = |z_1|^2 = 1 \text{ and } \overline{z}_1 = \frac{1}{z_1}.$$

Likewise, $\overline{z}_2 = \dfrac{1}{z_2}$. Hence denoting by $A$ the number in the problem we have

$$\overline{A} = \frac{\overline{z}_1 + \overline{z}_2}{1 + \overline{z}_1 \cdot \overline{z}_2} = \frac{\dfrac{1}{z_1} + \dfrac{1}{z_2}}{1 + \dfrac{1}{z_1} \cdot \dfrac{1}{z_2}} = \frac{z_1 + z_2}{1 + z_1z_2} = A,$$

so $A$ is a real number.

**Problem 3.** *Let $a$ be a positive real number and let*

$$M_a = \left\{ z \in \mathbb{C}^* : \left| z + \frac{1}{z} \right| = a \right\}.$$

*Find the minimum and maximum value of $|z|$ when $z \in M_a$.*

**Solution.** Squaring both sides of the equality $a = \left| z + \frac{1}{z} \right|$, we get

$$a^2 = \left| z + \frac{1}{z} \right|^2 = \left( z + \frac{1}{z} \right) \left( \overline{z} + \frac{1}{\overline{z}} \right) = |z|^2 + \frac{z^2 + (\overline{z})^2}{|z|^2} + \frac{1}{|z|^2}$$

$$= \frac{|z|^4 + (z + \overline{z})^2 - 2|z|^2 + 1}{|z|^2}.$$

Hence

$$|z|^4 - |z|^2 \cdot (a^2 + 2) + 1 = -(z + \overline{z})^2 \le 0$$

and consequently

$$|z|^2 \in \left[ \frac{a^2 + 2 - \sqrt{a^4 + 4a^2}}{2}, \frac{a^2 + 2 + \sqrt{a^4 + 4a^2}}{2} \right].$$

It follows that $|z| \in \left[ \frac{-a + \sqrt{a^2 + 4}}{2}, \frac{a + \sqrt{a^2 + 4}}{2} \right]$, so

$$\max |z| = \frac{a + \sqrt{a^2 + 4}}{2}, \quad \min |z| = \frac{-a + \sqrt{a^2 + 4}}{2}$$

and the extreme values are obtained for the complex numbers in $M$ satisfying $z = -\overline{z}$.

**Problem 4.** *Prove that for any complex number $z$,*

$$|z + 1| \ge \frac{1}{\sqrt{2}} \text{ or } |z^2 + 1| \ge 1.$$

**Solution.** Suppose by way of contradiction that

$$|1 + z| < \frac{1}{\sqrt{2}} \text{ and } |1 + z^2| < 1.$$

Setting $z = a + bi$, with $a, b \in \mathbb{R}$ yields $z^2 = a^2 - b^2 + 2abi$. We obtain

$$(1 + a^2 - b^2)^2 + 4a^2b^2 < 1 \text{ and } (1 + a)^2 + b^2 < \frac{1}{2},$$

and consequently

$$(a^2 + b^2)^2 + 2(a^2 - b^2) < 0 \text{ and } 2(a^2 + b^2) + 4a + 1 < 0.$$

Summing these inequalities implies

$$(a^2 + b^2)^2 + (2a + 1)^2 < 0,$$

which is a contradiction.

**Problem 5.** *Prove that*

$$\sqrt{\frac{7}{2}} \le |1 + z| + |1 - z + z^2| \le 3\sqrt{\frac{7}{6}}$$

*for all complex numbers with* $|z| = 1$.

**Solution.** Let $t = |1 + z| \in [0, 2]$. We have

$$t^2 = (1 + z) \cdot (1 + \overline{z}) = 2 + 2\,\mathrm{Re}(z), \text{ so } \mathrm{Re}(z) = \frac{t^2 - 2}{2}.$$

Then $|1 - z + z^2| = \sqrt{|7 - 2t^2|}$. It suffices to find the extreme values of the function

$$f : [0, 2] \to \mathbb{R}, \quad f(t) = t + \sqrt{|7 - 2t^2|}.$$

We obtain

$$f\left(\sqrt{\frac{7}{2}}\right) = \sqrt{\frac{7}{2}} \le t + \sqrt{|7 - 2t^2|} \le f\left(\sqrt{\frac{7}{6}}\right) = 3\sqrt{\frac{7}{6}}$$

as we can see from the figure below.

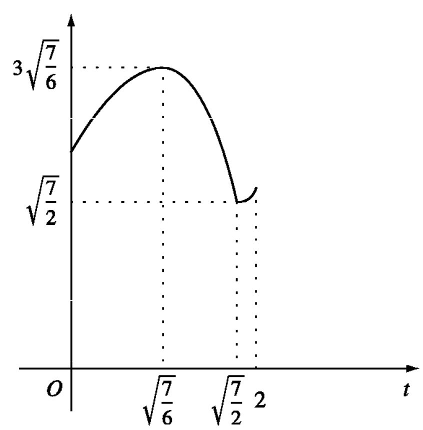


**Figure 1.1.**

**Problem 6.** *Consider the set*

$$H = \{z \in \mathbb{C} : z = x - 1 + xi, \quad x \in \mathbb{R}\}.$$

*Prove that there is a unique number* $z \in H$ *such that* $|z| \le |w|$ *for all* $w \in H$.

**Solution.** Let $\omega = y - 1 + yi$, with $y \in \mathbb{R}$.

It suffices to prove that there is a unique number $x \in \mathbb{R}$ such that

$$(x-1)^2 + x^2 \le (y-1)^2 + y^2$$

for all $y \in \mathbb{R}$.

In other words, $x$ is the minimum point of the function

$$f : \mathbb{R} \to \mathbb{R},\ f(y) = (y-1)^2 + y^2 = 2y^2 - 2y + 1 = 2\left(y - \frac{1}{2}\right)^2 + \frac{1}{2},$$

hence $x = \dfrac{1}{2}$ and $z = -\dfrac{1}{2} + \dfrac{1}{2}i$.

**Problem 7.** *Let $x, y, z$ be distinct complex numbers such that*

$$y = tx + (1-t)z, \quad t \in (0,1).$$

*Prove that*

$$\frac{|z| - |y|}{|z - y|} \ge \frac{|z| - |x|}{|z - x|} \ge \frac{|y| - |x|}{|y - x|}.$$

**Solution.** The relation $y = tx + (1-t)z$ is equivalent to

$$z - y = t(z - x).$$

The inequality

$$\frac{|z| - |y|}{|z - y|} \ge \frac{|z| - |x|}{|z - x|}$$

becomes

$$|z| - |y| \ge t(|z| - |x|),$$

and consequently

$$|y| \le (1-t)|z| + t|x|.$$

This is the triangle inequality for

$$y = (1-t)z + tx.$$

The second inequality can be proved similarly, writing the equality

$$y = tx + (1-t)z$$

as

$$y - x = (1-t)(z - x).$$

### 1.1.8 Solving quadratic equations

We are now able to solve the quadratic equation with real coefficients

$$ax^2 + bx + c = 0, \quad a \neq 0$$

in the case when its discriminant $\Delta = b^2 - 4ac$ is negative.

By completing the square, we easily get the equivalent form

$$a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{-\Delta}{4a^2}\right] = 0.$$

Therefore

$$\left(x + \frac{b}{2a}\right)^2 - i^2\left(\frac{\sqrt{-\Delta}}{2a}\right)^2 = 0,$$

and so $x_1 = \dfrac{-b + i\sqrt{-\Delta}}{2a}$, $x_2 = \dfrac{-b - i\sqrt{-\Delta}}{2a}$.

Observe that the roots are conjugate complex numbers and the factorization formula

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

holds even in the case $\Delta < 0$.

Let us consider now the general quadratic equation with complex coefficients

$$az^2 + bz + c = 0, \quad a \neq 0.$$

Using the same algebraic manipulation as in the case of real coefficients, we get

$$a\left[\left(z + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right] = 0.$$

This is equivalent to

$$\left(z + \frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2}$$

or

$$(2az + b)^2 = \Delta,$$

where $\Delta = b^2 - 4ac$ is also called the discriminant of the quadratic equation. Setting $y = 2az + b$, the equation is reduced to

$$y^2 = \Delta = u + vi,$$

where $u$ and $v$ are real numbers.

This equation has the solutions

$$y_{1,2} = \pm\left(\sqrt{\frac{r+u}{2}} + (\operatorname{sgn} v)\sqrt{\frac{r-u}{2}}i\right),$$

where $r = |\Delta|$ and $\operatorname{sign} v$ is the sign of the real number $v$.

The roots of the initial equation are

$$z_{1,2} = \frac{1}{2a}(-b + y_{1,2}).$$

Observe that the relations between roots and coefficients

$$z_1 + z_2 = -\frac{b}{a}, \quad z_1 z_2 = \frac{c}{a},$$

as well as the factorization formula

$$az^2 + bz + c = a(z - z_1)(z - z_2)$$

are also preserved when the coefficients of the equation are elements of the field of complex numbers $\mathbb{C}$.

**Problem 1.** *Solve, in complex numbers, the quadratic equation*

$$z^2 - 8(1-i)z + 63 - 16i = 0.$$

**Solution.** We have

$$\Delta' = (4-4i)^2 - (63-16i) = -63 - 16i$$

and

$$r = |\Delta'| = \sqrt{63^2 + 16^2} = 65,$$

where $\Delta' = \left(\frac{b}{2}\right)^2 - ac$.

The equation

$$y^2 = -63 - 16i$$

has the solution $y_{1,2} = \pm\left(\sqrt{\frac{65-63}{2}} + i\sqrt{\frac{65+63}{2}}\right) = \pm(1-8i)$. It follows that $z_{1,2} = 4 - 4i \pm (1-8i)$. Hence

$$z_1 = 5 - 12i \text{ and } z_2 = 3 + 4i.$$

**Problem 2.** *Let $p$ and $q$ be complex numbers with $q \neq 0$. Prove that if the roots of the quadratic equation $x^2 + px + q^2 = 0$ have the same absolute value, then $\frac{p}{q}$ is a real number.*

(1999 Romanian Mathematical Olympiad – Final Round)

**Solution.** Let $x_1$ and $x_2$ be the roots of the equation and let $r = |x_1| = |x_2|$. Then

$$\frac{p^2}{q^2} = \frac{(x_1 + x_2)^2}{x_1 x_2} = \frac{x_1}{x_2} + \frac{x_2}{x_1} + 2 = \frac{x_1\overline{x_2}}{r^2} + \frac{x_2\overline{x_1}}{r^2} + 2 = 2 + \frac{2}{r^2}\mathrm{Re}(x_1\overline{x_2})$$

is a real number. Moreover,

$$\mathrm{Re}(x_1\overline{x_2}) \geq -|x_1\overline{x_2}| = -r^2, \text{ so } \frac{p^2}{q^2} \geq 0.$$

Therefore $\frac{p}{q}$ is a real number, as claimed.

**Problem 3.** *Let $a, b, c$ be distinct nonzero complex numbers with $|a| = |b| = |c|$.*

*a) Prove that if a root of the equation $az^2 + bz + c = 0$ has modulus equal to 1, then $b^2 = ac$.*

*b) If each of the equations*

$$az^2 + bz + c = 0 \quad \textit{and} \quad bz^2 + cz + a = 0$$

*has a root having modulus 1, then $|a - b| = |b - c| = |c - a|$.*

**Solution.** a) Let $z_1, z_2$ be the roots of the equation with $|z_1| = 1$. From $z_2 = \frac{c}{a} \cdot \frac{1}{z_1}$ it follows that $|z_2| = \left|\frac{c}{a}\right| \cdot \frac{1}{|z_1|} = 1$. Because $z_1 + z_2 = -\frac{b}{a}$ and $|a| = |b|$, we have $|z_1 + z_2|^2 = 1$. This is equivalent to

$$(z_1 + z_2)(\overline{z_1} + \overline{z_2}) = 1, \text{ i.e., } (z_1 + z_2)\left(\frac{1}{z_1} + \frac{1}{z_2}\right) = 1.$$

We find that

$$(z_1 + z_2)^2 = z_1 z_2, \text{ i.e., } \left(-\frac{b}{a}\right)^2 = \frac{c}{a},$$

which reduces to $b^2 = ac$, as desired.

b) As we have already seen, we have $b^2 = ac$ and $c^2 = ab$. Multiplying these relations yields $b^2c^2 = a^2bc$, hence $a^2 = bc$. Therefore

$$a^2 + b^2 + c^2 = ab + bc + ca. \tag{1}$$

Relation (1) is equivalent to

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 0,$$

i.e.,

$$(a - b)^2 + (b - c)^2 + 2(a - b)(b - c) + (c - a)^2 = 2(a - b)(b - c).$$

It follows that $(a-c)^2 = (a-b)(b-c)$. Taking absolute values we find $\beta^2 = \gamma\alpha$, where $\alpha = |b-c|$, $\beta = |c-a|$, $\gamma = |a-b|$. In an analogous way we obtain $\alpha^2 = \beta\gamma$ and $\gamma^2 = \alpha\beta$. Adding these relations yields $\alpha^2 + \beta^2 + \gamma^2 = \alpha\beta + \beta\gamma + \gamma\alpha$, i.e., $(\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2 = 0$. Hence $\alpha = \beta = \gamma$.

### 1.1.9 Problems

**1.** Consider the complex numbers $z_1 = (1, 2)$, $z_2 = (-2, 3)$ and $z_3 = (1, -1)$. Compute the following complex numbers:

a) $z_1 + z_2 + z_3$; b) $z_1z_2 + z_2z_3 + z_3z_1$; c) $z_1z_2z_3$;

d) $z_1^2 + z_2^2 + z_3^2$; e) $\dfrac{z_1}{z_2} + \dfrac{z_2}{z_3} + \dfrac{z_3}{z_1}$; f) $\dfrac{z_1^2 + z_2^2}{z_2^2 + z_3^2}$.

**2.** Solve the equations:

a) $z + (-5, 7) = (2, -1)$; b) $(2, 3) + z = (-5, -1)$;

c) $z \cdot (2, 3) = (4, 5)$; d) $\dfrac{z}{(-1, 3)} = (3, 2)$.

**3.** Solve in $\mathbb{C}$ the equations:

a) $z^2 + z + 1 = 0$; b) $z^3 + 1 = 0$.

**4.** Let $z = (0, 1) \in \mathbb{C}$. Express $\displaystyle\sum_{k=0}^{n} z^k$ in terms of the positive integer $n$.

**5.** Solve the equations:

a) $z \cdot (1, 2) = (-1, 3)$; b) $(1, 1) \cdot z^2 = (-1, 7)$.

**6.** Let $z = (a, b) \in \mathbb{C}$. Compute $z^2$, $z^3$ and $z^4$.

**7.** Let $z_0 = (a, b) \in \mathbb{C}$. Find $z \in \mathbb{C}$ such that $z^2 = z_0$.

**8.** Let $z = (1, -1)$. Compute $z^n$, where $n$ is a positive integer.

**9.** Find real numbers $x$ and $y$ in each of the following cases:

a) $(1-2i)x + (1+2i)y = 1 + i$; b) $\dfrac{x-3}{3+i} + \dfrac{y-3}{3-i} = i$;

c) $(4-3i)x^2 + (3+2i)xy = 4y^2 - \dfrac{1}{2}x^2 + (3xy - 2y^2)i$.

**10.** Compute:

a) $(2-i)(-3+2i)(5-4i)$; b) $(2-4i)(5+2i) + (3+4i)(-6-i)$;

c) $\left(\dfrac{1+i}{1-i}\right)^{16} + \left(\dfrac{1-i}{1+i}\right)^{8}$; d) $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^{6} + \left(\dfrac{1-i\sqrt{7}}{2}\right)^{6}$;

e) $\dfrac{3+7i}{2+3i} + \dfrac{5-8i}{2-3i}$.

**11.** Compute:

a) $i^{2000} + i^{1999} + i^{201} + i^{82} + i^{47}$;

b) $E_n = 1 + i + i^2 + i^3 + \cdots + i^n$ for $n \geq 1$;

c) $i^1 \cdot i^2 \cdot i^3 \cdots i^{2000}$;

d) $i^{-5} + (-i)^{-7} + (-i)^{13} + i^{-100} + (-i)^{94}$.

**12.** Solve in $\mathbb{C}$ the equations:

a) $z^2 = i$; b) $z^2 = -i$; c) $z^2 = \frac{1}{2} - i\frac{\sqrt{2}}{2}$.

**13.** Find all complex numbers $z \neq 0$ such that $z + \frac{1}{z} \in \mathbb{R}$.

**14.** Prove that:

a) $E_1 = (2 + i\sqrt{5})^7 + (2 - i\sqrt{5})^7 \in \mathbb{R}$;

b) $E_2 = \left(\frac{19 + 7i}{9 - i}\right)^n + \left(\frac{20 + 5i}{7 + 6i}\right)^n \in \mathbb{R}$.

**15.** Prove the following identities:

a) $|z_1 + z_2|^2 + |z_2 + z_3|^2 + |z_3 + z_1|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_1 + z_2 + z_3|^2$;

b) $|1 + z_1\overline{z}_2|^2 + |z_1 - z_2|^2 = (1 + |z_1|^2)(1 + |z_2|^2)$;

c) $|1 - z_1\overline{z}_2|^2 - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2)$;

d) $|z_1 + z_2 + z_3|^2 + |-z_1 + z_2 + z_3|^2 + |z_1 - z_2 + z_3|^2 + |z_1 + z_2 - z_3|^2$
$= 4(|z_1|^2 + |z_2|^2 + |z_3|^2)$.

**16.** Let $z \in \mathbb{C}^*$ such that $\left|z^3 + \frac{1}{z^3}\right| \leq 2$. Prove that $\left|z + \frac{1}{z}\right| \leq 2$.

**17.** Find all complex numbers $z$ such that

$$|z| = 1 \text{ and } |z^2 + \overline{z}^2| = 1.$$

**18.** Find all complex numbers $z$ such that

$$4z^2 + 8|z|^2 = 8.$$

**19.** Find all complex numbers $z$ such that $z^3 = \overline{z}$.

**20.** Consider $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 1$. Prove that

$$\left|\frac{1}{z} - \frac{1}{2}\right| < \frac{1}{2}.$$

**21.** Let $a, b, c$ be real numbers and $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Compute

$$(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega).$$

**22.** Solve the equations:

a) $|z| - 2z = 3 - 4i$;

b) $|z| + z = 3 + 4i$;

c) $z^3 = 2 + 11i$, where $z = x + yi$ and $x, y \in \mathbb{Z}$;

d) $iz^2 + (1 + 2i)z + 1 = 0$;

e) $z^4 + 6(1 + i)z^2 + 5 + 6i = 0$;

f) $(1 + i)z^2 + 2 + 11i = 0$.

**23.** Find all real numbers $m$ for which the equation

$$z^3 + (3 + i)z^2 - 3z - (m + i) = 0$$

has at least a real root.

**24.** Find all complex numbers $z$ such that

$$z' = (z - 2)(\overline{z} + i)$$

is a real number.

**25.** Find all complex numbers $z$ such that $|z| = \left|\frac{1}{z}\right|$.

**26.** Let $z_1, z_2 \in \mathbb{C}$ be complex numbers such that $|z_1 + z_2| = \sqrt{3}$ and $|z_1| = |z_2| = 1$. Compute $|z_1 - z_2|$.

**27.** Find all positive integers $n$ such that

$$\left(\frac{-1 + i\sqrt{3}}{2}\right)^n + \left(\frac{-1 - i\sqrt{3}}{2}\right)^n = 2.$$

**28.** Let $n > 2$ be an integer. Find the number of solutions to the equation

$$z^{n-1} = i\overline{z}.$$

**29.** Let $z_1, z_2, z_3$ be complex numbers with

$$|z_1| = |z_2| = |z_3| = R > 0.$$

Prove that

$$|z_1 - z_2| \cdot |z_2 - z_3| + |z_3 - z_1| \cdot |z_1 - z_2| + |z_2 - z_3| \cdot |z_3 - z_1| \le 9R^2.$$

**30.** Let $u, v, w, z$ be complex numbers such that $|u| < 1$, $|v| = 1$ and $w = \dfrac{v(u - z)}{\overline{u} \cdot z - 1}$. Prove that $|w| \le 1$ if and only if $|z| \le 1$.

**31.** Let $z_1, z_2, z_3$ be complex numbers such that

$$z_1 + z_2 + z_3 = 0 \quad \text{and} \quad |z_1| = |z_2| = |z_3| = 1.$$

Prove that

$$z_1^2 + z_2^2 + z_3^2 = 0.$$

**32.** Consider the complex numbers $z_1, z_2, \ldots, z_n$ with

$$|z_1| = |z_2| = \cdots = |z_n| = r > 0.$$

Prove that the number

$$E = \frac{(z_1 + z_2)(z_2 + z_3)\cdots(z_{n-1} + z_n)(z_n + z_1)}{z_1 \cdot z_2 \cdots z_n}$$

is real.

**33.** Let $z_1, z_2, z_3$ be distinct complex numbers such that

$$|z_1| = |z_2| = |z_3| > 0.$$

If $z_1 + z_2z_3$, $z_2 + z_1z_3$ and $z_3 + z_1z_2$ are real numbers, prove that $z_1z_2z_3 = 1$.

**34.** Let $x_1$ and $x_2$ be the roots of the equation $x^2 - x + 1 = 0$. Compute:

a) $x_1^{2000} + x_2^{2000}$; b) $x_1^{1999} + x_2^{1999}$; c) $x_1^n + x_2^n$, for $n \in \mathbb{N}$.

**35.** Factorize (in linear polynomials) the following polynomials:

a) $x^4 + 16$; b) $x^3 - 27$; c) $x^3 + 8$; d) $x^4 + x^2 + 1$.

**36.** Find all quadratic equations with real coefficients that have one of the following roots:

a) $(2 + i)(3 - i)$; b) $\dfrac{5 + i}{2 - i}$; c) $i^{51} + 2i^{80} + 3i^{45} + 4i^{38}$.

**37.** (Hlawka's inequality) Prove that the following inequality

$$|z_1 + z_2| + |z_2 + z_3| + |z_3 + z_1| \le |z_1| + |z_2| + |z_3| + |z_1 + z_2 + z_3|$$

holds for all complex numbers $z_1, z_2, z_3$.

## 1.2 Geometric Interpretation of the Algebraic Operations

### 1.2.1 Geometric interpretation of a complex number

We have defined a complex number $z = (x, y) = x + yi$ to be an ordered pair of real numbers $(x, y) \in \mathbb{R} \times \mathbb{R}$, so it is natural to let a complex number $z = x + yi$ correspond to a point $M(x, y)$ in the plane $\mathbb{R} \times \mathbb{R}$.

For a formal introduction, let us consider $P$ to be the set of points of a given plane $\Pi$ equipped with a coordinate system $xOy$. Consider the bijective function $\varphi : \mathbb{C} \to P$, $\varphi(z) = M(x, y)$.

**Definition.** The point $M(x, y)$ is called the *geometric image* of the complex number $z = x + yi$.

The complex number $z = x + yi$ is called the *complex coordinate* of the point $M(x, y)$. We will use the notation $M(z)$ to indicate that the complex coordinate of $M$ is the complex number $z$.

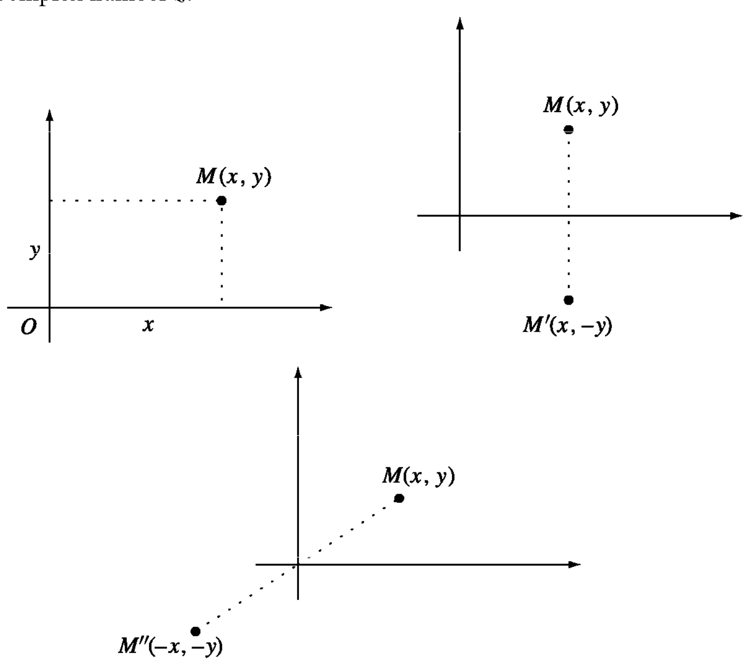


**Figure 1.2.**

The geometric image of the complex conjugate $\bar{z}$ of a complex number $z = x + yi$ is the reflection point $M'(x, -y)$ across the $x$-axis of the point $M(x, y)$ (see Fig. 1.2).

The geometric image of the additive inverse $-z$ of a complex number $z = x + yi$ is the reflection $M''(-x, -y)$ across the origin of the point $M(x, y)$ (see Fig. 1.2).

The bijective function $\varphi$ maps the set $\mathbb{R}$ onto the $x$-axis, which is called the *real axis*. On the other hand, the imaginary complex numbers correspond to the $y$-axis, which is called the *imaginary axis*. The plane $\Pi$, whose points are identified with complex numbers, is called the *complex plane*.

On the other hand, we can also identify a complex number $z = x + yi$ with the vector $\overrightarrow{v} = \overrightarrow{OM}$, where $M(x, y)$ is the geometric image of the complex number $z$.

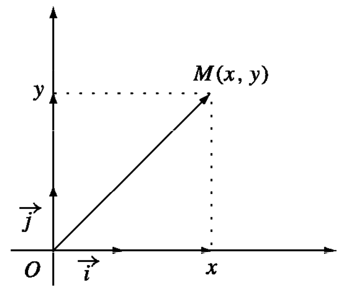


**Figure 1.3.**

Let $V_0$ be the set of vectors whose initial points are the origin $O$. Then we can define the bijective function

$$\varphi' : \mathbb{C} \to V_0, \quad \varphi'(z) = \overrightarrow{OM} = \overrightarrow{v} = x\overrightarrow{i} + y\overrightarrow{j},$$

where $\overrightarrow{i}$, $\overrightarrow{j}$ are the vectors of the $x$-axis and $y$-axis, respectively.

## 1.2.2 Geometric interpretation of the modulus

Let us consider a complex number $z = x + yi$ and the geometric image $M(x, y)$ in the complex plane. The Euclidean distance $OM$ is given by the formula

$$OM = \sqrt{(x_M - x_O)^2 + (y_M - y_O)^2},$$

hence $OM = \sqrt{x^2 + y^2} = |z| = |\overrightarrow{v}|$. In other words, the absolute value $|z|$ of a complex number $z = x + yi$ is the length of the segment $OM$ or the magnitude of the vector $\overrightarrow{v} = x\overrightarrow{i} + y\overrightarrow{j}$.

**Remarks.** a) For a positive real number $r$, the set of complex numbers with moduli $r$ corresponds in the complex plane to $\mathcal{C}(O; r)$, our notation for the circle $\mathcal{C}$ with center $O$ and radius $r$.

b) The complex numbers $z$ with $|z| < r$ correspond to the interior points of circle $\mathcal{C}$; on the other hand, the complex numbers $z$ with $|z| > r$ correspond to the points in the exterior of circle $\mathcal{C}$.

**Example.** The numbers $z_k = \pm\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, $k = 1, 2, 3, 4$, are represented in the complex plane by four points on the unit circle centered on the origin, since

$$|z_1| = |z_2| = |z_3| = |z_4| = 1.$$

### 1.2.3 Geometric interpretation of the algebraic operations

a) **Addition and subtraction.** Consider the complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ and the corresponding vectors $\vec{v}_1 = x_1\vec{i} + y_1\vec{j}$ and $\vec{v}_2 = x_2\vec{i} + y_2\vec{j}$. Observe that the sum of the complex numbers is

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i,$$

and the sum of the vectors is

$$\vec{v}_1 + \vec{v}_2 = (x_1 + x_2)\vec{i} + (y_1 + y_2)\vec{j}.$$

Therefore, the sum $z_1 + z_2$ corresponds to the sum $\vec{v}_1 + \vec{v}_2$.

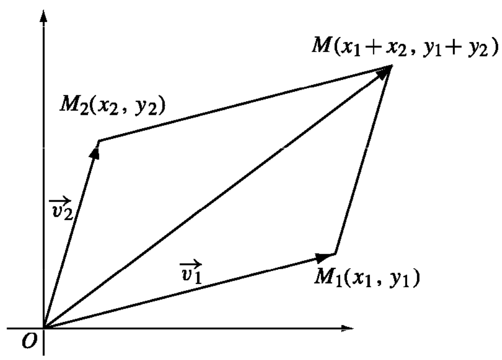


**Figure 1.4.**

**Examples.** 1) We have $(3 + 5i) + (6 + i) = 9 + 6i$; hence the geometric image of the sum is given in Fig. 1.5.

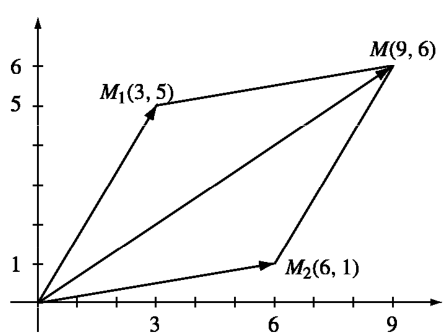


**Figure 1.5.**

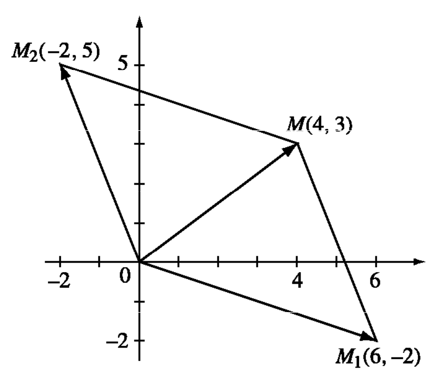


**Figure 1.6.**

2) Observe that $(6 - 2i) + (-2 + 5i) = 4 + 3i$. Therefore the geometric image of the sum of these two complex numbers is the point $M(4, 3)$ (see Fig. 1.6).

On the other hand, the difference of the complex numbers $z_1$ and $z_2$ is

$$z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2)i,$$

and the difference of the vectors $v_1$ and $v_2$ is

$$\vec{v}_1 - \vec{v}_2 = (x_1 - x_2)\vec{i} + (y_1 - y_2)\vec{j}.$$

Hence, the difference $z_1 - z_2$ corresponds to the difference $\vec{v}_1 - \vec{v}_2$.

3) We have $(-3 + i) - (2 + 3i) = (-3 + i) + (-2 - 3i) = -5 - 2i$; hence the geometric image of difference of these two complex numbers is the point $M(-5, -2)$ given in Fig. 1.7.

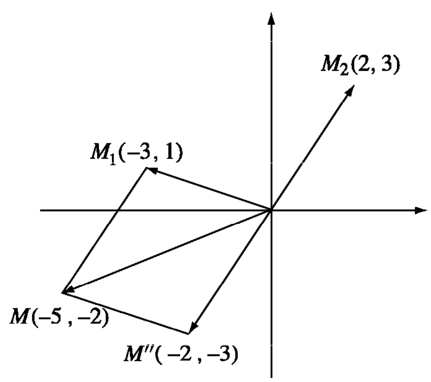


**Figure 1.7.**

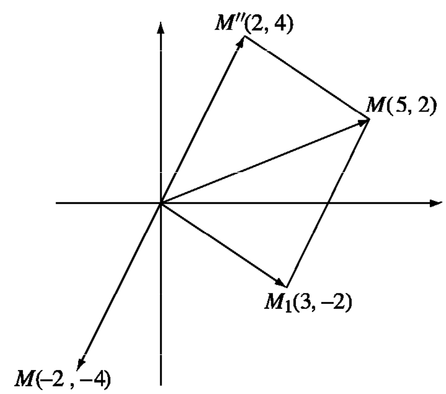


**Figure 1.8.**

4) Note that $(3 - 2i) - (-2 - 4i) = (3 - 2i) + (2 + 4i) = 5 + 2i$, and obtain the point $M(-2, -4)$ as the geometric image of the difference of these two complex numbers (see Fig. 1.8).

**Remark.** The distance $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ is equal to the modulus of the complex number $z_1 - z_2$ or to the length of the vector $\vec{v}_1 - \vec{v}_2$. Indeed,

$$|M_1M_2| = |z_1 - z_2| = |\vec{v}_1 - \vec{v}_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

b) **Real multiples of a complex number.** Consider a complex number $z = x + iy$ and the corresponding vector $\vec{v} = x\vec{i} + y\vec{j}$. If $\lambda$ is a real number, then the real multiple $\lambda z = \lambda x + i\lambda y$ corresponds to the vector

$$\lambda\vec{v} = \lambda x\vec{i} + \lambda y\vec{j}.$$

Note that if $\lambda > 0$ then the vectors $\lambda\vec{v}$ and $\vec{v}$ have the same orientation and

$$|\lambda\vec{v}| = \lambda|\vec{v}|.$$

When $\lambda < 0$, the vector $\lambda\overrightarrow{v}$ changes to the opposite orientation and $|\lambda\overrightarrow{v}| = -\lambda|\overrightarrow{v}|$. Of course, if $\lambda = 0$, then $\lambda\overrightarrow{v} = \overrightarrow{0}$.

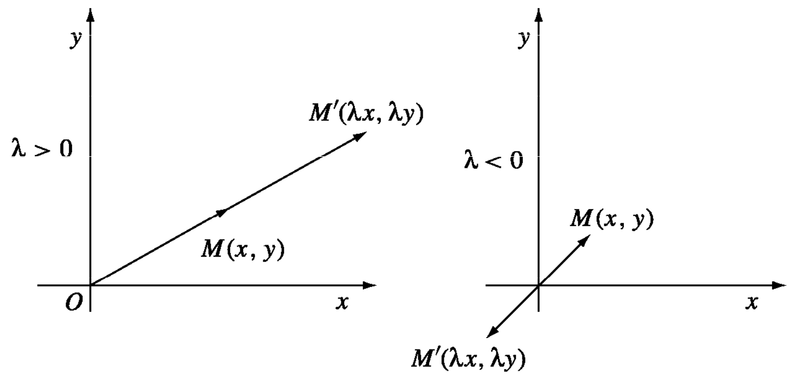


**Figure 1.9.**

**Examples.** 1) We have $3(1 + 2i) = 3 + 6i$; therefore $M'(3, 6)$ is the geometric image of the product of 3 and $z = 1 + 2i$.

2) Observe that $-2(-3 + 2i) = 6 - 4i$, and obtain the point $M'(6, -4)$ as the geometric image of the product of $-2$ and $z = -3 + 2i$.

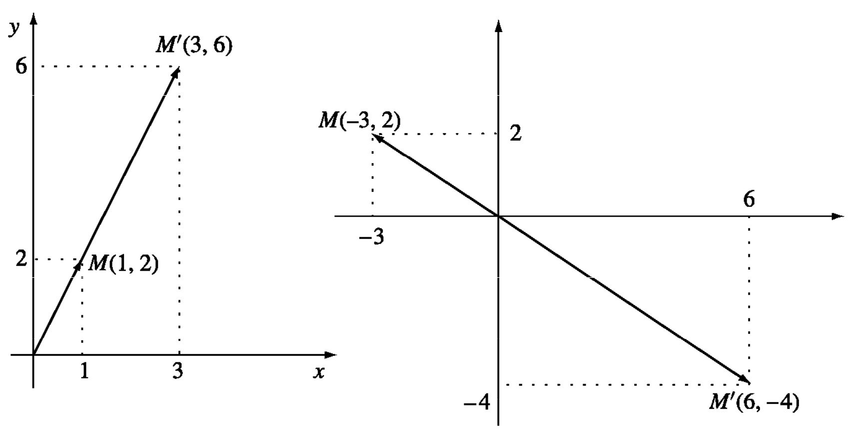


**Figure 1.10.**

### 1.2.4 Problems

**1.** Represent the geometric images of the following complex numbers:

$z_1 = 3 + i$; $z_2 = -4 + 2i$; $z_3 = -5 - 4i$; $z_4 = 5 - i$;
$z_5 = 1$; $z_6 = -3i$; $z_7 = 2i$; $z_8 = -4$.

**2.** Find the geometric interpretation for the following equalities:

a) $(-5 + 4i) + (2 - 3i) = -3 + i$;
b) $(4 - i) + (-6 + 4i) = -2 + 3i$;
c) $(-3 - 2i) - (-5 + i) = 2 - 3i$;
d) $(8 - i) - (5 + 3i) = 3 - 4i$;
e) $2(-4 + 2i) = -8 + 4i$;
f) $-3(-1 + 2i) = 3 - 6i$.

**3.** Find the geometric image of the complex number $z$ in each of the following cases:

a) $|z - 2| = 3$; b) $|z + i| < 1$; c) $|z - 1 + 2i| > 3$;
d) $|z - 2| - |z + 2| < 2$; e) $0 < \operatorname{Re}(iz) < 1$; f) $-1 < \operatorname{Im}(z) < 1$;
g) $\operatorname{Re}\left(\dfrac{z-2}{z-1}\right) = 0$; h) $\dfrac{1+\overline{z}}{z} \in \mathbb{R}$.

**4.** Find the set of points $P(x, y)$ in the complex plane such that

$$|\sqrt{x^2 + 4} + i\sqrt{y - 4}| = \sqrt{10}.$$

**5.** Let $z_1 = 1 + i$ and $z_2 = -1 - i$. Find $z_3 \in \mathbb{C}$ such that triangle $z_1, z_2, z_3$ is equilateral.

**6.** Find the geometric images of the complex numbers $z$ such that the triangle with vertices at $z$, $z^2$ and $z^3$ is right-angled.

**7.** Find the geometric images of the complex numbers $z$ such that

$$\left|z + \frac{1}{z}\right| = 2.$$

# 2

# Complex Numbers in Trigonometric Form

## 2.1 Polar Representation of Complex Numbers

### 2.1.1 Polar coordinates in the plane

Let us consider a coordinate plane and a point $M(x, y)$ that is not the origin.

The real number $r = \sqrt{x^2 + y^2}$ is called the *polar radius* of the point $M$. The direct angle $t^* \in [0, 2\pi)$ between the vector $\overrightarrow{OM}$ and the positive $x$-axis is called the *polar argument* of the point $M$. The pair $(r, t^*)$ is called the *polar coordinates* of the point $M$. We will write $M(r, t^*)$. Note that the function $h : \mathbb{R} \times \mathbb{R} \setminus \{(0, 0)\} \to (0, \infty) \times [0, 2\pi)$, $h((x, y)) = (r, t^*)$ is bijective.

The origin $O$ is the unique point such that $r = 0$; the argument $t^*$ of the origin is not defined.

For any point $M$ in the plane there is a unique intersection point $P$ of the ray $(OM$ with the unit circle centered at the origin. The point $P$ has the same polar argument $t^*$. Using the definition of the sine and cosine functions we find that

$$x = r \cos t^* \text{ and } y = r \sin t^*.$$

Therefore, it is easy to obtain the cartesian coordinates of a point from its polar coordinates.

Conversely, let us consider a point $M(x, y)$. The polar radius is $r = \sqrt{x^2 + y^2}$. To determine the polar argument we study the following cases:

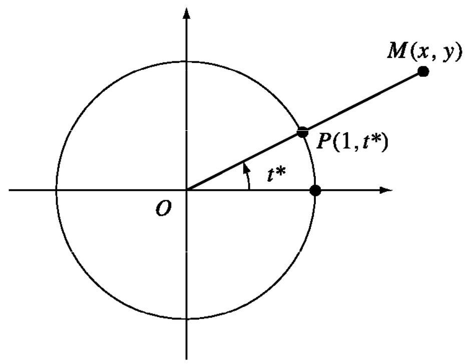


**Figure 2.1.**

a) If $x \neq 0$, from $\tan t^* = \frac{y}{x}$ we deduce that

$$t^* = \arctan \frac{y}{x} + k\pi,$$

where

$$k = \begin{cases} 0, & \text{for} \quad x > 0 \text{ and } y \geq 0 \\ 1, & \text{for} \quad x < 0 \text{ and any } y \\ 2, & \text{for} \quad x > 0 \text{ and } y < 0. \end{cases}$$

b) If $x = 0$ and $y \neq 0$, then

$$t^* = \begin{cases} \pi/2, & \text{for} \quad y > 0 \\ 3\pi/2, & \text{for} \quad y < 0. \end{cases}$$

**Examples.** 1. Let us find the polar coordinates of the points $M_1(2, -2)$, $M_2(-1, 0)$, $M_3(-2\sqrt{3}, -2)$, $M_4(\sqrt{3}, 1)$, $M_5(3, 0)$, $M_6(-2, 2)$, $M_7(0, 1)$ and $M_8(0, -4)$.

In this case we have $r_1 = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$; $t_1^* = \arctan(-1) + 2\pi = -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$, so $M_1\left(2\sqrt{2}, \frac{7\pi}{4}\right)$.

Observe that $r_2 = 1$, $t_2^* = \arctan 0 + \pi = \pi$, so $M_2(1, \pi)$.

We have $r_3 = 4$, $t_3^* = \arctan \frac{\sqrt{3}}{3} + \pi = \frac{\pi}{6} + \pi = \frac{7\pi}{6}$, so $M_3\left(4, \frac{7\pi}{6}\right)$.

Note that $r_4 = 2$, $t_4^* = \arctan \frac{\sqrt{3}}{3} = \frac{\pi}{6}$, so $M_4\left(2, \frac{\pi}{6}\right)$.

We have $r_5 = 3$, $t_5^* = \arctan 0 + 0 = 0$, so $M_5(3, 0)$.

We have $r_6 = 2\sqrt{2}$, $t_6^* = \arctan(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$, so $M_6\left(2\sqrt{2}, \frac{3\pi}{4}\right)$.

Note that $r_7 = 1$, $t_7^* = \frac{\pi}{2}$, so $M_7\left(1, \frac{\pi}{2}\right)$.

Observe that $r_8 = 4$, $t_8^* = \dfrac{3\pi}{2}$, so $M_8\left(1, \dfrac{3\pi}{2}\right)$.

2. Let us find the cartesian coordinates of the points $M_1\left(2, \dfrac{2\pi}{3}\right)$, $M_2\left(3, \dfrac{7\pi}{4}\right)$ and $M_3(1, 1)$.

We have $x_1 = 2\cos\dfrac{2\pi}{3} = 2\left(-\dfrac{1}{2}\right) = -1$, $y_1 = 2\sin\dfrac{2\pi}{3} = 2\dfrac{\sqrt{3}}{2} = \sqrt{3}$, so $M_1(-1, \sqrt{3})$.

Note that $x_2 = 3\cos\dfrac{7\pi}{4} = \dfrac{3\sqrt{2}}{2}$, $y_2 = 3\sin\dfrac{7\pi}{4} = -\dfrac{3\sqrt{2}}{2}$, so $M_2\left(\dfrac{3\sqrt{2}}{2}, -\dfrac{3\sqrt{2}}{2}\right)$.

Observe that $x_3 = \cos 1$, $y_2 = \sin 1$, so $M_3(\cos 1, \sin 1)$.

### 2.1.2 Polar representation of a complex number

For a complex number $z = x + yi$ we can write the polar representation

$$z = r(\cos t^* + i \sin t^*),$$

where $r \in [0, \infty)$ and $t^* \in [0, 2\pi)$ are the polar coordinates of the geometric image of $z$.

The polar argument $t^*$ of the geometric image of $z$ is called the argument of $z$, denoted by arg $z$. The polar radius $r$ of the geometric image of $z$ is equal to the modulus of $z$. For $z \neq 0$, the modulus and argument of $z$ are uniquely determined.

Consider $z = r(\cos t^* + i \sin t^*)$ and let $t = t^* + 2k\pi$ for an integer $k$. Then

$$z = r[\cos(t - 2k\pi) + i \sin(t - 2k\pi)] = r(\cos t + i \sin t),$$

i.e., any complex number $z$ can be represented as $z = r(\cos t + i \sin t)$, where $r \geq 0$ and $t \in \mathbb{R}$. The set $\operatorname{Arg} z = \{t : t^* + 2k\pi,\ k \in \mathbb{Z}\}$ is called the *extended argument* of the complex number $z$.

Therefore, two complex numbers $z_1, z_2 \neq 0$ represented as

$$z_1 = r_1(\cos t_1 + i \sin t_1) \text{ and } z_2 = r_2(\cos t_2 + i \sin t_2)$$

are equal if and only if $r_1 = r_2$ and $t_1 - t_2 = 2k\pi$, for an integer $k$.

**Example 1.** Let us find the polar representation of the numbers:
a) $z_1 = -1 - i$,
b) $z_2 = 2 + 2i$,
c) $z_3 = -1 + i\sqrt{3}$,
d) $z_4 = 1 - i\sqrt{3}$
and determine their extended argument.

a) As in the figure below the geometric image $P_1(-1, -1)$ lies in the third quadrant. Then $r_1 = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$ and

$$t_1^* = \arctan \frac{y}{x} + \pi = \arctan 1 + \pi = \frac{\pi}{4} + \pi = \frac{5\pi}{4}.$$

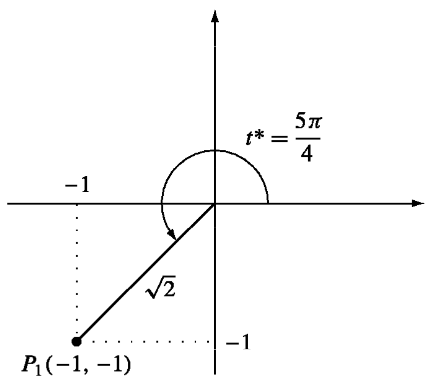


**Figure 2.2.**

Hence

$$z_1 = \sqrt{2}\left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right)$$

and

$$\operatorname{Arg} z_1 = \left\{\frac{5\pi}{4} + 2k\pi \mid k \in \mathbb{Z}\right\}.$$

b) The point $P_2(2, 2)$ lies in the first quadrant, so we can write

$$r_2 = \sqrt{2^2 + 2^2} = 2\sqrt{2} \text{ and } t_2^* = \arctan 1 = \frac{\pi}{4}.$$

Hence

$$z_2 = 2\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$$

and

$$\operatorname{Arg} z = \left\{\frac{\pi}{4} + 2k\pi \mid k \in \mathbb{Z}\right\}.$$

c) The point $P_3(-1, \sqrt{3})$ lies in the second quadrant, so

$$r_3 = 2 \text{ and } t_3^* = \arctan(-\sqrt{3}) + \pi = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}.$$

Therefore,

$$z_3 = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$$

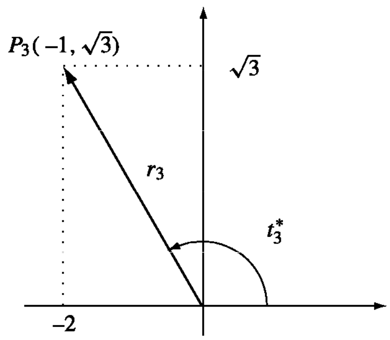


**Figure 2.3.**

and

$$\operatorname{Arg} z_3 = \left\{ \frac{2\pi}{3} + 2k\pi \mid k \in \mathbb{Z} \right\}.$$

d) The point $P_4(1, -\sqrt{3})$ lies in the fourth quadrant (Fig. 2.4), so

$$r_4 = 2 \text{ and } t_4^* = \arctan(-\sqrt{3}) + 2\pi = -\frac{\pi}{3} + 2\pi = \frac{5\pi}{3}.$$

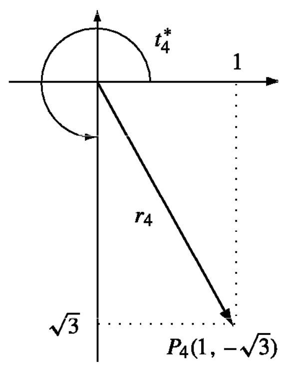


**Figure 2.4.**

Hence

$$z_4 = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right),$$

and

$$\operatorname{Arg} z_4 = \left\{ \frac{5\pi}{3} + 2k\pi \mid k \in \mathbb{Z} \right\}.$$

**Example 2.** Let us find the polar representation of the numbers

a) $z_1 = 2i$,

b) $z_2 = -1$,

c) $z_3 = 2$,

d) $z_4 = -3i$

and determine their extended argument.

a) The point $P_1(0, 2)$ lies on the positive $y$-axis, so

$$r_1 = 2, \quad t_1^* = \frac{\pi}{2}, \quad z_1 = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

and

$$\operatorname{Arg} z_1 = \left\{\frac{\pi}{2} + 2k\pi \mid k \in \mathbb{Z}\right\}.$$

b) The point $P_2(-1, 0)$ lies on the negative $x$-axis, so

$$r_2 = 1, \quad t_2^* = \pi, \quad z_2 = \cos\pi + i\sin\pi$$

and

$$\operatorname{Arg} z_2 = \{\pi + 2k\pi \mid k \in \mathbb{Z}\}.$$

c) The point $P_3(2, 0)$ lies on the positive $x$-axis, so

$$r_3 = 2, \quad t_3^* = 0, \quad z_3 = 2(\cos 0 + i\sin 0)$$

and

$$\operatorname{Arg} z_3 = \{2k\pi \mid k \in \mathbb{Z}\}.$$

d) The point $P_4(0, -3)$ lies on the negative $y$-axis, so

$$r_4 = 3, \quad t_4^* = \frac{3\pi}{2}, \quad z_3 = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

and

$$\operatorname{Arg} z_4 = \left\{\frac{3\pi}{2} + 2k\pi \mid k \in \mathbb{Z}\right\}.$$

**Remark.** The following formulas should be memorized:

$$1 = \cos 0 + i\sin 0; \quad i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2};$$

$$-1 = \cos\pi + i\sin\pi; \quad -i = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}.$$

**Problem 1.** *Find the polar representation of the complex number*

$$z = 1 + \cos a + i\sin a, \quad a \in (0, 2\pi).$$

**Solution.** The modulus is

$$|z| = \sqrt{(1+\cos a)^2 + \sin^2 a} = \sqrt{2(1+\cos a)} = \sqrt{4\cos^2\frac{a}{2}} = 2\left|\cos\frac{a}{2}\right|.$$

The argument of $z$ is determined as follows:

a) If $a \in (0, \pi)$, then $\frac{a}{2} \in \left(0, \frac{\pi}{2}\right)$ and the point $P(1+\cos a, \sin a)$ lies on the first quadrant. Hence

$$t^* = \arctan\frac{\sin a}{1+\cos a} = \arctan\left(\tan\frac{a}{2}\right) = \frac{a}{2},$$

and in this case

$$z = 2\cos\frac{a}{2}\left(\cos\frac{a}{2} + i\sin\frac{a}{2}\right).$$

b) If $a \in (\pi, 2\pi)$, then $\frac{a}{2} \in \left(\frac{\pi}{2}, \pi\right)$ and the point $P(1+\cos a, \sin a)$ lies on the fourth quadrant. Hence

$$t^* = \arctan\left(\tan\frac{a}{2}\right) + 2\pi = \frac{a}{2} - \pi + 2\pi = \frac{a}{2} + \pi$$

and

$$z = -2\cos\frac{a}{2}\left(\cos\left(\frac{a}{2}+\pi\right) + i\sin\left(\frac{a}{2}+\pi\right)\right).$$

c) If $a = \pi$, then $z = 0$.

**Problem 2.** *Find all complex numbers $z$ such that $|z| = 1$ and*

$$\left|\frac{z}{\overline{z}} + \frac{\overline{z}}{z}\right| = 1.$$

**Solution.** Let $z = \cos x + i\sin x$, $x \in [0, 2\pi)$. Then

$$\begin{aligned} 1 = \left|\frac{z}{\overline{z}} + \frac{\overline{z}}{z}\right| &= \frac{|z^2 + \overline{z}^2|}{|z|^2} \\ &= |\cos 2x + i\sin 2x + \cos 2x - i\sin 2x| \\ &= 2|\cos 2x| \end{aligned}$$

hence

$$\cos 2x = \frac{1}{2} \text{ or } \cos 2x = -\frac{1}{2}.$$

If $\cos 2x = \frac{1}{2}$, then

$$x_1 = \frac{\pi}{6}, \quad x_2 = \frac{5\pi}{6}, \quad x_3 = \frac{7\pi}{6}, \quad x_4 = \frac{11\pi}{6}.$$

If $\cos 2x = -\frac{1}{2}$, then

$$x_5 = \frac{\pi}{3}, \quad x_6 = \frac{2\pi}{3}, \quad x_7 = \frac{4\pi}{3}, \quad x_8 = \frac{5\pi}{3}.$$

Hence there are eight solutions

$$z_k = \cos x_k + i \sin x_k, \quad k = 1, 2, \ldots, 8.$$

### 2.1.3 Operations with complex numbers in polar representation

**1. Multiplication**

**Proposition.** *Suppose that*

$$z_1 = r_1(\cos t_1 + i \sin t_1) \textit{ and } z_2 = r_2(\cos t_2 + i \sin t_2).$$

*Then*

$$z_1 z_2 = r_1 r_2(\cos(t_1 + t_2) + i \sin(t_1 + t_2)). \tag{1}$$

*Proof.* Indeed,

$$z_1 z_2 = r_1 r_2(\cos t_1 + i \sin t_1)(\cos t_2 + i \sin t_2)$$

$$= r_1 r_2((\cos t_1 \cos t_2 - \sin t_1 \sin t_2) + i(\sin t_1 \cos t_2 + \sin t_2 \cos t_1))$$

$$= r_1 r_2(\cos(t_1 + t_2) + i \sin(t_1 + t_2)).$$ □

**Remarks.** a) We find again that $|z_1 z_2| = |z_1| \cdot |z_2|$.
b) We have $\arg(z_1 z_2) = \arg z_1 + \arg z_2 - 2k\pi$, where

$$k = \begin{cases} 0, & \text{for} \quad \arg z_1 + \arg z_2 < 2\pi, \\ 1, & \text{for} \quad \arg z_1 + \arg z_2 \geq 2\pi. \end{cases}$$

c) Also we can write $\operatorname{Arg}(z_1 z_2) = \{\arg z_1 + \arg z_2 + 2k\pi \, : \, k \in \mathbb{Z}\}$.
d) Formula (1) can be extended to $n \geq 2$ complex numbers. If $z_k = r_k(\cos t_k + i \sin t_k)$, $k = 1, \ldots, n$, then

$$z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n(\cos(t_1 + t_2 + \cdots + t_n) + i \sin(t_1 + t_2 + \cdots + t_n)).$$

The proof by induction is immediate. This formula can be written as

$$\prod_{k=1}^{n} z_k = \prod_{k=1}^{n} r_k \left( \cos \sum_{k=1}^{n} t_k + i \sin \sum_{k=1}^{n} t_k \right). \tag{2}$$

**Example.** Let $z_1 = 1 - i$ and $z_2 = \sqrt{3} + i$. Then

$$z_1 = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right), \quad z_2 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

and

$$\begin{aligned} z_1 z_2 &= 2\sqrt{2}\left[\cos\left(\frac{7\pi}{4} + \frac{\pi}{6}\right) + i\sin\left(\frac{7\pi}{4} + \frac{\pi}{6}\right)\right] \\ &= 2\sqrt{2}\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right). \end{aligned}$$

**2. The power of a complex number**

**Proposition.** (De Moivre[1]) *For* $z = r(\cos t + i\sin t)$ *and* $n \in \mathbb{N}$, *we have*

$$z^n = r^n(\cos nt + i\sin nt). \tag{3}$$

*Proof.* Apply formula (2) for $z = z_1 = z_2 = \cdots = z_n$ to obtain

$$\begin{aligned} z^n &= \underbrace{r \cdot r \cdots r}_{n \text{ times}}(\cos(\underbrace{t + t + \cdots + t}_{n \text{ times}}) + i\sin(\underbrace{t + t + \cdots + t}_{n \text{ times}})) \\ &= r^n(\cos nt + i\sin nt). \end{aligned}$$ □

**Remarks.** a) We find again that $|z^n| = |z|^n$.

b) If $r = 1$, then $(\cos t + i\sin t)^n = \cos nt + i\sin nt$.

c) We can write $\operatorname{Arg} z^n = \{n \arg z + 2k\pi \,:\, k \in \mathbb{Z}\}$.

**Example.** Let us compute $(1+i)^{1000}$.

The polar representation of $1 + i$ is $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. Applying de Moivre's formula we obtain

$$\begin{aligned} (1+i)^{1000} &= (\sqrt{2})^{1000}\left(\cos 1000\frac{\pi}{4} + i\sin 1000\frac{\pi}{4}\right) \\ &= 2^{500}(\cos 250\pi + i\sin 250\pi) = 2^{500}. \end{aligned}$$

**Problem.** *Prove that*

$$\sin 5t = 16\sin^5 t - 20\sin^3 t + 5\sin t;$$

$$\cos 5t = 16\cos^5 t - 20\cos^3 t + 5\cos t.$$

[1] Abraham de Moivre (1667–1754), French mathematician, a pioneer in probability theory and trigonometry.

**Solution.** Using de Moivre's theorem to expand $(\cos t + i \sin t)^5$, then using the binomial theorem, we have

$$\cos 5t + i \sin 5t = \cos^5 t + 5i \cos^4 t \sin t + 10i^2 \cos^3 t \sin^2 t$$
$$+ 10i^3 \cos^2 t \sin^3 t + 5i^4 \cos t \sin^4 t + i^5 \sin^5 t.$$

Hence

$$\cos 5t + i \sin 5t = \cos^5 t - 10 \cos^3 t (1 - \cos^2 t) + 5 \cos t (1 - \cos^2 t)^2$$
$$+ i (sint (1 - \sin^2 t)^2 \sin t - 10(1 - \sin^2 t) \sin^3 t + \sin^5 t).$$

Simple algebraic manipulation leads to the desired result.

**3. Division**

**Proposition.** *Suppose that*

$$z_1 = r_1(\cos t_1 + i \sin t_2), \quad z_2 = r_2(\cos t_2 + i \sin t_2) \neq 0.$$

*Then*

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(t_1 - t_2) + i \sin(t_1 - t_2)].$$

*Proof.* We have

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1(\cos t_1 + i \sin t_1)}{r_2(\cos t_2 + i \sin t_2)} = \\ &= \frac{r_1(\cos t_1 + i \sin t_1)(\cos t_2 - i \sin t_2)}{r_2(\cos^2 t_2 + \sin^2 t_2)} \\ &= \frac{r_1}{r_2}[(\cos t_1 \cos t_2 + \sin t_1 \sin t_2) + i(\sin t_1 \cos t_2 - \sin t_2 \cos t_1)] \\ &= \frac{r_1}{r_2}(\cos(t_1 - t_2) + i \sin(t_1 - t_2)).\end{aligned}$$ □

**Remarks.** a) We have again $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$;

b) We can write $\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \{\arg z_1 - \arg z_2 + 2k\pi : k \in \mathbb{Z}\}$;

c) For $z_1 = 1$ and $z_2 = z$,

$$\frac{1}{z} = z^{-1} = \frac{1}{r}(\cos(-t) + i \sin(-t));$$

d) De Moivre's formula also holds for negative integer exponents $n$, i.e., we have

$$z^n = r^n(\cos nt + i \sin nt).$$

**Problem.** *Compute*

$$z = \frac{(1-i)^{10}(\sqrt{3}+i)^5}{(-1-i\sqrt{3})^{10}}.$$

**Solution.** We can write

$$\begin{aligned} z &= \frac{(\sqrt{2})^{10}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)^{10} \cdot 2^5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5}{2^{10}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)^{10}} \\ &= \frac{2^{10}\left(\cos\frac{35\pi}{2} + i\sin\frac{35\pi}{2}\right)\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}{2^{10}\left(\cos\frac{40\pi}{3} + i\sin\frac{40\pi}{3}\right)} \\ &= \frac{\cos\frac{55\pi}{3} + i\sin\frac{55\pi}{3}}{\cos\frac{40\pi}{3} + i\sin\frac{40\pi}{3}} = \cos 5\pi + i\sin 5\pi = -1. \end{aligned}$$

### 2.1.4 Geometric interpretation of multiplication

Consider the complex numbers

$$z_1 = r_1(\cos t_1^* + i\sin t_1^*), \quad z_2 = r_2(\cos t_2^* + i\sin t_2^*)$$

and their geometric images $M_1(r_1, t_1^*)$, $M_2(r_2, t_2^*)$. Let $P_1$, $P_2$ be the intersection points of the circle $\mathcal{C}(O; 1)$ with the rays $(OM_1$ and $(OM_2$. Construct the point $P_3 \in \mathcal{C}(O; 1)$ with the polar argument $t_1^* + t_2^*$ and choose the point $M_3 \in (OP_3$ such that $OM_3 = OM_1 \cdot OM_2$. Let $z_3$ be the complex coordinate of $M_3$. The point $M_3(r_1r_2, t_1^* + t_2^*)$ is the geometric image of the product $z_1 \cdot z_2$.

Let $A$ be the geometric image of the complex number 1. Because

$$\frac{OM_3}{OM_1} = \frac{OM_2}{1}, \quad \text{i.e.,} \quad \frac{OM_3}{OM_2} = \frac{OM_2}{OA}$$

and $\widehat{M_2OM_3} = \widehat{AOM_1}$, it follows that triangles $OAM_1$ and $OM_2M_3$ are similar.

In order to construct the geometric image of the quotient, note that the image of $\dfrac{z_3}{z_2}$ is $M_1$.

### 2.1.5 Problems

**1.** Find the polar coordinates for the following points, given their cartesian coordinates:

a) $M_1(-3, 3)$; b) $M_2(-4\sqrt{3}, -4)$; c) $M_3(0, -5)$;

d) $M_4(-2, -1)$; e) $M_5(4, -2)$.

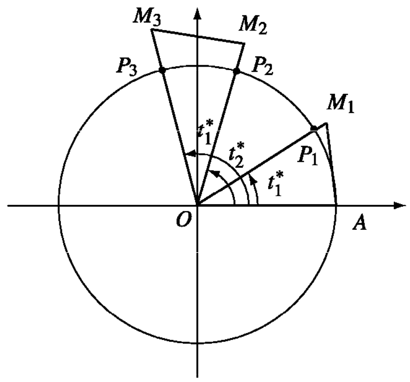


**Figure 2.5.**

**2.** Find the cartesian coordinates for the following points, given their polar coordinates:

a) $P_1\left(2, \frac{\pi}{3}\right)$; b) $P_2\left(4, 2\pi - \arcsin\frac{3}{5}\right)$; c) $P_3(2, \pi)$;

d) $P_4(3, -\pi)$; e) $P_5\left(1, \frac{\pi}{2}\right)$; f) $P_6\left(4, \frac{3\pi}{2}\right)$.

**3.** Express $\arg(\overline{z})$ and $\arg(-z)$ in terms of $\arg(z)$.

**4.** Find the geometric images for the complex numbers $z$ in each of the following cases:

a) $|z| = 2$; b) $|z + i| \geq 2$; c) $|z - i| \leq 3$;

d) $\pi < \arg z < \frac{5\pi}{4}$; e) $\arg z \geq \frac{3\pi}{2}$; f) $\arg z < \frac{\pi}{2}$;

g) $\arg(-z) \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$; h) $|z + 1 + i| < 3$ and $0 < \arg z < \frac{\pi}{6}$.

**5.** Find polar representations for the following complex numbers:

a) $z_1 = 6 + 6i\sqrt{3}$; b) $z_2 = -\frac{1}{4} + i\frac{\sqrt{3}}{4}$; c) $z_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$;

d) $z_4 = 9 - 9i\sqrt{3}$; e) $z_5 = 3 - 2i$; f) $z_6 = -4i$.

**6.** Find polar representations for the following complex numbers:

a) $z_1 = \cos a - i \sin a, \quad a \in [0, 2\pi)$;

b) $z_2 = \sin a + i(1 + \cos a), \quad a \in [0, 2\pi)$;

c) $z_3 = \cos a + \sin a + i(\sin a - \cos a), \quad a \in [0, 2\pi)$;

d) $z_4 = 1 - \cos a + i \sin a, \quad a \in [0, 2\pi)$.

**7.** Compute the following products using the polar representation of a complex number:

a) $\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)(-3 + 3i)(2\sqrt{3} + 2i)$; b) $(1 + i)(-2 - 2i) \cdot i$;

c) $-2i \cdot (-4 + 4\sqrt{3}i) \cdot (3 + 3i)$; d) $3 \cdot (1 - i)(-5 + 5i)$.

Verify your results using the algebraic form.

**8.** Find $|z|$, $\arg z$, $\operatorname{Arg} z$, $\arg \overline{z}$, $\arg(-z)$ for

a) $z = (1 - i)(6 + 6i)$; b) $z = (7 - 7\sqrt{3}i)(-1 - i)$.

**9.** Find $|z|$ and $\arg z$ for

a) $z = \dfrac{(2\sqrt{3} + 2i)^8}{(1 - i)^6} + \dfrac{(1 + i)^6}{(2\sqrt{3} - 2i)^8}$;

b) $z = \dfrac{(-1 + i)^4}{(\sqrt{3} - i)^{10}} + \dfrac{1}{(2\sqrt{3} + 2i)^4}$;

c) $z = (1 + i\sqrt{3})^n + (1 - i\sqrt{3})^n$.

**10.** Prove that de Moivre's formula holds for negative integer exponents.

**11.** Compute:

a) $(1 - \cos a + i \sin a)^n$ for $a \in [0, 2\pi)$ and $n \in \mathbb{N}$;

b) $z^n + \dfrac{1}{z^n}$, if $z + \dfrac{1}{z} = \sqrt{3}$.

## 2.2 The $n^{\text{th}}$ Roots of Unity

### 2.2.1 Defining the $n^{\text{th}}$ roots of a complex number

Consider a positive integer $n \geq 2$ and a complex number $z_0 \neq 0$. As in the field of real numbers, the equation

$$Z^n - z_0 = 0 \tag{1}$$

is used for defining the $n^{\text{th}}$ roots of number $z_0$. Hence we call any solution $Z$ of the equation (1) an *$n^{th}$ root of the complex number $z_0$*.

**Theorem.** *Let $z_0 = r(\cos t^* + i \sin t^*)$ be a complex number with $r > 0$ and $t^* \in [0, 2\pi)$.*

*The number $z_0$ has $n$ distinct $n^{th}$ roots, given by the formulas*

$$Z_k = \sqrt[n]{r}\left(\cos \frac{t^* + 2k\pi}{n} + i \sin \frac{t^* + 2k\pi}{n}\right),$$

$k = 0, 1, \ldots, n - 1$.

*Proof.* We use the polar representation of the complex number $Z$ with the extended argument

$$Z = \rho(\cos \varphi + i \sin \varphi).$$

By definition, we have $Z^n = z_0$ or equivalently

$$\rho^n(\cos n\varphi + i \sin n\varphi) = r(\cos t^* + i \sin t^*).$$

We obtain $\rho^n = r$ and $n\varphi = t^* + 2k\pi$ for $k \in \mathbb{Z}$; hence $\rho = \sqrt[n]{r}$ and $\varphi_k = \dfrac{t^*}{n} + k \cdot \dfrac{2\pi}{n}$ for $k \in \mathbb{Z}$.

So far the roots of equation (1) are

$$Z_k = \sqrt[n]{r}(\cos\varphi_k + i\sin\varphi_k) \text{ for } k \in \mathbb{Z}.$$

Now observe that $0 \le \varphi_0 < \varphi_1 < \cdots < \varphi_{n-1} < 2\pi$, so the numbers $\varphi_k$, $k \in \{0, 1, \ldots, n-1\}$, are reduced arguments, i.e., $\varphi_k^* = \varphi_k$. Until now we had $n$ distinct roots of $z_0$:

$$Z_0, Z_1, \ldots, Z_{n-1}.$$

Consider some integer $k$ and let $r \in \{0, 1, \ldots, n-1\}$ be the residue of $k$ modulo $n$. Then $k = nq + r$ for $q \in \mathbb{Z}$, and

$$\varphi_k = \frac{t^*}{n} + (nq + r)\frac{2\pi}{n} = \frac{t^*}{n} + r\frac{2\pi}{n} + 2q\pi = \varphi_r + 2q\pi.$$

It is clear that $Z_k = Z_r$. Hence

$$\{Z_k : k \in \mathbb{Z}\} = \{Z_0, Z_1, \ldots, Z_{n-1}\}.$$

In other words, there are exactly $n$ distinct $n^{\text{th}}$ roots of $z_0$, as claimed. □

The geometric images of the $n^{\text{th}}$ roots of a complex number $z_0 \neq 0$ are the vertices of a regular $n$-gon inscribed in a circle with center at the origin and radius $\sqrt[n]{r}$.

To prove this, denote $M_0, M_1, \ldots, M_{n-1}$ the points with complex coordinates $Z_0$, $Z_1, \ldots, Z_{n-1}$. Because $OM_k = |Z_k| = \sqrt[n]{r}$ for $k \in \{0, 1, \ldots, n-1\}$, it follows that the points $M_k$ lie on the circle $\mathcal{C}(O; \sqrt[n]{r})$. On the other hand, the measure of the arc $\overset{\frown}{M_k M_{k+1}}$ is equal to

$$\arg Z_{k+1} - \arg Z_k = \frac{t^* + 2(k+1)\pi - (t^* + 2k\pi)}{n} = \frac{2\pi}{n},$$

for all $k \in \{0, 1, \ldots, n-2\}$ and the remaining arc $\overset{\frown}{M_{n-1}M_0}$ is

$$\frac{2\pi}{n} = 2\pi - (n-1)\frac{2\pi}{n}.$$

Because all of the arcs $\overset{\frown}{M_0M_1}$, $\overset{\frown}{M_1M_2}, \ldots, \overset{\frown}{M_{n-1}M_0}$ are equal, the polygon $M_0M_1 \cdots M_{n-1}$ is regular.

**Example.** Let us find the third roots of the number $z = 1 + i$ and represent them in the complex plane.

The polar representation of $z = 1 + i$ is

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

The cube roots of the number $z$ are

$$Z_k = \sqrt[6]{2}\left(\cos\left(\frac{\pi}{12} + k\frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{12} + k\frac{2\pi}{3}\right)\right),\ k = 0, 1, 2,$$

or, in explicit form,

$$Z_0 = \sqrt[6]{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right),$$

$$Z_1 = \sqrt[6]{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

and

$$Z_2 = \sqrt[6]{2}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right).$$

Using polar coordinates, the geometric images of the numbers $Z_0$, $Z_1$, $Z_2$ are

$$M_0\left(\sqrt[6]{2}, \frac{\pi}{12}\right),\ M_1\left(\sqrt[6]{2}, \frac{3\pi}{4}\right),\ M_2\left(\sqrt[6]{2}, \frac{17\pi}{12}\right).$$

The resulting equilateral triangle $M_0M_1M_2$ is shown in the following figure:

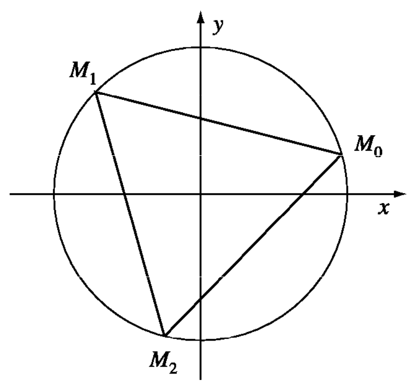


**Figure 2.6.**

## 2.2.2 The $n^{\text{th}}$ roots of unity

The roots of the equation $Z^n - 1 = 0$ are called the $n^{th}$ *roots of unity*. Since $1 = \cos 0 + i\sin 0$, from the formulas for the $n^{\text{th}}$ roots of a complex number we derive that the $n^{\text{th}}$ roots of unity are

$$\varepsilon_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k \in \{0, 1, 2, \ldots, n-1\}.$$

Explicitly, we have

$$\varepsilon_0 = \cos 0 + i\sin 0 = 1;$$

$$\varepsilon_1 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = \varepsilon;$$

$$\varepsilon_2 = \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n} = \varepsilon^2;$$

$$\dots$$

$$\varepsilon_{n-1} = \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n} = \varepsilon^{n-1}.$$

The set $\{1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{n-1}\}$ is denoted by $U_n$. Observe that the set $U_n$ is generated by the element $\varepsilon$, i.e., the elements of $U_n$ are the powers of $\varepsilon$.

As stated before, the geometric images of the $n^{\text{th}}$ roots of unity are the vertices of a regular polygon with $n$ sides inscribed in the unit circle with one of the vertices at 1.

We take a brief look at some particular values of $n$.

i) For $n = 2$, the equation $Z^2 - 1 = 0$ has the roots $-1$ and 1, which are the square roots of unity.

ii) For $n = 3$, the cube roots of unity, i.e., the roots of equation $Z^3 - 1 = 0$ are given by

$$\varepsilon_k = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3} \quad \text{for } k \in \{0, 1, 2\}.$$

Hence

$$\varepsilon_0 = 1, \quad \varepsilon_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \varepsilon$$

and

$$\varepsilon_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \varepsilon^2.$$

They form an equilateral triangle inscribed in the circle $\mathcal{C}(O; 1)$ as in the figure below.

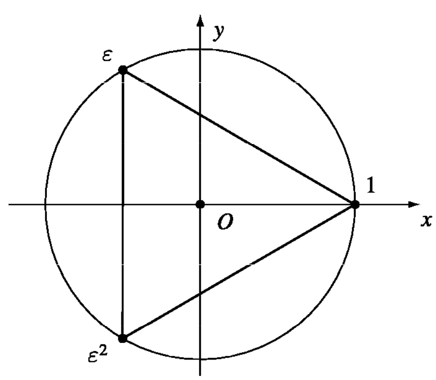


**Figure 2.7.**

iii) For $n = 4$, the fourth roots of unity are

$$\varepsilon_k = \cos\frac{2k\pi}{4} + i\sin\frac{2k\pi}{4} \text{ for } k = 0, 1, 2, 3.$$

In explicit form, we have

$$\varepsilon_0 = \cos 0 + i\sin 0 = 1; \quad \varepsilon_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i;$$

$$\varepsilon_2 = \cos\pi + i\sin\pi = -1 \text{ and } \varepsilon_3 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i.$$

Observe that $U_4 = \{1, i, i^2, i^3\} = \{1, i, -1, -i\}$. The geometric images of the fourth roots of unity are the vertices of a square inscribed in the circle $\mathcal{C}(O; 1)$.

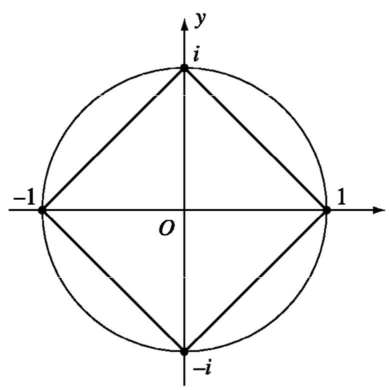


**Figure 2.8.**

The root $\varepsilon_k \in U_n$ is called *primitive* if for all positive integer $m < n$ we have $\varepsilon_k^m \neq 1$.

**Proposition 1.** *a) If $n|q$, then any root of $Z^n - 1 = 0$ is a root of $Z^q - 1 = 0$.*

*b) The common roots of $Z^m - 1 = 0$ and $Z^n - 1 = 0$ are the roots of $Z^d - 1 = 0$, where $d = \gcd(m, n)$, i.e., $U_m \cap U_n = U_d$.*

*c) The primitive roots of $Z^m - 1 = 0$ are $\varepsilon_k = \cos\frac{2k\pi}{m} + i\sin\frac{2k\pi}{m}$, where $0 \leq k \leq m$ and $\gcd(k, m) = 1$.*

*Proof.* a) If $q = pn$, then $Z^q - 1 = (Z^n)^p - 1 = (Z^n - 1)(Z^{(p-1)n} + \cdots + Z^n + 1)$ and the conclusion follows.

b) Consider $\varepsilon_p = \cos\frac{2p\pi}{m} + i\sin\frac{2p\pi}{m}$ a root of $Z^m - 1 = 0$ and $\varepsilon'_q = \cos\frac{2q\pi}{n} + i\sin\frac{2q\pi}{n}$ a root of $Z^n - 1 = 0$. Since $|\varepsilon_p| = |\varepsilon'_q| = 1$, we have $\varepsilon_p = \varepsilon'_q$ if and only if $\arg \varepsilon_p = \arg \varepsilon'_q$, i.e., $\dfrac{2p\pi}{m} = \dfrac{2q\pi}{n} + 2r\pi$ for some integer $r$. The last relation is equivalent to $\dfrac{p}{m} - \dfrac{q}{n} = r$, that is, $pn - qm = rmn$.

On the other hand we have $m = m'd$ and $n = n'd$, where $\gcd(m', n') = 1$. From the relation $pn - qm = rmn$ we find $n'p - m'q = rm'n'd$. Hence $m'|n'p$, so $m'|p$. That is, $p = p'm'$ for some positive integer $p'$ and

$$\arg \varepsilon_p = \frac{2p\pi}{m} = \frac{2p'm'\pi}{m'd} = \frac{2p'\pi}{d} \text{ and } \varepsilon_p^d = 1.$$

Conversely, since $d|m$ and $d|n$ (from property a), any root of $Z^d - 1 = 0$ is a root of $Z^m - 1 = 0$ and $Z^n - 1 = 0$.

c) First we will find the smallest positive integer $p$ such that $\varepsilon_k^p = 1$. From the relation $\varepsilon_k^p = 1$ it follows that $\dfrac{2kp\pi}{m} = 2k'\pi$ for some positive integer $k'$. That is, $\dfrac{kp}{m} = k' \in \mathbb{Z}$. Consider $d = \gcd(k, m)$ and $k = k'd$, $m = m'd$, where $\gcd(k', m') = 1$. We obtain $\dfrac{k'pd}{m'd} = \dfrac{k'p}{m'} \in \mathbb{Z}$. Since $k'$ and $m'$ are relatively primes, we get $m'|p$. Therefore, the smallest positive integer $p$ with $\varepsilon_k^p = 1$ is $p = m'$. Substituting in the relation $m = m'd$, it follows that $p = \dfrac{m}{d}$, where $d = \gcd(k, m)$.

If $\varepsilon_k$ is a primitive root of unity, then from relation $\varepsilon_k^p = 1$, $p = \dfrac{m}{\gcd(k, m)}$, it follows that $p = m$, i.e., $\gcd(k, m) = 1$. □

**Remark.** From Proposition 1.b) one obtains that the equations $Z^m - 1 = 0$ and $Z^n - 1 = 0$ have the unique common root 1 if and only if $\gcd(m, n) = 1$.

**Proposition 2.** *If $\varepsilon \in U_n$ is a primitive root of unity, then the roots of the equation $z^n - 1 = 0$ are $\varepsilon^r, \varepsilon^{r+1}, \ldots, \varepsilon^{r+n-1}$, where $r$ is an arbitrary positive integer.*

*Proof.* Let $r$ be a positive integer and consider $h \in \{0, 1, \ldots, n-1\}$. Then $(\varepsilon^{r+h})^n = (\varepsilon^n)^{r+h} = 1$, i.e., $\varepsilon^{r+h}$ is a root of $Z^n - 1 = 0$.

We need only prove that $\varepsilon^r, \varepsilon^{r+1}, \ldots, \varepsilon^{r+n-1}$ are distinct. Assume by way of contradiction that for $r + h_1 \neq r + h_2$ and $h_1 > h_2$, we have $\varepsilon^{r+h_1} = \varepsilon^{r+h_2}$. Then $\varepsilon^{r+h_2}(\varepsilon^{h_1-h_2} - 1) = 0$. But $\varepsilon^{r+h_2} \neq 0$ implies $\varepsilon^{h_1-h_2} = 1$. Taking into account that $h_1 - h_2 < n$ and $\varepsilon$ is a primitive root of $Z^n - 1 = 0$, we get a contradiction. □

**Proposition 3.** *Let $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}$ be the $n^{th}$ roots of unity. For any positive integer $k$ the following relation holds:*

$$\sum_{j=0}^{n-1} \varepsilon_j^k = \begin{cases} n, & \text{if } n|k, \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* Consider $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Then $\varepsilon \in U_n$ is a primitive root of unity, hence $\varepsilon^m = 1$ if and only if $n|m$. Assume that $n$ does not divides $k$. We have

$$\sum_{j=0}^{n-1}\varepsilon_j^k = \sum_{j=0}^{n-1}(\varepsilon^j)^k = \sum_{j=0}^{n-1}(\varepsilon^k)^j = \frac{1-(\varepsilon^k)^n}{1-\varepsilon^k} = \frac{1-(\varepsilon^n)^k}{1-\varepsilon^k} = 0.$$

If $n|k$, then $k = qn$ for some positive integer $q$, and we obtain

$$\sum_{j=0}^{n-1}\varepsilon_j^k = \sum_{j=0}^{n-1}\varepsilon_j^{qn} = \sum_{j=0}^{n-1}(\varepsilon_j^n)^q = \sum_{j=0}^{n-1}1 = n. \qquad \square$$

**Proposition 4.** *Let $p$ be a prime number and let $\varepsilon = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}$. If $a_0, a_1, \ldots, a_{p-1}$ are nonzero integers, the relation*

$$a_0 + a_1\varepsilon + \cdots + a_{p-1}\varepsilon^{p-1} = 0$$

*holds if and only if $a_0 = a_1 = \cdots = a_{p-1}$.*

*Proof.* If $a_0 = a_1 = \cdots = a_{p-1}$, then the above relation is clearly true.

Conversely, define the polynomials $f, g \in \mathbb{Z}[X]$ by $f = a_1 + a_1X + \cdots + a_{p-1}X^{p-1}$ and $g = 1 + X + \cdots + X^{p-1}$. If the polynomials $f, g$ have common zeros, then $\gcd(f, g)$ divides $g$. But it is well known (for example by Eisenstein's irreducibility criterion) that $g$ is irreducible over $\mathbb{Z}$. Hence $\gcd(f, g) = g$, so $g|f$ and we obtain $g = kf$ for some nonzero integer $k$, i.e., $a_0 = a_1 = \cdots = a_{n-1}$. $\square$

**Problem 1.** *Find the number of ordered pairs $(a, b)$ of real numbers such that $(a + bi)^{2002} = a - bi$.*

(American Mathematics Contest 12A, 2002, Problem 24)

**Solution.** Let $z = a + bi$, $\overline{z} = a - bi$, and $|z| = \sqrt{a^2 + b^2}$. The given relation becomes $z^{2002} = \overline{z}$. Note that

$$|z|^{2002} = |z^{2002}| = |\overline{z}| = |z|,$$

from which it follows that

$$|z|(|z|^{2001} - 1) = 0.$$

Hence $|z| = 0$, and $(a, b) = (0, 0)$, or $|z| = 1$. In the case $|z| = 1$, we have $z^{2002} = \overline{z}$, which is equivalent to $z^{2003} = \overline{z} \cdot z = |z|^2 = 1$. Since the equation $z^{2003} = 1$ has 2003 distinct solutions, there are altogether $1 + 2003 = 2004$ ordered pairs that meet the required conditions.

**Problem 2.** *Two regular polygons are inscribed in the same circle. The first polygon has 1982 sides and the second has 2973 sides. If the polygons have any common vertices, how many such vertices will there be?*

**Solution.** The number of common vertices is given by the number of common roots of $z^{1982} - 1 = 0$ and $z^{2973} - 1 = 0$. Applying Proposition 1.b), the desired number is $d = \gcd(1982, 2973) = 991$.

**Problem 3.** *Let $\varepsilon \in U_n$ be a primitive root of unity and let $z$ be a complex number such that $|z - \varepsilon^k| \le 1$ for all $k = 0, 1, \ldots, n-1$. Prove that $z = 0$.*

**Solution.** From the given condition it follows that $(z - \varepsilon^k)\overline{(z - \varepsilon^k)} \le 1$, yielding $|z|^2 \le z\overline{(\varepsilon^k)} + \overline{z} \cdot \varepsilon^k$, $k = 0, 1, \ldots, n-1$. By summing these relations we obtain

$$n|z|^2 \le z\overline{\left(\sum_{k=0}^{n-1} \varepsilon^k\right)} + \overline{z} \cdot \sum_{k=0}^{n-1} \varepsilon^k = 0.$$

Thus $z = 0$.

**Problem 4.** *Let $P_0 P_1 \cdots P_{n-1}$ be a regular polygon inscribed in a circle of radius 1. Prove that:*

*a)* $P_0P_1 \cdot P_0P_2 \cdots P_0P_{n-1} = n$;

*b)* $\sin\frac{\pi}{n} \sin\frac{2\pi}{n} \cdots \sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$;

*c)* $\sin\frac{\pi}{2n} \sin\frac{3\pi}{2n} \cdots \sin\frac{(2n-1)\pi}{2n} = \frac{1}{2^{n-1}}$.

**Solution.** a) Without loss of generality we may assume that the vertices of the polygon are the geometric images of the $n^{\text{th}}$ roots of unity, and $P_0 = 1$. Consider the polynomial $f = z^n - 1 = (z-1)(z-\varepsilon)\cdots(z-\varepsilon^{n-1})$, where $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Then it is clear that

$$n = f'(1) = (1-\varepsilon)(1-\varepsilon^2)\cdots(1-\varepsilon^{n-1}).$$

Taking the modulus of each side, the desired result follows.

b) We have

$$1 - \varepsilon^k = 1 - \cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n} = 2\sin^2\frac{k\pi}{n} - 2i\sin\frac{k\pi}{n}\cos\frac{k\pi}{n}$$
$$= 2\sin\frac{k\pi}{n}\left(\sin\frac{k\pi}{n} - i\cos\frac{k\pi}{n}\right),$$

hence $|1 - \varepsilon^k| = 2\sin\frac{k\pi}{n}$, $k = 1, 2, \ldots, n-1$, and the desired trigonometric identity follows from a).

c) Consider the regular polygon $Q_0Q_1 \cdots Q_{2n-1}$ inscribed in the same circle whose vertices are the geometric images of the $(2n)^{\text{th}}$ roots of unity. According to a),

$$Q_0Q_1 \cdot Q_0Q_2 \cdots Q_0Q_{2n-1} = 2n.$$

Now taking into account that $Q_0Q_2\cdots Q_{n-2}$ is also a regular polygon, we deduce from a) that

$$Q_0Q_2 \cdot Q_0Q_4 \cdots Q_0Q_{2n-2} = n.$$

Combining the last two relations yields

$$Q_0Q_1 \cdot Q_0Q_3 \cdots Q_0Q_{2n-1} = 2.$$

A similar computation to the one in b) leads to

$$Q_0Q_{2k-1} = 2\sin\frac{(2k-1)\pi}{2n}, \quad k = 1, 2, \ldots, n,$$

and the desired result follows.

Let $n$ be a positive integer and let $\varepsilon_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. The $n^{th}$-*cyclotomic polynomial* is defined by

$$\phi_n(x) = \prod_{\substack{1\le k\le n-1 \\ \gcd(k,n)=1}} (x - \varepsilon_n^k).$$

Clearly the degree of $\phi_n$ is $\varphi(n)$, where $\varphi$ is the Euler "totient" function. $\phi_n$ is a monic polynomial with integer coefficients and is irreducible over $\mathbb{Q}$. The first sixteen cyclotomic polynomials are given below:

$\phi(x) = x - 1$
$\phi_2(x) = x + 1$
$\phi_3(x) = x^2 + x + 1$
$\phi_4(x) = x^2 + 1$
$\phi_5(x) = x^4 + x^3 + x^2 + x + 1$
$\phi_6(x) = x^2 - x + 1$
$\phi_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
$\phi_8(x) = x^4 + 1$
$\phi_9(x) = x^6 + x^3 + 1$
$\phi_{10}(x) = x^4 - x^3 + x^2 - x + 1$
$\phi_{11}(x) = x^{10} + x^9 + x^8 + \cdots + x + 1$
$\phi_{12}(x) = x^4 - x^2 + 1$
$\phi_{13}(x) = x^{12} + x^{11} + x^{10} + \cdots + x + 1$
$\phi_{14}(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$
$\phi_{15}(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$
$\phi_{16}(x) = x^8 + 1$

The following properties of cyclotomic polynomials are well known:

1) If $q > 1$ is an odd integer, then $\phi_{2q}(x) = \phi_q(-x)$.

2) If $n > 1$, then

$$\phi_n(1) = \begin{cases} p, & \text{when } n \text{ is a power of a prime } p, \\ 1, & \text{otherwise.} \end{cases}$$

The next problem extends the trigonometric identity in Problem 4.b).

**Problem 5.** *The following identities hold:*

*a)* $\displaystyle\prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} \sin\frac{k\pi}{n} = \frac{1}{2^{\varphi(n)}}$, *whenever $n$ is not a power of a prime;*

*b)* $\displaystyle\prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} \cos\frac{k\pi}{n} = \frac{(-1)^{\frac{\varphi(n)}{2}}}{2^{\varphi(n)}}$, *for all odd positive integers $n$.*

**Solution.** a) As we have seen in Problem 4.b),

$$1-\varepsilon_n^k = 2\sin\frac{k\pi}{n}\left(\sin\frac{k\pi}{n} - i\cos\frac{k\pi}{n}\right) = \frac{2}{i}\sin\frac{k\pi}{n}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right).$$

We have

$$1 = \phi_n(1) = \prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} (1-\varepsilon_n^k) = \prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} \frac{2}{i}\sin\frac{k\pi}{n}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right)$$

$$= \frac{2^{\varphi(n)}}{i^{\varphi(n)}}\left(\prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} \sin\frac{k\pi}{n}\right)\left(\cos\frac{\varphi(n)}{2}\pi + i\sin\frac{\varphi(n)}{2}\pi\right)$$

$$= \frac{2^{\varphi(n)}}{(-1)^{\frac{\varphi(n)}{2}}}\left(\prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} \sin\frac{k\pi}{n}\right)(-1)^{\frac{\varphi(n)}{2}},$$

where we have used the fact that $\varphi(n)$ is even, and also the well-known relation

$$\sum_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} k = \frac{1}{2}n\varphi(n).$$

The conclusion follows.

b) We have

$$1+\varepsilon_n^k = 1+\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = 2\cos^2\frac{k\pi}{n} + 2i\sin\frac{k\pi}{n}\cos\frac{k\pi}{n}$$

$$= 2\cos\frac{k\pi}{n}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1.$$

Because $n$ is odd, from the relation $\phi_{2n}(x) = \phi_n(-1)$ it follows that $\phi_n(-1) = \phi_{2n}(1) = 1$. Then

$$1 = \phi_n(-1) = \prod_{\substack{1 \le k \le n-1 \\ \gcd(k,n)=1}} (1 - \varepsilon_n^k) = (-1)^{\varphi(n)} \prod_{\substack{1 \le k \le n-1 \\ \gcd(k,n)=1}} (1 + \varepsilon_n^k)$$

$$= (-1)^{\varphi(n)} \prod_{\substack{1 \le k \le n-1 \\ \gcd(k,n)=1}} 2 \cos \frac{k\pi}{n} \left( \cos \frac{k\pi}{n} + i \sin \frac{k\pi}{n} \right)$$

$$= (-1)^{\varphi(n)} 2^{\varphi(n)} \left( \prod_{\substack{1 \le k \le n-1 \\ \gcd(k,n)=1}} \cos \frac{k\pi}{n} \right) \left( \cos \frac{\varphi(n)}{2}\pi + i \sin \frac{\varphi(n)}{2}\pi \right)$$

$$= (-1)^{\frac{\varphi(n)}{2}} 2^{\varphi(n)} \prod_{\substack{1 \le k \le n-1 \\ \gcd(k,n)=1}} \cos \frac{k\pi}{n},$$

yielding the desired identity.

### 2.2.3 Binomial equations

A binomial equation is an equation of the form $Z^n + a = 0$, where $a \in \mathbb{C}^*$ and $n \ge 2$ is an integer.

Solving for $Z$ means finding the $n^{\text{th}}$ roots of the complex number $-a$. This is in fact a simple polynomial equation of degree $n$ with complex coefficients. From the well-known fundamental theorem of algebra it follows that it has exactly $n$ complex roots, and it is obvious that the roots are distinct.

**Example.** 1) Let us find the roots of $Z^3 + 8 = 0$.

We have $-8 = 8(\cos \pi + i \sin \pi)$, so the roots are

$$Z_k = 2\left( \cos \frac{\pi + 2k\pi}{3} + i \sin \frac{\pi + 2k\pi}{3} \right), \quad k \in \{0, 1, 2\}.$$

2) Let us solve the equation $Z^6 - Z^3(1+i) + i = 0$.

Observe that the equation is equivalent to

$$(Z^3 - 1)(Z^3 - i) = 0.$$

Solving for $Z$ the binomial equations $Z^3 - 1 = 0$ and $Z^3 - i = 0$, we obtain the solutions

$$\varepsilon_k = \cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3} \text{ for } k \in \{0, 1, 2\}$$

and

$$Z_k = \cos \frac{\frac{\pi}{2} + 2k\pi}{3} + i \sin \frac{\frac{\pi}{2} + 2k\pi}{3} \text{ for } k \in \{0, 1, 2\}.$$

### 2.2.4 Problems

**1.** Find the square roots of the following complex numbers:

a) $z = 1 + i$; b) $z = i$; c) $z = \dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$;

d) $z = -2(1 + i\sqrt{3})$; e) $z = 7 - 24i$.

**2.** Find the cube roots of the following complex numbers:

a) $z = -i$; b) $z = -27$; c) $z = 2 + 2i$;

d) $z = \dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$; e) $z = 18 + 26i$.

**3.** Find the fourth roots of the following complex numbers:

a) $z = 2 - i\sqrt{12}$; b) $z = \sqrt{3} + i$; c) $z = i$;

d) $z = -2i$; e) $z = -7 + 24i$.

**4.** Find the fifth, sixth, seventh, eighth, and twefth roots of the complex numbers given above.

**5.** Let $U_n = \{\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}\}$. Prove that:

a) $\varepsilon_j \cdot \varepsilon_k \in U_n$, for all $j, k \in \{0, 1, \ldots, n-1\}$;

b) $\varepsilon_j^{-1} \in U_n$, for all $j \in \{0, 1, \ldots, n-1\}$.

**6.** Solve the equations:

a) $z^3 - 125 = 0$; b) $z^4 + 16 = 0$;

c) $z^3 + 64i = 0$; d) $z^3 - 27i = 0$.

**7.** Solve the equations:

a) $z^7 - 2iz^4 - iz^3 - 2 = 0$; b) $z^6 + iz^3 + i - 1 = 0$;

c) $(2 - 3i)z^6 + 1 + 5i = 0$; d) $z^{10} + (-2 + i)z^5 - 2i = 0$.

**8.** Solve the equation

$$z^4 = 5(z - 1)(z^2 - z + 1).$$

# 3

# Complex Numbers and Geometry

## 3.1 Some Simple Geometric Notions and Properties

### 3.1.1 The distance between two points

Suppose that the complex numbers $z_1$ and $z_2$ have the geometric images $M_1$ and $M_2$. Then the distance between the points $M_1$ and $M_2$ is given by

$$M_1M_2 = |z_1 - z_2|.$$

The distance function $d : \mathbb{C} \times \mathbb{C} \to [0, \infty)$ is defined by

$$d(z_1, z_2) = |z_1 - z_2|,$$

and it satisfies the following properties:

a) (positiveness and nondegeneration):

$$d(z_1, z_2) \geq 0 \text{ for all } z_1, z_2 \in \mathbb{C};$$

$$d(z_1, z_2) = 0 \text{ if and only if } z_1 = z_2.$$

b) (symmetry):

$$d(z_1, z_2) = d(z_2, z_1) \text{ for all } z_1, z_2 \in \mathbb{C}.$$

c) (triangle inequality):

$$d(z_1, z_2) \leq d(z_1, z_3) + d(z_3, z_2) \text{ for all } z_1, z_2, z_3 \in \mathbb{C}.$$

To justify c) let us observe that

$$|z_1 - z_2| = |(z_1 - z_3) + (z_3 - z_2)| \leq |z_1 - z_3| + |z_3 - z_2|,$$

from the modulus property. Equality holds if and only if there is a positive real number $k$ such that

$$z_3 - z_1 = k(z_2 - z_3).$$

### 3.1.2 Segments, rays and lines

Let $A$ and $B$ be two distinct points with complex coordinates $a$ and $b$. We say that the point $M$ with complex coordinate $z$ is between the points $A$ and $B$ if $z \neq a$, $z \neq b$ and the following relation holds:

$$|a - z| + |z - b| = |a - b|.$$

We use the notation $A - M - B$.

The set $(AB) = \{M : A - M - B\}$ is called the *open segment* determined by the points $A$ and $B$. The set $[AB] = (AB) \cup \{A, B\}$ represents the *closed segment* defined by the points $A$ and $B$.

**Theorem 1.** *Suppose $A(a)$ and $B(b)$ are two distinct points. The following statements are equivalent:*

*1) $M \in (AB)$;*

*2) there is a positive real number $k$ such that $z - a = k(b - z)$;*

*3) there is a real number $t \in (0, 1)$ such that $z = (1 - t)a + tb$, where $z$ is the complex coordinate of $M$.*

*Proof.* We first prove that 1) and 2) are equivalent. Indeed, we have $M \in (AB)$ if and only if $|a - z| + |z - b| = |a - b|$. That is, $d(a, z) + d(z, b) = d(a, b)$, or equivalently there is a real $k > 0$ such that $z - a = k(b - z)$.

To prove that 2) $\Leftrightarrow$ 3), set $t = \dfrac{k}{k+1} \in (0, 1)$ or $k = \dfrac{t}{1-t} > 0$. Then we have $z - a = k(b - z)$ if and only if $z = \dfrac{1}{k+1}a + \dfrac{k}{k+1}b$. That is, $z = (1 - t)a + tb$ and we are done. □

The set $(AB = \{M \mid A - M - B \text{ or } A - B - M\}$ is called the *open ray* with endpoint $A$ that contains $B$.

**Theorem 2.** *Suppose $A(a)$ and $B(b)$ are two distinct points. The following statements are equivalent:*

*1) $M \in (AB$;*

*2) there is a positive real number $t$ such that $z = (1 - t)a + tb$, where $z$ is the complex coordinate of $M$;*

*3)* $\arg(z-a) = \arg(b-a)$*;*

*4)* $\dfrac{z-a}{b-a} \in \mathbb{R}^+$.

*Proof.* It suffices to prove that 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 1).

1) $\Rightarrow$ 2). Since $M \in (AB$ we have $A-M-B$ or $A-B-M$. There are numbers $t, l \in (0,1)$ such that

$$z = (1-t)a + tb \text{ or } b = (1-l)a + lz.$$

In the first case we are done; for the second case set $t = \dfrac{1}{l}$, hence

$$z = tb - (t-1)a = (1-t)a + tb,$$

as claimed.

2) $\Rightarrow$ 3). From $z = (1-t)a + tb$, $t > 0$ we obtain

$$z - a = t(b-a),\ \ t > 0.$$

Hence

$$\arg(z-a) = \arg(b-a).$$

3) $\Rightarrow$ 4). The relation

$$\arg\frac{z-a}{b-a} = \arg(z-a) - \arg(b-a) + 2k\pi \text{ for some } k \in \mathbb{Z}$$

implies $\arg\dfrac{z-a}{b-a} = 2k\pi$, $k \in \mathbb{Z}$. Since $\arg\dfrac{z-a}{b-a} \in [0, 2\pi)$, it follows that $k = 0$ and $\arg\dfrac{z-a}{b-a} = 0$. Thus $\dfrac{z-a}{b-a} \in \mathbb{R}^+$, as desired.

4) $\Rightarrow$ 1). Let $t = \dfrac{z-a}{b-a} \in \mathbb{R}^*$. Hence

$$z = a + t(b-a) = (1-t)a + tb,\ \ t > 0.$$

If $t \in (0,1)$, then $M \in (AB) \subset (AB$.

If $t = 1$, then $z = b$ and $M = B \in (AB$. Finally, if $t > 1$ then, setting $l = \dfrac{1}{t} \in (0,1)$, we have

$$b = lz + (1-l)a.$$

It follows that $A-B-M$ and $M \in (AB$.

The proof is now complete. □

**Theorem 3.** *Suppose $A(a)$ and $B(b)$ are two distinct points. The following statements are equivalent:*

*1) $M(z)$ lies on the line $AB$.*

*2) $\dfrac{z-a}{b-a} \in \mathbb{R}$.*

*3) There is a real number $t$ such that $z = (1-t)a + tb$.*

*4) $\begin{vmatrix} z-a & \overline{z}-\overline{a} \\ b-a & \overline{b}-\overline{a} \end{vmatrix} = 0$;*

*5) $\begin{vmatrix} z & \overline{z} & 1 \\ a & \overline{a} & 1 \\ b & \overline{b} & 1 \end{vmatrix} = 0$.*

*Proof.* To obtain the equivalences 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3) observe that for a point $C$ such that $C - A - B$ the line $AB$ is the union $(AB \cup \{A\} \cup (AC$. Then apply Theorem 2.

Next we prove the equivalences 2) $\Leftrightarrow$ 4) $\Leftrightarrow$ 5).

Indeed, we have $\dfrac{z-a}{b-a} \in \mathbb{R}$ if and only if $\dfrac{z-a}{b-a} = \overline{\left(\dfrac{z-a}{b-a}\right)}$.

That is, $\dfrac{z-a}{b-a} = \dfrac{\overline{z}-\overline{a}}{\overline{b}-\overline{a}}$, or, equivalently, $\begin{vmatrix} z-a & \overline{z}-\overline{a} \\ b-a & \overline{b}-\overline{a} \end{vmatrix} = 0$, so we obtain that 2) is equivalent to 4).

Moreover, we have

$$\begin{vmatrix} z & \overline{z} & 1 \\ a & \overline{a} & 1 \\ b & \overline{b} & 1 \end{vmatrix} = 0 \text{ if and only if } \begin{vmatrix} z-a & \overline{z}-\overline{a} & 0 \\ a & \overline{a} & 1 \\ b-a & \overline{b}-\overline{a} & 0 \end{vmatrix} = 0$$

The last relation is equivalent to

$$\begin{vmatrix} z-a & \overline{z}-\overline{a} \\ b-a & \overline{b}-\overline{a} \end{vmatrix} = 0,$$

so we obtain that 4) is equivalent to 5), and we are done. □

**Problem 1.** *Let $z_1, z_2, z_3$ be complex numbers such that $|z_1| = |z_2| = |z_3| = R$ and $z_2 \neq z_3$. Prove that*

$$\min_{a \in \mathbb{R}} |az_2 + (1-a)z_3 - z_1| = \frac{1}{2R}|z_1 - z_2| \cdot |z_1 - z_3|.$$

(Romanian Mathematical Olympiad – Final Round, 1984)

**Solution.** Let $z = az_2 + (1-a)z_3$, $a \in \mathbb{R}$ and consider the points $A_1$, $A_2$, $A_3$, $A$ of complex coordinates $z_1$, $z_2$, $z_3$, $z$, respectively. From the hypothesis it follows that the circumcenter of triangle $A_1A_2A_3$ is the origin of the complex plane. Notice that point $A$ lies on the line $A_2A_3$, so $A_1A = |z - z_1|$ is greater than or equal to the altitude $A_1B$ of the triangle $A_1A_2A_3$.

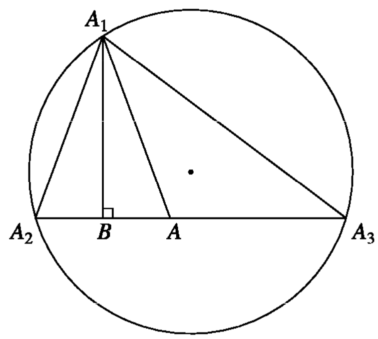


**Figure 3.1.**

It suffices to prove that

$$A_1B = \frac{1}{2R}|z_1 - z_2||z_1 - z_3| = \frac{1}{2R}A_1A_2 \cdot A_1A_3.$$

Indeed, since $R$ is the circumradius of the triangle $A_1A_2A_3$, we have

$$A_1B = \frac{2\text{area}[A_1A_2A_3]}{A_2A_3} = \frac{2\dfrac{A_1A_2 \cdot A_2A_3 \cdot A_3A_1}{4R}}{A_2A_3} = \frac{A_1A_2 \cdot A_3A_1}{2R},$$

as claimed.

### 3.1.3 Dividing a segment into a given ratio

Consider two distinct points $A(a)$ and $B(b)$. A point $M(z)$ on the line $AB$ divides the segments $AB$ into the ratio $k \in \mathbb{R} \setminus \{1\}$ if the following vectorial relation holds:

$$\overrightarrow{MA} = k \cdot \overrightarrow{MB}.$$

In terms of complex numbers this relation can be written as

$$a - z = k(b - z) \text{ or } (1 - k)z = a - kb.$$

Hence, we obtain

$$z = \frac{a - kb}{1 - k}.$$

Observe that for $k < 0$ the point $M$ lies on the line segment joining the points $A$ and $B$. If $k \in (0, 1)$, then $M \in (AB \setminus [AB]$. Finally, if $k > 1$, then $M \in (BA \setminus [AB]$.

As a consequence, note that for $k = -1$ we obtain that the coordinate of the midpoint of segment $[AB]$ is given by $z_M = \dfrac{a+b}{2}$.

**Example.** Let $A(a)$, $B(b)$, $C(c)$ be noncollinear points in the complex plane. Then the midpoint $M$ of segment $[AB]$ has the complex coordinate $z_M = \dfrac{a+b}{2}$. The centroid $G$ of triangle $ABC$ divides the median $[CM]$ into $2 : 1$ internally, hence its complex coordinate is given by $k = -2$, i.e.,

$$z_G = \frac{c + 2z_M}{1+2} = \frac{a+b+c}{3}.$$

### 3.1.4 Measure of an angle

Recall that a triangle is oriented if an ordering of its vertices is specified. It is positively or directly oriented if the vertices are oriented counterclockwise. Otherwise, we say that the triangle is negatively oriented. Consider two distinct points $M_1(z_1)$ and $M_2(z_2)$, other than the origin of a complex plane. The angle $\widehat{M_1OM_2}$ is oriented if the points $M_1$ and $M_2$ are ordered counterclockwise (Fig. 3.2 below).

**Proposition.** *The measure of the directly oriented angle* $\widehat{M_1OM_2}$ *equals* $\arg \dfrac{z_2}{z_1}$.

*Proof.* We consider the following two cases.

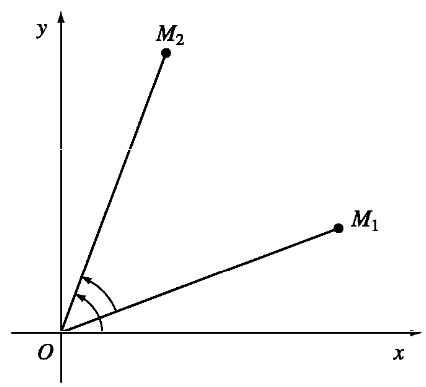


**Figure 3.2.**

a) If the triangle $M_1OM_2$ is negatively oriented (Fig. 3.2), then

$$\widehat{M_1OM_2} = \widehat{xOM_2} - \widehat{xOM_1} = \arg z_2 - \arg z_1 = \arg \frac{z_2}{z_1}.$$

b) If the triangle $M_1OM_2$ is positively oriented (Fig. 3.3), then

$$\widehat{M_1OM_2} = 2\pi - \widehat{M_2OM_1} = 2\pi - \arg \frac{z_2}{z_1},$$

since the triangle $M_2OM_1$ is negatively oriented. Thus

$$\widehat{M_1OM_2} = 2\pi - \arg\frac{z_1}{z_2} = 2\pi - \left(2\pi - \arg\frac{z_2}{z_1}\right) = \arg\frac{z_2}{z_1},$$

as claimed. □

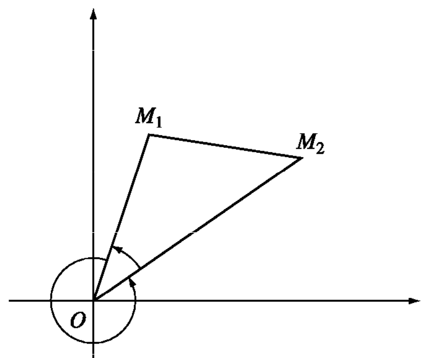


**Figure 3.3.**

**Remark.** The result also holds if the points $O$, $M_1$, $M_2$ are collinear.

**Examples.** a) Suppose that $z_1 = 1 + i$ and $z_2 = -1 + i$. Then (see Fig. 3.4)

$$\frac{z_2}{z_1} = \frac{-1+i}{1+i} = \frac{(-1+i)(1-i)}{2} = i,$$

so

$$\widehat{M_1OM_2} = \arg i = \frac{\pi}{2} \text{ and } \widehat{M_2OM_1} = \arg(-i) = \frac{3\pi}{2}.$$

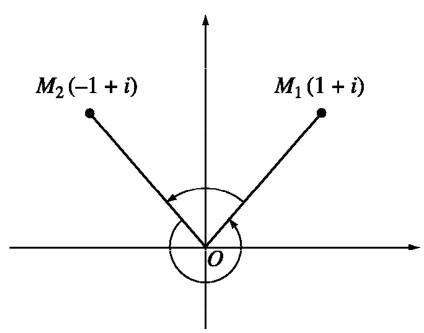


**Figure 3.4.**

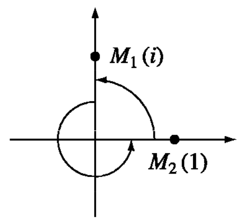


**Figure 3.5.**

b) Suppose that $z_1 = i$ and $z_2 = 1$. Then $\dfrac{z_2}{z_1} = \dfrac{1}{i} = -i$, so (see Fig. 3.5)

$$\widehat{M_1 O M_2} = \arg(-i) = \frac{3\pi}{2} \text{ and } \widehat{M_2 O M_1} = \arg(i) = \frac{\pi}{2}.$$

**Theorem.** *Consider three distinct points $M_1(z_1)$, $M_2(z_2)$ and $M_3(z_3)$. The measure of the oriented angle $\widehat{M_2 M_1 M_3}$ is* $\arg \dfrac{z_3 - z_1}{z_2 - z_1}$.

*Proof.* The translation with the vector $-z_1$ maps the points $M_1, M_2, M_3$ into the points $O, M_2', M_3'$, with complex coordinates $O, z_2 - z_1, z_3 - z_1$. Moreover, we have $\widehat{M_2 M_1 M_3} = \widehat{M_2' O M_3'}$. By the previous result, we obtain

$$\widehat{M_2' O M_3'} = \arg \frac{z_3 - z_1}{z_2 - z_1},$$

as claimed. □

**Example.** Suppose that $z_1 = 4 + 3i$, $z_2 = 4 + 7i$, $z_3 = 8 + 7i$. Then

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{4i}{4 + 4i} = \frac{i(1 - i)}{2} = \frac{1 + i}{2},$$

so

$$\widehat{M_3 M_1 M_2} = \arg \frac{1 + i}{2} = \frac{\pi}{4}$$

and

$$\widehat{M_2 M_1 M_3} = \arg \frac{2}{1 + i} = \arg(1 - i) = \frac{7\pi}{4}.$$

**Remark.** Using polar representation, from the above result we have

$$\frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| \left(\cos\left(\arg \frac{z_3 - z_1}{z_2 - z_1}\right) + i \sin\left(\arg \frac{z_3 - z_1}{z_2 - z_1}\right)\right)$$

$$= \left|\frac{z_3 - z_1}{z_2 - z_1}\right| (\cos \widehat{M_2 M_1 M_3} + i \sin \widehat{M_2 M_1 M_3}).$$

### 3.1.5 Angle between two lines

Consider four distinct points $M_i(z_i)$, $i \in \{1, 2, 3, 4\}$. The measure of the angle determined by the lines $M_1M_3$ and $M_2M_4$ equals $\arg \dfrac{z_3 - z_1}{z_4 - z_2}$ or $\arg \dfrac{z_4 - z_2}{z_3 - z_1}$. The proof is obtained following the same ideas as in the previous subsection.

### 3.1.6 Rotation of a point

Consider an angle $\alpha$ and the complex number given by

$$\varepsilon = \cos \alpha + i \sin \alpha.$$

Let $z = r(\cos t + i \sin t)$ be a complex number and $M$ its geometric image.

Form the product $z\varepsilon = r(\cos(t + \alpha) + i \sin(t + \alpha))$ and let us observe that $|z\varepsilon| = r$ and

$$\arg(z\varepsilon) = \arg z + \alpha.$$

It follows that the geometric image $M'$ of $z\varepsilon$ is the rotation of $M$ with respect to the origin by the angle $\alpha$.

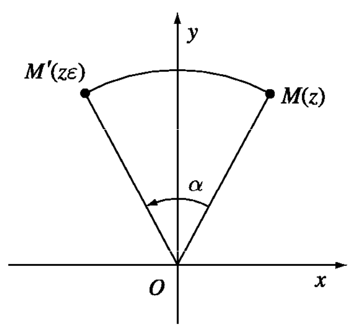


**Figure 3.6.**

Now we have all the ingredients to establish the following result:

**Proposition.** *Suppose that the point C is the rotation of B with respect to A by the angle $\alpha$.*

*If $a, b, c$ are the coordinates of the points $A, B, C$, respectively, then*

$$c = a + (b - a)\varepsilon, \ \text{where } \varepsilon = \cos \alpha + i \sin \alpha.$$

*Proof.* The translation with vector $-a$ maps the points $A, B, C$ into the points $O, B', C'$, with complex coordinates $O, b - a, c - a$, respectively (see Fig. 3.7). The point $C'$ is the image of $B'$ under rotation about the origin through the angle $\alpha$, so $c - a = (b - a)\varepsilon$, or $c = a + (b - a)\varepsilon$, as desired. □

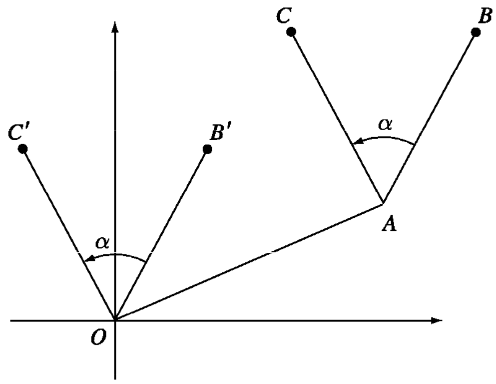


**Figure 3.7.**

We will call the formula in the above proposition the *rotation formula*.

**Problem 1.** *Let $ABCD$ and $BNMK$ be two nonoverlapping squares and let $E$ be the midpoint of $AN$. If point $F$ is the foot of the perpendicular from $B$ to the line $CK$, prove that points $E$, $F$, $B$ are collinear.*

**Solution.** Consider the complex plane with origin at $F$ and the axis $CK$ and $FB$, where $FB$ is the imaginary axis.

Let $c, k, bi$ be the complex coordinates of points $C$, $K$, $B$ with $c, k, b \in \mathbb{R}$. The rotation with center $B$ through the angle $\theta = \dfrac{\pi}{2}$ maps point $C$ to $A$, so $A$ has the complex coordinate $a = b(1-i) + ci$. Similarly, point $N$ is obtained by rotating point $K$ around $B$ through the angle $\theta = -\dfrac{\pi}{2}$ and its complex coordinate is

$$n = b(1+i) - ki.$$

The midpoint $E$ of segment $AN$ has the complex coordinate

$$e = \frac{a+n}{2} = b + \frac{c-k}{2}i,$$

so $E$ lies on the line $FB$, as desired.

**Problem 2.** *On the sides $AB$, $BC$, $CD$, $DA$ of quadrilateral $ABCD$, and exterior to the quadrilateral, we construct squares of centers $O_1$, $O_2$, $O_3$, $O_4$, respectively. Prove that*

$$O_1O_3 \perp O_2O_4 \quad \textit{and} \quad O_1O_3 = O_2O_4.$$

**Solution.** Let $ABMM'$, $BCNN'$, $CDPP'$ and $DAQQ'$ be the constructed squares with centers $O_1$, $O_2$, $O_3$, $O_4$, respectively.

Denote by a lowercase letter the coordinate of each of the points denoted by an uppercase letter, i.e., $o_1$ is the coordinate of $O_1$, etc.

Point $M$ is obtained from point $A$ by a rotation about $B$ through the angle $\theta = \frac{\pi}{2}$; hence $m = b + (a - b)i$. Likewise,

$$n = c + (b - c)i, \quad p = d + (c - d)i \text{ and } q = a + (d - a)i.$$

It follows that

$$o_1 = \frac{a+m}{2} = \frac{a+b+(a-b)i}{2}, \quad o_2 = \frac{b+c+(b-c)i}{2},$$

$$o_3 = \frac{c+d+(c-d)i}{2} \text{ and } o_4 = \frac{d+a+(d-a)i}{2}.$$

Then

$$\frac{o_3 - o_1}{o_4 - o_2} = \frac{c+d-a-b+i(c-d-a+b)}{a+d-b-c+i(d-a-b+c)} = -i \in i\mathbb{R}^*,$$

so $O_1O_3 \perp O_2O_4$. Moreover,

$$\left|\frac{o_3 - o_1}{o_4 - o_2}\right| = |-i| = 1;$$

hence $O_1O_3 = O_2O_4$, as desired.

**Problem 3.** *In the exterior of the triangle $ABC$ we construct triangles $ABR$, $BCP$, and $CAQ$ such that*

$$m(\widehat{PBC}) = m(\widehat{CAQ}) = 45°,$$

$$m(\widehat{BCP}) = m(\widehat{QCA}) = 30°,$$

*and*

$$m(\widehat{ABR}) = m(\widehat{RAB}) = 15°.$$

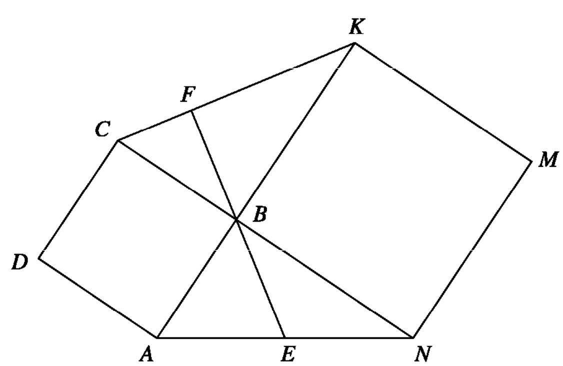


**Figure 3.8.**

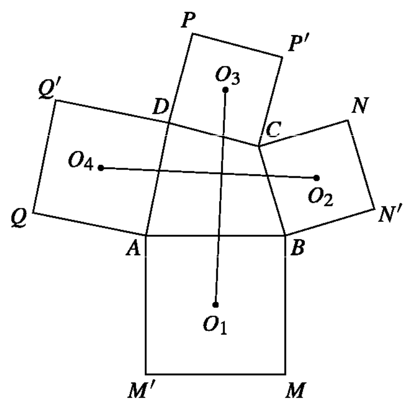


Figure 3.9.

*Prove that*

$$m(\widehat{QRP}) = 90^\circ \text{ and } RQ = RP.$$

**Solution.** Consider the complex plane with origin at point $R$ and let $M$ be the foot of the perpendicular from $P$ to the line $BC$.

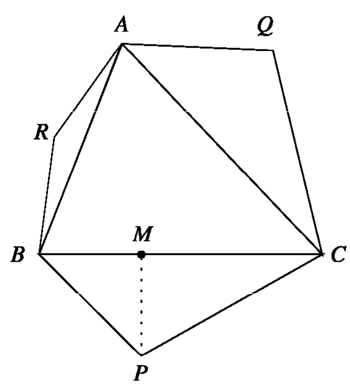


Figure 3.10.

Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter. From $MP = MB$ and $\dfrac{MC}{MP} = \sqrt{3}$ it follows that

$$\frac{p-m}{b-m} = i \text{ and } \frac{c-m}{p-m} = i\sqrt{3},$$

hence

$$p = \frac{c + \sqrt{3}b}{1 + \sqrt{3}} + \frac{b - c}{1 + \sqrt{3}} i.$$

Likewise,

$$q = \frac{c + \sqrt{3}a}{1 + \sqrt{3}} + \frac{a - c}{1 + \sqrt{3}} i.$$

Point $B$ is obtained from point $A$ by a rotation about $R$ through an angle $\theta = 150°$, so

$$b = a \left( -\frac{\sqrt{3}}{2} + \frac{1}{2} i \right).$$

Simple algebraic manipulations show that $\frac{p}{q} = i \in i\mathbb{R}^*$, hence $QR \perp PR$. Moreover, $|p| = |iq| = |q|$, so $RP = RQ$ and we are done.

## 3.2 Conditions for Collinearity, Orthogonality and Concyclicity

In this section we consider four distinct points $M_i(z_i)$, $i \in \{1, 2, 3, 4\}$.

**Proposition 1.** *The points $M_1, M_2, M_3$ are collinear if and only if*

$$\frac{z_3 - z_1}{z_2 - z_1} \in \mathbb{R}^*.$$

*Proof.* The collinearity of the points $M_1, M_2, M_3$ is equivalent to $\widehat{M_2 M_1 M_3} \in \{0, \pi\}$. It follows that $\arg \frac{z_3 - z_1}{z_2 - z_1} \in \{0, \pi\}$ or equivalently $\frac{z_3 - z_1}{z_2 - z_1} \in \mathbb{R}^*$, as claimed. □

**Proposition 2.** *The lines $M_1M_2$ and $M_3M_4$ are orthogonal if and only if*

$$\frac{z_1 - z_2}{z_3 - z_4} \in i\mathbb{R}^*.$$

*Proof.* We have $M_1M_2 \perp M_3M_4$ if and only if $(M_1M_2, M_3M_4) \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$. This is equivalent to $\arg \frac{z_1 - z_2}{z_3 - z_4} \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$. We obtain $\frac{z_1 - z_2}{z_3 - z_4} \in i\mathbb{R}^*$. □

**Remark.** Suppose that $M_2 = M_4$. Then $M_1M_2 \perp M_3M_2$ if and only if $\frac{z_1 - z_2}{z_3 - z_2} \in i\mathbb{R}^*$.

**Examples.** 1) Consider the points $M_1(2-i)$, $M_2(-1+2i)$, $M_3(-2-i)$, $M_4(1+2i)$. Simple algebraic manipulation shows that

$$\frac{z_1 - z_2}{z_3 - z_4} = i, \text{ hence } M_1M_2 \perp M_3M_4.$$

2) Consider the points $M_1(2-i)$, $M_2(-1+2i)$, $M_3(1+2i)$, $M_4(-2-i)$. Then we have $\frac{z_1-z_2}{z_3-z_4} = -i$ hence $M_1M_2 \perp M_3M_4$.

**Problem 1.** *Let $z_1, z_2, z_3$ be the coordinates of vertices $A, B, C$ of a triangle. If $w_1 = z_1 - z_2$ and $w_2 = z_3 - z_1$, prove that $\widehat{A} = 90^\circ$ if and only if* $\mathrm{Re}(w_1 \cdot \overline{w_2}) = 0$.

**Solution.** We have $\widehat{A} = 90^\circ$ if and only if $\frac{z_2-z_1}{z_3-z_1} \in i\mathbb{R}$, which is equivalent to $\frac{w_1}{-w_2} \in i\mathbb{R}$, i.e., $\mathrm{Re}\left(\frac{w_1}{-w_2}\right) = 0$. The last relation is equivalent to $\mathrm{Re}\left(\frac{w_1 \cdot \overline{w_2}}{-|w_2|^2}\right) = 0$, i.e., $\mathrm{Re}(w_1 \cdot \overline{w_2}) = 0$, as desired.

**Proposition 3.** *The distinct points $M_1(z_1)$, $M_2(z_2)$, $M_3(z_3)$, $M_4(z_4)$ are concyclic or collinear if and only if*

$$k = \frac{z_3-z_2}{z_1-z_2} : \frac{z_3-z_4}{z_1-z_4} \in \mathbb{R}^*.$$

*Proof.* Assume that the points are collinear. We can arrange four points on a circle in $(4-1)! = 3! = 6$ different ways. Consider the case when $M_1, M_2, M_3, M_4$ are given in this order. Then $M_1, M_2, M_3, M_4$ are concyclic if and only if

$$\widehat{M_1M_2M_3} + \widehat{M_1M_4M_3} \in \{3\pi, \pi\}.$$

That is,

$$\arg\frac{z_3-z_2}{z_1-z_2} + \arg\frac{z_1-z_4}{z_3-z_4} \in \{3\pi, \pi\}.$$

We obtain

$$\arg\frac{z_3-z_2}{z_1-z_2} - \arg\frac{z_3-z_4}{z_1-z_4} \in \{3\pi, \pi\},$$

i.e., $k < 0$.

For any other arrangements of the four points the proof is similar. Note that $k > 0$ in three cases and $k < 0$ in the other three. □

The number $k$ is called the *cross ratio* of the four points $M_1(z_1)$, $M_2(z_2)$, $M_3(z_3)$ and $M_4(z_4)$.

**Remarks.** 1) The points $M_1, M_2, M_3, M_4$ are collinear if and only if

$$\frac{z_3-z_2}{z_1-z_2} \in \mathbb{R}^* \text{ and } \frac{z_3-z_4}{z_1-z_4} \in \mathbb{R}^*.$$

2) The points $M_1, M_2, M_3, M_4$ are concyclic if and only if

$$k = \frac{z_3-z_2}{z_1-z_2} : \frac{z_3-z_4}{z_1-z_4} \in \mathbb{R}^*, \text{ but } \frac{z_3-z_2}{z_1-z_2} \notin \mathbb{R} \text{ and } \frac{z_3-z_4}{z_1-z_4} \notin \mathbb{R}.$$

**Examples.** 1) The geometric images of the complex numbers $1, i, -1, -i$ are concyclic. Indeed, we have the cross ratio $k = \frac{-1-i}{1-i} : \frac{-1+i}{1+i} = -1 \in \mathbb{R}^*$ and clearly $\frac{-1-i}{1-i} \notin \mathbb{R}$ and $\frac{-1+i}{1+i} \notin \mathbb{R}$.

2) The points $M_1(2-i)$, $M_2(3-2i)$, $M_3(-1+2i)$ and $M_4(-2+3i)$ are collinear. Indeed, $k = \frac{-4+4i}{-1+i} : \frac{1-i}{4-4i} = 1 \in \mathbb{R}^*$ and $\frac{-4+4i}{-1+i} = 4 \in \mathbb{R}^*$.

**Problem 2.** *Find all complex numbers $z$ such that the points of complex coordinates $z, z^2, z^3, z^4$ – in this order – are the vertices of a cyclic quadrilateral.*

**Solution.** If the points of complex coordinates $z, z^2, z^3, z^4$ – in this order – are the vertices of a cyclic quadrilateral, then

$$\frac{z^3 - z^2}{z - z^2} : \frac{z^3 - z^4}{z - z^4} \in \mathbb{R}^*.$$

It follows that

$$-\frac{1+z+z^2}{z} \in \mathbb{R}^*, \text{ i.e., } -1-\left(z+\frac{1}{z}\right) \in \mathbb{R}^*.$$

We obtain $z + \frac{1}{z} \in \mathbb{R}$, i.e., $z + \frac{1}{z} = \overline{z} + \frac{1}{\overline{z}}$. Hence $(z - \overline{z})(|z|^2 - 1) = 0$, hence $z \in \mathbb{R}$ or $|z| = 1$.

If $z \in \mathbb{R}$, then the points of complex coordinates $z, z^2, z^3, z^4$ are collinear, hence it is left to consider the case $|z| = 1$.

Let $t = \arg z \in [0, 2\pi)$. We prove that the points of complex coordinates $z, z^2, z^3, z^4$ lie in this order on the unit circle if and only if $t \in \left(0, \frac{2\pi}{3}\right) \cup \left(\frac{4\pi}{3}, 2\pi\right)$. Indeed,

a) If $t \in \left(0, \frac{\pi}{2}\right)$, then $0 < t < 2t < 3t < 4t < 2\pi$ or

$$0 < \arg z < \arg z^2 < \arg z^3 < \arg z^4 < 2\pi.$$

b) If $t \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right)$, then $0 \le 4t - 2\pi < t < 2t < 3t < 2\pi$ or

$$0 \le \arg z^4 < \arg z < \arg z^2 < \arg z^3 < 2\pi.$$

c) If $t \in \left[\frac{2\pi}{3}, \pi\right)$, then $0 \le 3t - 2\pi < t \le 4t - 2\pi < 2t < 2\pi$ or

$$0 \le \arg z^3 < \arg z \le \arg z^4 < \arg z^2.$$

In the same manner we can analyze the case $t \in [\pi, 2\pi)$.

To conclude, the complex numbers satisfying the desired property are

$$z = \cos t + i \sin t, \text{ with } t \in \left(0, \frac{2\pi}{3}\right) \cup \left(\frac{4\pi}{3}, \pi\right).$$

## 3.3 Similar Triangles

Consider six points $A_1(a_1)$, $A_2(a_2)$, $A_3(a_3)$, $B_1(b_1)$, $B_2(b_2)$, $B_3(b_3)$ in the complex plane. We say that the triangles $A_1A_2A_3$ and $B_1B_2B_3$ are similar if the angle at $A_k$ is equal to the angle at $B_k$, $k \in \{1, 2, 3\}$.

**Proposition 1.** *The triangles $A_1A_2A_3$ and $B_1B_2B_3$ are similar, having the same orientation, if and only if*

$$\frac{a_2 - a_1}{a_3 - a_1} = \frac{b_2 - b_1}{b_3 - b_1}. \tag{1}$$

*Proof.* We have $\triangle A_1A_2A_3 \sim \triangle B_1B_2B_3$ if and only if $\frac{A_1A_2}{A_1A_3} = \frac{B_1B_2}{B_1B_3}$ and $\widehat{A_3A_1A_2} \equiv \widehat{B_3B_1B_2}$. This is equivalent to $\frac{|a_2 - a_1|}{|a_3 - a_1|} = \frac{|b_2 - b_1|}{|b_3 - b_1|}$ and $\arg \frac{a_2 - a_1}{a_3 - a_1} = \arg \frac{b_2 - b_1}{b_3 - b_1}$. We obtain $\frac{a_2 - a_1}{a_3 - a_1} = \frac{b_2 - b_1}{b_3 - b_1}$. □

**Remarks.** 1) The condition (1) is equivalent to

$$\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0.$$

2) The triangles $A_1(0)$, $A_2(1)$, $A_3(2i)$ and $B_1(0)$, $B_2(-i)$, $B_3(-2)$ are similar, but opposite oriented. In this case the condition (1) is not satisfied. Indeed,

$$\frac{a_2 - a_1}{a_3 - a_1} = \frac{1 - 0}{2i - 0} = \frac{1}{2i} \neq \frac{b_2 - b_1}{b_3 - b_1} = \frac{-i - 0}{-2 - 0} = \frac{i}{2}.$$

**Proposition 2.** *The triangles $A_1A_2A_3$ and $B_1B_2B_3$ are similar, having opposite orientation, if and only if*

$$\frac{a_2 - a_1}{a_3 - a_1} = \frac{\overline{b_2} - \overline{b_1}}{\overline{b_3} - \overline{b_1}}.$$

*Proof.* Reflection across the $x$-axis maps the points $B_1, B_2, B_3$ into the points $M_1(\overline{b_1})$, $M_2(\overline{b_2})$, $M_3(\overline{b_3})$. The triangles $B_1B_2B_3$ and $M_1M_2M_3$ are similar and have opposite orientation, hence triangles $A_1A_2A_3$ and $M_1M_2M_3$ are similar with the same orientation. The conclusion follows from the previous proposition. □

**Problem 1.** *On sides $AB$, $BC$, $CA$ of a triangle $ABC$ we draw similar triangles $ADB$, $BEC$, $CFA$, having the same orientation. Prove that triangles $ABC$ and $DEF$ have the same centroid.*

**Solution.** Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter.

Triangles $ADB$, $BEC$, $CFA$ are similar with the same orientation, hence

$$\frac{d-a}{b-a} = \frac{e-b}{c-b} = \frac{f-c}{a-c} = z,$$

and consequently

$$d = a + (b-a)z, \quad e = b + (c-b)z, \quad f = c + (a-c)z.$$

Then

$$\frac{d+c+f}{3} = \frac{a+b+c}{3},$$

so triangles $ABC$ and $DEF$ have the same centroid.

**Problem 2.** *Let $M, N, P$ be the midpoints of sides $AB$, $BC$, $CA$ of triangle $ABC$. On the perpendicular bisectors of segments $[AB]$, $[BC]$, $[CA]$ points $C'$, $A'$, $B'$ are chosen inside the triangle such that*

$$\frac{MC'}{AB} = \frac{NA'}{BC} = \frac{PB'}{CA}.$$

*Prove that $ABC$ and $A'B'C'$ have the same centroid.*

**Solution.** Note that from

$$\frac{MC'}{AB} = \frac{NA'}{BC} = \frac{PB'}{CA}$$

it follows that $\tan(\widehat{C'AB}) = \tan(\widehat{A'BC}) = \tan(\widehat{B'CA})$. Hence triangles $AC'B$, $BA'C$, $CB'A$ are similar and we can proceed as in the previous problem.

**Problem 3.** *Let $ABO$ be an equilateral triangle with center $S$ and let $A'B'O$ be another equilateral triangle with the same orientation and $S \neq A'$, $S \neq B'$. Consider $M$ and $N$ the midpoints of the segments $A'B$ and $AB'$.*

*Prove that triangles $SB'M$ and $SA'N$ are similar.*

(30th IMO – Shortlist)

**Solution.** Let $R$ be the circumradius of the triangle $ABO$ and let

$$\varepsilon = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}.$$

Consider the complex plane with origin at point $S$ such that point $O$ lies on the positive real axis. Then the coordinates of points $O$, $A$, $B$ are $R$, $R\varepsilon$, $R\varepsilon^2$, respectively.

Let $R+z$ be the coordinate of point $B'$, so $R - z\varepsilon$ is the coordinate of point $A'$. It follows that the midpoints $M$, $N$ have the coordinates

$$z_M = \frac{z_B + z_{A'}}{2} = \frac{R\varepsilon^2 + R - z\varepsilon}{2} = \frac{R(\varepsilon^2+1) - z\varepsilon}{2}$$

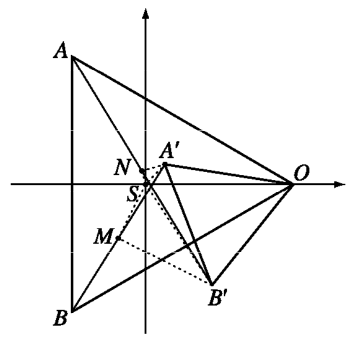


**Figure 3.11.**

$$= \frac{-R\varepsilon - z\varepsilon}{2} = \frac{-\varepsilon(R+z)}{2}$$

and

$$z_N = \frac{z_A + z_{B'}}{2} = \frac{R\varepsilon + R + z}{2} = \frac{R(\varepsilon+1)+z}{2} = \frac{-E\varepsilon^2 + z}{2}$$

$$= \frac{z - \frac{R}{\varepsilon}}{2} = \frac{R - z\varepsilon}{-2\varepsilon}.$$

Now we have

$$\frac{z_{B'} - z_S}{z_M - z_S} = \frac{\overline{z_{A'}} - \overline{z_S}}{\overline{z_N} - \overline{z_S}}$$

if and only if

$$\frac{R+z}{\frac{-\varepsilon(R+z)}{2}} = \frac{\overline{R - z\varepsilon}}{\overline{\left(\frac{R - z\varepsilon}{-2\varepsilon}\right)}}.$$

The last relation is equivalent to $\varepsilon \cdot \overline{\varepsilon} = 1$, i.e., $|\varepsilon|^2 = 1$. Hence the triangles $SB'M$ and $SA'N$ are similar, with opposite orientation.

## 3.4 Equilateral Triangles

**Proposition 1.** *Suppose $z_1, z_2, z_3$ are the coordinates of the vertices of the triangle $A_1A_2A_3$. The following statements are equivalent:*

*a) $A_1A_2A_3$ is an equilateral triangle;*

*b) $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$;*

*c) $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$;*

*d) $\frac{z_2 - z_1}{z_3 - z_2} = \frac{z_3 - z_2}{z_1 - z_2}$;*

*e)* $\dfrac{1}{z-z_1}+\dfrac{1}{z-z_2}+\dfrac{1}{z-z_3}=0$, *where* $z=\dfrac{z_1+z_2+z_3}{3}$;

*f)* $(z_1+\varepsilon z_2+\varepsilon^2 z_3)(z_1+\varepsilon^2 z_2+\varepsilon z_3)=0$, *where* $\varepsilon=\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}$;

*g)* $\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1 \end{vmatrix}=0.$

*Proof.* The triangle $A_1A_2A_3$ is equilateral if and only if $A_1A_2A_3$ is similar with same orientation with $A_2A_3A_1$, or

$$\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1 \end{vmatrix}=0,$$

thus a) $\Leftrightarrow$ g).

Computing the determinant we obtain

$$\begin{aligned} 0 &= \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1 \end{vmatrix} \\ &= z_1z_2+z_2z_3+z_3z_1-(z_1^2+z_2^2+z_3^2) \\ &= -(z_1+\varepsilon z_2+\varepsilon^2 z_3)(z_1+\varepsilon^2 z_2+\varepsilon z_3), \end{aligned}$$

hence g) $\Leftrightarrow$ c) $\Leftrightarrow$ f).

Simple algebraic manipulation shows that d) $\Leftrightarrow$ c). Since a) $\Leftrightarrow$ b) is obvious, we leave for the reader to prove that a) $\Leftrightarrow$ e). □

The next results bring some refinements to this issue.

**Proposition 2.** *Let $z_1, z_2, z_3$ be the coordinates of the vertices $A_1, A_2, A_3$ of a positively oriented triangle. The following statements are equivalent.*

*a) $A_1A_2A_3$ is an equilateral triangle;*

*b)* $z_3-z_1=\varepsilon(z_2-z_1)$, *where* $\varepsilon=\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}$;

*c)* $z_2-z_1=\varepsilon(z_3-z_1)$, *where* $\varepsilon=\cos\dfrac{5\pi}{3}+i\sin\dfrac{5\pi}{3}$;

*d)* $z_1+\varepsilon z_2+\varepsilon^2 z_3=0$, *where* $\varepsilon=\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}$.

*Proof.* $A_1A_2A_3$ is equilateral and positively oriented if and only if $A_3$ is obtained from $A_2$ by rotation about $A_1$ through an angle of $\dfrac{\pi}{3}$. That is,

$$z_3=z_1+\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)(z_2-z_1),$$

hence a) $\Leftrightarrow$ b).

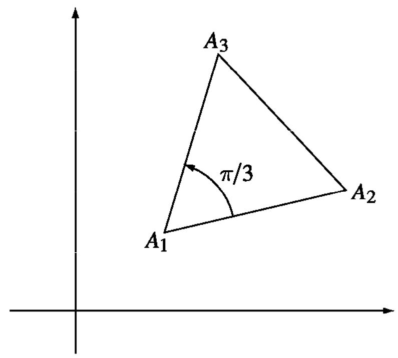


**Figure 3.12.**

The rotation about $A_1$ through an angle of $\frac{5\pi}{3}$ maps $A_3$ into $A_2$. Similar considerations show that a) $\Leftrightarrow$ c).

To prove that b) $\Leftrightarrow$ d), observe that b) is equivalent to

b′) $z_3 = z_1 + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)(z_2 - z_1) = \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) z_1 + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_2$. Hence

$$z_1 + \varepsilon z_2 + \varepsilon^2 z_3 = z_1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_2 + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) z_3$$

$$= z_1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_2$$

$$+ \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left[\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) z_1 + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_2\right]$$

$$= z_1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_2 - z_1 + \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) z_2 = 0,$$

or b) $\Leftrightarrow$ d). □

**Proposition 3.** *Let $z_1, z_2, z_3$ be the coordinates of the vertices $A_1, A_2, A_3$ of a negatively oriented triangle.*

*The following statements are equivalent:*

*a) $A_1A_2A_3$ is an equilateral triangle;*

*b) $z_3 - z_1 = \varepsilon(z_2 - z_1)$, where $\varepsilon = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$;*

*c) $z_2 - z_1 = \varepsilon(z_3 - z_1)$, where $\varepsilon = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$;*

*d) $z_1 + \varepsilon^2 z_2 + \varepsilon z_3 = 0$, where $\varepsilon = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$.*

*Proof.* Equilateral triangle $A_1A_2A_3$ is negatively oriented if and only if $A_1A_3A_2$ is a positively oriented equilateral triangle. The rest follows from the previous proposition. □

**Proposition 4.** *Let $z_1, z_2, z_3$ be the coordinates of the vertices of equilateral triangle $A_1A_2A_3$. Consider the statements:*

*1) $A_1A_2A_3$ is an equilateral triangle;*

*2) $z_1 \cdot \overline{z}_2 = z_2 \cdot \overline{z}_3 = z_3 \cdot \overline{z}_1$;*

*3) $z_1^2 = z_2 \cdot z_3$ and $z_2^2 = z_1 \cdot z_3$.*

*Then 2) $\Rightarrow$ 1), 3) $\Rightarrow$ 1) and 2) $\Leftrightarrow$ 3).*

*Proof.* 2) $\Rightarrow$ 1). Taking the modulus of the terms in the given relation we obtain

$$|z_1| \cdot |\overline{z}_2| = |z_2| \cdot |\overline{z}_3| = |z_3| \cdot |\overline{z}_1|,$$

or equivalently

$$|z_1| \cdot |z_2| = |z_2| \cdot |z_3| = |z_3| \cdot |z_1|.$$

This implies

$$r = |z_1| = |z_2| = |z_3|$$

and

$$\overline{z}_1 = \frac{r^2}{z_1}, \quad \overline{z}_2 = \frac{r^2}{z_2}, \quad \overline{z}_3 = \frac{r^2}{z_3}.$$

Returning to the given relation we have

$$\frac{z_1}{z_2} = \frac{z_2}{z_3} = \frac{z_3}{z_1},$$

or

$$z_1^2 = z_2z_3, \quad z_2^2 = z_3z_1, \quad z_3^2 = z_1z_2.$$

Summing up these relations yields

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1,$$

so triangle $A_1A_2A_3$ is equilateral.

Observe that we have also proved that 2) $\Rightarrow$ 3) and that the arguments are reversible; hence 2) $\Leftrightarrow$ 3). As a consequence, 3) $\Rightarrow$ 1) and we are done. □

**Problem 1.** *Let $z_1, z_2, z_3$ be nonzero complex coordinates of the vertices of the triangle $A_1A_2A_3$. If $z_1^2 = z_2z_3$ and $z_2^2 = z_1z_3$, show that triangle $A_1A_2A_3$ is equilateral.*

**Solution.** Multiplying the relations $z_1^2 = z_2z_3$ and $z_2^2 = z_1z_3$ yields $z_1^2z_2^2 = z_1z_2z_3^2$, and consequently $z_1z_2 = z_3^2$. Thus

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1,$$

so triangle $A_1A_2A_3$ is equilateral, by Proposition 1 in this section.

**Problem 2.** *Let $z_1, z_2, z_3$ be the coordinates of the vertices of triangle $A_1A_2A_3$. If $|z_1| = |z_2| = |z_3|$ and $z_1 + z_2 + z_3 = 0$, prove that triangle $A_1A_2A_3$ is equilateral.*

**Solution.** The following identity holds for any complex numbers $z_1$ and $z_2$ (see Problem 1 in Subsection 1.1.7):

$$|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2). \tag{1}$$

From $z_1 + z_2 + z_3 = 0$ it follows that $z_1 + z_2 = -z_3$, so $|z_1 + z_2| = |z_3|$. Using the relations $|z_1| = |z_2| = |z_3|$ and (1) we get $|z_1 - z_2|^2 = 3|z_1|^2$. Analogously, we find the relations $|z_2 - z_3|^2 = 3|z_1|^2$ and $|z_3 - z_1|^2 = 3|z_1|^2$. Therefore $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$, i.e., triangle $A_1A_2A_3$ is equilateral.

**Alternative solution 1.** If we pass to conjugates, then we obtain $\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3} = 0$. Combining this with the hypothesis yields $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1 = 0$, from which the desired conclusion follows by Proposition 1.

**Alternative solution 2.** Taking into account the hypotheses $|z_1| = |z_2| = |z_3|$ it follows that we can consider the complex plane with its origin at the circumcenter of triangle $A_1A_2A_3$. Then, the coordinate of orthocenter $H$ is $z_H = z_1 + z_2 + z_3 = 0 = z_O$. Hence $H = O$, and triangle $A_1A_2A_3$ is equilateral.

**Problem 3.** *In the exterior of triangle $ABC$ three positively oriented equilateral triangles $AC'B$, $BA'C$ and $CB'A$ are constructed. Prove that the centroids of these triangles are the vertices of an equilateral triangle.*

(Napoleon's problem)

**Solution.**

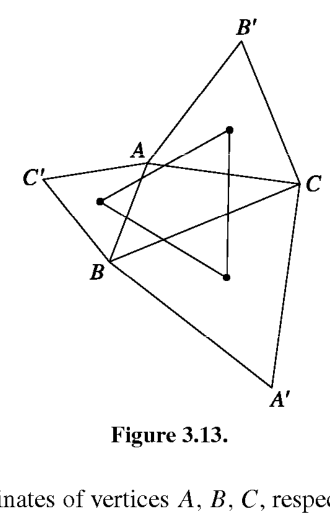


**Figure 3.13.**

Let $a$, $b$, $c$ be the coordinates of vertices $A$, $B$, $C$, respectively.

Using Proposition 2, we have

$$a + c'\varepsilon + b\varepsilon^2 = 0, \quad b + a'\varepsilon + c\varepsilon^2 = 0, \quad c + b'\varepsilon + a\varepsilon^2 = 0, \tag{1}$$

where $a'$, $b'$, $c'$ are the coordinates of points $A'$, $B'$, $C'$.

The centroids of triangles $A'BC$, $AB'C$, $ABC'$ have the coordinates

$$a'' = \frac{1}{3}(a' + b + c), \quad b'' = \frac{1}{3}(a + b' + c), \quad c'' = \frac{1}{3}(a + b + c'),$$

respectively. We have to check that $c'' + a''\varepsilon + b''\varepsilon^2 = 0$. Indeed,

$$\begin{aligned} 3(c'' + a''\varepsilon + b''\varepsilon^2) &= (a + b + c') + (a' + b + c)\varepsilon + (a + b' + c)\varepsilon^2 \\ &= (b + a'\varepsilon + c\varepsilon^2) + (c + b'\varepsilon + a\varepsilon^2)\varepsilon + (a + c'\varepsilon + b\varepsilon^2)\varepsilon^2 = 0. \end{aligned}$$

**Problem 4.** *On the sides of the triangle $ABC$ we draw three regular $n$-gons, external to the triangle. Find all values of $n$ for which the centers of the $n$-gons are the vertices of an equilateral triangle.*

(Balkan Mathematical Olympiad 1990 – Shortlist)

**Solution.** Let $A_0$, $B_0$, $C_0$ be the centers of the regular $n$-gons constructed externally on the sides $BC$, $CA$, $AB$, respectively.

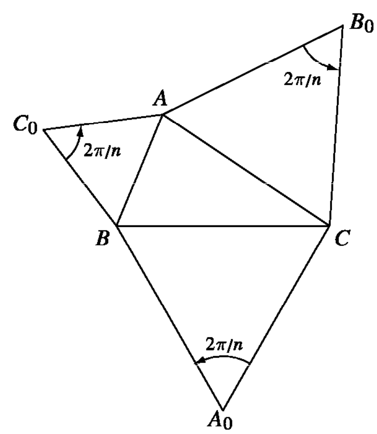


**Figure 3.14.**

The angles $\widehat{AC_0B}$, $\widehat{BA_0C}$, $\widehat{AB_0C}$ have the measures of $\dfrac{2\pi}{n}$. Let

$$\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

and denote by $a$, $b$, $c$, $a_0$, $b_0$, $c_0$ the coordinates of the points $A$, $B$, $C$, $A_0$, $B_0$, $C_0$, respectively.

Using the rotation formula, we obtain

$$a = c_0 + (b - c_0)\varepsilon;$$

$$b = a_0 + (c - a_0)\varepsilon;$$

$$c = b_0 + (a - b_0)\varepsilon.$$

Thus

$$a_0 = \frac{b - c\varepsilon}{1 - \varepsilon}, \quad b_0 = \frac{c - a\varepsilon}{1 - \varepsilon}, \quad c_0 = \frac{a - b\varepsilon}{1 - \varepsilon}.$$

Triangle $A_0B_0C_0$ is equilateral if and only if

$$a_0^2 + b_0^2 + c_0^2 = a_0b_0 + b_0c_0 + c_0a_0.$$

Substituting the above values of $a_0, b_0, c_0$ we obtain

$$(b - c\varepsilon)^2 + (c - a\varepsilon)^2 + (a - b\varepsilon)^2$$

$$= (b - c\varepsilon)(c - a\varepsilon) + (c - a\varepsilon)(a - b\varepsilon) + (a - b\varepsilon)(c - a\varepsilon).$$

This is equivalent to

$$(1 + \varepsilon + \varepsilon^2)[(a - b)^2 + (b - c)^2 + (c - a)^2] = 0.$$

It follows that $1 + \varepsilon + \varepsilon^2 = 0$, i.e., $\frac{2\pi}{n} = \frac{2\pi}{3}$ and we get $n = 3$. Therefore $n = 3$ is the only value with the desired property.

## 3.5 Some Analytic Geometry in the Complex Plane

### 3.5.1 Equation of a line

**Proposition 1.** *The equation of a line in the complex plane is*

$$\overline{\alpha} \cdot \overline{z} + \alpha z + \beta = 0,$$

*where* $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{R}$ *and* $z = x + iy \in \mathbb{C}$.

*Proof.* The equation of a line in the cartesian plane is

$$Ax + By + C = 0,$$

where $A, B, C \in \mathbb{R}$ and $A^2 + B^2 \neq 0$. If we set $z = x + iy$, then $x = \frac{z + \overline{z}}{2}$ and $y = \frac{z - \overline{z}}{2i}$. Thus,

$$A\frac{z + \overline{z}}{2} - Bi\frac{z - \overline{z}}{2} + C = 0,$$

or equivalently

$$\overline{z}\left(\frac{A+Bi}{2}\right)+z\frac{A-Bi}{2}+C=0.$$

Let $\alpha=\dfrac{A-Bi}{2}\in\mathbb{C}^*$ and $\beta=C\in\mathbb{R}$. Then

$$\overline{\alpha}\cdot\overline{z}+\alpha z+\beta=0,$$

as claimed. □

If $\alpha=\overline{\alpha}$, then $B=0$ and we have a vertical line. If $\alpha\neq\overline{\alpha}$, then we define the *angular coefficient* of the line as

$$m=-\frac{A}{B}=\frac{\alpha+\overline{\alpha}}{\frac{\alpha-\overline{\alpha}}{i}}=\frac{\alpha+\overline{\alpha}}{\alpha-\overline{\alpha}}i.$$

**Proposition 2.** *Consider the lines $d_1$ and $d_2$ with equations*

$$\overline{\alpha}_1\cdot\overline{z}+\alpha_1\cdot z+\beta_1=0$$

*and*

$$\overline{\alpha}_2\cdot\overline{z}+\alpha_2\cdot z+\beta_2=0,$$

*respectively.*

*Then the lines $d_1$ and $d_2$ are:*

*1) parallel if and only if* $\dfrac{\overline{\alpha}_1}{\alpha_1}=\dfrac{\overline{\alpha}_2}{\alpha_2}$;

*2) perpendicular if and only if* $\dfrac{\overline{\alpha}_1}{\alpha_2}+\dfrac{\overline{\alpha}_2}{\alpha_2}=0$;

*3) concurrent if and only if* $\dfrac{\overline{\alpha}_1}{\alpha_1}\neq\dfrac{\overline{\alpha}_2}{\alpha_2}$.

*Proof.* 1) We have $d_1\|d_2$ if and only if $m_1=m_2$. Therefore $\dfrac{\alpha_1+\overline{\alpha}_1}{\alpha_1-\overline{\alpha}_1}i=\dfrac{\alpha_2+\overline{\alpha}_2}{\alpha_2-\overline{\alpha}_2}i$, so $\alpha_2\overline{\alpha}_1=\alpha_1\overline{\alpha}_2$ and we get $\dfrac{\overline{\alpha}_1}{\alpha_1}=\dfrac{\overline{\alpha}_2}{\alpha_2}$.

2) We have $d_1\perp d_2$ if and only if $m_1m_2=-1$. That is, $\alpha_2\overline{\alpha}_1+\alpha_2\overline{\alpha}_2=0$, or $\dfrac{\overline{\alpha}_1}{\alpha}+\dfrac{\overline{\alpha}_2}{\alpha_2}=0$.

3) The lines $d_1$ and $d_2$ are concurrent if and only if $m_1\neq m_2$. This condition yields $\dfrac{\overline{\alpha}_1}{\alpha_1}\neq\dfrac{\overline{\alpha}_2}{\alpha_2}$.

The results for *angular coefficient* correspond to the properties of *slope*. □

The ratio $m_d=-\dfrac{\overline{\alpha}}{\alpha}$ is called the complex angular coefficient of the line $d$ of equation

$$\overline{\alpha}\cdot\overline{z}+\alpha\cdot z+\beta=0.$$

### 3.5.2 Equation of a line determined by two points

**Proposition.** *The equation of a line determined by the points* $P_1(z_1)$ *and* $P_2(z_2)$ *is*

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z & \overline{z} & 1 \end{vmatrix} = 0.$$

*Proof.* The equation of a line determined by the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the cartesian plane is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix} = 0.$$

Using complex numbers we have

$$\begin{vmatrix} \dfrac{z_1 + \overline{z_1}}{2} & \dfrac{z_1 - \overline{z_1}}{2i} & 1 \\ \dfrac{z_2 + \overline{z_2}}{2} & \dfrac{z_2 - \overline{z_2}}{2i} & 1 \\ \dfrac{z_1 + \overline{z}}{2} & \dfrac{z - \overline{z}}{2i} & 1 \end{vmatrix} = 0$$

if and only if

$$\frac{1}{4i}\begin{vmatrix} z_1 + \overline{z_1} & z_1 - \overline{z_1} & 1 \\ z_2 + \overline{z_2} & z_2 - \overline{z_2} & 1 \\ z + \overline{z} & z - \overline{z} & 1 \end{vmatrix} = 0.$$

That is,

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z & \overline{z} & 1 \end{vmatrix} = 0,$$

as desired. □

**Remarks.** 1) The points $M_1(z_1)$, $M_2(z_2)$, $M_3(z_3)$ are collinear if and only if

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = 0.$$

2) The complex angular coefficient of a line determined by the points with coordinates $z_1$ and $z_2$ is

$$m = \frac{z_2 - z_1}{\overline{z_2} - \overline{z_1}}.$$

Indeed, the equation is

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = 0 \Leftrightarrow z_1\overline{z_2} + z_2 z + z\overline{z_1} - z\overline{z_2} - z_1\overline{z} - z_2\overline{z_1} = 0$$

$$\Leftrightarrow \overline{z}(z_2 - z_1) - z(\overline{z_2} - \overline{z_1}) + z_1\overline{z_2} - z_2\overline{z_1} = 0.$$

Using the definition of the complex angular coefficient we obtain

$$m = \frac{z_2 - z_1}{\overline{z_2} - \overline{z_1}}.$$

### 3.5.3 The area of a triangle

**Theorem.** *The area of triangle $A_1A_2A_3$ whose vertices have coordinates $z_1, z_2, z_3$ is equal to the absolute value of the number*

$$\frac{i}{4}\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix}. \tag{1}$$

*Proof.* Using cartesian coordinates, the area of a triangle with vertices $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ is equal to the absolute value of the determinant

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Since

$$x_k = \frac{z_k + \overline{z_k}}{2}, \quad y_k = \frac{z_k - \overline{z_k}}{2i}, \quad k = 1, 2, 3$$

we obtain

$$\Delta = \frac{1}{8i}\begin{vmatrix} z_1 + \overline{z_1} & z_1 - \overline{z_1} & 1 \\ z_2 + \overline{z_2} & z_2 - \overline{z_2} & 1 \\ z_3 + \overline{z_3} & z_3 - \overline{z_3} & 1 \end{vmatrix} = -\frac{1}{4i}\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix}$$

$$= \frac{i}{4}\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix},$$

as claimed. □

It is easy to see that for positively oriented triangle $A_1A_2A_3$ with vertices with coordinates $z_1$, $z_2$, $z_3$ the following inequality holds:

$$\frac{i}{4}\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} > 0.$$

**Corollary.** *The area of a directly oriented triangle $A_1A_2A_3$ whose vertices have coordinates $z_1$, $z_2$, $z_3$ is*

$$\text{area}[A_1A_2A_3] = \frac{1}{2}\,\text{Im}(\overline{z_1}z_2 + \overline{z_2}z_3 + \overline{z_3}z_1). \tag{2}$$

*Proof.* The determinant in the above theorem is

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = (z_1\overline{z_2} + z_2\overline{z_3} + z_3\overline{z_1} - \overline{z_2}z_3 - z_1\overline{z_3} - z_2\overline{z_1})$$

$$= [(z_1\overline{z_2} + z_2\overline{z_3} + z_3\overline{z_1}) - \overline{(z_1\overline{z_2} + z_2\overline{z_3} + z_3\overline{z_1})}]$$

$$= 2i\,\text{Im}(z_1\overline{z_2} + z_2\overline{z_3} + z_3\overline{z_1}) = -2i\,\text{Im}(\overline{z_1}z_2 + \overline{z_2}z_3 + \overline{z_3}z_1).$$

Replacing this value in (1), the desired formula follows. □

We will see that formula (2) can be extended to a convex directly oriented polygon $A_1A_2\cdots A_n$ (see Section 4.3).

**Problem 1.** *Consider the triangle $A_1A_2A_3$ and the points $M_1$, $M_2$, $M_3$ situated on lines $A_2A_3$, $A_1A_3$, $A_1A_2$, respectively. Assume that $M_1$, $M_2$, $M_3$ divide segments $[A_2A_3]$, $[A_3A_1]$, $[A_1A_2]$ into ratios $\lambda_1$, $\lambda_2$, $\lambda_3$, respectively. Then*

$$\frac{\text{area}[M_1M_2M_3]}{\text{area}[A_1A_2A_3]} = \frac{1 - \lambda_1\lambda_2\lambda_3}{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)}. \tag{3}$$

**Solution.** The coordinates of the points $M_1$, $M_2$, $M_3$ are

$$m_1 = \frac{a_2 - \lambda_1 a_3}{1-\lambda_1}, \quad m_2 = \frac{a_3 - \lambda_2 a_1}{1-\lambda_2}, \quad m_3 = \frac{a_1 - \lambda_3 a_2}{1-\lambda_3}.$$

Applying formula (2) we find that

$$\begin{aligned}
\text{area}[M_1M_2M_3] &= \frac{1}{2}\operatorname{Im}(\overline{m_1}m_2 + \overline{m_2}m_3 + \overline{m_3}m_1) \\
&= \frac{1}{2}\operatorname{Im}\Bigg[\frac{(\overline{a_2} - \lambda_1\overline{a_3})(a_3 - \lambda_2 a_1)}{(1-\lambda_1)(1-\lambda_2)} + \frac{(\overline{a_3} - \lambda_2\overline{a_1})(a_1 - \lambda_3 a_2)}{(1-\lambda_2)(1-\lambda_3)} \\
&\qquad + \frac{(\overline{a_1} - \lambda_3\overline{a_2})(a_2 - \lambda_1 a_3)}{(1-\lambda_3)(1-\lambda_1)}\Bigg] \\
&= \frac{1}{2}\operatorname{Im}\left[\frac{1-\lambda_1\lambda_2\lambda_3}{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)}(\overline{a_1}a_2 + \overline{a_2}a_3 + \overline{a_3}a_1)\right] \\
&= \frac{1-\lambda_1\lambda_2\lambda_3}{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)}\text{area}[A_1A_2A_3].
\end{aligned}$$

**Remark.** From formula (3) we derive the well-known theorem of Menelaus: *The points $M_1, M_2, M_3$ are collinear if and only if $\lambda_1\lambda_2\lambda_3 = 1$, i.e.,*

$$\frac{M_1A_2}{M_1A_3} \cdot \frac{M_2A_3}{M_2A_1} \cdot \frac{M_3A_1}{M_3A_2} = 1$$

**Problem 2.** *Let $a, b, c$ be the coordinates of the vertices $A, B, C$ of a triangle. It is known that $|a| = |b| = |c| = 1$ and that there exists $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that $a + b\cos\alpha + c\sin\alpha = 0$. Prove that*

$$1 < \text{area}[ABC] \le \frac{1+\sqrt{2}}{2}.$$

(Romanian Mathematical Olympiad – Final Round, 2003)

**Solution.** Observe that

$$\begin{aligned}
1 = |a|^2 &= |b\cos\alpha + c\sin\alpha|^2 \\
&= (b\cos\alpha + c\sin\alpha)(\overline{b}\cos\alpha + \overline{c}\sin\alpha) \\
&= |b|^2\cos^2\alpha + |c|^2\sin^2\alpha + (b\overline{c} + \overline{b}c)\sin\alpha\cos\alpha \\
&= 1 + \frac{b^2+c^2}{bc}\cos\alpha\sin\alpha.
\end{aligned}$$

It follows that $b^2 + c^2 = 0$, hence $b = \pm ic$. Applying formula (2) we obtain

$$\text{area}[ABC] = \frac{1}{2}|\operatorname{Im}(\overline{a}b + \overline{b}c + \overline{c}a)|$$

$$= \frac{1}{2}|\operatorname{Im}[(-\overline{b}\cos\alpha - \overline{c}\sin\alpha)b + \overline{b}c - \overline{c}(b\cos\alpha + c\sin\alpha)]|$$

$$= \frac{1}{2}|\operatorname{Im}(-\cos\alpha - \sin\alpha - b\overline{c}\sin\alpha - b\overline{c}\cos\alpha + \overline{b}c)|$$

$$= \frac{1}{2}|\operatorname{Im}[\overline{b}c - (\sin\alpha + \cos\alpha)b\overline{c}]| = \frac{1}{2}|\operatorname{Im}[(1 + \sin\alpha + \cos\alpha)\overline{b}c]|$$

$$= \frac{1}{2}(1 + \sin\alpha + \cos\alpha)|\operatorname{Im}(\overline{b}c)| = \frac{1}{2}(1 + \sin\alpha + \cos\alpha)|\operatorname{Im}(\pm i c\overline{c})|$$

$$= \frac{1}{2}(1 + \sin\alpha + \cos\alpha)|\operatorname{Im}(\pm i)| = \frac{1}{2}(1 + \sin\alpha + \cos\alpha)$$

$$= \frac{1}{2}\left[1 + \sqrt{2}\left(\frac{\sqrt{2}}{2}\sin\alpha + \frac{\sqrt{2}}{2}\cos\alpha\right)\right] = \frac{1}{2}\left(1 + \sqrt{2}\sin\left(\alpha + \frac{\pi}{4}\right)\right).$$

Taking into account that $\frac{\pi}{4} < \alpha + \frac{\pi}{4} < \frac{3\pi}{4}$ we get that $\frac{\sqrt{2}}{2} < \sin\left(\alpha + \frac{\pi}{4}\right) \leq 1$ and the conclusion follows.

### 3.5.4 Equation of a line determined by a point and a direction

**Proposition 1.** *Let $d : \overline{\alpha}\overline{z} + \alpha \cdot z + \beta = 0$ be a line and let $P_0(z_0)$ be a point. The equation of a line parallel to $d$ and passing through point $P_0$ is*

$$z - z_0 = -\frac{\overline{\alpha}}{\alpha}(\overline{z} - \overline{z_0}).$$

*Proof.* Using cartesian coordinates, the line parallel to $d$ and passing through point $P_0(x_0, y_0)$ has the equation

$$y - y_0 = i\frac{\alpha + \overline{\alpha}}{\alpha - \overline{\alpha}}(x - x_0).$$

Using complex numbers the equation takes the form

$$\frac{z - \overline{z}}{2i} - \frac{z_0 - \overline{z_0}}{2i} = i\frac{\alpha + \overline{\alpha}}{\alpha - \overline{\alpha}}\left(\frac{z + \overline{z}}{2} - \frac{z_0 + \overline{z_0}}{2}\right).$$

This is equivalent to $(\alpha - \overline{\alpha})(z - z_0 - \overline{z} + \overline{z_0}) = (\alpha + \overline{\alpha})(z + \overline{z} - z_0 - \overline{z_0})$, or $\alpha(z - z_0) = -\overline{\alpha}(\overline{z} - \overline{z_0})$. We obtain $z - z_0 = -\frac{\overline{\alpha}}{\alpha}(\overline{z} - \overline{z_0})$. □

**Proposition 2.** *Let $d : \overline{\alpha}\overline{z} + \alpha \cdot z + \beta = 0$ be a line and let $P_0(z_0)$ be a point. The line passing through point $P_0$ and perpendicular to $d$ has the equation $z - z_0 = \frac{\overline{\alpha}}{\alpha}(\overline{z} - \overline{z_0})$.*

*Proof.* Using cartesian coordinates, the line passing through point $P_0$ and perpendicular to $d$ has the equation

$$y - y_0 = -\frac{1}{i} \cdot \frac{\alpha - \overline{\alpha}}{\alpha + \overline{\alpha}}(x - x_0).$$

Then we obtain

$$\frac{z - \overline{z}}{2i} - \frac{z_0 - \overline{z_0}}{2i} = i \cdot \frac{\alpha - \overline{\alpha}}{\alpha + \overline{\alpha}} \left( \frac{z + \overline{z}}{2} - \frac{z_0 + \overline{z_0}}{2} \right).$$

That is, $(\alpha + \overline{\alpha})(z - z_0 - \overline{z} + \overline{z_0}) = -(\alpha - \overline{\alpha})(z - z_0 + \overline{z} - \overline{z_0})$ or

$$(z - z_0)(\alpha + \overline{\alpha} + \alpha - \overline{\alpha}) = (\overline{z} - \overline{z_0})(-\alpha + \overline{\alpha} + \alpha + \overline{\alpha}).$$

We obtain $\alpha(z - z_0) = \overline{\alpha}(\overline{z} - \overline{z_0})$ and $z - z_0 = \frac{\overline{\alpha}}{\alpha}(\overline{z} - \overline{z_0})$. □

### 3.5.5 The foot of a perpendicular from a point to a line

**Proposition.** *Let $P_0(z_0)$ be a point and let $d : \overline{\alpha z} + \alpha z + \beta = 0$ be a line. The foot of the perpendicular from $P_0$ to $d$ has the coordinate*

$$z = \frac{\alpha z_0 - \overline{\alpha}\, \overline{z_0} - \beta}{2\alpha}.$$

*Proof.* The point $z$ is the solution of the system

$$\begin{cases} \overline{\alpha} \cdot \overline{z} + \alpha \cdot z + \beta = 0, \\ \alpha(z - z_0) = \overline{\alpha}(\overline{z} - \overline{z_0}). \end{cases}$$

The first equation gives

$$\overline{z} = \frac{-\alpha z - \beta}{\overline{\alpha}}.$$

Substituting in the second equation yields

$$\alpha z - \alpha z_0 = -\alpha z - \beta - \overline{\alpha} \cdot \overline{z_0}.$$

Hence

$$z = \frac{\alpha z_0 - \overline{\alpha}\, \overline{z_0} - \beta}{2\alpha},$$

as claimed. □

### 3.5.6 Distance from a point to a line

**Proposition.** *The distance from a point $P_0(z_0)$ to a line $d : \overline{\alpha} \cdot \overline{z} + \alpha \cdot z + \beta = 0$, $\alpha \in \mathbb{C}^*$ is equal to*

$$D = \frac{|\alpha z_0 + \overline{\alpha} \cdot \overline{z_0} + \beta|}{2\sqrt{\alpha \cdot \overline{\alpha}}}.$$

*Proof.* Using the previous result, we can write

$$D = \left| \frac{\alpha z_0 - \overline{\alpha} \cdot \overline{z_0} - \beta}{2\alpha} - z_0 \right| = \left| \frac{-\alpha z_0 - \overline{\alpha z_0} - \beta}{2\alpha} \right|$$

$$= \frac{|\alpha \cdot z_0 + \overline{\alpha z_0} + \beta|}{2|\alpha|} = \frac{|\alpha z_0 + \overline{\alpha z_0} + \beta|}{2\sqrt{\alpha\overline{\alpha}}}. \qquad \square$$

# 3.6 The Circle

## 3.6.1 Equation of a circle

**Proposition.** *The equation of a circle in the complex plane is*

$$z \cdot \overline{z} + \alpha \cdot z + \overline{\alpha} \cdot \overline{z} + \beta = 0,$$

*where* $\alpha \in \mathbb{C}$ *and* $\beta \in \mathbb{R}$.

*Proof.* The equation of a circle in the cartesian plane is

$$x^2 + y^2 + mx + ny + p = 0,$$

$m, n, p \in \mathbb{R}$, $p < \dfrac{m^2 + n^2}{4}$.

Setting $x = \dfrac{z + \overline{z}}{2}$ and $y = \dfrac{z - \overline{z}}{2i}$ we obtain

$$|z|^2 + m\frac{z + \overline{z}}{2} + n\frac{z - \overline{z}}{2i} + p = 0$$

or

$$z \cdot \overline{z} + z\frac{m - ni}{2} + \overline{z}\frac{m + ni}{2} + p = 0.$$

Take $\alpha = \dfrac{m - ni}{2} \in \mathbb{C}$ and $\beta = p \in \mathbb{R}$ in the above equation and the claim is proved. $\square$

Note that the radius of the circle is equal to

$$r = \sqrt{\frac{m^2}{4} + \frac{n^2}{4} - p} = \sqrt{\alpha\overline{\alpha} - \beta}.$$

Then the equation is equivalent to

$$(\overline{z} + \alpha)(z + \overline{\alpha}) = r^2.$$

Setting

$$\gamma = -\overline{\alpha} = -\frac{m}{2} - \frac{n}{2}i$$

the equation of the circle with center at $\gamma$ and radius $r$ is

$$(\overline{z}-\overline{\gamma})(z-\gamma)=r^2.$$

**Problem.** *Let $z_1, z_2, z_3$ be the coordinates of the vertices of triangle $A_1A_2A_3$. The coordinate $z_O$ of the circumcenter of triangle $A_1A_2A_3$ is*

$$z_O=\frac{\begin{vmatrix}1 & 1 & 1\\ z_1 & z_2 & z_3\\ |z_1|^2 & |z_2|^2 & |z_3|^2\end{vmatrix}}{\begin{vmatrix}1 & 1 & 1\\ z_1 & z_2 & z_3\\ \overline{z_1} & \overline{z_2} & \overline{z_3}\end{vmatrix}}. \tag{1}$$

**Solution.** The equation of the line passing through $P(z_0)$ which is perpendicular to the line $A_1A_2$ can be written in the form

$$z(\overline{z_1}-\overline{z_2})+\overline{z}(z_1-z_2)=z_0(\overline{z_1}-\overline{z_2})+\overline{z_0}(z_1-z_2). \tag{2}$$

Applying this formula for the midpoints of the sides $[A_2A_3]$, $[A_1A_3]$ and for the lines $A_2A_3$, $A_1A_3$, we find the equations

$$z(\overline{z_2}-\overline{z_3})+\overline{z}(z_2-z_3)=|z_2|^2-|z_3|^2$$
$$z(\overline{z_3}-\overline{z_1})+\overline{z}(z_3-z_1)=|z_3|^2-|z_1|^2.$$

By eliminating $\overline{z}$ from these two equations, it follows that

$$z[(\overline{z_2}-\overline{z_3})+(\overline{z_3}-\overline{z_1})(z_2-z_3)]$$
$$=(z_1-z_3)(|z_2|^2-|z_3|^2)+(z_2-z_3)(|z_3|^2-|z_1|^2),$$

hence

$$z\begin{vmatrix}1 & 1 & 1\\ z_1 & z_2 & z_3\\ \overline{z_1} & \overline{z_2} & \overline{z_2}\end{vmatrix}=\begin{vmatrix}1 & 1 & 1\\ z_1 & z_2 & z_3\\ |z_1|^2 & |z_2|^2 & |z_3|^2\end{vmatrix}$$

and the desired formula follows.

**Remark.** We can write this formula in the following equivalent form:

$$z_O=\frac{z_1\overline{z_1}(z_2-z_3)+z_2\overline{z_2}(z_3-z_1)+z_3\overline{z_3}(z_1-z_2)}{\begin{vmatrix}1 & 1 & 1\\ \overline{z_1} & \overline{z_2} & \overline{z_3}\\ z_1 & z_2 & z_3\end{vmatrix}}. \tag{3}$$

### 3.6.2 The power of a point with respect to a circle

**Proposition.** *Consider a point $P_0(z_0)$ and a circle with equation*

$$z \cdot \overline{z} + \alpha \cdot z + \overline{\alpha} \cdot \overline{z} + \beta = 0,$$

*for $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{R}$.*

*The power of $P_0$ with respect to the circle is*

$$\rho(z_0) = z_0 \cdot \overline{z_0} + \alpha z_0 + \overline{\alpha} \cdot \overline{z_0} + \beta.$$

*Proof.* Let $O(-\overline{\alpha})$ be the center of the circle. The power of $P_0$ with respect to the circle of radius $r$ is defined by $\rho(z_0) = OP_0^2 - r^2$. In this case we obtain

$$\rho(z_0) = OP_0^2 - r^2 = |z_0 + \overline{\alpha}|^2 - r^2 = z_0 \cdot \overline{z_0} + \alpha z_0 + \overline{\alpha z_0} + \alpha\overline{\alpha} - \alpha\overline{\alpha} + \beta$$

$$= z_0 \cdot \overline{z_0} + \alpha z_0 + \overline{\alpha} \cdot \overline{z_0} + \beta,$$

as claimed. □

Given two circles of equations

$$z \cdot \overline{z} + \alpha_1 \cdot z + \overline{\alpha_1} \cdot \overline{z} + \beta_1 = 0 \quad \text{and} \quad z \cdot \overline{z} + \alpha_2 \cdot z + \overline{\alpha_2} \cdot \overline{z} + \beta_2 = 0,$$

where $\alpha_1, \alpha_2 \in \mathbb{C},\ \beta_1, \beta_2 \in \mathbb{R}$, their *radical axis* is the locus of points having equal powers with respect to the circles. If $P(z)$ is a point of this locus, then

$$z \cdot \overline{z} + \alpha_1 z + \overline{\alpha_1} \cdot \overline{z} + \beta_1 = z \cdot \overline{z} + \alpha_2 z + \overline{\alpha_2} \cdot z + \beta_2,$$

or equivalently $(\alpha_1 - \alpha_2)z + (\overline{\alpha_1} - \overline{\alpha_2})\overline{z} + \beta_1 - \beta_2 = 0$, which is the equation of a line.

### 3.6.3 Angle between two circles

The angle between two circles with equations

$$z \cdot \overline{z} + \alpha_1 \cdot z + \overline{\alpha_1} \cdot \overline{z} + \beta_1 = 0$$

and

$$z \cdot \overline{z} + \alpha_2 \cdot z + \overline{\alpha_2} \cdot \overline{z} + \beta_2 = 0, \quad \alpha_1, \alpha_2 \in \mathbb{C}, \quad \beta_1, \beta_2 \in \mathbb{R},$$

is the angle $\theta$ determined by the tangents to the circles at a common point.

**Proposition.** *The following formula*

$$\cos\theta = \left| \frac{\beta_1 + \beta_2 - (\alpha_1\overline{\alpha_2} + \overline{\alpha_1}\alpha_2)}{2r_1r_2} \right|$$

*holds.*

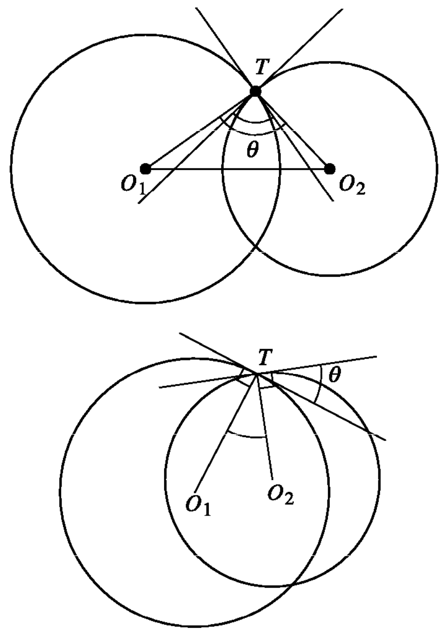


**Figure 3.15.**

*Proof.* Let $T$ be a common point and let $O_1(-\overline{\alpha_1})$, $O_2(-\overline{\alpha_2})$ be the centers of the circles.

The angle $\theta$ is equal to $\widehat{O_1TO_2}$ or $\pi - \widehat{O_1TO_2}$, hence

$$\cos\theta = |\cos\widehat{O_1TO_2}| = \frac{|r_1^2 + r_2^2 - O_1O_2^2|}{2r_1r_2}$$

$$= \frac{|\alpha_1\overline{\alpha_1} - \beta_1 + \alpha_2\overline{\alpha_2} - \beta_2 - |\overline{\alpha_1} - \overline{\alpha_2}|^2|}{2r_1r_2}$$

$$= \frac{|\alpha_1\overline{\alpha_1} + \alpha_2\overline{\alpha_2} - \beta_1 - \beta_2 - \overline{\alpha_1}\alpha_1 - \alpha_2\overline{\alpha_2} + \overline{\alpha_1}\alpha_2 + \alpha_1\overline{\alpha_2}|}{2r_1r_2}$$

$$= \frac{|\beta_1 + \beta_2 - (\alpha_1\overline{\alpha_2} + \overline{\alpha_1}\alpha_2)|}{2r_1r_2},$$

as claimed. □

Note that the circles are orthogonal if and only if

$$\beta_1 + \beta_2 = \alpha_1\overline{\alpha_2} + \overline{\alpha_1}\alpha_2.$$

**Problem 1.** *Let $a, b$ be real numbers such that $|b| \le 2a^2$. Prove that the set of points with coordinates $z$ such that*

$$|z^2 - a^2| = |2az + b|$$

*is the union of two orthogonal circles.*

**Solution.** The relation

$$|z^2 - a^2| = |2az + b|$$

is equivalent to

$$|z^2 - a^2|^2 = |2az + b|^2, \text{ i.e.,}$$
$$(z^2 - a^2)(\overline{z}^2 - \overline{a}^2) = (2az + b)(2a\overline{z} + b).$$

We can rewrite the last relation as

$$|z|^4 - a^2(z^2 + \overline{z}^2) + a^4 = 4a^2|z|^2 + 2ab(z + \overline{z}) + b^2, \text{ i.e.,}$$
$$|z|^4 - a^2[(z + \overline{z})^2 - 2|z|^2] + a^4 = 4a^2|z|^2 + 2ab(z + \overline{z}) + b^2.$$

Hence

$$|z|^4 - 2a^2|z|^2 + a^4 = a^2(z + \overline{z})^2 + 2ab(z + \overline{z}) + b^2, \text{ i.e.,}$$
$$(|z|^2 - a^2)^2 = (a(z + \overline{z}) + b)^2.$$

It follows that

$$z \cdot \overline{z} - a^2 = a(z + \overline{z}) + b \text{ or } z \cdot \overline{z} - a^2 = -a(z + \overline{z}) - b.$$

This is equivalent to

$$(z - a)(\overline{z} - a) = 2a^2 + b \text{ or } (z + a)(\overline{z} + a) = 2a^2 - b.$$

Finally

$$|z - a|^2 = 2a^2 + b \text{ or } |z + a|^2 = 2a^2 - b. \tag{1}$$

Since $|b| \le 2a^2$, it follows that $2a^2 + b \ge 0$ and $2a^2 - b \ge 0$. Hence the relations (1) are equivalent to

$$|z - a| = \sqrt{2a^2 + b} \text{ or } |z + a| = \sqrt{2a^2 - b}.$$

Therefore, the points with coordinates $z$ that satisfy $|z^2 - a^2| = |2az + b|$ lie on two circles of centers $C_1$ and $C_2$, whose coordinates $a$ and $-a$, and with radii $R_1 = \sqrt{2a^2 + b}$ and $R_2 = \sqrt{2a^2 - b}$. Furthermore,

$$C_1C_2^2 = 4a^2 = (\sqrt{2a^2 + b})^2 + (\sqrt{2a^2 - b})^2 = R_1^2 + R_2^2,$$

hence the circles are orthogonal, as claimed.

# 4

# More on Complex Numbers and Geometry

## 4.1 The Real Product of Two Complex Numbers

The concept of the scalar product of two vectors is well known. In what follows we will introduce this concept for complex numbers. We will see that in many situations use of this product simplifies the solution to the problem considerably.

Let $a$ and $b$ be two complex numbers.

**Definition.** We call the *real product* of complex numbers $a$ and $b$ the number given by

$$a \cdot b = \frac{1}{2}(\overline{a}b + a\overline{b}).$$

It is easy to see that

$$\overline{a \cdot b} = \frac{1}{2}(a\overline{b} + \overline{a}b) = a \cdot b;$$

hence $a \cdot b$ is a real number, which justifies the name of this product.

The following properties are easy to verify.

**Proposition 1.** *For all complex numbers $a, b, c, z$ the following relations hold:*

*1) $a \cdot a = |a|^2$.*

*2) $a \cdot b = b \cdot a$; (the real product is commutative).*

*3) $a \cdot (b + c) = a \cdot b + a \cdot c$; (the real product is distributive with respect to addition).*

*4) $(\alpha a) \cdot b = \alpha(a \cdot b) = a \cdot (\alpha b)$ for all $\alpha \in \mathbb{R}$.*

*5)* $a \cdot b = 0$ *if and only if* $OA \perp OB$, *where* $A$ *has coordinate* $a$ *and* $B$ *has coordinate* $b$.

*6)* $(az) \cdot (bz) = |z|^2(a \cdot b)$.

**Remark.** Suppose that $A$ and $B$ are points with coordinates $a$ and $b$. Then the real product $a \cdot b$ is equal to the power of the origin with respect to the circle of diameter $AB$.

Indeed, let $M\left(\dfrac{a+b}{2}\right)$ be the midpoint of $[AB]$, hence the center of this circle, and let $r = \dfrac{1}{2}AB = \dfrac{1}{2}|a-b|$ be the radius of this circle. The power of the origin with respect to the circle is

$$OM^2 - r^2 = \left|\frac{a+b}{2}\right|^2 - \left|\frac{a-b}{2}\right|^2$$
$$= \frac{(a+b)(\overline{a}+\overline{b})}{4} - \frac{(a-b)(\overline{a}-\overline{b})}{4} = \frac{a\overline{b}+b\overline{a}}{2} = a \cdot b,$$

as claimed.

**Proposition 2.** *Suppose that* $A(a)$, $B(b)$, $C(c)$ *and* $D(d)$ *are four distinct points. The following statements are equivalent:*

*1)* $AB \perp CD$;

*2)* $(b-a) \cdot (c-d) = 0$;

*3)* $\dfrac{b-a}{d-c} \in i\mathbb{R}^*$ *(or, equivalently,* $\mathrm{Re}\left(\dfrac{b-a}{d-c}\right) = 0$*).*

*Proof.* Take points $M(b-a)$ and $N(d-c)$ such that $OABM$ and $OCDN$ are parallelograms. Then we have $AB \perp CD$ if and only if $OM \perp ON$. That is, $m \cdot n = (b-a) \cdot (d-c) = 0$, using property 5) of the real product.

The equivalence 2) $\Leftrightarrow$ 3) follows immediately from the definition of the real product. □

**Proposition 3.** *The circumcenter of triangle* $ABC$ *is at the origin of the complex plane. If* $a, b, c$ *are the coordinates of vertices* $A, B, C$, *then the orthocenter* $H$ *has the coordinate* $h = a + b + c$.

*Proof.* Using the real product of the complex numbers, the equations of the altitudes $AA'$, $BB'$, $CC'$ of the triangle are

$$AA' : (z-a) \cdot (b-c) = 0, \;\; BB' : (z-b) \cdot (c-a) = 0, \;\; CC' : (z-c) \cdot (a-b) = 0.$$

We will show that the point with coordinate $h = a+b+c$ lies on all three altitudes. Indeed, we have $(h-a) \cdot (b-c) = 0$ if and only if $(b+c) \cdot (b-c) = 0$. The last relation is equivalent to $b \cdot b - c \cdot c = 0$, or $|b|^2 = |c|^2$. Similarly, $H \in BB'$ and $H \in CC'$, and we are done. □

**Remark.** If the numbers $a, b, c, o, h$ are the coordinates of the vertices of triangle $ABC$, the circumcenter $O$ and the orthocenter $H$ of the triangle, then $h = a+b+c-2o$.

Indeed, taking $A'$ diametrically opposite $A$ in the circumcircle of triangle $ABC$, the quadrilateral $HBA'C$ is a parallelogram. If $\{M\} = HA' \cap BC$, then

$$z_M = \frac{b+c}{2} = \frac{z_H + z_{A'}}{2} = \frac{z_H + 2o - a}{2}, \text{ i.e., } z_H = a + b + c - 2o.$$

**Problem 1.** *Let $ABCD$ be a convex quadrilateral. Prove that*

$$AB^2 + CD^2 = AD^2 + BC^2$$

*if and only if $AC \perp BD$.*

**Solution.** Using the properties of the real product of complex numbers, we have

$$AB^2 + CD^2 = BC^2 + DA^2$$

if and only if

$$(b-a)\cdot(b-a) + (d-c)\cdot(d-c) = (c-b)\cdot(c-b) + (a-d)\cdot(a-d).$$

That is,

$$a \cdot b + c \cdot d = b \cdot c + d \cdot a$$

and finally

$$(c-a)\cdot(d-b) = 0,$$

or, equivalently, $AC \perp BD$, as required.

**Problem 2.** *Let $M, N, P, Q, R, S$ be the midpoints of the sides $AB$, $BC$, $CD$, $DE$, $EF$, $FA$ of a hexagon. Prove that*

$$RN^2 = MQ^2 + PS^2$$

*if and only if $MQ \perp PS$.*

(Romanian Mathematical Olympiad – Final Round, 1994)

**Solution.** Let $a, b, c, d, e, f$ be the coordinates of the vertices of the hexagon. The points $M, N, P, Q, R, S$ have the coordinates

$$m = \frac{a+b}{2}, \quad n = \frac{b+c}{2}, \quad p = \frac{c+d}{2},$$
$$q = \frac{d+e}{2}, \quad r = \frac{e+f}{2}, \quad s = \frac{f+a}{2},$$

respectively.

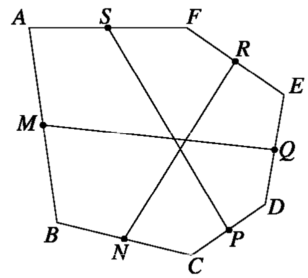


**Figure 4.1.**

Using the properties of the real product of complex numbers, we have

$$RN^2 = MQ^2 + PS^2$$

if and only if

$$(e + f - b - c) \cdot (e + f - b - c)$$
$$= (d + e - a - b) \cdot (d + e - a - b) + (f + a - c - d) \cdot (f + a - c - d).$$

That is,

$$(d + e - a - b) \cdot (f + a - c - d) = 0;$$

hence $MQ \perp PS$, as claimed.

**Problem 3.** *Let $A_1A_2 \cdots A_n$ be a regular polygon inscribed in a circle of center $O$ and radius $R$. Prove that for all points $M$ in the plane the following relation holds:*

$$\sum_{k=1}^{n} MA_k^2 = n(OM^2 + R^2).$$

**Solution.** Consider the complex plane with origin at point $O$ and let $R\varepsilon_k$ be the coordinate of vertex $A_k$, where $\varepsilon_k$ are the $n^{\text{th}}$-roots of unity, $k = 1, \ldots, n$. Let $m$ be the coordinate of $M$.

Using the properties of the real product of the complex numbers, we have

$$\begin{aligned}\sum_{k=1}^{n} MA_k^2 &= \sum_{k=1}^{n} (m - R\varepsilon_k) \cdot (m - R\varepsilon_k)\\ &= \sum_{k=1}^{n} (m \cdot m - 2R\varepsilon_k \cdot m + R^2\varepsilon_k \cdot \varepsilon_k)\\ &= n|m|^2 - 2R\Big(\sum_{k=1}^{n} \varepsilon_k\Big) \cdot m + R^2 \sum_{k=1}^{n} |\varepsilon_k|^2\\ &= n \cdot OM^2 + nR^2 = n(OM^2 + R^2),\end{aligned}$$

since $\sum_{k=1}^{n} \varepsilon_k = 0$.

**Remark.** If $M$ lies on the circumcircle of the polygon, then

$$\sum_{k=1}^{n} MA_k^2 = 2nR^2.$$

**Problem 4.** *Let $O$ be the circumcenter of the triangle $ABC$, let $D$ be the midpoint of the segment $AB$, and let $E$ is the centroid of triangle $ACD$. Prove that lines $CD$ and $OE$ are perpendicular if and only if $AB = AC$.*

(Balkan Mathematical Olympiad, 1985)

**Solution.** Let $O$ be the origin of the complex plane and let $a, b, c, d, e$ be the coordinates of points $A, B, C, D, E$, respectively. Then

$$d = \frac{a+b}{2} \quad \text{and} \quad e = \frac{a+c+d}{3} = \frac{3a+b+2c}{6}.$$

Using the real product of complex numbers, if $R$ is the circumradius of triangle $ABC$, then

$$a \cdot a = b \cdot b = c \cdot c = R^2.$$

Lines $CD$ and $DE$ are perpendicular if and only if $(d - c) \cdot e = 0$ That is,

$$(a + b - 2c) \cdot (3a + b + 2c) = 0.$$

The last relation is equivalent to

$$3a \cdot a + a \cdot b + 2a \cdot c + 3a \cdot b + b \cdot b + 2b \cdot c - 6a \cdot c - 2b \cdot c - 4c \cdot c = 0,$$

that is,

$$a \cdot b = a \cdot c. \tag{1}$$

On the other hand, $AB = AC$ is equivalent to

$$|b - a|^2 = |c - a|^2.$$

That is,

$$(b - a) \cdot (b - a) = (c - a) \cdot (c - a)$$

or

$$b \cdot b - 2a \cdot b + a \cdot a = c \cdot c - 2a \cdot c + a \cdot a,$$

hence

$$a \cdot b = a \cdot c. \tag{2}$$

The relations (1) and (2) show that $CD \perp OE$ if and only if $AB = AC$.

**Problem 5.** *Let $a, b, c$ be distinct complex numbers such that $|a| = |b| = |c|$ and $|b + c - a| = |a|$.*

*Prove that $b + c = 0$.*

**Solution.** Let $A, B, C$ be the geometric images of the complex numbers $a, b, c$, respectively. Choose the circumcenter of triangle $ABC$ as the origin of the complex plane and denote by $R$ the circumradius of triangle $ABC$. Then

$$a\overline{a} = b\overline{b} = c\overline{c} = R^2,$$

and using the real product of the complex numbers, we have

$$|b + c - a| = |a| \text{ if and only if } |b + c - a|^2 = |a|^2.$$

That is,

$$(b + c - a) \cdot (b + c - a) = |a|^2, \text{ i.e.,}$$

$$|a|^2 + |b|^2 + |c|^2 + 2b \cdot c - 2a \cdot c - 2a \cdot b = |a|^2.$$

We obtain

$$2(R^2 + b \cdot c - a \cdot c - a \cdot b) = 0, \text{ i.e.,}$$

$$a \cdot a + b \cdot c - a \cdot c - a \cdot b = 0.$$

It follows that $(a - b) \cdot (a - c) = 0$, hence $AB \perp AC$, i.e., $\widehat{BAC} = 90°$. Therefore, $[BC]$ is the diameter of the circumcircle of triangle $ABC$, so $b + c = 0$.

**Problem 6.** *Let $E, F, G, H$ be the midpoints of sides $AB, BC, CD, DA$ of the convex quadrilateral $ABCD$. Prove that lines $AB$ and $CD$ are perpendicular if and only if*

$$BC^2 + AD^2 = 2(EG^2 + FH^2).$$

**Solution.** Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter. Then

$$e = \frac{a+b}{2}, \quad f = \frac{b+c}{2}, \quad g = \frac{c+d}{2}, \quad h = \frac{d+a}{2}.$$

Using the real product of the complex numbers, the relation

$$BC^2 + AD^2 = 2(EG^2 + FH^2)$$

becomes

$$(c - b) \cdot (c - b) + (d - a) \cdot (d - a)$$

$$= \frac{1}{2}(c + d - a - b) \cdot (c + d - a - b) + \frac{1}{2}(a + d - b - c) \cdot (a + d - b - c).$$

This is equivalent to

$$c \cdot c + b \cdot b + d \cdot d + a \cdot a - 2b \cdot c - 2a \cdot d$$
$$= a \cdot a + b \cdot b + c \cdot c + d \cdot d - 2a \cdot c - 2b \cdot d,$$

or

$$a \cdot d + b \cdot c = a \cdot c + b \cdot d.$$

The last relation shows that $(a - b) \cdot (d - c) = 0$ if and only if $AB \perp CD$, as desired.

**Problem 7.** *Let $G$ be the centroid of triangle $ABC$ and let $A_1$, $B_1$, $C_1$ be the midpoints of sides $BC$, $CA$, $AB$, respectively. Prove that*

$$MA^2 + MB^2 + MC^2 + 9MG^2 = 4(MA_1^2 + MB_1^2 + MC_1^2)$$

*for all points $M$ in the plane.*

**Solution.** Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter. Then

$$g = \frac{a+b+c}{3}, \quad a_1 = \frac{b+c}{2}, \quad b_1 = \frac{c+a}{2}, \quad c_1 = \frac{a+b}{2}.$$

Using the real product of the complex numbers, we have

$$MA^2 + MB^2 + MC^2 + 9MG^2$$
$$= (m - a) \cdot (m - a) + (m - b) \cdot (m - b) + (m - c) \cdot (m - c)$$
$$+ 9\left(m - \frac{a+b+c}{3}\right) \cdot \left(m - \frac{a+b+c}{3}\right)$$
$$= 12|m|^2 - 8(a + b + c) \cdot m + 2(|a|^2 + |b|^2 + |c|^2) + 2a \cdot b + 2b \cdot c + 2c \cdot a.$$

On the other hand,

$$4(MA_1^2 + MB_1^2 + MC_1^2)$$
$$= 4\left[\left(m - \frac{b+c}{2}\right) \cdot \left(m - \frac{b+c}{2}\right) + \left(m - \frac{c+a}{2}\right)\right.$$
$$\left.\cdot \left(m - \frac{c+a}{2}\right) + \left(m - \frac{a+b}{2}\right) \cdot \left(m - \frac{a+b}{2}\right)\right]$$
$$= 12|m|^2 - 8(a + b + c) \cdot m + 2(|a|^2 + |b|^2 + |c|^2) + 2a \cdot b + 2b \cdot c + 2c \cdot a,$$

so we are done.

**Remark.** The following generalization can be proved similarly.

Let $A_1A_2\cdots A_n$ be a polygon with the centroid $G$ and let $A_{ij}$ be the midpoint of the segment $[A_iA_j]$, $i < j$, $i, j \in \{1, 2, \ldots, n\}$.

Then

$$(n-2)\sum_{k=1}^{n} MA_k^2 + n^2 MG^2 = 4\sum_{i<j} MA_{ij}^2,$$

for all points $M$ in the plane. A nice generalization is given in Theorem 5, Section 4.11.

## 4.2 The Complex Product of Two Complex Numbers

The cross product of two vectors is a central concept in vector algebra, with numerous applications in various branches of mathematics and science. In what follows we adapt this product to complex numbers. The reader will see that this new interpretation has multiple advantages in solving problems involving area or collinearity.

Let $a$ and $b$ be two complex numbers.

**Definition.** The complex number

$$a \times b = \frac{1}{2}(\overline{a}b - a\overline{b})$$

is called the *complex product* of the numbers $a$ and $b$.

Note that

$$a \times b + \overline{a \times b} = \frac{1}{2}(\overline{a}b - a\overline{b}) + \frac{1}{2}(a\overline{b} - \overline{a}b) = 0,$$

so $\operatorname{Re}(a \times b) = 0$, which justifies the definition of this product.

The following properties are easy to verify:

**Proposition 1.** *Suppose that $a, b, c$ are complex numbers. Then:*

*1) $a \times b = 0$ if and only if $a = 0$ or $b = 0$ or $a = \lambda b$, where $\lambda$ is a real number.*

*2) $a \times b = -b \times a$; (the complex product is* anticommutative*).*

*3) $a \times (b + c) = a \times b + a \times c$ (the complex product is* distributive *with respect to addition).*

*4) $\alpha(a \times b) = (\alpha a) \times b = a \times (\alpha b)$, for all real numbers $\alpha$.*

*5) If $A(a)$ and $B(b)$ are distinct points other than the origin, then $a \times b = 0$ if and only if $O, A, B$ are collinear.*

**Remarks.** a) Suppose $A(a)$ and $B(b)$ are distinct points in the complex plane, different from the origin.

The complex product of the numbers $a$ and $b$ has the following useful geometric interpretation:

$$a \times b = \begin{cases} 2i \cdot \text{area}[AOB], & \text{if triangle } OAB \text{ is positively oriented;} \\ -2i \cdot \text{area}[AOB], & \text{if triangle } OAB \text{ is negatively oriented.} \end{cases}$$

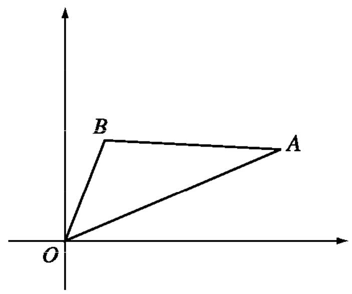


**Figure 4.2.**

Indeed, if triangle $OAB$ is positively (directly) oriented, then

$$\begin{aligned} 2i \cdot \text{area}[OAB] &= i \cdot OA \cdot OB \cdot \sin(\widehat{AOB}) \\ &= i|a| \cdot |b| \cdot \sin\left(\arg \frac{b}{a}\right) = i \cdot |a| \cdot |b| \cdot \operatorname{Im}\left(\frac{b}{a}\right) \cdot \frac{|a|}{|b|} \\ &= \frac{1}{2}|a|^2\left(\frac{b}{a} - \frac{\overline{b}}{\overline{a}}\right) = \frac{1}{2}(\overline{a}b - a\overline{b}) = a \times b. \end{aligned}$$

In the other case, note that triangle $OBA$ is positively oriented, hence

$$2i \cdot \text{area}[OBA] = b \times a = -a \times b.$$

b) Suppose $A(a)$, $B(b)$, $C(c)$ are three points in the complex plane.

The complex product allows us to obtain the following useful formula for the area of the triangle $ABC$:

$$\text{area}[ABC] = \begin{cases} \dfrac{1}{2i}(a \times b + b \times c + c \times a), \\ \text{if triangle } ABC \text{ is positively oriented;} \\ \\ -\dfrac{1}{2i}(a \times b + b \times c + c \times a), \\ \text{if triangle } ABC \text{ is negatively oriented.} \end{cases}$$

Moreover, simple algebraic manipulation shows that

$$\text{area}[ABC] = \frac{1}{2}\operatorname{Im}(\overline{a}b + \overline{b}c + \overline{c}a)$$

if triangle $ABC$ is directly (positively) oriented.

To prove the above formula, translate points $A$, $B$, $C$ with vector $-c$. The images of $A$, $B$, $C$ are points $A'$, $B'$, $O$ with coordinates $a - c$, $b - c$, $0$, respectively. Triangles $ABC$ and $A'B'O$ are congruent with the same orientation. If $ABC$ is positively oriented, then

$$\begin{aligned}\text{area}[ABC] &= \text{area}[OA'B'] = \frac{1}{2i}((a-c)\times(b-c))\\ &= \frac{1}{2i}((a-c)\times b-(a-c)\times c) = \frac{1}{2i}(c\times(a-c)-b\times(a-c))\\ &= \frac{1}{2i}(c\times a-c\times c-b\times a+b\times c) = \frac{1}{2i}(a\times b+b\times c+c\times a),\end{aligned}$$

as claimed.

The other situation can be similarly solved.

**Proposition 2.** *Suppose $A(a)$, $B(b)$ and $C(c)$ are distinct points. The following statements are equivalent:*

*1) Points $A, B, C$ are collinear.*

*2) $(b-a)\times(c-a)=0$.*

*3) $a\times b+b\times c+c\times a=0$.*

*Proof.* Points $A, B, C$ are collinear if and only if area$[ABC]=0$, i.e., $a\times b+b\times c+c\times a=0$. The last equation can be written in the form $(b-a)\times(c-a)=0$. □

**Proposition 3.** *Let $A(a)$, $B(b)$, $C(c)$, $D(d)$ be four points, no three of which are collinear. Then $AB\|CD$ if and only if $(b-a)\times(d-c)=0$.*

*Proof.* Choose the points $M(m)$ and $N(n)$ such that $OABM$ and $OCDN$ are parallelograms; then $m=b-a$ and $n=d-c$.

Lines $AB$ and $CD$ are parallel if and only if points $O, M, N$ are collinear. Using property 5, this is equivalent to $0=m\times n=(b-a)\times(d-c)$. □

**Problem 1.** *Points $D$ and $E$ lie on sides $AB$ and $AC$ of the triangle $ABC$ such that*

$$\frac{AD}{AB}=\frac{AE}{AC}=\frac{3}{4}.$$

*Consider points $E'$ and $D'$ on the rays $(BE$ and $(CD$ such that $EE'=3BE$ and $DD'=3CD$. Prove that:*

*1) points $D', A, E'$ are collinear;*

*2) $AD'=AE'$.*

**Solution.** The points $D, E, D', E'$ have the coordinates: $d=\dfrac{a+3b}{4}$, $e=\dfrac{a+3c}{4}$,

$$e'=4e-3b=a+3c-3b \text{ and } d'=4d-3c=a+3b-3c,$$

respectively.

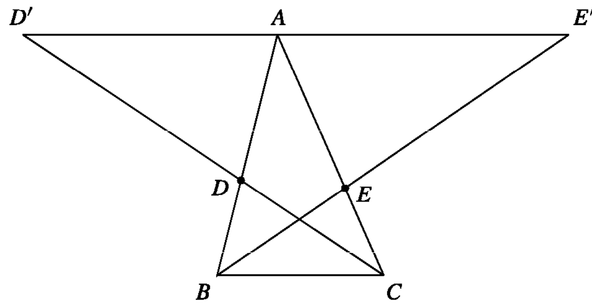


**Figure 4.3.**

1) Since

$$(a - d') \times (e' - d') = (3c - 3b) \times (6c - 6b) = 18(c - b) \times (c - b) = 0,$$

using Proposition 2 it follows that the points $D'$, $A$, $E'$ are collinear.

2) Note that

$$\frac{AD'}{D'E'} = \left|\frac{a - d'}{e' - d'}\right| = \frac{1}{2},$$

so $A$ is the midpoint of segment $D'E'$.

**Problem 2.** *Let $ABCDE$ be a convex pentagon and let $M$, $N$, $P$, $Q$, $X$, $Y$ be the midpoints of the segments $BC$, $CD$, $DE$, $EA$, $MP$, $NQ$, respectively.*

*Prove that $XY \| AB$.*

**Solution.** Let $a$, $b$, $c$, $d$, $e$ be the coordinates of vertices $A$, $B$, $C$, $D$, $E$, respectively.

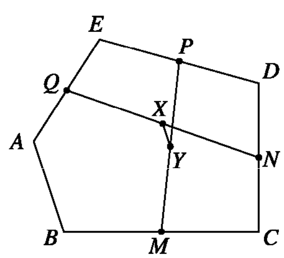


**Figure 4.4.**

Points $M$, $N$, $P$, $Q$, $X$, $Y$ have the coordinates

$$m = \frac{b+c}{2}, \quad n = \frac{c+d}{2}, \quad p = \frac{d+e}{2},$$
$$q = \frac{e+a}{2}, \quad x = \frac{b+c+d+e}{4}, \quad y = \frac{c+d+e+a}{4},$$

respectively. Then

$$\frac{y-x}{b-a} = \frac{\frac{a-b}{4}}{b-a} = -\frac{1}{4} \in \mathbb{R},$$

hence

$$(y-x) \times (b-a) = -\frac{1}{4}(b-a) \times (b-a) = 0.$$

From Proposition 3 it follows that $XY \| AB$.

## 4.3 The Area of a Convex Polygon

We say that the convex polygon $A_1 A_2 \cdots A_n$ is *directly* (or *positively*) *oriented* if for any point $M$ situated in the interior of the polygon the triangles $MA_k A_{k+1}$, $k = 1, 2, \ldots, n$, are directly oriented, where $A_{n+1} = A_1$.

**Theorem.** *Consider a directly oriented convex polygon $A_1 A_2 \cdots A_n$ with vertices with coordinates $a_1, a_2, \ldots, a_n$. Then*

$$\text{area}[A_1 A_2 \cdots A_n] = \frac{1}{2} \operatorname{Im}(\overline{a_1} a_2 + \overline{a_2} a_3 + \cdots + \overline{a_{n-1}} a_n + \overline{a_n} a_1).$$

*Proof.* We use induction on $n$. The base case $n = 3$ was proved above using the complex product. Suppose that the claim holds for $n = k$ and note that

$$\text{area}[A_1 A_2 \cdots A_k A_{k+1}] = \text{area}[A_1 A_2 \cdots A_k] + \text{area}[A_k A_{k+1} A_1]$$

$$= \frac{1}{2} \operatorname{Im}(\overline{a_1} a_2 + \overline{a_2} a_3 + \cdots + \overline{a_{k-1}} a_k + \overline{a_k} a_1) + \frac{1}{2} \operatorname{Im}(\overline{a_k} a_{k+1} + \overline{a_{k+1}} a_1 + \overline{a_1} a_k)$$

$$= \frac{1}{2} \operatorname{Im}(\overline{a_1} a_2 + \overline{a_2} a_3 + \cdots + \overline{a_{k-1}} a_k + \overline{a_k} a_{k+1} + \overline{a_{k+1}} a_1)$$

$$+ \frac{1}{2} \operatorname{Im}(\overline{a_k} a_1 + \overline{a_1} a_k) = \frac{1}{2} \operatorname{Im}(\overline{a_1} a_2 + \overline{a_2} a_3 + \cdots + \overline{a_k} a_{k+1} + \overline{a_{k+1}} a_1),$$

$$\text{since} \qquad \operatorname{Im}(\overline{a_k} a_1 + \overline{a_1} a_k) = 0.$$

*Alternative proof.* Choose a point $M$ in the interior of the polygon. Applying the formula (2) in Subsection 3.5.3 we have

$$\begin{aligned}
\text{area}[A_1A_2\cdots A_n] &= \sum_{k=1}^{n}\text{area}[MA_kA_{k+1}]\\
&= \frac{1}{2}\sum_{k=1}^{n}\text{Im}(\overline{z}a_k+\overline{a_k}a_{k+1}+\overline{a_{k+1}}z)\\
&= \frac{1}{2}\sum_{k=1}^{n}\text{Im}(\overline{a_k}a_{k+1})+\frac{1}{2}\sum_{k=1}^{n}\text{Im}(\overline{z}a_k+\overline{a_{k+1}}z)\\
&= \frac{1}{2}\text{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+1}\right)+\frac{1}{2}\text{Im}\left(\overline{z}\sum_{k=1}^{n}a_k+z\sum_{j=1}^{n}\overline{a_j}\right)=\frac{1}{2}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+1}\right),
\end{aligned}$$

since for any complex numbers $z$, $w$ the relation $\text{Im}(\overline{z}w+z\overline{w})=0$ holds. □

**Remark.** From the above formula it follows that the points $A_1(a_1)$, $A_2(a_2),\ldots,$ $A_n(a_n)$ are collinear if and only if

$$\text{Im}(\overline{a}_1a_2+\overline{a}_2a_3+\cdots+\overline{a}_{n-1}a_n+\overline{a}_na_1)=0.$$

**Problem 1.** *Let $P_0P_1\cdots P_{n-1}$ be the polygon whose vertices have coordinates $1,\varepsilon,\ldots,\varepsilon^{n-1}$ and let $Q_0Q_1\cdots Q_{n-1}$ be the polygon whose vertices have coordinates $1, 1+\varepsilon,\ldots,1+\varepsilon+\cdots+\varepsilon^{n-1}$, where $\varepsilon=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$. Find the ratio of the areas of these polygons.*

**Solution.** Consider $a_k=1+\varepsilon+\cdots+\varepsilon^k$, $k=0,1,\ldots,n-1$, and observe that

$$\begin{aligned}
\text{area}[Q_0Q_1\cdots Q_{n-1}] &= \frac{1}{2}\text{Im}\left(\sum_{k=0}^{n-1}\overline{a_k}a_{k+1}\right)\frac{1}{2}\text{Im}\left(\sum_{k=0}^{n-1}\frac{(\overline{\varepsilon})^{k+1}-1}{\overline{\varepsilon}-1}\cdot\frac{\varepsilon^{k+2}-1}{\varepsilon-1}\right)\\
&= \frac{1}{2|\varepsilon-1|^2}\text{Im}\left[\sum_{k=0}^{n-1}(\varepsilon-(\overline{\varepsilon})^{k+1}-\varepsilon^{k+2}+1)\right]\\
&= \frac{1}{2|\varepsilon-1|^2}\text{Im}(n\varepsilon+n)=\frac{1}{2|\varepsilon-1|^2}n\sin\frac{2\pi}{n}\\
&= \frac{n}{8\sin^2\frac{\pi}{n}}2\sin\frac{\pi}{n}\cos\frac{\pi}{n}=\frac{n}{4}\text{cotan}\frac{\pi}{n},
\end{aligned}$$

$$\text{since}\quad \sum_{k=0}^{n-1}\overline{\varepsilon}^{k+1}=0\quad\text{and}\quad\sum_{k=0}^{n-1}\varepsilon^{k+2}=0.$$

On the other hand, it is clear that

$$\text{area}[P_0P_1\cdots P_{n-1}]=n\,\text{area}[P_0OP_1]=\frac{n}{2}\sin\frac{2\pi}{n}=n\sin\frac{\pi}{n}\cos\frac{\pi}{n}.$$

We obtain

$$\frac{\text{area}[P_0P_1\cdots P_{n-1}]}{\text{area}[Q_0Q_1\cdots Q_{n-1}]} = \frac{n\sin\dfrac{\pi}{n}\cos\dfrac{\pi}{n}}{\dfrac{n}{4}\cot\text{an}\dfrac{\pi}{n}} = 4\sin^2\frac{\pi}{n}. \tag{1}$$

**Remark.** We have $Q_kQ_{k+1} = |a_{k+1} - a_k| = |\varepsilon^{k+1}| = 1$, and $P_kP_{k+1} = |\varepsilon^{k+1} - \varepsilon^k| = |\varepsilon^k(\varepsilon - 1)| = |\varepsilon^k||1 - \varepsilon| = |1 - \varepsilon| = 2\sin\dfrac{\pi}{n}$, $k = 0, 1, \ldots, n-1$. It follows that

$$\frac{P_kP_{k+1}}{Q_kQ_{k+1}} = 2\sin\frac{\pi}{n}, \quad k = 0, 1, \ldots, n-1.$$

That is, the polygons $P_0P_1\cdots P_{n-1}$ and $Q_0Q_1\cdots Q_{n-1}$ are similar and the result in (1) follows.

**Problem 2.** *Let $A_1A_2\cdots A_n$ $(n \geq 5)$ be a convex polygon and let $B_k$ be the midpoint of the segment $[A_kA_{k+1}]$, $k = 1, 2, \ldots, n$, where $A_{n+1} = A_1$. Then the following inequality holds:*

$$\text{area}[B_1B_2\cdots B_n] \geq \frac{1}{2}\text{area}[A_1A_2\cdots A_n].$$

**Solution.** Let $a_k$ and $b_k$ be the coordinates of points $A_k$ and $B_k$, $k = 1, 2, \ldots, n$. It is clear that the polygon $B_1B_2\cdots B_n$ is convex and if we assume that $A_1A_2\cdots A_n$ is positively oriented, then $B_1B_2\cdots B_n$ also has this property. Choose as the origin $O$ of the complex plane a point situated in the interior of polygon $A_1A_2\cdots A_n$.

We have $b_k = \dfrac{1}{2}(a_k + a_{k+1})$, $k = 1, 2, \ldots, n$, and

$$\begin{aligned}
\text{area}[B_1B_2\cdots B_n] &= \frac{1}{2}\operatorname{Im}\left(\sum_{k=1}^{n}\overline{b_k}b_{k+1}\right) = \frac{1}{8}\operatorname{Im}\sum_{k=1}^{n}(\overline{a_k} + \overline{a_{k+1}})(a_{k+1} + a_{k+2}) \\
&= \frac{1}{8}\operatorname{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+1}\right) + \frac{1}{8}\operatorname{Im}\left(\sum_{k=1}^{n}\overline{a_{k+1}}a_{k+2}\right) + \frac{1}{8}\operatorname{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+2}\right) \\
&= \frac{1}{2}\text{area}[A_1A_2\cdots A_n] + \frac{1}{8}\operatorname{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+2}\right) \\
&= \frac{1}{2}\text{area}[A_1A_2\cdots A_n] + \frac{1}{8}\sum_{k=1}^{n}\operatorname{Im}(\overline{a_k}a_{k+2}) \\
&= \frac{1}{2}\text{area}[A_1A_2\cdots A_n] + \frac{1}{8}\sum_{k=1}^{n}OA_k\cdot OA_{k+2}\sin\widehat{A_kOA_{k+2}} \\
&\geq \frac{1}{2}\text{area}[A_1A_2\cdots A_n].
\end{aligned}$$

We have used the relations

$$\operatorname{Im}\left(\sum_{k=1}^{n} \overline{a_k} a_{k+1}\right) = \operatorname{Im}\left(\sum_{k=1}^{n} \overline{a_{k+1}} a_{k+2}\right) = 2\,\text{area}[A_1 A_2 \cdots A_n],$$

and $\sin \widehat{A_k O A_{k+2}} \geq 0$, $k = 1, 2, \ldots, n$, where $A_{n+2} = A_2$.

## 4.4 Intersecting Cevians and Some Important Points in a Triangle

**Proposition 1.** *Consider the points $A'$, $B'$, $C'$ on the sides $BC$, $CA$, $AB$ of the triangle $ABC$ such that $AA'$, $BB'$, $CC'$ intersect at point $Q$ and let*

$$\frac{BA'}{A'C} = \frac{p}{n}, \quad \frac{CB'}{B'A} = \frac{m}{p}, \quad \frac{AC'}{C'B} = \frac{n}{m}.$$

*If $a, b, c$ are the coordinates of points $A, B, C$, respectively, then the coordinate of point $Q$ is*

$$q = \frac{ma + nb + pc}{m + n + p}.$$

*Proof.* The coordinates of $A', B', C'$ are $a' = \dfrac{nb + pc}{n + p}$, $b' = \dfrac{ma + pc}{m + p}$ and $c' = \dfrac{ma + nb}{m + n}$, respectively. Let $Q$ be the point with coordinate $q = \dfrac{ma + nb + pc}{m + n + p}$. We prove that $AA', BB', CC'$ meet at $Q$.

The points $A, Q, A'$ are collinear if and only if $(q - a) \times (a' - a) = 0$. This is equivalent to

$$\left(\frac{ma + nb + pc}{m + n + p} - a\right) \times \left(\frac{nb + pc}{n + p} - a\right) = 0$$

or $(nb + pc - (n + p)a) \times (nb + pc - (n + p)a) = 0$, which is clear by definition of the complex product.

Likewise, $Q$ lies on lines $BB'$ and $CC'$, so the proof is complete. □

**Some important points in a triangle.** 1) If $Q = G$, the centroid of the triangle $ABC$, we have $m = n = p = 1$. Then we obtain again that the coordinate of $G$ is

$$z_G = \frac{a + b + c}{3}.$$

2) Suppose that the lengths of the sides of triangle $ABC$ are $BC = \alpha$, $CA = \beta$, $AB = \gamma$. If $Q = I$, the incenter of triangle $ABC$, then, using the known result concerning the angle bisector, it follows that $m = \alpha$, $n = \beta$, $p = \gamma$. Therefore the coordinate of $I$ is

$$z_I = \frac{\alpha a + \beta b + \gamma c}{\alpha + \beta + \gamma} = \frac{1}{2s}[\alpha a + \beta b + \gamma c],$$

where $s = \frac{1}{2}(\alpha + \beta + \gamma)$.

3) If $Q = H$, the orthocenter of the triangle $ABC$, we easily obtain the relations

$$\frac{BA'}{A'C} = \frac{\tan C}{\tan B}, \quad \frac{CB'}{B'A} = \frac{\tan A}{\tan C}, \quad \frac{AC'}{C'B} = \frac{\tan B}{\tan A}.$$

It follows that $m = \tan A$, $n = \tan B$, $p = \tan C$, and the coordinate of $H$ is given by

$$z_H = \frac{(\tan A)a + (\tan B)b + (\tan C)c}{\tan A + \tan B + \tan C}.$$

**Remark.** The above formula can also be extended to the limiting case when the triangle $ABC$ is a right triangle. Indeed, assume that $A \to \frac{\pi}{2}$. Then $\tan A \to \pm\infty$ and $\frac{(\tan B)b + (\tan C)c}{\tan A} \to 0$, $\frac{\tan B + \tan C}{\tan A} \to 0$. In this case $z_H = a$, i.e., the orthocenter of triangle $ABC$ is the vertex $A$.

4) The Gergonne[1] point $J$ is the intersection of the cevians $AA'$, $BB'$, $CC'$, where $A'$, $B'$, $C'$ are the points of tangency of the incircle to the sides $BC$, $CA$, $AB$, respectively. Then

$$\frac{BA'}{A'C} = \frac{\frac{1}{s-\gamma}}{\frac{1}{s-\beta}}, \quad \frac{CB'}{B'A} = \frac{\frac{1}{s-\alpha}}{\frac{1}{s-\gamma}}, \quad \frac{AC'}{C'B} = \frac{\frac{1}{s-\beta}}{\frac{1}{s-\alpha}},$$

and the coordinate $z_J$ is obtained from the same proposition, where

$$z_J = \frac{r_\alpha a + r_\beta b + r_\gamma c}{r_\alpha + r_\beta + r_\gamma}.$$

Here $r_\alpha$, $r_\beta$, $r_\gamma$ denote the radii of the three excircles of triangle. It is not difficult to show that the following formulas hold:

$$r_\alpha = \frac{K}{s-\alpha}, \quad r_\beta = \frac{K}{s-\beta}, \quad r_\gamma = \frac{K}{s-\gamma},$$

where $K = \text{area}[ABC]$ and $s = \frac{1}{2}(\alpha + \beta + \gamma)$.

5) The Lemoine[2] point $K$ is the intersection of the symmedians of the triangle (the symmedian is the reflection of the bisector across the median). Using the notation from

[1]Joseph-Diaz Gergonne (1771–1859), French mathematician, founded the journal *Annales de Mathématiques Pures et Appliquées* in 1810.

[2]Emile Michel Hyacinthe Lemoine (1840–1912), French mathematician, made important contributions to geometry.

the proposition we obtain

$$\frac{BA'}{A'C}=\frac{\gamma^2}{\beta^2},\quad \frac{CB'}{B'A}=\frac{\alpha^2}{\gamma^2},\quad \frac{AC'}{C'B}=\frac{\beta^2}{\alpha^2}.$$

It follows that

$$z_K=\frac{\alpha^2 a+\beta^2 b+\gamma^2 c}{\alpha^2+\beta^2+\gamma^2}.$$

6) The Nagel[3] point $N$ is the intersection of the cevians $AA'$, $BB'$, $CC'$, where $A'$, $B'$, $C'$ are the points of tangency of the excircles with the sides $BC$, $CA$, $AB$, respectively. Then

$$\frac{BA'}{A'C}=\frac{s-\gamma}{s-\beta},\quad \frac{CB'}{B'A}=\frac{s-\alpha}{s-\gamma},\quad \frac{AC'}{C'B}=\frac{s-\beta}{s-\alpha},$$

and the proposition mentioned before gives the coordinate $z_N$ of the Nagel point $N$,

$$z_N=\frac{(s-\alpha)a+(s-\beta)b+(s-\gamma)c}{(s-\alpha)+(s-\beta)+(s-\gamma)}=\frac{1}{s}[(s-\alpha)a+(s-\beta)b+(s-\gamma)c]$$
$$=\left(1-\frac{\alpha}{s}\right)a+\left(1-\frac{\beta}{s}\right)b+\left(1-\frac{\gamma}{s}\right)c.$$

**Problem.** *Let $\alpha,\beta,\gamma$ be the lengths of sides $BC$, $CA$, $AB$ of triangle $ABC$ and suppose $\alpha<\beta<\gamma$. If points $O, I, H$ are the circumcenter, the incenter and the orthocenter of the triangle $ABC$, respectively. Prove that*

$$\text{area}[OIH]=\frac{1}{8r}(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha),$$

*where $r$ is the inradius of $ABC$.*

**Solution.** Consider triangle $ABC$, directly oriented in the complex plane centered at point $O$.

Using the complex product and the coordinates of $I$ and $H$, we have

$$\text{area}[OIH]=\frac{1}{2i}(I\times h)=\frac{1}{2i}\left[\frac{\alpha a+\beta b+\gamma c}{\alpha+\beta+\gamma}\times(a+b+c)\right]$$
$$=\frac{1}{4si}[(\alpha-\beta)a\times b+(\beta-\gamma)b\times c+(\gamma-\alpha)c\times a]$$
$$=\frac{1}{2s}[(\alpha-\beta)\cdot\text{area}[OAB]+(\beta-\gamma)\cdot\text{area}[OBC]+(\gamma-\alpha)\cdot\text{area}[OCA]]$$
$$=\frac{1}{2s}\left[(\alpha-\beta)\frac{R^2\sin 2C}{2}+(\beta-\gamma)\frac{R^2\sin 2A}{2}+(\gamma-\alpha)\frac{R^2\sin 2B}{2}\right]$$

[3]Christian Heinrich von Nagel (1803–1882), German mathematician. His contributions to triangle geometry were included in the book *The Development of Modern Triangle Geometry* [13].

$$= \frac{R^2}{4s}[(\alpha - \beta)\sin 2C + (\beta - \gamma)\sin 2A + (\gamma - \alpha)\sin 2B]$$

$$= \frac{1}{8r}(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha),$$

as desired.

## 4.5 The Nine-Point Circle of Euler

Given a triangle $ABC$, choose its circumcenter $O$ to be the origin of the complex plane and let $a, b, c$ be the coordinates of the vertices $A, B, C$. We have seen in Section 2.22, Proposition 3, that the coordinate of the orthocenter $H$ is $z_H = a + b + c$.

Let us denote by $A_1, B_1, C_1$ the midpoints of sides $BC, CA, AB$, by $A', B', C'$ the feet of the altitudes and by $A'', B'', C''$ the midpoints of segments $AH, BH, CH$, respectively.

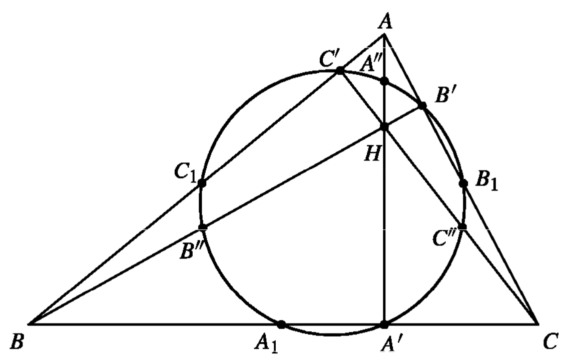


**Figure 4.5.**

It is clear that for the points $A_1, B_1, C_1, A'', B'', C''$ we have the following coordinates:

$$z_{A_1} = \frac{1}{2}(b + c), \quad z_{B_1} = \frac{1}{2}(c + a), \quad z_{C_1} = \frac{1}{2}(a + b),$$

$$z_{A''} = a + \frac{1}{2}(b + c), \quad z_{B''} = b + \frac{1}{2}(c + a), \quad z_{C''} = c + \frac{1}{2}(a + b).$$

It is not so easy to find the coordinates of $A', B', C'$.

**Proposition 1.** *Consider the point $X(x)$ in the plane of triangle $ABC$. Let $P$ be the projection of $X$ onto line $BC$. Then the coordinate of $P$ is given by*

$$p = \frac{1}{2}\left(x - \frac{bc}{R^2}\overline{x} + b + c\right)$$

*where $R$ is the circumradius of triangle $ABC$.*

*Proof.* Using the complex product and the real product we can write the equations of lines $BC$ and $XP$ as follows:

$$BC : (z - b) \times (c - b) = 0,$$

$$XP : (z - x) \cdot (c - b) = 0.$$

The coordinate $p$ of $P$ satisfies both equations; hence we have

$$(p - b) \times (c - b) = 0 \quad \text{and} \quad (p - x) \cdot (c - b) = 0.$$

These equations are equivalent to

$$(p - b)(\overline{c} - \overline{b}) - (\overline{p} - \overline{b})(c - b) = 0$$

and

$$(p - x)(\overline{c} - \overline{b}) + (\overline{p} - \overline{x})(c - b) = 0.$$

Adding the above relations we find

$$(2p - b - x)(\overline{c} - \overline{b}) + (\overline{b} - \overline{x})(c - b) = 0.$$

It follows that

$$p = \frac{1}{2}\left[b + x + \frac{c - b}{\overline{c} - \overline{b}}(\overline{x} - \overline{b})\right] = \frac{1}{2}\left[b + x + \frac{c - b}{\dfrac{R^2}{c} - \dfrac{R^2}{b}}(\overline{x} - \overline{b})\right]$$

$$= \frac{1}{2}\left[b + x - \frac{bc}{R^2}(\overline{x} - \overline{b})\right] = \frac{1}{2}\left(x - \frac{bc}{R^2}\overline{x} + b + c\right). \qquad \square$$

From the above Proposition 1, the coordinates of $A'$, $B'$, $C'$ are

$$z_{A'} = \frac{1}{2}\left(a + b + c - \frac{bc\overline{a}}{R^2}\right),$$

$$z_{B'} = \frac{1}{2}\left(a + b + c - \frac{ca\overline{b}}{R^2}\right),$$

$$z_{C'} = \frac{1}{2}\left(a + b + c - \frac{ab\overline{c}}{R^2}\right).$$

**Theorem 2.** (The nine-point circle.) *In any triangle $ABC$ the points $A_1$, $B_1$, $C_1$, $A'$, $B'$, $C'$, $A''$, $B''$, $C''$ are all on the same circle, whose center is at the midpoint of the segment $OH$, and the radius is one-half of the circumcircle.*

*Proof.* Denote by $O_9$ the midpoint of the segment $OH$. Using our initial assumption, it follows that $z_{O_9} = \frac{1}{2}(a+b+c)$. Also we have $|a| = |b| = |c| = R$, where $R$ is the circumradius of triangle $ABC$.

Observe that $O_9A_1 = |z_{A_1} - z_{O_9}| = \frac{1}{2}|a| = \frac{1}{2}R$, and also $O_9B_1 = O_9C_1 = \frac{1}{2}R$.

We can write $O_9A'' = |z_{A''} - z_{O_0}| = \frac{1}{2}|a| = \frac{1}{2}R$, and also $O_0B'' = O_9C'' = \frac{1}{2}R$.

The distance $O_9A'$ is also not difficult to compute:

$$O_9A' = |z_{A'} - z_{O_9}| = \left|\frac{1}{2}\left(a+b+c-\frac{bc\overline{a}}{R^2}\right) - \frac{1}{2}(a+b+c)\right|$$
$$= \frac{1}{2R^2}|bc\overline{a}| = \frac{1}{2R^2}|\overline{a}||b||c| = \frac{R^3}{2R^2} = \frac{1}{2}R.$$

Similarly, we get $O_9B' = O_9C' = \frac{1}{2}R$. Therefore $O_9A_1 = O_0B_1 = O_9C_1 = O_9A' = O_9B' = O_9C' = O_9A'' = O_9B'' = O_9C'' = \frac{1}{2}R$ and the desired property follows. □

**Theorem 3.** 1) (Euler[4] line of a triangle.) *In any triangle $ABC$ the points $O, G, H$ are collinear.*

2) (Nagel line of a triangle.) *In any triangle $ABC$ the points $I, G, N$ are collinear.*

*Proof.* 1) If the circumcenter $O$ is the origin of the complex plane, we have $z_O = 0$, $z_G = \frac{1}{3}(a+b+c)$, $z_H = a+b+c$. Hence these points are collinear by Proposition 2 in Section 2.22.

2) We have $z_I = \frac{\alpha}{2s}a + \frac{\beta}{2s}b + \frac{\gamma}{2s}c$, $z_G = \frac{1}{3}(a+b+c)$, and $z_N = \left(1-\frac{\alpha}{s}\right)a + \left(1-\frac{\beta}{s}\right)b + \left(1-\frac{\gamma}{s}\right)c$ and we can write $z_N = 3z_G - 2z_I$.

Applying the result mentioned above and properties of the complex product we obtain $(z_G - z_I) \times (z_N - z_I) = (z_G - z_I) \times [3(z_G - z_I)] = 0$; hence the points $I, H, N$ are collinear. □

**Remark.** Note that $NG = 2GI$, hence the triangles $OGI$ and $HGN$ are similar. It follows that the lines $OI$ and $NH$ are parallel and we have the following basic configuration of triangle $ABC$ (in Figure 4.6):

[4]Leonhard Euler (1707–1783), one of the most important mathematicians, created a good deal of analysis, and revised almost all the branches of pure mathematics which were then known, adding proofs, and arranging the whole in a consistent form. Euler wrote an immense number of memoirs on all kinds of mathematical subjects. We recommend William Dunham's book *Euler: The Master of Us All* (The Mathematical Association of America, 1999) for more details concerning Euler's contributions to mathematics.

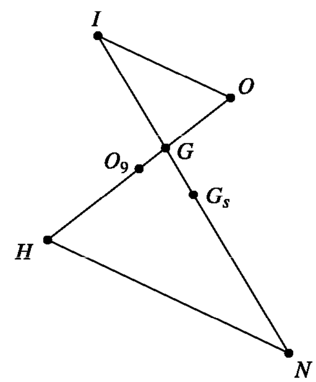


**Figure 4.6.**

If $G_s$ is the midpoint of segment $[IN]$, then its coordinate is

$$z_{G_s} = \frac{1}{2}(z_I + z_N) = \frac{(\beta+\gamma)}{4s}a + \frac{(\gamma+\alpha)}{4s}b + \frac{(\alpha+\beta)}{4s}c.$$

The point $G_s$ is called the *Spiecker point* of triangle $ABC$ and it is easy to verify that it is the incenter of the medial triangle $A_1B_1C_1$.

**Problem 1.** *Consider a point $M$ on the circumcircle of the triangle $ABC$. Prove that the nine-point centers of the triangles $MBC$, $MCA$, $MAB$ are the vertices of a triangle similar to triangle $ABC$.*

**Solution.** Let $A'$, $B'$, $C'$ be the nine-point centers of the triangles $MBC$, $MCD$, $MAB$, respectively. Take the origin of the complex plane to be at the circumcenter of triangle $ABC$. Denote by a lowercase letter the coordinate of the point denoted by an uppercase letter. Then

$$a' = \frac{m+b+c}{2}, \quad b' = \frac{m+c+a}{2}, \quad c' = \frac{m+a+b}{2},$$

since $M$ lies on the circumcircle of triangle $ABC$. Then

$$\frac{b'-a'}{c'-a'} = \frac{a-b}{a-c} = \frac{b-a}{c-a},$$

and hence triangles $A'B'C'$ and $ABC$ are similar.

**Problem 2.** *Show that triangle $ABC$ is a right triangle if and only if its circumcircle and its nine-point circle are tangent.*

**Solution.** Take the origin of the complex plane to be at circumcenter $O$ of triangle $ABC$ and denote by $a, b, c$ the coordinates of vertices $A$, $B$, $C$, respectively. Then the circumcircle of triangle $ABC$ is tangent to the nine-point circle of triangle $ABC$ if and only if $OO_9 = \frac{R}{2}$. This is equivalent to $OO_9^2 = \frac{R^2}{4}$, that is, $|a+b+c|^2 = R^2$.

Using properties of the real product, we have

$$\begin{aligned}|a+b+c|^2 &= (a+b+c)\cdot(a+b+c) = a^2+b^2+c^2+2(a\cdot b+b\cdot c+c\cdot a)\\ &= 3R^2+2(a\cdot b+b\cdot c+c\cdot a) = 3R^2+(2R^2-\alpha^2+2R^2-\beta^2+2R^2-\gamma^2)\\ &= 9R^2-(\alpha^2+\beta^2+\gamma^2),\end{aligned}$$

where $\alpha, \beta, \gamma$ are the lengths of the sides of triangle $ABC$. We have used the formulas $a\cdot b = R^2 - \frac{\gamma^2}{2}$, $b\cdot c = R^2 - \frac{\alpha^2}{2}$, $c\cdot a = R^2 - \frac{\beta^2}{2}$, which can be easily derived from the definition of the real product of complex numbers (see also the lemma in Subsection 4.6.2).

Therefore, $\alpha^2+\beta^2+\gamma^2 = 8R^2$, which is the same as $\sin^2 A + \sin^2 B + \sin^2 C = 2$. We can write the last relation as $1 - \cos 2A + 1 - \cos 2B + 1 - \cos 2C = 4$. This is equivalent to $2\cos(A+B)\cos(A-B) + 2\cos^2 C = 0$, i.e., $4\cos A\cos B\cos C = 0$, and the desired conclusion follows.

**Problem 3.** *Let $ABCD$ be a cyclic quadrilateral and let $E_a, E_b, E_c, E_d$ be the nine-point centers of triangles $BCD$, $CDA$, $DAB$, $ABC$, respectively. Prove that the lines $AE_a$, $BE_b$, $CE_c$, $DE_d$ are concurrent.*

**Solution.** Take the origin of the complex plane to be the center $O$ of the circumcircle of $ABCD$. Then the coordinates of the nine-point centers are

$$e_a = \frac{1}{2}(b+c+d), \quad e_b = \frac{1}{2}(c+d+a), \quad e_c = \frac{1}{2}(d+a+b), \quad e_d = \frac{1}{2}(a+b+c).$$

We have $AE_a : z = ka + (1-k)e_a$, $k \in \mathbb{R}$, and the analogous equations for the lines $BE_b$, $CE_c$, $DE_d$. Observe that the point with coordinate $\frac{1}{3}(a+b+c+d)$ lies on all of the four lines $\left(k = \frac{1}{3}\right)$, and we are done.

# 4.6 Some Important Distances in a Triangle

## 4.6.1 Fundamental invariants of a triangle

Consider the triangle $ABC$ with sides $\alpha, \beta, \gamma$, the semiperimeter $s = \frac{1}{2}(\alpha+\beta+\gamma)$, the inradius $r$ and the circumradius $R$. The numbers $s, r, R$ are called the *fundamental invariants* of triangle $ABC$.

**Theorem 1.** *The sides $\alpha, \beta, \gamma$ are the roots of the cubic equation*

$$t^3 - 2st^2 + (s^2+r^2+4Rr)t - 4sRr = 0.$$

*Proof.* Let us prove that $\alpha$ satisfies the equation. We have

$$\alpha = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2} \text{ and } s-\alpha = r\text{cotan}\frac{A}{2} = r\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}},$$

hence

$$\cos^2\frac{A}{2} = \frac{\alpha(s-\alpha)}{4Rr} \text{ and } \sin^2\frac{A}{2} = \frac{\alpha r}{4R(s-\alpha)}.$$

From the formula $\cos^2\frac{A}{2} + \sin^2\frac{A}{2} = 1$, it follows that

$$\frac{\alpha(s-\alpha)}{4Rr} + \frac{\alpha r}{4R(s-\alpha)} = 1.$$

That is, $\alpha^3 - 2s\alpha^2 + (s^2 + r^2 + 4Rr)\alpha - 4sRr = 0$. We can show analogously that $\beta$ and $\gamma$ are roots of the above equation. □

From the above theorem, by using the relations between the roots and the coefficients, it follows that

$$\alpha + \beta + \gamma = 2s,$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = s^2 + r^2 + 4Rr,$$

$$\alpha\beta\gamma = 4sRr.$$

**Corollary 2.** *In any triangle $ABC$, the following formulas hold:*

$$\alpha^2 + \beta^2 + \gamma^2 = 2(s^2 - r^2 - 4Rr),$$

$$\alpha^3 + \beta^3 + \gamma^3 = 2s(s^2 - 3r^2 - 6Rr).$$

*Proof.* We have

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = 4s^2 - 2(s^2 + r^2 + 4Rr)$$

$$= 2s^2 - 2r^2 - 8Rr = 2(s^2 - r^2 - 4Rr).$$

In order to prove the second identity, we can write

$$\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) + 3\alpha\beta\gamma$$

$$= 2s(2s^2 - 2r^2 - 8Rr - s^2 - r^2 - 4Rr) + 12sRr = 2s(s^2 - 3r^2 - 6Rr).$$ □

### 4.6.2 The distance $OI$

Assume that the circumcenter $O$ of the triangle $ABC$ is the origin of the complex plane and let $a, b, c$ be the coordinates of the vertices $A, B, C$, respectively.

**Lemma.** *The real products $a \cdot b, b \cdot c, c \cdot a$ are given by*

$$a \cdot b = R^2 - \frac{\gamma^2}{2}, \quad b \cdot c = R^2 - \frac{\alpha^2}{2}, \quad c \cdot a = R^2 - \frac{\beta^2}{2}.$$

*Proof.* Using the properties of the real product we have

$$\gamma^2 = |a - b|^2 = (a - b) \cdot (a - b) = a^2 - 2a \cdot b - b^2 = 2R^2 - 2a \cdot b,$$

and the first formula follows. □

**Theorem 4.** (Euler) *The following formula holds:*

$$OI^2 = R^2 - 2Rr.$$

*Proof.* The coordinate of the incenter is given by

$$z_I = \frac{\alpha}{2s}a + \frac{\beta}{2s}b + \frac{\gamma}{2s}c$$

so we can write

$$OI^2 = |z_I|^2 = \left(\frac{\alpha}{2s}a + \frac{\beta}{2s}b + \frac{\gamma}{2s}c\right) \cdot \left(\frac{\alpha}{2s}a + \frac{\beta}{2s}b + \frac{\gamma}{2s}c\right)$$

$$= \frac{1}{4s^2}(\alpha^2 + \beta^2 + \gamma^2)R^2 + 2\frac{1}{4s^2}\sum_{\text{cyc}}(\alpha\beta)a \cdot b.$$

Using the lemma above we find that

$$OI^2 = \frac{1}{4s^2}(\alpha^2 + \beta^2 + \gamma^2)R^2 + \frac{2}{4s^2}\sum_{\text{cyc}}\alpha\beta\left(R^2 - \frac{\gamma^2}{2}\right)$$

$$= \frac{1}{4s^2}(\alpha + \beta + \gamma)^2R^2 - \frac{1}{4s^2}\sum_{\text{cyc}}\alpha\beta\gamma^2 = R^2 - \frac{1}{4s^2}\alpha\beta\gamma(\alpha + \beta + \gamma)$$

$$= R^2 - \frac{1}{2s}\alpha\beta\gamma = R^2 - 2\frac{\alpha\beta\gamma}{4K} \cdot \frac{K}{s} = R^2 - 2Rr,$$

where the well-known formulas

$$R = \frac{\alpha\beta\gamma}{4K}, \quad r = \frac{K}{s},$$

are used. Here $K$ is the area of triangle $ABC$. □

**Corollary 5.** (Euler's inequality.) *In any triangle $ABC$ the following inequality holds:*

$$R \geq 2r.$$

*We have equality if and only if the triangle $ABC$ is equilateral.*

*Proof.* From Theorem 4 we have $OI^2 = R(R-2r) \geq 0$, hence $R \geq 2r$. The equality $R - 2r = 0$ holds if and only if $OI^2 = 0$, i.e., $O = I$. Therefore triangle $ABC$ is equilateral. □

### 4.6.3 The distance *ON*

**Theorem 6.** *If $N$ is the Nagel point of triangle $ABC$, then*

$$ON = R - 2r.$$

*Proof.* The coordinate of the Nagel point of the triangle is given by

$$z_N = \left(1 - \frac{\alpha}{s}\right)a + \left(1 - \frac{\beta}{s}\right)b + \left(1 - \frac{\gamma}{s}\right)c.$$

Therefore

$$\begin{aligned} ON^2 = |z_N|^2 = z_N \cdot z_N &= R^2 \sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right)^2 + 2\sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right)\left(1 - \frac{\beta}{s}\right) a \cdot b \\ &= R^2 \sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right)^2 + 2\sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right)\left(1 - \frac{\beta}{s}\right)\left(R^2 - \frac{\gamma^2}{2}\right) \\ &= R^2 \left(3 - \frac{\alpha+\beta+\gamma}{s}\right)^2 - \sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right)\left(1 - \frac{\beta}{s}\right)\gamma^2 \\ &= R^2 - \sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right)\left(1 - \frac{\beta}{s}\right)\gamma^2 = R^2 - E. \end{aligned}$$

To calculate $E$ we note that

$$\begin{aligned} E &= \sum_{\text{cyc}} \left(1 - \frac{\alpha+\beta}{s} + \frac{\alpha\beta}{s^2}\right)\gamma^2 = \sum_{\text{cyc}} \gamma^2 - \frac{1}{s}\sum_{\text{cyc}} (\alpha+\beta)\gamma^2 + \frac{1}{s^2}\sum_{\text{cyc}} \alpha\beta\gamma^2 \\ &= \sum_{\text{cyc}} \gamma^2 - \frac{1}{s}\sum_{\text{cyc}} (2s-\gamma)\gamma^2 + \frac{2\alpha\beta\gamma}{s} = -\sum_{\text{cyc}} \alpha^2 + \frac{1}{s}\sum_{\text{cyc}} \alpha^3 + 8\frac{\alpha\beta\gamma}{4K} \cdot \frac{K}{s} \\ &= -\sum_{\text{cyc}} \alpha^2 + \frac{1}{s}\sum_{\text{cyc}} \alpha^3 + 8Rr. \end{aligned}$$

Applying the formula in Corollary 2, we conclude that

$$E = -2(s^2 - r^2 - 4Rr) + 2(s^2 - 3r^2 - 6Rr) + 8Rr = -4r^2 + 4Rr.$$

Hence $ON^2 = R^2 - E = R^2 - 4Rr + 4r^2 = (R - 2r)^2$ and the desired formula is proved by Euler's inequality. □

**Theorem 7.** (Feuerbach[5]) *In any triangle the incircle and the nine-point circle of Euler are tangent.*

*Proof.* Using the configuration in Section 4.5 we observe that

$$\frac{1}{2} = \frac{GI}{GN} = \frac{GO_9}{GO}.$$

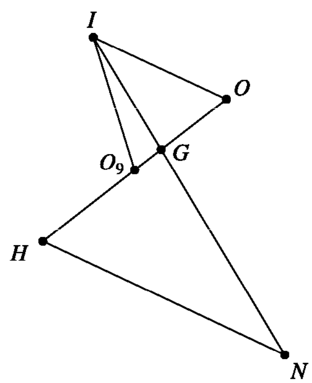


**Figure 4.7.**

Therefore triangles $GIO_9$ and $GNO$ are similar. It follows that the lines $IO_9$ and $ON$ are parallel and $IO_9 = \frac{1}{2}ON$. Applying Theorem 6 we get $IO_9 = \frac{1}{2}(R - 2r) = \frac{R}{2} - r = R_9 - r$, hence the incircle is tangent to the nine-point circle. □

The point of tangency of these two circles is denoted by $\varphi$ and is called the *Feuerbach point* of triangle.

### 4.6.4 The distance *OH*

**Theorem 8.** *If $H$ is the orthocenter of triangle $ABC$, then*

$$OH^2 = 9R^2 + 2r^2 + 8Rr - 2s^2.$$

*Proof.* Assuming that the circumcenter $O$ is the origin of the complex plane, the coordinate of $H$ is

$$z_H = a + b + c.$$

[5]Karl Wilhelm Feuerbach (1800–1834), German geometer, published the result of Theorem 7 in 1822.

Using the real product we can write

$$OH^2 = |z_H|^2 = z_H \cdot z_H = (a+b+c)\cdot(a+b+c)$$
$$= \sum_{\text{cyc}} |a|^2 + 2\sum ab = 3R^2 + 2\sum_{\text{cyc}} a \cdot b.$$

Applying the formulas in the lemma (p. 112) and then the first formula in Corollary 2, we obtain

$$OH^2 = 3R^2 + 2\sum_{\text{cyc}}\left(R^2 - \frac{\gamma^2}{2}\right) = 9R^2 - (\alpha^2+\beta^2+\gamma^2)$$
$$= 9R^2 - 2(s^2 - r^2 - 4Rr) = 9R^2 + 2r^2 + 8Rr - 2s^2. \qquad \square$$

**Corollary 9.** *The following formulas hold:*
*1)* $OG^2 = R^2 + \frac{2}{9}r^2 + \frac{8}{9}Rr - \frac{2}{9}s^2$*;*
*2)* $OO_9^2 = \frac{9}{4}R^2 + \frac{1}{2}r^2 + 2Rr - \frac{1}{2}s^2$.

**Corollary 10.** *In any triangle $ABC$ the inequality*

$$\alpha^2 + \beta^2 + \gamma^2 \le 9R^2$$

*is true. Equality holds if and only if the triangle is equilateral.*

## 4.7 Distance between Two Points in the Plane of a Triangle

### 4.7.1 Barycentric coordinates

Consider a triangle $ABC$ and let $\alpha, \beta, \gamma$ be the lengths of sides $BC, CA, AB$, respectively.

**Proposition 1.** *Let $a, b, c$ be the coordinates of vertices $A, B, C$ and let $P$ be a point in the plane of triangle. If $z_P$ is the coordinate of $P$, then there exist unique real numbers $\mu_a, \mu_b, \mu_c$ such that*

$$z_P = \mu_a a + \mu_b b + \mu_c c \text{ and } \mu_a + \mu_b + \mu_c = 1.$$

*Proof.* Assume that $P$ is in the interior of triangle $ABC$ and consider the point $A'$ such that $AP \cap BC = \{A'\}$. Let $k_1 = \frac{PA}{PA'}$, $k_2 = \frac{A'B}{A'C}$ and observe that

$$z_P = \frac{a + k_1 z_{A'}}{1+k_1}, \quad z_{A'} = \frac{b + k_2 c}{1+k_2}.$$

Hence in this case we can write

$$z_P = \frac{1}{1+k_1}a + \frac{k_1}{(1+k_1)(1+k_2)}b + \frac{k_1k_2}{(1+k_1)(1+k_2)}c.$$

Moreover, if we consider

$$\mu_a = \frac{1}{1+k_1}, \quad \mu_b = \frac{k_1}{(1+k_1)(1+k_2)}, \quad \mu_c = \frac{k_1k_2}{(1+k_1)(1+k_2)}$$

we have

$$\mu_a + \mu_b + \mu_c = \frac{1}{1+k_1} + \frac{k_1}{(1+k_1)(1+k_2)} + \frac{k_1k_2}{(1+k_1)(1+k_2)}$$
$$= \frac{1+k_1+k_2+k_1k_2}{(1+k_1)(1+k_2)} = 1.$$

We proceed in an analogous way in the case when the point $P$ is situated in the exterior of triangle $ABC$.

If the point $P$ is situated on the support line of a side of triangle $ABC$ (i.e., the line determined by two vertices)

$$z_P = \frac{1}{1+k}b + \frac{k}{1+k}c = 0 \cdot a + \frac{1}{1+k}b + \frac{k}{1+k}c,$$

where $k = \dfrac{PB}{PC}$. □

The real numbers $\mu_a, \mu_b, \mu_c$ are called the *absolute barycentric coordinates* of $P$ with respect to the triangle $ABC$.

The signs of numbers $\mu_a, \mu_b, \mu_c$ depend on the regions of the plane where the point $P$ is situated. Triangle $ABC$ determines seven such regions.

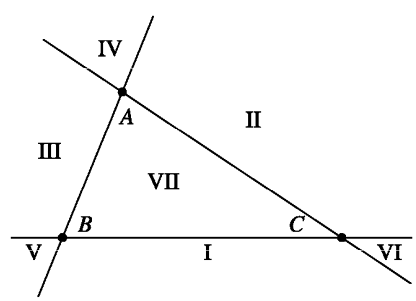


**Figure 4.8.**

In the next table we give the signs of $\mu_a, \mu_b, \mu_c$:

| | I | II | III | IV | V | VI | VII |
|---|---|---|---|---|---|---|---|
| $\mu_a$ | $-$ | $+$ | $+$ | $+$ | $-$ | $-$ | $+$ |
| $\mu_b$ | $+$ | $-$ | $+$ | $-$ | $+$ | $-$ | $+$ |
| $\mu_c$ | $+$ | $+$ | $-$ | $-$ | $-$ | $+$ | $+$ |

### 4.7.2 Distance between two points in barycentric coordinates

In what follows, in order to simplify the formulas, we will use the symbol called "cyclic sum." That is, $\sum_{\text{cyc}} f(x_1, x_2, \ldots, x_n)$, the sum of terms considered in the cyclic order. The most important example for our purposes is

$$\sum_{\text{cyc}} f(x_1, x_2, x_3) = f(x_1, x_2, x_3) + f(x_2, x_3, x_1) + f(x_3, x_1, x_2).$$

**Theorem 2.** *In the plane of triangle $ABC$ consider the points $P_1$ and $P_2$ with coordinates $z_{P_1}$ and $z_{P_2}$, respectively. If $z_{P_k} = \alpha_k a + \beta_k b + \gamma_k c$, where $\alpha_k, \beta_k, \gamma_k$ are real numbers such that $\alpha_k + \beta_k + \gamma_k = 1$, $k = 1, 2$, then*

$$P_1P_2^2 = -\sum_{\text{cyc}}(\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\gamma^2.$$

*Proof.* Choose the origin of the complex plane at the circumcenter $O$ of the triangle $ABC$. Using properties of the real product, we have

$$\begin{aligned}
P_1P_2^2 &= |z_{P_2} - z_{P_1}|^2 = |(\alpha_2 - \alpha_1)a + (\beta_2 - \beta_1)b + (\gamma_2 - \gamma_1)c|^2 \\
&= \sum_{\text{cyc}}(\alpha_2 - \alpha_1)^2 a \cdot a + 2\sum_{\text{cyc}}(\alpha_2 - \alpha_1)(\beta_2 - \beta_1)a \cdot b \\
&= \sum_{\text{cyc}}(\alpha_2 - \alpha_1)^2 R^2 + 2\sum_{\text{cyc}}(\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\left(R^2 - \frac{\gamma^2}{2}\right) \\
&= R^2(\alpha_2 + \beta_2 + \gamma_2 - \alpha_1 - \beta_1 - \gamma_1)^2 - \sum_{\text{cyc}}(\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\gamma^2 \\
&= -\sum_{\text{cyc}}(\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\gamma^2,
\end{aligned}$$

since $\alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = 1$. □

**Theorem 3.** *The points $A_1, A_2, B_1, B_2, C_1, C_2$ are situated on the sides $BC$, $CA$, $AB$ of triangle $ABC$ such that lines $AA_1, BB_1, CC_1$ meet at point $P_1$ and lines $AA_2, BB_2, CC_2$ meet at point $P_2$. If*

$$\frac{BA_k}{A_kC} = \frac{p_k}{n_k}, \quad \frac{CB_k}{B_kA} = \frac{m_k}{p_k}, \quad \frac{AC_k}{C_kB} = \frac{n_k}{m_k}, \quad k = 1, 2$$

*where $m_k, n_k, p_k$ are nonzero real numbers, $k = 1, 2$, and $S_k = m_k+n_k+p_k$, $k = 1, 2$, then*

$$P_1P_2^2 = \frac{1}{S_1^2S_2^2}\left[S_1S_2\sum_{\text{cyc}}(n_1p_2 + p_1n_2)\alpha^2 - S_1^2\sum_{\text{cyc}} n_2p_2\alpha^2 - S_2^2\sum_{\text{cyc}} n_1p_1\alpha^2\right].$$

*Proof.* The coordinates of points $P_1$ and $P_2$ are

$$z_{P_k} = \frac{m_ka + n_kb + p_kc}{m_k + n_k + p_k}, \quad k = 1, 2.$$

It follows that in this case the absolute barycentric coordinates of points $P_1$ and $P_2$ are given by

$$\alpha_k = \frac{m_k}{m_k + n_k + p_k} = \frac{m_k}{S_k}, \quad \beta_k = \frac{n_k}{m_k + n_k + p_k} = \frac{n_k}{S_k},$$

$$\gamma_k = \frac{p_k}{m_k + n_k + p_k} = \frac{p_k}{S_k}, \quad k = 1, 2.$$

Substituting in the formula in Theorem 2 we find

$$\begin{aligned} P_1P_2^2 &= -\sum_{\text{cyc}}\left(\frac{n_2}{S_2} - \frac{n_1}{S_1}\right)\left(\frac{p_2}{S_2} - \frac{p_1}{S_1}\right)\alpha^2 \\ &= -\frac{1}{S_1^2S_2^2}\sum_{\text{cyc}}(S_1n_2 - S_2n_1)(S_1p_2 - S_2p_1)\alpha^2 \\ &= -\frac{1}{S_1^2S_2^2}\sum_{\text{cyc}}[S_1^2n_2p_2 + S_2^2n_1p_1 - S_1S_2(n_1p_2 + n_2p_1)]\alpha^2 \\ &= \frac{1}{S_1^2S_2^2}\left[S_1S_2\sum_{\text{cyc}}(n_1p_2 + p_1n_2)\alpha^2 - S_1^2\sum_{\text{cyc}} n_2p_2\alpha^2 - S_2^2\sum_{\text{cyc}} n_1p_1\alpha^2\right] \end{aligned}$$

and the desired formula follows. □

**Corollary 4.** *For any real numbers $\alpha_k, \beta_k, \gamma_k$ with $\alpha_k + \beta_k + \gamma_k = 1$, $k = 1, 2$, the following inequality holds:*

$$\sum_{\text{cyc}}(\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\gamma^2 \le 0,$$

*with equality if and only if $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$.*

**Corollary 5.** *For any nonzero real numbers $m_k, n_k, p_k$, $k = 1, 2$, with $S_k = m_k + n_k + p_k$, $k = 1, 2$, the lengths of sides $\alpha, \beta, \gamma$ of triangle $ABC$ satisfy the inequality*

$$\sum_{\text{cyc}}(n_1p_2 + p_1n_2)^2 \ge \frac{S_1}{S_2}\sum_{\text{cyc}} n_2p_2\alpha^2 + \frac{S_2}{S_1}\sum_{\text{cyc}} n_1p_1\alpha^2$$

*with equality if and only if $\frac{p_1}{n_1} = \frac{p_2}{n_2}$, $\frac{m_1}{p_1} = \frac{m_2}{p_2}$, $\frac{n_1}{m_1} = \frac{n_2}{m_2}$.*

**Applications.** 1) Let us use the formula in Theorem 3 to compute the distance $GI$, where $G$ is the centroid and $I$ is the incenter of the triangle.

We have $m_1 = n_1 = p_1 = 1$ and $m_2 = \alpha$, $n_2 = \beta$, $p_2 = \gamma$; hence

$$S_1 = \sum_{\text{cyc}} m_1 = 3; \quad S_2 = \sum_{\text{cyc}} m_2 = \alpha + \beta + \gamma = 2s;$$

$$\sum_{\text{cyc}} (n_1 p_2 + n_2 p_1)\alpha^2 = (\beta + \gamma)\alpha^2 + (\gamma + \alpha)\beta^2 + (\alpha + \beta)\gamma^2$$

$$= (\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - 3\alpha\beta\gamma = 2s(s^2 + r^2 + 4rR) - 12sRr$$

$$= 2s^3 + 2sr^2 - 4sRr.$$

On the other hand,

$$\sum_{\text{cyc}} n_2 p_2 \alpha^2 = \alpha^2\beta\gamma + \beta^2\gamma\alpha + \gamma^2\alpha\beta = \alpha\beta\gamma(\alpha + \beta + \gamma) = 8s^2 Rr$$

and

$$\sum_{\text{cyc}} n_1 p_1 \alpha^2 = \alpha^2 + \beta^2 + \gamma^2 = 2s^2 - 2r^2 - 8Rr.$$

Then

$$GI^2 = \frac{1}{9}(s^2 + 5r^2 - 16Rr).$$

2) Let us prove that in any triangle $ABC$ with sides $\alpha, \beta, \gamma$, the following inequality holds:

$$\sum_{\text{cyc}} (2\alpha - \beta - \gamma)(2\beta - \alpha - \gamma)\gamma^2 \leq 0.$$

In the inequality in Corollary 4 we consider the points $P_1 = G$ and $P_2 = I$. Then $\alpha_1 = \beta_1 = \gamma_1 = \frac{1}{3}$ and $\alpha_2 = \frac{\alpha}{2s}$, $\beta_2 = \frac{\beta}{2s}$, $\gamma_2 = \frac{\gamma}{2s}$, and the above inequality follows. We have equality if and only if $P_1 = P_2$; that is, $G = I$, so the triangle is equilateral.

## 4.8 The Area of a Triangle in Barycentric Coordinates

Consider the triangle $ABC$ with $a, b, c$ the coordinates of its vertices, respectively. Let $\alpha, \beta, \gamma$ be the lengths of sides $BC$, $CA$ and $AB$.

**Theorem 1.** *Let $P_j(z_{P_j})$, $j = 1, 2, 3$, be three points in the plane of triangle $ABC$ with $z_{P_j} = \alpha_j a + \beta_j b + \gamma_j c$, where $\alpha_j, \beta_j, \gamma_j$ are the barycentric coordinates of $P_j$. If the triangles $ABC$ and $P_1P_2P_3$ have the same orientation, then*

$$\frac{\text{area}[P_1P_2P_3]}{\text{area}[ABC]} = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

*Proof.* Suppose that the triangles $ABC$ and $P_1P_2P_3$ are positively oriented. If $O$ denotes the origin of the complex plane, then using the complex product we can write

$$2i\ \text{area}[P_1OP_2] = z_{P_1} \times z_{P_2} = (\alpha_1 a + \beta_1 b + \gamma_1 c) \times (\alpha_2 a + \beta_2 b + \gamma_2 c)$$

$$= (\alpha_1\beta_2 - \alpha_2\beta_1)a \times b + (\beta_1\gamma_2 - \beta_2\gamma_1)b \times c + (\gamma_1\alpha_2 - \gamma_2\alpha_1)c \times a$$

$$= \begin{vmatrix} a\times b & b\times c & c\times a \\ \gamma_1 & \alpha_1 & \beta_1 \\ \gamma_2 & \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a\times b & b\times c & 2i\ \text{area}[ABC] \\ \gamma_1 & \alpha_1 & 1 \\ \gamma_2 & \alpha_2 & 1 \end{vmatrix}.$$

Analogously, we find

$$2i\ \text{area}[P_2OP_3] = \begin{vmatrix} a\times b & b\times c & 2i\ \text{area}[ABC] \\ \gamma_2 & \alpha_2 & 1 \\ \gamma_3 & \alpha_3 & 1 \end{vmatrix},$$

$$2i\ \text{area}[P_3OP_1] = \begin{vmatrix} a\times b & b\times c & 2i\ \text{area}[ABC] \\ \gamma_3 & \alpha_3 & 1 \\ \gamma_1 & \alpha_1 & 1 \end{vmatrix}.$$

Assuming that the origin $O$ is situated in the interior of triangle $P_1P_2P_3$, it follows that

$$\text{area}[P_1P_2P_3] = \text{area}[P_1OP_2] + \text{area}[P_2OP_3] + \text{area}[P_3OP_1]$$

$$= \frac{1}{2i}(\alpha_1 - \alpha_2 + \alpha_2 - \alpha_3 + \alpha_3 - \alpha_1)a \times b - \frac{1}{2i}(\gamma_1 - \gamma_2 + \gamma_2 - \gamma_3 + \gamma_3 - \gamma_1)b \times c$$

$$+ (\gamma_1\alpha_2 - \gamma_2\alpha_1 + \gamma_2\alpha_3 - \gamma_3\alpha_2 + \gamma_3\alpha_1 - \gamma_1\alpha_3)\text{area}[ABC]$$

$$= (\gamma_1\alpha_2 - \gamma_2\alpha_1 + \gamma_2\alpha_3 - \gamma_3\alpha_2 + \gamma_3\alpha_1 - \gamma_1\alpha_3)\text{area}[ABC]$$

$$= \text{area}[ABC]\begin{vmatrix} 1 & \gamma_1 & \alpha_1 \\ 1 & \gamma_2 & \alpha_2 \\ 1 & \gamma_3 & \alpha_3 \end{vmatrix} = \text{area}[ABC]\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

and the desired formula is obtained. □

**Corollary 2.** *Consider the triangle $ABC$ and the points $A_1, B_1, C_1$ situated on the lines $BC, CA, AB$, respectively, such that*

$$\frac{A_1B}{A_1C} = k_1, \quad \frac{B_1C}{B_1A} = k_2, \quad \frac{C_1A}{C_1B} = k_3.$$

*If $AA_1 \cap BB_1 = \{P_1\}$, $BB_1 \cap CC_1 = \{P_2\}$ and $CC_1 \cap AA_1 = \{P_3\}$, then*

$$\frac{\text{area}[P_1P_2P_3]}{\text{area}[ABC]} = \frac{(1 - k_1k_2k_3)^2}{(1 + k_1 + k_1k_2)(1 + k_2 + k_2k_3)(1 + k_3 + k_3k_1)}.$$

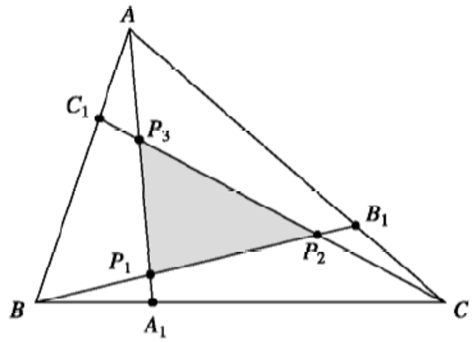


**Figure 4.9.**

*Proof.* Applying Menelaus's well-known theorem in triangle $AA_1B$ we find that

$$\frac{C_1A}{C_1B} \cdot \frac{CB}{CA_1} \cdot \frac{P_3A_1}{P_3A} = 1.$$

Hence

$$\frac{P_3A}{P_3A_1} = \frac{C_1A}{C_1B} \cdot \frac{CB}{CA_1} = k_3(1+k_1).$$

The coordinate of $P_3$ is given by

$$z_{P_3} = \frac{a + k_3(1+k_1)z_{A_1}}{1+k_3(1+k_1)} = \frac{a + k_3(1+k_1)\dfrac{b+k_1c}{1+k_1}}{1+k_3+k_3k_1} = \frac{a+k_3b+k_3k_1c}{1+k_3+k_3k_1}.$$

In an analogous way we find that

$$z_{P_1} = \frac{k_1k_2a + b + k_1c}{1+k_1+k_1k_2} \quad \text{and} \quad z_{P_2} = \frac{k_2a + k_2k_3b + c}{1+k_2+k_2k_3}.$$

The triangles $ABC$ and $P_1P_2P_3$ have the same orientation; hence by applying the formula in Theorem 1 we find that

$$\frac{\text{area}[P_1P_2P_3]}{\text{area}[ABC]}$$

$$= \frac{1}{(1+k_1+k_1k_2)(1+k_2+k_2k_3)(1+k_3+k_3k_1)} \begin{vmatrix} k_1k_2 & 1 & k_1 \\ k_2 & k_2k_3 & 1 \\ 1 & k_3 & k_3k_1 \end{vmatrix}$$

$$= \frac{(1-k_1k_2k_3)^2}{(1+k_1+k_1k_2)(1+k_2+k_2k_3)(1+k_3+k_3k_1)}. \qquad \square$$

**Remark.** When $k_1 = k_2 = k_3 = k$, from Corollary 2 we obtain Problem 3 from the $23^{rd}$ Putnam Mathematical Competition.

Let $A_j, B_j, C_j$ be points on the lines $BC, CA, AB$, respectively, such that

$$\frac{BA_j}{A_jC} = \frac{p_j}{n_j}, \quad \frac{CB_j}{B_jA} = \frac{m_j}{p_j}, \quad \frac{AC_j}{C_jB} = \frac{n_j}{m_j}, \quad j = 1, 2, 3.$$

**Corollary 3.** *If $P_j$ is the intersection point of lines $AA_j, BB_j, CC_j$, $j = 1, 2, 3$, and the triangles $ABC$, $P_1P_2P_3$ have the same orientation, then*

$$\frac{\text{area}[P_1P_2P_3]}{\text{area}[ABC]} = \frac{1}{S_1S_2S_3}\begin{vmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ m_3 & n_3 & p_3 \end{vmatrix}$$

*where $S_j = m_j + n_j + p_j$, $j = 1, 2, 3$.*

*Proof.* In terms of the coordinates of the triangle, the coordinates of the points $P_j$ are

$$z_{P_j} = \frac{m_ja + n_jb + p_jc}{m_j + n_j + p_j} = \frac{1}{S_j}(m_ja + n_jb + p_jc), \quad j = 1, 2, 3.$$

The formula above follows directly from Theorem 1. □

**Corollary 4.** *In triangle $ABC$ let us consider the cevians $AA'$, $BB'$ and $CC'$ such that*

$$\frac{A'B}{A'C} = m, \quad \frac{B'C}{B'A} = n, \quad \frac{C'A}{C'B} = p.$$

*Then the following formula holds:*

$$\frac{\text{area}[A'B'C']}{\text{area}[ABC]} = \frac{1 + mnp}{(1 + m)(1 + n)(1 + p)}.$$

*Proof.* Observe that the coordinates of $A'$, $B'$, $C'$ are given by

$$z_{A'} = \frac{1}{1+m}b + \frac{m}{1+m}c, \quad z_{B'} = \frac{1}{1+n}c + \frac{n}{1+n}a, \quad z_{C'} = \frac{1}{1+p}a + \frac{p}{1+p}b.$$

Applying the formula in Corollary 3 we obtain

$$\frac{\text{area}[A'B'C']}{\text{area}[ABC]} = \frac{1}{(1 + m)(1 + n)(1 + p)}\begin{vmatrix} 0 & 1 & m \\ n & 0 & 1 \\ 1 & p & 0 \end{vmatrix}$$

$$= \frac{1 + mnp}{(1 + m)(1 + n)(1 + p)}.$$ □

**Applications.** 1) (Steinhaus[6]) Let $A_j, B_j, C_j$ be points on lines $BC, CA, AB$, respectively, $j = 1, 2, 3$. Assume that

$$\frac{BA_1}{A_1C} = \frac{2}{4}, \quad \frac{CB_1}{B_1A} = \frac{1}{2}, \quad \frac{AC_1}{C_1B} = \frac{4}{1};$$

[6]Hugo Dyonizy Steinhaus (1887–1972), Polish mathematician, made important contributions in functional analysis and other branches of modern mathematics.

$$\frac{BA_2}{A_2C} = \frac{4}{1}, \quad \frac{CB_2}{B_2A} = \frac{2}{4}, \quad \frac{AC_2}{C_2B} = \frac{1}{2};$$
$$\frac{BA_3}{A_3C} = \frac{1}{2}, \quad \frac{CB_3}{B_3A} = \frac{4}{1}, \quad \frac{AC_3}{C_3B} = \frac{2}{4}.$$

If $P_j$ is the intersection point of lines $AA_j, BB_j, CC_j$, $j = 1, 2, 3$, and triangles $ABC$, $P_1P_2P_3$ are of the same orientation, then from Corollary 3 we obtain

$$\frac{\text{area}[P_1P_2P_3]}{\text{area}[ABC]} = \frac{1}{7 \cdot 7 \cdot 7} \begin{vmatrix} 1 & 4 & 2 \\ 2 & 1 & 4 \\ 4 & 2 & 1 \end{vmatrix} = \frac{49}{7^3} = \frac{1}{7}.$$

2) If the cevians $AA'$, $BB'$, $CC'$ are concurrent at point $P$, let us denote by $K_P$ the area of triangle $A'B'C'$. We can use the formula in Corollary 4 to compute the areas of some triangles determined by the feet of the cevians of some remarkable points in a triangle.

(i) If $I$ is the incenter of triangle $ABC$ we have

$$K_I = \frac{1 + \dfrac{\gamma}{\beta} \cdot \dfrac{\beta}{\alpha} \cdot \dfrac{\alpha}{\gamma}}{\left(1 + \dfrac{\gamma}{\beta}\right)\left(1 + \dfrac{\beta}{\alpha}\right)\left(1 + \dfrac{\alpha}{\gamma}\right)} \text{area}[ABC]$$
$$= \frac{2\alpha\beta\gamma}{(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)} \text{area}[ABC] = \frac{2\alpha\beta\gamma s r}{(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)}.$$

(ii) For the orthocenter $H$ of the acute triangle $ABC$ we obtain

$$K_H = \frac{1 + \dfrac{\tan C}{\tan B} \cdot \dfrac{\tan B}{\tan A} \cdot \dfrac{\tan A}{\tan C}}{\left(1 + \dfrac{\tan C}{\tan B}\right)\left(1 + \dfrac{\tan B}{\tan A}\right)\left(1 + \dfrac{\tan A}{\tan C}\right)} \text{area}[ABC]$$
$$= (2 \cos A \cos B \cos C)\text{area}[ABC] = (2 \cos A \cos B \cos C) sr.$$

(iii) For the Nagel point of triangle $ABC$ we can write

$$K_N = \frac{1 + \dfrac{s - \gamma}{s - \beta} \cdot \dfrac{s - \alpha}{s - \gamma} \cdot \dfrac{s - \beta}{s - \alpha}}{\left(1 + \dfrac{s - \gamma}{s - \beta}\right)\left(1 + \dfrac{s - \alpha}{s - \gamma}\right)\left(1 + \dfrac{s - \beta}{s - \alpha}\right)} \text{area}[ABC]$$
$$= \frac{2(s - \alpha)(s - \beta)(s - \gamma)}{\alpha\beta\gamma} \text{area}[ABC] = \frac{4\text{area}^2[ABC]}{2s\alpha\beta\gamma} \text{area}[ABC]$$
$$= \frac{r}{2R} \text{area}[ABC] = \frac{sr^2}{2R}.$$

If we proceed in the same way for the Gergonne point $J$ we find the relation

$$K_J = \frac{r}{2R}\text{area}[ABC] = \frac{sr^2}{2R}.$$

**Remark.** Two cevians $AA'$ and $AA''$ are *isotomic* if the points $A'$ and $A''$ are symmetric with respect to the midpoint of the segment $BC$. Assuming that

$$\frac{A'B}{A'C} = m, \quad \frac{B'C}{B'A} = n, \quad \frac{C'A}{C'B} = p,$$

then for the corresponding isotomic cevians we have

$$\frac{A''B}{A''C} = \frac{1}{m}, \quad \frac{B''C}{B''A} = \frac{1}{n}, \quad \frac{C''A}{C''B} = \frac{1}{p}.$$

Applying the formula in Corollary 4, it follows that

$$\frac{\text{area}[A'B'C']}{\text{area}[ABC]} = \frac{1+mnp}{(1+m)(1+n)(1+p)}$$

$$= \frac{1+\dfrac{1}{mnp}}{\left(1+\dfrac{1}{m}\right)\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{1}{p}\right)} = \frac{\text{area}[A''B''C'']}{\text{area}[ABC]}.$$

Therefore $\text{area}[A'B'C'] = \text{area}[A''B''C'']$. A special case of this relation is $K_N = K_J$, since the points $N$ and $J$ are isotomic (i.e., these points are intersections of isotomic cevians).

3) Consider the excenters $I_\alpha, I_\beta, I_\gamma$ of triangle $ABC$. It is not difficult to see that the coordinates of these points are

$$z_{I_\alpha} = -\frac{\alpha}{2(s-\alpha)}a + \frac{\beta}{2(s-\beta)}b + \frac{\gamma}{2(s-\gamma)}c,$$

$$z_{I_\beta} = \frac{\alpha}{2(s-\alpha)}a - \frac{\beta}{2(s-\beta)}b + \frac{\gamma}{2(s-\gamma)}c,$$

$$z_{I_\gamma} = \frac{\alpha}{2(s-\alpha)}a + \frac{\beta}{2(s-\beta)}b - \frac{\gamma}{2(s-\gamma)}c.$$

From the formula in Theorem 1, it follows that

$$\text{area}[I_\alpha I_\beta I_\gamma] = \begin{vmatrix} -\dfrac{\alpha}{2(s-\alpha)} & \dfrac{\beta}{2(s-\beta)} & \dfrac{\gamma}{2(s-\gamma)} \\ \dfrac{\alpha}{2(s-\alpha)} & -\dfrac{\beta}{2(s-\beta)} & \dfrac{\gamma}{2(s-\gamma)} \\ \dfrac{\alpha}{2(s-\alpha)} & \dfrac{\beta}{2(s-\beta)} & -\dfrac{\gamma}{2(s-\gamma)} \end{vmatrix} \text{area}[ABC]$$

$$= \frac{\alpha\beta\gamma}{8(s-\alpha)(s-\beta)(s-\gamma)} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \operatorname{area}[ABC]$$

$$= \frac{s\alpha\beta\gamma \operatorname{area}[ABC]}{2s(s-\alpha)(s-\beta)(s-\gamma)} = \frac{s\alpha\beta\gamma \operatorname{area}[ABC]}{2\operatorname{area}^2[ABC]} = \frac{2s\alpha\beta\gamma}{4\operatorname{area}[ABC]} = 2sR.$$

4) (Nagel line.) Using the formula in Theorem 1, we give a different proof for the so-called Nagel line: the points $I$, $G$, $N$ are collinear. We have seen that the coordinates of these points are

$$z_I = \frac{\alpha}{2s}a + \frac{\beta}{2s}b + \frac{\gamma}{2s}c,$$

$$z_G = \frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c,$$

$$z_N = \left(1 - \frac{\alpha}{s}\right)a + \left(1 - \frac{\beta}{s}\right)b + \left(1 - \frac{\gamma}{s}\right)c.$$

Then

$$\operatorname{area}[IGN] = \begin{vmatrix} \frac{\alpha}{2s} & \frac{\beta}{2s} & \frac{\gamma}{2s} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 - \frac{\alpha}{s} & 1 - \frac{\beta}{s} & 1 - \frac{\gamma}{s} \end{vmatrix} \cdot \operatorname{area}[ABC] = 0,$$

hence the points $I$, $G$, $N$ are collinear.

## 4.9 Orthopolar Triangles

### 4.9.1 The Simson–Wallance line and the pedal triangle

Consider the triangle $ABC$, and let $M$ be a point situated in the triangle plane. Let $P$, $Q$, $R$ be the projections of $M$ onto lines $BC$, $CA$, $AB$, respectively.

**Theorem 1.** (The Simson[7] line[8]) *The points $P$, $Q$, $R$ are collinear if and only if $M$ is on the circumcircle of triangle $ABC$.*

[7]Robert Simson (1687–1768), Scottish mathematician.

[8]This line was attributed to Simson by Poncelet, but is now frequently known as the Simson–Wallance line since it does not actually appear in any work of Simson. William Wallance (1768–1843) was also a Scottish mathematician, who possibly published the theorem above concerning the Simson line in 1799.

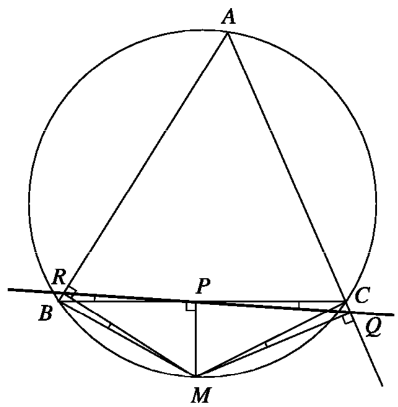


**Figure 4.10.**

*Proof.* We will give a standard geometric argument.

Suppose that $M$ lies on the circumcircle of triangle $ABC$. Without loss of generality we may assume that $M$ is on the arc $\overset{\frown}{BC}$. In order to prove the collinearity of $R$, $P$, $Q$, it suffices to show that the angles $\widehat{BPR}$ and $\widehat{CPQ}$ are congruent. The quadrilaterals $PRBM$ and $PCQM$ are cyclic (since $\widehat{BRM} \equiv \widehat{BPM}$ and $\widehat{MPC} + \widehat{MQC} = 180°$), hence we have $\widehat{BPR} \equiv \widehat{BMR}$ and $\widehat{CPQ} \equiv \widehat{CMQ}$. But $\widehat{BMR} = 90° - \widehat{ABM} = 90° - \widehat{MCQ}$, since the quadrilateral $ABMC$ is cyclic too. Finally, we obtain $\widehat{BMR} = 90° - \widehat{MCQ} = \widehat{CMQ}$, so the angles $\widehat{BPR}$ and $\widehat{CPQ}$ are congruent.

To prove the converse, we note that if the points $P$, $Q$, $R$ are collinear, then the angles $\widehat{BPR}$ and $\widehat{CPQ}$ are congruent, hence $\widehat{ABM} + \widehat{ACM} = 180°$, i.e., the quadrilateral $ABMC$ is cyclic. Therefore the point $M$ is situated on the circumcircle of triangles $ABC$. □

When $M$ lies on the circumcircle of triangle $ABC$, the line in the above theorem is called the *Simson–Wallance line of M* with respect to triangle $ABC$.

We continue with a nice generalization of the property contained in Theorem 1. For an arbitrary point $X$ in the plane of triangle $ABC$ consider its projections $P$, $Q$ and $R$ on the lines $BC$, $CA$ and $AB$, respectively.

The triangle $PQR$ is called the *pedal triangle* of point $X$ with respect to the triangle $ABC$. Let us choose the circumcenter $O$ of triangle $ABC$ as the origin of the complex plane.

**Theorem 2.** *The area of the pedal triangle of $X$ with respect to the triangle $ABC$ is given by*

$$\text{area}[PQR] = \frac{\text{area}[ABC]}{4R^2}|x\overline{x} - R^2| \qquad (1)$$

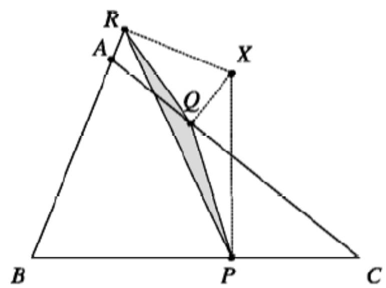


**Figure 4.11.**

*where $R$ is the circumradius of triangle $ABC$.*

*Proof.* Applying the formula in Proposition 1, Section 4.5, we obtain the coordinates $p, q, r$ of the points $P$, $Q$, $R$, respectively:

$$p = \frac{1}{2}\left(x - \frac{bc}{R^2}\overline{x} + b + c\right),$$

$$q = \frac{1}{2}\left(x - \frac{ca}{R^2}\overline{x} + c + a\right),$$

$$r = \frac{1}{2}\left(x - \frac{ab}{R^2}\overline{x} + a + b\right).$$

Taking into account the formula in Section 2.5.3 we have

$$\text{area}[PQR] = \frac{i}{4}\begin{vmatrix} p & \overline{p} & 1 \\ q & \overline{q} & 1 \\ r & \overline{r} & 1 \end{vmatrix} = \frac{i}{4}\begin{vmatrix} q-p & \overline{q}-\overline{p} \\ r-p & \overline{r}-\overline{p} \end{vmatrix}.$$

For the coordinates $p, q, r$ we obtain

$$\overline{p} = \frac{1}{2}\left(\overline{x} - \frac{\overline{b}\,\overline{c}}{R^2}x + \overline{b} + \overline{c}\right),$$

$$\overline{q} = \frac{1}{2}\left(\overline{x} - \frac{\overline{c}\,\overline{a}}{R^2}x + \overline{c} + \overline{a}\right),$$

$$\overline{r} = \frac{1}{2}\left(\overline{x} - \frac{\overline{a}\,\overline{b}}{R^2}x + \overline{a} + \overline{b}\right).$$

It follows that

$$q - p = \frac{1}{2}(a-b)\left(1 - \frac{c\overline{x}}{R^2}\right) \text{ and } r - p = \frac{1}{2}(a-c)\left(1 - \frac{b\overline{x}}{R^2}\right), \tag{2}$$

$$\overline{q} - \overline{p} = \frac{1}{2abc}(a-b)(x-c)R^2 \quad \text{and} \quad \overline{r} - \overline{p} = \frac{1}{2abc}(a-c)(x-b)R^2.$$

Therefore

$$\begin{aligned}
\text{area}[PQR] &= \frac{i}{4}\begin{vmatrix} q-p & \overline{q}-\overline{p} \\ r-p & \overline{r}-\overline{p} \end{vmatrix} \\
&= \frac{i(a-b)(a-c)}{16abc}\begin{vmatrix} 1-\dfrac{c\overline{x}}{R^2} & (x-c)R^2 \\ & \\ 1-\dfrac{b\overline{x}}{R^2} & (x-b)R^2 \end{vmatrix} \\
&= \frac{i(a-b)(a-c)}{16abc}\begin{vmatrix} R^2-c\overline{x} & x-c \\ R^2-b\overline{x} & x-b \end{vmatrix} \\
&= \frac{i(a-b)(a-c)}{16abc}\begin{vmatrix} (b-c)\overline{x} & b-c \\ R^2-b\overline{x} & x-b \end{vmatrix} \\
&= \frac{i(a-b)(b-c)(a-c)}{16abc}\begin{vmatrix} \overline{x} & 1 \\ R^2-b\overline{x} & x-b \end{vmatrix} \\
&= \frac{i(a-b)(b-c)(a-c)}{16abc}(x\overline{x}-R^2).
\end{aligned}$$

Proceeding to moduli we find that

$$\begin{aligned}
\text{area}[PQR] &= \frac{|a-b||b-c||c-a|}{16|a||b||c|}|x\overline{x}-R^2| = \frac{\alpha\beta\gamma}{16R^3}|x\overline{x}-R^2| \\
&= \frac{\text{area}[ABC]}{4R^2}|x\overline{x}-R^2|,
\end{aligned}$$

where $\alpha$, $\beta$, $\gamma$ are the length of sides of triangle $ABC$. □

**Remarks.** 1) The formula in Theorem 2 contains the Simson–Wallance line property. Indeed, points $P$, $Q$, $R$ are collinear if and only if $\text{area}[PQR] = 0$. That is, $|x\overline{x} - R^2| = 0$, i.e., $x\overline{x} = R^2$. It follows that $|x| = R$, so $X$ lies on the circumcircle of triangle $ABC$.

2) If $X$ lies on a circle of radius $R_1$ and center $O$ (the circumcenter of triangle $ABC$), then $x\overline{x} = R_1^2$, and from Theorem 2 we obtain

$$\text{area}[PQR] = \frac{\text{area}[ABC]}{4R^2}|R_1^2 - R^2|.$$

It follows that the area of triangle $PQR$ does not depend on the point $X$.

The converse is also true. The locus of all points $X$ in the plane of triangle $ABC$ such that $\text{area}[PQR] = k$ (constant) is defined by

$$|x\overline{x} - R^2| = \frac{4R^2k}{\text{area}[ABC]}.$$

This is equivalent to

$$|x|^2 = R^2 \pm \frac{4R^2k}{\text{area}[ABC]} = R^2\left(1 \pm \frac{4k}{\text{area}[ABC]}\right).$$

If $k > \frac{1}{4}\text{area}[ABC]$, then the locus is a circle of center $O$ and radius $R_1 = R\sqrt{1 + \frac{4k}{\text{area}[ABC]}}$.

If $k \leq \frac{1}{4}\text{area}[ABC]$, then the locus consists of two circles of center $O$ and radii $R\sqrt{1 \pm \frac{4k}{\text{area}[ABC]}}$, one of which degenerated to $O$ when $k = \frac{1}{4}\text{area}[ABC]$.

**Theorem 3.** *For any point $X$ in the plane of triangle $ABC$, we can construct a triangle with sides $AX \cdot BC$, $BX \cdot CA$, $CX \cdot AB$. This triangle is then similar to the pedal triangle of point $X$ with respect to the triangle $ABC$.*

*Proof.* Let $PQR$ be the pedal triangle of $X$ with respect to triangle $ABC$. From formula (2) we obtain

$$q - p = \frac{1}{2}(a-b)(x-c)\frac{R^2 - c\overline{x}}{R^2(x-c)}. \tag{3}$$

Proceeding to moduli in (3), it follows that

$$|q - p| = \frac{1}{2R^2}|a-b||x-c|\left|\frac{R^2 - c\overline{x}}{x-c}\right|. \tag{4}$$

On the other hand,

$$\left|\frac{R^2 - c\overline{x}}{x-c}\right|^2 = \frac{R^2 - c\overline{x}}{x-c} \cdot \frac{R^2 - \overline{c}x}{\overline{x}-\overline{c}} = \frac{R^2 - c\overline{x}}{x-c} \cdot \frac{R^2 - \overline{c}x}{\overline{x} - \frac{R^2}{c}}$$

$$= \frac{R^2 - c\overline{x}}{x-c} \cdot \frac{R^2(c-x)}{c\overline{x} - R^2} = R^2,$$

hence from (4) we derive the relation

$$|q - p| = \frac{1}{2R}|a-b||x-c|. \tag{5}$$

Therefore

$$\frac{PQ}{CX \cdot AB} = \frac{QR}{AX \cdot BC} = \frac{RP}{BX \cdot CA} = \frac{1}{2R}, \tag{6}$$

and the conclusion follows. □

**Corollary 4.** *In the plane of triangle $ABC$ consider the point $X$ and denote by $A'B'C'$ the triangle with sides $AX \cdot BC$, $BX \cdot CA$, $CX \cdot AB$. Then*

$$\text{area}[A'B'C'] = \text{area}[ABC]|x\overline{x} - R^2|. \tag{7}$$

*Proof.* From formula (6) it follows that $\text{area}[A'B'C'] = 4R^2\text{area}[PQR]$, where $PQR$ is the pedal triangle of $X$ with respect to triangle $ABC$. Replacing this result in (1), we find the desired formula. □

**Corollary 5.** (Ptolemy's inequality) *For any quadrilateral $ABCD$ the following inequality holds:*

$$AC \cdot BD \leq AB \cdot CD + BC \cdot AD. \tag{8}$$

**Corollary 6.** (Ptolemy's theorem) *The convex quadrilateral $ABCD$ is cyclic if and only if*

$$AC \cdot BD = AB \cdot CD + BC \cdot AD. \tag{9}$$

*Proof.* If the relation (9) holds, then triangle $A'B'C'$ in Corollary 4 is degenerate; i.e., $\text{area}[A'B'C'] = 0$. From formula (7) it follows that $d \cdot \overline{d} = R^2$, where $d$ is the coordinate of $D$ and $R$ is the circumradius of triangle $ABC$. Hence the point $D$ lies on the circumcircle of triangle $ABC$.

If quadrilateral $ABCD$ is cyclic, then the pedal triangle of point $D$ with respect to triangle $ABC$ is degenerate. From (6) we obtain the relation (9). □

**Corollary 7.** (Pompeiu's Theorem[9]) *For any point $X$ in the plane of the equilateral triangle $ABC$, three segments with lengths $XA$, $XB$, $XC$ can be taken as the sides of a triangle.*

*Proof.* In Theorem 3 we have $BC = CA = AB$ and the desired conclusion follows. □

The triangle in Corollary 7 is called the *Pompeiu triangle* of $X$ with respect to the equilateral triangle $ABC$. This triangle is degenerate if and only if $X$ lies on the circumcircle of $ABC$. Using the second part of Theorem 3 we find that Pompeiu's triangle of point $X$ is similar to the pedal triangle of $X$ with respect to triangle $ABC$ and

$$\frac{CX}{PQ} = \frac{AX}{QR} = \frac{BX}{RP} = \frac{2R}{\alpha} = \frac{2\sqrt{3}}{3}. \tag{10}$$

**Problem 1.** *Let $A$, $B$ and $C$ be equidistant points on the circumference of a circle of unit radius centered at $O$, and let $X$ be any point in the circle's interior. Let $d_A$, $d_B$, $d_C$ be the distances from $X$ to $A$, $B$, $C$, respectively. Show that there is a triangle with*

[9]Dimitrie Pompeiu (1873–1954), Romanian mathematician, made important contributions in the fields of mathematical analysis, functions of a complex variable, and rational mechanics.

*sides $d_A, d_B, d_C$, and the area of this triangle depends only on the distance from $X$ to $O$.*

(2003 Putnam Mathematical Competition)

**Solution.** The first assertion is just the property contained in Corollary 7. Taking into account the relations (10), it follows that the area of Pompeiu's triangle of point $X$ is $\frac{2}{3}\text{area}[PQR]$. From Theorem 2 we get that $\text{area}[PQR]$ depends only on the distance from $P$ to $O$, as desired.

**Problem 2.** *Let $X$ be a point in the plane of the equilateral triangle $ABC$ such that $X$ does not lie on the circumcircle of triangle $ABC$, and let $XA = u$, $XB = v$, $XC = w$. Express the length side $\alpha$ of triangle $ABC$ in terms of real numbers $u, v, w$.*

(1978 GDR Mathematical Olympiad)

**Solution.** The segments $[XA], [XB], [XC]$ are the sides of Pompeiu's triangle of point $X$ with respect to equilateral triangle $ABC$. Denote this triangle by $A'B'C'$. From relations (10) and from Theorem 2 it follows that

$$\text{area}[A'B'C'] = \left(\frac{2\sqrt{3}}{3}\right)^2 \text{area}[PQR] = \frac{1}{3R^2}\text{area}[ABC]|x \cdot \overline{x} - R^2|$$

$$= \frac{1}{3R^2} \cdot \frac{\alpha^2\sqrt{3}}{4}|x \cdot \overline{x} - R^2| = \frac{\sqrt{3}}{4}|XO^2 - R^2|. \qquad (11)$$

On the other hand, using the well-known formula of Hero we obtain, after a few simple computations:

$$\text{area}[A'B'C'] = \frac{1}{4}\sqrt{(u^2 + v^2 + w^2)^2 - 2(u^4 + v^4 + w^4)}.$$

Substituting in (11) we find

$$|XO^2 - R^2| = \frac{1}{\sqrt{3}}\sqrt{(u^2 + v^2 + w^2)^2 - 2(u^4 + v^4 + w^4)}. \qquad (12)$$

Now we consider the following two cases:

**Case 1.** If $X$ lies in the interior of the circumcircle of triangle $ABC$, then $XO^2 < R^2$. Using the relation (see also formula (4) in Section 4.11)

$$XO^2 = \frac{1}{3}(u^2 + v^2 + w^2 - 3R^2),$$

from (12) we find that

$$2R^2 = \frac{1}{3}(u^2 + v^2 + w^2) + \frac{1}{\sqrt{3}}\sqrt{(u^2 + v^2 + w^2)^2 - 2(u^4 + v^4 + w^4)},$$

hence

$$\alpha^2 = \frac{1}{2}(u^2 + v^2 + w^2) + \frac{\sqrt{3}}{2}\sqrt{(u^2 + v^2 + w^2)^2 - 2(u^4 + v^4 + w^4)}.$$

**Case 2.** If $X$ lies in the exterior of circumcircle of triangle $ABC$, then $XO^2 > R^2$ and after some similar computations we find

$$\alpha^2 = \frac{1}{2}(u^2 + v^2 + w^2) - \frac{\sqrt{3}}{2}\sqrt{(u^2 + v^2 + w^2)^2 - 2(u^4 + v^4 + w^4)}.$$

### 4.9.2 Necessary and sufficient conditions for orthopolarity

Consider a triangle $ABC$ and points $X, Y, Z$ situated on its circumcircle. Triangles $ABC$ and $XYZ$ are called *orthopolar triangles* (or *S-triangles*)[10] if the Simson–Wallance line of point $X$ with respect to triangle $ABC$ is perpendicular (orthogonal) to line $YZ$.

Let us choose the circumcenter $O$ of triangle $ABC$ at the origin of the complex plane. Points $A$, $B$, $C$, $X$, $Y$, $Z$ have the coordinates $a, b, c, x, y, z$ with

$$|a| = |b| = |c| = |x| = |y| = |z| = R,$$

where $R$ is the circumradius of the triangle $ABC$.

**Theorem 3.** *Triangles $ABC$ and $XYZ$ are orthopolar triangles if and only if $abc = xyz$.*

*Proof.* Let $P$, $Q$, $R$ be the feet of the orthogonal lines from the point $X$ to the lines $BC$, $CA$, $AB$, respectively.

Points $P$, $Q$, $R$ are on the same line; that is, the Simson–Wallance line of point $X$ with respect to triangle $ABC$.

The coordinates of $P$, $Q$, $R$ are denoted by $p, q, r$, respectively. Using the formula in Proposition 1, Section 4.5, we have

$$p = \frac{1}{2}\left(x - \frac{bc}{R^2}\overline{x} + b + c\right)$$
$$q = \frac{1}{2}\left(x - \frac{ca}{R^2}\overline{x} + c + a\right),$$
$$r = \frac{1}{2}\left(x - \frac{ab}{R^2}\overline{x} + a + b\right).$$

We study two cases.

---

[10] This definition was given in 1915 by Romanian mathematician Traian Lalescu (1882–1929). He is famous for his book *La géometrie du triangle* published by Librairie Vuibert, Paris, 1937.

**Case 1.** Point $X$ is not a vertex of triangle $ABC$.

Then $PQ$ is orthogonal to $YZ$ if and only if $(p-q)\cdot(y-z)=0$. That is,

$$\left[(b-a)\left(1-\frac{c\overline{x}}{R^2}\right)\right]\cdot(y-z)=0$$

or

$$(\overline{b}-\overline{a})(R^2-\overline{c}x)(y-z)+(b-a)(R^2-c\overline{x})(\overline{y}-\overline{z})=0.$$

We obtain

$$\left(\frac{R^2}{b}-\frac{R^2}{a}\right)\left(R^2-\frac{R^2}{c}x\right)(y-z)+(b-a)\left(R^2-c\frac{R^2}{x}\right)\left(\frac{R^2}{y}-\frac{R^2}{z}\right)=0,$$

hence

$$\frac{1}{abc}(a-b)(c-x)(y-z)-\frac{1}{xyz}(a-b)(c-x)(y-z)=0.$$

The last relation is equivalent to

$$(abc-xyz)(a-b)(c-x)(y-z)=0$$

and finally we get $abc=xyz$, as desired.

**Case 2.** Point $X$ is a vertex of triangle $ABC$. Without loss of generality, assume that $X=B$.

Then the Simson–Wallance line of point $X=B$ is the orthogonal line from $B$ to $AC$. It follows that $BQ$ is orthogonal to $YZ$ if and only if lines $AC$ and $YZ$ are parallel. This is equivalent to $ac=yz$. Because $b=x$, we obtain $abc=xyz$, as desired. □

**Remark.** Due to the symmetry of the relation $abc=xyz$, we observe that the Simson–Wallance line of any vertex of triangle $XYZ$ with respect to $ABC$ is orthogonal to the opposite side of the triangle $XYZ$. Moreover, the same property holds for the vertices of triangle $ABC$.

Hence $ABC$ and $XYZ$ are orthopolar triangles if and only if $XYZ$ and $ABC$ are orthopolar triangles. Therefore the orthopolarity relation is symmetric.

**Problem 1.** *The median and the orthic triangles of a triangle $ABC$ are orthopolar in the nine-point circle.*

**Solution.** Consider the origin of the complex plane at the circumcenter $O$ of triangle $ABC$. Let $M, N, P$ be the midpoints of $AB, BC, CA$ and let $A', B', C'$ be the feet of the altitudes of triangles $ABC$ from $A, B, C$, respectively.

If $m, n, p, a', b', c'$ are coordinates of $M, N, P, A', B', C'$ then we have

$$m=\frac{1}{2}(a+b),\quad n=\frac{1}{2}(b+c),\quad p=\frac{1}{2}(c+a)$$

and

$$a' = \frac{1}{2}\left(a + b + c - \frac{bc}{R^2}\overline{a}\right) = \frac{1}{2}\left(a + b + c - \frac{bc}{a}\right),$$

$$b' = \frac{1}{2}\left(a + b + c - \frac{ca}{b}\right), \quad c' = \frac{1}{2}\left(a + b + c - \frac{ab}{2}\right).$$

The nine-point center $O_9$ is the midpoint of the segment $OH$, where $H(a + b + c)$ is the orthocenter of triangle $ABC$. The coordinate of $O_9$ is $\omega = \frac{1}{2}(a + b + c)$.

Now observe that

$$(a - \omega)(b - \omega)(c - \omega) = (m - \omega)(n - \omega)(p - \omega) = \frac{1}{8}abc,$$

and the claim is proved.

**Problem 2.** *The altitudes of triangle $ABC$ meet its circumcircle at points $A_1, B_1, C_1$, respectively. If $A_1', B_1', C_1'$ are the antipodal points of $A_1, B_1, C_1$ on the circumcircle $ABC$, then $ABC$ and $A_1'B_1'C_1'$ are orthopolar triangles.*

**Solution.** The coordinates of $A_1, B_1, C_1$ are $-\frac{bc}{a}, -\frac{ca}{b}, -\frac{ab}{c}$, respectively. Indeed, the equation of line $AH$ in terms of the real product is $AH : (z - a) \cdot (b - c) = 0$. It suffices to show that the point with coordinate $-\frac{bc}{a}$ lies both on $AH$ and on the circumcircle of triangle $ABC$. First, let us note that $\left|-\frac{bc}{a}\right| = \frac{|b||c|}{|a|} = \frac{R \cdot R}{R} = R$, hence this point is situated on the circumcircle of triangle $ABC$. Now, we show that the complex number $-\frac{bc}{a}$ satisfies the equation of the line $AH$. This is equivalent to

$$\left(\frac{bc}{a} + a\right) \cdot (b - c) = 0.$$

Using the definition of the real product, this reduces to

$$\left(\frac{\overline{b}\,\overline{c}}{\overline{a}} + \overline{a}\right)(b - c) + \left(\frac{bc}{a} + a\right)(\overline{b} - \overline{c}) = 0$$

or

$$\left(\frac{a\overline{b}\,\overline{c}}{R^2} + \overline{a}\right)(b - c) + \left(\frac{bc}{a} + a\right)\left(\frac{R^2}{b} - \frac{R^2}{c}\right) = 0.$$

Finally, this comes down to

$$(b - c)\left(\frac{a\overline{bc}}{R^2} + \overline{a} - \frac{R^2}{a} - \frac{aR^2}{bc}\right) = 0,$$

a relation that is clearly true.

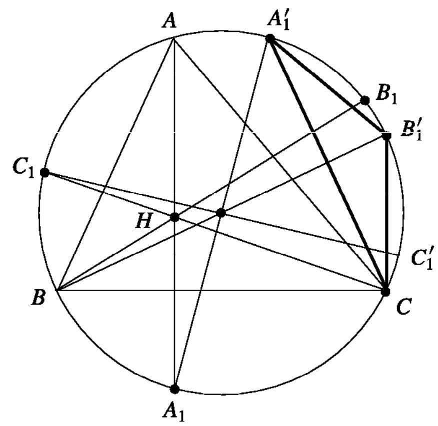


**Figure 4.12.**

It follows that $A_1'$, $B_1'$, $C_1'$ have coordinates $\frac{bc}{a}$, $\frac{ca}{b}$, $\frac{ab}{c}$, respectively. Because

$$\frac{bc}{a} \cdot \frac{ca}{c} \cdot \frac{ab}{c} = abc,$$

we obtain that the triangles $ABC$ and $A_1'B_1'C_1'$ are orthopolar.

**Problem 3.** *Let $P$ and $P'$ be distinct points on the circumcircle of triangle $ABC$ such that lines $AP$ and $AP'$ are symmetric with respect to the bisector of angle $\widehat{BAC}$. Then triangles $ABC$ and $APP'$ are orthopolar.*

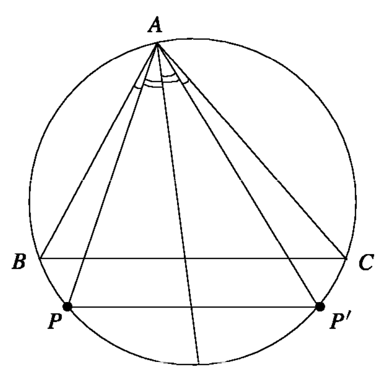


**Figure 4.13.**

**Solution.** Let us consider $p$ and $p'$ the coordinates of points $P$ and $P'$, respectively. It is clear that the lines $PP'$ and $BC$ are parallel. Using the complex product, it follows that $(p - p') \times (b - c) = 0$. This relation is equivalent to

$$(p - p')(\overline{b} - \overline{c}) - (\overline{p} - \overline{p'})(b - c) = 0.$$

Considering the origin of the complex plane at the circumcenter $O$ of triangle $ABC$, we have

$$(p - p')\left(\frac{R^2}{b} - \frac{R^2}{c}\right) - \left(\frac{R^2}{p} - \frac{R^2}{p'}\right)(b - c) = 0,$$

so

$$R^2(p - p')(b - c)\left(\frac{1}{bc} - \frac{1}{pp'}\right) = 0.$$

Therefore $bc = pp'$, i.e., $abc = app'$. From Theorem 3 it follows that $ABC$ and $APP'$ are orthopolar triangles.

## 4.10 Area of the Antipedal Triangle

Consider a triangle $ABC$ and a point $M$. The perpendicular lines from $A, B, C$ to $MA, MB, MC$, respectively, determine a triangle; we call this triangle the *antipedal* triangle of $M$ with respect to $ABC$.

Recall that $M'$ is the *isogonal point* of $M$ if the pairs of lines $AM, AM'$; $BM, BM'$; $CM, CM'$ are isogonal, i.e., the following relations hold: $\widehat{MAC} \equiv \widehat{M'AB}$, $\widehat{MBC} \equiv \widehat{M'BA}$, $\widehat{MCA} \equiv \widehat{M'CB}$.

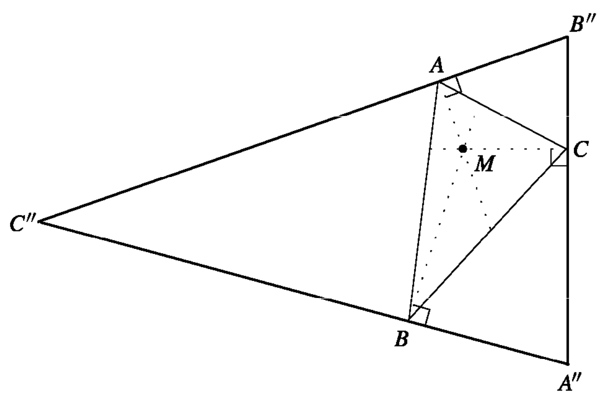


**Figure 4.14.**

**Theorem.** *Consider $M$ a point in the plane of triangle $ABC$, $M'$ the isogonal point of $M$ and $A''B''C''$ the antipedal triangle of $M$ with respect to $ABC$. Then*

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|R^2 - OM'^2|}{4R^2} = \frac{|\rho(M')|}{4R^2},$$

*where $\rho(M')$ is the power of $M'$ with respect to the circumcircle of triangle $ABC$.*

*Proof.* Consider point $O$ the origin of the complex plane and let $m, a, b, c$ be the coordinates of $M, A, B, C$. Then

$$R^2 = a\overline{a} = b\overline{b} = c\overline{c} \text{ and } \rho(M) = R^2 - m\overline{m}. \tag{1}$$

Let $O_1, O_2, O_3$ be the circumcenters of triangles $BMC, CMA, AMB$, respectively. It is easy to verify that $O_1, O_2, O_3$ are the midpoints of segments $MA''$, $MB''$, $MC''$, respectively, and so

$$\frac{\text{area}[O_1O_2O_3]}{\text{area}[A''B''C'']} = \frac{1}{4}. \tag{2}$$

The coordinate of the circumcenter of the triangle with vertices with coordinates $z_1, z_2, z_3$ is given by the following formula (see formula (1) in Subsection 3.6.1):

$$z_O = \frac{z_1\overline{z_1}(z_2 - z_3) + z_2\overline{z_2}(z_3 - z_1) + z_3\overline{z_3}(z_1 - z_2)}{\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix}}.$$

The bisector line of the segment $[z_1, z_2]$ has the following equation in terms of real product: $\left[z - \frac{1}{2}(z_1 + z_2)\right] \cdot (z_1 - z_2) = 0$. It is sufficient to check that $z_O$ satisfies this equation as this implies, by symmetry, that $z_O$ belongs to the perpendicular bisectors of segments $[z_2, z_3]$ and $[z_3, z_1]$.

The coordinate of $O_1$ is

$$z_{O_1} = \frac{m\overline{m}(b - c) + b\overline{b}(c - m) + c\overline{c}(m - b)}{\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}}$$

$$= \frac{(R^2 - m\overline{m})(c - b)}{\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}} = \frac{\rho(M)(c - b)}{\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}}.$$

Let

$$\Delta = \begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}$$

and consider

$$\alpha = \frac{1}{\Delta}\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}, \quad \beta = \frac{1}{\Delta}\begin{vmatrix} m & \overline{m} & 1 \\ c & \overline{c} & 1 \\ a & \overline{a} & 1 \end{vmatrix},$$

and

$$\gamma = \frac{1}{\Delta}\begin{vmatrix} m & \overline{m} & 1 \\ a & \overline{a} & 1 \\ b & \overline{b} & 1 \end{vmatrix}.$$

With this notation we obtain

$$\begin{aligned}
&(\alpha a + \beta b + \gamma c)\cdot\Delta \\
&= \sum_{\text{cyc}} m(a\overline{b} - a\overline{c}) - \sum_{\text{cyc}} \overline{m}(ab - ac) + \sum_{\text{cyc}} a(b\overline{c} - \overline{b}c) \\
&= m\Delta - \overline{m}\cdot 0 + \sum_{\text{cyc}} a\left(b\frac{R^2}{c} - \frac{R^2}{c}a\right) \\
&= m\Delta + R^2\sum_{\text{cyc}}\left(\frac{ab}{c} - \frac{ac}{b}\right) = m\Delta,
\end{aligned}$$

and consequently

$$\alpha a + \beta b + \gamma c = m,$$

since it is clear that $\Delta \neq 0$.

We note that $\alpha, \beta, \gamma$ are real numbers and $\alpha + \beta + \gamma = 1$, so $\alpha, \beta, \gamma$ are the barycentric coordinates of point $M$.

Since

$$z_{O_1} = \frac{(c-b)\cdot\rho(M)}{\alpha\cdot\Delta}, \quad z_{O_2} = \frac{(c-a)\cdot\rho(M)}{\beta\Delta}, \quad z_{O_3} = \frac{(a-b)\cdot\rho(M)}{\gamma\cdot\Delta},$$

we have

$$\frac{\text{area}[O_1O_2O_3]}{\text{area}[ABC]} = \left|\frac{\frac{i}{4}\begin{vmatrix} z_{O_1} & \overline{z_{O_1}} & 1 \\ z_{O_2} & \overline{z_{O_2}} & 1 \\ z_{O_3} & \overline{z_{O_3}} & 1 \end{vmatrix}}{\frac{i}{4}\cdot\Delta}\right|$$

$$= \left|\frac{1}{\Delta}\cdot\frac{\rho^2(M)}{\Delta^2}\cdot\frac{1}{\alpha\beta\gamma}\begin{vmatrix} b-c & \overline{b}-\overline{c} & \alpha \\ c-a & \overline{c}-\overline{a} & \beta \\ a-b & \overline{a}-\overline{b} & \gamma \end{vmatrix}\right|$$

$$= \left|\frac{\rho^2(M)}{\Delta^3}\cdot\frac{1}{\alpha\beta\gamma}\cdot\begin{vmatrix} c-a & \overline{c}-\overline{a} \\ a-b & \overline{a}-\overline{b} \end{vmatrix}\right|$$

$$= \left|\frac{\rho^2(M)}{\Delta^3}\cdot\frac{1}{\alpha\beta\gamma}\cdot\Delta\right| = \left|\frac{\rho^2(M)}{\Delta^2}\cdot\frac{1}{\alpha\beta\gamma}\right|. \tag{3}$$

Relations (2) and (3) imply that

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|\Delta^2\alpha\beta\gamma|}{4\rho^2(M)}. \tag{4}$$

Because $\alpha$, $\beta$, $\gamma$ are the barycentric coordinates of $M$, it follows that

$$z_M = \alpha z_A + \beta z_B + \gamma z_C.$$

Using the real product we find that

$$\begin{aligned}
OM^2 = z_M \cdot z_M &= (\alpha z_A + \beta z_B + \gamma z_C) \cdot (\alpha z_A + \beta z_B + \gamma z_C) \\
&= (\alpha^2 + \beta^2 + \gamma^2)R^2 + 2\sum_{\text{cyc}} \alpha\beta z_A \cdot z_B \\
&= (\alpha^2 + \beta^2 + \gamma^2)R^2 + 2\sum_{\text{cyc}} \alpha\beta \left(R^2 - \frac{AB^2}{2}\right) \\
&= (\alpha + \beta + \gamma)^2 R^2 - \sum_{\text{cyc}} \alpha\beta AB^2 = R^2 - \sum_{\text{cyc}} \alpha\beta AB^2.
\end{aligned}$$

Therefore the power of $M'$ with respect to the circumcircle of triangle $ABC$ can be expressed in the form

$$\rho(M) = R^2 - OM^2 = \sum_{\text{cyc}} \alpha\beta AB^2.$$

On the other hand, if $\alpha$, $\beta$, $\gamma$ are the barycentric coordinates of the point $M$, then its isogonal point $M'$ has the barycentric coordinates given by

$$\alpha' = \frac{\beta\gamma BC^2}{\beta\gamma BC^2 + \alpha\gamma CA^2 + \alpha\beta AB^2}, \quad \beta' = \frac{\gamma\alpha CA^2}{\beta\gamma BC^2 + \alpha\gamma CA^2 + \alpha\beta AB^2},$$

$$\gamma' = \frac{\alpha\beta AB^2}{\beta\gamma BC^2 + \alpha\gamma CA^2 + \alpha\beta AB^2}.$$

Therefore

$$\begin{aligned}
\rho(M') &= \sum_{\text{cyc}} \alpha'\beta' AB^2 \\
&= \frac{\alpha\beta\gamma AB^2 \cdot BC^2 \cdot CA^2}{(\beta\gamma BC^2 + \alpha\gamma CA^2 + \alpha\beta AB^2)^2} = \frac{\alpha\beta\gamma AB^2 BC^2 CA^2}{\rho^2(M)}.
\end{aligned} \tag{5}$$

On the other hand, we have

$$\Delta^2 = \left|\left(\frac{4}{i} \cdot \frac{i}{4}\Delta\right)^2\right| = \left|\frac{4}{i} \cdot \text{area}[ABC]\right|^2 = \frac{AB^2 \cdot BC^2 \cdot CA^2}{R^2}. \tag{6}$$

The desired conclusion follows from the relations (4), (5), and (6). □

**Applications.** 1) If $M$ is the orthocenter $H$, then $M'$ is the circumcenter $O$ and

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{R^2}{4R^2} = \frac{1}{4}.$$

2) If $M$ is the circumcenter $O$, then $M'$ is the orthocenter $H$ and we obtain

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|R^2 - OH^2|}{4R^2}.$$

Using the formula in Theorem 8, Subsection 4.6.4, it follows that

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|(2R + r)^2 - s^2|}{2R^2}.$$

3) If $M$ is the Lemoine point $K$, then $M'$ is the centroid $G$ and

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|R^2 - OG^2|}{4R^2}.$$

Applying the formula in Corollary 9, Subsection 4.6.4, then the first formula in Corollary 2, Subsection 4.6.1, it follows that

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{2(s^2 - r^2 - 4Rr)}{36R^2} = \frac{\alpha^2 + \beta^2 + \gamma^2}{36R^2}$$

where $\alpha$, $\beta$, $\gamma$ are the sides of triangle $ABC$.

From the inequality $\alpha^2 + \beta^2 + \gamma^2 \leq 9R^2$ (Corollary 10, Subsection 4.6.4) we obtain

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} \leq \frac{1}{4}.$$

4) If $M$ is the incenter $I$ of triangle $ABC$, then $M' = I$ and using Euler's formula $OI^2 = R^2 - 2Rr$ (see Theorem 4 in Subsection 4.6.2) we find that

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|R^2 - OI^2|}{4R^2} = \frac{2Rr}{4R^2} = \frac{r}{4R}.$$

Applying Euler's inequality $R \geq 2r$ (Corollary 5 in Subsection 4.6.2) it follows that

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} \leq \frac{1}{4}.$$

## 4.11 Lagrange's Theorem and Applications

Consider the distinct points $A_1(z_1), \ldots, A_n(z_n)$ in the complex plane. Let $m_1, \ldots, m_n$ be nonzero real numbers such that $m_1 + \cdots + m_n \neq 0$. Let $m = m_1 + \cdots + m_n$.